Airfoil - Gust Interaction
Historical Survey

P. Joseph
Two classical problems

• Wagner problem - Airfoil undergoing step change in angle of attack

• Sears problem - Airfoil encountering a gust
Flat-plate airfoil theory
Hierarchy of problems

Increasing complexity

\[
\begin{align*}
2D \text{ gust} & \quad \downarrow \quad 3D \text{ gust} \\
2D \text{ gust} & \quad \downarrow \quad 3D \text{ gust}
\end{align*}
\]

\[
\begin{align*}
\text{Incompressible Flow} & \\
\text{Compressible Flow}
\end{align*}
\]
1938 - 1952

Incompressible flow over a two-dimensional flat-plate airfoil.

von Karman (1938) Sears (1939), Kemp (1952)
Assumptions

• Small perturbations. Boundary condition of zero normal velocity applied at the boundary \( z = 0 \)

• Kutta condition. Pressure jump across the airfoil set equal to zero at the trailing edge (TE)

• Vortices shed from the TE because of the application of the Kutta condition are assumed to lie in the plane \( z = 0 \) and move downstream from the TE with velocity \( U \).
Solution and its interpretation

Pressure on the airfoil at $x$ due to upwash at element $dx_1$ may be expressed in the form

\[
dp = dp_{ap} + dp_{qs} + dp_w
\]

\[dp_{ap} = \pm \frac{\rho_0}{2\pi} bL(x, x_1) \frac{\partial w(x_1, t)}{\partial t} dx_1 \quad \text{Apparent mass}
\]

\[dp_{qs} = \pm \frac{\rho_0 dx_1}{\pi} U \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1+x_1}{1-x_1}} \frac{w(x_1, t)}{x_1-x} dx_1 \quad \text{Quasi-steady}
\]

\[dp_w = \pm \frac{\rho_0 U dx_1}{\pi b} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1+x_1}{1-x_1}} (C(k) - 1) w(x_1) e^{i\omega t} \quad \text{Shed vorticity (harmonic result)}
\]
where

\[
L(x, x_1) = \ln \left[ \frac{(x - x_1)^2 + \sqrt{1 - x^2} - \sqrt{1 - x_1^2})^2}{(x - x_1)^2 + \sqrt{1 - x^2} + \sqrt{1 - x_1^2}^2} \right]
\]

and

\[
C(k) = \frac{H_1^{(1)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \quad \text{The Theodorsen function}
\]

\[
k = \frac{\omega b}{U} \quad \text{Reduced frequency}
\]
The Sears Problem
Gust moving at free stream velocity

Assume harmonic gust being convected with the free stream

\[ w_g(x_1, t) = w_0 \, e^{i(\omega t - kx)} \]

\[ w(x_1, t) = -w_0 \, e^{i(\omega t - kx_1)} \]  \quad \text{Zero normal-velocity b.c.}

Substituting for \( w \) into the expression for \( dp(x_1) \) and integrating over \( x_1 \)

\[ p = \mp \rho_0 U w_0 \sqrt{\frac{1 - x}{1 + x}} S(k)e^{i\omega t} \]  \quad \text{S is the Sears function given by}

**Fan/Gust workshop**
22 - 23 August 2002
(Incompressible) Sears function

\[ S(k) = C(k)[J_0(k) - iJ_1(k)] + iJ_1(k) \]

\[ S(k) = \frac{2}{\pi k} \frac{1}{H_0^{(2)}(k) - iH_1^{(2)}(k)} \]
(Incompressible) Sears function

\[ S(k) \]

\[ S(k)e^{ik} \]

Response relative to centre  
Response relative to LE
Comments

• Pressure variation with $x$, $\sqrt{\frac{1-x}{1+x}}$ independent of $k$. 2D incompressible linear airfoil theory predicts that the pressure on an airfoil passing through any form of 2D gust moving with the free stream is distributed proportional to $\sqrt{\frac{1-x}{1+x}}$

• Contribution of shed vorticity will always give the pressure distribution $\sqrt{\frac{1-x}{1+x}}$ whereas the quasi-steady and inertial contributions will not
Total lift $L(t)$ and moment $M(t)$

\[ L(t) = -2b \int_{-1}^{1} p(x, t) dx \]
\[ M(t) = -2b^2 \int_{-1}^{1} x p(x, t) dx \]

\[ L(t) = 2\pi\rho_0 bUw_0 S(k)e^{i\omega t} \]
\[ M(t) = -\frac{b}{2} L(t) \]

Centre of lift acts through quarter chord point
Limiting high and low (reduced) frequency behaviour

\[ S(k) \to 1, \quad k \to 0 \]

- \( L(t) \) reduces to quasi-steady \((qs)\) approximation. Difference between general result and \(qs\) result is the lift due to lift-force required to accelerate the surrounding fluid plus the lift generated by vorticity generated in the wake acting back on the airfoil

\[ S(k) \to \frac{\exp[-i(k - \pi/4)]}{\sqrt{2\pi k}}, \quad k \to \infty \]

- Airfoil relatively unaffected by gusts of high (reduced) frequency
Two-dimensional incompressible theory
Horizontal gust

In a typical airfoil - gust interaction, both vertical and horizontal velocity components are present.
Horizontal gust

The horizontal-gust problem has been studied by Horlock (1968)

\[ u_g = u_0 e^{i(\omega t - kx)} \]

Horizontal gust resolved in vertical direction

\[ L_u(t) = 2\pi \rho_0 b U (u_0 \alpha T(k)) e^{i\omega t} \]

\[ T(k) = S(k) + J_0(k) + J_1(k) \]

For \( u_0 \approx w_0 \) horizontal gust contribution significant for \( \alpha > 10^\circ \)
1969 - 1971

Three-dimensional incompressible theories
Three-dimensional incompressible theories

- Problem solved exactly by Graham (1970) for an airfoil of infinite span. Solution in the form of infinite series.

- Approximate analytical solution obtained by Filotas (1969).

- Solutions reduce to 2D result for $\theta = \pi/2$.

- Mugridge (1971) derived an approximate multiplicative factor for the 2D strip theory approximation.
Approach by Filotas

\[ w_g(x, y) = w_0 e^{ik(x \cos \theta + y \sin \theta)} \]

Seek solution of the form \( \Delta p = g(x)e^{ik(x \cos \theta + y \sin \theta)} \).

Filotas solves for the limiting cases \( k \to 0, \infty \). A function is then postulated that predicts the correct limiting behaviour, with the assumption that it gives reasonable predictions for intermediate \( k \)-values.
Solution for large $k$

$$p \rightarrow \pm \frac{\rho_0 U w_0}{\sqrt{2\pi k}} \sqrt{\frac{1+x}{1-x}} e^{-i\theta/2} e^{ikx-k_y(x+iy)}, \quad k \rightarrow \infty$$

- For $k_y \neq 0$ the pressure decays exponentially with distance from the LE. Loading should behave as a delta function for $k\cos \theta >> 1$
- Shape of the loading distribution depends on $k_y$ but not on $k_x$
- $k_x$ affects the amplitude of the distribution but not its shape
- Centre of lift is not fixed at quarter-chord point but approaches the LE as frequency increases
Solution of Mugridge

\[ C_L^2 = 4\pi^2 \frac{w_0^2}{U^2} \frac{1}{1 + 2\pi k_x} \frac{k_x^2 + 2/\pi^2}{k_x^2 + k_y^2 + 2/\pi^2} \]

Good agreement with the exact solution of Graham for \( k_x < 1 \), but breaks down rapidly at frequencies above this.

Unlike the Filotas solution, the Mugridge results does not have the correct \( k \to \infty \) asymptote.

The Mugridge prediction for the pressure distribution similar to \( \sqrt{(1-x)/(1+x)} \) but Filotas has shown that this behaviour breaks down for large \( k_y \).
Fluid incompressibility implies that disturbances at TE are felt at the LE with no time delay. For this to be a good approximation,

\[
\frac{2b}{c - U} < \frac{2\pi}{\omega}
\]

Assuming \( \max \frac{\Delta t}{T} = \frac{1}{\pi} \) incompressible unsteady theory limited to

\[
\frac{kM}{1 - M} < 1 \quad (k = \omega/c)
\]
1970 - 1971
Numerical solutions of Graham and Adamczyk

Graham (1970) and Adamczyk (1971) have obtained numerical solutions for the unsteady lift in a two-dimensional compressible flow.

Their analyses differ but their solutions are exact within the limitations of small perturbation theory.

Graham presents Sears-type lift coefficients for $0 \leq k \leq 6$

Adamczyk analysis is applicable to any Mach number, reduced frequency and gust convection speed
Analytic solutions of Sears (1971) and Osborne (1973)

Ignoring terms of order \( (Mk\beta^2)^2 \)

\[
p = \mp \rho_0 \frac{U}{\beta} w_0 \sqrt{\frac{1 - x}{1 + x}} S(k') e^{i(\omega t + k'M^2 x)}, \quad k' = k / \beta^2, \quad \beta^2 = 1 - M^2
\]

Modified Sears function which accounts for the first order effects of finite chord/wavelength ratio. Valid at low \( k \)

\[
L(t) = 2\pi\rho_0 \frac{bU}{\beta} w_0 K(k', M) e^{i\omega t}
\]

\[
K(k', M) = S(k') \left[ J_0 \left( M^2 k' \right) - iJ_1 \left( M^2 k' \right) \right]
\]
Amiet correction to the Osborne-Sears formulation (1971)

The Osborne-Sears formulation implies exactness to order $\varepsilon = Mk/\beta$. This assumes that the exact solution may be expressed as a series expansion in $\varepsilon$. The validity of this assertion has been questioned by Amiet for two-dimensional problems with an infinite wake. By direct expansion of the exact integral equation, the following corrections to the Osborne-Sears results are obtained:

$$
\begin{align*}
 p_{Am}(x,t) &= p_{Os}(x,t) \left( \frac{k}{k + \beta^2} \right), \quad f(M) = (1 - \beta)\ln M + \beta\ln(1 + \beta) - \ln 2 \\
 L_{Am}(t) &= L_{Os}(t)
\end{align*}
$$

Discrepancy is of $O(k)$
Adamczyk solution for large $kM$

In general the compressible gust problem must be solved numerically. Adamczyk (1971) has obtained an exact solution in the limit of high frequency $k$.

At sufficiently high $k$ at finite Mach number the gust and acoustic wavelength become much smaller than the chord. The airfoil may therefore be modelled as a *semi-infinite plate*. With the LE at $x = 0$

$$p_1 = \mp \frac{\rho_0 U w_0}{\sqrt{(1 + M)\pi kx}} e^{i\left(\pi - \frac{M k x}{1 + M} - \frac{\pi}{4}\right)} , \quad kM \rightarrow \infty$$

This solution does not satisfy the Kutta condition at $x = 2$. 

Fan/Gust workshop
22 - 23 August 2002
Adamczyk solution cont

Total lift is calculated from

\[ L_1(t) = 2\pi\rho_0 b Uw_0 \frac{1}{\beta} S_1(k, M) e^{i\omega t} \]

where

\[ S_1(k, M) = \frac{\sqrt{2\beta}}{\pi k \sqrt{M}} \left[ C\left(\frac{\sqrt{2kM}}{1 + M}\right) - iS\left(\frac{\sqrt{2kM}}{1 + M}\right) \right] e^{i(k-\pi/4)} \]

Note that:

Setting \( M = 1 \) in this expression reduces to the lift expression for transonic unsteady flow derived by Landahl (1961). In the high reduced frequency limit,

\[ S_1(t) \rightarrow \frac{\beta}{\pi k \sqrt{M}} e^{i(k-\pi/2)} , \quad k \rightarrow \infty \]

\( cf \ k^{-1/2} \) and \( e^{i\pi/4} \) for asymptotic Sears function
The Landahl (1972) iterative correction procedure

The previous solution violates the TE boundary condition. Landhal proposed a procedure which corrects the downstream b.c. but violates the upstream b.c. Applying this procedure iteratively generates an infinite series in powers of $k^{1/2}$ that converges for all $k$.

Adamczyk (1972) has calculated the second term in the series that sets $p_1 + p_2 = 0$ at the TE.

$$S_2(t) = \frac{\beta \sqrt{1+M}}{iM(\pi k)^{3/2}} \left[ E^* \left( \frac{2}{\beta} \sqrt{kM} \right) - \frac{1-i}{2} + \left[ \frac{1-i}{2} - \sqrt{\frac{2}{1+M}} E^* \left( \sqrt{2kM} \right) \right] e^{\frac{-i2kM}{1+M}} \right] e^{ik}$$

$S_2$ is a generally small term at practical frequencies and Mach numbers can be neglected.
Three-dimensional compressible theories
Graham’s Similarity Principle

\[ V = U \sec \theta \]

2D gust in compressible fluid

3D gust in incompressible fluid

Fan/Gust workshop
22 - 23 August 2002
Subsonic trace velocity

\( V < c \)

Application of the Prandtl-Glauert transform the governing equations to an incompressible problem. Graham shows that the boundary condition on the airfoil and the wake transform properly. The case \( V < c \) is therefore equivalent to the 3D incompressible problem discussed previously.

\[
\tilde{k}_x = k_x / \beta^2, \quad \tilde{k}_y = k_y \left(1 - \sigma^2\right) / \beta, \quad \sigma = M k_x / \beta k_y, \quad \tilde{M} = 0
\]

\[
C_p (M, k_x, k_y, x) = \frac{1}{\beta} \tilde{C}_p (0, \tilde{k}_x, \tilde{k}_y, x) e^{i \left[M^2 \tilde{k}_x x + (\tilde{k}_y - k_y) y\right]}
\]

Actual flow \hspace{1cm} Transformed flow
Supersonic trace velocity

\[ V > c \ (\sigma > 1) \]

The transformation for this case is less well known to convert the 3D problem to an equivalent 2D problem

\[
\bar{M} = M \sqrt{1 - 1/\sigma^2}, \quad \bar{k}_x = k_x \left(1 + k_y^2 / k_x^2\right) = k_x \bar{\beta}^2 / \beta^2,
\]

\[
\bar{\beta} = \sqrt{1 - \bar{M}^2}, \quad \bar{k}_y = 0, \quad \sigma = M k_x / \beta k_y
\]

\[
C_p(M, k_x, k_y, x) = \sqrt{1 + k_x^2 / k_y^2} \cdot \bar{C}_p(\bar{M}, \bar{k}_x, 0, x) e^{i [x k_x / k_y - y]}
\]

Note, putting \( V = c \ (\sigma = 1) \) in either similarity relation recovers the Osborne result.