Faithful Action on the Space of Global Differentials of an Algebraic Curve

BERNHARD KÖCK

Abstract. Given a faithful action of a finite group G on an algebraic curve of genus at least 2, we prove that the induced action on the space of global holomorphic differentials is faithful as well unless the characteristic of the base field is 2 and G contains a hyperelliptic involution.

Mathematics Subject Classification 2000. 14H30; 14F10; 11R32.

Let X be a connected smooth projective algebraic curve over an algebraically closed field k equipped with a faithful action of a finite group G of order n. Then G also acts on the vector space $H^0(X, \Omega_X)$ of global holomorphic differentials on X. A widely studied problem is to determine the structure of $H^0(X, \Omega_X)$ as module over the group ring $k[G]$. It goes back to the Chevalley-Weil when $k = \mathbb{C}$, see [CW]. If the canonical projection $\pi : X \to Y$ from X to the quotient curve $Y = X/G$ is tamely ramified, a fairly explicit answer to this problem has been given in 1986 by Kani in [Ka]. For more recent answers to (related) questions in more general situations the reader is referred to the papers [Bo] and [FWK]. In the case of arbitrary wild ramification the explicit calculation of the $k[G]$-isomorphism class of $H^0(X, \Omega_X)$ is still an open problem.

This note is concerned with the weaker question whether G acts faithfully on the space $H^0(X, \Omega_X)$. We give the following answer to this question. Let g_X and g_Y denote the genus of X and Y, respectively, and let p denote the characteristic of k. We recall that a hyperelliptic involution of X is an automorphism σ of X of order 2 such that the quotient curve $X/\langle\sigma\rangle$ is isomorphic to $\mathbb{P}^1_k$.

Theorem. *We assume that $g_X \geq 2$. Then G acts faithfully on $H^0(X, \Omega_X)$ if and only if $p \neq 2$ or if G does not contain any hyperelliptic involution.*

The proof of this theorem will be given after the proof of Proposition 1 below.

Corollary. *Let $g_X \geq 2$. If G does not act faithfully on $H^0(X, \Omega_X)$ then $p = 2$, $g_Y = 0$ and the projection π is not tamely ramified.*

Proof. By the previous theorem we have $p = 2$ and there exists a hyperelliptic involution $\sigma \in G$. Then the Hurwitz formula (see Corollary 2.4 on p. 301 in [Ha]) applied to the projection $X \to X/\langle\sigma\rangle \cong \mathbb{P}^1_k$ shows that $X \to X/\langle\sigma\rangle$ is not unramified and hence not tamely ramified; then π is not tamely ramified either. And the Hurwitz formula applied to the projection $\mathbb{P}^1_k \cong X/\langle\sigma\rangle \to Y$ shows that the genus of Y is 0 as well.

The following example describes some (mostly trivial) cases when the action of G on $H^0(X, \Omega_X)$ is in fact trivial.

Example.
(a) If $g_X = 0$ then G obviously acts trivially on $H^0(X, \Omega_X) = \{0\}$.
(b) If $g_X = 1$ (that is if X is an elliptic curve) and if G is a finite subgroup of $X(k)$ acting on X by translations then G leaves invariant any global non-vanishing holomorphic differential and hence G acts trivially on $H^0(X, \Omega_X)$.
(c) Let $p = 2$. If $n = 2$ and $g_Y = 0$, then $G \cong \mathbb{Z}/2\mathbb{Z}$ acts trivially on $H^0(X, \Omega_X)$ by Proposition 2 below. For instance, let r be an odd natural number, let $k(x, y)$ be the extension of the rational function field $k(x)$ given by the Artin-Schreier equation $y^2 - y = x^r$ and let $\pi : X \to \mathbb{P}^1_k$ be the corresponding cover of nonsingular curves over k; then G acts trivially on the vector space $H^0(X, \Omega_X)$ whose dimension is $\frac{r-1}{2}$ by Example 2.5 on p. 1095 in [Kö].

The next lemma is crucial for the proof of Proposition 1 which in turn is the main idea for the proof of our theorem. We begin by introducing some notations. For any G-invariant divisor D on X let $\mathcal{O}_X(D)$ denote the corresponding equivariant invertible $\mathcal{O}_X$-module, as usual. Furthermore let $\pi_*^G(\mathcal{O}_X(D))$ denote the subsheaf of the direct image $\pi_*(\mathcal{O}_X(D))$ fixed by the obvious action of G on $\pi_*(\mathcal{O}_X(D))$ and let $\left\lfloor \frac{\pi_*(D)}{n} \right\rfloor$ denote the divisor on Y obtained from the push-forward $\pi_*(D)$ by replacing the coefficient m_Q of Q in $\pi_*(D)$ with the integral part $\left\lfloor \frac{m_Q}{n} \right\rfloor$ of $\frac{m_Q}{n}$ for every $Q \in Y$. The function fields of X and Y are denoted by $K(X)$ and $K(Y)$, respectively. Finally, for any $P \in X$, let e_P denote the ramification index of π at P and let ord_P and ord_Q denote the respective valuations of $K(X)$ and $K(Y)$ at P and $Q := \pi(P)$.

Lemma. *Let D be a G-invariant divisor on X. Then the sheaves $\pi_*^G(\mathcal{O}_X(D))$ and $\mathcal{O}_Y\left(\left\lfloor \frac{\pi_*(D)}{n} \right\rfloor\right)$ are equal as subsheaves of the constant sheaf $K(Y)$ on Y. In particular the sheaf $\pi_*^G(\mathcal{O}_X(D))$ is an invertible $\mathcal{O}_Y$-module.*

For the reader's convenience we include a proof of this lemma although it may already exist in the literature.

Proof. For every open subset V of Y we have

$$\pi_*^G(\mathcal{O}_X(D))(V) = \mathcal{O}_X(D)(\pi^{-1}(V))^G \subseteq K(X)^G = K(Y).$$

In particular both sheaves are subsheaves of the constant sheaf $K(Y)$ as stated. It therefore suffices to check that their stalks are equal. Let $Q \in Y$, let $P \in \pi^{-1}(Q)$ and let n_P denote the coefficient of D at P. Then we have

$$\begin{aligned}
\pi_*^G\left(\mathcal{O}_X(D)\right)_Q &= \mathcal{O}_X(D)_P \cap K(Y) \\
&= \{f \in K(Y) : \mathrm{ord}_P(f) \geq -n_P\} \\
&= \left\{ f \in K(Y) : \mathrm{ord}_Q(f) \geq -\frac{n_P}{e_P} \right\} \\
&= \left\{ f \in K(Y) : \mathrm{ord}_Q(f) \geq -\left\lfloor \frac{n_P}{e_P} \right\rfloor \right\} \\
&= \mathcal{O}_Y\left(\left\lfloor \frac{\pi_*(D)}{n} \right\rfloor\right)_Q,
\end{aligned}$$

as desired. □

Let $R := \sum_{P \in X} \dim_k(\Omega_{X/Y})[P]$ denote the ramification divisor of π. The following proposition computes the dimension of the subspace of $H^0(X, \Omega_X)$ fixed by G.

Proposition 1.

$$\dim_k \left(H^0(X, \Omega_X)^G\right) = \begin{cases} g_Y & \text{if } \deg \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor = 0 \\ g_Y - 1 + \deg \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor & \text{if } \deg \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor > 0. \end{cases}$$

Proof. Let K_X be a G-invariant canonical divisor on X, that is we have an equivariant isomorphism $\mathcal{O}_X(K_X) \cong \Omega_X$. Let the divisor K_Y on Y be defined by the equality $\pi^*(\Omega_Y) = \mathcal{O}_X(\pi^*(K_Y))$ of subsheaves of $\mathcal{O}_X(K_X)$. Note that we consider $\pi^*(\Omega_Y)$ as a subsheaf of $\Omega_X \cong \mathcal{O}_X(K_X)$ and that we have a short exact sequence

$$0 \to \pi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

In particular we have $K_X = \pi^* K_Y + R$ and hence

$$\left\lfloor \frac{\pi_*(K_X)}{n} \right\rfloor = \left\lfloor \frac{\pi_*\pi^*(K_Y) + \pi_*(R)}{n} \right\rfloor = K_Y + \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor.$$

Using the previous lemma we conclude that $\pi_*^G(\Omega_X) \cong \mathcal{O}_Y \left(K_Y + \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor\right)$ and finally that

$$\begin{aligned} &\dim_k \left(H^0(X, \Omega_X)^G\right) \\ &= \dim_k \left(H^0\left(Y, \pi_*^G(\Omega_X)\right)\right) \\ &= \dim_k \left(H^0\left(Y, \mathcal{O}_Y\left(K_Y + \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor\right)\right)\right). \end{aligned}$$

If $\deg \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor = 0$ then $\left\lfloor \frac{\pi_*(R)}{n} \right\rfloor$ is the zero divisor and we conclude that

$$\dim_k \left(H^0(X, \Omega_X)^G\right) = \dim_k \left(H^0(Y, \Omega_Y)\right) = g_Y,$$

as desired. If $\deg \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor > 0$ the divisor $K_Y + \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor$ is non-special and using the Riemann-Roch theorem (see Theorem 1.3 on p. 295 and Example 1.3.4 on p. 296 in [Ha]) we obtain

$$\begin{aligned} &\dim_k \left(H^0(X, \Omega_X)^G\right) \\ &= \deg\left(K_Y + \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor\right) + 1 - g_Y \\ &= g_Y - 1 + \deg \left\lfloor \frac{\pi_*(R)}{n} \right\rfloor, \end{aligned}$$

as stated. □

Proof of Theorem. To prove the if-direction we suppose that G does not act faithfully on $H^0(X,\Omega_X)$. By replacing G with the (non-trivial) kernel H of the action of G on $H^0(X,\Omega_X)$ we may assume that G is non-trivial and that G acts trivially on $H^0(X,\Omega_X)$.

We first prove that π is not tamely ramified. Suppose that π is tamely ramified. Then we have $R=\sum_{P\in X}(e_P-1)[P]$ by Proposition 2.2(c) on p. 300 in [Ha]; hence $\left\lfloor\frac{\pi_*(R)}{n}\right\rfloor$ is the zero divisor. Therefore we obtain

$$g_X=\dim_k\left(H^0(X,\Omega_X)\right)=\dim\left(H^0(X,\Omega_X)^G\right)=g_Y$$

by Proposition 1. Substituting this equality into the Hurwitz formula

$$2(g_X-1)=2n(g_Y-1)+\deg(R)$$

yields the desired contradiction because $n\geq 2$, $g_X\geq 2$ and $\deg(R)\geq 0$.

We next prove that $g_Y=0$. By the argument used in the previous paragraph we know that $\left\lfloor\frac{\pi_*(R)}{n}\right\rfloor$ is not the zero divisor. Then Proposition 1 tells us that

$$g_X=g_Y-1+\deg\left\lfloor\frac{\pi_*(R)}{n}\right\rfloor.$$

Substituting this equality into the Hurwitz formula we obtain

$$2\left(g_Y-1+\deg\left\lfloor\frac{\pi_*(R)}{n}\right\rfloor-1\right)=2n(g_Y-1)+\deg(R).$$

For any $Q\in Y$ let n_Q denote the coefficient of the ramification divisor R at any $P\in\pi^{-1}(Q)$ and let $e_Q:=e_P$ for any $P\in\pi^{-1}(Q)$. Rewriting the previous equation yields

$$\begin{aligned}(2n-2)g_Y&=2n-4+2\deg\left\lfloor\frac{\pi_*(R)}{n}\right\rfloor-\deg(R)\\&=2\left(n-2+\sum_{Q\in Y}\left(\left\lfloor\frac{n}{e_Q}\frac{n_Q}{n}\right\rfloor-\frac{n}{e_Q}\frac{n_Q}{2}\right)\right)\\&=2\left(n-2+\sum_{Q\in Y}\left(\left\lfloor\frac{n_Q}{e_Q}\right\rfloor-\frac{n_Q}{e_Q}\frac{n}{2}\right)\right)\\&\leq 2(n-2)\end{aligned}$$

because $\frac{n}{2}\geq 1$ and $\left\lfloor\frac{n_Q}{e_Q}\right\rfloor\leq\frac{n_Q}{e_Q}$ for all $Q\in Y$. Hence we obtain $g_Y\leq\frac{n-2}{n-1}<1$ and therefore $g_Y=0$, as desired.

As π is not tamely ramified, the characteristic of k is positive and we may furthermore replace G by a cyclic subgroup of G of order p (that is still supposed to act trivially on $H^0(X,\Omega_X)$). In the previous paragraph we have shown that $g_Y=0$. To finish the proof of the if-direction of our theorem it therefore suffices to show that $p=2$. This and the other direction follow from the following proposition. □

Proposition 2. *Let $p > 0$ and let G be cyclic of order p. We furthermore assume that $g_Y = 0$. Then G acts trivially on $H^0(X, \Omega_X)$ if and only if one of the following three conditions holds:*
(i) $p = 2$.
(ii) $g_X = 0$.
(iii) $p = 3$ *and* $g_X = 1$.

Proof. Let $P_1, \ldots, P_r \in X$ be the ramified points of $\pi : X \to Y$ and, for $i = 1, \ldots, r$, define $N_i \in \mathbb{N}$ by $\mathrm{ord}_{P_i}(\sigma(\pi_i) - \pi_i) = N_i + 1$ where π_i is a local parameter at P_i and σ is a generator of G. From Lemma 1 on p. 87 in [Na] we know that p does not divide N_i, a fact we will use several times below. The ramification divisor R of π is equal to $\sum_{i=1}^r (N_i + 1)(p-1)[P_i]$ by Hilbert's formula for the order of the different (see Prop. 4, §1, Ch. IV on p. 72 in [Se]). Let $N := \sum_{i=1}^r N_i$. Using the Hurwitz formula we obtain

$$2g_X - 2 = -2p + (N + r)(p - 1)$$

and hence

$$\dim_k\left(H^0(X, \Omega_X)\right) = g_X = \frac{(N + r - 2)(p - 1)}{2}.$$

Since $g_X \geq 0$ we obtain $r \geq 1$; that is, π is not unramified. Therefore we have

$$\deg \left\lfloor \frac{\pi_*(R)}{p} \right\rfloor = \sum_{i=1}^r \left\lfloor \frac{(N_i + 1)(p - 1)}{p} \right\rfloor \geq \sum_{i=1}^r \left\lfloor \frac{2(p - 1)}{p} \right\rfloor = r > 0.$$

From Proposition 1 we then conclude that

$$\begin{aligned} \dim_k\left(H^0(X, \Omega_X)^G\right) &= g_Y - 1 + \deg \left\lfloor \frac{\pi_*(R)}{p} \right\rfloor \\ &= -1 + \sum_{i=1}^r \left\lfloor \frac{(N_i + 1)(p - 1)}{p} \right\rfloor \\ &= -1 + N + r + \sum_{i=1}^r \left\lfloor -\frac{N_i + 1}{p} \right\rfloor. \end{aligned}$$

If $p = 2$ the dimension of both $H^0(X, \Omega_X)$ and $H^0(X, \Omega_X)^G$ is therefore equal to $\frac{N+r-2}{2}$. If $g_X = 0$ both dimensions are obviously equal to 0. If $p = 3$ and $g_X = 1$ we obtain $N + r = 3$ and hence $r = 1$ and $N = 2$; thus both dimensions are equal to 1. Therefore in all three of these cases G acts trivially on $H^0(X, \Omega_X)$. This finishes the proof of the if-direction in Proposition 2.

To prove the other direction we now assume that G acts trivially on $H^0(X, \Omega_X)$ and that $p \geq 3$ and prove that condition (ii) or condition (iii) holds. For each $i = 1, \ldots, r$, we write $N_i = s_i p + t_i$ with $s_i \in \mathbb{N}$ and $t_i \in \{1, \ldots, p - 1\}$. We furthermore put $S := \sum_{i=1}^r s_i$ and $T := \sum_{i=1}^r t_i \geq r$. Then we have

$$\frac{(N + r - 2)(p - 1)}{2} = \dim_k(H^0(X, \Omega_X)) = \dim_k\left(H^0(X, \Omega_X)^G\right) = N - S - 1.$$

Rearranging this equation we obtain

$$(3-p)N - 2S = (r-2)(p-1) + 2$$

and hence

$$(-p^2 + 3p - 2)S = (r-2)(p-1) + 2 - (3-p)T.$$

Since $-p^2 + 3p - 2 = -(p-1)(p-2)$ and $p \geq 3$ this equation implies that

$$S = \frac{(r-2)(1-p) - 2 + T(3-p)}{(p-1)(p-2)}.$$

Because $S \geq 0$ the numerator of this fraction is non-negative, that is

$$\begin{aligned} 0 &\leq (r-2)(1-p) - 2 + T(3-p) \\ &\leq (r-2)(1-p) - 2 + r(3-p) \\ &= 2(r-1)(2-p). \end{aligned}$$

Hence we have $r = 1$ and that numerator is 0. We conclude that $S = 0$ and hence that $T = 1$ or $p = 3$. If $T = 1$ we also have $N = 1$ and finally

$$g_X = \frac{(N+r-2)(p-1)}{2} = 0,$$

i.e. condition (ii) holds. If $T \neq 1$ and $p = 3$ we obtain $N = T = 2$ and finally

$$g_X = \frac{(N+r-2)(p-1)}{2} = 1,$$

i.e. condition (iii) holds. □

Acknowledgments. The question underlying this paper goes back to Michel Matignon. I would like to thank Niels Borne for communicating this question to me, for pointing me to the above lemma and for sketching a proof of the fact that G acts faithfully on $H^0(X, \Omega_X)$ in the tamely ramified case.

References

[Bo] N. Borne, Cohomology of G-sheaves in positive characteristic, *Adv. Math.* **201** (2006), 454-515.

[CW] C. Chevalley and A. Weil, Über das Verhalten der Integrale erster Gattung bei Automorphismen des Funktionenkörpers, *Hamb. Abh.* **10** (1934), 358-361.

[FWK] H. Fischbacher-Weitz and B. Köck, Equivariant Riemann-Roch theorems for curves over perfect fields, *Manuscripta Math.* **128** (2009), 89-105.

[Ha] R. Hartshorne, Algebraic Geometry, *Grad. Texts in Math.*, vol. 52, Springer, New York 1977.

[Ka] E. Kani, The Galois-module structure of the space of holomorphic differentials of a curve, *J. Reine Angew. Math.* **367** (1986), 187-206.

[Kö] B. Köck, Galois structure of Zariski cohomology for weakly ramified covers of curves, *American Journal of Mathematics* **126** (2004), 1085-1107.

[Na] S. Nakajima, Action of an automorphism of order p on cohomology groups of an algebraic curve, *J. Pure Appl. Algebra* **42** (1986), 85-94.

[Se] J.-P. Serre, Corps locaux, *Publications de l'Institut de Mathématique de l'Université de Nancago VIII*, Hermann, Paris 1962.

School of Mathematics, University of Southampton, Southampton SO17 1BJ, UK.
E-mail: B.Koeck@soton.ac.uk.