ACQUIRING ABSTRACT GEOMETRICAL CONCEPTS: THE INTERACTION BETWEEN THE FORMAL AND THE INTUITIVE.

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The acquiring of formal, abstract mathematical concepts by students may be said to be one of the major goals of mathematics teaching. How are such abstract concepts acquired? How does this formal knowledge interact with the students' intuitive knowledge of mathematics? How does the transition from informal mathematical knowledge to formal mathematical knowledge take place? This paper reports on a research project which is examining the nature of the interaction and possible conflict between the formal and the intuitive components of mathematical activity. Details are presented of an initial study in which mathematics graduates, who could be considered to have acquired formal mathematical concepts, tackled a series of geometrical problems. The study indicates the complex nature of the interaction between formal and intuitive concepts of mathematics. The plans for the next stage in the research project are outlined.

Introduction

A major goal of mathematics teaching is the acquisition by those learning mathematics of formal, abstract mathematical concepts. According to the Piagetian model of development this is only possible when the learners have reached the stage of formal operational thinking after the age of about 11 years or so. After this stage is reached, says the Piagetian view, deduction and logical argument are possible. Of course, the Piagetian viewpoint has come under critical review over the past twenty years. Much has been made of research showing that Piaget may have misinterpreted children he studied, either underestimating or overestimating what they can do, and that Piagetian theory does not take sufficient account of social interaction and the development of language.

In terms of learning mathematics it is important that we understand how pupils make the transition from the informal explorations that characterise the early mathematical experiences of pupils to the more formalised processes studied in the secondary school and later. There seems to be a number of ways of looking at this transition from intuitive to formal mathematical thinking. Piaget (1966 p 225), for instance, seems to suggest a hierarchy when he writes:

> Although effective at all stages and remaining fundamental from the point of view of invention, the cognitive role of intuition diminishes (in a relative sense) during development. .... there then results an internal tendency towards formalisation which, without ever being able to cut itself off entirely from its intuitive roots, progressively limits the field of intuition (in the sense of non-formalised operational thought).

Fischbein, however, amongst others, suggests either a plurality or a dialectic. In his most recent article, Fischbein (1994 p 244) claims that:

> The interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood.

Thirdly, Papert, for example, tends to argue that the intuitive mode is the natural one; analytical thinking is then merely a useful tool. Recently he has written that “The basic kind of thought is intuitive: formal logical thinking is an artificial, though certainly enormously useful, construct” (Papert 1993 p 167). This paper describes the first stage of a research project designed to investigate the nature of the relationship between the formal and the intuitive components of mathematical activity. The research project is examining a number of questions. How are abstract geometrical concepts acquired? How does this formal knowledge interact with the students' intuitive knowledge of mathematics? How does the transition from informal mathematical knowledge to formal mathematical knowledge take place?

Research Focus

In planning the first stage of the project a suitable focus was required. In Schoenfeld's study of recent graduates (and others) solving geometrical construction problems (Schoenfeld 1985) he comments that “insight and intuition come from drawing” and that two factors dominate in generating and rank ordering hypotheses for solution, the first being what Schoenfeld refers to as the “intuitive apprehensibility” of a solution. In other words, Schoenfeld's study shows that geometric construction tasks may be ones in which it may be possible to discern the role of geometrical intuition. However, Schoenfeld makes little other reference to the nature and role of intuition in problem solving.

The role of intuition in geometrical understanding has been emphasised for some time (for instance see Hilbert 1932 and Van Heile 1986). Indeed Fischbein's work stresses the role of visualisation in the generation of intuitions (Fischbein 1987). In addition, the advent of dynamic geometry packages for the computer, such as *Cabri-Géomètre*, provides new opportunities for studying the approaches used in solving geometrical problems. Indeed, such use of the computer may make intuition more accessible for study. Papert, for instance, claims that the computer can “help bridge the gap between formal knowledge and intuitive understanding” (Papert 1980 p 145).

Hence the focus for the initial study was the solving of geometrical construction problems by pairs of recent mathematics graduates using the dynamic geometry package *Cabri-Géomètre*. This led to the following research questions:

Is it possible to discern the effect of geometrical intuition during the solving of geometrical construction problems?
What is the influence of geometrical intuition on the solving of these problems?

Research Framework

Fischbein's definition of intuition is that it is a cognition characterised by the following properties (Fischbein 1987 p 43-56): *self-evidence and immediacy* (in that extrinsic justification is not needed), *intrinsic certainty* (note that self-evidence and certainty are not the same), *perseverance* (so that intuitions are stable), *coerciveness, theory status, extrapolativeness, globality* (in that intuitions offer a unitary global view), *implicitness* (so that although this is the result of selection, globalisation and inference, intuitions will appear to be implicit). The approach I adopted in my research was then to analyse the three geometrical construction problems which Schoenfeld provides (Schoenfeld 1985) together with transcripts of pairs of students attempting to solve them. The objective is to provide a way of discerning the nature and role of geometrical intuition in the solving of the three problems.

Step 1: analyse Schoenfeld's transcripts of students working on the geometrical construction problems to help discern *critical decisions in the solution of these problems*. The conjecture being proposed here is that geometrical intuition plays a part in the critical decisions that problem-solvers make when tackling geometrical problems.

Step 2: use the *categories of intuition* proposed by Fischbein to discern examples of the use of geometrical intuition by Schoenfeld's subjects at the moments of critical decision in the solution of each of the problems ascertained in step 1 above.

This process provided a framework for analysing the protocols resulting from video-recording three pairs of recent mathematics graduates solving the same problems as used by Schoenfeld but in this study the students were using *Cabri-Géomètre*.

Research Design

The students were provided with a two hour introductory session using *Cabri* when they worked through a series of activities. The initial activities were heavily structured, later activities were more open, and were designed to introduce the students to the software. At all stages of this introductory session the students were encouraged both to ask their own questions and pose their own problems, and to consider notions of proof. This both fits in with normal practice on their postgraduate course, and provides the appropriate experience prior to the experimental sessions for this study. Thus, although in some respects this is a laboratory study, in other respects it has features of the regular classroom experience of the students. The experimental sessions took place in the students' regular teaching room. The researcher is their regular teacher. The approach is one that fits in reasonably well with their normal experience of teaching sessions on their postgraduate course.

The students (mathematics graduates) were briefed that the focus of the study is the reasons they do what they do when they were solving problems; in other words, that their success or otherwise in solving the problems is not of paramount concern. The experimental procedure was based on that of Schoenfeld in that the pairs of students worked through the series of three problems given by Schoenfeld (and illustrated below). Unlike Schoenfeld's approach, however, a time limit was not set and the student pairs could work on each problem for as long or as little as they chose. All the problem solving sessions using *Cabri* were videotaped. Later the researcher and the students reviewed the videotape and the students were encouraged to talk about the reasons they made the decisions they did during their problem solving session. These review sessions were audio-recorded.

Results

Problem 1

You are given two intersecting straight lines and a point *P* marked on one of them. Show how to construct a circle that is tangent to both lines and has point *P* as its point of tangency to one of the lines.

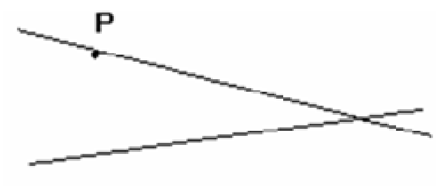

Figure 1

Critical decisions in the solution of Problem 1 are as follows:

1. Constructing a perpendicular line through P
2. Constructing the angle bisector of the angle between the two intersecting lines
   or constructing a circle centred at the intersection and passing through P giving an intersection with the second line; a perpendicular line through this point intersects the perpendicular through P at the centre of the required circle.

While all the pairs I studied made the first critical decision almost immediately, only one student saw that decision 2 was also critical. The other pairs constructed a second perpendicular, perpendicular to the lower of the intersecting lines, and proceeded to drag this line with the mouse into approximately the correct position. This prompted them into drawing the angle bisector of the intersecting lines and the problem was solved. No students constructed a circle centred at the intersection of the given lines and passing through P.

For one particular pair the suggestion to draw the angle bisector was made quite tentatively:

TC: Yes ... Ah! Now would the centre of the circle lie .. I'm just thinking something slightly different now, because I'm just trying to think, there must be a way of securing the centre accurately .. and I'm thinking .. does the centre of the circle ..sit on the bisector of the angle that's made by those two lines ..

This is how the student in this case accounted for his idea:

TC: .. [long pause] .. well, partly previous knowledge. I wasn't .. completely sure. I wasn't saying `Oh, yes. This is what does happen'. I just had a sneaky feeling that we were missing something and I couldn't work out what it was, but I thought, well I'm sure the angle .. there must be some connection between the angle between the two lines and the centre [of the circle]. So, let's put the line in and see what happens.

It turned out to be right, but it was just a sort of stab .. well, it wasn't a stab in the dark completely ...

I can't think why, but I was sure we should be bisecting the angle.

More details about the tackling of this problem are given in Jones 1993.

Problem 2

You are given a triangle *T* with base *B*. Construct a line parallel to *B* that divides *T* into two parts of equal area.
Similarly, divide *T* into five parts of equal area.

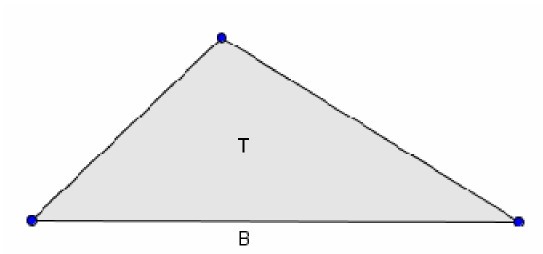

Figure 2

The critical decisions for this problem are:

1. recognising the problem as a ratio problem involving similar triangles or as an enlargement with a fractional scale factor
2. working out either the ratio of the sides of the similar triangles or the scale factor
3. working out how to apply a $1,1,\sqrt{2}$ triangle to the problem or how to divide a line in a given ratio

Each of the student pairs worked on this problem for about two hours. Only one pair successfully solved the problem. At one point, one student from the successful pair says:

GT: These are similar, these two triangles. So you can say, if .. what is it? .. the ratio of the edges? What is the ratio .. of the edges and the area? How does it work? .. For similar triangles ..

What made her say that?

GT: I thought it [the problem] was hard. I thought of calculus and then I thought of similar triangles .. and ratios.

Later, her partner says:

AW: Should we work out what √2 is ... across here? ... and we half it [this divides the edge of the triangle in the ratio 1:√2/2].

He explains:

AW: I was dead [exactly] right looking at it, but I didn't know what I was talking about at the time. At the time I didn't know how I was going to get √2 there. I just thought, that's what we had to do. We had to get a line of length √2 down there.

Problem 3

> Three points are chosen on the circumference of a circle of radius R,and the triangle containing them is drawn. What choice of points results in the triangle with the largest possible area? Justify your answer as well as you can.

All the pairs who tackled this problem conjectured quickly that the triangle with the largest area would be equilateral. None of the pairs produced a satisfactory justification.

Discussion

Fischbein suggests (Fischbein 1994) that, in analysing students' mathematical behaviour, three aspects have to be taken into account: the formal (definitions, theorems etc), the algorithmic (solving techniques and standard strategies), and the intuitive (the subjective acceptance of a mathematical concept, theorem or solution). Sometimes, Fischbein argues, these three components converge, but more often conflicting interactions may appear. However, he claims that most of the time the intuitive interpretation, based on limited but strongly rooted individual experience "annihilates the formal control ...and thus distorts or even blocks a correct mathematical reaction".

I would suggest that the results of my study given above support Fischbein's case. In each of the examples when the solution to the problem was not quickly apparent to the students (and after all that is a common definition of a mathematical problem), the students adopted a range of strategies none of which could be said to involve formal analysis. This is particularly evident in those cases when the problem was not solved. For example, with problem 1, one pair drew a line parallel to the base line and, by adjusting the height of this parallel line, they made the areas equal. Thus they had an empirical solution. With the parallel line in position they proceeded to try various methods of fixing its position, including using the midpoints of the sides, where the lines joining the midpoints intersected the angle bisectors and so on. None of these approaches proved successful and, eventually, the pair gave up on the problem.

This reliance on intuitive acceptability (which can lead one astray, as indicated in the preceding paragraph, as much as it can lead one to a correct solution; see Jones 1994) is also evident in the justification given by the students for their actions even when they came to a correct solution. One student refers to a "sneaky feeling" [a fairly strong suspicion, usually ungrounded], another says that "I was .. right ... but I didn't know what I was talking about at the time". These are examples, I would argue, of geometrical intuition since they display a number of the properties such as coerciveness, theory status and implicitness given by Fischbein as aspects of mathematical intuition.

The students in this study were mathematics graduates, chosen for two reasons. First, to allow Schoenfeld's work to provide a framework for the research and secondly, because one would have expected mathematics graduates to have acquired formal mathematical concepts sufficient to solve these particular geometrical problems. As is illustrated above, these particular graduates found the problems far from straightforward suggesting that their grasping of geometrical concepts may be quite fragile. Davis summarises a range of studies which, he suggests, "show in detail how many students, believed to be successful when one judges by typical tests, are revealed as seriously confused when one looks closely at how they think about the subject" (Davis 1984 p 349). It is just such a close look at the acquiring of abstract geometrical concepts by secondary age pupils that is in progress now as the next stage in this research project.

Future Research

The newly published UK National Curriculum for mathematics (DfE and Welsh Office 1995) identifies three specific opportunities for learning geometrical concepts that should be given to pupils during secondary schooling (*ibid* p 16). One of these is that "pupils should be given the opportunity to: use computers to generate and transform graphic images and to solve problems" (*op cit*). The aim of this current project is to investigate the nature of the learning of abstract geometrical concepts by secondary-school age pupils in the classroom when using *Cabri-Géomètre*. A particular focus is the relationship between abstract and intuitive thinking in learning geometrical concepts. The study is longitudinal and is scheduled to last for 12 months.

The empirical evidence provided in this paper shows something of the complex nature of the interaction between the formal and intuitive components of mathematical activity. From the evidence collected so far in this project, it is by no means certain that there is, as Piaget suggests, an "internal tendency towards formalisation" (Piaget 1966 p 225). Perhaps, in the words of Papert, "the basic kind of thought is intuitive" (Papert 1993 p 167). In that case, formalisation is something we can use as a tool to solve routine problems or exercises in mathematics or to communicate mathematical results, usually in the form of proofs. The impact of formalisation on the learning of mathematics remains an area for further research.

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