The Carcinogenic Example

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Abstract

This paper will introduce a new model for Linear Logic, the IE model (I for Internal E for External). Intuitively the model assumes there is a global environment which is divided up into an internal resource and an external resource. Such an assumption makes for an elementary definition of linear negation. The model results in a fairly natural semantics for the multiplicative operators and the constants of Linear Logic. A particular example of the model is given which gives a proper technical account of the cigarette example ([**7**], [**9**]).

1 Introduction

The intuition behind linear logic [**7**] is founded on the metaphor: Proofs are Consumers. Proof hypotheses represent tangible entities which are consumed during the proof process. One might visualise this by creating an initial internal environment containing all the available hypothesis statements. When a statement is consumed during the proof it is placed in an external environment where it is on show as part of the final conclusion. Expanding the metaphor, a proof describes how to correctly consume the hypothesis resource in a manner that validates the conclusion.

Many of the concepts associated with resource consumerism are encoded within linear logic. It seems reasonable then to hope for a model of linear logic which naturally describes such consumerism.

Phase semantics for linear logic were invented by Girard in his seminal paper [**7**]. A rather more straightforward account for phase semantics can be found in [**4**]. This model provides an adequate (sound and complete) description of linear logic, but does not naturally capture the idea of resource consumerism. Quantales [**1**], [**15**] provide a more generic model with closer links to mainstream mathematics, such as ring theory. Effectively a quantale is a monoid in the category of complete semi-lattices.

Currently the literature seems to be concentrated on the semantics of proof, as opposed to the semantics of formulae. The geometry of interaction, ([**8**] or [**10**] section V for example), explains how one can interpret proofs as proof nets, and tries to establish a link between these and Hilbert spaces. This connects resource consumerism with proof in linear logic, but does not highlight how the formulae themselves express properties of resource allocation. More recently, [**2**] and [**11**], some game theory models have been invented which give a good description of the multiplicative connectives in the context of proof semantics.

Resource semantics [**3**] provides a model for linear logic formulae which goes a long way towards a model of resource consumerism. It does not quite embody all the ideas of consumerism required for linear logic since it does not give an interpretation to linear negation $\perp$. This model is in the same style as used to model relevance logic [**14**], [**5**], which is a precursor of linear logic. It is possible to attach sufficient categorical contraptions to resource semantics to turn it into an adequate model, [**3**] contains the details.

The purpose of this paper is to describe a model which provides a natural description of consumerism as practised by linear logic. The proposed model is called the IE model. The intuition is that there is a global resource available to demonstrate that a formula is satisfied. Initially all of this is allocated to an internal environment. As the formula is satisfied this environment interacts with an external environment representing consumed resource. Linear negation in this model represents being forbidden to swap the internal and external environments. In this respect the IE model is similar to de Piavas Dielectica categories [**13**]. That paper uses the same idea for negation but models the other connectives differently.

Section 3 gives the details of the IE model and proves that it is sound. Section 4 defines an adequate class of IE-quantales for $M\mathcal{L}\mathcal{L}$ (linear logic without the modal operators). These quantales use the same intuition of internal and external resources to model $\otimes$ (multiplicative and) and $\multimap$ (linear implication). The IE model of section 3 is a member of this class. Section 3.2 describes the cigarette example and shows that there is an IE model which exactly describes the intended semantics behind that example.

2 Classical Linear Logic

This short section recalls the definition of classical linear logic [**7**]. The following account is taken from [**4**], which is recommended as a good starting point for the uninitiated.

Formulae of $\mathcal{L}\mathcal{L}$ are made from the propositions $\mathcal{P}$, the constants $\mathbf{1}$, $\perp$, $\mathbf{0}$, $\mathbb{T}$, the binary infix connectives $\otimes$, $⅋$, $\&$, $\oplus$ and the unary operator $\perp$. Also there are modal unary operators ! and ?.

An $\mathcal{L}\mathcal{L}$ sequent is a sequence of the form

$$\vdash \varphi_1, \ldots, \varphi_k$$

where each φ_i is a formula in $\mathcal{L}\mathcal{L}$. Capital Greek letters (e.g Γ, Δ) are always used to denote sequences of formulae. The axioms of $\mathcal{L}\mathcal{L}$ are

- identity $\vdash \varphi, \varphi^{\perp}$
- additive $\vdash \mathbb{T}, \Delta$
- multiplicative $\vdash \mathbf{1}$

The identity axiom usually restricts φ to being a proposition, and then it is proved to hold for an arbitrary formula.

The inference rules of $\mathcal{L}\mathcal{L}$ are

- structural

$$\frac{\vdash \Delta}{\vdash \sigma(\Delta)}\ \text{Exch} \qquad \frac{\vdash \varphi, \Delta \quad \vdash \varphi^{\perp}, \Gamma}{\vdash \Delta, \Gamma}\ \text{Cut}$$

  where σ is any permutation of Δ.

- additive

$$\frac{\vdash \varphi, \Delta \quad \vdash \psi, \Delta}{\vdash \varphi \& \psi, \Delta}\ (\&) \qquad \frac{\vdash \varphi, \Delta}{\vdash \varphi \oplus \psi, \Delta}\ (1\oplus) \qquad \frac{\vdash \varphi, \Delta}{\vdash \psi \oplus \varphi, \Delta}\ (2\oplus)$$

- multiplicative

$$\frac{\vdash \varphi,\, \Delta \quad \vdash \psi,\, \Gamma}{\vdash \varphi \otimes \psi,\, \Delta, \Gamma}\,(\otimes) \quad \frac{\vdash \varphi,\, \psi,\, \Delta}{\vdash \varphi \invamp \psi,\, \Delta}\,(\invamp) \quad \frac{\vdash \Delta}{\vdash \bot,\, \Delta}\,(\bot)$$

- modalities

$$\frac{\vdash \Delta}{\vdash ?\varphi,\, \Delta}\,(TH?) \quad \frac{\vdash ?\varphi,\, ?\varphi,\, \Delta}{\vdash ?\varphi,\, \Delta}\,(C?) \quad \frac{\vdash \varphi,\, \Delta}{\vdash ?\varphi,\, \Delta}\,(D?) \quad \frac{\vdash \varphi,\, ?\Delta}{\vdash !\varphi,\, ?\Delta}\,(!)$$

when Δ abbreviates $\varphi_1, \ldots \varphi_k$, then $?\Delta$ abbreviates $?\varphi_1, \ldots ?\varphi_k$

Linear negation is defined syntactically in $\mathcal{LL}$ by the identities:

$$\begin{array}{ll}
\mathbf{1}^\bot = \bot & \bot^\bot = \mathbf{1} \\
\mathbb{T}^\bot = \mathbf{0} & \mathbf{0}^\bot = \mathbb{T} \\
(\varphi \otimes \psi)^\bot = \varphi^\bot \invamp \psi^\bot & (\varphi \invamp \psi)^\bot = \varphi^\bot \otimes \psi^\bot \\
(\varphi \& \psi)^\bot = \varphi^\bot \oplus \psi^\bot & (\varphi \oplus \psi)^\bot = \varphi^\bot \& \psi^\bot \\
(!\varphi)^\bot = ?\varphi^\bot & (?\varphi)^\bot = !\varphi^\bot
\end{array}$$

also, for $p \in \mathcal{P}$, $p^{\bot\bot} = p$. One can prove by induction that for any $\varphi \in \mathcal{LL}$, $\varphi^{\bot\bot} = \varphi$. Finally linear implication $\multimap$ is a derived operator given by $(\varphi \multimap \psi) = (\varphi^\bot \invamp \psi)$.

When defining a satisfaction relation $\models$ (sometimes referred to as a forcing relation), because of the multiplicative rule $(\invamp)$, it is only necessary to define satisfaction of a single formula rather than for a sequence of formulae. This is how $\models$ is defined for the IE model in the next section.

3 The IE model

This section will define the IE model and prove that the model is sound for classical linear logic. Throughout this section M is assumed to be a commutative monoid with product $+$ and identity element 0. Let M^2 denote the set of pairs $(m, n) \in M \times M$. Further let $\models_0$ be a binary relation between pairs from M^2 and $\mathcal{P}$ the set of $\mathcal{LL}$ propositions. This relation is extended to the set of $\mathcal{LL}$ formulae by the following definition. We say that a tuple $(M, +, 0, \models_0)$ is an IE model.

Definition 3.1
For $\phi \in \mathcal{LL}$ and $(m, n) \in M^2$ define the satisfiability relation $(m, n) \models \phi$ inductively as follows. If ϕ is a proposition then $(m, n) \models \phi$ iff $(m, n) \models_0 \phi$.

The multiplicative connectives are given by

- $(m, n) \models \varphi \otimes \psi$ if $\exists\ n_1,\ n_2$ such that $n = n_1 + n_2$ and $(m + n_1, n_2) \models \varphi$ and $(m + n_2, n_1) \models \psi$
- $(m, n) \models \varphi \invamp \psi$ if $\forall\ m_1,\ m_2$, if $m = m_1 + m_2$ then $(m_1, n + m_2) \models \varphi$ or $(m_2, n + m_1) \models \psi$
- $(m, n) \models p^\bot$ if $(n, m) \not\models p$, for $p \in \mathcal{P}$

The additive connectives are given by

- $(m, n) \models \varphi \& \psi$ if $(m, n) \models \varphi$ and $(m, n) \models \psi$
- $(m, n) \models \varphi \oplus \psi$ if $(m, n) \models \varphi$ or $(m, n) \models \psi$

The constants are given by

- $(m,n) \models \mathbf{1}$ if $n = 0$
- $(m,n) \models \perp$ if $m \neq 0$
- $(m,n) \models \mathbb{T}$, i.e. $\mathbb{T}$ is just True
- $(m,n) \not\models \mathbf{0}$, i.e. $\mathbf{0}$ is just False

The modal operators ! and ? and linear implication $\multimap$ are derived operators in the IE model, they are defined by

$$\begin{array}{lcl} (\varphi \multimap \psi) & = & (\varphi^{\perp} \parr \psi) \\ (?\varphi) & = & (\perp \oplus \varphi) \\ (!\varphi) & = & (\mathbf{1} \& \varphi) \end{array}$$

Any pair $(m,0)$ is said to be *initial*. A formula φ of $\mathcal{LL}$ is *valid* if for every initial $(m,0)$, $(m,0) \models \varphi$.

We write $(M,+,0,\models_0) \models \varphi$ to denote that φ is valid. When φ is valid for every choice of $\models_0$ we write $M \models \varphi$.

Notice that the definitions of ! and ? are equivalent to

$$\begin{array}{lcl} (m,n) \models !\varphi & \Leftrightarrow & n = 0 \ \text{ and } \ (m,n) \models \varphi \\ (m,n) \models ?\varphi & \Leftrightarrow & m \neq 0 \ \text{ or } \ (m,n) \models \varphi \end{array}$$

This definition generates a semantics for linear negation, as demonstrated by the next result.

Proposition 3.2
For any formula $\varphi \in \mathcal{LL}$,

$$(m,n) \models \varphi^{\perp} \Leftrightarrow (n,m) \not\models \varphi$$

Proof
The proof is a straightforward structural induction. We illustrate it with the first interesting case. Consider

$$\begin{array}{lcl} (m,n) \models (\varphi \otimes \psi)^{\perp} & \Leftrightarrow & (m,n) \models \varphi^{\perp} \parr \psi^{\perp} \\ & & (\text{ by definition of } \ \perp) \\ & \Leftrightarrow & \forall\, m = m_1 + m_2,\ (m_1, n + m_2) \models \varphi^{\perp} \ \text{ or } \ (m_2, n + m_1) \models \psi^{\perp} \\ & & (\text{ by definition of } \ \parr) \\ & \Leftrightarrow & \forall\, m = m_1 + m_2,\ (n + m_2, m_1) \not\models \varphi \ \text{ or } \ (n + m_1, m_2) \not\models \psi \\ & & (\text{ by induction hypothesis }) \\ & \Leftrightarrow & \neg \exists\, m_1,\, m_2 : m = m_1 + m_2 \\ & & \quad \text{and } \ (n + m_2, m_1) \models \varphi \ \text{ and } \ (n + m_1, m_2) \models \psi \\ & \Leftrightarrow & \neg (n,m) \models \varphi \otimes \psi \\ & & (\text{ by definition of } \otimes) \end{array}$$

The other cases follow in the same manner. □

3.1 Motivation

Intuitively this model is motivated by regarding the pair (m, n) as a division of a global resource $m + n$ into an internal environment m and an external environment n. The internal environment is used to keep track of the amount of free resource. Resource which is not yet allocated. The external environment is used to keep track of the amount of resource which is currently allocated. There is a conservation principle that resources are never lost or created, so that the sum of the internal and external environment is always constant. The definition of $\models$ can be interpreted in the following way.

When $(m, n) \models \varphi \otimes \psi$ there is some way of internally dividing up the external environment between φ and ψ to satisfy them both. If allocating resource worth n_1 satisfies φ and allocating resource worth n_2 satisfies ψ then allocating $n_1 + n_2 = n$ will satisfy $\varphi \otimes \psi$.

When $(m, n) \models \varphi \parr \psi$ then no matter how the internal environment is redistributed externally between φ and ψ at least one of them is satisfied. Thus n represents some kind of lower bound on the amount of allocated resource necessary to satisfy one of φ and ψ.

When $(m, n) \models \varphi^{\perp}$ it can not be the case that swapping the internal and external environment will satisfy φ.

The additive connectives are modelled in the classical way which allows the IE model to be a special case of a resource model. This of course prevents the IE model from being complete.

The constant $\mathbf{1}$ is satisfied only by initial models, that is a pair which represents an internal environment equal to the global resource and an empty external environment. Constant $\perp$ is satisfied whenever the internal environment is not zero.

Perhaps rather surprisingly the $\mathcal{LL}$ modal operators are derived in the IE model. The formula $!\varphi = \mathbf{1} \& \varphi$ is interpreted as: φ is true and the internal environment is the whole of the global resource.

Notice that this makes $!\varphi$ affect the context in which it occurs. For example if $(m, n) \models !\varphi \otimes \psi$, then unravelling the definitions, it must be that $(m + n, 0) \models !\varphi$ and $(m, n) \models \psi$. If one thinks of m as unconsumed resource, and n as the amount of consumed resource, then satisfying $!\varphi$ has no cost. You may therefore use $!\varphi$ as often as you like since it comes for free.

The formula $?\varphi = \perp \oplus \varphi$ is interpreted as: either the internal environment is not empty or φ is true. From the definitions if $(m, n) \models ?\varphi \parr \psi$ then either $(0, m + n) \models ?\varphi$ or $(m, n) \models \psi$. In this case $?\varphi$ is the exact opposite of free, the whole global resource must be spent to get it.

Valid formulae are the ones satisfied by all initial pairs which is analogous to phase semantics where valid formulae are those facts which contain 1.

3.2 The Carcinogenic Example

In motivating linear logic various papers (e.g. [**7**], [**9**]) look at the example of spending a dollar to buy a packet of camel cigarettes. The example amounts to considering a formula $\$1 \multimap C$, where C represents the act of owning the desired goods, and \$1 is the act of spending a dollar. This formula therefore represents, I can exchange my dollar for goods C. This intuitive explanation can be derived precisely in the right IE model.

Let $\mathbb{N}$ be the natural numbers. Let (m, n) represent the idea that the internal environment m consists of the amount of money in my pocket, and n is the amount of money I have spent since some initial point (pay day?). Choose an initial forcing relation $\models_0$ where $(m, n) \models_0 \$k$ iff $n = k$, and for goods G, $(m, n) \models_0 G$ iff the cost of $G = \$n$.

Intuitively $(m,n) \models_0 \$k$ if I have spent k dollars since pay day, $(m,n) \models_0 G$ if the goods G are what I could have spent my money $\$n$ on. Consider when it is possible for $(m,0) \models \$1 \multimap C$. This represents the first act of pay day, no money has yet been spent.

$$\begin{array}{ll}
(m,0) \models \$1 \multimap C & \Leftrightarrow \\
(m,0) \models \$1^{\perp} \invamp C & \Leftrightarrow \\
\forall\, m = m_1 + m_2,\ (m_1,m_2) \models \$1^{\perp} \text{ or } (m_2,m_1) \models C & \Leftrightarrow \\
\forall\, m = m_1 + m_2,\ (m_2,m_1) \not\models \$1 \text{ or } (m_2,m_1) \models C & \Leftrightarrow \\
(m-1,1) \models C \ \text{ (since } (m_2,m_1) \not\models \$1 \text{ whenever } m_1 \neq 1) &
\end{array}$$

That is $(m,0) \models \$1 \multimap C$ iff $(m-1,1) \models C$. Therefore $\$1 \multimap C$ is interpreted by the model $(m,0)$ as transferring one dollar from my pocket to the external environment in order to buy goods C.

Suppose there are also goods D, which cost a dollar. That is, $(m-1,1) \models C$ and $(m-1,1) \models D$. Therefore, from the definition of $\otimes$, $(m-2,2) \models C \otimes D$. Therefore $C \otimes D$ has a cost of two dollars. An analogous calculation to the one above proves that

$$(m,0) \models \$1 \otimes \$1 \multimap C \otimes D \ \Leftrightarrow\ (m-2,2) \models C \otimes D$$

Therefore formula $\$1 \otimes \$1 \multimap C \otimes D$ is interpreted by the model $(m,0)$ as transferring two dollars from my pocket to the external environment in exchange for C and D.

Another calculation proves that

$$(m,0) \models \$1 \multimap (C \& D) \ \Leftrightarrow\ ((m-1,1) \models C \ \text{ and } \ (m-1,1) \models D)$$

Therefore $\$1 \multimap (C\&D)$ is interpreted as, I can spend my dollar on buying C, and I can spend my dollar on buying D, so I can choose to spend my dollar on C or D.

One can also prove

$$(m,0) \models \$1 \multimap (C \oplus D) \ \Leftrightarrow\ ((m-1,1) \models C \ \text{ or } \ (m-1,1) \models D)$$

Therefore $\$1 \multimap (C \oplus D)$ is interpreted as, I can spend my dollar on buying C or D, but I don't know in advance which one I will get. The choice in this case is made by some outside agent.

This example demonstrates how the various intuitive illustrations provided by the cigarette example have a natural and precise meaning within the above IE model.

3.3 Soundness of the IE model

This section proves that the IE model is sound for $\mathcal{LL}$. The axioms of $\mathcal{LL}$ are proved to hold in the IE model as a series of propositions, one for each axiom. Before dealing with the axioms it is necessary to prove that $\otimes$ and $\invamp$ are associative. This follows immediately from the following lemma.

Lemma 3.3
$(m,n) \models \varphi_1 \otimes \cdots \otimes \varphi_k$ iff there are $n_1, \ldots, n_k$ such that $n = n_1 + \cdots n_k$ and for $1 \leq i \leq k$, $(m+n_i, n-n_i) \models \varphi_i$.

$(m,n) \models \varphi_1 \invamp \cdots \invamp \varphi_k$ iff for all $m_1, \ldots, m_k$ if $m = m_1 + \cdots m_k$ then for some $1 \leq i \leq k$, $(m-m_i, n+m_i) \models \varphi_i$.

The proof is so straightforward it is left out. The following propositions prove that each of the axioms are valid in the IE model. Recall that a formula is valid if it holds in every initial pair $(m,0)$.

The Multiplicative Axioms

First each axiom concerning a multiplicative connective is dealt with.

Proposition 3.4
For any initial pair $(m, 0)$, if $(m, 0) \models \varphi \parr \Delta$ and $(m, 0) \models \psi \parr \Gamma$ then

$$(m, 0) \models (\varphi \otimes \psi) \parr \Delta \parr \Gamma$$

Proof
From lemma 3.3 $(m, 0) \models \varphi \otimes \psi \parr \Delta \parr \Gamma$ if and only if for every $m = m_1 + m_2 + m_3$

1. $(m_1, m_2 + m_3) \models \varphi \otimes \psi$ or
2. $(m_2, m_1 + m_3) \models \Delta$ or
3. $(m_3, m_1 + m_2) \models \Gamma$

Consider a particular $m_1 + m_2 + m_3$, if either of the later two cases hold then the conclusion is true for this particular triple. Suppose that they are false. It is necessary in this case to prove that $(m_1, m_2 + m_3) \models \varphi \otimes \psi$.

From the definition of $(m, 0) \models \varphi \parr \Delta$, since $(m_2, m_1 + m_3) \not\models \Delta$, $(m_1 + m_3, m_2) \models \varphi$. Similarly, from the definition of $(m, 0) \models \psi \parr \Gamma$, since $(m_3, m_1 + m_2) \not\models \Gamma$, $(m_1 + m_2, m_3) \models \psi$. Therefore from the definition, $(m_1, m_2 + m_3) \models \varphi \otimes \psi$. □

Proposition 3.5
If $(m, 0) \models \varphi \parr \Delta$ and $(m, 0) \models \varphi^{\perp} \parr \Gamma$ then

$$(m, 0) \models \Gamma \parr \Delta$$

Proof
It is necessary to prove that for all $m = m_1 + m_2$, either $(m_1, m_2) \models \Delta$ or $(m_2, m_1) \models \Gamma$.

For any particular $m_1 + m_2 = m$, either $(m_1, m_2) \models \Delta$ or $(m_1, m_2) \not\models \Delta$. In the first case there is nothing to prove, so assume that the later holds.

From the definition of $\parr$, the above assumption and that $(m, 0) \models \varphi \parr \Delta$, it must be that $(m_2, m_1) \models \varphi$. From the definition of $\perp$ therefore, $(m_1, m_2) \not\models \varphi^{\perp}$. Since $(m, 0) \models \varphi^{\perp} \parr \Gamma$ we have $(m_2, m_1) \models \Gamma$. Thus under the above assumption $(m, 0) \models \Gamma \parr \Delta$ which completes the proof. □

Proposition 3.6
$(m, 0) \models \varphi \parr \varphi^{\perp}$

Proof
Suppose that $m = m_1 + m_2$. Either $(m_1, m_2) \models \varphi$ or $(m_1, m_2) \not\models \varphi$. Clearly $(m_1, m_2) \not\models \varphi$ iff $(m_2, m_1) \models \varphi^{\perp}$. Therefore by the definition of $\parr$, $(m, 0) \models \varphi \parr \varphi^{\perp}$ as required.
□

Proposition 3.7
If $(m, 0) \models \Delta$ then $(m, 0) \models \Delta \parr \perp$

Proof
Suppose that $m = m_1 + m_2$. If $m_2 \neq 0$ then by the definition of $\perp$, $(m_2, m_1) \models \perp$. If $m_2 = 0$, then $m_1 = m$ and $(m_1, m_2) \models \Delta$.

$\square$

Proposition 3.8
$(m, 0) \models \mathbf{1}$

The Additive Axioms

The following propositions show the axioms for the additive connectives hold in the IE model.

Proposition 3.9
If $(m, 0) \models \Gamma \parr \varphi$ and $(m, 0) \models \Gamma \parr \psi$ then

$$(m, 0) \models \Gamma \parr (\varphi \& \psi)$$

Proof
It is necessary to prove that for all $m_1 + m_2 = m$, either $(m_1, m_2) \models \Gamma$ or $(m_2, m_1) \models \varphi \& \psi$.

Consider a particular $m_1 + m_2 = m$. If $(m_1, m_2) \models \Gamma$ then there is nothing left to prove so assume $(m_1, m_2) \not\models \Gamma$. Since $(m, 0) \models \Gamma \parr \varphi$ it follows that $(m_2, m_1) \models \varphi$. Also since $(m, 0) \models \Gamma \parr \psi$ it follows that $(m_2, m_1) \models \psi$. Therefore by definition of $\&$, $(m_2, m_1) \models \varphi \& \psi$. This completes the proof.

$\square$

The following propositions are so easy to prove there proofs are not given.

Proposition 3.10
If $(m, 0) \models \Gamma \parr \varphi$ then $(m, 0) \models \Gamma \parr (\varphi \oplus \psi)$

Proposition 3.11
$(m, 0) \models \varphi \oplus \mathbb{T}$

Modalities

The following propositions show the axioms for the modal connectives hold in the IE model.

Proposition 3.12
If $(m, 0) \models \Delta$ then $(m, 0) \models ?\varphi \parr \Delta$

Proof
It is necessary to prove that $(m, 0) \models (\perp \oplus \varphi) \parr \Delta$.

Suppose that $m = m_1 + m_2$. If $m_1 = 0$ then by the hypothesis, $(m_2, m_1) \models \Delta$. If $m_1 \neq 0$ then $(m_1, m_2) \models \perp$ by the definition of $\perp$. Therefore, from the definition of $\oplus$, $(m_2, m_1) \models \perp \oplus \varphi$. This completes the proof.

$\square$

Proposition 3.13
If $(m, 0) \models \Delta \parr \varphi$ then $(m, 0) \models \Delta \parr ?\varphi$

Proof

Suppose that $m = m_1 + m_2$. If $(m_1, m_2) \models \Delta$ then there is nothing left to prove, so assume that $(m_1, m_2) \not\models \Delta$. Therefore, from the definition of $(m, 0) \models \Delta \invamp \varphi$, $(m_2, m_1) \models \varphi$. The definition of $\oplus$ immediately gives $(m_2, m_1) \models \varphi \oplus \perp$ as required.

□

Proposition 3.14

If $(m, 0) \models ?\varphi \invamp ?\varphi \invamp \Delta$ then $(m, 0) \models ?\varphi \invamp \Delta$

Proof

First notice that whenever $a \neq 0$, $(a, b) \models ?\varphi$. So from the definitions of $\invamp$, $(m, 0) \models ?\varphi \invamp \Delta$ iff $(0, m) \models ?\varphi$ or $(m, 0) \models \Delta$.

Applying this argument once more to the original hypothesis, since $(m, 0) \models ?\varphi \invamp ?\varphi \invamp \Delta$, then $(0, m) \models ?\varphi$ or $(0, m) \models ?\varphi$ or $(m, 0) \models \Delta$. This completes the proof.

□

Proposition 3.15

If $(m, 0) \models \varphi \invamp ?\Delta$ then $(m, 0) \models !\varphi \invamp ?\Delta$

Proof

Recall that whenever $a \neq 0$, $(a, b) \models ?\Delta$. Thus $(m, 0) \models !\varphi \invamp ?\Delta$ iff
$(m, 0) \models !\varphi$ or $(0, m) \models \Delta$, iff
$(m, 0) \models \mathbf{1} \& \varphi$ or $(0, m) \models \Delta$, iff
$(m, 0) \models \varphi$ or $(0, m) \models \Delta$.

However, from the definition of $\invamp$, since $(m, 0) \models \varphi \invamp ?\Delta$, and taking $m = m + 0$, we have

$$(m, 0) \models \varphi \quad \text{or} \quad (0, m) \models ?\Delta$$

which holds iff

$$(m, 0) \models \varphi \quad \text{or} \quad (0, m) \models \Delta$$

Which completes the proof.

□

These propositions prove that the IE model is sound, it is not however complete. It is the additive connectives which are the primary reason for this failure. The additive constants are simply defined as the classical $\wedge$ and $\vee$, so of course in the IE model $a \& (b \oplus c)$ is equivalent to $a \& b \oplus a \& c$ which is not a theorem of linear logic.

Exercise for the reader

Consider the IE model generated by the natural numbers $\mathbb{N}$ with multiplication $\cdot$ and identity 1. Recall that $\mathbb{N} \models \varphi$ denotes that for every $n \in \mathbb{N}$, $(n, 1) \models \varphi$, i.e. φ is valid in the $(\mathbb{N}, \cdot, 1, \models_0)$ IE model for all choices of $\models_0$.

Define $\mathbf{Mu}\mathcal{LL}$ to be the set of formulae constructed only from $\mathcal{P}$, constants $\mathbf{1}$ and $\perp$ and multiplicative connectives $\otimes$, $\invamp$, and linear negation $\varphi^{\perp}$.

Conjecture 3.16

A formula $\varphi \in \mathbf{Mu}\mathcal{LL}$ is provable if and only if $\mathbb{N} \models \varphi$.

4 IE-quantales

This section defines IE-quantales and proves that they are complete for $M\mathcal{LL}$. The section starts with a short review of quantales and their semantics for $M\mathcal{LL}$.

Definition 4.1
A quantale is a tuple $\langle Q, \leq, \cdot, 1, 0, \mathbf{T}\rangle$ such that

- $\langle Q, \leq, 0, \mathbf{T}\rangle$ is a complete lattice
- $\langle Q, \cdot, 1\rangle$ is a commutative monoid with identity
- $a \cdot \bigvee_i b_i = \bigvee_i a \cdot b_i$, i.e. multiplication distributes over suprema

For $x,\ y \in Q$ define the quantic implication of x to y to be $x \multimap y = \bigvee\{z \mid z \cdot x \leq y\}$.

Quantales can be used to provide an algebraic semantics for $M\mathcal{LL}$.

Definition 4.2
Let $\langle Q, \leq, \cdot, 1, 0, \mathbf{T}\rangle$ be a quantale. Let $[\![\ ]\!]$ be a function $[\![\ ]\!] : \mathcal{P} \cup \{\perp\} \longrightarrow Q$. This is extended to $[\![\ ]\!] : M\mathcal{LL} \longrightarrow Q$ as follows

$$\begin{array}{lcl} [\![\varphi \multimap \psi]\!] & = & [\![\varphi]\!] \multimap [\![\psi]\!] \\ [\![\varphi^\perp]\!] & = & [\![\varphi]\!] \multimap [\![\perp]\!] \\ [\![\varphi \otimes \psi]\!] & = & ([\![\varphi]\!] \cdot [\![\psi]\!])^{\perp\perp} \\ [\![\varphi \oplus \psi]\!] & = & ([\![\varphi]\!] \vee [\![\psi]\!])^{\perp\perp} \\ [\![\mathbf{1}]\!] & = & 1^{\perp\perp} \end{array}$$

Operators $\varphi \parr \psi$ and $\varphi \& \psi$ are defined as $(\varphi^\perp \otimes \psi^\perp)^\perp$ and $(\varphi^\perp \oplus \psi^\perp)^\perp$ respectively. Constants $\mathbb{T}$ and $\mathbf{0}$ map to $\mathbf{T}$ and $0^{\perp\perp}$ respectively.

A function $[\![\ ]\!] : \mathcal{P} \cup \{\perp\} \longrightarrow Q$ is an *evaluation* if $[\![p]\!] = [\![p]\!]^{\perp\perp}$ for every proposition p. For an evaluation one can prove that for every $M\mathcal{LL}$ formula φ, $[\![\varphi]\!] = [\![\varphi]\!]^{\perp\perp}$.

A formula φ is valid for a given evaluation $[\![\ ]\!]$ iff

$$[\![\mathbf{1}]\!] \leq [\![\varphi]\!]$$

Quantales provide an adequate description of $M\mathcal{LL}$ in that φ is provable iff for every evaluation of every quantale φ is valid.

The above definition is equivalent to a semantics given by the quantic nucleus $X \longrightarrow X^{\perp\perp}$ (see [**12**]). The style of presentation here has been deliberately chosen to fit in with the semantics of phase spaces since they will be involved in the completeness proof.

Let $\mathcal{P}(A)$ denote the power set of the set A. Let $(M, \cdot, 1)$ be a commutative monoid with identity. For $X, Y \in \mathcal{P}(M)$ let $X \cdot Y = \{x \cdot y \mid x \in X,\ y \in Y\}$. It is a straightforward exercise to show that the tuple

$$\langle \mathcal{P}(M), \subseteq, \cdot, \{1\}, \emptyset, M\rangle$$

is a quantale. Notice that for $X, Y \in \mathcal{P}(M)$, $X \vee Y = X \cup Y$, and
$X \multimap Y = \{v \mid x \in X \Rightarrow vx \in Y\}$. In this case the quantale semantics given by an evaluation $[\![\ ]\!]$ are exactly the phase space semantics defined by $[\![\ ]\!]$ on $(M, \cdot, 1)$.

4.1 IE-quantales: Definition

Throughout this section let $(M, \cdot, 1)$ be a commutative monoid with identity 1, let M^2 be the set of pairs of elements from M, i.e. $M^2 = M \times M$. Define operators $\odot$ and $\multimap$ on $\mathcal{P}(M^2)$ as follows. Let A, $B \subseteq M^2$,

$$\begin{array}{lcl} A \odot B & = & \{(x, y) \mid \exists\, u,\, v \in M.\ y = u \cdot v \ \text{ and } \ (x \cdot u, v) \in A \ \text{ and } \ (x \cdot v, u) \in B\} \\ A \multimap B & = & \{(x, y) \mid \forall\, u,\, v \in M \ \text{ s.t. } x = u \cdot v,\ (y \cdot u, v) \in A \Rightarrow (u, y \cdot v) \in B\} \end{array}$$

The $\multimap$ operator is known as the residuation operator due to the following lemma.

Lemma 4.3
The operator $\multimap$ above is a residuation with respect to the multiplication $\odot$ and the inequality $\subseteq$. That is for A, B, $C \subseteq M^2$,

$$(A \odot B \subseteq C) \Leftrightarrow (A \subseteq B \multimap C)$$

Proof
"$\Rightarrow$"
Suppose that $(a, b) \in A$ and that $a = u \cdot v$. If $(ub, v) \notin B$ then $(a, b) \in B \multimap C$, so assume that $(ub, v) \in B$. By the definition of $\odot$, $(u, vb) \in A \odot B$, hence by definition of $\multimap$, $(a, b) \in B \multimap C$.
"$\Leftarrow$"
Let $(x, y) \in A \odot B$. Therefore there are a, b where $y = ab$ and $(xa, b) \in A$ and $(xb, a) \in B$. From the hypothesis $(xa, b) \in B \multimap C$, hence from the definition of $\multimap$, $(x, ba) \in C$. That is $(x, y) \in C$.

$\square$

Convention We will write $X \times u$ to denote $X \times \{u\}$, where $X \subseteq M^2$ and $u \in M$. Similarly we write $X - u$ for $X - \{u\} = \{x \in X \mid x \neq u\}$. So for A, $B \subseteq M^2$, and u, $v \in M$, $(A - u) \times (B - v)$ makes sense.

Proposition 4.4
The tuple $\langle \mathcal{P}(M^2), \subseteq, \odot, M \times 1, \emptyset, M^2 \rangle$ is a quantale, the quantic implication for this tuple is the residuation operator $\multimap$ defined above.

Proof
The proof that $M \times 1$ is the unit is a straightforward exercise. The fact that $\subseteq$ gives a complete lattice is a standard result.

Let $\mathcal{A} \subseteq \mathcal{P}(M^2)$. In this tuple

$$\bigvee \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

From this and lemma 4.3 it follows that the residuation operator $A \multimap B$ is equal to $\bigvee \{Z \mid Z \odot A \subseteq B\}$. The distributivity law for $\odot$ over $\bigvee$ is proved in a straightforward manner by using the residuation property of lemma 4.3.

$\square$

A quantale of the form $\langle \mathcal{P}(M^2), \subseteq, \odot, M \times 1, \emptyset, M^2 \rangle$ is called an *IE-quantale*.

4.2 The IE model derived from an IE-quantale

The IE model semantics of section 3 can be defined in terms of IE-quantales. Let $\perp_F = \{(x, y) \in M^2 \mid x \neq 1\}$.

Lemma 4.5
For $A \in \mathcal{P}(M^2)$, $(A \multimap \perp_F) = \{(x, y) \mid (y, x) \notin A\}$. Further for any $A \subseteq M^2$, $A = A^{\perp\perp}$, where $A^{\perp} = A \multimap \perp_F$.

Proof
If $(a, b) \in A \multimap \perp_F$ then for all $a = uv$,

$$(ub, v) \in A \Rightarrow (u, bv) \in \perp_F$$

In particular with $u = 1$, $v = a$, if $(b, a) \in A$ then $(1, ba) \in \perp_F$. Hence $(b, a) \notin A$.

Next suppose we have a pair (a, b) where $(b, a) \notin A$. If $a = uv$, and $(ub, v) \in A$ then trivially $u \neq 1$ since we are given $(b, a) \notin A$, hence $(u, bv) \in \perp_F$ as required.

It is immediate from the first part of the lemma that $A = A^{\perp\perp}$. □

Let $(M, \cdot, 1, \models_0)$ be an IE model (see definition 3.1). Let

$$[\![p]\!]_{IE} = \{(x, y) \mid (x, y) \models_0 p\}$$

and let $[\![\perp]\!]_{IE} = \perp_F$. The IE-quantale $\langle \mathcal{P}(M^2), \subseteq, \odot, M \times 1, \emptyset, M^2 \rangle$ extends this evaluation function to define $[\![\theta]\!]_{IE}$ for any $M\mathcal{LL}$ formula.

Corollary 4.6
For all x, $y \in M$,

$$((x, y) \models \theta) \Leftrightarrow ([\![\mathbf{1}]\!]_{IE} \subseteq [\![\theta]\!]_{IE})$$

The proof of this comes directly from the definitions of $\odot$ and $\multimap$ for $\mathcal{P}(M^2)$ and lemma 4.5. This demonstrates that the IE model semantics is one particular example of an interesting class of quantales; all of which use the intuition of internal/external resource allocation. The next section proves that this class is complete for $M\mathcal{LL}$.

4.3 IE-quantale semantics is complete

From now on we assume that M has a zero element 0 (i.e. $x \cdot 0 = 0$ for $x \in M$), such that if $0 = u \cdot v$ then $u = 0$ or $v = 0$. (Note this is not to be confused with the 0 bottom element associated with a quantale). Such a monoid is called an *integral* monoid.

Define a 0-evaluation to be any evaluation $[\![\ ]\!] : \mathcal{P} \cup \{\perp\} \longrightarrow \mathcal{P}(M)$ such that $\forall p \in \mathcal{P}$, $0 \in [\![p]\!]$ and $0 \in [\![\perp]\!]$. Given a 0-evaluation $[\![\ ]\!]$, for any $\theta \in M\mathcal{LL}$, $0 \in [\![\theta]\!]$.

This section will prove that every 0-evaluation defined over such a monoid corresponds to an evaluation for an IE-quantale. This class of phase spaces is easy to prove complete by suitably manipulating the formulae of $M\mathcal{LL}$, and so the corresponding class of IE-quantales is also complete.

Let $\sim$ be the equivalence relation on $M\mathcal{LL}$ generated by the equations

$$\begin{array}{lcl} x \otimes (y \otimes z) & = & (x \otimes y) \otimes z \\ x \otimes y & = & y \otimes x \\ x \otimes \mathbf{1} & = & x \\ x \otimes 0 & = & 0 \end{array}$$

Define $M\mathcal{LL}_0$ to be the set of $\sim$ equivalence classes. Clearly $M\mathcal{LL}_0$ is an integral monoid.

Let $[\![\perp]\!]_{ph} = \{\Gamma \mid (\vdash \Gamma \multimap \perp)\} = \{\Gamma \mid \vdash \Gamma^{\perp}\}$, and for propositions p let

$$[\![p]\!]_{ph} = \{\Gamma \mid (\vdash \Gamma \multimap p)\}$$

This defines an evaluation $[\![\ ]\!]_{ph}$ for the quantale

$$\langle \mathcal{P}(M\mathcal{L}\mathcal{L}_0), \subseteq, \otimes, \{1\}, \emptyset, M\mathcal{L}\mathcal{L}_0 \rangle$$

which, as mentioned earlier, is just the phase space given by $[\![\ ]\!]_{ph}$ on $(M\mathcal{L}\mathcal{L}_0, \otimes, 1)$. It is a straightforward exercise to prove that for $\theta \in M\mathcal{L}\mathcal{L}$

$$[\![\theta]\!]_{ph} = \{\Gamma \mid \vdash \Gamma \multimap \theta\ \}$$

Notice that $0 \in [\![\theta]\!]_{ph}$ and $0 \in [\![\perp]\!]_{ph}$, that is $[\![\ ]\!]_{ph}$ is a 0-evaluation.

It is easy to see that $[\![\mathbf{1}]\!]_{ph} \subseteq [\![\theta]\!]_{ph}$ iff $\vdash \theta$. This proves that the class of phase spaces given by integral monoids is complete. Moreover it is complete just for the case of 0-evaluations. This is of course only a very minor modification to Girrards proof of completeness in [**7**].

Definition 4.7
Let $\eta : \mathcal{P}(M) \longrightarrow \mathcal{P}(M^2)$ be defined by

$$\eta(X) = M \times 0 \cup (M - 0) \times (M - 0) \cup 0 \times X$$

This function embeds the phase space semantics of an integral monoid into the IE-quantale semantics for $\mathcal{P}(M^2)$. This will be made precise after the following lemmas. These lemmas prove that η is very like a quantale morphism.

From now on when $X, Y \subseteq M$, we write $X \multimap Y$ to denote the phase space implication given by $\{y \mid \forall v \in M.\, v \in X \Rightarrow yv \in Y\}$. Recall that when A, $B \subseteq M^2$, $A \multimap B$ is defined at the beginning of section 4.1.

Lemma 4.8
For $X, Y \subseteq M$, $\eta(X \multimap Y) = (\eta(X) \multimap \eta(Y))$ whenever $0 \in X \cap Y$.

Proof
Let $A = \eta(X)$ and $B = \eta(B)$. Suppose that $(x, y) \in (A \multimap B)$, that is

$$\forall x = uv.\ (uy, v) \in A \Rightarrow (u, yv) \in B \tag{1}$$

To decide if any particular $(x, y) \in A \multimap B$ there are various cases to consider depending on whether x and y are 0. Suppose there are u, v such that $x = uv$.

1. $x \neq 0$, $y \neq 0$. Since M is an integral monoid this implies all of u, v, uy and yv are not 0, hence $(uy, v) \in A$ and $(u, yv) \in B$ by the definition of η.

2. $y = 0$. In this case (1) holds iff for every $x = uv$

$$(0, v) \in A \Rightarrow (u, 0) \in B$$

This is always true because $M \times 0 \subseteq B$.

3. $x = 0$. In this case either $u = 0$ or $v = 0$, so (1) splits into two cases.

When $u = 0$ we require for every $v \in M$

$$(0, v) \in A \Rightarrow (0, yv) \in B$$

That is $y \in X \multimap Y$

When $v = 0$ we require for every $u \in M$

$$(uy, 0) \in A \Rightarrow (u, 0) \in B$$

This is always true since $M \times 0 \subseteq B$.

This completes the proof. □

Lemma 4.9
For $X, Y \subseteq M$, $\eta(X \odot Y) = \eta(X) \odot \eta(Y)$ whenever $0 \in X \cap Y$.

Proof
Let $A = \eta(X)$ and $B = \eta(B)$. $(x, y) \in A \odot B$ iff

$$\exists y = uv.\ (xu, v) \in A \quad \text{and} \quad (xv, u) \in B \tag{2}$$

To decide if any particular $(x, y) \in A \odot B$ there are various cases to consider depending on whether x and y are 0.

1. $x \neq 0$, $y \neq 0$. Then $(x, y) \in A \cap B$ by the definition of η. Take $u = 1$, and $v = y$. Hence $(xu, v) = (x, y) \in A$ and $(xv, u) = (xy, 1) \in B$ since $(M - 0) \times (M - 0) \subseteq B$.
2. $y = 0$. Choose $u = v = 0$. Then $(xu, v) = (0, 0) \in A$ and $(xv, u) = (0, 0) \in B$.
3. $x = 0$. In this case (2) holds iff there are u and v with $y = uv$ and $(0, u) \in A$ and $(0, v) \in B$. That is $u \in X$ and $v \in Y$, i.e. $y \in X \cdot Y$.

This completes the proof. □

Let $[\![\ ]\!]_{ph} : \mathcal{P} \cup \{\perp\} \longrightarrow \mathcal{P}(M)$ be a 0-evaluation on the quantale (phase space) $\langle \mathcal{P}(M), \subseteq, \cdot, \{1\}, \emptyset, M \rangle$. This defines a quantale (phase space) semantics $[\![\theta]\!]_{ph} \subseteq M$ for $\theta \in M\mathcal{LL}$.

For the quantale $\langle \mathcal{P}(M^2), \subseteq, \odot, M \times 1, \emptyset, M^2 \rangle$ let $[\![\ ]\!]_{ie} : \mathcal{P} \cup \{\perp\} \longrightarrow \mathcal{P}(M^2)$ be the evaluation given by

$$[\![\perp]\!]_{ie} = \eta([\![\perp]\!]_{ph}),\ [\![p]\!]_{ie} = \eta([\![p]\!]_{ph})$$

From lemmas 4.8 and 4.9 it immediately follows that

$$[\![\theta]\!]_{ie} = \eta([\![\theta]\!]_{ph})$$

and $[\![\mathbf{1}]\!]_{ie} = [\![\mathbf{1}]\!]_{ph}$. Therefore θ is valid with respect to $[\![\ ]\!]_{ph}$ if and only if θ is valid with respect to $[\![\ ]\!]_{ie}$. Since the class of integral phase spaces and 0-evaluations is complete this proves that the class of IE-quantales is also complete.

The map η has proved a stronger result than completeness. It has provided an embedding of any phase space generated by an integral monoid and 0-evaluation into an IE-quantale. It is interesting to note that the IE model semantics $[\![\ ]\!]_{IE}$ (see section 4.2) is not in the image of η. This implies that IE model semantics represents a distinct departure from standard phase space semantics.

5 Conclusion

During the 1980-s consumerism became the dominant theme amongst western economies. It also had its effect on logicians, for Girard introduced the world to linear logic in [**7**]. The flavour of linear logic is to suppose that hypotheses are consumed during a proof, so that proofs must manage exactly the available resources to reach a successful outcome.

The IE model proposed here uses a pair (m, n) of elements from a commutative monoid as a model for $\mathcal{LL}$ formulae. The total resource available to satisfy a formula φ is represented by $m + n$. This resource is divided into an internal environment m and an external environment n. The various operators of $\mathcal{LL}$ express how resources may be moved between these two environments.

The IE model attempts to describe $M\mathcal{LL}$ connectives in a natural way. In particular linear negation is modelled as a kind of twisted classical negation. To the author it seems important to have a model which ties in with the accepted intuition which drives the logic. With linear logic the most quoted motivating example is the infamous cigarette example. IE semantics are capable of describing precisely the informal semantics of this example (see section 3.2).

The IE model is an IE-quantale. All these quantales model $M\mathcal{LL}$ formulae in terms of pairs (m, n). This class of quantales is complete for $M\mathcal{LL}$.

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