Airfoil - Gust Interaction Historical Survey

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Two classical problems

- Wagner problem - Airfoil undergoing step change in angle of attack

- Sears problem - Airfoil encountering a gust

Flat-plate airfoil theory
Hierarchy of problems

Increasing complexity ↓

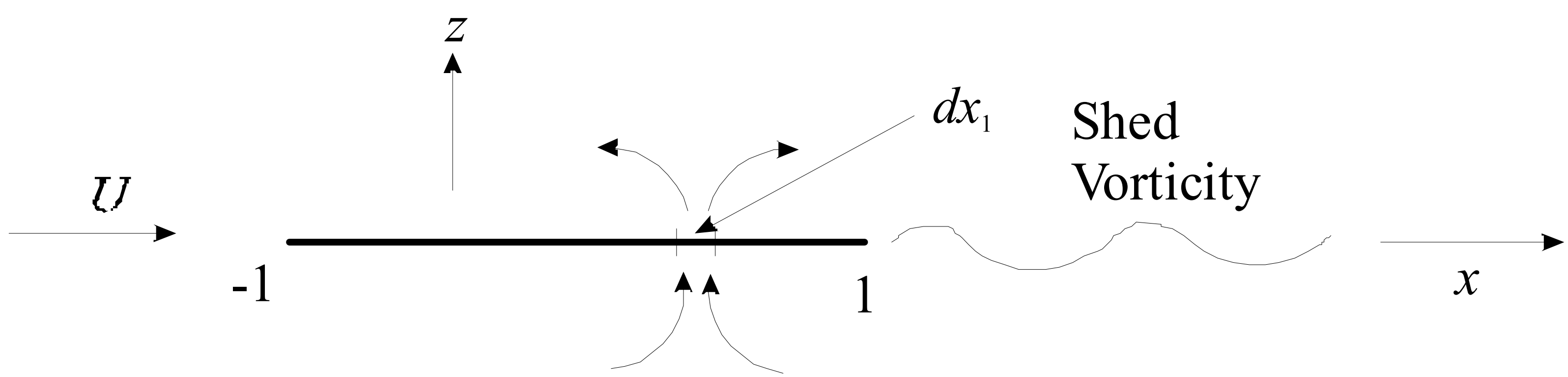
$$\left.\begin{array}{l} 2D\,\text{gust} \\ \\ 3D\,\text{gust} \end{array}\right\}\ \text{Incompressible Flow}$$

$$\left.\begin{array}{l} 2D\,\text{gust} \\ \\ 3D\,\text{gust} \end{array}\right\}\ \text{Compressible Flow}$$

1938 - 1952

Incompressible flow over a two-dimensional flat-plate airfoil.

von Karman (1938) Sears (1939), Kemp (1952)



Assumptions

- Small perturbations. Boundary condition of zero normal velocity applied at the boundary $z = 0$
- Kutta condition. Pressure jump across the airfoil set equal to zero at the trailing edge (TE)
- Vortices shed from the TE because of the application of the Kutta condition are assumed to lie in the plane $z = 0$ and move downstream from the TE with velocity U.

Solution and its interpretation

Pressure on the airfoil at x due to upwash at element dx_1 may be expressed in the form

$$dp = dp_{ap} + dp_{qs} + dp_w$$

$$dp_{ap} = \pm \frac{\rho_0}{2\pi} bL(x, x_1) \frac{\partial w(x_1, t)}{\partial t} dx_1$$

Apparent mass

$$dp_{qs} = \pm \frac{\rho_0 dx_1}{\pi} U \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1+x_1}{1-x_1}} \frac{w(x_1, t)}{x_1 - x} dx_1$$

Quasi-steady

$$dp_w = \pm \frac{\rho_0 U dx_1}{\pi b} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1+x_1}{1-x_1}} (C(k) - 1) w(x_1) \mathrm{e}^{i\omega t}$$

Shed vorticity (harmonic result)

where

$$L(x,x_1)=\ln\left[\frac{(x-x_1)^2+\left(\sqrt{1-x^2}-\sqrt{1-x_1^2}\right)^2}{(x-x_1)^2+\left(\sqrt{1-x^2}+\sqrt{1-x_1^2}\right)^2}\right]$$

and

$$C(k)=\frac{H_1^{(1)}(k)}{H_1^{(2)}(k)+iH_0^{(2)}(k)}$$ The Theodorsen function

$$k=\frac{\omega b}{U}$$ Reduced frequency

The Sears Problem
Gust moving at free stream velocity

Assume harmonic gust being convected with the free stream

$$w_g(x_1,t) = w_0\,\mathrm{e}^{i(\omega t - kx)}$$

$$w(x_1,t) = -w_0\,\mathrm{e}^{i(\omega t - kx_1)}$$ Zero normal-velocity b.c.

Substituting for w into the expression for $dp(x_1)$ and integrating over x_1

$$p = \mp\rho_0 U w_0 \sqrt{\frac{1-x}{1+x}}\,S(k)\mathrm{e}^{i\omega t}$$ S is the Sears function given by

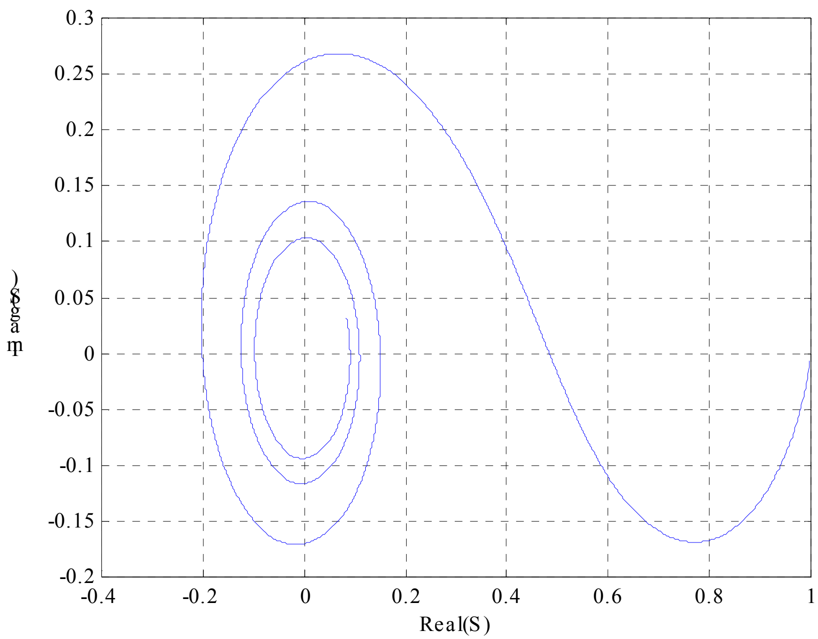
(Incompressible) Sears function

$$S(k) = C(k)\left[J_0(k) - iJ_1(k)\right] + iJ_1(k)$$

$$S(k) = \frac{2}{\pi k} \frac{1}{H_0^{(2)}(k) - iH_1^{(2)}(k)}$$

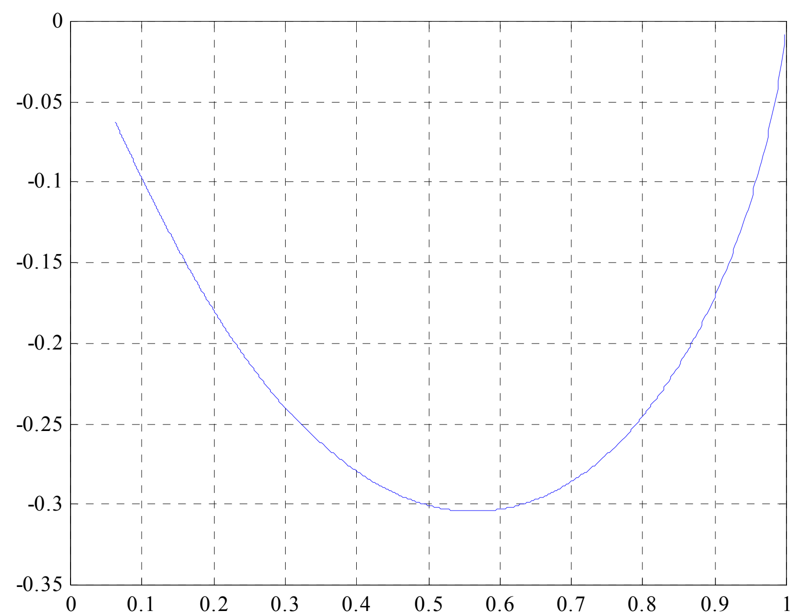
(Incompressible) Sears function

$S(k)$



Response relative to centre

$S(k)e^{ik}$



Response relative to LE

Comments

- Pressure variation with x, $\sqrt{(1-x)/(1+x)}$ independent of k. 2D incompressible linear airfoil theory predicts that the pressure on an airfoil passing through any form of 2D gust moving with the free stream is distributed proportional to $\sqrt{(1-x)/(1+x)}$

- Contribution of shed vorticity will always give the pressure distribution $\sqrt{(1-x)/(1+x)}$ whereas the quasi-steady and inertial contributions will not

Total lift $L(t)$ and moment $M(t)$

$$L(t) = -2b\int_{-1}^{1} p(x,t)\,dx$$

$$M(t) = -2b^2\int_{-1}^{1} x\,p(x,t)\,dx$$

$$L(t) = 2\pi\rho_0 b U w_0 S(k)\mathrm{e}^{i\omega t}$$

$$M(t) = -\frac{b}{2}L(t)$$

Centre of lift acts through quarter chord point

Limiting high and low (reduced) frequency behaviour

$$S(k) \to 1, \qquad k \to 0$$

- $L(t)$ reduces to quasi-steady (qs) approximation. Difference between general result and qs result is the lift due to lift-force required to accelerate the surrounding fluid plus the lift generated by vorticity generated in the wake acting back on the airfoil

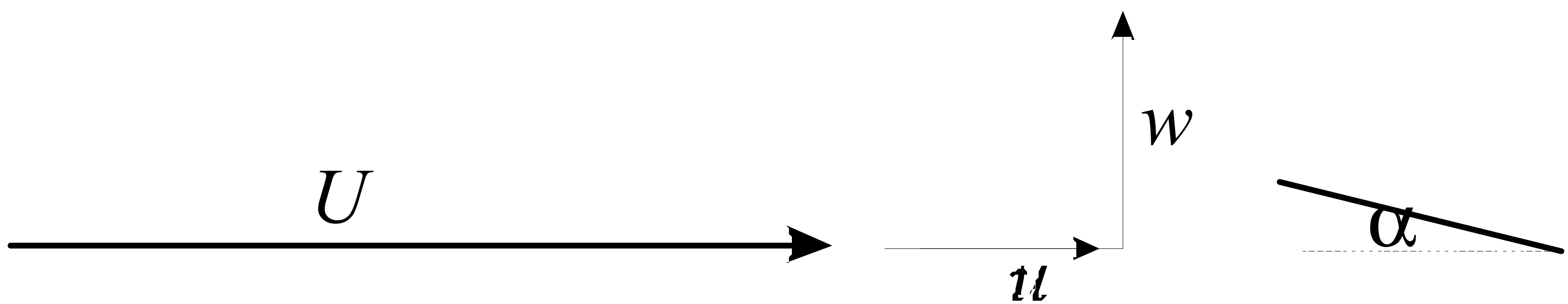
$$S(k) \to \frac{\exp[-i(k-\pi/4)]}{\sqrt{2\pi k}}, \qquad k \to \infty$$

- Airfoil relatively unaffected by gusts of high (reduced) frequency

1968

Two-dimensional incompressible theory
Horizontal gust

In a typical airfoil - gust interaction, both vertical and horizontal velocity components are present.



Horizontal gust

The horizontal-gust problem has been studied by Horlock (1968)

$$u_g = u_0 \, \mathrm{e}^{i(\omega t - kx)}$$

Horizontal gust resolved in vertical direction

$$L_u(t) = 2\pi\rho_0 bU(u_0\alpha)T(k)\mathrm{e}^{i\omega t}$$

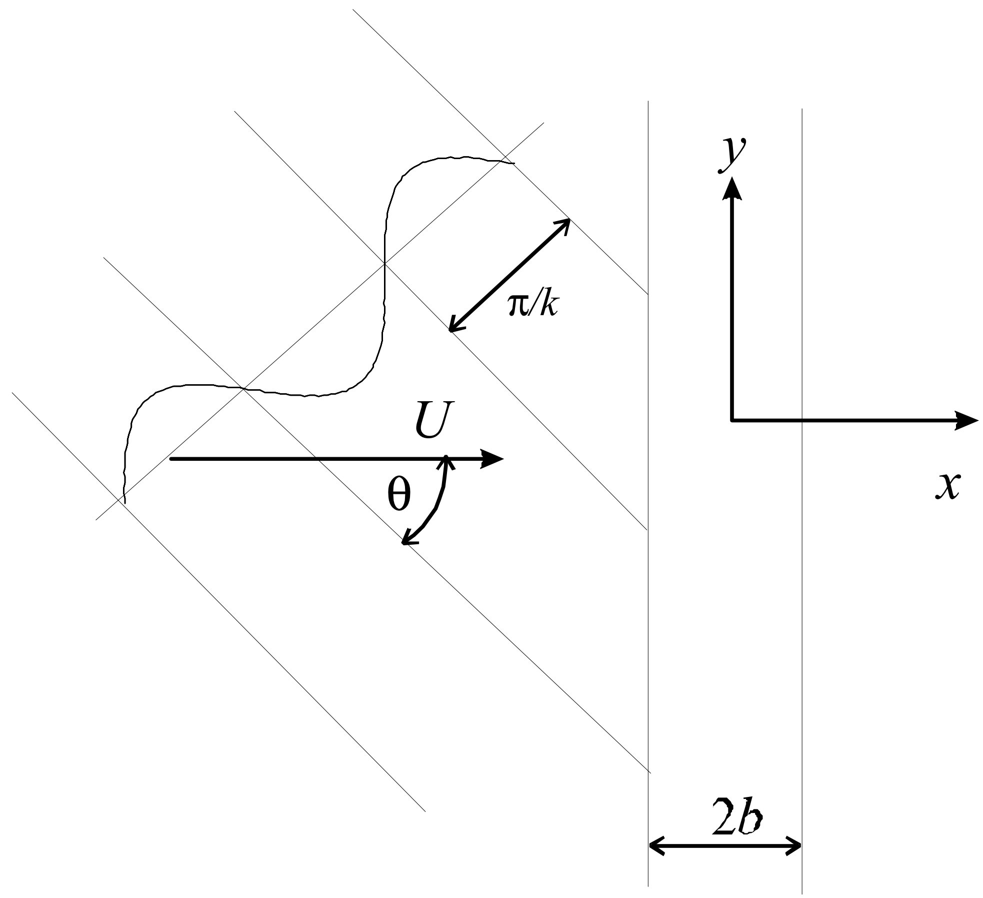
$$T(k) = S(k) + J_0(k) + J_1(k)$$

$$\frac{L_u}{L_w} = \frac{u_0}{w_0}\alpha\underbrace{\frac{T(k)}{S(k)}}_{\approx 2}$$

For $u_0 \approx w_0$ horizontal gust contribution significant for $\alpha > 10^\circ$

1969 - 1971

Three - dimensional incompressible theories



Three - dimensional incompressible theories

- Problem solved exactly by Graham (1970) for an airfoil of infinite span. Solution in the form of infinite series
- Approximate analytical solution obtained by Filotas (1969)
- Solutions reduce to 2D result for $\theta = \pi/2$
- Mugridge (1971) derived an approximate muliplicative factor for the 2D strip theory approximation

Approach by Filotas

$$w_g(x, y) = w_0\, \mathrm{e}^{ik(x\cos\theta + y\sin\theta)}$$

Seek solution of the form $\Delta p = g(x)\mathrm{e}^{ik(x\cos\theta + y\sin\theta)}$.

Filotas solves for the limiting cases $k \to 0, \infty$. A function is then postulated that predicts the correct limiting behaviour, with the assumption that it gives reasonable predictions for intermediate *k*-values.

Solution for large *k*

$$p \to \pm \frac{\rho_0 U w_0}{\sqrt{2\pi k}} \sqrt{\frac{1+x}{1-x}} \mathrm{e}^{-i\theta/2} \mathrm{e}^{ikx - k_y(x+1+iy)}, \qquad k \to \infty$$

- For $k_y \neq 0$ the pressure decays exponentially with distance from the LE. Loading should behave as a delta function for $k\cos\theta >> 1$
- Shape of the loading distribution depends on k_y but not on k_x
- k_x affects the amplitude of the distribution but not its shape
- Centre of lift is not fixed at quarter-chord point but approaches the LE as frequency increases

Solution of Mugridge

$$C_L^2 = 4\pi^2 \frac{w_0^2}{U^2} \frac{1}{1+2\pi k_x} \frac{k_x^2 + 2/\pi^2}{k_x^2 + k_y^2 + 2/\pi^2}$$

Good agreement with the exact solution of Graham for $k_x < 1$, but breaks down rapidly at frequencies above this.

Unlike the Filotas solution, the Mugridge results does not have the correct $k \to \infty$ asymptote.

The Mugridge prediction for the pressure distribution similar to $\sqrt{(1-x)/(1+x)}$ but Filotas has shown that this behaviour breaks down for large k_y.

1970 - 1971

Two - dimensional compressible theories

Fluid incompressibility implies that disturbances at TE are felt at the LE with no time delay. For this to be a good approximation,

$$\frac{2b}{c-U} < \frac{2\pi}{\omega}$$

Assuming $\max \frac{\Delta t}{T} = \frac{1}{\pi}$ incompressible unsteady theory limited to

$$\frac{kM}{1-M} < 1 \qquad (k = \omega/c)$$

1970 - 1971
Numerical solutions of Graham and Adamczyk

Graham (1970) and Adamczyk (1971) have obtained numerical solutions for the unsteady lift in a two-dimensional compressible flow.

Their analyses differ but their solutions are exact within the limitations of small perturbation theory.

Graham presents Sears-type lift coefficients for $0 \leq k \leq 6$

Adamczyk analysis is applicable to any Mach number, reduced frequency and gust convection speed

Analytic solutions of Sears (1971) and Osborne (1973)

Ignoring terms of order $\left(Mk\beta^2\right)^2$

$$p = \mp\rho_0 \frac{U}{\beta} w_0 \sqrt{\frac{1-x}{1+x}} S(k') \mathrm{e}^{i\left(\omega t + k' M^2 x\right)}, \quad k' = k/\beta^2, \qquad \beta^2 = 1 - M^2$$

$$L(t) = 2\pi\rho_0 \frac{bU}{\beta} w_0 K(k', M) \mathrm{e}^{i\omega t}$$

Modified Sears function which accounts for the first order effects of finite chord/wavelength ratio. Valid at low k

$$K(k', M) = S(k')\left[J_0\left(M^2 k'\right) - iJ_1\left(M^2 k'\right)\right]$$

Amiet correction to the Osborne-Sears formulation (1971)

The Osborne-Sears formulation implies exactness to order $\varepsilon = Mk/\beta$. This assumes that the exact solution may be expressed as a series expansion in ε. The validity of this assertion has been questioned by Amiet for two-dimensional problems with an infinite wake. By direct expansion of the exact integral equation, the following corrections to the Osborne-Sears results are obtained:

$$\left.\begin{aligned} p_{Am}(x,t) &= p_{Os}(x,t) \\ L_{Am}(t) &= L_{Os}(t) \end{aligned}\right\} \mathrm{e}^{ikf(M)/\beta^2}, \quad f(M) = (1-\beta)\ln M + \beta \ln(1+\beta) - \ln 2$$

Discrepancy is of $O(k)$

Adamczyk solution for large *kM*

In general the compressible gust problem must be solved numerically. Adamczyk (1971) has obtained an exact solution in the limit of high frequency *k*.

At sufficiently high *k* at finite Mach number the gust and acoustic wavelength become much smaller than the chord. The airfoil may therefore be modelled as a *semi-infinite plate*. With the LE at $x = 0$

$$p_1 = \mp \frac{\rho_0 U w_0}{\sqrt{(1+M)\pi k x}} \mathrm{e}^{i\left(\omega t - \frac{Mkx}{1+M} - \pi/4\right)}, \qquad kM \to \infty$$

This solution does not satisfy the Kutta condition at $x = 2$

Adamczyk solution *cont*

Total lift is calculated from

$$L_1(t) = 2\pi\rho_0 b U w_0 \frac{1}{\beta} S_1(k, M) \mathrm{e}^{i\omega t}$$

where

$$S_1(k, M) = \frac{\sqrt{2}\beta}{\pi k \sqrt{M}} \left[C\left(\sqrt{\frac{2kM}{1+M}}\right) - iS\left(\sqrt{\frac{2kM}{1+M}}\right) \right] \mathrm{e}^{i(k-\pi/4)}$$

Note that:

Setting $M = 1$ in this expression reduces to the lift expression for transonic unsteady flow derived by Landahl (**1961**). In the high reduced frequency limit,

$$S_1(t) \to \frac{\beta}{\pi k \sqrt{M}} \mathrm{e}^{i(k-\pi/2)}, \qquad k \to \infty$$

cf $k^{-1/2}$ and $\mathrm{e}^{i\pi/4}$ for asymptotic Sears function

The Landahl (1972) iterative correction procedure

The previous solution violates the TE boundary condition. Landhal proposed a procedure which corrects the downstream b.c. but violates the upstream b.c. Applying this procedure iteratively generates an infinite series in powers of $k^{-1/2}$ that converges for all k.

Adamczyk (1972) has calculated the second term in the series that sets $p_1 + p_2 =$ 0 at the TE.

$$S_2(t) = \frac{\beta\sqrt{1+M}}{iM(\pi k)^{3/2}}\left[E^*\left(\frac{2}{\beta}\sqrt{kM}\right) - \frac{1-i}{2} + \left[\frac{1-i}{2} - \sqrt{\frac{2}{1+M}}E^*\left(\sqrt{\frac{2kM}{1-M}}\right)\right]\mathrm{e}^{-i\frac{2kM}{1+M}}\right]\mathrm{e}^{ik}$$

S_2 is a generally small term at practical frequencies and Mach numbers can be neglected

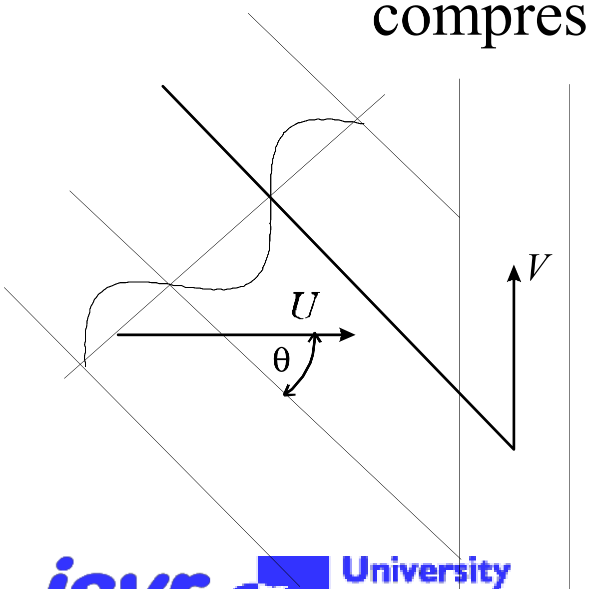
1970 - 1971

Three - dimensional compressible theories
Graham's Similarity Principle

3D gust in compressible fluid

$V > c$ → 2D gust in compressible fluid

$V < c$ → 3D gust in incompressible fluid



$$V = U \sec \theta$$

Subsonic trace velocity

$V < c$

Application of the Prandtl-Glauert transform the governing equations to an incompressible problem. Graham shows that the boundary condition on the airfoil and the wake transform properly. The case $V < c$ is therefore equivalent to the 3D incompressible problem discussed previously.

$$\tilde{k}_x = k_x / \beta^2, \quad \tilde{k}_y = k_y\left(1-\sigma^2\right)/\beta, \quad \sigma = Mk_x/\beta k_y, \quad \tilde{M} = 0$$

$$C_p\left(M, k_x, k_y, x\right) = \frac{1}{\beta}\tilde{C}_p\left(0, \tilde{k}_x, \tilde{k}_y, x\right)\mathrm{e}^{i\left[M^2\tilde{k}_x x + \left(\tilde{k}_y - k_y\right)y\right]}$$

Actual flow Transformed flow

Supersonic trace velocity
$V > c\ (\sigma > 1)$

The transformation for this case is less well known to convert the 3D problem to an equivalent 2D problem

$$\overline{M} = M\sqrt{1 - 1/\sigma^2}, \quad \overline{k}_x = k_x\left(1 + k_y^2 / k_x^2\right) = k_x \overline{\beta}^2 / \beta^2,$$

$$\overline{\beta} = \sqrt{1 - \overline{M}^2}, \quad \overline{k}_y = 0, \quad \sigma = Mk_x / \beta k_y$$

$$C_p\left(M, k_x, k_y, x\right) = \sqrt{1 + k_x^2 / k_y^2}\ .\overline{C}_p\left(\overline{M}, \overline{k}_x, 0, x\right) \mathrm{e}^{i\left[x k_x / k_y - y\right]}$$

Note, putting $V = c\ (\sigma = 1)$ in either similarity relation recovers the Osborne result