4 THE PLACE OF EXPERIMENTAL TASKS IN GEOMETRY TEACHING: LEARNING FROM THE TEXTBOOKS DESIGN OF THE EARLY 20TH CENTURY

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The dual nature of geometry, in that it is a theoretical domain and *an area of practical experience, presents mathematics teachers with opportunities and dilemmas. Opportunities exist to link theory with the everyday knowledge of pupils but the dilemmas are that learners very often find the dual nature of geometry a chasm that is very difficult to bridge. With research continuing to focus on understanding the nature of this problem, with a view to developing better pedagogical techniques, this paper examines the place of experimental tasks in the process of learning geometry. In particular, the paper provides some results from an analysis of innovative geometry textbooks designed in the early part of the 20th Century, a time when significant efforts were being made to improve the teaching and learning of geometry. The analysis suggests that experimental tasks have a vital role to play and that a potent tool for informing the design of such tasks, so that they build effectively on pupils' geometrical intuition, is the notion of the* geometrical eye, *a term coined by Charles Godfrey in 1910 as the power of seeing geometrical properties detach themselves from a figure.*

INTRODUCTION

Geometry is one of the most important components of the school mathematics curriculum yet designing a suitable geometry curriculum remains an elusive task (see, Clausen-May, Jones, McLean, and Rollands, 2000; Mammana and Villani, 1998; Royal Society, 2001). A recent comparative study of geometry curricula found considerable variation in current approaches to school geometry across different countries (Hoyles, Foxman and Küchemann, 2002). For example, a 'realistic' or practical approach is apparent in Holland, while a theoretical approach is evident in France and Japan. In the UK, over recent years, the specification of the National Curriculum for mathematics has been adjusted to clarify, for example, the requirements in geometry (especially the balance between applications and theoretical components) and this is now exemplified for 11-14 year olds in the Key Stage 3 Framework for mathematics (DfEE, 2001).

A characteristic feature of geometry is its dual nature, in that it is both a theoretical domain and perhaps the most concrete, reality-linked part of mathematics. This dual nature has dual consequences for the teaching and learning of geometry. While, hypothetically, the dual nature of geometry should help teachers to link mathematical theory to their pupils' lived experience, in practice for many pupils the dual nature is experienced as a gap that they find very difficult to bridge. Thus, research continues to focus on the difficulties that pupils have in developing an understanding of geometrical theory and making the transition to formal proofs in geometry in lower secondary schools (see, for example, Arzarello *et al*, 1998; Malara and Iadorosa, 1997; Miyazaki, 2000).

While the use of software tools, such as dynamic geometry, is proving to be helpful (for recent research evidence, see the special issue of *Educational Studies in Mathematics* edited by Jones *et al*, 2000), there is an urgent need to develop more effective pedagogical theory for geometry so that such tools can be integrated more successfully in mathematics classrooms. With a view to informing the development of better pedagogical models, this paper reports some of the findings from a study of forms of innovative geometry textbooks published in the early part of the 20th Century, a time when significant efforts were being made to improve the teaching and learning of geometry. The analysis of curriculum materials and associated teaching methods undertaken as part of this study focus, in part, on ways of bridging the gap between practical and deductive geometry. The analysis suggests a vital role for experimental tasks in geometry education. Such tasks need very careful design. This paper argues that, in informing the design of experimental tasks, much promise lies in the notion of the *geometrical eye*, a term coined by one of the major movers behind the reform of the geometry teaching in the early 20^{th} Century, Charles Godfrey (1910). Godfrey defined the *geometrical eye* as the power of seeing geometrical properties detach themselves from a figure (*ibid*, p. 197). This paper argues that this notion might be a potent tool for informing the design of experimental tasks in geometry textbooks so that they build effectively on geometrical intuition.

THE STUDY OF TEXTBOOKS IN MATHEMATICS EDUCATION

Various studies, including the Third International Mathematics and Science Study, have demonstrated that textbooks, together with documents for use in classrooms as teaching aids, such as resources of exercises, remain important tools in today’s classrooms (see, for example, Foxman, 1999; Valverde *et al*: 2002). Recent studies also recognise the importance of the study of textbooks in mathematics education research. For example, Haggarty and Pepin consider that “a textbook reflects national curricular goals and, further, reflects and legitimises national cultural traditions” (Haggarty and Pepin, 2002, p. 568). Sutherland, Winter and Harries suggest that “pupils’ construction of knowledge cannot be separated from the multifaceted external representations of this knowledge which envelope the learning pupil” (Sutherland, Winter and Harries, 2001, p. 155). This implies that textbooks, one such external representation, can influence and ‘shape’ students’ mathematical knowledge (also see, Healy and Hoyles, 1999).

The geometry textbooks chosen for analysis in this study are those by Godfrey and Siddons published in the early 20^{th} Century. These texts were selected because Godfrey and Siddons were major players in the reform of the teaching of mathematics at that time, and their textbooks are widely recognised as being very important and influential (see, Price, 1976; Howson, 1982; Quadling, 1996; Griffiths, 1998).

It could be argued that old textbooks are unlikely to be useful in informing ways of improving current teaching practice yet, as Herbst suggests, "the history of how instruction responded to past curriculum change efforts can serve as a source of information, encouragement, and caution" (Herbst, 2002, p. 285). Furthermore, UK curriculum developments in geometry in the early 20th Century are often viewed as a 'Golden Age', a time when a great effort was made to improve geometry teaching (Howson, 1982; Price, 1994, Griffiths, 1998; also see Price, 2001, p. 217). The issue of how geometry, especially deductive geometry, should be taught was a major concern of reformers at that time (e.g. Perry's address in Glasgow in 1901 and the Annual Meeting of the Mathematical Association in 1902). In particular, the introduction of experimental tasks such as drawing and measurement were discussed, and these discussions were reflected on the textbooks published at that time. Thus, examining textbooks and other documents published during this period should provide interesting and informative perspectives on the relationship between experimental and deductive geometry.

METHODOLOGICAL CONSIDERATIONS

As a methodological guide to the analysis of textbooks, Schubring (1987) has proposed the study of textbook authors and their offerings, especially the production of revised versions as examining these revisions should provide a variety of information on changing trends and ideas in teaching. Data sources selected for analysis come from the geometry textbooks written by Godfrey and Siddons during 1903-1920. The books included: *Elementary Geometry, Practical and Theoretical* (1903), *Modern Geometry* (1908), *Geometry for Beginners* (1909), *Solid Geometry* (1909), *A Shorter Geometry* (1912) and *Practical and Theoretical Geometry* (1920). Many of these books were major sellers, with for example, *Elementary Geometry*, selling "13000 [copies] of the complete book and 9000 of Volume I in the first ten months. A further 8000 of the complete book and 3500 of Volume I followed in the next twelve months" (Siddons, 1952, p. 9). Indeed, Howson reports that this particular textbook remained in print until 1973 and sold over a million copies (Howson, 1982, p. 268).

All of these textbooks are intended for use in secondary schools, except *Modern Geometry*, which 'covers the schedule of Modern Plane Geometry required for the Special Examination in Mathematics for the Ordinary B.A. Degree at Cambridge' (Godfrey and Siddons; 1908, Preface). It is also worth noting at this point that an initial analysis of the content of these textbooks showed that *Geometry for Beginners* corresponds closely to the first and second sections of *A Shorter Geometry*, and so is not treated in any depth in this paper, and that *Solid Geometry* is the only textbook by Godfrey and Siddons that is about 3-D figures. In order to utilise Schubring's methodological approach of examining revisions to textbooks, this paper mainly focuses on 2-D figures as these are dealt with over a number of the books written by Godfrey and Siddons so this latter textbook is not considered in this paper.

The experimental tasks in the three main textbooks by Godfrey and Siddons, *Elementary Geometry* (1903), *A Shorter Geometry* (1912) and *Practical and Theoretical Geometry* (1920) were selected for detailed analysis. Particularly attention was paid to common features of these tasks in these three texts in order to examine the relationship developed between experimental and deductive geometry. Schubring suggest that the revisions of these textbooks can be used as mirrors which reflect on trends in teaching practice such as other teachers' thinking (Schubring, 1987, p. 41). This entails examining what has changed and what has been retained *unchanged*. The latter is anticipated to be important, as unchanged items are likely to be those that the textbook authors consider as essential in the teaching of geometry. The analysis is framed by the following procedure:

1. Descriptions of the design of the textbooks by Godfrey and Siddons, and the roles of experimental tasks;
2. Descriptions of the common features of these tasks in the three geometry textbooks analysed;
3. Consideration of the relationship between experimental and deductive geometry of Godfrey and Siddons, by referring to their articles in periodical journals and their book *The Teaching of Elementary Mathematics*[1] (1931).

EXPERIMENTAL TASKS IN THE TEXTBOOKS OF GODFREY AND SIDDONS

The role of experimental tasks in *Elementary Geometry* (1903)

Elementary Geometry (1903) by Godfrey and Siddons was published as a result of the reform in the teaching of mathematics in 1901-3. With regard to reforming the teaching of geometry, the main issue at the time was how Euclid's *Elements*, widely used as a geometry textbook up to then, should be amended in order that the teaching of geometrical concepts was more effective (for example, see Howson, 1982; Price, 1994). The introduction of experimental tasks was a topic of particular discussion and various proposals were made by, for example, Perry (1902) and the Mathematical Association (1902). Unlike Euclid's *Elements*, *Elementary Geometry* consists of two parts: Part I. Experimental Geometry and Part II. Theoretical Geometry. 'Experimental Geometry' mainly contains experimental tasks such as measurement or drawing dealing with both plane and solid figures. In contrast, 'Theoretical Geometry", the main part of this text, consisted of propositions from Euclid's *Elements* with four continuous books: Book I: Straight lines, Book II: Areas, Book III: Circles, and Book IV Similarity. The design of this textbook reflected a significant development at the time, given the failure of the earlier reforms of geometry teaching initiated by the Association for the Improvement of Geometry (the fore-runner of the Mathematical Association) in the late 19th Century (Brock, 1975).

The analysis of the book shows that throughout *Elementary Geometry*, experimental tasks can be seen. In the first section of the book, ‘Experimental Geometry’, these tasks are aimed at introducing students to various geometrical instruments and figures as well as leading them to discover various facts in geometry, which are proved in the later sections (Godfrey and Siddons, 1903, Preface). For example, through the exercise below, students learn how to measure angles (Godfrey and Siddons, 1903, p. 12):

> Ex. 37. Measure the angles of your set square (i) directly, (ii) by making a copy on paper and measuring the copy.
>
> Ex.123. Cut out a paper triangle, mark its angles, tear off the corners and fit them together with their vertices at one point. What relation between the angles of a triangle is suggested by this experiment?

In ‘Theoretical Geometry’, although the main emphasis is deductive reasoning (see Langley’s review in the MA, 1971, p. 239-40), experimental tasks are often located before (and sometimes after) theorems. For example, before the Pythagorean theorem is introduced, a right-angled isosceles triangle, a 3-4-5 right-angled triangle and Perigal’s dissection are studied practically (Godfrey and Siddons, 1903, p. 187-9).

In some of the practical exercises in the theoretical section, students apply chosen theorems in a practical way. For example, the following exercise is undertaken after the students have learned that there is one circle, and one only, which passes through three given points not in a straight line (Godfrey and Siddons, 1903, p. 224, note that squared paper is used in this exercises):

> Ex. 1162. (using graph paper.) Draw a circle to pass through the points (0, 3), (2, 0), (-1, 0), and measure its radius. Does this circle pass through (i) (0, -3), (ii) (1, 3), (iii) (0, -2/3)?

In addition, some of the practical exercises are included to justify geometrical facts through work on experimental tasks. For example:

> Ex.192. Draw a parallelogram having sides=9.2cm. and 4.3cm., and one angle=125. Draw its diagonals, and measure their parts.
>
> Ex. 193. Repeat the last Ex. with the following measurements, 8.6cm., 6.8cm., 68, test your facts you noted in that Ex.
>
> (Godfrey and Siddons, 1931, p. 39)

Godfrey discusses the importance of testing conjectures in one of his published articles:

> Uneducated people are apt to omit the last step of the process, the testing of their hypothesis, the habit of testing hypotheses seems to be one of those habits that education can cultivate. Let us therefore cultivate it in teaching mathematics. All that we have to do is to form the habit of saying Have I verified this hypothesis? Perhaps this good habit will have a better chance of spreading from mathematical to general

> activities if it is pointed out explicitly, to boys of suitable age, that they are not to leave behind them in the class-room any good habits that they may have acquired there, it is not good to conceal from boys that their work at school has some bearing on their after life, and it might be well if many teachers kept this point more constantly in view.
>
> (Godfrey and Siddons, 1931, p.24)

Overall, the analysis shows that the experimental tasks in *Elementary Geometry* were introduced by the authors for several reasons: a) to make students familiar with geometrical instruments and figures, b) to lead students to discover geometrical facts, c) to apply the theorems to practical problems, and d) to justify the geometrical facts by experimental tasks.

Experimental tasks in A Shorter Geometry (1912) and Practical and Theoretical Geometry (1920)

Some differences can be observed in the overall structure of *A Shorter Geometry* (1912), the next geometry title by Godfrey and Siddons, as compared to that present in *Elementary Geometry*. In *A Shorter Geometry* there are three parts: First Stage: introductory course, Second Stage: discovery of the fundamental geometrical facts by experiment and intuition, and Third Stage: deductive development of theorems (Godfrey and Siddons: 1912, Preface). The third stage consists, like *Elementary Geometry*, of four continuous books: Book I: Straight lines, Book II: Areas, Book III: Circles, and Book IV Similarity. The content of the second stage, Books I, II, III and IV, of *A Shorter Geometry* almost correspond to those in Books I, II, III and IV of *Elementary Geometry*. The four roles for experimental tasks described above can be seen in this textbook, i.e. plenty of drawing and measurement are still required, and a heuristic approach is used in the deductive stages. One change is that the number of these tasks is somewhat reduced. For example, the number of experimental tasks before the Pythagorean theorem is reduced from 8 to 3 in *A Shorter Geometry*. It seems, in *A Shorter Geometry*, that Godfrey and Siddons tried to make the roles of these exercises clearer for teachers and learners, retaining essential exercises and omitting others. In fact, Godfrey stated in the 1930s:

> You might make a boy measure the sides of 100 different right-angled triangles; but he might stare at the results forever without evolving any hypothesis likely to harmonise them. … If we begin with a picture like the first figure [a right angled isosceles triangle] and follow it up with the second [a 3-4-5 right angled triangle], most boys will induce the general theorem.
>
> (Godfrey and Siddons; 1931, p. 25)

Twenty years of teaching practice led Godfrey and Siddons to revise their geometry textbooks further[2] (Godfrey, 1920). *Practical and Theoretical Geometry* (1920) consists of four stages: Stage I. Introductory practical work. Stage II. Intuitive treatment of a few fundamental propositions. Stage III. A free treatment of the whole field of elementary plane geometry, by methods sometimes formal, sometimes informal. and Stage IV. A logical chain of propositions in plane geometry, in Euclidean form (Godfrey and Siddons, 1920, p. viii). Most of the facts and theorems in *Elementary Geometry* and *A Shorter Geometry* are introduced in stages II and III (for pupils aged from 14-16), whereas the order of theorems (e.g. 'Similar figures' are introduced before 'Area') and methods of proof (e.g. the proof of the Pythagorean theorems) are changed, and the content from *Solid Geometry* (1909) is also included at the end of the stage III. Nevertheless, similar exercises in the former two texts can be seen throughout *Practical Geometry*, and from the point of view of the roles of experimental tasks, again, the roles in this text are the same as those in the other texts by Godfrey and Siddons, although it should be noted that experimental tasks are omitted in *Theoretical Geometry*.

THE PLACE OF EXPERIMENTAL TASKS: DEVELOPING THE GEOMETRICAL EYE

Having described the roles of experimental tasks in the textbooks by Godfrey and Siddons, it is important to turn to the relationship between experimental and deductive aspects in their geometry teaching. Articles written by Godfrey and Siddons (e.g. Godfrey, 1910) show that they were unhappy with the ways in which their textbooks were being used. Siddons, for example, wrote that 'Godfrey and I were very dissatisfied with the way in which some teachers were using the book [*Elementary Geometry*] working though all the Practical part first before starting the Theoretical' (Siddons, 1952, p. 9). Godfrey also questioned the strict distinction between experimental and deductive geometry:

> … all experiment at first, all theory later. This I am sure is a mistaken view, it springs from a deep-seated if undefined belief that mathematical thought is solely deductive, and that any other element in mathematical teaching is a rather disreputable intrusion, inevitable perhaps, but a thing to be got over and done with as early as may be, like measles and mumps. What is really needed is a sensible blend at each stage, and anyone will come to this conclusion who will take the trouble to examine the workings of his own mind when a new study is undertaken.
>
> (Godfrey and Siddons, 1931, p. 21)

Godfrey considered that mathematics should not be undertaken only by logic (Godfrey, 1910, p. 197). He wrote that another important power is necessary for solving mathematical problems, what he called "geometrical power" and which he describes as "the power we exercise when we solve a rider [a difficult geometrical problem or proof] (Godfrey, 1910, p. 197). To develop this geometrical power, Godfrey argues, it would be essential to train student's "geometrical eye", which he defined as "the power of seeing geometrical properties detach themselves from a figure" (Godfrey, 1910, p. 197). This 'geometrical eye' is illustrated in the following example: if A, B are the mid-points of the equal sides XY, XZ of an isosceles triangle, see Figure 1, prove that AZ=BY (Godfrey and Siddons, 1903, p. 94).

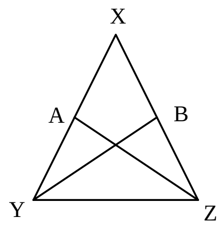

Figure 1. Isosceles triangle XYZ

In tackling this problem, someone would not be able to prove that AZ=BY unless they could see, first of all that, for example, triangle AYZ and triangle BZY are likely to be congruent.

Godfrey stated that this kind of power would be essential to solve geometrical problems, and it was experimental tasks that would it make it possible to train the geometrical eye at any stage in geometry:

> There must be a good foundation of practical work, and recourse to practical and experimental illustration wherever this can be introduced naturally into the later theoretical course. Only in this way can the average boy develop what I will call the geometrical eye.
>
> (Godfrey, 1910, p. 197)

Thus, in addition to the exercises which would lead students to discover geometrical facts, various types of exercises immediately preceded (or followed) some theorems[3]. For example, the exercises before theorem 2 in Book II of *A Shorter Geometry*, 'Triangles on the same base and between the same parallels (or, of the same altitude) are equivalent' were as follows (Godfrey and Siddons, 1912, p. 120):

> Ex. 698. Draw an acute-angled triangle and draw the three altitudes. (Freehand.)
>
> Ex. 699. Repeat Ex. 698 for a right-angled triangle. (Freehand.)
>
> Ex. 700. Repeat Ex. 698 for an obtuse-angled triangle. (Freehand.)
>
> Ex. 701. In what case are two of the altitudes of a triangle equal?

These exercises would make students pay attention to the height of triangles, which would be important to understand the theorem above (notice that these exercises require discussion between the teacher and students).

Another example is that the following exercises were studied before theorem 1 in Book III of *A Shorter Geometry*: a straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord, (Godfrey and Siddons, 1912, pp. 151-2), see Figure 2:

> Ex. 877. Draw a circle of about 3 in. radius, draw freehand a set of parallel chords (about 6), bisect each chord by eye. What is the locus of the mid-points of the chord?

Ex. 878. Draw a circle and a diameter. This is an axis of symmetry. Mark four pairs of corresponding points. Is there any case in which a pair of corresponding points coincide? (*Freehand*.)

Ex. 879. What axes of symmetry has (i) a sector, (ii) a segment, (iii) an arc, of a circle?

These exercises would help students become aware of the symmetry of the circle as well as leading them to discover the theorem. Also, to prove this theorem, it is necessary to show that triangles OAD and OBD are congruent, and the exercises would help students to see the congruency of the triangles. This illustrates how Godfrey and Siddons use experimental tasks to help develop students' geometrical eye.

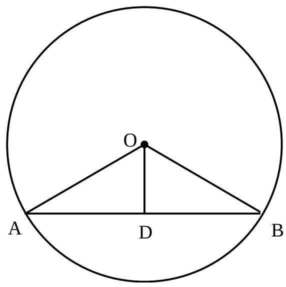

Figure 2. Circle and triangle

A further example is that before the construction of an inscribed circle in a given triangle the following exercise is studied (Godfrey and Siddons, 1912, p. 171) - see Figure 3.

Ex. 974. What is the locus of the centres of circles touching two lines which cross an angle of 60. Draw a number of such circles.

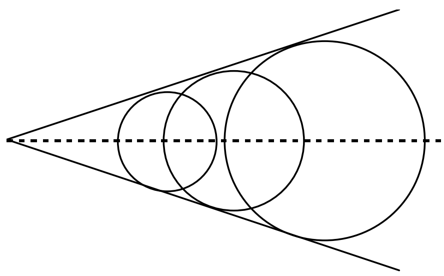

Figure 3. Circles touching two lines

This task would help students understand the proof of this construction, but also in solving the following theoretical exercise: prove that the bisectors of the three angles of a triangle meet in a point (Godfrey and Siddons, 1912, p. 172).

From the point of view of developing the geometrical eye, the design of the experimental tasks in their textbooks can be summarised as follows: the experimental exercises were carefully chosen and designed leading to showing and requiring a proof; using this design the aim of Godfrey and Siddons was to develop what they called the ‘geometrical eye’. Thus, the place of these tasks in the teaching of geometry by Godfrey and Siddons is very important, not only for the sake of discovery, but also for the developing the geometrical eye of students.

DISCUSSION

Geometry is an area of mathematics in which intuition is frequently mentioned. Views vary, however, about the role and nature of geometrical intuition, and how it might help or hinder the learning of geometry (and other areas of mathematics). Whereas Piaget or van Hiele gave intuition a relatively minor role in their models of the latter stages of learning geometry, professional geometers, nevertheless, tend to recognise the importance of geometrical intuition. For example, Poincaré wrote:

> It is by logic one demonstrates, by intuition that one invents.... Logic tells us that on such and such a way we may be sure not to meet any obstacle; it does not say which way leads to the end. For that it is necessary to see from afar, the faculty that teaches us to see is intuition.
>
> (Poincaré, 1913, p216 & 217)

Hilbert was of the view that “it is as still as true today as it ever was that intuitive understanding plays a major role in geometry” (1932, p. iii) and, most recently, Atiyah observed that:

> … spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool...
>
> (Atiyah, 2001, p. 50)

Despite these views, much additional research is needed on the relations between intuitive, inductive and deductive approaches to geometrical objects, the role and impact of practical experiments, and the age at which geometrical concepts should be introduced. As Fischbein stated;

> The interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood.
>
> (Fischbein, 1994, p. 244)

In relation to these interactions and conflicts, the notion of mathematisation, taken as the ability to perceive mathematical relationships and to idealise them into purely mental material (Wheeler, 1976, reprinted 2001; Gattegno, 1988), may be useful. Godfrey’s geometrical eye might be considered as a specialised version of this mathematisation, as a sort of intuitive skill in geometry.

Other educators have also discussed ideas that are related to Godfrey's idea of the geometrical eye. For example, Fischbein (1993) has proposed the notion of *figural concept*. As Fischbein observes, while a geometrical figure such as a square can be described as having intrinsic conceptual properties (in that it is controlled by geometrical theory), it is not solely a concept, it is an image too: " it possesses a property which usual concepts do not possess, namely it includes the mental representation of a space property" (Fischbein, 1993, p141). This means, Fischbein argues, that all geometrical figures represent mental constructs which simultaneously possess conceptual and figural properties. According to this notion of figural concepts, geometrical reasoning is characterised *by the interaction between these two aspects, the figural and the conceptual*. These interactions are likely to be important aspects of the 'geometrical eye'.

A further related idea is that of "seeing mathematically", and the notion of the "inner screen" contained in some of the writings of Mason (see, for example, Mason, 1991; Mason and Heal, 1995). Mason suggests that diagrams, such as the ones contained in this paper, can be thought of as a means, first and foremost, for awakening mental imagery and, secondly, as ways of augmenting, extending and strengthening mental imagery and hence mathematical thinking (Mason, 1991, p84). Mason argues that:

> Geometrical activity is one excellent way of gaining access to that world [of mathematics], through the power to form mental images, through seeing *through* diagrams to the world of generality which can be read into them. It is one way to encounter the discipline of mathematics, where convincing people *why* something must be a fact is as important as finding out what the fact is.
>
> (Mason, 1991, p84, emphasis in the original)

All the above ideas are aspects that need developing if the notion of the geometrical eye, coined by Charles Godfrey in 1910, is going to be useful in tackling some of the issues in contemporary geometry teaching.

CONCLUDING COMMENT

The Royal Society report on the teaching and learning of geometry recommends that, "we need to develop a completely new pedagogy in geometry" (RS/JMC 2001, p11). One major component of an innovative geometry pedagogy would be to improve on appeals to develop geometrical intuition by linking such intuition more directly with geometrical theory. This would entail developing pedagogical methods that mean that a deductive and an intuitive approach are mutually reinforcing when solving geometrical problems (see, Jones 1998).

This paper argues that Godfrey's notion of the *geometrical eye* might be a potent tool for building effectively on geometrical intuition. As demonstrated in the analysis provided above, Godfrey and Siddons considered that, in the teaching of geometry, practical and deductive geometry should be combined. Godfrey considered the *geometrical eye* to be essential for successfully solving geometrical problems, and that it should be trained by practical tasks at all stages of geometry. It is illuminating that innovative teachers 100 years ago pointed to the importance and roles of visual images in geometry and geometrical thinking.

Further research is needed to examine whether it would be possible to define more clearly the notion of the *geometrical eye*, what the relationships are between the difficulties of learning to prove in geometry and the *geometrical eye*, and how (or whether) it would be possible to develop students *geometrical eye* though practical tasks. Such research could make an important contribution to providing a firmer theoretical basis for formulating new curricular and pedagogic models for geometry.

NOTES

1. Although *The Teaching of Elementary Mathematics* was published in 1931, the first chapter (only found after the death of Godfrey in 1924), was originally written by Godfrey in 1911 (Godfrey and Siddons; 1931, 'Preface'), and therefore, this chapter is important to understand Godfrey's pedagogy around the 1910s.
2. Obviously this section is too brief to describe the 20 years' development. A further historical investigation will be needed with regard to the development of 1903-20. Also, it is necessary to look at other geometry textbooks by other reformers to obtain a wider perspective of the experimental tasks at that time.
3. These tasks can be seen in *Elementary Geometry* (1903) and *Practical Geometry* (1920). However, the locations of some exercises in *Practical Geometry* are slightly different. For example, the exercises 698-701 in *A Shorter Geometry* also appeared in *Practical Geometry* (Godfrey and Siddons, 1920, p. 105), but they are introduced immediately after the theorem.
4. However, the experimental tasks are omitted in *Theoretical Geometry* (1920), designed as the fourth stage in their geometry teaching in the 1920s. This could be seen to contradict the views of Godfrey expressed in 1910. Probably, Godfrey and Siddons considered the combination of experimental and deductive geometry in the third stage in *Practical Geometry* (1920) would be enough to develop students' *geometrical eye*, but a further (historical) examination will be needed with regard to this question.

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