Paper presented to Topic Study Group 29 (TSG29) at the 10th International Congress on Mathematical Education (ICME-10). Copenhagen, Denmark; 4--11 July 2004.

THE ROLE OF INTUITION IN GEOMETRY EDUCATION: LEARNING FROM THE TEACHING PRACTICE IN THE EARLY 20TH CENTURY

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Intuition is often regarded as essential in the learning of geometry, but questions remain about how we might effectively develop students' such skills. This paper provides some results from analyses of innovative geometry teaching in the early part of the 20th century, a time when significant efforts were being made to improve the teaching and learning of geometry. As examples, we examine the tasks for students that can be found in Treutlein's "Geometrical Intuitive Instruction" (Germany) and Godfrey's geometry textbook (England). The analyses suggest that educators at that time attempted to develop students' intuitive skills through various practical tasks such as drawing, measurement, and imagining and manipulating figures, which could be useful for current geometry teaching. We also identify different approaches taken to the development mathematics teaching in Germany and England.

INTRODUCTION

Geometry is an attractive area in mathematics and it can provide us with interesting and challenging course content for the curriculum from primary to university level. However, the place of geometry has been a matter of debate for some considerable time. In the ICMI study in the teaching and learning of geometry, Villani concluded that 'it would be improper to claim that it is possible to elaborate a geometry curriculum having universal validity' (Villani, 1998, p. 321). A recent comparative study of geometry curricula (Hoyles, Foxman and Küchemann, 2002) also found considerable variations in current approaches to the design of the school geometry curriculum across a range of countries around the world. The study concludes by noting 'there is evidence of a state of flux in the geometry curriculum, with most countries looking to change' (*op cit* p. 121).

However, this state of flux means that there are opportunities to improve the specification of the curriculum for geometry. This paper focuses on the role of intuition in geometry for students in lower secondary schools, where intuition is usually considered as an essential skill in geometry (we discuss this later in this paper). It might be difficult to define 'intuition' precisely, but for the purposes of this paper we regard it as a skill to 'see' geometrical figures and solids, creating and manipulating them in the mind to solve problems in geometry (for more on this, see Fujita, Jones, and Yamamoto, 2004). Our approach is an historical reflection so that the purpose of this paper is to examine the roles of intuition in geometry discussed by educators in the early 20th century, a time when a great effort was made to improve the teaching of geometry. In particular, we report some findings from the analysis of writings by P. Treutlein (Germany) and a textbook by C. Godfrey (England), who acted to improve the teaching of geometry in the early 20th century. In what follows, we start from a brief description of the teaching of geometry around the 1900s, focusing on the discussion of practical approaches in geometry. We then analyse the tasks for students to be found in the writings and textbooks of Treutlein and Godfrey, and end with a general discussion about the importance of intuitive skills and how we might effectively develop such skills in the teaching and learning geometry.

THE TEACHING OF GEOMETRY IN THE EARLY 20TH CENTURY

In England, up to the beginning of 20th Century, the teaching of geometry always meant the direct teaching of Euclid. In 1871, the Association for the Improvement of Geometrical Teaching (the fore-runner of the Mathematical Association) was founded by University mathematicians and teachers from public schools with the aim of improving the teaching of geometry. To offer an alternative to Euclid's *Elements*, the methods of proof and the order of theorems were discussed by the members of this association. However, these efforts failed to change the teaching of geometry, partly because Cayley, the most powerful Cambridge mathematician at that time, opposed radical reform, and more importantly, because the Examination Boards such as Oxford and Cambridge were reluctant to revise their examination requirements (Siddons: 1936, p. 18, Griffiths: 1998, p. 196).

Yet the case for reform in the teaching of geometry remained current, mainly due to J. Perry, Professor of Engineering at the Royal College of Science, who gave an address 'The Teaching of Mathematics' at the British Association for the Advancement of Science (BAAS) meeting in Glasgow, UK in 1901. In the address, he denounced contemporary mathematics teaching, and he questioned the educational value of Euclidean geometry for all students. He emphasised the value of the introduction of experimental tasks in the early stages (Perry, 1902, pp. 158-81). Perry's address immediately caused discussions between mathematicians and teachers not only in the UK but also across the world (Griffiths and Howson, 1974, p. 15). For example, in the UK it was soon agreed that a practical and intuitive approach should be introduced in the early stages in geometry teaching in secondary schools, since students would attain the foundation of skills to tackle 'geometry' based on deductive proofs in later stages. Examples from individuals or subject associations include, for example Langlay, 1901; Mathematical Association (MA), 1902a, 1902b, 1902c; Godfrey, 1902; Godfrey and Siddons *et al*, 1902; Lodge, 1903 (also see in Price, 1994).

In Germany, the leading mathematician F. Klein also advocated that mathematics in schools be reorganised from a 'functional thinking' point of view (Klein, 1907). His suggestion had great influence on the 'Reform report on the teaching of mathematics and natural science' (Reformvorschlage fur den mathematischen und naturwissenschaftlichen Unterricht, Die Gessellschaft Deutscher Naturforscher und Arzte, 1905), usually called 'The Meran Report' in 1905 (Coleman, 1942, p. 63). In 'The Meran Report', practical and intuitive approaches were considered important for learning geometrical concepts, and the early stages of geometry were prescribed in the 'Propaedeutic geometry' (Propadeutische Raumlehre)' and 'Geometry' (Raumlehre). The former was for 11-year-old students, and started with observations of things in everyday life, and then the study of the basic concept of figures, the use of rulers and compasses, and drawing and measurement (*ibid*, 1905). More theoretical studies of geometrical figures followed, which was regarded as a formal geometry course for students aged from 13 to 17. The emphasis of 'Propedeutic geometry' and 'Geometry' was not only to develop the students' logical ways of thinking, but also included developing students' ability to see the real world mathematically. Thus, the development of the abilities such as 'spatial intuitive ability' (das raumliche Anschauungsvermogen) and 'functional thinking' (das funktionale Denken) were also emphasised.

TREUTLEIN'S GEOMETRICAL INTUITIVE INSTRUCTION

P. Treutlein[1] (1845-1912), a head master of the Realgymnasium in Karlsruhe, attempted to design his own 'Propedeutic Geometry' (Propadeutische Raumlehre), suggested by the Meran Report in 1905. Examples of his instruction can be found in *Geometrical Intuitive Instruction* (*Der geometrische Anschauungsunterricht als Unterstufe eines zweistufigen geometrischen Unterrichtes an unseren hohen Schulen*, 1911). According to the instruction, the teaching of geometry was organised for students aged

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[1] It might be strange that we, not from Germany, pay attention to Treutlein. In fact, his instruction was imported by Japanese teachers in the early 20th century, and strongly influenced Japanese geometry teaching in primary and lower secondary schools (cf.Yamamoto, 1999).

from 10 to 12, and comprised the practical study of plane and solid geometry, i.e. 'Propaedeutic Geometry' suggested in the Meran Report was extended to a two years course. Treutlein argued that it was very important to train students' 'imagination' through studying geometry, and in fact his *Geometrical Intuitive Instruction* particularly aimed at developing 'spatial intuitive skills' (das raumliche Anschauungsvermogen), which he considered as essential skills in geometry as well as in everyday life (*ibid*, S. 82). An example by which Treutlein attained this aim was an exercise on the 'Formation of new figures' (bilden neuer Formen) (see figure 1. below, *ibid*, S. 150[2]).

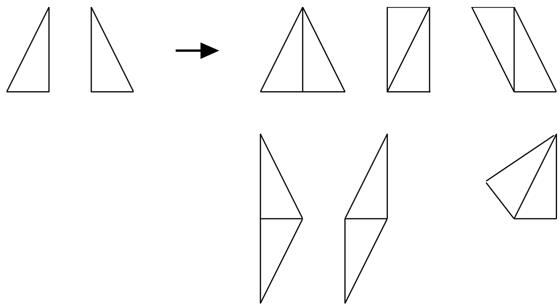

Figure 1. Example of 'Formation of new figures'

In the example in Figure 1, students were required to make new figures by using triangles. It is particularly important that they had to manipulate them in their head. Treutlein stated that 'these tasks aim to achieve the developmental introduction of new figures, and an utmost stimulation of students' spatial imagination (eine moglichste Anregung der Raumphantasie bei den Schulern)' (*ibid*, S. 111). Also, students would have to turn over figures (reflection) when they tried to make new figures, and it would make them be aware of thinking about plane figures in space (*ibid*, S. 119). Through the activity, they may be able to imagine all combinations of figures, or manipulate such activities in the mind, which Treutlein considered as important elements of spatial imagination[3].

Another example of the development of students' imagination skills is the study of the 'air cube' (der Luftwürfel), an imagined cube in the mind. For example, after studying a concrete model of a cube, students were required to imagine an 'air cube', and by using their hands to undertake tasks such as finding parallel sides, relational positions of sides etc (e.g. *ibid*, S. 122-5). Treutlein considered that students' geometrical images would be further enriched through activities with the 'air cube' after studying concrete models of cubes (*ibid*, S. 110).

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[2] A similar task can be found in Goldenberg *et al*, 1998, p. 10.

[3] Treutlien used the words 'die kombinierende Phantasie' and 'die anschauliche Phantasie'.

An interesting feature in Treutlein's instruction is that he strongly recommended that the teaching of geometry be started from concrete models of figures rather than observations of things in everyday life. Treutlein considered that it would not be suitable to use things in everyday life (e.g. desks, windows, etc.) to develop 'spatial intuitive skills' (das raumliche Anschauungsvermogen), because, firstly, such things would disturb students' concentration, and secondly, students would not find geometrical figures in the things unless they had images of such figures already in their minds (*ibid*, S. 100-1). That is to say, the things in everyday life could not be 'outer intuition' (ausser Anschauung), which would then directly form 'inner intuition (inner Anschauung)' in students' minds. The second point of Treutlein can be explained by using, for example, the case of doctors. Whereas doctors can diagnose various medical problems from X-ray pictures, we cannot. This is because they have prior knowledge about disease, and we do not. Treutlein hypothesised that unless students have the images of figures in their minds, they would not be able to find figures in things in everyday life. Thus, the teaching of geometry should start with the models of figures, and it was appropriate to give students the tasks by using the 'air cube' or 'making new figures', which would create 'inner intuition' in the mind.

GODFREY'S 'GEOMETRICAL EYE'

In the reform after Perry's address, C. Godfrey[4] (1876-1924) acted promptly to improve the teaching of (Euclidean style) geometry in England (e.g. Howson, 1973a, 1982). In 1903, Godfrey wrote a geometry textbook *Elementary Geometry* with A. W. Siddons (1876-1959), which soon became one of the most popular texts in secondary schools at that time in England[5]. Unlike Euclid's *Elements*, *Elementary Geometry* consists of two parts: Part I. Experimental Geometry and Part II. Theoretical Geometry. 'Experimental Geometry', which aimed at consolidating concepts of geometrical figures as well as leading students to discover various facts in geometry, mainly contains experimental tasks such as measurement or

[4] Again, it is worth considering the teaching of geometry by Godfrey in an international context. He had influenced on not only British, but also Japanese mathematics teaching in the early 20th century. The textbook *Elementary Geometry* written with A. W. Siddons, in 1903 was translated into Hindi, Bengali and Urdu (Howson, 1973b).

[5] At that time in England, secondary schools were for a few elite boys. According to Godfey and Siddons (1931), the teaching of geometry started between the ages of 10 and 12 (the first part of this book about teaching was written by Godfrey in around 1911, and discovered by Siddons after Godfrey's death in 1924).

drawing dealing with both plane and solid figures. In contrast, 'Theoretical Geometry', the main part of this text, consisted of propositions from Euclid's *Elements* with four consecutive books:: Book I Straight lines; Book II Areas; Book III Circles; Book IV Similarity[6]. The analysis of these books shows that there are plenty of experimental tasks not only in 'Experimental Geometry', but also 'Theoretical Geometry'. Most of them are introduced for the sake of discovery of geometrical facts as well as consolidating students' knowledge of geometrical figures.

The overall design of this textbook reflects Godfrey's pedagogical background. First of all, Godfrey agreed that mathematics could provide us with an opportunity to develop deductive reasoning (Godfrey and Siddons, 1931, p. 18). He considered, however, that a sensible blend of experimental, intuitive and deductive approaches should be achieved in the teaching of geometry (Godfrey and Siddons, 1931, pp. 20-1). In relation to 'intuition', it is interesting to see that he distinguished experimental and intuitive approaches.

> Experimental and intuitional methods are not identical. … Take the equality of vertically opposite angles. If I measure the angles I am proceeding experimentally; if I open out two sticks crossed in the form of an X, and say that it is obvious to me that the amount of opening is equal on the two sides, then I am using intuition.
>
> (Godfrey and Siddons, 1931, p. 21)

What Godfrey tried to describe here is that we could 'see' properties without depending either on actual measurements or deductive reasoning, but on 'intuition'. It is useful to pay attention to his notion of the 'geometrical eye' to clarify what he meant. He emphasised that we need to exercise 'logical power' as well as 'geometrical power' for solving mathematical problems. He describes 'geometrical power' as 'the power we exercise when we solve a rider [a difficult geometrical problem or proof]' (Godfrey, 1910, p. 197). To develop this geometrical power, Godfrey argued, it would be essential to train each student's 'geometrical eye', which he defined as 'the power of seeing geometrical properties detach themselves from a figure' (Godfrey, 1910, p. 197). The nature of this 'geometrical eye' is illustrated in the following example (Figure 2): if A, B are the mid-points of the equal sides XY, XZ of an isosceles triangle, prove that AZ=BY (Godfrey and Siddons, 1903, p. 94).

[6] With regard to the design of the theoretical part of *Elementary Geometry*, see Fujita (2001).

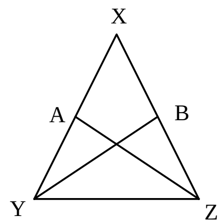

Figure 2. Isosceles triangle XYZ

In tackling this problem, someone would not be able to prove that AZ=BY unless, first of all, they could see that, for example, triangle AYZ and triangle BZY are likely to be congruent. Godfrey stated that this kind of power, i.e. creating proper images in the mind, would be essential to solve geometrical problems. In the example of the vertically opposite angles quoted above, Godfrey could see that the vertically opposite angles were equal to each other when he opened two sticks, because he had (and exercised) his 'geometrical eye'.

Godfrey also suggested that we train the 'geometrical eye' by experimental tasks at any stage in geometry (Godfrey, 1910, p. 197). Thus, we can find tasks that would develop students' geometrical eye in Godfrey's textbook *Elementary Geometry*. For example, the exercises before theorem 2 in Book II of *Elementary Geometry*, 'The area of a triangle is measured by half the product of the base and the altitude' were as follows (Godfrey and Siddons, 1903, p. 172):

Ex. 932. Draw an acute-angled triangle and draw the three altitudes.

Ex. 933. Repeat Ex. 932 for a right-angled triangle.

Ex. 934. Repeat Ex. 932 for an obtuse-angled triangle.

Ex. 935. In what case are two of the altitudes of a triangle equal?

These exercises would make students pay attention to the height of triangles, which would be important in understanding the theorem above.

Another example is that before the construction of an inscribed circle in a given triangle, exercise 1256, is studied (Godfrey and Siddons, 1903, p. 241), - see Figure 3 and exercise below.

Ex. 1256. What is the locus of the centres of circles touching two lines which cross at an angle of 60^0. Draw a number of such circles.

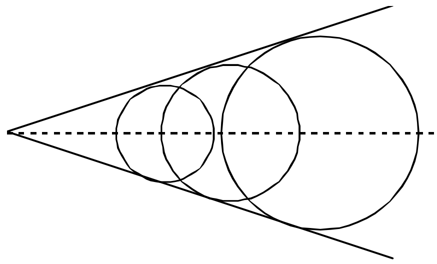

Figure 3. Circles touching two lines

This task would help students understand the proof of this construction, but also in solving the following theoretical exercise: prove that the bisectors of the three angles of a triangle meet in a point (Godfrey and Siddons, 1903, p. 243).

From our analysis it is clear that the experimental exercises in the textbooks by Godfrey and Siddons were carefully chosen and designed so that they would lead to the requirement of a proof and then show that proof. Using this design, the aim of Godfrey and Siddons was to develop what they called the 'geometrical eye'. Thus, the placing of experimental tasks in the teaching of geometry by Godfrey and Siddons is very important, not only for the sake of discovery, but also for the developing the 'geometrical eye' of students (also see in Fujita and Jones, 2003a).

DISCUSSION

As we have seen, Treutlein considered that it would be very important to develop spatial imagination skills in geometry. Godfrey also discussed the importance of intuition in geometry by using the notion of the 'geometrical eye'. In summary, both of them suggested that it would be essential to develop the skills first to create and manipulate geometrical images in the mind, and then to apply them to analyse geometrical problems. They also suggested that such skills should be actively developed through appropriate tasks in geometry. In this section, we discuss further the idea of intuition in geometry, in particular focusing on a) the importance of intuitive skills suggested by Treutlein and Godfrey in terms of recent suggestions by various researchers, and b) some approaches by which we could effectively develop students' intuitive skills in geometry.

Importance of intuition in the teaching and learning of geometry

Geometry is an area of mathematics in which intuition is frequently mentioned. Views vary, however, about the role and nature of geometrical intuition, and how it might help or hinder the learning of geometry (and other areas of mathematics). Piaget or van Hiele considered 'intuitive thoughts' as important in their models of the early stages. Professional geometers, nevertheless, tend to recognise the importance of geometrical intuition in the latter stages too (e.g. Poincaré, 1913; Hilbert, 1932, p. iii; Atiyah, 2001, p. 50, see also Schmalz, 1988)

Like Treutlein and Godfrey in the early 20th century, recent educators have suggested that intuition would have crucial roles in the teaching and learning of geometry. Fischbein (1993) has proposed the notion of 'figural concept'. As Fischbein observes, while a geometrical figure such as a square can be described as having intrinsic conceptual properties (in that it is controlled by geometrical theory), it is not solely a concept, it is an image too: 'it possesses a property which usual concepts do not possess, namely it includes the mental representation of a space property' (Fischbein, 1993, p. 141). According to this notion of figural concepts, geometrical reasoning is characterised *by the interaction between these two aspects, the figural and the conceptual*. He also suggested that 'the *process of building figural concepts in the students' mind should not be considered a spontaneous effect of usual geometry courses*' (Fischbein, 1993, p. 156, emphasis in the original). Mason suggests that diagrams can be thought of as a means, first and foremost, for awakening mental imagery and, secondly, as ways of augmenting, extending and strengthening mental imagery and hence mathematical thinking (e.g. Mason, 1991, p. 84). Goldenberg, Cuoco, and Mark considered that visualisations in geometry are very important when we solve problems in geometry, and suggested that the "prerequisite to that is the ability to take a figure apart in the mind, see the individual elements, and make sufficiently good conjectures about their relationships to guide the choice of further experimental and analytic tools' (p. 7), a notion that is very similar to Godfrey's 'geometrical eye'.

In relation to these different views of intuition, the notion of 'mathematisation' may be useful. Mathematisation is taken as the ability to perceive mathematical relationships and to idealise them into purely mental material (Wheeler, 1976, reprinted 2001; Gattegno, 1988). In terms of 'mathematisation', the suggestions of Treutlein and Godfrey can be understood as the need to create idealised mental images of geometrical figures which are clearly related to appropriate properties (and theorems) when solving problems in geometry.

How we could effectively develop students' intuitive skills in geometry?

From the consideration of these theoretical discussions by recent educators, it appears that the suggestions by Treutlein and Godfrey are not outdated. In fact they can provide us with many interesting questions to explore and research. One of these is how we could effectively develop such skills in geometry. In other words, what kind of tasks would be appropriate for this effective development? A problem is that mental activities in geometry are often neglected in the current geometry curricula (for the case of England, see Jones and Mooney, p. 9). We also examined the designs of current textbooks in lower in secondary schools in Japan and part of the UK (Scotland), focusing on the content of geometry (Fujita and Jones, 2003b). We found that the designs of the textbooks in these two countries have their own strengths and weaknesses, and suggested that it may be necessary to revise their design from the point of view of the development of intuitive skills.

As we have seen in the previous sections, both Treutlein and Godfrey particularly designed some practical tasks for the development of intuitive skills. It is very interesting to notice that their approaches had different aspects. Treutlein regarded developing students' intuitive skills as the most important goal in his instruction. He also considered that it would be important to imagine geometrical figures as well as manipulating them in the mind. On the other hand, Godfrey emphasised the former skill mainly to solve what he called 'riders', theoretical exercises in geometry. Thus, Treutlein actively included the tasks such as creating new figures or the 'air cube' which actively required students to imagine figures in the mind, but Godfrey concentrated on drawing and measurement of geometrical figures, which were directly related to definitions, theorems and riders in his textbook. From the considerations of the current designs of geometry curricula, the tasks by Treutlein and Godfrey can provide us with opportunities to address the issue of the development of intuition in geometry.

CONCLUDING COMMENT

In most countries, although not all, the comparative study by Hoyles, Foxman and Küchemann (2002) found that elements of proof and proving were included in the curricula specifications for geometry. Yet even here there are variations too, with some countries favouring an approach with congruence as a central element, while other used similarity and transformations. For example, in Japan, the teaching and learning of deductive reasoning remains a major problem. Research indicates that while most 14-15 year-old students (Japanese secondary 3rd grade) can write down a proof, around 70% cannot understand why proofs are needed (Miyazaki, 1999; Kunimune, 2000). One major component of an innovative geometry pedagogy would be to improve on appeals to develop geometrical intuition by linking such intuition more directly with geometrical theory. This would entail developing pedagogical methods that mean that a deductive and an intuitive approach are mutually reinforcing when solving geometrical problems (for example, see Jones 1998). In this paper, we examined the development of the teaching of geometry in the early 20th century, focusing on the role of intuition and the tasks for students in the writings by Treutlen and Godfrey. In the reform of the teaching of geometry at that time, both of these educators considered that intuitive skills would be essential. Their effort was devoted to the effective development of such skills and, through our analysis we find that their ideas are still very much worth discussing today for the improvement of the teaching and learning of geometry. It is illuminating that innovative teachers about 100 years ago pointed to the importance and roles of visual images in geometry and geometrical thinking.

To close this paper, we shall consider some future tasks with regard to the issues of intuition in geometry. First, there need to be further discussion of the design of tasks that would effectively develop students' imagination skills in geometry. In relation to this, there should be consideration of what the relationships are between the difficulties of learning to prove in geometry and intuitive skills, and how (or whether) it would be possible to develop students' intuition through practical tasks which can be found in writings by Treutlein and Godfrey. Finally, a further historical investigation is necessary. Treutlein and Godfrey were both influenced by a great educator J. F. Herbart (1776-1841), who suggested that the development of imaginative skills would be very important (Herbart, 1802; Treutlein, 1911; Godfrey and Siddons, 1931). In relation to the teaching practices of Treutlein and Godfrey, we should examine the pedagogy of Herbart for the further understanding of the roles of intuition at that time. Such research could make an important contribution to providing a firmer theoretical basis for formulating new curricular and pedagogic models for geometry.

ACKNOWLEDEMENT

We would like to thank the reviewers of this paper for their insightful advice and the organisers of the TSG29, ICME-10, for their attention to detail. We also thank Prof. J. McGonigal, University of Glasgow, who kindly commented on an early draft of this paper.

Author's note: All the authors of this paper contributed equally. The order of authorship is strictly alphabetical.

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