CONFIGURATION SPACES AND POLYHEDRAL PRODUCTS

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Abstract. This paper aims to find the most general combinatorial conditions under which a moment-angle complex \((D^2, S^1)^K\) is a co-\(H\)-space, thus splitting unstably in terms of its full subcomplexes. Our main tool is a certain generalization of the theory of labelled configuration spaces.

1. Introduction

Polyhedral products have been the subject of quite a bit of interest recently, beginning with their appearance as homotopy theoretical generalizations of various objects studied in toric topology. Of particular importance are the polyhedral products \((D^2, S^1)^K\) and \((\mathbb{C}P^\infty, *)^K\), known as moment-angle complexes and Davis-Januszkiewicz spaces respectively. The homotopy theory of these spaces has many applications - from complex and symplectic geometry (c.f. [6, 24, 23]), to combinatorial and homological algebra (c.f. [29, 1]). For example, moment-angle manifolds appear as intersection of quadrics or as quasitoric manifolds after taking a certain orbit space. Stanley-Reisner rings of simplicial complexes are realized by the cohomology of Davis-Januszkiewicz spaces (equivalently, the equivariant cohomology ring of moment-angle complex), while the cohomology ring of moment-angle complexes is closely related to the study of the cohomology of a local ring (c.f. [8, 7]). One would like to know how the combinatorics of the underlying simplicial complex \(K\) encodes geometrical and topological properties of polyhedral products, and vice-versa. The case of Golod complexes is especially relevant. A ring \(R = \mathbb{k}[v_1, \ldots, v_n]/I\) for \(I\) a homogeneous ideal is said to be \textit{Golod} if all products and higher Massey products in \(\text{Tor}^\mathbb{k}_{[v_1, \ldots, v_n]}((R, \mathbb{k})\) vanish. Golod [14] showed that the Poincaré series of the homology ring \(\text{Tor}_R(\mathbb{k}, \mathbb{k})\) of \(R\) represents a rational function whenever \(R\) is Golod. From the context of combinatorics, a simplicial complex \(K\) on vertex set \([n] = \{1, \ldots, n\}\) is said to be \textit{Golod} over \(\mathbb{k}\) if the \textit{Stanley-Reisner ring} \(\mathbb{k}[K]\) is Golod, and if this is true for all fields \(\mathbb{k}\) and \(\mathbb{k} = \mathbb{Z}\), we simply say that \(K\) is Golod. Fixing \(\mathbb{k}\) to be a field or \(\mathbb{Z}\), by [7, 12, 3, 19] there are isomorphisms of graded commutative algebras

\[H^*(((D^2, S^1)^K; \mathbb{k}) \cong \text{Tor}_{\mathbb{k}_{[v_1, \ldots, v_n]}}(\mathbb{k}[K], \mathbb{k}) \cong \bigoplus_{I \subseteq [n]} \hat{H}^*(\Sigma^{[I]+1}|K_I|; \mathbb{k})\]

where \(\mathbb{k}[K]\) is the Stanley-Reisner ring of \(K\), \(K_I\) is the restriction of \(K\) to vertex set \(I \subseteq [n]\), and the (cup product) multiplication is realized by maps

\[\psi_{I,J}: |K_{I,J}| \rightarrow |K_I* K_J| \cong |K_I|* |K_J| \cong \Sigma |K_I| \wedge |K_J|,\]

induced by the canonical inclusions \(K_{I,J} \rightarrow K_I * K_J\) whenever \(I\) and \(J\) are non-empty and disjoint. The Golod condition can then be reinterpreted as \(\psi_{I,J}\) inducing trivial maps on \(\mathbb{k}\)-cohomology for disjoint non-empty \(I\) and \(J\) together with Massey products vanishing in \(\hat{H}^*((D^2, S^1)^K; \mathbb{k})\). This topological interetation has been a starting point for applying the homotopy theory of moment-angle complexes to the problem of determining which simplicial complexes \(K\) are Golod, see [15] for example. In the opposite direction, a moment-angle complex \((D^2, S^1)^K\) takes its simplest algebraic form when we restrict to Golod \(K\). Golod complexes are therefore a natural starting point for studying the homotopy types of moment-angle complexes.

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Considerable work has been done on the homotopy theory of moment-angle complexes over Golod complexes [16, 17, 18, 21, 20, 15], culminating in the following conjectured topological characterization of the Golod complexes.

**Conjecture 1.1.** A moment-angle complex \((D^2, S^1)^K\) is a co-\(H\)-space if and only if \(K\) is Golod.

The right-hand implication follows from the general fact that cohomology rings of co-\(H\)-spaces have trivial cup and Massey products. The left-hand implication, if true (or to whichever extent it is true), immediately gives a nice explicit combinatorial description for the homotopy type of \((D^2, S^1)^K\). Namely, \((D^2, S^1)^K\) is a co-\(H\)-space if and only if

\[
(D^2, S^1)^K \simeq \bigvee_{I \subseteq [n]} \Sigma^{|I|+1} |K_I|.
\]

This was shown by Iriye and Kishimoto [20], and follows from a general suspension splitting due to Bahri, Bendersky, Cohen, and Gitler (the BCG splitting) [2]

\[
\Sigma (D^2, S^1)^K \simeq \bigvee_{I \subseteq [n]} \Sigma^{|I|+2} |K_I|.
\]

Our goal in this paper is to determine the extent to which Conjecture 1.1 holds. To do this we consider several simplifications, the first of these being localization. We confirm a large primes version of Conjecture 1.1, thus reproducing the rational result of Berglund [4] without making any significant use of rational homotopy theory.

**Theorem 1.2.** Localized at any sufficiently large prime \(p\), \((D^2, S^1)^K\) is a co-\(H\)-space if and only if \(K\) is Golod over \(\mathbb{Z}_p\).

Another simplification is to strengthen the hypothesis on a simplicial complex being Golod. One way of doing so is (ignoring Massey products) to require some suspension of each inclusion \(\iota_{I,J}\) to be null-homotopic (instead of only inducing trivial maps on cohomology) for every disjoint and non-empty \(I, J \notin [n]\). In Proposition 4.4 we show that after appropriate suspension of \(\iota_{I,J}\), this is a necessary condition for \((D^2, S^1)^K\) to be a co-\(H\)-space, and we call this the homotopy Golod condition:

**Theorem 1.3.** If \((D^2, S^1)^K\) to be a co-\(H\)-space, then it is homotopy Golod.

Moreover, we will see that under certain additional coherence conditions on these null-homotopies, the homotopy Golod condition becomes sufficient, though possible no longer necessary. We outline this coherence condition as follows. Take, for example, the natural inclusion \(\iota_{I,J}:|K_{I} \cup J| \hookrightarrow |K_I * K_J|\) for disjoint \(I, J\), and where \(I, J_1, J_2\) are disjoint and non-empty and \(\iota_I\) is the identity \(|K_I| \hookrightarrow |K_I|\), consider that there is a commutative diagram

\[
\begin{array}{ccc}
|K_{I} \cup J_1 * K_{J_2}| & \xrightarrow{\iota_{I, J_1, J_2}} & |K_{I} * K_{J_1} * K_{J_2}| \\
\downarrow \iota \downarrow \downarrow \iota & & \downarrow \iota \downarrow \downarrow \iota \downarrow \downarrow \iota \\
|K_{I} \cup J_1 \cup J_2| & \xrightarrow{\iota_{I, J_1 \cup J_2}} & |K_{I} * K_{J_1} * K_{J_2}| \\
\end{array}
\]

Then in defining the coherent homotopy Golod condition, we require the composites

\[
\text{Cone}(\Sigma |K_{I} \cup J_1 \cup J_2|) \xrightarrow{\iota_{I, J_1, J_2}} \Sigma |K_{I} \cup J_1 * K_{J_2}| \xrightarrow{\Sigma (\iota_{I, J_1}) * \iota_{J_2}} \Sigma |K_{I} * K_{J_1} * K_{J_2}|
\]

\[
\text{Cone}(\Sigma |K_{I} \cup J_1 \cup J_2|) \xrightarrow{\iota_{I, J_1 \cup J_2}} \Sigma |K_{I} * K_{J_1} * K_{J_2}| \xrightarrow{\Sigma (\iota_{I, J_1} * \iota_{J_2})} \Sigma |K_{I} * K_{J_1} * K_{J_2}|
\]

to be homotopic to to each other via a homotopy that is fixed on the base \(\Sigma |K_{I} \cup J_1 \cup J_2|\) of the cone \(\text{Cone}(\Sigma |K_{I} \cup J_1 \cup J_2|)\), where \(\iota_{I, J_1, J_2}\) and \(\iota_{I, J_1 \cup J_2}\) are the extensions given by the nullhomotopies of
The main idea in this paper is the following well-known fact due to Ganea: a space $Y$ is a co-$H$-space if and only if the evaluation map $\Sigma\Omega Y \to Y$ has a right homotopy inverse. Our method for constructing right homotopy inverses of $\Sigma\Omega(D^2, S^1)^K \to (D^2, S^1)^K$ depends on a construction of a certain scanning map $\gamma: C((D^1, S^0)^K) \to \Omega(D^2, S^1)^K$ together with a suspension splitting $\Sigma C((D^1, S^0)^K) \simeq \bigvee_i \Sigma D_i((D^1, S^0)^K)$. Here $C((D^1, S^0)^K)$ is a configuration space that generalizes classical labelled configuration spaces by allowing particles to collide under certain rules, and $D_i((D^1, S^0)^K)$ is a certain quotiented configuration space. We remark that there are various constructions of configuration spaces in the literature that are related to these [9, 22, 26, 10, 11]. The upshot of this is that the summands $\Sigma D_i((D^1, S^0)^K)$ in the splitting of $\Sigma\Omega(D^2, S^1)^K$ are in a sense more combinatorial in their behaviour in comparison to polyhedral products, allowing us to obtain combinatorial statements about the homotopy type of $(D^2, S^1)^K$. This is analogous to the summands in the BCG splitting (1) being combinatorial objects - each of them is known to be homeomorphic to certain quotiented moment-angle complexes.

Throughout this paper, $-1$ is taken to be the basepoint of $D^1 = [-1, 1]$. Points in $S^1$ are thought of as real numbers in $D^1$ with $-1$ and 1 identified as the basepoint. The suspension $\Sigma X$ of a space $X$ is taken to be the reduced suspension $D^1 \times X/((-1, 1) \times X \cup D^1 \times \{\ast\})$ whenever $X$ is basepointed with basepoint $\ast$. Otherwise it is the unreduced suspension, in other words, the quotient space of $D^1 \times X$ under identifications $(-1, x) \sim x_{-1}$ and $(1, x) \sim x_1$. In any case, $\Sigma X$ is always basepointed, in the unreduced case the basepoint taken to be $\ast_{-1}$.

2. Coordinate Suspensions

Fix a product of connected basepointed spaces $X = X_1 \times \cdots \times X_n$ and a subspace $W \subseteq X$. For any product $Y = Y_1 \times \cdots \times Y_n$ of basepointed spaces, define the coordinate smash $Y \wedge_X W$ of $Y$ and $W$ in $X$ to be the subspace of $Y \wedge_X X = (Y_1 \wedge X_1) \times \cdots \times (Y_n \wedge X_n)$ given by

$$Y \wedge_X W = \{((y_1, x_1), \ldots, (y_n, x_n)) \in Y \wedge_X X \mid (x_1, \ldots, x_n) \in W\}.$$ 

In particular, when each $Y_i$ is an $\ell$-sphere $S^i$, we call $Y \wedge_X W$ the $\ell$-fold coordinate suspension of $W$ in $X$ and denote it by $W^\ell$. This is the subspace of $X^\ell = (\Sigma^i X_1) \times \cdots \times (\Sigma^i X_n)$ given by

$$W^\ell = \{((t_1, x_1), \ldots, (t_n, x_n)) \in X^\ell \mid (x_1, \ldots, x_n) \in W\}$$

for pairs $(t_i, x_i)$ in the reduced suspension $\Sigma^i X_i = D^i \times X_i/(\partial D^i \times X_i \cup D^i \times \{\ast\})$, $D^i$ being the unit $\ell$-disk. Notice $W^0 = W$ and $(W^t_1)^{t_2} = W^{t_1 + t_2}$. 

Theorem 1.4. If $K$ is coherently homotopy Golod, then $(D^2, S^1)^K$ is a co-$H$-space. In particular, coherently homotopy Golod complexes are Golod.
Example 2.1. Let \( W \) be some union of product subspaces \( A_1 \times \cdots \times A_n \). Then \( W^\ell \) is a union of the subspaces \( \Sigma^\ell A_1 \times \cdots \times \Sigma^\ell A_n \). In particular, thinking of the \( A_i \)'s as cells in \( X_i \), if \( W \) is a subcomplex of \( X \), then we see that \( W^\ell \) is a subcomplex of \( X^\ell \).

Example 2.2. If \((X,A) = ((X_1,A_1), \ldots, (X_n,A_n))\) is a sequence of pairs of spaces, \( K \) is a simplicial complex on \( n \) vertices, and \( W \) is the polyhedral product

\[
(X,A)^K = \bigcup_{\sigma \in K} Y^\sigma_1 \times \cdots \times Y^\sigma_n \subseteq X_1 \times \cdots \times X_n,
\]

where

\[
Y^\sigma_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma, \end{cases}
\]

then \( W^\ell \) is the polyhedral product \((\Sigma^\ell X, \Sigma^\ell A)^K \subseteq X^\ell\) of the \( n \)-tuple \((\Sigma^\ell X, \Sigma^\ell A)\) of pairs of spaces \((\Sigma^\ell X_1, \Sigma^\ell A_1), \ldots, (\Sigma^\ell X_n, \Sigma^\ell A_n)\).

2.1. Cofibrations and Splittings. Given any subset \( I = \{i_1, \ldots, i_k\} \subseteq [n] = \{1, \ldots, n\} \), let \( W_I^f \) be the image of \( W^\ell \) under the projection \( X^\ell \to X_I^f \), and \( W_I^f \) the image of \( W_I^f \) under the quotient map \( X_I^f \to \hat{X}_I^f \), where

\[
X_I^f = \Sigma^\ell X_{i_1} \times \cdots \times \Sigma^\ell X_{i_k},
\]

\[
\hat{X}_I^f = \Sigma^\ell X_{i_1} \wedge \cdots \wedge \Sigma^\ell X_{i_k}.
\]

Definition 2.3. We say \( W \) satisfies the simplicial property if given any point \( x \in w \), the point \( x' \in X \) obtained from \( x \) by replacing some coordinate in \( x \) with a basepoint, is also a point in \( W \).

Example 2.4. Polyhedral products have the simplicial property. If \( \ell \geq 1 \), then \( W^\ell \) is always has the simplicial property as a subspace of \( X^\ell \). Also, if each \( X_i \) is a basepoint CW-complex (i.e. with a fixed basepoint, built up by attaching cells of dimension \( \geq 1 \) via basepoint preserving attaching maps), and the zero skeleton of \( X_i \) contains a single point (the basepoint), then \( W \) has the simplicial property.

Suppose \( W \) satisfies the simplicial property. Then

\[
W_I^f \cong \{(x_1, \ldots, x_n) \in W^\ell \mid x_i = * \text{ if } i \notin I\}.
\]

We will often think of \( W_I^f \) as this subspace of \( W^\ell \). Thus, for any \( 0 \leq k \leq n \), we can take the union

\[
W_k^f = \bigcup_{I \subseteq [n], |I| = k} W_I^f.
\]

In other words,

\[
W_k^f = \{(x_1, \ldots, x_n) \in W^\ell \mid \text{at least } n - k \text{ coordinates } x_i \text{ are the basepoint } * \in X_i\}.
\]

The equality \((W^\ell)_k = (W_k^f)^\ell\) is clear, so we forget brackets. Take the quotient spaces

\[
\hat{W}_k^f = \frac{W_k^f}{W_{k-1}^f},
\]

and \( \hat{V}^f_I = W_I^f/(W_I^f)_{|I|-1} \). Applying these constructions to \( X \) in place of \( W \), since \( X_I^f \cap X_J^f = X^f_{I \cap J} \) as subspaces of \( X^\ell \), and \( X_I^f \cap X_{k-1}^f = (X^f_I)_{|I|-1} \) if \( |I| = k \), then

\[
\hat{X}_k^f = \frac{X_k^f}{X_{k-1}^f} = \bigvee_{I \subseteq [n], |I| = k} \hat{X}_I^f
\]

as subspaces of \( \hat{X}_k^f \). (Note not to confuse \( X_I^f \) with the factors \( X_i \) of \( X \)). One can think of \( \hat{W}_k^f \) as the image of \( W_k^f \) under the quotient map \( X_k^f \to \hat{X}_k^f \), and this restricts to \( \hat{W}_I^f \to \hat{V}^f_I \) for any \( |I| = k \). Thus

\[
\hat{W}_k^f = \bigvee_{I \subseteq [n], |I| = k} \hat{W}_I^f.
\]
Mapping $\tilde{W}_I^\ell$ via the homeomorphism $\tilde{X}_I^\ell \cong \Sigma^{\ell+1} \tilde{X}_I$ that sends $((t_1, x_1), \ldots, (t_{|I|}, x_{|I|}))$ to the point $((t_1, \ldots, t_{|I|}), (x_1, \ldots, x_{|I|}))$, we see that

$$\tilde{W}_I^\ell \cong \Sigma^{\ell+1} \tilde{W}_I,$$

$$\tilde{W}_I^\ell \cong \Sigma^{\ell+1} \tilde{W}_I.$$

Note $W_I^\ell$ and $W_k^\ell$ are subcomplexes of $W^\ell$ if each $X_i$ is a CW-complex, and $W$ is a subcomplex of $X$.

For the remainder of this section we assume each $X_i$ is a connected basepointed CW-complex, and $W$ is a connected subcomplex of their product $X$ satisfying the simplicial property. The next two propositions generalize the polyhedral product splittings in [2] and [20]. For example, we may recover the BBCG suspension splitting of $W^1 = (D^2, S^1)^K$ by applying the homeomorphism $\tilde{W}_I \cong \Sigma |K_I|$ given in [2].

**Proposition 2.5.** For each $\ell \geq 0$ and $1 \leq k \leq n$ there exists a suspension splitting

$$\Sigma W_k^\ell \cong \bigvee_{I \subseteq [k]} \Sigma W_I^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \Sigma W_I^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \Sigma^{\ell+1} \tilde{W}_I.$$

**Proof.** Since $W_k^\ell$ has the simplicial property, given a subset $I = \{i_1, \ldots, i_n\} \subseteq [n]$, we can define a map $q_I : W_k^\ell \to W_I^\ell \to \tilde{W}_I^\ell \cong \Sigma^{\ell+1} \tilde{W}_I$ as the restriction of the composite $X^\ell \to X_I^\ell \to \tilde{X}_I^\ell$ of the projection and quotient map to $W_k^\ell$. Notice that $q_I$ is the composite $W_k^\ell \to \tilde{W}_k^\ell \xrightarrow{\phi} W_I^\ell$ when $|I| = k$, where $\phi$ is the quotient onto the summand $W_I^\ell$ with respect to homeomorphism (4).

We proceed by induction on $k$. Assume that for some $m < k$ a co-$H$-space sum

$$h_m : \Sigma W_m^\ell \xrightarrow{\text{pinch}} \bigvee_{I \subseteq [n], |I| \leq m} \Sigma W_I^\ell \xrightarrow{q_I} \bigvee_{I \subseteq [n], |I| \leq m} \Sigma W_I^\ell$$

is a homotopy equivalence. This is trivial when $m = 1$. We assume the order in which these co-$H$-space sums $h_j$ are taken is such that the maps $\Sigma q_I$ are summed before the maps $\Sigma q_J$ when $|I| < |J|$. Then there is a homotopy commutative diagram of cofibration sequences

$$\begin{align*}
\Sigma W_m^\ell & \to \Sigma W_{m+1}^\ell \to \Sigma W_{m+1}^\ell \\
\bigvee_{I \subseteq [n], |I| \leq m} \Sigma W_I^\ell & \bigvee_{I \subseteq [n], |I| \leq m+1} \Sigma W_I^\ell \bigvee_{I \subseteq [n], |I| = m+1} \Sigma W_I^\ell
\end{align*}$$

(5)

where the bottom horizontal maps are respectively the summand-wise inclusion and the quotient map onto the appropriate summands. The top maps are the suspended inclusion and quotient. Since for $|I| = m + 1$ the composite of the first two maps in the sequence $X_m^\ell \to X_{m+1}^\ell \to \hat{X}_{m+1}^\ell \to \tilde{X}_I^\ell$ is the constant map (where the last map is the quotient map with respect to the splitting (3)), its restriction $W_m^\ell \to W_{m+1}^\ell \xrightarrow{q_I} \tilde{W}_I^\ell$ is the constant map. Then since the suspended inclusion $\Sigma W_m^\ell \to \Sigma W_{m+1}^\ell$ is a co-$H$-map, and $h_m$ and $h_{m+1}$ are the co-$H$-space sums of the maps $\Sigma q_I$ as defined above, we see that the left-hand square commutes up to homotopy. The right-hand map $\hat{h}_{m+1}$ is taken to be a co-$H$-space sum of the composites $\Sigma q_I : \Sigma W_{m+1}^\ell \to \Sigma W_I^\ell \xrightarrow{\text{pinch}} \Sigma W_I^\ell$ over all $|I| = m + 1$, where the first map is the suspension of the homeomorphism (4), and the last map the suspended quotient map. The first map being a homeomorphism, we see $\hat{h}_{m+1}$ induces an isomorphism on homology, thus is a homotopy equivalence. We assume the co-$H$-space sum $\hat{h}_{m+1}$ is taken in the same order as the last maps $\Sigma q_I$ for $|I| = m + 1$ are taken in the co-$H$-space sum $h_{m+1}$. Then since $\Sigma q_I$ is the composite of the suspended quotient map $\Sigma W_m^\ell \to \Sigma W_{m+1}^\ell$ and $\Sigma q_I$, and the suspended quotient map being a co-$H$-map, we see that the right-hand square commutes up to homotopy.
The left-hand and right-hand vertical maps being homotopy equivalences, \( h_{m+1} \) induces an isomorphism of homology groups. All spaces being simply connected CW-complexes, \( h_{m+1} \) is a homotopy equivalence. This completes the induction. \( \square \)

**Lemma 2.6.** \( W_k^\ell \) is simply connected for \( \ell \geq 2 \) and each \( k \leq n \). Moreover, if \( W \) is connected, then \( W_k^1 \) is also simply connected.

**Proof.** Since \( X^\ell \) has no 1-cells when \( \ell \geq 2 \), neither does its subcomplex \( W_k^\ell \), so it is simply-connected.

Suppose \( W \) is connected. Then each \( W_f \) is connected, and since \( W \) has the simplicial property, the \( W_f \)'s intersect at the basepoint, so \( W_k \) is also connected. Notice the 1-skeleton of \( \Sigma X \) consists of points \((t,y)\) such that \( y \) is in the 0-skeleton of \( X \). Since cells in \( X^1 \) are products of cells in each \( \Sigma X \), the 1-skeleton of the subcomplex \( W_k^1 \subseteq X^1 \) is contained in the subspace \( V \subseteq W_k^1 = (W_k)^1 \) consisting of points \( z = ((t_1,x_1),\ldots,(t_n,x_n)) \) such that \( x = (x_1,\ldots,x_n) \) is a point in the 0-skeleton of \( W_k \). Since \( W_k \) is connected, we can define a nullhomotopy of \( V \) in \( W_k^1 \) by picking a homotopy in \( W_k^1 \) from \( x \) to the basepoint \(*,\ldots,*\) to define a homotopy of \( z = ((t_1,x_1),\ldots,(t_n,x_n)) \) to the basepoint \(((t_1,1),\ldots,(t_n,1)) \sim * \). Notice the choice of homotopy does not matter. This is because the points \( x \) are disjoint points in the 0-skeleton of \( W_k \), so the points \( z \in V \) are disjoint when each \(-1 < t_i < 1 \), and otherwise are the basepoint and the homotopy is fixed. This therefore defines a nullhomotopy of the 1-skeleton in \( W_k^1 \). \( \square \)

The following well-known fact will be useful.

**Lemma 2.7.** If \( Y \) is a co-H-space, and \( \psi: Y \longrightarrow Y \vee Y \) its comultiplication, then \( \Sigma \psi \) is homotopic to the pinch map \( \Sigma Y \xrightarrow{\text{pinch}} \Sigma Y \vee \Sigma Y \).

**Proof.** Let \( H \) and \( H' \) be the homotopies to the identity of the composites \( Y \xrightarrow{\psi} Y \vee Y \xrightarrow{1_{\psi}} Y \) and \( Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\psi} Y \). The composite \( Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\psi} Y \times Y \), where \( \iota \) is the canonical inclusion, is homotopic to the diagonal map \( Y \xrightarrow{\Delta} Y \times Y \) via the homotopy \( G_{\iota}(y) = (H_{\iota}(y), H'_{\iota}(y)) \). Take the composite

\[
f: \Sigma(Y \times Y) \xrightarrow{\text{pinch}} \Sigma(Y \times Y) \vee \Sigma(Y \times Y) \longrightarrow \Sigma Y \vee \Sigma Y
\]

where the last map is the wedge sum of the suspended projection maps \( \Sigma Y \longrightarrow \Sigma Y \) onto the first and second factor. The composite \( f \) is a left homotopy inverse of \( \Sigma \iota \). Therefore \( \Sigma Y \xrightarrow{\Sigma \psi} \Sigma(Y \times Y) \xrightarrow{\Sigma \Delta} \Sigma Y \vee \Sigma Y \) is homotopic to \( \Sigma \psi \), and \( \Sigma Y \xrightarrow{\Sigma \Delta} \Sigma(Y \times Y) \xrightarrow{f} \Sigma Y \vee \Sigma Y \) is the pinch map \( \Sigma Y \xrightarrow{\text{pinch}} \Sigma Y \vee \Sigma Y \). These two composites are homotopic since \( \Sigma \iota \circ \Sigma \psi = \Sigma(\iota \circ \psi) \) is homotopic to \( \Sigma \Delta \). \( \square \)

**Proposition 2.8.** Let \( \ell \geq 0 \) and \( 1 \leq k \leq n \), and consider the following:

(a) \( W_k^\ell \) is a co-H-space;

(b) there is a splitting

\[
W_k^\ell \cong \bigvee_{0 \leq j \leq k} {\widehat W_j^\ell} \cong \bigvee_{I \subseteq [n], |I| \leq k} {\widehat W_I^\ell} \cong \bigvee_{I \subseteq [n], |I| \leq k} {\Sigma |I| {\widehat W_I}^\ell};
\]

(c) the quotient map \( W_m^\ell \longrightarrow {\widehat W_m^\ell} \) has a right homotopy inverse for each \( m \leq k \);

(d) \( W_k^1 \) is a simply connected co-H-space for each \( I \subseteq [n] \) with \(|I| \leq k \);

(e) the quotient map \( W_f^\ell \longrightarrow {\widehat W_f^\ell} \) has a right homotopy inverse for each \( I \subseteq [n] \) with \(|I| \leq k \).

Then (a) \( \iff \) (b) \( \iff \) (c) \( \iff \) (d) \( \iff \) (e) when \( \ell \geq 2 \), or when \( \ell = 1 \) and \( W \) is connected. Generally, (a) \( \Rightarrow \) (b) provided \( W_f^\ell \) is simply connected for \( j \leq k \).

**Proof.** When \( \ell \geq 2 \), or \( \ell = 1 \) and \( W \) is connected, Lemma 2.6 ensures that all our spaces are simply connected. The implication (b) \( \Rightarrow \) (a) is clear when \( \ell \geq 1 \). To see that (a) \( \Rightarrow \) (b) holds, for \( m \leq k \), consider the composite

\[
h'_m: W_m^\ell \longrightarrow W_k^\ell \xrightarrow{\psi} \bigvee_{I \subseteq [n], |I| \leq m} W_k^\ell \xrightarrow{\varphi_{\ell I}} \bigvee_{I \subseteq [n], |I| \leq m} {\widehat W_I^\ell}.
\]
where the first map is the inclusion and $\psi'$ is given by iterating the comultiplication $W_k^\ell \xrightarrow{\psi} W_k^\ell \lor W_k^\ell$ in some order. By Lemma 2.7, $\Sigma \psi'$ is homotopic to the iterated pinch map, so by naturality the composite of $\Sigma \psi'$ with the suspended inclusion $\Sigma W_m^\ell \to \Sigma W_m^\ell$ is an iterated pinch map. $\Sigma h_m'$ is then homotopic to the homotopy equivalence $h_m$ in the proof of Proposition 2.5, meaning $h_m'$ induces isomorphisms of homology groups, and since this is a map between simply-connected CW-complexes, $h_m'$ is a homotopy equivalence.

Suppose (b) holds. From the commutativity of the right square in diagram (5), we may take as a right homotopy inverse of the suspended quotient map $\Sigma W_m^\ell \to \Sigma W_m^\ell$ the composite $$f : \Sigma \hat{W}_m^\ell \xrightarrow{\varepsilon} \bigvee_{I \subseteq [n], |I| = m} \Sigma \hat{W}_I^\ell \to \bigvee_{J \subseteq [n], |J| = m} \Sigma W_J^\ell h_m^\ell \to \Sigma W_m^\ell,$$

where the second map is the inclusion. Since $h_m \cong \Sigma h_m'$ and $h_m'$ is a homotopy equivalence, we may take $h_m^{-1} = \Sigma(h_m')^{-1}$. All the maps in this composite being suspensions, $f$ desuspends to a map $f'$ such that the composite $\hat{W}_m^\ell f' W_m^\ell \to \hat{W}_m^\ell$ induces isomorphisms on homology groups. Since each of the summands in the splitting of $\hat{W}_m^\ell$ are summands in the splitting of $\Sigma W_m^\ell$, this last composite must be homotopic to the identity. Thus (b) $\Rightarrow$ (c) holds.

Since each of the cofibrations $W_{m-1}^\ell \to W_m^\ell \to \hat{W}_m^\ell$ for $m \leq k$ trivializes when there is right homotopy inverse of $W_m^\ell \to W_m^\ell$, (c) $\Rightarrow$ (b) holds. Thus, we have shown (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

The restriction $W_k^\ell \to W_k^\ell$ of the projection map $X^\ell \to X_k^\ell$ to $W_k^\ell$ has a right inverse $W_k^\ell \to W_k^\ell$ whenever $|I| \leq k$, given by restricting the inclusion $X_k^\ell \to X^\ell$ to $W_k^\ell$. Thus (a) $\Rightarrow$ (d).

Suppose (d) holds. By (a) $\Rightarrow$ (c) with $W_k^\ell$ in place of $W_m^\ell$, the quotient map $W_k^\ell \to \hat{W}_k^\ell$ has a right homotopy inverse. Thus (d) $\Rightarrow$ (e).

Suppose (c) holds. For any $|I| = m$ we have a commutative square $$\begin{array}{ccc}
W_k^\ell & \to & \hat{W}_k^\ell \\
\downarrow & & \downarrow \\
W_m^\ell & \to & \hat{W}_m^\ell \\
\end{array}$$

with the vertical maps the inclusions, the right-hand one being the inclusion into the summand $\hat{W}_m^\ell$. Therefore the wedge sum of maps $\hat{W}_j^\ell \to W_m^\ell$ over all $|I| = m$, where $s_j$ is our given right homotopy inverse, is a right homotopy inverse of $W_m^\ell \to \hat{W}_m^\ell$. Thus (e) $\Rightarrow$ (c).

\[\square\]

### 3. Configuration Spaces

Fix $W$ to be a subspace of a product $X = X_1 \times \cdots \times X_n$. Let $M$ be any path connected space, $N \subseteq M$ a subspace, and $Y$ a basepointed space with basepoint $\ast$. Let $D_0(M, N; Y) = \ast$ and take the quotient space $D_k(M, N; Y) = \coprod_{i=0}^{k} M^{\times i} \times Y^{\times i} / \sim$

where the equivalence relation $\sim$ is given by

$$((z_1, \ldots, z_i; x_1, \ldots, x_i) \sim (z_{\sigma(1), \ldots, z_{\sigma(i)}; x_{\sigma(1)}, \ldots, x_{\sigma(i)})$$

for any permutation $\sigma$ in the symmetric group $\Sigma_i$, and

$$((z_1, \ldots, z_i; x_1, \ldots, x_i) \sim (z_1, \ldots, z_{i-1}; x_1, \ldots, x_{i-1})$$

whenever $x_i$ is the basepoint $\ast$ or $z_i \in N$. Then

$$D(M, N; Y) = \bigcup_{k=0}^{\infty} D_k(M, N; Y).$$
Notice $D(M,N;Y)$ is just the infinite symmetric product $SP((M/N) \land Y)$. We may think of $D(M,N;Y)$ as the space of (possibly colliding) particles in $M$ with labels in $Y$ that are annihilated in $N$. We thus refer to points in $D(M,N;Y)$ as configurations of particles in $M$ with labels in $Y$, and by this we usually mean, without loss of generality, that the particles in such a configuration are non-degenerate, or in other words, that they are outside $N$ and their labels are not the basepoint. The empty configuration is vacuously non-degenerate. Since order does not matter, it will sometimes be convenient to think of a configuration $(z_1, \ldots, z_i; x_1, \ldots, x_i)$ as a set $\{(z_1, x_1) \ldots (z_i, x_i)\}$ where each of the pairs $(z_i, x_i)$ is considered distinct regardless of the values of $z_i$ or $x_i$.

**Definition 3.1.** A multiset of points $S = \{(x_1, \ldots, x_s)\} \in \coprod_i X_i$ is represented by a point in $W$ if there exists a $(x_1, \ldots, x_n) \in W$ and some injective function $f_S: [s] \rightarrow [n]$ such that $x_i = \bar{x}_{f_S(i)}$.

Equivalently, when $s > 0$, $S$ is represented by a point in $W$ if $(x_{\sigma(1)}, \ldots, x_{\sigma(s)})$ is in $W_1$ for some distinct permutation $\sigma \in \text{Aut}([s])$ and $I \subseteq [n]$ with $|I| = s$.

**Remark:** $S$ is not represented by a point if it contains two or more elements in the same $X_i$ for some $i$.

Let $X' = X_1 \lor \cdots \lor X_n$ be the wedge at basepoints $\ast \in X_i$. Define $C(M,N;W)$ to be the subspace of $D(M,N;X')$ given by the following rule: a configuration of non-degenerate particles $(z_1, \ldots, z_k; x_1, \ldots, x_k) \in D(M,N;X')$ is in $C(M,N;W)$ if and only if

- for each increasing integer sequence $1 \leq i_1 < \cdots < i_s \leq k$, if $z_{i_1} = \cdots = z_{i_s}$, then the multiset of labels $\{(x_{i_1}, \ldots, x_{i_s})\}$ is represented by some point in $W$.

The condition $z_{i_1} = \cdots = z_{i_s}$ is vacuously true when $s = 1$, thus, the label $x$ of each particle is an element of $W_{(i)} \subseteq X_i$ for some $i$. We filter $C(M,N;W)$ by the subspaces

$$C_k(M,N;W) = C(M,N;W) \cap D_k(M,N;X')$$

having no more than $k$ non-degenerate particles.

These configuration spaces generalize McDuff’s and Bödghheimer’s [25, 5], originally studied by Boardman, Vogt, May, Giffen, and Segal. Several other generalizations have appeared since then, some of these bearing certain similarities to these (see [9, 22, 26, 10, 11] for example).

**Example 3.2.** If $W = X_1 \lor \cdots \lor X_n$, then the space $C(M,N;W)$ is homeomorphic to the product of classical labelled configuration spaces $C(M,N;X_1) \times \cdots \times C(M,N;X_n)$. If $W = X_1 \lor \cdots \lor X_n$, then $C(M,N;W) = C(M,N;W) \lor D(M,N;X')$.

When referring to collided particles in a labelled configuration of particles in $M$, we mean a subset of all particles in that configuration sharing the same point in $M$. We can think of these configuration spaces as being like classical configuration spaces with labels in $X'$, but with certain collisions allowed governed by the way $W$ sits inside its ambient space $X$. Here a single particle is considered to collide with itself, and at most $n$ particles can collide.

### 3.1 Faithful Neighbourhoods.

Recall a CW-complex $Z$ is said regular if its attaching maps are embeddings. A mapping cylinder neighbourhood of a subspace $A \subseteq Z$ is a neighbourhood of $A$ in $Z$ homeomorphic to a mapping cylinder $M_f = A \cup ([0,1] \times B)/\sim$, where $(0,b) \sim f(b)$, for some map $f:B \rightarrow A$. An open mapping cylinder neighbourhood is a subspace of $M_f$ of the form $A \cup ([0,t] \times B)/\sim$ for some $0 < t < 1$. A sub-mapping cylinder of $M_f$ is a subspace of $M_f$ of the form $A \cup ([0,1] \times B')/\sim$ for some subspace $B' \subseteq B$, where $f'$ restricts $f$ to $B'$. We call the subspaces $(0,1] \times \{b\} \subseteq M_f$ for $b \in B$ the fibers of $M_f$.

It is easy to see via a cell-by-cell induction that any subcomplex of a (geometric) simplicial complex has a mapping cylinder neighbourhood. Moreover, each of its fibers lie entirely in the interior a single simplex. Since the cells of a regular CW-complex can be subdivided to give it the structure of a simplicial complex, subcomplexes of regular CW-complexes also have mapping cylinder neighbourhoods, with each of their fibers lying entirely in a single cell.

The following is a coordinate-wise version of the neighbourhood deformation retraction property of a pair:
Definition 3.3. We say $W$ has a **faithful neighbourhood** if it has an open neighbourhood $U \subseteq X$ with associated basepoint preserving homotopies $h_{i,t}: X_i \to X_i$ for $i = 1, \ldots, n$, such that: (1) $h_{i,0}$ is the identity and $h_{i,t}(*) = *$; (2) there is an open neighbourhood $U_i$ of the basepoint $* \in X_i$ such that $h_{1}(U_i) = *$, and $h_{2}(U_i)$ is an open neighbourhood of $*$ for each $t \in [0,1]$; and (3) the homotopy $h_t = h_{1,t} \times \cdots \times h_{n,t}: X \to X$ satisfies $h_t(W) \subseteq W$ for each $t \in [0,1]$, $h_t(U) = W$, and $h_t(U)$ is an open neighbourhood of $W$ for each $t \in [0,1]$.

Since any open subset of $X$ is a union of products of open subsets of each $X_i$, the projection $X \to X_I = \prod_{i \in I} X_i$ maps open subsets to open subsets for any $I \subseteq [n]$, one can then check that:

**Lemma 3.4.** The image of our faithful neighbourhood $U$ of $W$ under the projection $X \to X_I$ is a faithful neighbourhood of $W_I$ in $X_I$, inheriting from $U$ the associated homotopies $h_{i,t}$ where $i \in I$. □

**Example 3.5.** Given a simplicial complex $K$ on vertex set $[n]$, the real moment-angle complex $W = (D^1, S^0)^K$ has as a faithful neighbourhood the polyhedral product $U = (D^1, [0,\frac{1}{2}) \cup (\frac{3}{4},1])^K$ with each $h_{i,t}: D^1 \to D^1$ defined explicitly by $h_{i,t}(s) = s(1-t)$ for $s \in [0,\frac{1}{2})$, $h_{i,t}(s) = s + t(s-\frac{1}{2})$ for $s \in (\frac{1}{2},\frac{3}{4})$, and $h_{i,t}(s) = s + t(1-s)$ for $s \in [\frac{3}{4},1]$. Note $U$ and the $h_{i,t}$’s also form a faithful neighbourhood for each subcomplex $W_k$.

**Example 3.6.** More generally, let $Y$ be a regular CW-complex and $A$ a subcomplex. The polyhedral product $W = (Y, A)^K$ has as a faithful neighbourhood the polyhedral product $U = (Y, M_f)^K$, where $M_f$ is an open mapping cylinder neighbourhood $A \cup_f ([0,\frac{1}{2}) \times B)$ of $A$. Each homotopy $h_{i,t}: Y \to Y$ is given as follows. Define $g_{i,t}: Y \to Y$ by extending to $Y$ the deformation retraction of $M_f$ onto $A$ to give our homotopy $g_{i,t}$. Then define $g'_{i,t}: Y \to Y$ by taking an open mapping cylinder neighbourhood $M'_f$ of the basepoint $*$ in $A$, extending it to an open mapping cylinder $U_i = M_*$ of $*$ in $Y$, and extending the deformation retraction of $M_*$ onto $*$ to give our homotopy $g'_{i,t}$. Note since $M'_f$ is a sub-mapping cylinder of $M_*$, the deformation retraction of $M_*$ onto $*$ restricts to a deformation retraction in $A$ of $M'_f$ onto $*$, so we can take $g'_{i,t}$ so that it restricts to $g'_{i,t}: A \to A$ on $A$, which we do. Then let $h_{i,t} = g'_{i,t} \circ g_{i,t}$.

If $A$ is a disjoint union $A' \bigsqcup \{\ast\}$, so that $M_f$ is a disjoint union $M_f' \bigsqcup M_*$, then each $W_k \subseteq Z_k(Y, A)$ has the subspace $U_k = \{ (x_1, \ldots, x_n) \in Z_k(Y, M_f) | x_i \in M_* \text{ for at least } n-k \text{ coordinates } x_i \}$ as a faithful neighbourhood, with the same associated homotopies $h_{i,t}$.

### 3.2. The Case of a Disk and its Boundary

From now on we assume that each $X_i$ is a connected basepointed regular CW-complex, $W$ is a connected subcomplex of their product $X$ containing the basepoint, and $W$ has a faithful neighbourhood in $X$.

Consider the inclusion $\iota: W^\ell \to C(D^\ell, \partial D^\ell; W)$ given by sending $((z_1, x_1), \ldots, (z_n, x_n))$ to the configuration $(z_1, z_n; x_1, x_n)$. Notice $\iota$ is the restriction of the likewise defined map $X^\ell \to C(D^\ell, \partial D^\ell; X)$. This map in turn is just the product of inclusions $\Sigma^\ell X_i \to C(D^\ell, \partial D^\ell; X_i)$ into the classical configuration spaces, each of which is a homeomorphism of $\Sigma^\ell X_i$ onto its image [25, 5]. Therefore $\iota$ is homeomorphism of $W^\ell$ onto its image, and we think of $W^\ell$ as a subspace of $C(D^\ell, \partial D^\ell; W)$ under this inclusion. Notice that a configuration of non-degenerate particles is in the image of $\iota$ if and only if the multiset of labels of all its non-degenerate particles is represented by a point in $W$.

**Lemma 3.7.** There is a deformation retraction of $C(D^\ell, \partial D^\ell; W)$ onto the subspace $W^\ell$ given by the inclusion $\iota$.

**Proof.** We use a two-step radial expansion argument, generalizing the deformation retraction for the classical labelled configuration space $C(D^\ell, \partial D^\ell; Y)$ onto $\Sigma^\ell Y$. We think of $D^\ell$ as the closed unit-disk centered at the origin in $\mathbb{R}^\ell$, and the radial expansion $R_t: D^\ell \to D^\ell$ here is given by $R_t(x) = tx / |tx|$ if $|tx| < 1$ and $R_t(x) = x$ if $|tx| \geq 1$. 
Let $\mathcal{C}$ denote $\mathcal{C}(D^i, \partial D^i; W)$, and $\mathcal{C}'$ be the subspace of configurations in $\mathcal{C}$ whose non-degenerate particles have labels in distinct $X_i$’s (thus, there can be no more than $n$ of such particles). To define a deformation retraction of $\mathcal{C}$ onto $\mathcal{C}'$ we define a homotopy $H_t: \mathcal{C} \to \mathcal{C}$, for $t \in [0, \infty)$, by applying the radial expansion to non-degenerate particles in a given configuration $\omega \in \mathcal{C}$, while at the same time we homotope labels ‘near’ the basepoint towards the basepoint. We do this by picking a mapping cylinder neighbourhood $M_t$ of the basepoint in each $X_i$ (these exist since $X_i$ is a regular CW-complex), keeping a label fixed if it is the basepoint or outside the interior of some $M_i$, or else homotoping it towards the basepoint along the corresponding fiber $(0, 1) \times \{b\} \subseteq M_i$ of the mapping cylinder, this being done at a speed that tends to zero the closer the label is to the boundary of $M_i$. We may assume each of the fibers $(0, 1) \times \{b\}$ of $M_i$ lies in a single cell in $X_i$. Then since $W$ as a subcomplex of $X$ is a union of products of cells in the $X_i$’s, under this homotopy of labels, collided particles still have their set of labels represented by a point in $W$. This defines $H$.

Notice only particles that are collided at the center of $D^i$ are unaffected by the radial expansion, but then their multiset of labels is represented by a point in $W$, so in particular their labels must already be in distinct $X_i$’s. On the other hand, particles away from the center may eventually enter the boundary $\partial D^i$ under the radial expansion, or have their labels homotoped to the basepoint, in either case becoming degenerate, until we are left with at most $n$ non-degenerate particles with labels in distinct $X_i$’s. Thus, there is a smallest sufficiently large time $t_\omega$ depending on $\omega$ such that $H_{t_\omega}(\omega)$ is a configuration in $\mathcal{C}'$, at which point we stop the homotopy $H$. To see that the map $\mathcal{C} \to [0, \infty), \omega \mapsto t_\omega$, is continuous, notice our homotopy of labels near the basepoint towards it ensures $t_\omega$ varies continuously as we vary the labels by making them approach a basepoint. Then given an open neighbourhood $V \subseteq [0, \infty)$ of $t_\omega$, for sufficiently small open neighbourhoods $V' \subseteq \mathcal{C}$ of $\omega$, the only possible configurations $\mathcal{C}$ that could be in $V'$, and for which $t_\omega$ could not be in $V$, would have to have at least two particles collided at the origin with labels in the same $X_i$. But such configurations cannot exist in $\mathcal{C}$, since they would not be represented by a point in $W$. This defines our first deformation retraction.

Finally, to define a deformation retraction of $\mathcal{C}'$ onto $W^i$, notice the set of labels of non-degenerate particles in a configuration in $\omega \in \mathcal{C}'$ corresponds to a distinct point $x$ in $X$ whose coordinates are each either a basepoint or one of the labels. So we can think of the set of labels of $\omega$ as a point $x \in X$. We define a homotopy $G_t: \mathcal{C}' \to \mathcal{C}'$, $t \in [0, \infty)$, by again apply the radial expansion $R_t$ to $\omega$ starting at $t = 0$, meanwhile, homotoping the set of labels towards $W$ when it is ‘near’ $W$. We must be careful how we perform this homotopy of labels, since we must ensure labels of particles that are collided at a point remain represented by a point in $W$, and that the homotopy does not become discontinuous as a particle enters the boundary and becomes degenerate (in which case the coordinate in $x$ corresponding to the label of this particle becomes the basepoint). We therefore use a faithful neighbourhood $U$ of $W$ with associated homotopy $h_t: X \to X$, and homotope labels near $W$ towards $W$ by mapping $x$ to $h_{f(t)}(x)$ at each time $t$, where $f(t) = \frac{t}{1 + t}$. This takes care of both of these issues since $h_t$ maps $W$ to $W$, and is defined coordinate-wise as a product of basepoint preserving maps $h_{i,t}: X_i \to X_i$. In particular, for each $I \subseteq [n]$, the product $h_{f(t)}: X_I \to X_I$ of the $h_{i,t}$’s over $i \in I$ maps $W_I$ to $W_I$. So if a subset of $|I|$ non-degenerate particles in $\omega$, each with a label in $X_i$ for each $i \in I$, are represented by a point in $W$, that is, their set of labels are the coordinates of a distinct point in $W_I$, they remain so under the homotopy $h_{f(t)}$. This defines $G_t$.

As those particles unaffected by radial expansion are collided at the origin and already represented by a point in $W$, we see that there is a smallest sufficiently large time $t'_\omega$ depending on $\omega$ when enough particles are annihilated at the boundary, or have the set of labels homotoped to a point in $W$, so that the labels of the non-degenerate particles are represented by a point in $W$. That is, $G_{t'_\omega}(\omega)$ becomes a point in the image of $t$, at which point we stop the homotopy $G$. As before, $\mathcal{C}' \to [0, \infty)$ mapping $\omega \mapsto t'_\omega$ is continuous. Here, since $\omega$ is in $\mathcal{C}'$, the labels of any subset of particles in $\omega$ are the coordinates of a point $p \in X_I$ for some $I \subseteq [n]$, and by Lemma 3.4, the homotopy $h_{i,t}$, restricting to $h_{I,t}$ on the labels $p$ makes $p$ approach $W_I$ near $W_I$. Moreover, since each $h_{i,t}$ by definition homotopes
points close to the basepoint in $X_i$ towards it, any label near a basepoint is homotoped towards it by $h_{f(t)}$. So $t'_n$ varies continuously as we homotope a subset of labels of $\omega$ towards a point in $W_i$, or each towards a basepoint. Then given an open neighbourhood $V \subseteq [0, \infty)$, for sufficiently small open neighbourhoods $V' \subseteq C'$ of $\omega$, the only possible configurations $\pi$ that could be in $V'$, and for which $t'_n$ could not be in $V$, would have to have particles collided at the origin that are not represented by a point in $W$, but these do not exist in $C'$. This defines our second deformation retraction. \hfill \box

3.3. The Scanning Map. Let $C(W)$ denote $C(\mathbb{R}, \varnothing; W)$, and $C_i(W)$ the subspace $C_i(\mathbb{R}, \varnothing; W)$. We define a scanning map

$$\gamma: C(W) \xrightarrow{\sim} C(D^1; W) \xrightarrow{\sim} \text{map}(D^1, \partial D^1; W) \cong \Omega W^1$$

analogously to the one for classical labelled configuration spaces as follows. The first homotopy equivalence in this composite is induced by the inclusion $\mathbb{R} \rightarrow D^1$ sending $t \mapsto \frac{t}{1+|t|}$ (with a choice of homotopy inverse induced by the inclusion $D^1 \rightarrow \mathbb{R}$). Let $D_{\varepsilon, z}$ denote the closed interval $[z-\varepsilon, z+\varepsilon] \subseteq \mathbb{R}$. For any configuration $\omega \in C(W)$, $\gamma(\omega)$ is the map $D^1 \rightarrow W^1$ defined for any $z \in D^1$ by taking $\gamma(\omega)(z)$ to be the image of $\omega$ under the composite

$$(6) \quad C(W) \rightarrow C(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; W) \xrightarrow{\sim} C(D^1, \partial D^1; W) \xrightarrow{r} W^1$$

where the first map forgets those particles from $\omega$ that are in $\mathbb{R} - D_{\varepsilon, z}$, the second homeomorphism induced by the homeomorphism $D^1 \cong D_{\varepsilon, z}$, and $r$ is the retraction from Lemma 3.7.

With regards to labels, $C()$ is a functor from the category of subspaces of length $n$ products of basepointed $CW$-complexes, and coordinate-wise maps between them. More precisely, given $Y = Y_1 \times \cdots \times Y_n$ a product of basepointed $CW$-complexes, basepoint preserving maps $f_i: X_i \rightarrow Y_i$, and a subspace $V \subseteq Y$, suppose the coordinate-wise map $\bar{f} = (f_1 \times \cdots \times f_n): X \rightarrow Y$ restricts to a map $f: W \rightarrow V$. This induces a map

$$C(f): C(W) \rightarrow C(V)$$

sending $(z_1, \ldots, z_n; x_1, \ldots, x_n) \mapsto (z_1, \ldots, z_n; f_1(x_1), \ldots, f_n(x_n))$, and a map

$f^1: W^1 \rightarrow V^1$

sending $((z_1, x_1), \ldots, (z_n, x_n)) \mapsto ((z_1, f_1(x_1)), \ldots, (z_n, f_n(x_n)))$. The scanning map $\gamma$ is then a natural transformation in the homotopy category:

**Lemma 3.8.** There is a homotopy commutative square

$$\begin{array}{ccc}
C(W) & \xrightarrow{C(f)} & C(V) \\
\downarrow \gamma & & \downarrow \gamma \\
\Omega W^1 & \xrightarrow{\Omega f^1} & \Omega V^1.
\end{array}$$

**Proof.** Since the following square commutes

$$\begin{array}{ccc}
W^1 & \xrightarrow{f^1} & V^1 \\
\downarrow & & \downarrow \\
C(D^1, \partial D^1; W) & \xrightarrow{C(f)} & C(D^1, \partial D^1; V),
\end{array}$$

we may replace the homotopy equivalences $\iota$ with their homotopy inverse $r$ to obtain a homotopy commutative square. Thus we have a diagram

$$\begin{array}{ccc}
C(W) & \xrightarrow{C(f)} & C(D; W) \xrightarrow{\sim} C(D^1, \partial D^1; W) \xrightarrow{r} W^1 \\
\downarrow C(f) & & \downarrow C(f) \\
C(V) & \xrightarrow{C(f)} & C(D; V) \xrightarrow{\sim} C(D^1, \partial D^1; V) \xrightarrow{r} V^1.
\end{array}$$
the left square commuting strictly, and the homotopy $f^1 \circ r = r \circ C(f)$ in the right square being independent of $z \in \mathbb{R}$. This then gives a homotopy $\gamma C(f)(\omega) \simeq \Omega f^1 \circ \gamma(\omega)$ of maps $D^1 \to W^1$ independently of $z$ and $\omega$, so in turn, it defines a homotopy $\gamma \circ C(f) \simeq \Omega f^1 \circ \gamma$. \hfill \Box

3.4. Suspension Splittings. Our goal is to obtain a (single) suspension splitting of $C(W)$. Take the quotient space

$$D_m(W) = \frac{C_m(W)}{C_{m-1}(W)}.$$

**Lemma 3.9.** The pair $(C_m(W), C_{m-1}(W))$ is an NDR-pair. Thus, the sequence

$$C_{m-1}(W) \to C_m(W) \to D_m(W)$$

is a cofibration sequence.

Proof. Since $X_i$ is a regular CW-complex, as in the proof of Lemma 3.7, we can take a mapping cylinder neighbourhood $M_i$ of the basepoint $\ast \in X_i$ such that each of the fibers $(0,1] \times \{b\} \subseteq M_i$ lie entirely in a single cell in $X_i$. Let $f_i: X_i \to [0,1]$ be the map given by $f_i(\ast) = 0$, $f_i(x) = 1$ if $x \notin M_i$, and $f_i(x) = s$ if $x$ is at the point $(s, b)$ along a fiber $(0,1] \times \{b\}$. Define $f: X_1 \vee \cdots \vee X_n \to [0,1]$ by restricting to $f_i$ on $X_i$, and let $u: C_m(W) \to [0,1]$ be given for any $\omega = \{(t_1, x_1), \ldots, (t_m, x_m)\}$ by $u(\omega) = 0$ if $\omega \in C_{m-1}(W)$, otherwise $u(\omega) = \prod_{i=1}^m f(x_i)$. Notice $u^{-1}(0) = C_{m-1}(W)$. Define a homotopy $H: C_m(W) \times [0,1] \to C_m(W)$ by homotoping the labels of a configuration $\omega \in C_m(W)$ simultaneously as follows. For each particle $(t, x) \in \omega$, we either keep the label $x$ fixed if $x$ is a basepoint or $x$ is not in the interior of any mapping cylinder $M_i$, or else we homotope $x$ towards the basepoint along the corresponding fiber of $(0,1] \times \{b\}$ of the mapping cylinder. This is done a speed that tends to zero the closer $x$ is to the boundary of corresponding mapping cylinder. Since each of the fibers of $M_i$ lies in a single cell of $X_i$, and since $W$ as a subcomplex of $X$ is a union of products of cells in the $X_i$’s, then under this homotopy of labels collided particles still have their set of labels represented by a point in $W$ as necessary. We stop homotoping the labels of $\omega$ as soon as there are less than $m$ particles (among our homotoped particles in $\omega$) that have non-basepoint labels. Thus $H_t$ restricts to the identity on $C_{m-1}(W)$. Notice at least one of the labels is in the interior of a mapping cylinder $M_i$ when $u(\omega) < 1$, and $u(\omega)$ approaches 1 if the labels of $\omega$ all approach points outside these mapping cylinders. Then for any $(s, b) \in (0,1] \times \{b\}$, there is a sufficiently large time $t$ such that $H_t(u^{-1}((0,s)))$ is contained in $C_{m-1}(W)$. This therefore defines a neighbourhood deformation retraction for $C_{m-1}(W)$ in $C_m(W)$. \hfill \Box

Let

$$V_m(W) = \bigvee_{i=1}^m D_i(W).$$

We now define a map

$$\zeta: C(W) \to C(\mathbb{R}; V_\infty(W)),$$

where the right-hand space is the classical configuration space of non-overlapping particles in $\mathbb{R}$ with labels in $V_\infty(W)$. Since $X_1 \vee \cdots \vee X_n$ wedged at the basepoint $\ast$ is a CW-complex, we may pick a map

$$f: X_1 \vee \cdots \vee X_n \to [0,1]$$

such that $f(\ast) = 0$ and $f(x) > 0$ when $x \neq \ast$. Let $\zeta_0: C_0(W) \to C(\mathbb{R}; V_\infty(W))$ be given by mapping the empty configuration to the empty configuration, and define

$$\zeta_m: C_m(W) - C_{m-1}(W) \to C(\mathbb{R}; V_\infty(W))$$

for $m > 0$ as follows. Notice that any configuration $\omega = \{(t_1, x_1), \ldots, (t_m, x_m)\} \in C_m(W) - C_{m-1}(W)$ has only non-degenerate particles, that is, the label $x_i$ of each particle in $\omega$ is not the basepoint $\ast$. Since any two non-degenerate particles in a configuration in $C(W)$ cannot collide when their labels are in the same summand $X_i$, (so $t_i \neq t_j$ when $x_i, x_j \in X_i - \{\ast\}$), there is a strict total order on $\omega$ given for any $y = (t_i, x_i), y' = (t_j, x_j) \in \omega$ by the relation...
• \( y < y' \) if and only if either \( x_i \in X_i - \{ \ast \} \) and \( x_j \in X_j - \{ \ast \} \) such that \( i < j' \), or else both \( x_i, x_j \in X_l - \{ \ast \} \) for some \( l \) and \( t_i < t_j \).

Given \( y \in \omega \), let \( \sigma(y) \) be the integer such that \( y \) is the \( \sigma(y) \)th smallest element in \( \omega \) with respect to this ordering. Then take the strict total order on the set \( 2^\omega \) of subsets of \( \omega \), given for any \( S, S' \subseteq \omega \) by the relation

\[
S < S' \quad \text{if and only if} \quad \sum_{y \in S} 10^\sigma(y) < \sum_{y' \in S'} 10^\sigma(y').
\]

This total order does not change as we move these particles and vary their labels in such a way that \( \omega \) moves along any path in \( C_m(W) - C_{m-1}(W) \), and if we take a subconfiguration \( \omega' \in \omega \) in \( C_{m'}(W) - C_{m'-1}(W) \), \( m' < m \), then the ordering on \( 2^{\omega'} \subset 2^\omega \) is the one inherited from \( 2^\omega \). Write \( S \leq S' \) if either \( S < S' \) or \( S = S' \), and let

\[
\eta_{\omega, S'} = \sum_{S \leq S'} \left( \prod_{(t, x_i) \in S} f(x_i) \right).
\]

Since each \( x_i \neq \ast \), we have \( f(x_i) > 0 \), so \( \eta_{\omega, S} < \eta_{\omega, S'} \) whenever \( S < S' \). Then we may define

\[
\zeta_m(\omega) = \{(\eta_{\omega, S}, \chi_S) \mid S \subseteq \omega \} \in C(\mathbb{R}; V_\infty(W))
\]

where the label \( \chi_S \in V_\infty(W) \) is the image of the configuration \( S \subseteq C_{|S|}(W) \) under the composite of the quotient map and inclusion

\[
C_{|S|}(W) \longrightarrow D_{|S|}(W) \longrightarrow V_\infty(W)
\]

\((S \subseteq \omega \) is indeed a configuration in \( C_{|S|}(W) \), since every subset of a represented multiset of labels is itself represented by a point in \( W \). Notice that when \( \omega \) approaches some configuration \( \omega' \in C_{m'}(W) - C_{m'-1}(W) \) for some \( m' < m \), that is, when some subset \( L \) of at least \( m - m' \) labels \( x_i \) from \( \omega \) approaches the basepoint, then \( \prod_{(x_i, t_i) \in S} f(x_i) \) approaches 0 and \( \chi_S \) approaches \( \ast \in V_\infty(W) \) for all \( S \subseteq \omega \) that contain a particle with a label in \( L \). Therefore, we see that \( \zeta_m(\omega) \) approaches \( \zeta_{m'}(\omega') \). Thus, we can define \( \zeta \) by restricting to \( \zeta_m \) on \( C_m(W) - C_{m-1}(W) \), for any \( m > 0 \), and \( \zeta_0 \) on the empty configuration.

Notice \( \zeta_m \) maps to the subspace \( C(\mathbb{R}; V_m(W)) \subseteq C(\mathbb{R}; V_\infty(W)) \). Restricting \( \zeta \) to a subspace \( C_t(W) \), there is a commutative diagram

\[
\begin{array}{ccc}
C_{t-1}(W) & \longrightarrow & C_t(W) & \longrightarrow & D_t(W) \\
\downarrow \zeta & & \downarrow \zeta & & \\
C(\mathbb{R}; V_{t-1}(W)) & \longrightarrow & C(\mathbb{R}; V_t(W)) & \longrightarrow & C(\mathbb{R}; D_t(W))
\end{array}
\]

with the left-hand and right-hand horizontal maps the inclusions and quotient maps respectively, and the bottom maps induced by the inclusion \( V_{t-1}(W) \longrightarrow V_t(W) \) and the quotient map \( V_t(W) \longrightarrow D_t(W) \). Since the composite of these two maps is the constant map, the composite of the maps in the bottom sequence is the constant map, which gives the right-hand vertical extension. Notice this extension sends a (quotiented) configuration \( \omega \) to a configuration with a single particle, whose label is \( \omega \). Recall from the theory of classical labelled configuration spaces \([27]\) there is a scanning map \( \lambda: C(\mathbb{R}; V_\infty(W)) \longrightarrow \Omega \Sigma V_\infty(W) \),

which in fact is a homotopy equivalence. This is also just a special case of our scanning map constructed above. Here one takes \( \Sigma V_\infty(W) \) as the coordinate suspension of \( V_\infty(W) \) on a single coordinate, meaning \( C(\mathbb{R}; V_\infty(W)) = C(\mathbb{R}; V_\infty(W)) \). Composing the vertical maps in Diagram (7) with the corresponding scanning maps \( \lambda \), and taking the adjoints of these, we obtain a commutative
diagram of cofibration sequences

\[
\Sigma C_{i-1}(W) \longrightarrow \Sigma C_i(W) \longrightarrow \Sigma D_i(W)
\]

(\ref{eq:diagram}

the left and middle vertical maps being the restrictions of the adjoint \(\zeta; \Sigma C(W) \longrightarrow \Sigma V_\infty(W)\) of the composite \(\lambda \circ \zeta\), and the bottom cofibration sequence the composite of the inclusion and quotient onto the summand \(\Sigma(D_i(W))\). The right-hand vertical extension can be homotoped to the identity using a homotopy to the identity of the composite \(\Sigma V_\infty(W) \longrightarrow C(D^1, \partial D^1; V_\infty(W)) \longrightarrow \Sigma V_\infty(W)\) of the inclusion from Lemma 3.7 and its left homotopy inverse used to define the scanning map. Then the middle vertical \(\zeta\) is a homotopy equivalence whenever the left-hand \(\zeta\) is, so by induction, we see that \(\Sigma C(W) \xrightarrow{\zeta} \Sigma V_\infty(W)\) is a homotopy equivalence. Thus

**Proposition 3.10.** There is a splitting

\[
\Sigma C(W) \simeq \bigvee_{i \geq 1} \Sigma D_i(W).
\]

\[
\square
\]

Remark: If each \(X_i = Y\) is a regular CW-complex, and \(W\) is a polyhedral product of the form \((Y, \ast)^K\), then by [9], \(C(W)\) is homotopy equivalent to \(\Omega(\Sigma Y, \ast)^K\). This improves on the stable splitting of \(\Omega(\Sigma Y, \ast)^K\) in [9].

3.5. A Refinement. Suppose \(W\) has the simplicial property. Recall our filtration \(\ast = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = W\), where \(W_k = \bigcup_{j \in \{0,1,\ldots,n\}} W_j\). Again, \(C_m(W_k)\) denotes \(C_m(\mathbb{R}, \emptyset; W_k)\), these configuration spaces being constructed by thinking of \(W_k\) as a subspace of \(X\). We still refer to elements in \(D_m(W_k) = C_m(W_k)/C_{m-1}(W_k)\) as configurations of labelled particles, in the sense that they are under the image of the quotient map \(C_m(W_k) \longrightarrow D_m(W_k)\). They must have exactly \(i\) particles, and are identified with the basepoint when one of them is a degenerate particle, that is, when its label is a basepoint. The empty configuration is the basepoint. Since \(C_m(W_k)\) is a subspace of \(SP_m(Y), Y = (\mathbb{R} \times X^\vee)/(\mathbb{R} \times \{\ast\})\), then \(D_m(W_k)\) is a subspace of \(SP_m(Y) = SP_m(\mathbb{R})/SP_{m-1}(\mathbb{R})\).

We call a sequence \(S = (s_1, \ldots, s_n)\) an \((m, n)\)-partition of a non-negative integer \(m\) if \(s_1 + \cdots + s_n = m\) and each \(s_i \geq 0\). Given such a sequence, let \(D_{S}(W_k)\) be the subspace of \(D_m(W_k)\) of configurations \(\omega\) where exactly \(s_i\) non-degenerate particles in \(\omega\) have labels in \(X_i\) for each \(1 \leq i \leq n\), or else \(\omega\) is the basepoint. Notice \(D_m(W_k)\) is the union of \(D_{S}(W_k)\) over all \((m, n)\)-partitions. To homotope the label of a particle in a configuration in \(D_{m}(W_k)\) from one summand \(X_j\) to another summand \(X_i\), one must make at least one label go through the basepoint, so the entire configuration must go through the basepoint. Therefore \(D_{S}(W_k) \cap D_{S'}(W_k) = \{\ast\}\) for any \((m, n)\)-partitions \(S\) and \(S'\), and

\[
D_m(W_k) = \bigvee_{(m, n)\text{-partitions } S} D_{S}(W_k)
\]

wedged at the basepoint, and the inclusion

\[
t_{m,k}: D_m(W_{k-1}) \longrightarrow D_m(W_k)
\]

induced by \(W_{k-1} \longrightarrow W_k\), and extending \(t_{m,k}: C_m(W_{k-1}) \longrightarrow C_m(W_k)\), restricts to an inclusion

\[
t_{S,k}: D_{S}(W_{k-1}) \longrightarrow D_{S}(W_k)
\]

for each \((m, n)\)-partition \(S\). Given a basepoint preserving map \(f: X \longrightarrow Y\) of non-degenerately basepointed spaces, let \(M(f) = M_f/([0, 1] \times \{\ast\}) = (Y \cup f([0, 1] \times X))/([0, 1] \times \{\ast\})\) be the reduced mapping cylinder of \(f\), and \(C(f) = M(f)/(\{1\} \times X)\) the reduced mapping cone of \(f\) (or homotopy cofiber of \(f\)). We write points in \(C(f)\) as pairs \((t, y)\), where \(t \in [0, 1], y \in Y\) if \(t = 0, y \in X\) if \(t = 0, y \in X\) if \(t > 0\), and
(t, y) is the basepoint * when \( t = 1 \) or \( y \) is the basepoint. Since \( \iota_{m,k} \) is as a wedge sum of \( \iota_{S,k} \) over \((m,n)\)-partitions \( S \),

\[
\mathcal{C}(\iota_{m,k}) \cong \bigvee_{(m,n)\text{-partitions } S} \mathcal{C}(\iota_{S,k}).
\]

Notice that the following property holds when \( W \) has the simplicial property.

(A) For each \( m < k \) a multiset \( \{x_1, \ldots, x_m\} \) is represented by a point in \( W_k \) if and only it is represented by a point in \( W_m \).

**Lemma 3.11.** Assume \( W \) has the simplicial property. If at most \( k-1 \) elements in \( S \) are nonzero, then \( D_S(W_{k-1}) = D_S(W_k) \) as subspaces of \( SP_m(Y) \), and \( \iota_{S,k} \) is the identity. In particular, \( D_m(W_{k-1}) = D_m(W_k) \), \( \iota_{m,k} \) is the identity, when \( m \leq k - 1 \).

**Proof.** Any non-degenerate configuration \( \omega \in D_S(W_k) \) has \( m \) particles when \( S \) is an \((m,n)\)-partition, but since collisions are not allowed between particles with labels in the same \( X_i \), no more than \( k-1 \) can collide at a point when at most \( k-1 \) elements in \( S \) are nonzero. Since the labels of any \( j \leq k-1 \) particles in \( \omega \) that are collided at a point are represented by a point in \( W_k \), by Property (A) they are represented by a point in \( W_j \subseteq W_{k-1} \). Therefore \( \omega \in D_S(W_{k-1}) \).

3.6. **Comparing Filtrations.** We again assume each \( X_i \) is a connected basepointed regular CW-complex, \( W \) is a connected subcomplex of their product \( X \) containing the basepoint and having a faithful neighbourhood. Moreover, we assume \( W \) has the simplicial property, and that its subcomplex \( W_{n-1} \) also has a faithful neighbourhood. Note \( W_1 \) is simply connected by Lemma 2.6.

Take the \((n,n)\)-partition \( A = (1, \ldots, 1) \). Then \( D_A(W_k) \) is the subspace of \( D_n(W_k) \) containing those configurations of \( n \) particles whose labels are each in a distinct \( X_i \). Our goal will be to prove the following.

**Proposition 3.12.** There exists a homotopy commutative diagram

\[
\begin{array}{cccccc}
\mathcal{C}(\iota_{A,n}) & \xrightarrow{\partial} & \Sigma D_A(W_{n-1}) & \xrightarrow{\Sigma \iota_{A,n}} & \Sigma D_A(W_n) & \xrightarrow{\Sigma \iota_{A,n}} & \mathcal{C}(\iota_{A,n}) \\
\phi \cong & & \Sigma C_n(W_{n-1}) & \xrightarrow{\Sigma \iota_{n,n}} & \Sigma C_n(W_n) & \xrightarrow{\Sigma \iota_{n,n}} & \mathcal{C}(\iota_{n,n}) \\
\Sigma^{n-1} W_n & \xrightarrow{ev} & W^1_{n-1} & \xrightarrow{ev} & W^1_n & \xrightarrow{ev} & \tilde{W}^1_n \\
\end{array}
\]

where consecutive maps in each row form homotopy cofibration sequences, the right-hand vertical homotopy equivalences are extensions induced by the homotopy commutativity of the middle squares. The top second and third vertical maps are homotopy inverses of the quotient maps coming from the splitting in Proposition 3.10, and the bottom maps \( ev \) are the adjoints \( \Sigma C_n(W_i) \xrightarrow{\gamma} \Sigma \Omega W^1_i \xrightarrow{eval} W^1_i \) of the corresponding scanning map \( \gamma \). The map \( \partial \) is the connecting map, given by collapsing the subspace \( D_A(W_n) \) of \( \mathcal{C}(\iota_{A,n}) \).

To some extent, this reduces the study of the filtration of \( W^1_n = W^1_k \) by the spaces \( W^1_k \) to the corresponding filtration for \( D_A(W_n) \).

**Corollary 3.13.** Considering the following:

(a) \( W^1_n \to \tilde{W}^1_n \) has a right homotopy inverse;
(b) \( \Sigma D_A(W_n) \to \Sigma \mathcal{C}(\iota_{A,n}) \cong W^1_n \) has a right homotopy inverse;
(c) \( \mathcal{C}(\iota_{A,n}) \xrightarrow{\partial} \Sigma D_A(W_{n-1}) \) is nullhomotopic.

Then (c) \( \iff (b) \Rightarrow (a) \).
Proof. The implication \((b) \Rightarrow (a)\) holds by Proposition 3.12, while \((c) \Rightarrow (b)\) is a standard property of homotopy cofibration sequences. Suppose \((b)\) holds. Then the homotopy cofibration sequence

\[
\Sigma D_A(W_{n-1}) \xrightarrow{\Sigma i_{t,A,n}} \Sigma D_A(W_n) \rightarrow \Sigma C(t_{A,n})
\]

trivializes. In particular \(\Sigma i_{t,A,n}\) has a left homotopy inverse \(i\). Since \(C(t_{A,n}) \xrightarrow{\partial} \Sigma D_A(W_{n-1}) \xrightarrow{\Sigma i_{t,A,n}} \Sigma D_A(W_n)\) is a homotopy cofibration sequence, \(\Sigma i_{t,A,n} \circ \partial\) is nullhomotopic, so \((i \circ \Sigma i_{t,A,n}) \circ \partial \approx \partial\) is nullhomotopic. Thus \((b) \Rightarrow (c)\).

\[\square\]

Corollary 3.14. \(W^1_n\) is a co-H-space if one of the conditions in Corollary 3.13 hold for \(W_I\) in place of \(W\) for each \(I \subseteq [n]\).

Proof. Combine Corollary 3.13 with Proposition 2.8. \[\square\]

3.7. Proof of Proposition 3.12. Take a configuration \(\omega = (t_1, \ldots, t_n; x_1, \ldots, x_n) \in D_A(W_n)\). Configurations in \(D_A(W_n)\) have particles whose labels are each in a distinct \(X_i\), so we can and will assume without loss of generality that \(x_i \in X_i\). Given \(x = (x_1, \ldots, x_n) \in \hat{W}_n\), then \(x\) is represented by \((x_1, \ldots, x_n) \in W_n\), so there is an inclusion \(\hat{W}_n \rightarrow D_A(W_n)\) given by mapping \(x \mapsto (0, \ldots, 0; x_1, \ldots, x_n)\). We can extend this to an inclusion

\[\phi: \Sigma^{n-1}\hat{W}_n \rightarrow C(t_{A,n})\]

by mapping the basepoint to the basepoint, otherwise

\[((t_1, \ldots, t_{n-1}), (x_1, \ldots, x_n)) \mapsto (\max\{|t_1|, \ldots, |t_{n-1}|, 0\}, (t_1, \ldots, t_{n-1}, 0; x_1, \ldots, x_n)),\]

where each parameter \(t_i\) is in the unit 1-disk \(D^1 = [-1,1]\). Notice \(\phi\) is a homeomorphism of \(\Sigma^{n-1}\hat{W}_n\) onto its image, so we regard \(\Sigma^{n-1}\hat{W}_n\) as a subspace of \(C(t_{A,n})\) under this inclusion.

Remark: Since points in \(W_{n-1}\) have at least one coordinate a basepoint, no more than \(n-1\) non-degenerate particles in a configuration in \(D_A(W_{n-1})\) can collide at a point. Therefore any non-degenerate point \((t_1, \ldots, t_n; x_1, \ldots, x_n) \in C(t_{A,n})\) must have \(t = 0\) when \(t_1 = \cdots = t_n\), i.e., we must be at the base of the mapping cone \(D_A(W_n) \subseteq C(t_{A,n})\).

Lemma 3.15. There is a deformation retraction of \(C(t_{A,n})\) onto the subspace \(\Sigma^{n-1}\hat{W}_n\) given by the inclusion \(\phi\).

Proof. Let \(A\) be the subspace of \(C(t_{A,n})\) consisting of the points \((t_1, \ldots, t_n; x_1, \ldots, x_n)\) such that \(t_n = 0\) and \(t = \max\{|t_1|, \ldots, |t_{n-1}|, 0\} \leq 1\), as well as the basepoint. Define a deformation retraction of \(C(t_{A,n})\) onto \(A\) by keeping the basepoint fixed, otherwise homotoping a point \((t_1, \ldots, t_n; x_1, \ldots, x_n) \in C(t_{A,n})\) to a point in \(A\), first by shifting the \(t_i\)’s, by linearly homotoping each \(t_i\) to \(t_i = t_i - t_n\) via \(t_i,s = (1-s)t_i + st_n\). Then, depending on \(t\) and \(\beta = \max\{|t_1|, \ldots, |t_{n-1}|, 0\}\), we do one of three things: (i) if \(t = \beta\), do nothing since we are done; (ii) if \(t > \beta\), radially expand the \(t_i\)’s to \(\frac{t}{\beta}t_i\); (iii) if \(t < \beta\), linearly homotope \(t\) towards \(\beta\), stopping either when \(t = \beta\), or when \(t = 1\) (we are at the basepoint). Notice \(\beta\) cannot be 0 in case \((ii)\), otherwise the \(t_i\)’s are 0, and by our preceding remark \(t = 0\). The homotopy in case \((iii)\) is possible since \(\beta > t \geq 0\), so our configuration of particles \((t_i, x_i)\) is necessarily in \(D_A(W_{n-1})\). Note no new collisions between particles are created during these homotopies, and this homotopy is continuous in the neighbourhood of the basepoint.

Next, we define a deformation retraction of \(A\) onto \(\Sigma^{n-1}\hat{W}_n\), homotoping a given point \(\omega = (t_1, \ldots, t_{n-1}; x_1, \ldots, x_{n})\) in \(A\) as follows. If \(\omega\) is the basepoint, we are done, so assume \(\omega\) is not the basepoint. Then \(t < 1\) and each \(x_i\) is not the basepoint. Since we are in \(C(t_{A,n})\), we may assume without loss of generality that \(x_i \in X_i\), so \(\hat{x} = (x_1, \ldots, x_n)\) is a point in \(\hat{X} = X_1 \wedge \cdots \wedge X_n\), and we regard the set of labels \(x_i\) as the point \(\hat{x}\) in \(\hat{X}\). Since \(\hat{x}\) is not the basepoint, there is a unique \(x \in X\) such that \(q(x) = \hat{x}\), where \(X \rightarrow \hat{X}\) is the quotient map. To start off the homotopy, if \(t = 0\), we do nothing. If \(t > 0\), we homotope \(t = \max\{|t_1|, \ldots, |t_{n-1}|, 0\}\) linearly towards 1 via \(t_s = (1-s)t + s\), and homotope each \(t_i\) for \(i \leq n-1\) by radial expansion via \(t_i,s = \frac{1}{t}t_i\) (so that \(t_s = \max\{|t_1,s|, \ldots, |t_{n-1},s|, 0\}\), while at
the same time, homotoping the set of labels \( \hat{x} \) by homotoping \( x \) towards \( W_n \) when it is ‘near’ \( W_n \) to obtain \( x_t \) at each time \( t \), meanwhile letting \( \hat{x}_t = q(x_t) \) to obtain our homotopy \( \hat{x} \mapsto \hat{x}_t \). We perform this homotopy of \( x \) using a faithful neighbourhood of \( W_n \) and applying its associated homotopy \( h_t: X \to X \) to \( x \), letting \( x_t = h_t(x) \), exactly as in the proof of Lemma 3.7, thereby ensuring that labels of collided particles remain represented by a point in \( W_n \) during the homotopy of labels \( \hat{x} \).

Since \( h_t \) is defined coordinate-wise as a product of maps \( X_i \to X_i \) that are basepoint preserving, our homotopy for \( \hat{x} \) is continuous for \( \hat{x} \) in the neighbourhood of the basepoint. The homotopy stops as soon as \( x_t \) becomes a point in \( W_n \), in which case \( \hat{x}_t \in W_n \), or else when \( t \) becomes 1, in which case \( \omega \) becomes the basepoint. Notice there is no issue of continuity, or the homotopy not stopping, when \( t = \max\{|t_1|, \ldots, |t_n|\}, 0 \) is at or near 0. This is because all particles are collided when \( t = 0 \). Thus (as a set) \( \hat{x} \) must be represented by a point in \( W_n \), and since there are \( n \) labels, \( \hat{x} \) must be a point \( \hat{W}_n \) and \( x \) a point in \( W_n \), and the homotopy stops immediately. At the same time, points \( \omega \) approaching some point with \( t = 0 \) must have their label set \( \hat{x} \) approach a point in \( W_n \), so \( x \) approaches a point in \( W_n \), and our homotopy of \( x \) near \( W_n \) towards \( W_n \) ensures continuity. We let \( t_\omega \) be the smallest time at which this homotopy stops for \( \omega \). For the same reasons as in the second step of the proof of Lemma 3.7, \( t_\omega \) depends continuously on \( \omega \in A \). This then defines our second deformation retraction.

\[ \square \]

We will want to describe a right homotopy inverse of \( \Sigma \phi \) that it is compatible with the scanning map. To do so we will need to define a version of the scanning map for quotiented configuration spaces. Fix \( C^\partial = C(D^1, \partial D^1; W_n), C_i^\partial = C_i(D^1, \partial D^1; W_n), \) and \( D_i^\partial = C_i^\partial / C_{i-1}^\partial \). Consider the map

\[ i: \hat{W}_n^1 \to D_n^\partial \]

given by sending \(((t_1, x_1), \ldots, (t_n, x_n))\) to \( (t_1, \ldots, t_n; x_1, \ldots, x_n) \).

Since the inclusion \( W_n^1 \to C^\partial \) from Lemma 3.7 includes into the subspace \( C_n^\partial \), we have an inclusion \( i_n: W_n^1 \to C_n^\partial \). In turn, \( i_n \) restricts to a map \( i_{n-1}: W_n^1 \to C_{n-1}^\partial \). Notice then our maps fit into a commutative square

\[ \begin{array}{ccc} W_n^1 & \xrightarrow{q} & \hat{W}_n^1 \\ \downarrow{i_n} & & \downarrow{i} \\ C_n^\partial & \xrightarrow{p} & D_n^\partial, \end{array} \]

where \( p \) and \( q \) are the quotient maps. Like \( i \), both \( i_n \) and \( \hat{i} \) are homeomorphisms onto their images, and we think of \( W_n^1 \) and \( \hat{W}_n^1 \) as subspaces of \( C_n^\partial \) and \( D_n^\partial \) under these inclusions.

**Lemma 3.16.** There is a deformation retraction of \( C_n^\partial \) onto the subspace \( W_n^1 \) given by the inclusion \( i_n \). Moreover, it extends to a deformation retraction of \( D_n^\partial \) onto the subspace \( W_n^1 \) given by the inclusion \( i \). Therefore, \( i_n \) and \( \hat{i} \) are homotopy equivalences.

**Proof.** Recall that the deformation retraction \( H: C^\partial \times [0, 1] \to C^\partial \) of \( C^\partial \) onto \( W_n^1 \) in the proof of Lemma 3.7. This is defined via a two-step radial expansion. A configuration \( \omega \) is homotoped by radial expanding particles towards the boundary, meanwhile, in the first step, labels near the basepoint are homotoped towards it until there are no more than \( n \) particles left, and in the second step, the labels of the remaining particles are homotoped towards \( W_n \) when they are near \( W_n \), this being done via a coordinate-wise homotopy \( h: X \times [0, 1] \to X \) that is basepoint preserving on each coordinate.

From this, we see that for any \( \omega = (t_1, \ldots, t_m; x_1, \ldots, x_m) \) and each \( t \in [0, 1] \), \( H \) satisfies \( H_t(\omega) = (t'_1, \ldots, t'_m; x'_1, \ldots, x'_m) \) such that \( t_i \in \partial D^1 \Rightarrow t'_i \in \partial D^1 \) and \( x_i = * \Rightarrow x'_i = * \). Degenerate particles thereby remain degenerate during this homotopy. Thus \( H \) restricts to a deformation retraction \( H^i \) of \( C_i^\partial \) onto \( W_n^1 \) when \( i \geq n \), and a deformation retraction \( H^n \) of \( C_i^\partial \) onto \( W_n^1 \) when \( i \leq n-1 \). Since \( H^n: C_i^\partial \times [0, 1] \to C_n^\partial \) restricts to \( H^{n-1}: C_{n-1}^\partial \times [0, 1] \to C_{n-1}^\partial \), there is a well-defined extension \( H \) of
\(H^n\), as in the composite

\[
\tilde{H} : D_n^\partial \times [0, 1] \longrightarrow (C_n^\partial \times [0, 1])/((C_{n-1}^\partial \times [0, 1])) \xrightarrow{\tilde{H}} D_n^\partial,
\]

with the first map being the quotient map collapsing \(s \times [0, 1]\) onto the basepoint. Then \(\tilde{H}'\) is a deformation retraction of \(D_n^\partial\) onto \(\tilde{W}_n^1\). 

The quotiented scanning map

\[
\tilde{\gamma} : D_n(W_n) \longrightarrow \Omega \tilde{W}_n^1
\]

is defined as follows. Let \(\mathcal{D}_n(M, N; W_n) = C_n(M, N; W_n)/C_{n-1}(M, N; W_n)\). Fix any \(\varepsilon > 0\) (this will be the radius of our scanning interval). For any configuration \(\omega' \in D_n(W_n)\), we define \(\tilde{\gamma}(\omega') : \mathbb{R} \longrightarrow \tilde{W}_n^1\) as the map given for any \(z \in \mathbb{R}\) as the image of \(\omega'\) under the composite

\[
D_n(W_n) \longrightarrow D_n(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; W_n) \xrightarrow{\varepsilon} D_n^\partial \xrightarrow{\tilde{\gamma}} \tilde{W}_n^1
\]

where \(D_{\varepsilon, z} = [z - \varepsilon, z + \varepsilon] \subseteq \mathbb{R}\), and where the first map forgets those particles from \(\omega'\) that are in \(\mathbb{R} - D_{\varepsilon, z}\) (thus maps to the basepoint if at least one particle is in \(\mathbb{R} - D_{\varepsilon, z}\), the second homeomorphism is induced by the homeomorphism \(D_{\varepsilon, z} \cong D^1\), and \(\tilde{r}\) is the homotopy inverse of \(\tilde{r}\) taken as the retraction from Lemma 3.16. Recall that the regular scanning map \(\gamma\) is defined similarly. In the case of \(C(W_n)\), the restriction of \(\mathcal{C}(W_n) \longrightarrow \Omega W_n^1\) to \(\mathcal{C}_n(W_n)\) is defined by sending \(\omega \in \mathcal{C}_n(W_n)\) to the map \(\gamma(\omega) : \mathbb{R} \longrightarrow W_n^1\), given for any \(z \in \mathbb{R}\) as the image of \(\omega\) under

\[
\mathcal{C}_n(W_n) \longrightarrow \mathcal{C}_n(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; W_n) \xrightarrow{\varepsilon} \mathcal{C}_n^\partial \xrightarrow{\gamma} W_n^1
\]

where the first map forgets particles in \(\mathbb{R} - D_{\varepsilon, z}\) as before, and (since the deformation retraction \(H\) in the proof of Lemma 3.16 restricts as it does) \(\tilde{r}\) is the homotopy inverse of \(\tilde{r}\) taken as the retraction from Lemma 3.16. Replacing the homotopy equivalences \(\tilde{r}\) and \(\tilde{r}\) in Diagram (12) with their homotopy inverses \(\tilde{r}\) and \(\tilde{r}\) gives a homotopy commutative diagram, so we obtain a homotopy \(q \circ r \cong \tilde{r} \circ p\). Combining this homotopy with the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_n(W_n) & \longrightarrow & \mathcal{C}_n(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; W_n) \\
\downarrow{\gamma} & & \downarrow{\tilde{\gamma}} \\
\Omega W_n^1 & \longrightarrow & \Omega \tilde{W}_n^1
\end{array}
\]

where the vertical maps are the quotient maps, gives a homotopy of \(\tilde{\gamma}(p'(\omega)) \cong q(\gamma(\omega))\) that is independent of \(\omega\) and \(z\). Therefore:

Lemma 3.17. There is a homotopy commutative square

\[
\begin{array}{ccc}
\mathcal{C}_n(W_n) & \longrightarrow & \mathcal{D}_n(W_n) \\
\downarrow{\gamma} & & \downarrow{\tilde{\gamma}} \\
\Omega W_n^1 & \longrightarrow & \Omega \tilde{W}_n^1
\end{array}
\]

Let

\[
\tilde{e}v : \Sigma \mathcal{D}_n(W_n) \longrightarrow \tilde{W}_n^1
\]

be the adjoint of \(\mathcal{D}_n(W_n) \longrightarrow \Omega \tilde{W}_n^1\). The composite \(\Sigma \mathcal{D}_n(W_n) \xrightarrow{\Sigma \mathcal{r}_{n,n}} \Sigma \mathcal{D}_n(W_n) \xrightarrow{\tilde{e}v} \tilde{W}_n^1\) is nullhomotopic as follows. Since \(\tilde{\gamma}\) is given by scanning an \(\varepsilon\)-neighbourhood (for some fixed \(\varepsilon > 0\)) through a configuration \(\omega\) of labelled particles in \(\mathbb{R}\), while applying a radial expansion to particles in this neighbourhood to obtain a point along a based loop in \(\tilde{W}_n^1\) (as defined in the proof of Lemmas 3.7 and 3.16). If no \(\varepsilon\)-neighbourhood contains all \(n\) particles, then every point in the loop is the basepoint in \(\tilde{W}_n^1\), in which case \(\tilde{e}v\) will map \((t, \omega)\) to the basepoint in \(\tilde{W}_n^1\) for all \(t \in [-1, 1]\). Then since no
more than \( n - 1 \) particles in a configuration \( \omega \in \mathcal{D}_n(W_{n-1}) \) can collide at a point, we can homotope \( \bar{e}v \circ \Sigma_{n,n}(t, \omega) \) to the basepoint by homotopying \((t, \omega)\) via radial expansion of the \( n \) particles in \( \omega \) away from \( 0 \in \mathbb{R} \), until no \( \varepsilon \)-neighbourhood in \( \mathbb{R} \) contains them all (alternatively, one can scan ever smaller neighbourhoods). This defines a nullhomotopy of \( \bar{e}v \circ \Sigma_{t_{n,n}} \), and in turn, this nullhomotopy defines an extension

\[
\rho: \Sigma C(t_{n,n}) \rightarrow \hat{W}_n^1
\]

of \( \bar{e}v \). Take the composite

\[
\rho_A: \Sigma C(t_{A,n}) \rightarrow \Sigma C(t_{n,n}) \rightarrow \hat{W}_n^1
\]

where the first map is the inclusion. This is not the same map as the suspension of the retraction map onto \( W_n^1 \cong \Sigma^n W_n \) in Lemma 3.15, but it will be of use to us since it fits naturally with the maps \( ev \) and \( \bar{e}v \).

**Corollary 3.18.** The composite \( \rho_A \circ \Sigma \phi \) is a homotopy equivalence. Thus, \( \rho_A \) is a homotopy equivalence.

**Proof.** Take an element \( \omega = ((t_1, \ldots, t_n), (x_1, \ldots, x_n)) \in \hat{W}_n^1 \). Since \( x = (x_1, \ldots, x_n) \) is in \( W_n \), any subset of \( \{x_1, \ldots, x_n\} \) is represented the point \( x \) in \( W_n \). Then since the homeomorphism image of the inclusion \( W_n \rightarrow \mathcal{D}_n^0 \) in Lemma 3.16 is the subspace of configurations whose particles have labels represented by a point in \( W_n \), and since \( \bar{g} \) is defined in terms of a deformation retraction of \( \mathcal{D}_n^0 \) onto this image, then the mapping \( \omega \mapsto \rho_A \circ \Sigma \phi(\omega) \) does not change the labels \( x_i \) of \( \omega \), that is, \( \rho_A \circ \Sigma \phi(\omega) = ((t_1, \ldots, t_n), (x_1, \ldots, x_n)) \) for some \( t_i \). Moreover, one can check directly from the construction of \( \rho_A \) and \( \phi(\omega) \) that each \( t_i \) satisfies the following for each \( c = \pm 1 \) as \( (t_1, \ldots, t_n) \) approaches the boundary of \( D^n = [-1, 1]^n \); (1) \( \bar{t}_n \) approaches \( c \) as \( t_n \) approaches \( c \) (since the scanning maps scans a neighbourhood from right to left in \( \mathbb{R} \)); and (2) for \( i \leq n - 1 \), if \( t_i \) approaches \( c \), then either \( \bar{t}_i \) approaches \( c \), or else \( t_n \) approaches \( -c \) (as does \( \bar{t}_n \)). So if we define a linear homotopy from \( \bar{t}_i \) to \( t_i \) via \( t_{i,s} = (1 - s)\bar{t}_i + st_i \), then (1) and (2) hold with \( t_{i,s} \) in place of \( \bar{t}_i \), and we see that \( (t_{1,s}, \ldots, t_{n,s}) \) approaches the boundary of \( D^n \) for each \( s \) whenever \( (t_1, \ldots, t_n) \) does.

There are therefore no issues of continuity in defining a homotopy from \( \rho_A \circ \Sigma \phi \) to the identity by mapping \( \omega \) to \( ((t_{1,s}, \ldots, t_{n,s}), (x_1, \ldots, x_n)) \) at each time \( s \). Then using Lemma 3.15, \( \rho_A \) is a homotopy equivalence since both \( \rho_A \circ \Sigma \phi \) and \( \Sigma \phi \) are.

**Lemma 3.19.** Suppose we are given a homotopy commutative diagram of homotopy cofibration sequences

\[
\begin{array}{ccc}
Z & \xrightarrow{\partial} & X \\
\downarrow{\bar{g}} & & \downarrow{g} \\
A & \xrightarrow{h} & B
\end{array} \quad \Sigma Z \xrightarrow{\Sigma g} \Sigma Y \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma \partial} \Sigma Z
\]

where the homotopy equivalence \( \bar{g} \) is an induced map of homotopy cofibers, and all spaces are homotopy equivalent to simply connected CW-complexes. Then the bottom homotopy cofibration sequence extends to the left to a homotopy cofibration sequence

\[
\begin{array}{ccc}
\Sigma Z & \xrightarrow{\Sigma \bar{g}} & A \\
\downarrow{\Sigma \bar{g}} & & \downarrow{h} \\
Z & \xrightarrow{\partial} & X \\
\downarrow{\bar{g}} & & \downarrow{g} \\
A & \xrightarrow{h} & B
\end{array}
\]

**Proof.** Since the induced map \( \bar{g} \) is a homotopy equivalence, the left-most square \( Q \) in the above diagram is a homotopy pushout, that is, \( Q \) is equivalent in the homotopy category to a pushout square whose horizontal maps are cofibrations (the dual of Proposition 7.6.1 in [28]). Likewise, if we take \( B' \) to be the homotopy cofiber of \( \bar{g} \circ \partial \), we obtain a homotopy commutative diagram of homotopy cofibrations

\[
\begin{array}{ccc}
Z & \xrightarrow{\partial} & X \\
\downarrow{\bar{g}} & & \downarrow{g'} \\
A & \xrightarrow{h'} & B'
\end{array} \quad \Sigma Z \xrightarrow{\Sigma g'} \Sigma B' \xrightarrow{\Sigma h'} \Sigma Z
\]
where the last two vertical maps are iteratively induced maps of homotopy cofibers, and so we see the middle square $Q'$ here is a homotopy pushout as well. We obtain a map of homotopy pushouts $Q \xrightarrow{q} Q'$, and thus a map $B' \xrightarrow{p} B$, by mapping $X, Y$, and $A$ to themselves via the identity, and since these three are homotopy equivalences, so is $p$. From the cube diagram given by $q$ we see that $p \circ h'$ is homotopic to $h$, and we are done.

Proof of Proposition 3.12. Take the homotopy equivalences $\zeta'$ from (8). It is clear that they make left square commute in the diagram of homotopy cofibration sequences

\[
\begin{array}{c}
\Sigma C_n(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma C_n(W_n) \xrightarrow{\zeta'} \Sigma C(\iota_{n,n}) \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma V_n(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma V_n(W_n) \xrightarrow{\zeta'} \bigvee_{1 \leq i \leq n} \Sigma C(\iota_{i,n}),
\end{array}
\]

where the bottom sequence is the wedge sum of homotopy cofibration sequences $\Sigma D_i(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma D_i(W_n)$ for $1 \leq i \leq n$, thus giving the right vertical extension $\hat{\zeta}'$. Since the left and middle vertical maps are homotopy equivalences, so is $\zeta'$. But since $\iota_{m,k}$ is a homeomorphism when $m < k$ by Lemma 3.11, the only non-contractible homotopy cofibre $\Sigma C(\iota_{i,n})$ between $1 \leq i \leq n$ is $\Sigma C(\iota_{n,n})$, implying that the composite

\[
\hat{\nu}: \Sigma C(\iota_{n,n}) \xrightarrow{\text{include}} \bigvee_{1 \leq i \leq n} \Sigma C(\iota_{i,n}) \xrightarrow{(\zeta')^{-1}} \Sigma C(\iota_{n,n})
\]

is a homotopy equivalence, where $(\zeta')^{-1}$ is a homotopy inverse of $\zeta'$. Replacing the vertical homotopy equivalences $\zeta'$, $\hat{\zeta}'$ with their inverses, and restricting to the homotopy cofibration sequence $\Sigma D_n(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma D_n(W_n) \xrightarrow{\zeta'} \Sigma C(\iota_{n,n})$, gives a homotopy commutative diagram of homotopy cofibrations

\[
\begin{array}{c}
\Sigma D_n(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma D_n(W_n) \xrightarrow{\nu} \Sigma C(\iota_{n,n}) \\
\downarrow \quad \downarrow \quad \downarrow \hat{\nu} \\
\Sigma C_n(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma C_n(W_n) \xrightarrow{\zeta'} \Sigma C(\iota_{n,n}).
\end{array}
\]

In turn, the top homotopy cofibration sequence splits as the wedge sum of homotopy cofibrations sequences

\[
\Sigma D_S(W_{n-1}) \xrightarrow{\Sigma \iota_{S,n}} \Sigma D_S(W_n) \xrightarrow{\zeta'} \Sigma C(\iota_{S,n})
\]

over all $(n,n)$-partitions $S$, and since $\iota_{S,k}$ is a homeomorphism when at most $k - 1$ elements in an $(m,n)$-partition $S$ are non-zero, the only non-contractible homotopy cofibre in this splitting is $\Sigma C(\iota_{A,n})$, so the inclusion

\[
\Sigma C(\iota_{A,n}) \xrightarrow{\text{inclusion}} \Sigma C(\iota_{n,n})
\]

is a homotopy equivalence. Restricting to the sequence $S = A$, we obtain the top homotopy commutative diagram of homotopy cofibrations in the statement of the proposition.

As for the bottom diagram of homotopy cofibrations, note by adjoining the square in Lemma 3.17, we have a commutative diagram

\[
\begin{array}{c}
\Sigma C_n(W_n) \xrightarrow{\nu} \Sigma D_n(W_n) \xrightarrow{\zeta'} \Sigma C(\iota_{n,n}) \\
\downarrow \quad \downarrow \quad \downarrow \hat{\rho} \\
W_n^1 \xrightarrow{\nu} \Sigma D_n(W_n) \xrightarrow{\zeta'} W_n^1.
\end{array}
\]
The top left horizontal quotient map fits into a commutative diagram of homotopy cofibrations

\[
\begin{array}{ccc}
\Sigma\mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n-n}} & \Sigma\mathcal{C}_n(W_n) \\
& | & | \\
& q & | \\
& \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n-n}} & \Sigma\mathcal{D}_n(W_n) \\
\end{array}
\]

induced by the left square, where the vertical maps are the quotient maps. Combining the right square with the first diagram, we obtain a commutative diagram of homotopy cofibration sequences

\[(14)\]

\[
\begin{array}{ccc}
\Sigma\mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n-n}} & \Sigma\mathcal{C}_n(W_n) \\
& | & | \\
& ev & | \\
& W_{n-1}^1 & \xrightarrow{ev} & W_n^1 \\
\end{array}
\]

where \(\tilde{\rho}\) is the composite

\[\tilde{\rho}: \Sigma\mathcal{C}(i_{n,n}) \xrightarrow{q} \Sigma\mathcal{C}(i_{n,n}) \xrightarrow{\rho} \tilde{W}_n^1.\]

To see that \(\tilde{\rho}\) is a homotopy equivalence, note by Corollary 3.18 the composite

\[\rho_\Delta: \Sigma\mathcal{C}(i_{\Delta,n}) \xrightarrow{\nu} \Sigma\mathcal{C}(i_{n,n}) \xrightarrow{\rho} \tilde{W}_n^1\]

is a homotopy equivalence, and since we saw above that the first inclusion is a homotopy equivalence, then so is \(\rho\). The composite of \(\Sigma\mathcal{C}_n(W_n) \xrightarrow{\nu} \Sigma V_n(W_n)\) with the quotient map onto the summand \(\Sigma\mathcal{D}_n(W_n)\) is the left homotopy inverse of the map \(\nu\) in diagram (13), and by the right-hand commutative square in diagram (8) in Section 3.4, this composite is equal to the composite

\[\Sigma\mathcal{C}_n(W_n) \xrightarrow{q} \Sigma\mathcal{D}_n(W_n) \xrightarrow{ev} \Sigma\mathcal{D}_n(W_n)\]

of \(q\) and some self homotopy equivalence. Thus, \(q\) is also a left homotopy inverse of \(\nu\), and similarly for \(W_{n-1}\) in place of \(W_n\). Then composing (13) with the commutative diagram

\[
\begin{array}{ccc}
\Sigma\mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n-n}} & \Sigma\mathcal{C}_n(W_n) \\
& q & \ \\
& \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n-n}} & \Sigma\mathcal{D}_n(W_n) \\
\end{array}
\]

gives a commutative diagram of homotopy cofibrations

\[
\begin{array}{ccc}
\Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n-n}} & \Sigma\mathcal{D}_n(W_n) \\
& q\nu & \ \\
& \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n-n}} & \Sigma\mathcal{D}_n(W_n) \\
\end{array}
\]

Since both composites \(q \circ \nu\) are homotopic to the identity, \(\hat{q} \circ \hat{\nu}\) is a homotopy equivalence. But \(\hat{\nu}\) is a homotopy equivalence, so \(\hat{q}\) is homotopy equivalence, and therefore \(\tilde{\rho} = \rho \circ \hat{q}\) is a homotopy equivalence since \(\rho\) is. Thus, we have obtain the right-side of the bottom homotopy commutative diagram of homotopy cofibrations in the statement of the proposition.

It remains to show that the bottom cofibration sequence in the proposition extends to the left, and to obtain the left-hand square involving \(\phi\). This is clear when \(n = 1\), since \(W_{n-1}^1 = \ast\) in this case. Since \(\Sigma^{n-1}W_n\) is simply connected when \(n \geq 2\), and \(\phi\) is a homotopy equivalence by Lemma 3.15, the \(n \geq 2\) case follows by Lemma 3.19.
4. The Golod Property

From now on we fix $W = (D^1, S^0)^K$ as a subspace of $X = (D^1)^{\times m}$, where $K$ on vertex set $[n]$ has no ghost vertices. Thus, $W^1 = (D^2, S^1)^K$. Since $W$ is connected, has the simplicial property, and each $W_k$ has a faithful neighbourhood by Examples 3.6 and 3.5, all of our previous results apply.

The following confirms a large primes version of Conjecture 1.1, generalizing Berglund’s rational result in [4].

**Theorem 4.1.** Localized at any sufficiently large prime $p$, $(D^2, S^1)^K$ is homotopy equivalent to a wedge of spheres if and only if $K$ is Golod over $\mathbb{Z}_p$.

**Proof.** The right-hand implication is clear. For the left-hand, recall that a space $Y$ is a co-$H$-space if and only if the evaluation map $\Sigma \Omega Y \xrightarrow{ev} Y$ has a right homotopy inverse [13], and if $Y$ is homotopy equivalent to a finite CW-complex, then $\Sigma Y$ is homotopy equivalent to a wedge of spheres when localized at sufficiently large primes (depending on $Y$).

Since $W$ is a finite CW-complex, $\Omega W$ has finite type and the skeleta of $\Omega W$ are finite CW-complexes. Suspending, the skeleta of $\Sigma \Omega W$ are homotopy equivalent to wedges of spheres when localized at sufficiently large primes. Also, the evaluation map $\Sigma \Omega W^1 \xrightarrow{ev} W^1$ induces a surjection on $\mathbb{Z}_p$-homotopy whenever $H^*(W^1; \mathbb{Z}_p)$ has trivial cup products and Massey products, i.e. when $K$ is Golod over $\mathbb{Z}_p$ (see Theorem 3.2 and the proof of Theorem 3.1 in [15]). Then $H_*(W^1; \mathbb{Z}_p)$ being a finite dimensional vector space, for sufficiently large $k$ and prime $p$, we can construct a right homotopy inverse for $ev$ localized at $p$ by taking the inclusion of a skeleton $(\Sigma \Omega W)^{(k)} \xrightarrow{\text{inclusion}} \Sigma \Omega W$ and selecting the appropriate spheres from a $p$-local splitting of $(\Sigma \Omega W)^{(k)}$. In particular, $W^1$ is homotopy equivalent to a wedge of spheres localized at $p$. \qed

We now focus on the integral case. An ordered partition of a finite set $I$ is a sequence of disjoint non-empty subsets $I = (I_1, \ldots, I_m)$ of $I$ such that $I = I_1 \cup \cdots \cup I_m$. If $I$ is a subset of the integers $[n]$ with $k = |I|$ elements, and $S = (s_1, \ldots, s_k)$ is any sequence of real numbers, let $i_\ell$ denote the $\ell$th smallest element in $I$, $s'_j$ denote the $j$th smallest element in the set $S = \{s_1, \ldots, s_k\}$, and $m = |S|$ be the number of distinct elements in $S$. Then assign to $S$ the ordered partition $I_S = (I_1, \ldots, I_m)$ of $I$ where $I_j = \{i_\ell \in I \mid s_\ell = s'_j\}$. When $I = [n]$ we have $k = n$ and $I_j = \{\ell \mid s_\ell = s'_j\}$, so for example, if $I = [4]$ and $S = (-1, 0, -1, 0)$, then $[4]_S = \{(1, 3), (4, 2)\}$.

We think of the topological $(n-1)$-simplex $[\Delta^{n-1}]$ and its subcomplexes $[K]$ as being without basepoint. The first suspensions $\Sigma [\Delta^{n-1}]$ and $\Sigma [K]$ are therefore unreduced. Again, the suspensions $\Sigma [\Delta^{n-1}]$ and $\Sigma [K]$ are themselves basepointed, with the basepoint $\ast$ the tip of the double cone corresponding to the basepoint $-1 \in D^1 = [-1, 1]$. Higher suspensions are therefore reduced. This means that any point $(t_1, \ldots, t_n, z) \in \Sigma^n [K]$ is identified with the basepoint $\ast$ if and only if $t_n = -1$ or $t_i = \pm 1$ for some $i < n$.

Take the diagonal

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \cdots = x_n\}$$

and consider the following subspace of the smash product $P_n = (\mathbb{R}^n - \Delta_n) \wedge \Sigma [\Delta^{n-1}]$:

$$Q_K = \bigcup_{y \in (\mathbb{R}^n - \Delta_n), t_1, \ldots, t_m = [n]} \{y \wedge \Sigma [K_{I_1} \ast \cdots \ast K_{I_m}]\}$$

where $[n]_y$ is a partition $[n]_{y_1, \ldots, y_n}$ of $[n]$ as defined above. Here we took $\mathbb{R}^n - \Delta_n$ to be without a basepoint, so $P_n$ is the half-smash product $((\mathbb{R}^n - \Delta_n) \times \Sigma [\Delta^{n-1}])/(\mathbb{R}^n - \Delta_n) \times \{\ast\}$, and is basepointed.

**Definition 4.2.** A simplicial complex $K$ on vertex set $[n]$ is *weakly coherently homotopy Golod* if $K$ is a single vertex, or (recursively) $K \setminus \{i\}$ is weakly coherently homotopy Golod for each $i \in [n]$, and the map

$$\Phi_K : \Sigma^n [K] \to \Sigma Q_K$$

given for any $z \in [K]$, $t_1, \ldots, t_{n-1}, t \in [-1, 1]$, and $\beta = \max\{|t_1|, \ldots, |t_{n-1}|, 0\}$ by

$$\Phi_K(t_1, \ldots, t_{n-1}, t, z) = (2\beta - 1, (t_1, \ldots, t_{n-1}, 0), (t, z))$$
is nullhomotopic.

**Theorem 4.3.** \((D^2, S^1)^K\) is a co-H-space when \(K\) is weakly coherently homotopy Golod.

We will need a bit of setup before we can prove this. Recall that \(\bar{W}\) is the image of \(W\) under the quotient map \((D^1)^n \to (D^1)^n\). To prove this theorem we will need to use a simplicial description of \(\bar{W}\). In [2] the theory of diagrams of spaces was used to show the existence of a homeomorphism \(\Sigma[K] \cong \bar{W}\), but for our purposes, we give an explicit description of this homeomorphism as follows. Take the simplex \(\Delta^{n-1}\) to be on vertex set \([n]\), and think of its geometric realization as the subspace

\[|\Delta^{n-1}| = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_i \leq 1 \text{ and } t_1 + \cdots + t_n = 1\}\]

with each face \(\sigma \in \Delta^{n-1}\) corresponding to the subspace \(|\sigma| = \{(t_1, \ldots, t_n) \in |\Delta^{n-1}| \mid t_i = 0 \text{ if } i \notin \sigma\}\).

Take the subspace \(|\Delta^{n-1}|\) and \((D^1)^n\):

\[U_i = \{(t_1, \ldots, t_n) \in |\Delta^{n-1}| \mid t_i = \max\{t_1, \ldots, t_n\}\},\]

\[V_{i,t} = \{(t_1', \ldots, t_n') \in (D^1)^n \mid t_i' = t = \min\{t_1', \ldots, t_n'\}\} .\]

Consider a homeomorphism

\[(15) \quad h: \Sigma|\Delta^{n-1}| \xrightarrow{\cong} (D^1)^n\]

given as follows. For \(-1 < t < 1\), take the homeomorphism \(h_{i,t}: U_i \to V_{i,t}\) given by mapping

\[(t_1, \ldots, t_n) \mapsto \left( t + (1 - t) \frac{t_i - t_1}{t_i}, \ldots, t + (1 - t) \frac{t_i - t_n}{t_i} \right),\]

and its inverse homeomorphism \(h_{i,t}^{-1}: V_{i,t} \to U_i\) is given by mapping

\[(t_1', \ldots, t_n') \mapsto \left( \frac{1 - t_1'}{n - \sum_{k=1}^{n} t_k'}, \ldots, \frac{1 - t_n'}{n - \sum_{k=1}^{n} t_k'} \right)\]

(to see that \(h_{i,t} \circ h_{i,t}^{-1}\) and \(h_{i,t}^{-1} \circ h_{i,t}\) are the identity, use the fact that here \(t'_i = t\) and \(1 = t_1 + \cdots + t_n\)). Notice that \(h_{i,t}\) agrees with \(h_{j,t}\) on \(U_i \cap U_j\), and \(h_{i,t}^{-1}\) agrees with \(h_{j,t}^{-1}\) on \(V_{i,t} \cap V_{j,t}\). Thus, for each \(t \in (-1, 1)\), we can piece together the homeomorphisms \(h_{i,t}\) over all \(i \in [n]\) to give a homeomorphism 

\[h_t: |\Delta^{n-1}| = \left( \bigcup_{i=1}^{n} U_i \right) \xrightarrow{\cong} \left( \bigcup_{i=1}^{n} V_{i,t} \right) = V_t\]

defined unambiguously by mapping \(z \mapsto h_{i,t}(z)\) if \(z \in U_i\), with inverse \(h_t^{-1}\) defined by mapping \(z' \mapsto h_{i,t}^{-1}(z')\) if \(z' \in V_{i,t}\). Notice that \((D^1)^n\) is the disjoint union of \(V_t\) over all \(t \in [-1, 1]\), with \(V_1\) being a single point \((1, \ldots, 1)\), and \(V_{-1}\) being the basepoint, and the maps \(h_t\) vary continuously with respect to \(t\). Thus, we define the homeomorphism \(h\) for any \((t, z) \in \Sigma|\Delta^{n-1}|\) by

\[h(t, z) = \begin{cases} h_t(z) & \text{if } -1 < t < 1; \\
(1, \ldots, 1) & \text{if } t = 1; \\
* & \text{if } t = -1. \end{cases}\]

The crucial property of \(h\) is that it restricts on an \(m\)-dimensional face \(\sigma\) of \(\Delta^{n-1}\) to a homeomorphism

\[h_{|\sigma|}: \Sigma|\Delta^m| \cong \Sigma|\sigma| \xrightarrow{\cong} (Y_1^\sigma \wedge \cdots \wedge Y_n^\sigma) \cong (D^1)^{\wedge(n+1)},\]

where

\[Y_i^\sigma = \begin{cases} D^1 & \text{if } i \in \sigma \\
\{1\} & \text{if } i \notin \sigma. \end{cases}\]

Then piecing these together, for any subcomplex \(L\) of \(\Delta^{n-1}\), \(h\) restricts to a homeomorphism 

\[\Sigma|L| = \bigcup_{\sigma \in L} \Sigma|\sigma| \xrightarrow{\cong} \bigcup_{\sigma \in L} Y_1^\sigma \wedge \cdots \wedge Y_n^\sigma.\]
In particular, when $W = (D^1, S^0)^K$, 

$$\hat{W} = \bigcup_{\sigma \in K} Y_1^\sigma \wedge \cdots \wedge Y_n^\sigma,$$

so $h$ restricts to a homeomorphism $\Sigma|K| \cong \hat{W}$. More generally, for any partition $(I_1, \ldots, I_m)$ of $[n]$, we have an inclusion $\hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m}$ into $(D^1)^{\land n}$ given by mapping elements by permuting coordinates, so that if a $j^{th}$ coordinate is the coordinate in a factor $\hat{W}_{I_k}$ that corresponds to $i \in I_k$, upon mapping it becomes the $j^{th}$ coordinate. This then gives a homeomorphism onto its image

$$\hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m} \cong \bigcup_{\sigma_j \in K_{I_1}, \ldots, \sigma_m \in K_{I_m}} \Sigma(\sigma_1 \cup \cdots \cup \sigma_m) \wedge Y_n^{\sigma_1 \cup \cdots \cup \sigma_m},$$

so more generally $h$ restricts to a homeomorphism

$$(16) \quad \Sigma|K_{I_1} \ast \cdots \ast K_{I_m}| = \bigcup_{\sigma_1 \in K_{I_1}, \ldots, \sigma_m \in K_{I_m}} \Sigma|\sigma_1 \cup \cdots \cup \sigma_m| \xrightarrow{\hat{\sigma}} \hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m}.$$ 

In this manner, we will think of $\hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m}$ as a subspace of $(D^1)^{\land n}$.

**Proof of Theorem 4.3.** Let $W = (D^1, S^0)^K$. At most $n - 1$ particles collide in a configuration in $\mathcal{D}_A(W_{n-1})$, and if we look at the subspace of all configurations of $n$ particles fixed at some $t_1, \ldots, t_n \in \mathbb{R}$ (that is, all possible labels for particles fixed in this way), then this subspace is homeomorphic to $\{y\} \wedge \hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m}$, where $y = (t_1, \ldots, t_n)$, and $(I_1, \ldots, I_m) = [n]_y$. This in turn is homeomorphic to $\{y\} \wedge \Sigma|K_{I_1} \ast \cdots \ast K_{I_m}|$ via the restriction of the homeomorphism $\Sigma|\Delta^{n-1}| \xrightarrow{\hat{h}} (D^1)^{\land n}$ in (15).

Thus, we see that the map

$$\hat{h}: Q_K \rightarrow \mathcal{D}_A(W_{n-1})$$

sending $((t_1, \ldots, t_n), x) \mapsto (t_1, \ldots, t_n, x_1, \ldots, x_n)$, where $(x_1, \ldots, x_n) = h(x)$, is a homeomorphism. Notice this map fits into a commutative square

$$\begin{array}{ccc}
\Sigma^n|K| & \xrightarrow{\Phi_K} & \Sigma Q_K \\
\downarrow{\zeta} & & \downarrow{\zeta} \\
\Sigma^{n-1}h & \xrightarrow{\zeta} & \Sigma h \\
\Sigma^{n-1}\hat{W} & \xrightarrow{\phi} & C(\mathcal{L}_A) & \xrightarrow{\partial} & \Sigma \mathcal{D}_A(W_{n-1}),
\end{array}$$

where $\partial$ is the connecting map given by collapsing the subspace $\mathcal{D}_A(W_n)$, and $\phi$ the homotopy equivalence from Lemma 3.15.

Induct on number of vertices. Assume the statement holds for simplicial complexes on less than $n$ vertices. Suppose $K$ is weakly coherently homotopy Golod. Directly from definition, each $K_I$ is weakly coherently homotopy Golod, so by our inductive assumption $W_I$ is a co-$H$-space for $|I| < n$. Then using Proposition 2.8, the quotient map $W_I \rightarrow W_I^1$ has a right homotopy inverse for $|I| < n$, and to show $W^1$ is a co-$H$-space, it remains to show that $W^1 \rightarrow \hat{W}^1$ has a right homotopy inverse. Since $\Phi_K$ is nullhomotopic and $\phi$ is a homotopy equivalence, $\partial$ is nullhomotopic by the above commutative square, and so $W^1 \rightarrow \hat{W}^1$ has a right homotopy inverse by Corollary 3.13. 

The weakly coherent homotopy Golod condition does not look near enough to the Golod condition for our liking. With this in mind, we start by considering complexes $K$ for which each inclusion $[K_{I \cup J}] \hookrightarrow [K_I \ast K_J]$ is nullhomotopic after appropriate suspension. This is just a homotopy version of the Golod property without consideration for Massey products, so appropriately, we call it the *homotopy Golod* condition. It turns out it is a necessary condition for $(D^2, S^1)^K$ to be a co-$H$-space (and so it might be an good avenue for finding a counterexample to Conjecture 1.1).

**Proposition 4.4.** If $(D^2, S^1)^K$ is a co-$H$-space, then each one of the suspended inclusions

$$\Sigma^{|I \cup J|+1}I_{I \cup J}: \Sigma^{|I \cup J|+1}|K_{I \cup J}| \rightarrow \Sigma^{|I \cup J|+1}|K_I \ast K_J|$$

is nullhomotopic for every disjoint non-empty $I, J \subseteq [n]$. 

Proof. Recall that if \( Y \) is a co-\( \mathcal{H} \)-space with comultiplication \( Y \xrightarrow{\psi} Y \vee Y \) if and only if the diagonal map \( Y \xrightarrow{\Delta} Y \times Y \) is homotopic to the composite \( Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\text{include}} Y \times Y \). In particular, this implies the reduced diagonal map

\[
\bar{\Delta} : Y \xrightarrow{\Delta} Y \times Y \longrightarrow (Y \times Y)/(Y \vee Y) \xrightarrow{\overline{\psi}} Y \wedge Y
\]

is nullhomotopic whenever \( Y \) is a co-\( \mathcal{H} \)-space. As before, let \( W = W_n = (D^1,S^0)^K \), in which case \( W^1 = W_1 = (D^2,S^1)^K \), and suppose \( W^1 \) is a co-\( \mathcal{H} \)-space. By Proposition 2.8, \( W^1_{I,U,J} \) is a co-\( \mathcal{H} \)-space for each disjoint non-empty \( I,J \not\subseteq [n] \), and so \( W^1_{I,U,J} \xrightarrow{\Delta} W^1_{I,U,J} \wedge W^1_{I,U,J} \) is nullhomotopic. Then the composite

\[
W^1_{I,U,J} \xrightarrow{\bar{\Delta}} W^1_{I,U,J} \wedge W^1_{I,U,J} \longrightarrow W^1_I \wedge W^1_J \longrightarrow \hat{W}^1_I \wedge \hat{W}^1_J
\]

is nullhomotopic (where the second last map is the smash of the coordinate-wise projection maps onto \( W^1_I \) and \( W^1_J \), and the last map is the smash of quotient maps) and moreover, is equal to the composite

\[
W^1_{I,U,J} \xrightarrow{\text{quotient}} W^1_{I,U,J} \xrightarrow{\iota^1} \hat{W}^1_I \wedge \hat{W}^1_J
\]

where the last map \( \iota^1 \) is the coordinate-wise inclusion (given by mapping \( (x_1,\ldots,x_n) \) to the pair \( ((y_1,\ldots,y_{|I|}), (z_1,\ldots,z_{|J|})) \) such that \( y_i = x_k \) if \( k \in I \) is the \( i \)-th largest element, or else \( z_j = x_k \) if \( k \in J \) is the \( j \)-th largest element), so this composite is nullhomotopic as well. But since \( W^1_{I,U,J} \) is a co-\( \mathcal{H} \)-space, by Proposition 2.8 the quotient map \( W^1_{I,U,J} \longrightarrow \hat{W}^1_{I,U,J} \) has a right homotopy inverse, so \( \iota^1 \) must be nullhomotopic. Notice that with respect to the homeomorphisms \( \hat{W}^1_I \cong \Sigma^{|I|} W_I \), given in Section 3.4, \( \iota^1 \) is the \( (|I|+|J|) \)-fold suspension of the coordinate-wise inclusion \( \iota^0 : \hat{W}^1_{I,U,J} \longrightarrow \hat{W}^1_I \wedge \hat{W}^1_J \), and with respect to the homeomorphisms (16), \( \iota^0 \) is just the suspended inclusion \( \Sigma[K_{I,U,J}] \cong \Sigma[K_I \ast K_J] \). Thus, we see that \( \Sigma^{(|I|+|J|)+1} \iota^1_{I,U,J} \) is nullhomotopic.

We will strengthen a desuspension of the homotopy Golod condition so that it is sufficient for \( (D^2,S^1)^K \) to be a co-\( \mathcal{H} \)-space, though possibly no longer necessary. This is the coherent homotopy Golod condition defined below. It will also be useful to generalize the Golod and homotopy Golod conditions, and in turn the coherent homotopy Golod condition, to a relative context: if \( K \) is a subcomplex of \( L \), and \( I,J \not\subseteq [n] \) are disjoint, we may ask if the inclusions \( \iota^1_{I,U,J} : K_{I,U,J} \longrightarrow [L_I \ast L_J] \) induce a trivial map on cohomology or are nullhomotopic after some suspension.

In defining the coherent homotopy Golod condition, recall that we want the inclusions \( \Sigma \iota^1_{I,U,J} \) to be nullhomotopic for disjoint non-empty \( I,J \not\subseteq [n] \), and at the same time we want the coherence conditions between these nullhomotopies to hold as outlined in the introduction. These properties are encoded as follows.

\[\text{Figure 1. 1-skeleton of an order 4 permutohedron}\]
Consider the delta set \( K_n = \{ F_0, \ldots, F_{n-1} \} \), whose set of \((m-2)\)-faces \( F_{m-2} \) is the set of all length \( m \) ordered partitions \((I_1, \ldots, I_m)\) of \([n]\), with each \( I_j \neq \emptyset \), and whose face maps \( d_i : F_{m-2} \to F_{m-3} \) are given by \( d_i((I_1, \ldots, I_m)) = (I_1, \ldots, I_{i-1}, I_i \cup I_{i+1}, I_{i+2}, \ldots, I_m) \) for \( 1 \leq i \leq m-1 < n \). These satisfy the required identity \( d_i \circ d_j = d_{j-1} \circ d_i \) when \( i < j \). Since each ordered partition \( S \in F_{m-2} \) is uniquely determined by its \((m-3)\)-faces \( d_i(S), \ldots, d_{m-1}(S) \), then \( K_n \) is a simplicial complex of dimension \( n-2 \). We will see that it is a triangulation of \( S^{n-2} \). In fact, \( K_n \) is the dual of the order \( n \) permutohedron.

We will from now on assume \( K \) is a subcomplex of \( L \) on vertex set \([n]\), \( L \) has no ghost vertices, though \( K \) possibly has less than \( n \) vertices if \( K \neq L \). If \( K = L \), then \( K \) also has no ghost vertices. We fix \( W' = \mathbb{Z}_L(D^1, S^n) \), in which case \( W' \cong \Sigma[L] \). If \( K = L \), then \( W' = W \).

Let \( \text{Cone}(\Sigma[K]) = ([0,1] \times \Sigma[K]) / \sim \) under the identifications \((1, x) \sim (t, *_{-1}) \sim \ast \).

**Definition 4.5.** The pair \((L, K)\) on vertex set \([n]\) is **coherently homotopy Golod** if \( L \) is a single vertex or \( K = \emptyset \), or (recursively) \((L \setminus \{i\}, K \setminus \{i\})\) is coherently homotopy Golod on vertex set \([n]-\{i\}\) for each \( i \in [n] \), and there is a map

\[
\hat{\Psi}_{L,K} : [K_n] \times \text{Cone}(\Sigma[K]) \to \Sigma[D^{n-1}]
\]

such that for any \( \gamma \in [K_n] \), \( \hat{\Psi}_{L,K}(\gamma, \ast) \) is the basepoint \(*_{-1} \in \Sigma[D^{n-1}] \), and:

1. The restriction of \( \hat{\Psi}_{L,K} \) to \( \{ \gamma \} \times \{ 0 \} \times \Sigma[K] \) is the suspended inclusion \( \Sigma[K] \to \Sigma[L] \subseteq \Sigma[D^{n-1}] \);
2. If \( \gamma \in \Sigma[S] \) for some \( S = (I_1, \ldots, I_m) \in F_{m-2} \), then \( \hat{\Psi}_{L,K} \) maps \( \{ \gamma \} \times \text{Cone}(\Sigma[K]) \) to a subspace of \( \Sigma[L_{I_1} \ast \cdots \ast L_{I_m}] \subseteq \Sigma[D^{n-1}] \).

If \( K = L \), we simply say \( K \) is **coherently homotopy Golod**, and denote \( \hat{\Psi}_K = \hat{\Psi}_{K,K} \).

We will give two more equivalent definitions of coherently homotopy Golodness, which are a bit less directly related to the Golod condition, but which from our standpoint will be more workable. Recall that if \( S = (s_1, \ldots, s_n) \) is any sequence of real numbers, then \([n]_S\) is the ordered partition \((I_1, \ldots, I_m)\) of \([n]\), where \( I_j = \{ \ell \in [n] \mid s_\ell = s_j' \} \), \( s_j' \) is the \( j \)th smallest element in the set \( \{s_1, \ldots, s_n\} \), and \( m = |\{s_1, \ldots, s_n\}| \) is the number of distinct elements in \( S \).

**Definition 4.6** (Second definition of coherently homotopy Golodness). The pair \((L, K)\) on vertex set \([n]\) is **coherently homotopy Golod** if \( L \) is a single vertex or \( K = \emptyset \), or (recursively) \((L \setminus \{i\}, K \setminus \{i\})\) is coherently homotopy Golod on vertex set \([n]-\{i\}\) for each \( i \in [n] \), and for some choice of vertex ordering there is a basepoint preserving map

\[
\Psi_{L,K} : \Sigma^n[K] \to \Sigma[D^{n-1}]
\]

such that for any \( y = (t_1, \ldots, t_{n-1}) \in \mathbb{N}[\Sigma^n[K]] = \mathbb{N}[[1]]^{(n-1)}[\Sigma[n]] \) that is not the basepoint, \( \Psi_{L,K} \) maps the subspace \( \{y\} \times \Sigma[K] \) of \( \Sigma^n[K] \) as follows:

1. When \( t_1 = \cdots = t_{n-1} = 0 \), the restriction of \( \Psi_{L,K} \) to \( \{y\} \times \Sigma[K] \) is the suspended inclusion \( \Sigma[K] \to \Sigma[L] \subseteq \Sigma[D^{n-1}] \);
2. Letting \( S_y = (t_1, \ldots, t_{n-1}, 0) \) and \((I_1, \ldots, I_m) = [n]_{S_y} \), \( \Psi_{L,K} \) maps \( \{y\} \times \Sigma[K] \) to a subspace of \( \Sigma[L_{I_1} \ast \cdots \ast L_{I_m}] \subseteq \Sigma[D^{n-1}] \).

For the base case where \( K \) is a single vertex, \( \Psi_{L,K} \) can be taken as the constant map. If \( K = L \), we simply say \( K \) is **coherently homotopy Golod**, and denote \( \Psi_K = \Psi_{K,K} \).

**Remark:** If \( K \) is on vertex set \( J \subseteq [n] \), \( |J| = m \), recall the inclusion \( \Sigma[K] \to \Sigma[D^{n-1}] \) is up to homeomorphism the inclusion \( \hat{W} \to \Sigma[D^1]^{\wedge m} \), given as the restriction of the inclusion \( (D^1)^{\wedge m} \to (D^1)^{\wedge n} \) sending \((x_1, \ldots, x_m) \mapsto (x'_1, \ldots, x'_m)\), where \( x'_j = x_j \) when \( i \) is the \( j \)th largest element in \( J \), otherwise \( x'_j = 1 \).

**Definition 4.7** (Third definition of coherently homotopy Golodness). Same as the second definition, but with \( \Sigma[D^{n-1}] \) replaced with \( (D^1)^{\wedge n} \), the subspaces \( \Sigma[L_{I_1} \ast \cdots \ast L_{I_m}], \Sigma[L] \), and \( \Sigma[K] \) of \( \Sigma[D^{n-1}] \) replaced with the subspaces \( \hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m}, \hat{W}' \), and \( \hat{W} \) of \( (D^1)^{\wedge n} \), and \( \Psi_{L,K} \) becoming a map

\[
\Psi_{L,K} : \Sigma^{n-1}\hat{W} \to (D^1)^{\wedge n}.
\]
If \( K = L \), then \( \hat{W}' = \hat{W}'_t \cong \Sigma[K] \), and we denote \( \Psi_K = \Psi_{K,K} \).

**Proposition 4.8.** All three definitions of coherent homotopy Golodness are equivalent.

**Proof.** Since the homeomorphisms \( \Sigma[K] \cong \hat{W} \) and \( \Sigma[L_1 * \ldots * L_m] \cong \hat{W}'_1 \wedge \ldots \wedge \hat{W}'_m \) that we defined before are restrictions of the homeomorphism \( \Sigma[\Delta^{n-1}] \cong (D^1)^{\wedge n} \), the equivalence of the second and third definitions follows.

Think of \( S^{n-1} \) as the unit \((n-1)\)-cube quotient boundary \([-1,1]^{\wedge(n-1)}/\partial([-1,1]^{\wedge(n-1)}) \). We can write \( S^{n-1} = \bigcup_{t \in \mathbb{Z}} U_t \), where \( U_t = \{ \{t_1, \ldots, t_{n-1}\} \in S^{n-1} \mid \max\{t_1, \ldots, t_{n-1}\} = t \} \). When \( 0 < t < 1 \), \( U_t \) is homeomorphic to \( S^{n-2} \), while \( U_1 = \{ \ast \} \) and \( U_0 = \{(0, \ldots, 0)\} \), where \( \ast \in S^{n-1} \) is the basepoint.

We construct homeomorphisms

\[
\tau_t : [K_0] \longrightarrow U_t \cong S^{n-2}
\]

for \( 0 < t < 1 \) as follows. Given any face \( S = (I_1, \ldots, I_m) \in F_m \) of \( K_0 \), its vertices are the length 2 partitions \( S_1 = (I_1 \cup \ldots \cup I_{i_k} \cup \ldots \cup I_m) \) of \([n]\) for \( i = 1, \ldots, m - 1 \), so we can think of any point \( \gamma \in |S| \) as a formal sum

\[
\gamma = s_1 S_1 + \ldots + s_{m-1} S_{m-1}
\]

such that each \( s_i \geq 0 \) and \( s_1 + \ldots + s_{m-1} = 1 \), and \( \gamma \) is on a boundary face \( \partial d(S) \in |S| \) if and only if \( s_i = 0 \), in which case the summand \( s_i S_i \) vanishes.

Given \( j \in [n] \), let \( j_S \) be the integer such that \( j \in I_{j_S} \), and let \( t_{j_S} = s_0 + s_1 + \ldots + s_{j_S - 1} \) where \( s_0 = 0 \). Notice at least two of these must be distinct, so \( \beta_\gamma = \max\{t_1, \gamma - t_{n, \gamma}, \ldots, t_{n-1, \gamma} - t_{n, \gamma}\} > 0 \). Then define \( \tau_t \) by

\[
\tau_t(\gamma) = \left( t_{\beta_\gamma} - t_{j_S}, \frac{t_{\beta_\gamma} - t_{j_S}}{t_{\beta_\gamma} - t_{j_S}} (t_{\beta_\gamma} - t_{j_S}) \right). \]

One can see that \( \tau_t \) is well-defined by checking this definition agrees on each boundary face \( \partial d(S) \).

If \( \gamma \) is in the interior of some \( k \)-face \( |S'| \subseteq |S| \) for \( k \leq m \), then \([n]_{\tau_t(\gamma) \times (0)} = S' \). Then one can check that \( \tau_t \) is a homeomorphism with inverse \( \tau_t^{-1} : U_t \longrightarrow |K_0| \) given by

\[
\tau_t^{-1}((t_1, \ldots, t_{n-1})) = \left( \frac{t_{j_S} - t_1}{t_{j_S} - t_{j_S}} S_1 + \ldots + \left( \frac{t_{j_S} - t_{j_S - 1}}{t_{j_S} - t_{j_S}} S_{j_S - 1} \right) \right).
\]

where \( t_{j_S} \) is the \( \ell \)-th smallest integer in the set \( S = \{t_1, \ldots, t_{n-1}, 0\} \), \( k = |S| \), and \( S_k \) is the length 2 partition \((I_1 \cup \ldots \cup I_{j_S} \cup I_{j_S + 1} \cup \ldots \cup I_m) \), given that \( S' = (I_1, \ldots, I_{j_S}) \) denotes the partition \([n]_{(t_1, \ldots, t_{n-1}, 0)} \).

For \( t = 0 \) and \( t = 1 \), we define \( \tau_0 : |K_0| \longrightarrow U_0 \) and \( \tau_1 : |K_0| \longrightarrow U_1 \) to be the constant maps sending all points to \((0, \ldots, 0)\) and \( \ast \) respectively.

Take the quotient space

\[
Q = |K_0| \times Cone(\Sigma[K]) / \sim
\]

under the identifications \((\gamma, (0, x)) \sim (\gamma', (0, x)) \sim (0, x) \) and \((\gamma, \ast) \sim (0, x) \) and \((\gamma', \ast) \sim (0, x) \) for any \( \gamma, \gamma' \in |K_0| \) and \( x \in \Sigma[K] \). Let

\[
\tau_K : Q \longrightarrow S^{n-1} \setminus \Sigma[K] \cong \Sigma^n[K]
\]

be given by mapping \((\gamma, (t, x)) \mapsto (\tau_t(\gamma), x) \). This is a homeomorphism, with inverse

\[
\tau_K^{-1} : S^{n-1} \setminus \Sigma[K] \longrightarrow Q
\]

given by mapping \((0, \ldots, 0, x) \mapsto (0, x) \) and \((t_1, \ldots, t_{n-1}, 0) \mapsto (\tau_t^{-1}(t_1, \ldots, t_{n-1}), (t, x)) \) when \((t_1, \ldots, t_{n-1}) \in U_t \) for some \( 0 < t < 1 \). From the first definition of coherent homotopy Golodness, we see \( \Psi_{L,K} \) factors as

\[
\Psi_{L,K} : |K_0| \times Cone(\Sigma[K]) \overset{quotient}{\longrightarrow} Q \overset{\Psi_{L,K}}{\longrightarrow} \Sigma[\Delta^{n-1}].
\]

Thus, given \( \Psi_{L,K} \) exists as in the first definition, we can construct \( \Psi_{L,K} \) as in the second definition by taking it as the composite \( \Psi_{L,K} \circ \tau_K^{-1} \). Conversely, given \( \Psi_{L,K} \) exists, we can define \( \hat{\Psi}_{L,K} = \Psi_{L,K} \circ \tau_K \), and then take \( \hat{\Psi}_{L,K} \) to be the above composite of the quotient map and \( \hat{\Psi}_{L,K} \). Therefore, the first and second definitions are equivalent.
Notice each parameter \( t_i \) in the second and third definition of coherent homotopy Golodness corresponds to the vertex \( \{i\} \) in \( L \). But there are \( n \) vertices and only \( n-1 \) parameters \( t_i \), the vertex \( \{n\} \) being treated exceptionally here. Then both the second and third definitions depend on the choice of ordering of vertices in \( K \) and \( L \) by elements in \( [n] \). The first definition of coherent homotopy Golodness is however independent of vertex ordering, meaning it holds for all orderings if it holds for one. Moreover, in the proof of Proposition 4.8 the argument that the first definition implies the second and third works for any choice of ordering of vertices. We can thus go from the second or third definitions to the first definition, change the ordering, and then come back again with a new map \( \Psi_{L,K} \) corresponding to the new ordering. Therefore:

**Corollary 4.9.** If the second or third definitions of coherent homotopy Golodness hold for one vertex ordering, then they hold for all vertex orderings. \( \square \)

We will use the second and third definitions of coherent homotopy Golodness from now on.

**Theorem 4.10.** If \( K \) is coherently homotopy Golod, then \( (D^2,S^1)^K \) is a co-H-space.

**Proof.** Induct on number of vertices. The base case is trivial. Assume the statement holds for simplicial complexes on less than \( n \) vertices. Since by definition, each \( K_f \) is coherently homotopy Golod when \( K \) is, by our inductive assumption \( W_f^1 \) is a co-H-space for \( |I| < n \). Then using Proposition 2.8, the quotient map \( W_f^1 \to \hat{W_f^1} \) has a right homotopy inverse for \( |I| < n \), and to show \( W^1 \) is a co-H-space, it remains to show that \( W^1 \to \hat{W^1} \) has a right homotopy inverse.

Given our map \( \Psi_{K}; \Sigma^{n-1} W \to (D^1)^{\wedge n} \), define a basepoint preserving map \( g; \Sigma^{n-1} \hat{W} \to \mathcal{D}_A(W) \) by mapping a point \( \sigma = (t_1, \ldots, t_{n-1}, x) \) by

\[
g(\sigma) = (t_1, \ldots, t_{n-1}, 0; \Psi_K(\sigma)).
\]

The composite of \( g \) and the inclusion \( q; \mathcal{D}_A(W) \to C(\iota_{A,n}) \) is homotopic to the homotopy equivalence \( \Sigma^{n-1} \hat{W} \xrightarrow{\phi} C(\iota_{A,n}) \) from Lemma 3.15 as follows. Let \( \beta = \max\{|t_1|, \ldots, |t_{n-1}|, 0\} \). Since no more than \( n-1 \) particles in \( g(\sigma) \) collide at a point when \( \beta > 0 \), \( g(\sigma) \in \mathcal{D}_A(W_{n-1}) \) when \( \beta > 0 \), so we can homotope \( g \circ g(\sigma) = (0, g(\sigma)) \) to \((\beta, g(\sigma)) \) inside \( C(\iota_{A,n}) \) preserving the basepoint by homotoping \( 0 \) to \( \beta \) linearly. Then we homotope \((\beta, g(\sigma)) \) to \( \phi(\sigma) = (\beta, (t_1, \ldots, t_{n-1}, 0; x)) \) via the basepoint preserving homotopy

\[
H_t(\sigma) = \begin{cases} 
* & \text{if } \beta = 1; \\
(\beta, (t_1, \ldots, t_{n-1}, 0; \Psi_K((1-t)t_1, \ldots, (1-t)t_{n-1}, x))) & \text{if } \beta < 1.
\end{cases}
\]

Notice \( H_t(\sigma) \) is well-defined when \((t_1, \ldots, t_{n-1})\) is on the boundary of \([-1,1]^{(n-1)} \), since then \( \beta = 1 \), and is continuous as \((t_1, \ldots, t_{n-1})\) approaches the boundary of \([-1,1]^{(n-1)} \), since then \( \beta \) approaches 1, so \((\beta, (t_1, \ldots, t_{n-1}, 0; \Psi_K((1-t)t_1, \ldots, (1-t)t_{n-1}, x)))\) approaches the basepoint in \( C(\iota_{A,n}) \). Thus, since \( \phi \) is a homotopy equivalence, the composite of \( g \) and \( \mathcal{D}_A(W) \xrightarrow{q} C(\iota_{A,n}) \) is a homotopy equivalence, so \( q \) has a right homotopy inverse, and then by Corollary 3.13, so does \( W^1 \to \hat{W^1} \).

The following example will be useful.

**Example 4.11.** Suppose \( K \) with no ghost vertices is coherently homotopy Golod on vertex set \([n]\), and let \( L = K \cup \{n+1\} \) be \( K \) with a disjoint vertex \( \{n+1\} \) added. Then the pair \((L,K)\) is coherently homotopy Golod on vertex set \([n+1]\) as follows.

Assume this is true for complexes \( K \) with less than \( n \) vertices. The base case \( K = \emptyset \) holds by definition. Since \( L \setminus \{i\} = K \setminus \{i\} \cup \{n+1\} \) when \( i \neq n+1 \), then by induction each \((L \setminus \{i\}, K \setminus \{i\})\) is coherently homotopy Golod on vertex set \([n+1] - \{i\} \), while \( K \) being coherently homotopy Golod implies by definition that \((L \setminus \{n+1\}, K \setminus \{n+1\}) = (K,K)\) is. Thus we are left to show the existence of the map \( \Psi_{L,K} \).

As before, \( W' = Z_L(D^1, S^0) \) and \( W = Z_K(D^1, S^0) \). Given our map \( \Psi_K; \Sigma^{n-1} \hat{W} \to (D^1)^{\wedge n} \) from the third definition of coherent homotopy Golodness of \( K \), define \( \Psi_{L,K}; \Sigma^n \hat{W} \to (D^1)^{\wedge (n+1)} \) by
mapping the basepoint to the basepoint, and a non-basepoint \( \sigma = (t_1, \ldots, t_{n-1}, s, (x_1, \ldots, x_n)) \in \Sigma^n W \) to

\[
\Psi_{L,K}(\sigma) = (x'_1, \ldots, x'_n, y),
\]

where \( (x'_1, \ldots, x'_n) = \Psi_K(t_1, \ldots, t_{n-1}, (x_1, \ldots, x_n)) \),

\[
y = \begin{cases} 
1 & \text{if } \alpha \leq s \leq \beta; \\
2 \left( \frac{1-s}{1-\beta} \right) - 1 & \text{if } \beta < s < 1; \\
2 \left( \frac{s-\alpha}{2} \right) - 1 & \text{if } -1 < s < \alpha; \\
-1 & \text{if } |s| = 1,
\end{cases}
\]

and \( \alpha = \min\{t_1, \ldots, t_{n-1}, 0\} \) and \( \beta = \max\{t_1, \ldots, t_{n-1}, 0\} \). Notice there is a discontinuity here for \( y \) as both \( s \) and \( \beta \) approach 1, or \( s \) and \( \alpha \) approach -1. But this does affect the continuity of \( \Psi_{L,K} \) since \( \Psi_K \) being basepoint preserving \( (x'_1, \ldots, x'_n) \) approaches the basepoint as \( \alpha \) approaches -1 or \( \beta \) approaches 1, so \( \Psi_{L,K}(\sigma) \) approaches the basepoint here. Likewise, \( \Psi_{L,K}(\sigma) \) approaches the basepoint if (1) some \( x_i \) approaches -1 (since some \( x'_i \) approaches -1, since \( \Psi_K \) is basepoint preserving) or (2) \( s \) approaches 1 or -1 and \( \beta \) or \( \alpha \) respectively do not (since \( y \) approaches -1). In any case, \( \Psi_{L,K}(\sigma) \) approaches the basepoint as \( \sigma \) approaches the basepoint.

Since \( \Psi_K \) satisfies the first condition of coherent homotopy Golodness, \( \Psi_{L,K}(\sigma) = (x'_1, \ldots, x'_n, y) = (x_1, \ldots, x_n, 1) \) when the \( t_i \)’s and \( s \) are zero, so \( \Psi_{L,K} \) restricts to the inclusion \( W \to \hat{W} \) here, thus also satisfies the first condition.

Note \( \hat{W}_I' = \hat{W}_I \) if \( \{n+1\} \notin I \). Let \( \mathcal{S} = (t_1, \ldots, t_{n-1}, 0) \) and \( (I_1, \ldots, I_m) = [n] \mathcal{S} \). Since \( \Psi_K \) satisfies the second condition of coherent homotopy Golodness, \( (x'_1, \ldots, x'_n) \) is in \( A = \hat{W}_I_1 \wedge \cdots \wedge \hat{W}_I_m \). Let \( \mathcal{S}' = (t_1, \ldots, t_{n-1}, s, 0) \) and \( (I'_1, \ldots, I'_{m'}) = [n + 1] \mathcal{S} \). If \( s = 0 \) or \( s = t_i \) for some \( i \), then \( m' = m' \), \( I'_k = I_k \cup \{n + 1\} \) for some \( k \), \( I'_k = I_k \) for \( i \neq k \), and \( y = 1 \) (since then \( \alpha \leq s \leq \beta \)). In any case, each \( I_j \in I'_j \). Thus \( \Psi_{L,K}(\sigma) = (x'_1, \ldots, x'_n, 1) \) is in the subspace \( A' = \hat{W}_{I'_1} \wedge \cdots \wedge \hat{W}_{I'_m} \) given on each factor by the inclusions \( \hat{W}_{I_k} \to \hat{W}_{I'_k} \) as described in Remark 4, these being equalities when \( i \neq k \). On the other hand, if \( s \neq 0 \) and \( s \neq t_i \) for each \( i \), then \( m' = m + 1 \), \( I'_k = \{n + 1\} \) for some \( k \), and \( I'_i = I_i \) and \( I'_{i+1} = I_{i+1} \) for \( i < k < j \), so \( \hat{W}_{I_k} = D^1 \), \( \hat{W}_{I_j} = \hat{W}_{I'_j} \), and \( \hat{W}_{I_{j+1}} = \hat{W}_{I'_{j+1}} \) for \( i < k < j \). Then \( \Psi_{L,K}(\sigma) \) is in the (homeomorphic) subspace \( A \wedge D^1 \) of \( A' \). Thus, we see \( \Psi_{L,K} \) satisfies the second condition of coherent homotopy Golodness.

**Example 4.12.** Recall from [20] that a complex \( K \) on vertex set \([n]\) (without ghost vertices) is **extractible** if either \( K \setminus \{i\} \) is a simplex for some \( i \in [n] \), or else (recursively) \( K \setminus \{i\} \) is extractible for each \( i \in [n] \) and the wedge sum of inclusions \( \bigvee_{i \in [n]} \Sigma[K \setminus \{i\}] \to \Sigma[K] \) has a right homotopy inverse

\[
s : \Sigma[K] \to \bigvee_{i \in [n]} \Sigma[K \setminus \{i\}].
\]

We show that \( K \) is coherently homotopy Golod by inducting on number of vertices. Suppose any extractible complex on less than \( n \) vertices is coherently homotopy Golod, the base case being trivial. Then since each \( K \setminus \{i\} \) is extractible when \( K \) is, each \( K \setminus \{i\} \) is coherently homotopy Golod on vertex set \([n] - \{i\}\) by induction. Thus, to show \( K \) is coherently homotopy Golod, we are left only to construct the map \( \Psi_K : \Sigma^{n-1} W \to (D^1)^{\infty} \) (or \( \Psi_K : \Sigma^n K \to \Sigma[\Delta^{n-1}] \) from the second definition).

First consider the case where \( K \setminus \{i\} \) is a simplex for some \( i \in [n] \). Without loss of generality, assume \( i = n \). Let \( W = (D^1, S^n)K \) and define \( \Psi_K : \Sigma^{n-1} W \to (D^1)^{\infty} \) by mapping \( \sigma = (t_1, \ldots, t_{n-1}, (x_1, \ldots, x_n)) \) to

\[
\Psi_K(\sigma) = (x'_1, \ldots, x'_{n-1}, x_n).
\]

where \( x'_i = (1 - |t_i|)x_i - |t_i| \). The last coordinate \( x_n \) that is unchanged corresponds to the vertex \( \{n\} \). This clearly satisfies the first condition of coherent homotopy Golodness. One can see that it satisfies the second condition precisely because \( K_I \) is a simplex for any \( I \subseteq [n] - \{n\} \), that is,
\( \tilde{W}_I = (D^1)^{|I|} \). So for the second condition to hold, each of the first \( n-1 \) \( x_i' \) of \( \Psi_K(\sigma) \) still has the freedom to take on any value in \( D^1 \), as long as the corresponding \( t_i \) is non-zero.

For the general case, let \( K^i = K \setminus \{i\} \) and take the disjoint union \( L^i = (K^i) \cup \{i\} \). Since by induction each \( K^i \) is coherently homotopy Golod on vertex set \([n] \setminus \{i\}\), by Example 4.11 the pair \((L^i, K^i)\) is coherently homotopy Golod on vertex set \([n]\). Let \( \Psi_{L^i, K^i} : \Sigma^n|K^i| \to \Sigma|\Delta^{n-1}| \) be the associated map. By Corollary 4.9 we can assume each \( \Psi_{L^i, K^i} \) has been taken so that \( \{n\} \) is the exceptional vertex, and each of the first \( n-1 \) vertices \( \{i\} \in L^i \) correspond to the parameter \( t_i \) in the second definition of coherently homotopy Golodness. Consider the map

\[
\Psi'_K : \Sigma^n|K| \xrightarrow{\sum_{i=1}^n t_i} \bigvee_{i \notin [n]} \Sigma^n|K^i| \xrightarrow{\psi_{L^i, K^i}} \bigvee_{i \notin [n]} \Sigma|\Delta^{n-1}| \to \Sigma|\Delta^{n-1}|
\]

where the last map is the fold map. Notice that \( \Psi_K \) satisfies the second condition of coherently homotopy Golodness since each \( \Psi_{L^i, K^i} \) does, since the first map \( \sum_{i=1}^{n-1} t_i \) and second map are \((n-1)\)-fold suspensions (thus, do not change the first \( n-1 \) coordinates \( t_i \)), and since each \([L^i_1 \ast \cdots \ast L^i_m]\) is a subspace of \([K^i_1 \ast \cdots \ast K^i_m] \) in \( |\Delta^{n-1}| \) (since \( K \) has no ghost vertices). However, \( \Psi_K' \) does not necessarily satisfy the first condition. Instead, as each \( \Psi_{L^i, K^i} \) satisfies the first condition of coherently homotopy Golodness and \( \Sigma^{n-1} s \) is an \((n-1)\)-fold suspension, \( \Psi_K' \) maps the subspace \( \{(0, \ldots, 0) \} \wedge |K| \to |K| \) to \( |K| \) in \( |\Delta^{n-1}| \) via the composite \( \Sigma|K| \to \bigvee |K^i| \to |K| \), thus the restriction of \( \Psi_K' \) to this subspace is homotopic to the identity. Denote this homotopy by \( H : \Sigma|K| \times [0, 1] \to \Sigma|K| \). Pick some \( 0 < \varepsilon < 1 \). Let

\[
\chi(t) = \begin{cases} 0 & \text{if } |t| \leq \varepsilon; \\ \frac{t}{|t|} & \text{if } t > \varepsilon; \\ \frac{t}{|t|} & \text{if } t < -\varepsilon, \end{cases}
\]

and for any \( \sigma = (t_1, \ldots, t_{n-1}, x) \in \Sigma^n|K| \), where \( x \in \Sigma|K| \), let \( \beta = \max\{ |t_1|, \ldots, |t_{n-1}|, 0 \} \), and

\[
t_\sigma = \begin{cases} \frac{\varepsilon \beta}{\varepsilon} & \text{if } \beta \leq \varepsilon; \\ 0 & \text{if } \beta > \varepsilon. \end{cases}
\]

Note \( |\chi(t)| \) approaches 1 as \( |t| \) approaches 1. Define \( \Psi_K : \Sigma^n|K| \to \Sigma|\Delta^{n-1}| \) by

\[
\Psi_K(\sigma) = \Psi'_K(\chi(t_1), \ldots, \chi(t_{n-1}), H_{\varepsilon}(x)).
\]

Now \( \Psi_K \) satisfies the first condition of coherently homotopy Golodness. Since \( \chi(t_i) \leq \chi(t_j) \) when \( t_i < t_j \), \( \chi(t_i) = \chi(t_j) \) when \( t_i = t_j \), and \( \Psi_K \) is defined in terms of \( H_{\varepsilon} \) (which maps to \( \Sigma|K| \subseteq |\Delta^{n-1}| \)) and \( \Psi'_K \) (which satisfies the second condition of coherently homotopy Golodness), we see \( \Psi_K \) also satisfies the second condition.

**Example 4.13.** Define a simplicial complex \( K \) on vertex set \([n]\) to be \( m \)-neighbourly if every subset of at most \( m \) vertices in \([n]\) is a face in \( K \). We say \( K \) is \( n \)-neighbourly if it is \([\frac{n}{2}] \)-neighbourly.

Suppose \( K \) is \( n \)-neighbourly. Since \( |I| \leq |\frac{n}{2}| \) or \( |J| \leq |\frac{n}{2}| \) for any disjoint non-empty \( I, J \subseteq [n] \), then at least one of \( |K_I| \) or \( |K_J| \) is a simplex, implying \(|K_I \ast K_J| = |\Sigma|K\| \wedge |K|| \) is contractible. Therefore each inclusion \( \Sigma|K| \to \Sigma|K_I \ast K_J| \) is nullhomotopic, implying \( K \) is Golod. We show \( K \) is coherently homotopy Golod by inducting on number of vertices as follows.

Suppose any \( n \)-neighbourly complex on less than \( n \) vertices is coherently homotopy Golod. Since each \( K^i \{i\} \) is \( n \)-neighbourly when \( K \) is, \( K \setminus \{i\} \) is coherently homotopy Golod by induction. Now define

\[
\Psi_K : \Sigma^{n-1} \tilde{W} \to (D^1)^{\leq n}
\]

by mapping the basepoint to the basepoint, and a non-basepoint \( \sigma = (t_1, \ldots, t_{n-1}, (x_1, \ldots, x_n)) \) to

\[
\Psi_K(\sigma) = (f_1(x_1), \ldots, f_n(x_n)),
\]
where $\beta = \max\{|t_1|, \ldots, |t_{n-1}|, 0\}$,

$$f_i(x_i) = \begin{cases} (1 - 2\alpha_i\beta)(x_i + 1) - 1 & \text{if } 0 \leq \alpha_i < \frac{1}{2}, \\ (1 - \beta)(x_i + 1) - 1 & \text{if } \alpha_i \geq \frac{1}{2}, \end{cases}$$

and (letting $t_n = 0$)

$$\alpha_i = \min \left\{ \sum_{j \in S} |t_i - t_j| \left| S \subseteq [n] - \{i\} \right. \text{ and } |S| = \frac{n}{2} \right\}.$$ 

We must show $\Psi_K$ is continuous. Any possible discontinuity happens when $\beta$ approaches 1, in other words, when $(t_1, \ldots, t_{n-1})$ approaches the boundary of $[-1, 1]^{(n-1)}$, or when $x = (x_1, \ldots, x_n)$ approaches the basepoint. To see that there is no issue here, notice $S \cap S'$ is non-empty for any $S \subseteq [n] - \{i\}$ and $S' \subseteq [n] - \{n\}$ such that $i \neq n$ and $|S| = |S'| = \left\lfloor \frac{n}{2} \right\rfloor$, implying $\max\{\alpha_n, \alpha_i\} \geq \frac{|t_i - t_n|}{2}$.

So if $|t_i|$ approaches 1 for some $i \neq n$, since $t_n$ is always 0, at least one of $\alpha_n$ or $\alpha_i$ approaches some value $L \geq \frac{1}{2}$, so one of $f_i(x_i)$ or $f_n(x_n)$ approaches $-1$. Also, since $f(x_i)$ approaches $-1$ as $x_i$ does, $\Psi_K(\sigma)$ approaches the basepoint when $x$ does. In any case, $\Psi_K(\sigma)$ approaches the basepoint as $\sigma$ approaches the basepoint.

Clearly $\Psi_K$ satisfies the first condition of coherent homotopy Golodness. Since $\tilde{W}_I = (D^1)^{|I|}$ whenever $|I| \leq \left\lfloor \frac{n}{2} \right\rfloor$, in order for $\Psi_K$ to satisfy the second coherent homotopy Golod condition, for each $i$, $f_i(x_i)$ has freedom to take on any value in $[-1, 1]$ whenever there are no more than $\left\lfloor \frac{n}{2} \right\rfloor$ values of $j$ for which $t_j = t_i$, so defining $f_i$ as we did presents no difficulties here. On the other hand, when there are strictly more than $\left\lfloor \frac{n}{2} \right\rfloor$ values of $j$ for which $t_j = t_i$, we have $f_i(x_i) = x_i$ and each $f_j(x_j) = x_j$ (since $\alpha_i = \alpha_j = 0$). So because $x = (x_1, \ldots, x_n) \in \tilde{W}_I$, the restriction of $x$ to the coordinates corresponding to $x_i$ and each $x_j$ is an element in some $\tilde{W}_I$ (this does not give a continuous map $W \to \tilde{W}_I$), so there is no issue here either. Thus $\Psi_K$ satisfies the second condition, and $K$ is coherently homotopy Golod.

**Example 4.14.** Let $K = (\partial \Delta^2 \ast \partial \Delta^3) \cup_{\partial \Delta^4} \Delta^4$ on vertex set $[5]$. Since every pair of distinct vertices in $K$ is connected by an edge, $K$ is neighbourly, and therefore coherently homotopy Golod. But $H_2(|K|) \cong \mathbb{Z}$, while $H_2(|K\setminus\{\tilde{i}\}|) = 0$ for each $\tilde{i} \in [5]$, so $K$ cannot be extractable.

**Question 4.15.** Do any of the following implications hold: (1) $K$ Golod $\Rightarrow$ $K$ homotopy Golod; (2) $K$ homotopy Golod $\Rightarrow$ $K$ weakly coherently homotopy Golod; (3) $K$ weakly coherently homotopy Golod $\Rightarrow$ $K$ coherently homotopy Golod?

Conjecture 1.1 is false if (1) is false, and true if both (1) and (2) are true.

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