Cite as: Ding, L. and Jones, K. (2009) Instructional strategies in explicating the discovery function of proof for lower secondary school students. In, Lin, Fou-Lai, Feng-Jui, Hsieh, Hanna, Gila and de Villiers, Michael (eds.) *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*. Taipei, Taiwan, National Taiwan Normal University, volume 1, pp136-141.

INSTRUCTIONAL STRATEGIES IN EXPLICATING THE DISCOVERY FUNCTION OF PROOF FOR LOWER SECONDARY SCHOOL STUDENTS

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In this paper, we report on the analysis of teaching episodes selected from our pedagogical and cognitive research on geometry teaching that illustrate how carefully-chosen instructional strategies can guide Grade 8 students to see and appreciate the discovery function of proof in geometry.

INTRODUCTION

This paper focus on two of the issues in the teaching and learning of mathematical proof that have drawn educators' and researchers' serious efforts during the last two decades. The first is to do with students' cognitive development for acquiring proof capability, and the second is associated with the relationship of teaching to such development. To date, what the pedagogical factors are and how they may relate to the development of students' understanding of proof remain elusive. For instance, Martin, McCrone, Bower, and Dindyal (2005) found that the teacher's modeling of deductive reasoning was an effective means for helping students learn to develop arguments and provide appropriate justifications. Yet these researchers also found that students might still not be fully capable of understanding the abstract and general nature of arguments and representative diagrams. Next, the intervention studies conducted in Taiwan and in Germany by Heinze, Cheng, Ufer, Lin, and Reiss (2008) show that two totally different instructional approaches were successfully used to foster students' proof competence in the different classrooms and learning cultures of East Asia and Western Europe. Noticeably, the two teaching strategies had their own advantages and limits on learners with different levels of achievement on constructing a multi-step proof.

For these reasons, and others, it is vital to research new strategies for teaching deductive proof, particularly the teaching of the functions of deductive proof as a means of explanation and discovery to promote students' mathematical understanding. In this paper, we make a contribution to this central research question (raised in theme 5 of the ICMI Study 19 Discussion Document) of how teachers might develop effective strategies to help students see and appreciate the discovery function of proof – for example, deriving results deductively rather than experimentally.

THE COMPLEXITY OF TEACHERS' DIDACTICAL PRACTICES

One of our research aims is to develop fundamental understanding of the complexity of teachers' didactical practice in respect of the development of students' thinking for constructing proofs in secondary school geometry classes. In one component of our research, we linked the van Hiele theory to the instructional practice of a sample of expert teachers of geometry at Grade 8 (13-14 years old) in Shanghai, China (Ding & Jones, 2007). What we found was that although the practices of these teachers were linked to some of the van Hiele teaching phases, their instructional intentions were sometimes quite different. For example, in the Chinese classrooms we noted that the teacher played a significant role in building the bridge between students and the mathematical subject in the teaching of the solving of geometrical proof problems. Based on the analysis of our classroom observation data within the larger study (see Ding, 2008), we proposed a pedagogical framework to elucidate the unique instructional strategies and approaches that expert teachers apply to support the development of students' thinking in constructing proofs in geometry. Our proposed framework seeks to account for alternative pedagogies to the van Hiele-based instructional model.

In this paper, we focus on one aspect of this pedagogical framework – that of teaching with inductive and deductive approaches. In particular, we apply Polya's (1945) problem-solving framework to the analysis of the instructional strategies used by a selected case-study teacher that we call Lily (pseudonym) to help her students to see and appreciate the discovery function of proof (for further details of the teacher and her class, see Ding & Jones, 2007).

INSTRUCTIONAL STRATEGIES TO DEDUCTIVE PROOF

During the process of solving a problem, Polya (1945) highlights two essential stages: "working for better understanding" and "hunting for the helpful idea" (pp.33-36). In this section, we select two teaching episodes which exemplify the instructional strategies related to these two fundamental stages within the context of solving a proof problem.

Working for better understanding

Proof problem *Given: Triangle ABC and AED are equilateral triangles; CD=BF. Prove: Quadrilateral CDEF is a parallelogram.* Teacher Lily, in the second of a sequence of lessons on developing her students' understanding of this multi-step proof problem (see figure 1.1), first guided her students to consider a related problem that they had learned before (see figure 1.2).

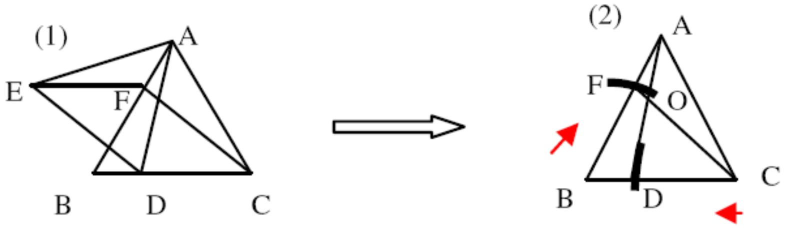

Figure 1.1 and 1.2

Thus, Lily began her instruction by redrawing part of the whole figure, equilateral triangle ABC, on the blackboard (in other words, figure 1.1 was redrawn as figure 1.2). Lily then turned to explaining the importance of the given (CD=BF) of the problem. She explained this as follows:

425. Lily: CD=BF. What does this mean?

426. Students do not respond

427 Lily: It means that D and F are dynamic points, aren't they? (The teacher repeated the question a couple of times, dialogue omitted).

430 Lily: OK. CD=BF. This means that D and F are dynamic points. D could be here, could be here, could be here, right? (The teacher recreated the figure by using compasses to draw D and F, making CD=BF, and then used a ruler to link C and F, A and D. see the result of her drawing in figure 1.2.)

434 Lily: D and F are dynamic points. Now they move such that CD=BF. So if D goes this way. F goes that way. … The different dynamic points go in different directions at the same speed, right? … So the length they (D and F) moved should be the same, shouldn't they? (The teacher put red arrows in the figure on the blackboard, see figure 1.2)

435 Lily: If you are told like this statement, you might understand that this means CD=BF. We could describe a problem in different way, yet the meaning could be same. In this problem, it means that CD=BF.

436 Lily: Well. Now, are you familiar with this figure? (see figure 1.2)

The teacher encouraged students in the whole class to observe and compare between figures 1.1 and 1.2 on the blackboard (#437-439).

440 Lily: You could think about this figure during the lesson break (see figure 1.2). You learnt about the equilateral triangle at Grade 7. In the process of the movement of D and F, D and F move regularly. Could you find what is never changed in the movement?

The instructional process at this stage is one of discovery, to involve students in finding the implicit relationship of a geometric figure (see #440). To prove quadrilateral CDEF is a parallelogram, one way is to prove CF=ED and CF//ED (see figure 1.1). Thus, the teacher's instructional intention here is to involve students in first exploring the facts that AD=CF and angle AOF=60°(see figure 1.2). The teaching process starts with developing the students' understanding of the principal parts of a "*problem to find*" (Polya, 1945, p.33), namely the unknown, the data, and the condition. Here, the teacher dynamically presents the static figure on the blackboard as an interesting way to interpret the data (the length of the sides of triangle ABC and the length of CD and BF. #430-435), and the conditions of the problem (equilateral triangle ABC and CD=BF). However, it is noted that the unknown is not of the construction of the figure, but of the hidden geometrical objects and properties of the figure (#440). Moreover, two types of teacher questioning simultaneously occur during this stage. The first type of question (see #440) requires students to make their own guess (Polya, 1945, p.99). The second type of question (see #436, 440) is like the questions such as "*Is it familiar to you? Have you seen it before?*" (Polya, 1945, p.110). Here, the teacher's main intention is to engage students in extracting relevant elements from their memory, and mobilizing the pertinent parts of their prior knowledge. Accordingly, students' thinking fostered during this teaching process can be linked to *hypothetical bridging* in the work of Heinze *et al*. (2008, p.445), the reasoning of which is similar – to construct the intermediary condition in a multi-steps proof.

Hunting for the helpful idea

Students continued to have difficulty in perceiving the hidden geometrical objects and properties of the figure (AD=CF and angle AOF=60°, see figure 1.2). So, at the beginning of the following lesson, Lily instigated a whole-class discussion of the problem:

37. Lily: In this figure, could you find what is not changed, when D and F are moving? (see figure 1.2) (Students discussed in the classroom (#38).)

40. Some students: DC=BF.

41. Lily: DC=BF? This is already given. Except this, what else is not changed?

42. Some students: Oh, AF=BD. Because AB=BC.

43. Lily: AB=BC? This is given, as it is an equilateral triangle (ABC).

More students discussed CF=AD in the class. Lily encouraged a boy student to stand up and to present his finding to the class (#44-49).

50. Wang WY (boy): Two triangles are congruent (probably ADC and CFB). AD and CF are always equal.

After CF=AD was made explicit in the class, Lily moved to draw students' attention to another hidden property of the figure – the location relationship of AD and CF.

58. Lily: Obviously, they (AD, CF) are not parallel. They are intersected, aren't they? How is the angle they formed? Will it change? You could use a protractor to measure the figure on your book. You could measure the angle before and after the movement. (see figure 1.2)

59.1. Some students: It will be the same. (One students responded 60º. (#57))

59.2. Liuliu (boy): (Noticed his classmate's response.) 60º, 60º. Only need to prove two parallel lines. (probably CF//ED in figure 1.1)

60. Lily: How do you explain that they are equal? No change? How much is the angle then?

More students like Liuliu suggested 60º of angles AOF and COD (#61-62).

64. Lily: If this angle (AOF) is 60º. How to prove? (The teacher used number 1 to represent angle AOF, see figure 1.1).

Some students like Beibei (girl) wondered why angle AOF is 60º, while Lily encouraged an explanation of the finding (#65-67).

68. Beibei: (asked Liuliu) Why is it 60º? Parallel?

69. Liuliu: If both of them are 60º, then they are always parallel. (Probably if angle AOF=angle ADE=60º, then FC//ED.)

70. Linlin (boy): Oh, in the middle, there is a pair of vertically opposite angles! (Probably angle AOF=angle COD)

The teacher invited a boy student to present his ideas to the whole class (#71).

72 Zheng YQ (boy): Because angle 1= angle DAC + angle ACF. (The teacher then used number 2 to represent angle DAC.)

75.1 Some students, Linlin and Liuliu: Ah? It is angle ACF? (Surprised tune)

76. Zheng YQ: Because of the congruent triangles (ADC and FBC), angle 2=angle FCB.

76.1 Some students: Oh, the bottom angle! (Probably angle ACD.) (Surprised tune)

The instructional process at this stage facilitates further discovery, involving students seeing and appreciating how new pieces of information are logically deduced by proof. Students are engaged in seeking the logic connection of the principal parts of a "*problem to prove*" (Polya, 1945, p.33), namely the hypothesis and the conclusion. Here, the hypothesis of the problem involves equilateral triangle ABC and CD=BF. There are several conclusions hidden in the static figure, such as congruent triangles ADC and FBC, CF=AD and angles AOF and COD are 60º. Moreover, the teacher varies two types of questions during this stage. The first type of questions (see #37, 58) encourages students to make their own guess. Mostly, however, the teacher addresses the second type of questions (see #60), which leads students to make a deductive reasoning for their conjectures. Noticeably, during this teaching process, some students (see #42) could only make *simple bridging* for a single-step proof (Heinze *et al.*, 2008, p.444). Some (see #59.2, 69, 70) constructed *hypothetical bridging*, yet goalless at the moment as they were not able to order the relationship of geometrical properties. A few students (see #50, 72, 76) demonstrated their *coordination* ability to approach the multi-steps proof (*ibid*, p.445).

DISCUSSION

In this paper, two factors characterize the instructional strategies for helping lower secondary school students to see and appreciate the discovery function of proof in geometry: one is the variation of mathematical problems; the other the variation of teaching questions.

Understanding of a proof problem is the instructional result of the variation of mathematical problems in the foregoing episodes. The instruction started from guiding students to understand the principles of a "*problem to find*" and achieved finally in engaging students to seek the logic connection of the principal parts of a "*problem to prove*". The instructional strategy here may be called *presenting a proof problem as an experimental problem*. Such instructional practice, in our opinion, represents an important didactical opportunity for enhancing students' capability in coping with the co-operation between experiment and proof. In such ways, students learn to mentally manipulate geometrical objects. Consequently, they learn to derive results not from operational construction and measurement of a figure, but from their previously acquired geometrical knowledge.

The episodes also provide evidence of the variation of teacher questions. On the one hand, the teacher's questioning involves students' geometrical intuition, as she encourages students to formulate a range of plausible reasons for the properties and relations of the geometric figure (for instance, see #57, 59.1, 70). On the other hand, the intention of the teacher's questioning is to increase students' awareness of the discovery function of deductive proof, in which deduction makes possible a discovery that is inaccessible to insight or empiricism. For instance, as shown in the two selected teaching episodes, though an empirical approach by itself can be a plausible way to perceive some facts and gather information for deduction (see #40, 42, 59.1, 59.2, 70), it would be insufficient to discover the relation of angles AOF, COD and ACD 60º (see #68, 75.1, 76.1).

The didactical practice identified in our study substantiates the hypothesis by Martin *et al*. (2005) that students can, with the appropriate instructional strategies, become more skilled in how to construct proofs on a *multi-tiered procedure* (p.122). In addition, students' diverse learning responses indicate the complex pedagogical situations the teacher created. A key task for our future research is to identify types of pedagogical situations which would systematically involve the dynamical co-operation between experiment and proof, towards their fusion in unitary mental objects for solving mathematical proof problems.

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