# A Primer for Neural Arithmetic Logic Modules

#### Bhumika Mistry

Department of Vision Learning, and Control Electronics and Computer Science University of Southampton Southampton, SO17 1BJ, United Kingdom

#### Katayoun Farrahi

Department of Vision Learning, and Control Electronics and Computer Science University of Southampton Southampton, SO17 1BJ, United Kingdom

#### Jonathon Hare

Department of Vision Learning, and Control Electronics and Computer Science University of Southampton Southampton, SO17 1BJ, United Kingdom K.FARRAHI@SOTON.AC.UK

BM4G15@SOTON.AC.UK

JSH2@ECS.SOTON.AC.UK

## Abstract

Neural Arithmetic Logic Modules have become a growing area of interest, though remain a niche field. These units are small neural networks which aim to achieve systematic generalisation in learning arithmetic operations such as  $\{+, -, \times, \div\}$  while also being interpretive in their weights. This paper is the first in discussing the current state of progress of this field, explaining key works, starting with the Neural Arithmetic Logic Unit (NALU). Focusing on the shortcomings of NALU, we provide an in-depth analysis to reason about design choices of recent units. A cross-comparison between units is made on experiment setups and findings, where we highlight inconsistencies in a fundamental experiment causing the inability to directly compare across papers. We finish by providing a novel discussion of existing applications for NALU and research directions requiring further exploration.

**Keywords:** Neural Arithmetic Logic Module, Interpretability, Systematic Generalisation, Extrapolation

### 1. Introduction

The ability to learn by composition of already known knowledge is a form of systematic generalisation Fodor et al. (1988), also termed as compositional generalisation Lake (2019). Humans can learn such generalisations for arithmetic. For example, combining primitive operations such as addition (a + b) and multiplication  $(a \times b)$  to produce more complex expressions (such as  $(a + b) \times (c + d)$ ). Humans can also transfer their skills in applying operations on simple numbers (e.g. between 1-10) to other various ranges of numbers (e.g. 50-100) which are outside the range they were taught on. This ability to extrapolate, i.e. generalise to out-of-distribution (OOD) data, is a desirable property for neural networks.

Research suggests neural networks struggle to extrapolate even for the simplest of tasks such as learning the identity function Trask et al. (2018). Rather than generalising, networks lean towards memorization in which the model memorises the training labels Zhang et al. (2020).

To address this issue, Trask et al. (2018) introduce the first in a new class of neural modules which we term **Neural Arithmetic Logic Modules (NALMs)**. Their unit, the NALU, aims to learn systematic generalisation for arithmetic computations. For example, learning the relation between input  $[x_1, x_2, x_3, x_4]$  and output  $o_1$  where the input elements are real numbers and output is expression  $x_1 + x_3 - x_2$ . To achieve this they incorporate an inductive learning bias such that discrete weight values can be interpreted as different primitive arithmetic operations. This form of interpretability is comparable to the definition of *decomposable transparency* by Lipton (2016). Though NALU shows promising improvements over networks such as Multilayer Perceptrons (MLPs) for extrapolation, the unit still presents various shortcomings in architecture, convergence, and transparency. These areas for improvement inspired the design of other units Heim et al. (2020); Madsen and Johansen (2020); Schlör et al. (2020); Rana et al. (2019). Due to the growing interest of NALMs, we believe it is important to have a resource, this paper, to explain current motivations, strengths, weaknesses and gaps in this line of research.

## **Contributions:**

- 1. We provide the first definition to describe this research field by defining a NALM to be a Neural Network with the ability to model arithmetic in a generalisable manner which encourages the weights of the network to be interpretable.
- 2. We explain how recent modules are designed to overcome various shortcomings of NALU including: inability to process negative inputs and outputs, lack of convergence and adhering to its inductive bias, weak modelling of the division operation, and lack of compositionality.
- 3. We highlight how a popular experiment for testing modules arithmetic capabilities is inconsistent between different papers with regards to hyperparameters and experiment setup.
- 4. We show the usefulness of NALUs in larger differentiable applications which require arithmetic and extrapolation capabilities, while also making aware situations in which NALU is sub-optimal.
- 5. We outline possible research directions regarding modelling division, robustness across diving training ranges, compositionality of modelled expressions, and affect when trained along with other types of neural architectures.

## **Outline:**

In this paper we begin by defining a NALM, motivating their aim and uses in Section 2. Section 3 and 4 explains the definitions of key NALMs: NALU, iNALU, NAU, NMU, and NPU to build understanding. Using the first NALM, the NALU, as a focal point, Section 5 provides an in-depth analysis of the shortcomings of NALU to understand the motivation behind design choices for more recent NALMs. Section 6 highlights inconsistencies in experiment setup and compares findings across existing modules. Additionally, we outline all experiments used to currently evaluate the modules. Section 7 shows the diversity in NALU's use in applications, while also indicating situations in which NALU is sub-optimal. Section 8 considers all discussed issues and outlines remaining gaps, suggesting possible research directions to take as a result.

## 2. What are NALMs and Why use them?

We begin by defining NALMs. More specifically, before we detail instances of NALMs, we first answer three questions: 1. What is a NALM? 2. What is the aim of a NALM? 3. Why is a NALM useful?

### 2.1 What is a NALM?

NALM stands for Neural Arithmetic Logic Module. Neural refers to neural networks. Arithmetic refers to the ability to learn to model arithmetic operations such as addition. Logic refers to the ability to learn operations such as selection, comparison and logic. Module refers to the neural units which model arithmetic. The term module encompasses both a single (sub-)unit and multiple (sub-)units combined together.

What kind of operations can be learnt? Existing work has tried to model arithmetic operations including addition, subtraction, multiplication, division, square, and square-root. Other operations include logic (e.g. conjunction) Reimann and Schwung (2019) and control (e.g.  $\langle = \rangle$ ) Faber and Wattenhofer (2020). Selection of relevant inputs to the modules is also learnt.

How are operations learnt? Because a NALM is a neural network, a module can model the relation between input and output vectors via supervised learning which trains weights through backpropogation. To learn the relation between input and output, requires learning to select relevant elements of the input and apply the relevant arithmetic operation/s to the selected input to create the output.

How is data represented? The input and outputs are both vectors. Each vector element is a floating point number. Each output element can learn a different arithmetic expression. For a single data sample, this can be illustrated in Figure 1 where we assume that the NALM used (made from two stacked sub-units) can learn addition, subtraction and multiplication. In practice data would be given in batch form.

## 2.2 What is the Aim of a NALM?

The main aim of NALMs is to be utilised in larger systems while remaining interpretable. A by-product of the interpretability enables NALMs to achieve systematic generalisation in learning arithmetic expressions and be extrapolative on OOD data.

What does Interpretability mean for NALMs? Currently, we say a NALM is interpretable if it has *decomposable transparency* Lipton (2016). Transparency means to understand how the model works. Decomposability is transparency at component level defined by Lipton (2016) as 'each part of the model - each input, parameter, and calculation - admits an intuitive explanation'. So far, only Heim et al. (2020) has considered their NALMs in terms of decomposable transparency. A consequence of NALMs achieving this form of interpretability results in parameters being discrete values and calculations being

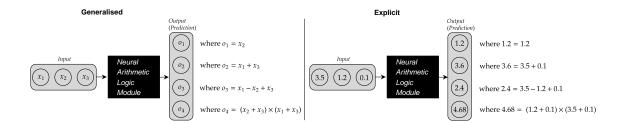


Figure 1: High-level example of the input output structure into a NALM. Both networks are the same. The generalised network defines the notation of each element in the input and output. The explicit network is an example of valid input and output values.

compositions of arithmetic operations. The discrete parameter values result in exact solutions which are valid regardless of the data distribution, enabling generalisation on OOD data.

What does Extrapolation on OOD data mean for NALMs? Once trained, a NALM should be able to predict the output of the input data which comes for a range outside the training range. Any loss in predictive accuracy will only occur due to numerical imprecisions due to hardware limitations.

## 2.3 Why is a NALM useful?

The ability to learn arithmetic seems trivial in comparison to other architectures such as LSTMs, CNNs or Transformers which can be used as standalone networks which learn tasks such as arithmetic, object recognition and language modeling. So, why care about NALMs?

Learning arithmetic, though it may seem a simple task, still remains unsolved for neural networks. To solve this problem requires precisely learning the underlying rules of arithmetic such that failure cases will not occur on cases of OOD data. Therefore, before considering more complex tasks, solving the simple tasks seems reasonable. Even though NALMs specialise in arithmetic there is no restriction in using them as part of larger end-to-end neural networks. For example, attaching a NALM to a CNN. In Section 7, we show a vast array of applications in which NALMs are being utilised. Being used as a sub-component in a larger network implies that the sub-component has the ability to learn regardless of the data distribution. Therefore, the ability to extrapolate is essential.

### 3. Overview of the NALU Architecture

The NALU, illustrated in Figure 2, provides the ability to model basic arithmetic operations, specifically: addition, subtraction, multiplication, division. NALU requires no indication of which operation to apply, and aims to learn weights that provide extrapolation capabilities if correctly converged. NALU comprises of two sub-units, a summative unit which models  $\{+, -\}$  and a multiplicative unit which models  $\{\times, \div\}$ . Following the notation of Madsen

<sup>1.</sup> The learned gate matrix  $(\mathbb{R}^{3\times 4})$  is mistakenly drawn as a vector  $\mathbb{R}^3$  (the 3 vertical circles in blue).

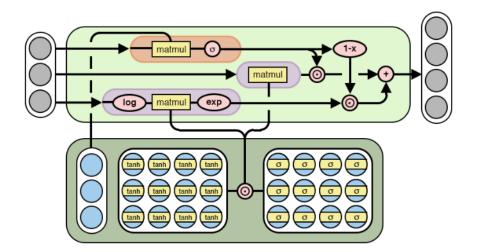


Figure 2: Original NALU architecture, taken from Trask et al.  $(2018)^1$ .

and Johansen (2020) we denote the sub-units as NAC<sub>+</sub> and NAC<sub>•</sub> respectively. Formally, NALU is expressed as:

$$\boldsymbol{W} = \tanh(\boldsymbol{\hat{W}}) \odot \operatorname{sigmoid}(\boldsymbol{\hat{M}}) \tag{1}$$

$$NAC_{+}: \boldsymbol{a} = \boldsymbol{W}\boldsymbol{x} \tag{2}$$

$$NAC_{\bullet}: \boldsymbol{m} = \exp \boldsymbol{W}(\log(|\boldsymbol{x}| + \epsilon)) \tag{3}$$

$$g = \text{sigmoid}(Gx)$$
 (4)

NALU: 
$$\hat{\boldsymbol{y}} = \boldsymbol{g} \odot \boldsymbol{a} + (\boldsymbol{1} - \boldsymbol{g}) \odot \boldsymbol{m}$$
 (5)

where  $\hat{W}, \hat{M} \in \mathbb{R}^{I \times O}$  are learnt matrices (where I and O represent input and output dimension sizes). A non-linear transformation is applied to each matrix and then both are combined via element-wise multiplication to form W (equation 1). Due to the range values of tanh and sigmoid, W aims to have a inductive bias towards values  $\{-1, 0, 1\}$  which can be interpreted as selecting a particular operation within a sub-unit (i.e. intra-subunit selection). For example, in  $NAC_{+}$  +1 is addition and -1 is subtraction, and in  $NAC_{\bullet}$ +1 is multiplication and -1 is division. In both sub-units, 0 represents not selecting (i.e. ignoring) an input element. A sigmoidal gating mechanism (equation 4) enables selection between the sub-units (inter-sub-unit), where an open gate, 1, selects the NAC<sub>+</sub> and closed gate, 0, selects the NAC<sub>•</sub>. Once trained the gating should ideally select a single sub-unit. G is learnt, and the gating vector g represents which sub-unit to use for each element in the output vector. Finally, equation 5 gates the sub-units and sums the result to give the output. NALU's gating only allows for each output element to have a mixture of operations from the same sub-unit. Therefore, each output element is an expression of a combination of operations from either  $\{+, -\}$  or  $\{\times, \div\}$  but not  $\{+, -, \times, \div\}$ . This issue is fixed by stacking NALUs such that the output of one NALU is the input of another. Next, we overview architectures of some recent units.

## 4. NALU Influenced Units

NALU has inspired the creation of other units/ sub-units including: Improved NALU (iN-ALU) Schlör et al. (2020), Neural Addition Units (NAU)/ Neural Multiplication Units (NMU) Madsen and Johansen (2020), Neural Power Units (NPU) Heim et al. (2020), Golden Ratio NALU (G-NALU) Rana et al. (2019), Neural Logic Rules (NLR) Reimann and Schwung (2019) and Neural Status Registers (NSR) Faber and Wattenhofer (2020). Existing unit illustrations are found in Appendix 10.

**iNALU** identifies key issues in NALU and modifies the unit to incorporate solutions (detailed in Section 5). They introduce methods to improve convergence and stability during training through regularisation, clipping, and decouple previously shared parameters between sub-units.

**NAU** and **NMU** are sub-units for addition/subtraction and multiplication respectively. Architecture and initialisations of the units have strong theoretical justifications and empirical results to validate design choices. The NAU and NMU definitions for calculating an output element indexed at o is:

$$NAU: a_o = \sum_{i=1}^{I} \left( W_{i,o} \cdot \mathbf{x}_i \right) \tag{6}$$

NMU: 
$$m_o = \prod_{i=1}^{I} (W_{i,o} \cdot \mathbf{x}_i + 1 - W_{i,o})$$
 (7)

Prior to applying the weights of a sub-unit to the input vector, each element of W is clamped between [-1,1] if using the NAU, or [0,1] if using the NMU.

The **NPU**, equation 8, focuses on improving the division ability of the NAC<sub>•</sub> by applying a complex log transformation and using real and complex weight matrices. A relevance gate (g) is also combined. g learns to convert values close to 0 to 1 to avoid the output multiplication becoming 0.

NPU := exp(
$$\boldsymbol{W}^{(\boldsymbol{r})} \log(\boldsymbol{r}) - \boldsymbol{W}^{(\boldsymbol{i})} \boldsymbol{k}$$
)  $\odot \cos(\boldsymbol{W}^{(\boldsymbol{i})} \log(\boldsymbol{r}) + \boldsymbol{W}^{(\boldsymbol{r})} \boldsymbol{k})$  (8)

where

$$\boldsymbol{r} = \boldsymbol{g} \odot (|\boldsymbol{x}| + \epsilon) + (\boldsymbol{1} - \boldsymbol{g}), \tag{9}$$

and

$$k_i = \begin{cases} 0 & x_i \ge 0\\ \pi \mathbf{g}_i & x_i < 0 \end{cases}$$
(10)

Additionally a simplified version of the NPU exists, named RealNPU, considering only real values of equation 8

$$\operatorname{RealNPU} := \exp(\boldsymbol{W}^{(\boldsymbol{r})} \log(\boldsymbol{r})) \odot \cos(\boldsymbol{W}^{(\boldsymbol{r})} \boldsymbol{k}).$$
(11)

**G-NALU** replaces the exponent in the tanh and sigmoid operations when calculating NALU's weight matrix with the golden ratio value.

**NLR**, influenced by inductive biases in Trask et al. (2018), creates a unit to express logic rules and simple arithmetic operations via modelling AND (conjunction), OR (disjunction) and NOT (negation).

**NSR**, models comparison based control logic:  $\langle , \rangle, ! =, =, \rangle =, \langle =$ . The NSR also use the inductive bias idea in Trask et al. (2018) to constrain the parameter space, and regularisation like Madsen and Johansen (2020) to enforce the biases.

## 5. NALU's Shortcomings and Existing Solutions

We detail the weaknesses of NALU and explain existing solutions. We focus on the iNALU, NAU, NMU and NPU when looking at solutions, as these modules focus on overcoming the shortcomings of NALU.

#### 5.1 Mixed Sign Inputs and Negative Outputs

The NAC<sub>•</sub> cannot deal with mixed sign inputs/negative outputs. Equation 3 requires converting negative inputs into their positive counterparts because the log transformation cannot evaluate negative values. Therefore the sign of the input is lost, causing the NAC<sub>•</sub> to be unable to have negative target values. The use of an exponent also causes the inability to have negative outputs, as the range of an exponent is  $\mathbb{R}_{>0}$ . To allow for negative targets, a unit can incorporate logic to deal with assigning the correct sign to the output such as the iNALU's sign correction mechanism Schlör et al. (2020) or the NPU's inherent sign retrieval Heim et al. (2020). The sign correction mechanism creates a mixed sign vector  $(\mathbf{msv})^2 \in \mathbb{R}^{O \times 1}$ ,

$$msv = \prod_{i=1}^{I} \left( \text{sign}(x_i) \cdot |W_{i,o}| + 1 - |W_{i,o}| \right),$$
(12)

consisting of elements  $\{-1,1\}$  (assuming W has converged to integers  $\{-1,0,1\}$ ), where each element represents the correct sign for each output element. The msv is multiplied to the end of equation 3, regaining the lost signs. The  $+1 - |W_{i,o}|$  gives non-selected inputs a msv value of 1 to avoid effecting the final sign value. In the case of a RealNPU, the latter half of its definition i.e.  $\odot \cos(W_r k)$  can be interpreted as a sign retrieval mechanism. k represents positive inputs as 0 and negative inputs as 1 (assuming the gate value converged to select the input). Assuming convergence,  $W_r$  values are  $\{1, -1\}$  representing  $\{\times, \div\}$ . Two outcomes are possible from evaluating the expression:  $-\cos(\pm\pi) = -1$  or  $\cos(0) = 1$ where the output value represents the sign of the input value.

Alternatively, it is possible to remove the need for transformations in the log/exponent space in equation 3, as Madsen and Johansen (2020) does for defining the NMU (equation 7). This means negative targets can be expressed because the sign is no longer removed from the input.

<sup>2.</sup> Notice the similarity in calculation between the NMU (equation 7) and iNALU's msv (equation 12).

## 5.2 Gating Parameter Convergence

The NALU gate, responsible for selection between the NAC<sub>+</sub> and NAC<sub>•</sub> units, is unable to converge reliably. This is due to the different convergence properties of the NAC<sub>+</sub> and NAC<sub>•</sub> Madsen and Johansen (2020). Partial convergence of gate values lead to a leaky gate effect, noted by Schlör et al. (2020), where the gate allows for the unit to incorrectly take both a multiplicative and summative route which can lead to exploding outputs. This issue is amplified when additional NALU layers are stacked. In cases where the correct gate is selected, the NALU unit still fails to converge consistently Madsen and Johansen (2020) implying additional architectural issues for the unit. Even with using the improved NAU and NMU sub-units, gating still leads to inferior results. Madsen and Johansen (2020) replace unit gating with unit stacking. Schlör et al. (2020) suggests using separate weights for the iNALU sub-units to improve convergence, and independent gating (removing  $\boldsymbol{x}$ from equation 4) so learning  $\boldsymbol{G}$  is no longer influenced by input. However this provides only minimal improvements for simple arithmetic tasks.

#### 5.3 Convergence and Inductive Biases

Good initialisations are crucial for convergence. Assuming the Madsen and Johansen (2020) implementation of NALU is used for initialisation, then weight matrices are from a uniform distribution with the range calculated from the fan values<sup>3</sup>, and the gate matrix from a Xavier uniform initialisation with a sigmoid gain<sup>4</sup>. This results in difficultly for both optimisation and robustness. Fragility results in the expected inductive bias of weight values converging to  $\{-1, 0, 1\}$  to be difficult to achieve Madsen and Johansen (2020). Unsparse solutions result in a lack of transparent and hence ungeneralisable solutions.

The weight biases are achieved by adding a regularisation term for sparsity Madsen and Johansen (2020); Schlör et al. (2020) and using weight clamping Madsen and Johansen (2020). Regularisation encourages weights to converge to the discrete values, activating and warming-up for a predefined period of time to avoid overpowering the main MSE loss term.

Madsen and Johansen (2020) use sparsity regularisation to enforce the relevant biases for both NAU  $\{-1, 0, 1\}$  and NMU  $\{0, 1\}$ :

$$\mathcal{R}_{sparse} = \frac{1}{I \cdot O} \sum_{o=1}^{O} \sum_{i=1}^{I} \min\left(|W_{i,o}|, 1 - |W_{i,o}|\right).$$
(13)

Note that the absolute of  $W_{i,o}$  is not necessary when using NMU. Clamping is applied to the weights beforehand, which clamps to the ranges of the desired biases. A scaling factor

$$\lambda = \hat{\lambda} \max\left(\min\left(\frac{iteration_i - \lambda_{start}}{\lambda_{end} - \lambda_{start}}, 1\right), 0\right), \tag{14}$$

is multiplied to  $\mathcal{R}_{sparse}$  to get the final value, where regularisation strength is scaled by a predefined  $\hat{\lambda}$ .

<sup>3.</sup> https://github.com/AndreasMadsen/stable-nalu/blob/2db888bf2dfcb1bba8d8065b94b7dab9dd178332/ stable\_nalu/layer/nac.py#L22

<sup>4.</sup> https://github.com/AndreasMadsen/stable-nalu/blob/2db888bf2dfcb1bba8d8065b94b7dab9dd178332/ stable\_nalu/layer/\_abstract\_nalu.py#L90

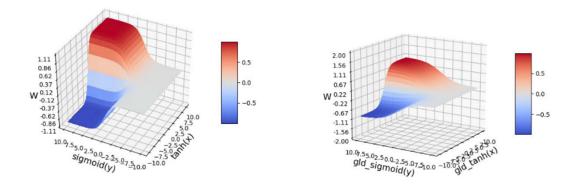


Figure 3: Figure taken from Rana et al. (2019). Left: NAC<sub>+</sub> W values over the domain of  $\hat{W}$  and  $\hat{M}$ . Right: NALU where the base value for non-linear functions (tanh and sigmoid) uses the golden ratio rather than exponential resulting in smoother value transition.

iNALU uses a piece-wise function for regularisation on weight  $(\hat{W}, \hat{M})$  and gate parameters (G),

$$\mathcal{R}_{\text{sparse}} = \frac{1}{t} \max(\min(-w, w)) + t, 0) \tag{15}$$

to encourage discrete values that do not converge to near-zero values. Rather than a warmup period, regularisation occurs only once the loss is under a pre-defined threshold and stops once a discretisation threshold t (=20) is met.

These methods alone would restrict the parameter space, but not the unit's output scale. To address this, Madsen and Johansen (2020) use a linear weight matrix construction (removing the need of non-linear transformations), allowing for easier optimisation, while Schlör et al. (2020) use clipping of the NAC<sub>•</sub> weights and gradients. The weights in equation 3 would be clipped between  $[\log(\max(|\boldsymbol{x}|, 10^{-7})), 20]$  before the exp is applied.

Rana et al. (2019) modify the non-linear activations, using G-NALU, for the weight matrices for smoother gradient propagation as shown by Figure 3. In contrast, in attempts to avoid falling into a local optima, iNALU allows multiple reinitialisations of a model during training. Through a grid search they find having the mean of the gate and NALU weight matrices  $\hat{M}$ ,  $\hat{W}$  initialised to be 0, -1 and 1 respectively, results in the most stable units. However, even when using such initialisations, the stability problem remains for division.

#### 5.4 Division

Division is NALU's weakest operation Trask et al. (2018). Having both division and multiplication in the same sub-unit causes optimisation difficulties. Madsen and Johansen (2020) highlight the singularity issue (division by 0 or values close to 0 bounded by an epsilon value) in the NAC• which causes exploding outputs (see Figure 4). This issue is amplified due to operations being applied in log space. NMU removes the use of log, therefore is not epsilon bound. Furthermore, the NMU is only designed for multiplication. NPU takes Madsen and Johansen (2020)'s interpretation of multiplication (using products of power functions),

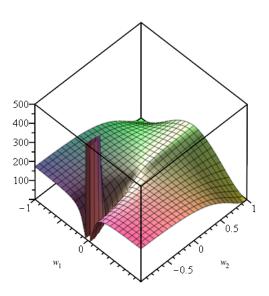


Figure 4: Taken from Madsen and Johansen (2020). Illustration of singularity issue arising in the NAC<sub> $\bullet$ </sub>.

but applies it in a complex space enabling division and multiplication Heim et al. (2020). Though the NPU cannot fully solve the singularity issue as a log transformation is still applied to the inputs, the relevance gate aids in smoothing the loss surface. Schlör et al. (2020) observe that reinitialising units numerous times during training can still lead to failure, implying that the issue lies in unit architecture as well as initialisation. Hence, division remains an open issue.

## 5.5 Compositionality

A single NALU is unable to output expressions whose operations are from both  $\{+, -\}$  and  $\{\times, \div\}$ , e.g.  $x_1 + x_2 * x_3$ . Bogin et al. (2020) hint at NALU's inflexibility to learn different expressions from same weights. Rana et al. (2019) develop CalcNet, a parsing algorithm, to decompose expressions before applying the NALU sub-units. However decomposition requires fixed rules and pre-trained sub-units which are undesirable.

## 5.6 Summary

A summary of the discussed NALU issues and proposed solutions is given in Table 1.

#### 6. Experiments and Findings of Units

To understand the evaluation of units, we go through the experiments used in the papers for: NALU, iNALU, NAU, NMU, and NPU. We indicate inconsistencies across papers for the two-layer arithmetic task setup, encouraging the need of task standardisation. Inter-unit comparison using existing findings is made to infer the best unit per operation.

Shortcoming	NMU	iNALU	NPU	CalcNet
$NAC_{\bullet}$ cannot	Remove log-	Sign correction	Sign re-	Fixed rules
have negative	exponent	(mixed sign	trieval	and sign
inputs/targets	transformation	vector)		parsing
Convergence of	Stacking	Separate gate	-	-
gate parameters		and weights		
		per sub unit		
Fragile initialisa-	Theoretically	Reinitialise	-	-
tion	valid initialisa-	model		
	tion scheme			
Weight inductive	Regularisation	Regularisation	-	-
bias of $\{-1,0,1\}$	loss term	loss term		
not met (non-		and weight		
discrete solutions)		clipping		
Unrestricted out-	Linear weight	Weight and	-	-
put scale	matrix	gradient clip		
Gradient propa-	-	Reinitialise	Relevance	Replace
gation		model	gating	sigmoid
				and tanh
				exponent's
				with golden
				ratio
Singularity (val-	Remove log-	-	Complex	-
ues close to 0)	exponent		space trans-	
	transformation		formation	
			and rel-	
			evance	
			gating	
Compositionality	-	-	-	Parsing al-
				gorithm

Table 1: Summarised NALU shortcomings and existing proposed solutions.

# 6.1 Why the Square and Square-Root Operations are not included in this Discussion?

Though mentioned in Trask et al. (2018) that NALU can learn to model square and squarerooting, we will purposefully avoid analysing the ability of the multiplicative units to do square  $(a^2)$  and square-root  $(\sqrt{a})$  operations.

The squared operation can be solved when using a multiplication unit. Firstly, there could be two input elements with the same value resulting in the operation  $a \times a$ . Secondly, the unit can set the weight value corresponding to the input to 2. The first way is a multiplication operation (which is separately tested), and the second requires breaking the

inductive bias assumption of discrete weights with a magnitude up to 1. Therefore, we avoid analysing the square operation.

For a multiplicative unit to solve the square-root operation such that the weights are interpretable requires a weight value of 0.5. Though this allows to model square-rooting as  $a^{\frac{1}{2}}$ , it contradicts the inductive bias of discrete weights with a magnitude up to 1. Therefore, we avoid analysis square-root operation.

#### 6.2 Two Layer Arithmetic Task

A task consistently used in all papers is the 'Static Simple Function Learning' experiment Trask et al. (2018), which evaluates the ability of a unit to learn a trivial two-operation function. Madsen and Johansen (2020) renames this task 'Arithmetic Datasets' and introduce their own experiment setup (including details for reproducibility). Specifically, given an input vector  $\mathbb{R}^{100}$  of floats, the first (addition) layer should learn to output two values (denoted a and b) which are the sums of two different partially overlapping slices (i.e. subsets) of the input, and the second layer should perform an operation on a and b. Figure 5 illustrates such an example. Due to the rigorous setup, evaluation metrics, and available code, we strongly suggest this experiment be used to test and compare new units. iNALU's experiments 4 ('Influence of Initalization') and 5 ('Simple

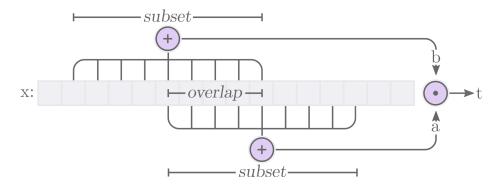


Figure 5: Taken from Madsen and Johansen (2020). Illustration on how to get from input vector to target scalar for the Dataset Arithmetic Task.

Function Learning Task') is a copy of the task but is different to the original. Experiment 4 calculates a and b differently to Madsen and Johansen (2020) by not allowing for overlap between the slices which form a and b, and experiment 5 use different interpolation and extrapolation ranges to the original experiments. Heim et al. (2020)'s claims that their 'Large Scale Arithmetic' task is equivalent to the Arithmetic Dataset task. However, as shown in Table 2 there are key distinctions between the two meaning the results from the two papers are not directly comparable.

## 6.3 Additional Experiments

The papers also carry out experiments on top of the two-layer arithmetic task. Trask et al. (2018) carries out a recurrent version of their static task experiment to test the NAC<sub>+</sub>,

Table 2:	Differences in the <i>Large Scale Arithmetic</i> ' task used in the papers Madsen and
	Johansen (2020) and Heim et al. (2020). 'a' and 'b' represent summed slices of
	the input, and are the expected output values for the addition unit.

Property	Madsen and Jo-	Heim et al. (2020)
	hansen $(2020)$	
Hidden size	2	100
Iterations for one	5,000,000	50,000
run		
Number of seeds	100	10
Learning rates	1e-3	1e-2 for addition and 5e-3 for all other op-
		erations
Subset and over-	0.25  and  0.5	0.5 and $0.25$ (for addition, subtraction,
lap ratios		and multiplication)
Division	a/b	1/a
Interpolation and	Uniform distributions,	Sobol(-1,1) for training addition, subtrac-
extrapolation	using $U[1,2)$ for train-	tion, and multiplication, $Sobol(0,0.5)$ for
ranges	ing all operations, test-	division. Testing uses Sobol(-4,4) for
	ing on $U[2,6)$ .	addition, subtraction and multiplication,
		Sobol(-0.5, 0.5) for division.
Regularisation	Biasing weight discriti-	L1 on all parameters
penalty	sation	
Programming	Python 3	Julia
language		

where the subsets a and b are accumulated over multiple timesteps. The purpose of this task is to generate much larger output values to test NALU on. As well as pure arithmetic tasks, Trask et al. (2018) tests NALU in other settings such as: translating numbers in text form into the numerical form (e.g. 'two hundred and one' to 201), a block grid-world which requires travelling from point A to B in exactly n timesteps, and program evaluation for programs with arithmetic and control operations. MNIST is also used to evaluate NALU's abilities on being part of end-to-end applications. This includes exploring counting the occurrence of different digits, addition of a sequence of digits, and parity prediction.

Madsen and Johansen (2020) also use MNIST for testing the unit's abilities to act as a recurrent unit for adding/ multiplying the digits. Madsen and Johansen (2020) additionally provide experiments to express the validity of their units. This includes modifying the number of redundant units, ablation on multiplication, stress testing the stacked NAU-NMU against difference input sizes, overlap ratios and subset ratios, showing the failure of gating in convergence, and parameter tuning regularisation parameters.

Schlör et al. (2020) provide three additional experiments. Experiment 1 ('Minimal Arithmetic Task') uses a single-layer to do a single operation with no redundancy to see the effect of different input distributions. Experiment 2 ('Input Magnitude') sees the effect of training data by controlling the magnitude of the interpolation data. NALU fails on magnitudes greater than 1. iNALU remains unaffected for addition and subtraction. Multiplication performance is coupled to magnitude where extrapolation error increases with magnitude. Division is uncorrelated to the input magnitude. To increase problem difficulty, experiment 3 ('Simple Arithmetic Task') introduces redundancy where from 10 inputs only 2 are relevant. NALU improves on performance for exponentially distributed data when redundant inputs are introduced. iNALU show improvements for multiplication where the unit is able to succeed on previously failed training ranges such as an exponential distribution with a scale parameter of 5 (i.e. lambda 0.2), but worsens for division.

Heim et al. (2020) highlights the relevance gate's use via a toy experiment to select one of the two inputs. Additionally, they demonstrate an application of a stacked NAU-NPU unit for equation discovery for an epidemiological model.

#### 6.4 Cross Unit Comparison

We compare existing findings across units. NALU is no longer considered the state-of-theart for neural arithmetic operation learning. For each operation the best sub-unit is as follows - **addition or subtraction**: NAU, **multiplication**: NMU, **division**: NPU (or RealNPU if the task is trivial).

iNALU generally outperforms NALU at the cost of additional parameters and complexities to the model. The magnitude of iNALU's improvement varies, as Schlör et al. (2020) claims vast improvements, while Heim et al. (2020) claim minor. For division both the iNALU and NALU performances remain comparable. Success on multiplication is dependent on the input training range. Heim et al. (2020) states the NMU outperforms iNALU on multiplication (as expected), but also addition and subtraction. The reason lies in the architecture used. The model is a stacked NAU-NMU meaning the addition/subtraction would be modelled by the NAU. Therefore, the NMU would only be required to act as a selector, selecting the output of the summation (i.e. have a single weight at 1 and the rest a 0). Therefore, if two NMUs are stacked together we expect the failure in a pure addition/subtraction task as shown in the Appendix C.7 in Madsen and Johansen (2020). Surprisingly the 2-layer NMU was able to get 56% success for subtraction, though 0% success for addition Madsen and Johansen (2020). Heim et al. (2020) is the only work (at the time of writing this paper) to experimentally compare the main units mentioned. Results show NPU outperforms iNALU for multiplication and division. When stacked on top of a NAU, the NPU performs similar to the NMU for addition and subtraction. The NPU is outperformed by the NMU for multiplication, however it is more consistent in convergence against different runs. For addition and subtraction, the NAU-NMU is the sparsest unit (having the least number of non-zero weights). Interpretive units require the weight and gate values to be discrete. Regularisation penalties have been a popular approach Madsen and Johansen (2020): Schlör et al. (2020) to achieve this. NPU uses L1 regularisation for arithmetic tasks, encouraging sparsity over discretisation. This may explain results from Heim et al. (2020) where NMU models are generally sparser than NPUs for multiplication.

## 7. Applications of NALU

This section describes uses of NALU as a sub-component in architectures to tackle practical problems outside the domain of solving arithmetic on numeric inputs. Success and failure cases are mentioned. We choose to focus on NALU applications on the basis that the improved units discussed above can be applied in place of NALU to provide additional performance gains to the mentioned applications.

### 7.1 Existing Applications

Before discussing applications, we raise awareness of a case where the NALU is not utilised for its capabilities as a NALM. The *Language to Number Translation* task in Trask et al. (2018) converts numbers in their string form to their numerical form (such as 'twenty one' to '21'). The NALU is applied to an LSTM's hidden state vector; therefore it is questionable on if the arithmetic capabilities of NALU is being used, as the NALU may also have to decode the numerical values from the LSTM vector.

Xiao et al. (2020) insert a NALU layer between a two-layer Gated Recurrent Unit (GRU) and dense layer to predict vehicle trajectory of complex road sections (containing constantly changing directions). NALU improves extrapolation capabilities to deal with abnormal input cases outside the range of the GRU hidden states output.

Raj et al. (2020) combine NAC<sub>+</sub> sub-units before LSTM cells for *fast* training in the extraction of temporal features to classify videos for badminton strokes. They further experiment in using NAC<sub>+</sub> units with a dense layer to learn temporal transformations, finding better performance than the LSTM based module and the dense modules being quicker to train. They justify the use of the NAC<sub>+</sub> as a way to produce sparse representations of frames, as non-relevant pixels would not be selected by the NAC<sub>+</sub> resulting in 0 values, while relevant pixels accumulate.

Zhang et al. (2019a) use deep reinforcement learning to learn to schedule views on content-delivery-networks (CDNs) for crowdsourced-live-streaming (CLS). NALU's extrapolative ability alleviates the issue of data bias (which is the failure of models outside the training range) by using NALU to build a offline simulator to train the agent when learning to choose actions. The simulator is composed of a 2-layer LSTM with a NALU layer attached to the end. Zhang et al. (2019b) propose a novel framework (named Livesmart) for cost-efficient CLS scheduling on CDNs with a quality-of-service (QoS) guarantee. Two components required in Livesmart contain models using NALU. The first component (named new viewer predictor) uses a stacked LSTM-NALU to predict workloads from new viewers. The second component (named QoS characterizer) predicts the QoS of a CDN provider. This component uses a stack of Convolutional Neural Networks (CNNs), LSTM and NALU. Both components use NALU's ability to capture OOD data to aid in dealing with rare events/ unexpected data.

Wu et al. (2020) combines layers of  $NAC_+$  to learn to do addition and subtraction on vector embeddings to form novel compositions for creating analogies. Units are applied to the output of an attention module (scoring candidate analogies) that is passed through a MLP. The output of the  $NAC_+$  units is passed to a LSTM producing the final analogy encoding.

NALU has also been used with CNNs. Rajaa and Sahoo (2019) applies stacked NALUs to the end of convolution units to predict stock future stock prices. Rana et al. (2020) utilises the NAC<sub>+</sub>/NALU as residual connections modules to larger convolutional networks such as U-Net and a fully convolutional regression networks for cell counting in images.

Such connections enable better generalisation when transitioning to data with higher cell counts to the training data. However, no observations are made to what the units learn which lead to an improvement on cell counting over the baseline models.

Chennupati et al. (2020) uses NALU as part of a larger architecture to predict the runtime of code on different hardware devices configured using hyperparameters. NALU predicts the reuse profile of the program, keeping track of the count of memory references accessed in the execution trace. NALU outperforms a Genetic Programming approach for doing such a prediction.

Teitelman et al. (2020) explores the problem domain of cloning black-box functionality in a generalisable and interpretable way. A decision tree is trained to differentiate between different tasks of the black box. Each leaf of the tree is assigned a neural network comprising of stacked dense layers with a NALU layer between them. Each neural network is able to learn the black-box behaviour for a particular task. Like Xiao et al. (2020), results showed that NALU is required to learn the more complex tasks.

Finally, Sestili et al. (2018) suggests NALU has potential use in networks which predict security defects in code. This is due to the unit's ability to work with numerical inputs in a generalisable manner, instead of limiting the application to be bound to a fixed token vocabulary requiring lookups.

#### 7.2 Applications Where NALU Is Inferior

There exist situations where alternate architectures are favoured over NALU. Madsen and Johansen (2020) show that the NAU/NMU outperforms NALU in the MNIST sequence task for both addition and multiplication. Dai and Muggleton (2020) show the arithmetic ability (named background knowledge) of NALU is incapable in performing the MNIST task for addition or products when combined with a LSTM. Instead, they show a neural model for symbolic learning, which learns logic programs using pre-defined rules as background knowledge, can perform with over 95% accuracy. However, we question whether the failure is a result of NALU or due to the misuse of its abilities from combining it with a LSTM. Jacovi et al. (2019) show that in black box cloning for the Trask et al. (2018) MNIST addition task, their EstiNet model which captures non-differentiable models outperforms NALU. Though it can be argued that a more relevant comparison would test the NAC<sub>+</sub> or the NAU which are solely designed for addition. Joseph-Rivlin et al. (2019) show that although the NAC $_{\bullet}$  can learn the order for a polynomial transformation to a high accuracy, it is still outperformed by a pre-defined order two polynomial model. Results suggest that the NAC<sub>•</sub> may not have fully converged to express integer orders. Dobbels et al. (2020)found NALU was unable to extrapolate for the task of predicting far-infrared radiation fluxes from ultraviolet-mid-infrared fluxes. Though no clear reason was stated, the lack of extrapolation could be attributed to the co-dependence of features because of applying a fully connected layers prior to the unit. Jia et al. (2020) considers NALU as a hardware component concluding that NALU has too high an area and power cost to be feasible for practical use. Implementing for addition costs 17 times the area of a digital adder, and the memory requirements for weight storage is energy inefficient for doing CPU operations.

## 8. Remaining Gaps

This section discusses areas which remain to be fully addressed. We focus on: *division*, *robustness*, *compositionality*, *and interpretability of more complex architectures*.

**Division** remains a challenge. To date no unit has been able to reliably solve division. Currently the NPU by Heim et al. (2020) is the best unit to use, though it would struggle with input values close to zero. Madsen and Johansen (2020) argues modelling division is not possible due to the singularity issue. One suggestion for dealing with the zero case is to take influence from Reimann and Schwung (2019) which can have an option for showing an output which is invalid (or in their case all off values).

One goal of these units is to be able to extrapolate. To achieve this, a unit should be **robust** to being trained on any input range. Madsen and Johansen (2020) show that units are unable to achieve full success of all tested ranges (with the stacked NAU-NMU failing on a training range of [1.1,1.2), being unable to obtain a single success). Reinitialisation of weights Schlör et al. (2020) during training could provide a solution, however this seems to be a unlikely given Madsen and Johansen (2020) tests against 100 model initialisations.

**Compositionality** is desirable. A model should be flexible, having the option to select different types of operations and model complex mathematical expressions. Currently the two popular approaches are gating and stacking. Gating has been found to not work as expected and give convergence issues. Stacking, though more reliable, has less options in operation selection than gating. Deep stacking of units (in a non-recurrent fashion) remains untested.

It remains to be understood **how units influence learning of other modules** (such as recurrent networks and CNNs) in their representations. For example, seeing if representations are more interpretable because of being trained with a unit.

## 9. Related Work

We outline alternate research in neural models for solving arithmetic tasks. Such works require components such as convolutions Kaiser and Sutskever (2016), or Transformers Saxton et al. (2019); Lample and Charton (2020). Neural GPUs can extrapolate to long sequence lengths (2000) from being trained on length 20 inputs, but use binary inputs rather than real numbers Kaiser and Sutskever (2016). Furthermore only a few models generalise to such a long sequence, but this has been improved on in Freivalds and Liepins (2017). Even more complex architectures such as Transformers which can process numerical values, remain unsuccessful for extrapolation tasks which are simple e.g. arithmetic using multiplication Saxton et al. (2019), or complex e.g. integration Lample and Charton (2020). Other approaches which can process raw numerical inputs include using reinforcement learning or non-specialised architectures. The Chen et al. (2018) hierarchical reinforcement learning approach requires arithmetic operation/s to be defined in the input. Non-specialised architectures from Nollet et al. (2020) trains using task decomposition and active learning but is not fully robust to noisy redundant inputs. In short, though various alternates to NALMs exist, each have their own shortcomings in regard to input format, extrapolation, and robustness.

# 10. Conclusion

NALMs are a promising area of research for systematic generalisation. Focusing on the first Neural Arithmetic Unit, NALU, we explained the unit's limitations along with existing solutions from other units: iNALU, NAU, NMU, NPU, and CalcNet. There exists a range of applications for NALU, though some uses remain questionable. Cross-comparing units suggest inconsistencies with experiment methodology and limitations existing in the current state-of-the-art units. Finally, we outline remaining research gaps regarding: solving division, robustness, compositionality and interpretability of complex architectures.

# Acknowledgments

We would like to thank Andreas Madsen for informative discussions and explanations regarding the Neural Arithmetic Units.

# Unit Illustrations

Table 3 displays unit illustrations given in their respective papers, displayed chronologically.

# References

- Ben Bogin, Sanjay Subramanian, Matt Gardner, and Jonathan Berant. Latent compositional representations improve systematic generalization in grounded question answering. arXiv preprint arXiv:2007.00266, 2020. URL https://arxiv.org/pdf/2007.00266.pdf.
- Kaiyu Chen, Yihan Dong, Xipeng Qiu, and Zitian Chen. Neural arithmetic expression calculator, 2018. URL https://arxiv.org/pdf/1809.08590.pdf.
- Gopinath Chennupati, Nandakishore Santhi, Phill Romero, and Stephan Eidenbenz. Machine learning enabled scalable performance prediction of scientific codes. arXiv preprint arXiv:2010.04212, 2020. URL https://arxiv.org/pdf/2010.04212.pdf.
- Wang-Zhou Dai and Stephen H. Muggleton. Abductive knowledge induction from raw data, 2020. URL https://arxiv.org/pdf/2010.03514.pdf.
- Wouter Dobbels, Maarten Baes, Sébastien Viaene, S Bianchi, JI Davies, V Casasola, CJR Clark, J Fritz, M Galametz, F Galliano, et al. Predicting the global far-infrared sed of galaxies via machine learning techniques. *Astronomy & Astrophysics*, 634:A57, 2020. URL https://arxiv.org/pdf/1910.06330.pdf.
- Lukas Faber and Roger Wattenhofer. Neural status registers. arXiv preprint arXiv:2004.07085, 2020. URL https://arxiv.org/pdf/2004.07085.pdf.
- Jerry A Fodor, Zenon W Pylyshyn, et al. Connectionism and cognitive architecture: A critical analysis. *Cognition*, 28(1-2):3-71, 1988. URL https://uh.edu/~garson/F&P1. PDF.

- Karlis Freivalds and Renars Liepins. Improving the neural gpu architecture for algorithm learning. arXiv preprint arXiv:1702.08727, 2017. URL https://arxiv.org/pdf/1702. 08727.pdf.
- Niklas Heim, Tomáš Pevný, and Václav Šmídl. Neural power units. Advances in Neural Information Processing Systems, 33, 2020. URL https://papers.nips.cc/paper/2020/ file/48e59000d7dfcf6c1d96ce4a603ed738-Paper.pdf.
- Alon Jacovi, Guy Hadash, Einat Kermany, Boaz Carmeli, Ofer Lavi, George Kour, and Jonathan Berant. Neural network gradient-based learning of black-box function interfaces. In *International Conference on Learning Representations*, 2019. URL https: //openreview.net/forum?id=r1e13s05YX.
- T. Jia, Y. Ju, R. Joseph, and J. Gu. Ncpu: An embedded neural cpu architecture on resource-constrained low power devices for real-time end-to-end performance. In 2020 53rd Annual IEEE/ACM International Symposium on Microarchitecture (MICRO), pages 1097-1109, 2020. doi: 10.1109/MICRO50266.2020.00091. URL https://ieeexplore. ieee.org/document/9251958.
- M. Joseph-Rivlin, A. Zvirin, and R. Kimmel. Momenêt: Flavor the moments in learning to classify shapes. In 2019 IEEE/CVF International Conference on Computer Vision Workshop (ICCVW), pages 4085-4094, 2019. URL https://ieeexplore.ieee.org/ stamp/stamp.jsp?tp=&arnumber=9022223.
- Lukasz Kaiser and Ilya Sutskever. Neural GPUs learn algorithms. In 4th International Conference on Learning Representations, ICLR 2016 - Conference Track Proceedings. International Conference on Learning Representations, ICLR, 2016. URL http://arxiv. org/abs/1511.08228.
- Brenden M Lake. Compositional generalization through meta sequence-tosequence learning. In Advances in Neural Information Processing Systems, pages 9791-9801, 2019. URL https://proceedings.neurips.cc/paper/2019/file/ f4d0e2e7fc057a58f7ca4a391f01940a-Paper.pdf.
- Guillaume Lample and François Charton. Deep learning for symbolic mathematics. In *International Conference on Learning Representations*, 2020. URL https://openreview.net/forum?id=S1eZYeHFDS.
- Zachary C. Lipton. The Mythos of Model Interpretability. Communications of the ACM, 61(10):35-43, jun 2016. URL http://arxiv.org/abs/1606.03490.
- Andreas Madsen and Alexander Rosenberg Johansen. Neural arithmetic units. In International Conference on Learning Representations, 2020. URL https://openreview.net/ forum?id=H1gNOeHKPS.
- Bastien Nollet, Mathieu Lefort, and Frédéric Armetta. Learning arithmetic operations with a multistep deep learning. In 2020 International Joint Conference on Neural Networks (IJCNN), pages 1-8. IEEE, 2020. URL https://ieeexplore.ieee.org/stamp/stamp. jsp?tp=&arnumber=9206963.

- Aditya Raj, Pooja Consul, and Sakar K Pal. Fast neural accumulator (nac) based badminton video action classification. In *Proceedings of SAI Intelligent Systems Conference*, pages 452-467. Springer, 2020. URL https://link.springer.com/chapter/10.1007/ 978-3-030-55180-3\_34.
- Shangeth Rajaa and Jajati Keshari Sahoo. Convolutional feature extraction and neural arithmetic logic units for stock prediction. In *International Conference on Advances* in Computing and Data Sciences, pages 349–359. Springer, 2019. URL https://link. springer.com/chapter/10.1007/978-981-13-9939-8\_31.
- Ashish Rana, Avleen Malhi, and Kary Främling. Exploring numerical calculations with calcnet. In 2019 IEEE 31st International Conference on Tools with Artificial Intelligence (ICTAI), pages 1374–1379. IEEE, 2019. URL https://ieeexplore.ieee.org/stamp/ stamp.jsp?tp=&arnumber=8995315.
- Ashish Rana, Taranveer Singh, Harpreet Singh, Neeraj Kumar, and Prashant Singh Rana. Systematically designing better instance counting models on cell images with neural arithmetic logic units, 2020. URL https://arxiv.org/pdf/2004.06674.pdf.
- Jan Niclas Reimann and Andreas Schwung. Neural logic rule layers. arXiv preprint arXiv:1907.00878, 2019. URL https://arxiv.org/pdf/1907.00878.pdf.
- David Saxton, Edward Grefenstette, Felix Hill, and Pushmeet Kohli. Analysing mathematical reasoning abilities of neural models. In *International Conference on Learning Representations*, 2019. URL https://openreview.net/forum?id=H1gR5iR5FX.
- Daniel Schlör, Markus Ring, and Andreas Hotho. inalu: Improved neural arithmetic logic unit. Frontiers in Artificial Intelligence, 3:71, 2020. ISSN 2624-8212. doi: 10.3389/ frai.2020.00071. URL https://www.frontiersin.org/article/10.3389/frai.2020. 00071.
- Carson D Sestili, William S Snavely, and Nathan M VanHoudnos. Towards security defect prediction with ai. *arXiv preprint arXiv:1808.09897*, 2018. URL https://arxiv.org/pdf/1808.09897.pdf.
- Daniel Teitelman, I. Naeh, and Shie Mannor. Stealing black-box functionality using the deep neural tree architecture. ArXiv, abs/2002.09864, 2020. URL https://arxiv.org/ pdf/2002.09864.pdf.
- Andrew Trask, Felix Hill, Scott E Reed, Jack Rae, Chris Dyer, and Phil Blunsom. Neural arithmetic logic units. In *Advances in Neural Information Processing Systems*, pages 8035–8044, 2018. URL https://openreview.net/pdf?id=H1gNOeHKPS.
- Bo Wu, Haoyu Qin, Alireza Zareian, Carl Vondrick, and Shih-Fu Chang. Analogical reasoning for visually grounded language acquisition. *arXiv preprint arXiv:2007.11668*, 2020. URL https://arxiv.org/pdf/2007.11668.pdf.
- Zhu Xiao, Fancheng Li, Ronghui Wu, Hongbo Jiang, Yupeng Hu, Ju Ren, Chenglin Cai, and Arun Iyengar. Trajdata: On vehicle trajectory collection with commodity plug-and-play

obu devices. *IEEE Internet of Things Journal*, 2020. URL https://ieeexplore.ieee.org/document/9115028.

- Chiyuan Zhang, Samy Bengio, Moritz Hardt, Michael C. Mozer, and Yoram Singer. Identity crisis: Memorization and generalization under extreme overparameterization. In International Conference on Learning Representations, 2020. URL https://openreview.net/ forum?id=B116y0VFPr.
- Rui-Xiao Zhang, Tianchi Huang, M. Ma, Haitian Pang, Xin Yao, Chenglei Wu, and L. Sun. Enhancing the crowdsourced live streaming: a deep reinforcement learning approach. Proceedings of the 29th ACM Workshop on Network and Operating Systems Support for Digital Audio and Video, 2019a. URL https://dl.acm.org/doi/10.1145/3304112. 3325607.
- Ruixiao Zhang, M. Ma, Tianchi Huang, Haitian Pang, X. Yao, Chenglei Wu, J. Liu, and L. Sun. Livesmart: A qos-guaranteed cost-minimum framework of viewer scheduling for crowdsourced live streaming. *Proceedings of the 27th ACM International Conference on Multimedia*, 2019b. URL https://dl.acm.org/doi/10.1145/3343031.3351013.

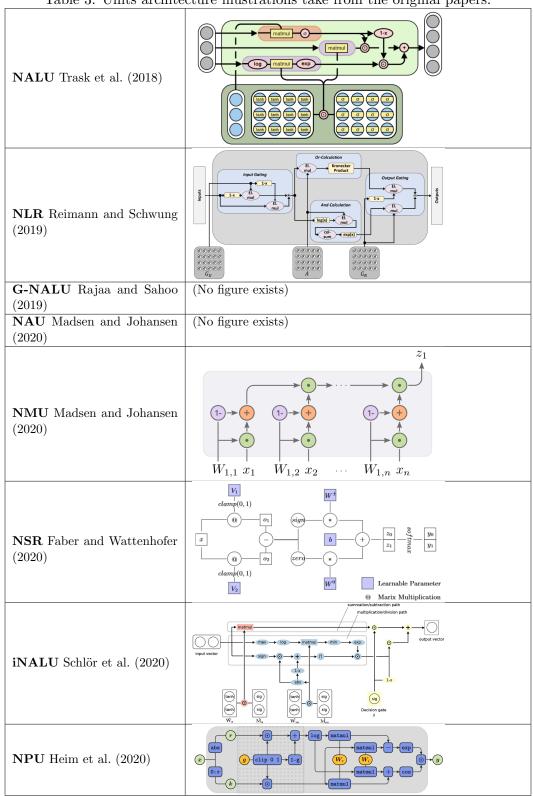


Table 3: Units architecture illustrations take from the original papers.