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by

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CONTENTS

1 Introduction

Unlike isotropic metallic plates, composite plates exhibit quite different nonlinear behavior. For example, the geometric nonlinear effects could be very significant even at small loads and deflections, depending on the lamination scheme and boundary conditions[1][2]. So, it is necessary to investigate the geometric nonlinear effects of composite laminates

In reference[3], authors have derived the stiffness matrix of an 8-node laminated plate taking into account higher-order shear deformation. Based on this, we will adopt nonlinear strain-displacement relations and nonlinear equilibrium equations to derive the stiffness matrix of an 8-node laminated plate element taking into account both higher-order shear deformation and geometric nonlinearity. In the end of this paper the stiffness factors of an 8-node laminated plate element taking into account higher-order shear deformation and geometric nonlinearity are presented.

2 Stress-Strain Relation

Consider a laminated plate element of N layers with thickness h , length $2a$, and width $2b$,as shown in Fig.1 . Each layer is taken to be macroscopically homogeneous and orthotropic.

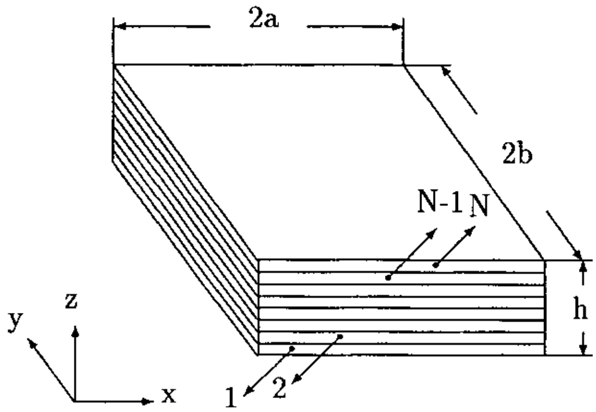

Figure 1

Based on the ***Duhamel − Neumann*** law, the stress-strain relation of the $\boldsymbol{k}$th layer is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{1}$$

where

$$\begin{aligned} Q_{11} &= E_1/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21}) = \nu_{21}E_1/(1-\nu_{12}\nu_{21}) = Q_{13} \\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21}) \\ Q_{44} &= G_{23} \end{aligned} \tag{2}$$

$$Q_{55} = G_{12} = G_{13} = Q_{66}$$

$$Q_{23} = \nu_{23}E_2/(1-\nu_{23}^2) = \nu_{32}E_3/(1-\nu_{32}^2)$$

In the derivation of Eq.(1), the stresses and strains are defined in the principal material directions for that orthotropic lamina. However, in angle-ply laminated plates, the principal directions of orthotropy of each individual lamina do not coincide with the geometrical coordinate frame. It is necessary to use the transformed reduced stiffness

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & o & 0 & \overline{Q}_{26} \\ \overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{36} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{3}$$

The thirteen constants $\overline{Q}_{ij}$ are related to the nine Q_{ij} through the following transformation formulae

$$\overline{Q}_{11} = Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4)$$

$$\overline{Q}_{13} = Q_{13}m^2 + Q_{23}n^2$$

$$\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})m^3n + (Q_{12} - Q_{22} + 2Q_{66})mn^3$$

$$\overline{Q}_{22} = Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4$$

$$
\begin{aligned}
\overline{Q}_{23} &= Q_{13}n^2 + Q_{23}m^2 \\
\overline{Q}_{33} &= Q_{33} \qquad (4) \\
\overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})mn^3 + (Q_{12} - Q_{22} + 2Q_{66})m^3n \\
\overline{Q}_{36} &= (Q_{13} - Q_{23})mn \\
\overline{Q}_{44} &= Q_{44}m^2 + Q_{55}n^2 \\
\overline{Q}_{45} &= (Q_{45} - Q_{44})mn \\
\overline{Q}_{55} &= Q_{55}m^2 + Q_{44}n^2 \\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12})m^2n^2 + Q_{66}(m^2 - n^2)^2
\end{aligned}
$$

where

$$m = cos\theta_k \quad n = sin\theta_k \quad (see \;\; Fig.2) \qquad (5)$$

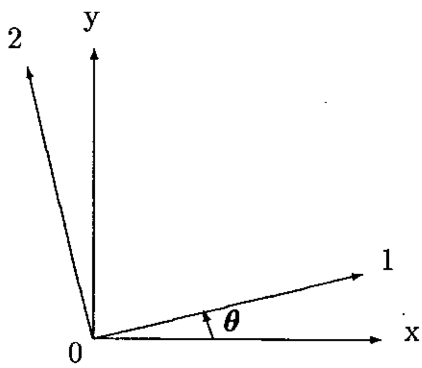

Figure 2

3 The Strain-Displacement relation

The displacement field of Reddy's higher-order shear deformation theory has been given in reference[3] as following

$$\begin{aligned}
u(x,y,z) &= u_0(x,y) + z\phi_1(x,y) - \frac{4z^3}{3h^2}[\phi_1(x,y) + \frac{\partial w}{\partial x}] \\
v(x,y,z) &= v_0(x,y) + z\phi_2(x,y) - \frac{4z^3}{3h^2}[\phi_2(x,y) + \frac{\partial w}{\partial y}] \\
w(x,y,z) &= w_0(x,y)
\end{aligned} \tag{6}$$

where u_0, v_0, w_0 are associated midplane displacements, and ϕ_1 and ϕ_2 are the rotations of the transverse normal in the xz and yz planes. The coordinate frame is chosen in such a way that the xy plane coincides with the midplane of plate.

Note that we restrict our study to problems that involve moderate rotation (say $10^0 - 15^0$). In this paper the following terms associated with rotations of transverse normal are considered

$$(\frac{\partial w}{\partial x})^2 \quad (\frac{\partial w}{\partial y})^2 \quad \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{7}$$

Once the displacement field of composite laminated plate is known the strains can be computed using nonlinear strain-displacement relation

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2 \\
\epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}(\frac{\partial w}{\partial y})^2
\end{aligned}$$

$$
\begin{aligned}
\epsilon_z &= \frac{\partial w}{\partial z} = 0 \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial y} \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}
\end{aligned} \tag{8}
$$

The strains in Eqs.(8) are called *von Karman* strains.

The strains associated with the displacement field given in Eqs.(6) can now be computed by means of *von Karman* strain-displacement relation shown in Eqs.(8)

$$
\begin{aligned}
\epsilon_x &= \epsilon_x^0 + zk_x^1 - \frac{4z^3}{3h^2}\frac{\partial \gamma_{xz}^1}{\partial x} + k_x^n \\
\epsilon_y &= \epsilon_y^0 + zk_y^1 - \frac{4z^3}{3h^2}\frac{\partial \gamma_{yz}^1}{\partial y} + k_y^n \\
\epsilon_z &= 0 \\
\gamma_{xy} &= \gamma_{xy}^0 + zk_{xy}^1 - \frac{4z^3}{3h^2}\left(\frac{\partial \gamma_{yz}^1}{\partial x} + \frac{\partial \gamma_{xz}^1}{\partial y}\right) + k_{xy}^n \\
\gamma_{xz} &= \gamma_{xz}^1\left(1 - \frac{4z^2}{h^2}\right) \\
\gamma_{yz} &= \gamma_{yz}^1\left(1 - \frac{4z^2}{h^2}\right)
\end{aligned} \tag{9}
$$

where

$$
\{\epsilon^0\} = \left\{ \begin{array}{c} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{array} \right\} \tag{10}
$$

$$\{k^1\} = \begin{Bmatrix} k_x^1 \\ k_y^1 \\ k_{xy}^1 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial y} \end{Bmatrix} \tag{11}$$

$$\{\gamma^1\} = \begin{Bmatrix} \gamma_{xz}^1 \\ \gamma_{yz}^1 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_0}{\partial x} + \phi_1 \\ \frac{\partial w_0}{\partial y} + \phi_2 \end{Bmatrix} \tag{12}$$

$$\{k^n\} = \begin{Bmatrix} k_x^n \\ k_y^n \\ k_{xy}^n \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(\frac{\partial w_0}{\partial x})^2 \\ \frac{1}{2}(\frac{\partial w_0}{\partial y})^2 \\ \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} \tag{13}$$

Substituting Eq.(9) into Eq.(3), one has the stresses expressed in terms of midplane strains $\{\epsilon^0\}$, rotations of transverse normal $\{k^1\}, \{\gamma^1\}$,and $\{k^n\}$.

$$\begin{aligned}
\sigma_x &= \overline{Q}_{11}\epsilon_x^0 + \overline{Q}_{12}\epsilon_y^0 + \overline{Q}_{16}\gamma_{xy}^0 + \overline{Q}_{11}k_x^1 z + \overline{Q}_{12}k_y^1 z + \overline{Q}_{16}k_{xy}^1 z \\
&\quad -\frac{4}{3h^2}[(\overline{Q}_{11}\frac{\partial}{\partial x} + \overline{Q}_{16}\frac{\partial}{\partial y})\gamma_{xz}^1]z^3 - \frac{4}{3h^2}[(\overline{Q}_{12}\frac{\partial}{\partial y} + \overline{Q}_{16}\frac{\partial}{\partial x})\gamma_{yz}^1]z^3 \\
&\quad +\overline{Q}_{11}k_x^n + \overline{Q}_{12}k_y^n + \overline{Q}_{16}k_{xy}^n \\
\sigma_y &= \overline{Q}_{12}\epsilon_x^0 + \overline{Q}_{22}\epsilon_y^0 + \overline{Q}_{26}\gamma_{xy}^0 + \overline{Q}_{12}k_x^1 z + \overline{Q}_{22}k_y^1 z + \overline{Q}_{26}k_{xy}^1 z \\
&\quad -\frac{4}{3h^2}[(\overline{Q}_{12}\frac{\partial}{\partial x} + \overline{Q}_{26}\frac{\partial}{\partial y})\gamma_{xz}^1]z^3 - \frac{4}{3h^2}[(\overline{Q}_{22}\frac{\partial}{\partial y} + \overline{Q}_{26}\frac{\partial}{\partial x})\gamma_{yz}^1]z^3 \\
&\quad +\overline{Q}_{12}k_x^n + \overline{Q}_{22}k_y^n + \overline{Q}_{26}k_{xy}^n \\
\sigma_{yz} &= \overline{Q}_{44}\gamma_{yz}^1(1 - \frac{4z^2}{h^2}) + \overline{Q}_{45}\gamma_{xz}^1(1 - \frac{4z^2}{h^2}) \qquad (14) \\
\sigma_{xz} &= \overline{Q}_{45}\gamma_{yz}^1(1 - \frac{4z^2}{h^2}) + \overline{Q}_{55}\gamma_{xz}^1(1 - \frac{4z^2}{h^2})
\end{aligned}$$

$$
\begin{aligned}
\sigma_{xy} \;=\; & \overline{Q}_{16}\epsilon_x^0 + \overline{Q}_{26}\epsilon_y^0 + \overline{Q}_{66}\gamma_{xy}^0 + \overline{Q}_{16}k_x^1 z + \overline{Q}_{26}k_y^1 z + \overline{Q}_{66}k_{xy}^1 z \\
& -\frac{4}{3h^2}[(\overline{Q}_{16}\frac{\partial}{\partial x} + \overline{Q}_{66}\frac{\partial}{\partial y})\gamma_{xz}^1]z^3 - \frac{4}{3h^2}[(\overline{Q}_{26}\frac{\partial}{\partial y} + \overline{Q}_{66}\frac{\partial}{\partial x})\gamma_{yz}^1]z^3 \\
& +\overline{Q}_{16}k_x^n + \overline{Q}_{26}k_y^n + \overline{Q}_{66}k_{xy}^n
\end{aligned}
$$

The stress resultants and moment resultants are defined as

$$
\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz \tag{15}
$$

$$
\{Q\} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz \tag{16}
$$

$$
\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} zdz \tag{17}
$$

where the geometrical notation is as shown below:

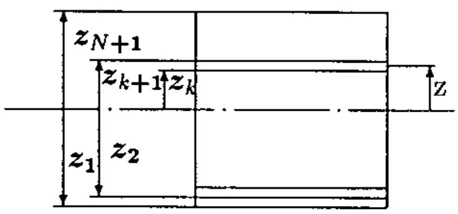

Figure 3

Substituting Eq. (14) into Eqs. (15) , (16) , (17) , we have

$$\left\{\begin{array}{c} N \\ M \end{array}\right\} = \left\{\begin{array}{c} N^1 \\ M^1 \end{array}\right\} + \left\{\begin{array}{c} N^3 \\ M^3 \end{array}\right\} + \left\{\begin{array}{c} N^n \\ M^n \end{array}\right\} \tag{18}$$

$$\left\{\begin{array}{c} Q_x \\ Q_y \end{array}\right\} = \left\{\begin{array}{c} Q_x^1 \\ Q_y^1 \end{array}\right\} + \left\{\begin{array}{c} Q_x^3 \\ Q_y^3 \end{array}\right\} \tag{19}$$

where

$$\left\{\begin{array}{c} N^1 \\ M^1 \end{array}\right\} = \left\{\begin{array}{c} N_x^1 \\ N_y^1 \\ N_{xy}^1 \\ M_x^1 \\ M_y^1 \\ M_{xy}^1 \end{array}\right\} = \left[\begin{array}{cccccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left\{\begin{array}{c} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ k_x^1 \\ k_y^1 \\ k_{xy}^1 \end{array}\right\} \tag{20}$$

$$\left\{\begin{array}{c} N^3 \\ M^3 \end{array}\right\} = \left\{\begin{array}{c} N_x^3 \\ N_y^3 \\ N_{xy}^3 \\ M_x^3 \\ M_y^3 \\ M_{xy}^3 \end{array}\right\} = \left[\begin{array}{cccc} E_{11} & E_{16} & E_{16} & E_{12} \\ E_{12} & E_{26} & E_{26} & E_{22} \\ E_{16} & E_{66} & E_{66} & E_{26} \\ F_{11} & F_{16} & F_{16} & F_{12} \\ F_{12} & F_{26} & F_{26} & F_{22} \\ F_{16} & F_{66} & F_{66} & F_{26} \end{array}\right] \left\{\begin{array}{c} \frac{\partial \gamma_{xz}^1}{\partial x} \\ \frac{\partial \gamma_{xz}^1}{\partial y} \\ \frac{\partial \gamma_{yz}^1}{\partial x} \\ \frac{\partial \gamma_{yz}^1}{\partial y} \end{array}\right\} \tag{21}$$

$$\left\{\begin{array}{c} N^n \\ M^n \end{array}\right\} = \left\{\begin{array}{c} N_x^n \\ N_y^n \\ N_{xy}^n \\ M_x^n \\ M_y^n \\ M_{xy}^n \end{array}\right\} = \left[\begin{array}{ccc} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \\ B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{array}\right] \left\{\begin{array}{c} k_x^n \\ k_y^n \\ k_{xy}^n \end{array}\right\} \tag{22}$$

$$\begin{aligned} \left\{\begin{array}{c} Q_x^1 \\ Q_y^1 \end{array}\right\} &= \left[\begin{array}{cc} A_{55} & A_{45} \\ A_{45} & A_{44} \end{array}\right] \left\{\begin{array}{c} \gamma_{xz}^1 \\ \gamma_{yz}^1 \end{array}\right\} \\ \left\{\begin{array}{c} Q_x^3 \\ Q_y^3 \end{array}\right\} &= -\frac{4}{h^2} \left[\begin{array}{cc} D_{55} & D_{45} \\ D_{45} & D_{44} \end{array}\right] \left\{\begin{array}{c} \gamma_{xz}^1 \\ \gamma_{yz}^1 \end{array}\right\} \end{aligned} \tag{23}$$

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1} - z_k) \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^2 - z_k^2) \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^3 - z_k^3) \\ E_{ij} &= -\frac{1}{3h^2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^4 - z_k^4) \\ F_{ij} &= -\frac{4}{15h^2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (z_{k+1}^5 - z_k^5) \end{aligned} \tag{24}$$

4 Interpolation Function

We adopt Serendipity 8-node element with five degrees of freedom for each node . The total number of degrees of freedom for the element is 40. The natural coordinate system (ξ, η) as shown in Fig.3 is taken to define the element geometry . The element has sides $\xi = \pm 1$ and $\eta = \pm 1$ (see Fig.3) . For the element of side 2a by 2b

$$\begin{aligned} \xi &= (x - x_c)/a \\ \eta &= (y - y_c)/b \end{aligned} \tag{25}$$

where (x_c, y_c) are the coordinates at the centre of the element . Thus we have

$$\frac{d\xi}{dx} = \frac{1}{a} \qquad \frac{d\eta}{dy} = \frac{1}{b} \tag{26}$$

and the element area of the rectangular element is give as

$$dxdy = abd\xi d\eta \tag{27}$$

To integrate any function f(x,y) over the element we transform to the natural coordinate system ,so that

$$\int\int_{\Omega^e} f(x,y)\ dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\ abd\xi d\eta \tag{28}$$

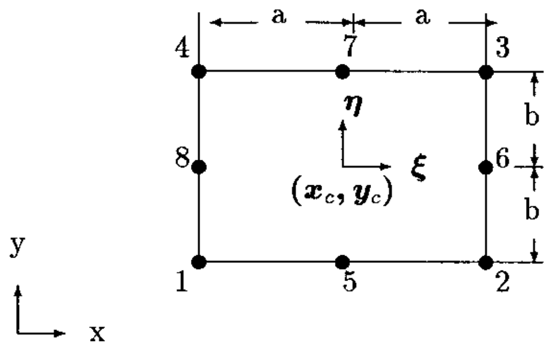

Figure 4

For the 8-node Serendipity element shown in Fig.4 the interpolation function has the following form for the corner and midside nodes

1. for the corner nodes

$$\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) \qquad i = 1,2,3,4 \tag{29}$$

2. for the midside nodes

$$\psi_i = \frac{\xi_i^2}{2}(1+\xi\xi_i)(1-\eta^2)+\frac{\eta_i^2}{2}(1+\eta\eta_i)(1-\xi^2) \qquad i = 5,6,7,8 \tag{30}$$

According to the values of 8 node coordinates we have from Eqs.(29) and (30) the interpolation function for each node as follows

$$\begin{aligned}
\psi_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1) \\
\psi_2 &= \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\
\psi_3 &= \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\
\psi_4 &= \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)
\end{aligned}$$

$$
\begin{aligned}
\psi_5 &= \frac{1}{2}(1-\eta)(1-\xi^2) \\
\psi_6 &= \frac{1}{2}(1+\xi)(1-\eta^2) \\
\psi_7 &= \frac{1}{2}(1+\eta)(1-\xi^2) \\
\psi_8 &= \frac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned} \tag{31}
$$

The derivatives of the interpolation functions are as follows

1. for the corner nodes

$$
\begin{aligned}
\frac{\partial\psi_i}{\partial\xi} &= \frac{1}{4}\xi_i(1+\eta\eta_i)(2\xi\xi_i+\eta\eta_i) \\
\frac{\partial\psi_i}{\partial\eta} &= \frac{1}{4}\eta_i(1+\xi\xi_i)(2\eta\eta_i+\xi\xi_i) \\
\frac{\partial^2\psi_i}{\partial\xi^2} &= \frac{1}{2}\xi_i^2(1+\eta\eta_i) \\
\frac{\partial^2\psi_i}{\partial\eta^2} &= \frac{1}{2}\eta_i^2(1+\xi\xi_i) \\
\frac{\partial^2\psi_i}{\partial\xi\partial\eta} &= \frac{\partial^2\psi_i}{\partial\eta\partial\xi} = \frac{1}{4}\xi_i\eta_i(1+2\eta\eta_i+2\xi\xi_i)
\end{aligned} \tag{32}
$$

2. for the midside nodes

$$
\begin{aligned}
\frac{\partial\psi_i}{\partial\xi} &= \frac{1}{2}\xi_i^3(1-\eta^2)-\eta_i^2(1+\eta\eta_i)\xi \\
\frac{\partial\psi_i}{\partial\eta} &= \frac{1}{2}\eta_i^3(1-\xi^2)-\xi_i^2(1+\xi\xi_i)\eta \\
\frac{\partial^2\psi_i}{\partial\xi^2} &= -\eta_i^2(1+\eta\eta_i) \\
\frac{\partial^2\psi_i}{\partial\eta^2} &= -\xi_i^2(1+\xi\xi_i) \\
\frac{\partial^2\psi_i}{\partial\xi\partial\eta} &= \frac{\partial^2\psi_i}{\partial\eta\partial\xi} = -\xi_i^3\eta-\xi\eta_i^3
\end{aligned} \tag{33}
$$

Note that the polynomial terms contained in this element are $\boldsymbol{1, x, y, x^2, xy,}$ $\boldsymbol{y^2, x^2y, xy^2}$. For this element, the interpolation functions have satisfied the conditions

$$\sum_i \psi_i(\xi\eta) = 1 \tag{34}$$

and

$$\psi_i(\xi_j\eta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases} \tag{35}$$

The displacement components are approximated by the product of the interpolation function matrix $[\psi_i]$ and the nodal displacement vector $\{q_i^e\} = [u_{0i}\ v_{0i}\ w_{0i}\ \phi_{1i}\ \phi_{2i}]^T$, i.e.,

$$\{q\} = \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \sum_{i=1}^{8} [\psi_i]\{q_i^e\} \tag{36}$$

the superscript e of $\{q_i^e\}$ denotes these variables are defined on the element and need to be determined .

5 The Stiffness Matrix of an 8-Node Laminated Plate Element

For the geometric nonlinear case, by means of the principle of virtual displacement, we have[3][4]

$$
\begin{aligned}
\int_{\Omega^e} -(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y})\delta u_0 dxdy &= 0 \\
\int_{\Omega^e} -(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y})\delta v_0 dxdy &= 0 \\
\int_{\Omega^e} -[\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}(N_x\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}) + \frac{\partial}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x} + N_y\frac{\partial w_0}{\partial y}) \\
+q]\delta w_0 dxdy &= 0 \qquad (37) \\
\int_{\Omega^e} -(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x)\delta\phi_1 dxdy &= 0 \\
\int_{\Omega^e} -(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y)\delta\phi_2 dxdy &= 0
\end{aligned}
$$

Recall that $N_x, N_{xy}, ..., M_{xy}$ are functions of the derivatives of the displacement $u_0, v_0, w_0, \phi_1, \phi_2$. To reduce the differentiability of the interpolation functions used in the finite element approximation of $u_0, v_0, w_0, \phi_1, \phi_2$ the differentiation on $N_x, N_{xy}, ..., M_{xy}$ is treated to weight function $\delta u_0, \delta v_0, \delta w_0, \delta\phi_1, \delta\phi_2$ by using integration-by-parts

$$
\int_{\Omega^e}(\frac{\partial \delta u_0}{\partial x}N_x + \frac{\partial \delta u_0}{\partial y}N_{xy})dxdy - \int_{\Gamma^e} N_n\delta u_0 ds = 0
$$

$$
\begin{aligned}
\int_{\Omega^e}(\frac{\partial\delta v_0}{\partial x}N_{xy}+\frac{\partial\delta v_0}{\partial y}N_y)dxdy-\int_{\Gamma^e}N_{ns}\delta v_0 ds &= 0\\
\int_{\Omega^e}[\frac{\partial\delta w_0}{\partial x}Q_x+\frac{\partial\delta w_0}{\partial y}Q_y+\frac{\partial\delta w_0}{\partial x}(N_x\frac{\partial w_0}{\partial x}+N_{xy}\frac{\partial w_0}{\partial y})&\\
+\frac{\partial\delta w_0}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x}+N_y\frac{\partial w_0}{\partial y})-q\delta w_0]dxdy&\\
-\int_{\Gamma^e}(Q_n\delta w_0+N_n\frac{\partial w_0}{\partial x}\delta w_0+N_{ns}\frac{\partial w_0}{\partial y}\delta w_0)ds &= 0 \qquad (38)\\
\int_{\Omega^e}(\frac{\partial\delta\phi_1}{\partial x}M_x+\frac{\partial\delta\phi_1}{\partial y}M_{xy}+Q_x\delta\phi_1)dxdy-\int_{\Gamma^e}M_n\delta\phi_1 ds &= 0\\
\int_{\Omega^e}(\frac{\partial\delta\phi_2}{\partial x}M_{xy}+\frac{\partial\delta\phi_2}{\partial y}M_y+Q_y\delta\phi_2)dxdy-\int_{\Gamma^e}M_{ns}\delta\phi_2 ds &= 0
\end{aligned}
$$

where

Γ^e ——— the boundary edge of the element domain Ω^e;

Ω^e ——— the element domain;

N_n, N_{ns} – the stress resultants at the boundary edge of the element;

Q_n ——— the transverse shear force at the boundary edge of the element;

M_n, M_{ns} – the moment resultants at the boundary edge of the element.

Substituting $\delta u_0=\psi_i$, $\delta v_0=\psi_i$, $\delta w_0=\psi_i$, $\delta\phi_1=\psi_i$ and $\delta\phi_2=\psi_i$ into Eqs. (38) , we have

$$
\begin{aligned}
\int_{\Omega^e}(\frac{\partial\psi_i}{\partial x}N_x+\frac{\partial\psi_i}{\partial y}N_{xy})dxdy-\int_{\Gamma^e}N_n\psi_i ds &= 0\\
\int_{\Omega^e}(\frac{\partial\psi_i}{\partial x}N_{xy}+\frac{\partial\psi_i}{\partial y}N_y)dxdy-\int_{\Gamma^e}N_{ns}\psi_i ds &= 0\\
\int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}Q_x+\frac{\partial\psi_i}{\partial y}Q_y+\frac{\partial\psi_i}{\partial x}(N_x\frac{\partial w_0}{\partial x}+N_{xy}\frac{\partial w_0}{\partial y})&\\
+\frac{\partial\psi_i}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x}+N_y\frac{\partial w_0}{\partial y})-q\psi_i]dxdy&
\end{aligned}
$$

$$-\int_{\Gamma^e}(Q_n\psi_i + N_n\frac{\partial w_0}{\partial x}\psi_i + N_{ns}\frac{\partial w_0}{\partial y}\psi_i)ds \;=\; 0 \tag{39}$$

$$\int_{\Omega^e}(\frac{\partial \psi_i}{\partial x}M_x + \frac{\partial \psi_i}{\partial y}M_{xy} + Q_x\psi_i)dxdy - \int_{\Gamma^e} M_n\psi_i ds \;=\; 0$$

$$\int_{\Omega^e}(\frac{\partial \psi_i}{\partial x}M_{xy} + \frac{\partial \psi_i}{\partial y}M_y + Q_y\psi_i)dxdy - \int_{\Gamma^e} M_{ns}\psi_i ds \;=\; 0$$

Substituting Eq.(36) for u_0, v_0, w_0, ϕ_1, and ϕ_2 into Eqs.(10), (11), (12), (13) we obtain $\{\epsilon^0\}, \{k^1\}, \{\gamma^1\}$, and $\{k^n\}$ represented by interpolation functions and nodal displacements. Then substituting $\{\epsilon^0\}, \{k^1\}, \{\gamma^1\}$ and $\{k^n\}$ into Eqs. (18), (19) , we have stress resultants, moment resultants and transverse shear forces $\{N\}, \{M\}$ and $\{Q\}$ represented by interpolation functions and nodal displacements . After substituting $\{N\}, \{M\}$ and $\{Q\}$ represented by interpolation functions and nodal displacements into Eq.(39) we obtain the finite element model taking into account higher-order shear deformation and geometric nonlinearity .

$$\sum_{j=1}^{8}\sum_{\beta=1}^{5} K_{ij}^{\alpha\beta}\Delta_j^{\beta} - F_i^{\alpha} = 0 \quad (\alpha = 1, 2, ..., 5) \tag{40}$$

or

$$[K^e]\{\Delta^e\} \;=\; \{F^e\} \tag{41}$$

where the variables Δ_j^{β}, the stiffness and force coefficients are defined by

$$\Delta_j^1 \;=\; u_{0j} \qquad \Delta_j^2 \;=\; v_{0j} \qquad \Delta_j^3 \;=\; w_{0j}$$

$$\Delta_j^4 \;=\; \phi_{1j} \qquad \Delta_j^5 \;=\; \phi_{2j} \tag{42}$$

$$K_{ij}^{1\alpha} \;=\; \int_{\Omega^e}[\frac{\partial \psi_i}{\partial x}(N_{1j}^{\alpha} + \overline{N}_{1j}^{\alpha} + N_1^{\alpha}) + \frac{\partial \psi_i}{\partial y}(N_{6j}^{\alpha} + \overline{N}_{6j}^{\alpha} + N_6^{\alpha})]dxdy$$

$$
\begin{aligned}
K_{ij}^{2\alpha} &= \int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}(N_{6j}^\alpha+\overline{N}_{6j}^\alpha+N_6^\alpha)+\frac{\partial\psi_i}{\partial y}(N_{2j}^\alpha+\overline{N}_{2j}^\alpha+N_2^\alpha)]dxdy \\
K_{ij}^{3\alpha} &= \int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}(Q_{1j}^\alpha+\overline{Q}_{1j}^\alpha)+\frac{\partial\psi_i}{\partial y}(Q_{2j}^\alpha+\overline{Q}_{2j}^\alpha) \\
&\quad +\frac{\partial\psi_i}{\partial x}\frac{\partial w_0}{\partial x}(N_{1j}^\alpha+\overline{N}_{1j}^\alpha)+\frac{\partial\psi_i}{\partial x}\frac{\partial w_0}{\partial y}(N_{6j}^\alpha+\overline{N}_{6j}^\alpha) \\
&\quad +\frac{\partial\psi_i}{\partial y}\frac{\partial w_0}{\partial x}(N_{6j}^\alpha+\overline{N}_{6j}^\alpha)+\frac{\partial\psi_i}{\partial y}\frac{\partial w_0}{\partial y}(N_{2j}^\alpha+\overline{N}_{2j}^\alpha) \\
&\quad +\frac{\partial\psi_i}{\partial x}(\frac{\partial\psi_j}{\partial x}Q_1^\alpha+\frac{\partial\psi_j}{\partial y}Q_6^\alpha)+\frac{\partial\psi_i}{\partial y}(\frac{\partial\psi_j}{\partial x}Q_6^\alpha+\frac{\partial\psi_j}{\partial y}Q_2^\alpha)]dxdy \qquad (43) \\
K_{ij}^{4\alpha} &= \int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}(M_{1j}^\alpha+\overline{M}_{1j}^\alpha+M_1^\alpha)+\frac{\partial\psi_i}{\partial y}(M_{6j}^\alpha+\overline{M}_{6j}^\alpha+M_6^\alpha) \\
&\quad +\psi_i(Q_{1j}^\alpha+\overline{Q}_{1j}^\alpha)]dxdy \\
K_{ij}^{5\alpha} &= \int_{\Omega^e}[\frac{\partial\psi_i}{\partial x}(M_{6j}^\alpha+\overline{M}_{6j}^\alpha+M_6^\alpha)+\frac{\partial\psi_i}{\partial y}(M_{2j}^\alpha+\overline{M}_{2j}^\alpha+M_2^\alpha) \\
&\quad +\psi_i(Q_{2j}^\alpha+\overline{Q}_{2j}^\alpha)]dxdy
\end{aligned}
$$

The coefficients $N_{Ij}^\alpha, \overline{N}_{Ij}^\alpha, N_1^\alpha, N_2^\alpha, N_6^\alpha, M_{Ij}^\alpha, \overline{M}_{Ij}^\alpha, M_1^\alpha, M_2^\alpha, M_6^\alpha, Q_{Ij}^\alpha, \overline{Q}_{Ij}^\alpha, Q_1^\alpha, Q_2^\alpha$, and Q_6^α for $\alpha = 1, 2, ..., 5$ and $I = 1, 2, 6$ are given by

$$
\begin{aligned}
N_{1j}^1 &= A_{11}\frac{\partial\psi_j}{\partial x}+A_{16}\frac{\partial\psi_j}{\partial y} & N_{1j}^2 &= A_{12}\frac{\partial\psi_j}{\partial y}+A_{16}\frac{\partial\psi_j}{\partial x} \\
N_{1j}^4 &= B_{11}\frac{\partial\psi_j}{\partial x}+B_{16}\frac{\partial\psi_j}{\partial y} & N_{1j}^5 &= B_{12}\frac{\partial\psi_j}{\partial y}+B_{16}\frac{\partial\psi_j}{\partial x} \\
N_{2j}^1 &= A_{12}\frac{\partial\psi_j}{\partial x}+A_{26}\frac{\partial\psi_j}{\partial y} & N_{2j}^2 &= A_{22}\frac{\partial\psi_j}{\partial y}+A_{26}\frac{\partial\psi_j}{\partial x} \qquad (44) \\
N_{2j}^4 &= B_{12}\frac{\partial\psi_j}{\partial x}+B_{26}\frac{\partial\psi_j}{\partial y} & N_{2j}^5 &= B_{22}\frac{\partial\psi_j}{\partial y}+B_{26}\frac{\partial\psi_j}{\partial x} \\
N_{6j}^1 &= A_{16}\frac{\partial\psi_j}{\partial x}+A_{66}\frac{\partial\psi_j}{\partial y} & N_{6j}^2 &= A_{26}\frac{\partial\psi_j}{\partial y}+A_{66}\frac{\partial\psi_j}{\partial x} \\
N_{6j}^4 &= B_{16}\frac{\partial\psi_j}{\partial x}+B_{66}\frac{\partial\psi_j}{\partial y} & N_{6j}^5 &= B_{26}\frac{\partial\psi_j}{\partial y}+B_{66}\frac{\partial\psi_j}{\partial x}
\end{aligned}
$$

$$
\begin{aligned}
\overline{N}^3_{1j} &= E_{11}\frac{\partial^2\psi_j}{\partial x^2} + 2E_{16}\frac{\partial^2\psi_j}{\partial x\partial y} + E_{12}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{N}^4_{1j} &= E_{11}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y} \qquad \overline{N}^5_{1j} = E_{12}\frac{\partial\psi_j}{\partial y} + E_{16}\frac{\partial\psi_j}{\partial x} \\
\overline{N}^3_{2j} &= E_{12}\frac{\partial^2\psi_j}{\partial x^2} + 2E_{26}\frac{\partial^2\psi_j}{\partial x\partial y} + E_{22}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{N}^4_{2j} &= E_{12}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y} \qquad \overline{N}^5_{2j} = E_{22}\frac{\partial\psi_j}{\partial y} + E_{26}\frac{\partial\psi_j}{\partial x} \\
\overline{N}^3_{6j} &= E_{16}\frac{\partial^2\psi_j}{\partial x^2} + 2E_{66}\frac{\partial^2\psi_j}{\partial x\partial y} + E_{26}\frac{\partial^2\psi_j}{\partial y^2} \\
\overline{N}^4_{6j} &= E_{16}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y} \qquad \overline{N}^5_{6j} = E_{26}\frac{\partial\psi_j}{\partial y} + E_{66}\frac{\partial\psi_j}{\partial x}
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
N^3_1 &= \frac{1}{2}[A_{11}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + A_{12}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + A_{16}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + A_{16}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x}] \\
N^3_2 &= \frac{1}{2}[A_{12}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + A_{22}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + A_{26}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + A_{26}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x}] \\
N^3_6 &= \frac{1}{2}[A_{16}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial x} + A_{26}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial y} + A_{66}\frac{\partial w_0}{\partial x}\frac{\partial\psi_j}{\partial y} + A_{66}\frac{\partial w_0}{\partial y}\frac{\partial\psi_j}{\partial x}]
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
Q^3_{1j} &= A_{55}\frac{\partial\psi_j}{\partial x} + A_{45}\frac{\partial\psi_j}{\partial y} \quad Q^4_{1j} = A_{55}\psi_j \quad Q^5_{1j} = A_{45}\psi_j \\
Q^3_{2j} &= A_{45}\frac{\partial\psi_j}{\partial x} + A_{44}\frac{\partial\psi_j}{\partial y} \quad Q^4_{2j} = A_{45}\psi_j \quad Q^5_{2j} = A_{44}\psi_j
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\overline{Q}^3_{1j} &= -\frac{4}{h^2}(D_{55}\frac{\partial\psi_j}{\partial x} + D_{45}\frac{\partial\psi_j}{\partial y}) \quad \overline{Q}^4_{1j} = -\frac{4}{h^2}D_{55}\psi_j \\
\overline{Q}^5_{1j} &= -\frac{4}{h^2}D_{45}\psi_j \\
\overline{Q}^3_{2j} &= -\frac{4}{h^2}(D_{45}\frac{\partial\psi_j}{\partial x} + D_{44}\frac{\partial\psi_j}{\partial y}) \quad \overline{Q}^4_{2j} = -\frac{4}{h^2}D_{45}\psi_j \\
\overline{Q}^5_{2j} &= -\frac{4}{h^2}D_{44}\psi_j
\end{aligned} \tag{48}
$$

$$
Q^3_1 = \frac{1}{2}A_{11}(\frac{\partial w_0}{\partial x})^2 + \frac{1}{2}A_{12}(\frac{\partial w_0}{\partial y})^2 + A_{16}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}
$$

$$
\begin{aligned}
Q_2^3 &= \frac{1}{2}A_{12}(\frac{\partial w_0}{\partial x})^2 + \frac{1}{2}A_{22}(\frac{\partial w_0}{\partial y})^2 + A_{26}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \\
Q_6^3 &= \frac{1}{2}A_{16}(\frac{\partial w_0}{\partial x})^2 + \frac{1}{2}A_{26}(\frac{\partial w_0}{\partial y})^2 + A_{66}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
M_{1j}^1 &= B_{11}\frac{\partial \psi_j}{\partial x} + B_{16}\frac{\partial \psi_j}{\partial y} & M_{1j}^2 &= B_{12}\frac{\partial \psi_j}{\partial y} + B_{16}\frac{\partial \psi_j}{\partial x} \\
M_{1j}^4 &= D_{11}\frac{\partial \psi_j}{\partial x} + D_{16}\frac{\partial \psi_j}{\partial y} & M_{1j}^5 &= D_{12}\frac{\partial \psi_j}{\partial y} + D_{16}\frac{\partial \psi_j}{\partial x} \\
M_{2j}^1 &= B_{12}\frac{\partial \psi_j}{\partial x} + B_{26}\frac{\partial \psi_j}{\partial y} & M_{2j}^2 &= B_{22}\frac{\partial \psi_j}{\partial y} + B_{26}\frac{\partial \psi_j}{\partial x} \\
M_{2j}^4 &= D_{12}\frac{\partial \psi_j}{\partial x} + D_{26}\frac{\partial \psi_j}{\partial y} & M_{2j}^5 &= D_{22}\frac{\partial \psi_j}{\partial y} + D_{26}\frac{\partial \psi_j}{\partial x} \\
M_{6j}^1 &= B_{16}\frac{\partial \psi_j}{\partial x} + B_{66}\frac{\partial \psi_j}{\partial y} & M_{6j}^2 &= B_{26}\frac{\partial \psi_j}{\partial y} + B_{66}\frac{\partial \psi_j}{\partial x} \\
M_{6j}^4 &= D_{16}\frac{\partial \psi_j}{\partial x} + D_{66}\frac{\partial \psi_j}{\partial y} & M_{6j}^5 &= D_{26}\frac{\partial \psi_j}{\partial y} + D_{66}\frac{\partial \psi_j}{\partial x}
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
\overline{M}_{1j}^3 &= F_{11}\frac{\partial^2 \psi_j}{\partial x^2} + 2F_{16}\frac{\partial^2 \psi_j}{\partial x \partial y} + F_{12}\frac{\partial^2 \psi_j}{\partial y^2} \\
\overline{M}_{1j}^4 &= F_{11}\frac{\partial \psi_j}{\partial x} + F_{16}\frac{\partial \psi_j}{\partial y} \qquad \overline{M}_{1j}^5 = F_{12}\frac{\partial \psi_j}{\partial y} + F_{16}\frac{\partial \psi_j}{\partial x} \\
\overline{M}_{2j}^3 &= F_{12}\frac{\partial^2 \psi_j}{\partial x^2} + 2F_{26}\frac{\partial^2 \psi_j}{\partial x \partial y} + F_{22}\frac{\partial^2 \psi_j}{\partial y^2} \\
\overline{M}_{2j}^4 &= F_{12}\frac{\partial \psi_j}{\partial x} + F_{26}\frac{\partial \psi_j}{\partial y} \qquad \overline{M}_{2j}^5 = F_{22}\frac{\partial \psi_j}{\partial y} + F_{26}\frac{\partial \psi_j}{\partial x} \\
\overline{M}_{6j}^3 &= F_{16}\frac{\partial^2 \psi_j}{\partial x^2} + 2F_{66}\frac{\partial^2 \psi_j}{\partial x \partial y} + F_{26}\frac{\partial^2 \psi_j}{\partial y^2} \\
\overline{M}_{6j}^4 &= F_{16}\frac{\partial \psi_j}{\partial x} + F_{66}\frac{\partial \psi_j}{\partial y} \qquad \overline{M}_{6j}^5 = F_{26}\frac{\partial \psi_j}{\partial y} + F_{66}\frac{\partial \psi_j}{\partial x}
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
M_1^3 &= \frac{1}{2}[B_{11}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial x} + B_{12}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial y} + B_{16}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial y} + B_{16}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial x}] \\
M_2^3 &= \frac{1}{2}[B_{12}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial x} + B_{22}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial y} + B_{26}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial y} + B_{26}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial x}] \\
M_6^3 &= \frac{1}{2}[B_{16}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial x} + B_{26}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial y} + B_{66}\frac{\partial w_0}{\partial x}\frac{\partial \psi_j}{\partial y} + B_{66}\frac{\partial w_0}{\partial y}\frac{\partial \psi_j}{\partial x}]
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
F_i^1 &= \int_{\Gamma^e} \psi_i N_n ds \\
F_i^2 &= \int_{\Gamma^e} \psi_i N_{ns} ds \\
F_i^3 &= \int_{\Omega^e} q\psi_i dxdy + \int_{\Gamma^e} \psi_i (Q_n + N_n \frac{\partial w_0}{\partial x} + N_{ns}\frac{\partial w_0}{\partial y}) ds \qquad (53) \\
F_i^4 &= \int_{\Gamma^e} \psi_i M_n ds \\
F_i^5 &= \int_{\Gamma^e} \psi_i M_{ns} ds
\end{aligned}
$$

All other coefficients are zero.

Equation (43) represents the stiffness factors of an 8-node laminated plate element taking into account higher-order shear deformation and geometric nonlinearity. It is not difficult to derive from the equations of this paper the stiffness factors of an 8-node laminated plate element taking into account first-order shear deformation and geometric nonlinearity . The solution in this paper can be reduced to a linear solution and so it has generality.

6 References

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