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Making space for geometry in primary mathematics

Keith Jones and Claire Mooney

The movie *Toy Story*, released in 1995, was the first full-length, fully computer animated feature film. What may be surprising is that most of the *Toy Story* characters (see Figure 1.1) began life as collections of simple geometrical shapes, like spheres and cylinders, created on computers. These basic shapes were then transformed geometrically to produce each final figure. Once all the characters had been created, the challenge was to get them to move in what looked like realistic ways. This involved even more geometry. The final result was an award-winning film.

Figure 1.1 Some of the cast of *Toy Story*

Figure 1.2 The Great Court at the British Museum: classical and modern geometry combined

While much of the sophisticated geometry used in the making of films like *Toy Story* is very new (getting the shading right is apparently the most difficult part), the roots of this new geometry are very old. Nevertheless, by the middle of the twentieth century geometry was in danger of becoming eclipsed by the inexorable rise in the sophistication of algebraic techniques. These advances in algebra strongly influenced the curricular reforms that began in the 1960s, one of the results of which was that the amount of geometry included in school curricula around the world was reduced, sometimes severely so, especially for pupils aged 11–19.

Recently, however, there has begun a renaissance in geometry. Fuelled to a large extent by technological developments, it is now becoming possible to model situations visually/geometrically with quite astonishing sophistication. Computer animation, used to great effect in many popular films (such as *Toy Story*), is one manifestation of these advances. Medical imaging such as MRI (magnetic resonance imaging), global positioning systems and developments in many other areas, from architecture to robotics, all use advanced geometrical ideas. If anything, with the advent of new materials and new technologies, such as those used in the design and construction illustrated in Figure 1.2, the emphasis on everything visual is increasing. All this means that it is becoming increasingly vital to reassess our approach to the teaching and learning of geometrical ideas at all levels.

An inquiry by the Royal Society (RS) and Joint Mathematical Council (JMC) found that the current specification of the mathematics National Curriculum at primary school level is reasonable, arguing that: 'provided that this curriculum is effectively implemented, then pupils transferring from primary schools to Year 7 in secondary schools should have a suitable basis on which to develop their study of geometry' (RS/JMC 2001: 13).

Nevertheless, while the RS/JMC report contains many very useful recommendations, the fact that the inquiry was largely restricted to the needs of students in the 11–19 age range reinforces the impression that geometry at the primary school level is, and remains, the poor relation in comparison to work on number. Indeed, as Fielker (1986: 124) describes, up to the 1960s the mathematics curriculum at primary level was almost entirely devoted to work on number.

While, as mentioned above, the curricular reforms of the 1960s reduced the amount of geometry in the secondary school curriculum, some efforts were made to provide an appropriate geometry curriculum for primary age pupils. In many respects, the current specification of the National Curriculum for mathematics (DfEE 1999a) could be said to continue these efforts by including 'shape, space and measures' as one of the three content areas of mathematics at the upper primary level (along with number/algebra and handling data) and as one of only two areas (with handling data included as part of number) at Key Stage 1. From this specification, it might be surmised that geometry could constitute as much as one third of the primary mathematics curriculum, yet the National Curriculum documentation provides little guidance on its implementation with respect to the relative importance of the specified content areas. This leaves the National Numeracy Strategy (NNS) free to specify the relative emphasis given to the various aspects of mathematics.

This chapter examines the structure and recommendations of the NNS with respect to the teaching of geometry at primary school, as exemplified in the NNS' own published materials. It looks at ways in which these recommendations might be best taken forward and whether there are important aspects of geometry that the NNS has omitted or to which it has paid too little attention. Given that children experience life on a solid planet in a 3D world and that much of this experience is sensed through visual stimuli, the chapter concludes by suggesting what it might mean to give geometry appropriate consideration at the primary level.

Geometry in primary school mathematics

Geometry, says the renowned UK mathematician Sir Michael Atiyah (2001: 50), is one of the two pillars of mathematics (the other being algebra). Understanding and making sense of the world, claims Atiyah, is a very important part of what it means to be human. Sir Michael writes:

> spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics – not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool . . .

The focus for this chapter is the development of spatial reasoning at the primary level. Work on measures, while often beginning with a spatial context, just as often very rapidly leaves this behind and is probably experienced by children as yet another way of doing calculations. By concentrating on *geometry*, the focus is on the development and application of *spatial concepts* through which children learn to represent and make sense of the world: 'Geometry is grasping space . . . that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it' (Freudenthal 1973: 403).

Within the mathematics National Curriculum (for England), geometry is found as part of the attainment target currently entitled 'shape, space and measures'. This specification lists the content for this part of mathematics under the following headings:

- using and applying shape, space and measures;
- understanding properties of shape;
- understanding properties of position and movement;
- understanding measures.

The next section examines how this curriculum specification is implemented within the NNS.

Geometry in the NNS

In the introduction to the main NNS document, the *Framework for Teaching Mathematics from Reception to Year 6* (DfEE 1999b: 4), the following definition of numeracy is offered:

> Numeracy is a proficiency which involves confidence and competence with numbers and measures. It requires an understanding of the number system, a repertoire of computational skills and an inclination and ability to solve number problems in a variety of contexts. Numeracy also demands practical understanding of the ways in which information is gathered by counting and measuring, and is presented in graphs, diagrams, charts and tables.

Disappointingly, perhaps, there is little sign of geometry in this definition. Nevertheless, the *Framework* does identify teaching objectives designed to cover all the requirements of the National Curriculum for mathematics, including geometry. Some of these objectives are identified as *key* objectives, ones that are 'more critical than others' (DfEE 1999b: 3). Based on these Key Objectives, Table 1.1 shows the geometrical priorities as set out in the *Framework*.

As is made clear in the NNS training materials, the priority at Key Stage 1 is to develop pupils' facility with the language associated with shape and space through practical exploration. At Key Stage 2, the training materials highlight the following three aspects of shape and space (DfEE 1999c: Ch. 7):

- 2D and 3D shapes and their properties;
- position and direction;
- transformations.

In terms of the progression of geometrical ideas, there is nothing in the NNS materials equivalent to the advice on number and algebra (for instance, there is nothing comparable to the sections on 'The approach to calculation' and 'Laying the foundation for algebra' in the *Framework* document). Nevertheless, it is possible to glean a reasonably clear model from a close study of the *Framework* (and its associated specification of suggested mathematical vocabulary). For example, the *Framework* suggests that children should be taught to identify and name certain geometric shapes, ranging from cube, pyramid, sphere, cone, circle, triangle, square and rectangle in Reception through to dodecahedron, rhombus, kite, parallelogram and trapezium in Year 6. A progression can also be traced in the topological and rectilinear properties that children are taught to identify and use in order to classify shapes. These range from descriptive terms in Reception (flat, curved, straight, hollow, solid, corner, face, side, edge, end) through to congruent, concentric and intersecting (in Years 5 and 6), and from language to describe position and pattern

Table 1.1 The geometrical priorities in the NNS

Year	*Key objectives*	*Additional objectives given in the yearly teaching programme*
R	Position Geometrical language	Symmetrical patterns
1	Geometrical language	Properties of solids and flat shapes 2D representation Turning, position, direction
2	Sorting/classifying Position, direction, movement	Line symmetry Visualizing objects Recognizing turns
3	Line symmetry	Classifying and describing 2D and 3D shapes Compass directions and turns Straight line as two right angles
4	Classifying Symmetry properties	Describing and visualizing 2D and 3D shapes Nets of solids Whole turn as 360°
5	Parallel/perpendicular lines Properties of plane shapes	Classifying triangles Visualizing 3D shapes Nets of a cube Reflective symmetry Translation
6	Position	Describing and visualizing 3D shapes Reflection, rotation Sum of the angles of a triangle

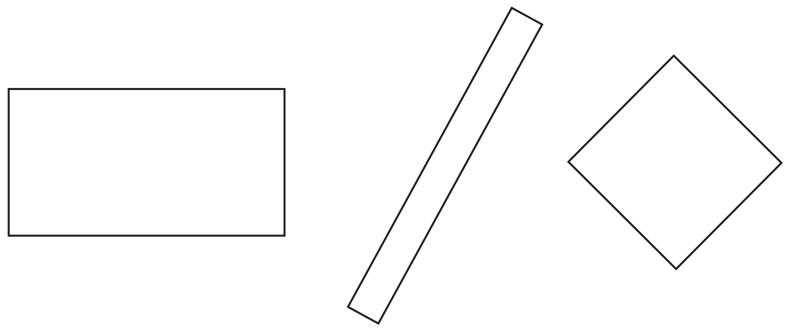

Figure 1.3 Examples of rectangles in different orientations

(e.g. on, in, symmetrical pattern, repeating pattern) in Reception through to more technical terms such as axis of symmetry, reflective symmetry, parallel, perpendicular, rotation, acute, obtuse and reflex in Year 6.

As the RS/JMC report indicates, this is a reasonably good geometry curriculum. The question to address now is how this curriculum is experienced by primary school pupils.

Children's experience of geometry in the NNS

According to the Office for Standards in Education (Ofsted 2002), the NNS had a positive impact on the standards attained in mathematics and on the quality of teaching in primary schools. For example, government pupil achievement targets were almost being reached and there were, says Ofsted, discernible increases in pupils' enjoyment of, and confidence and involvement in, mathematics lessons. In terms of pupil experience of geometry, an analysis of specially commissioned tests taken by pupils at a national sample of 300 schools (reported in Ofsted 2002: 7–8) indicated that among the topics with which pupils made the greatest amount of progress between 2001 and 2002 were 'recognition of squares and triangles' (Year 3) and 'finding the perimeter of a shape' (Year 4). Geometric topics with which pupils were still having the greatest difficulty were 'identification of parallel and perpendicular lines' and 'finding the coordinates of a missing vertex of a rectangle' (both Year 5 topics). At the same time, and as a result of the 2001 Key Stage 2 national tests taken by all 11-year-old pupils, the Qualifications and Curriculum Authority (QCA) recommended that teachers should give children opportunities to increase their familiarity with angle facts and with using precise geometrical terms for shapes, and associating these terms with their related properties (QCA 2002). The QCA also recommended that children needed more practice at working with shapes in less familiar orientations, as illustrated by the range of orientations given in Figure 1.3, and experience of, for instance, reflecting shapes in lines that are not solely parallel or perpendicular to a vertical mirror line.

While neither Ofsted nor the QCA appeared to comment on the balance of mathematics in the NNS, some mathematics teachers had the opportunity to do so in a survey of the perceived impact of the NNS at Key Stage 3 (Barnes *et al.* 2003: 38). While many secondary school teachers, the survey found, took the view that 'the constant practice of number work is necessary and beneficial', many others argued that equally significant aspects of the mathematics curriculum were being sidelined through the emphasis on number. The report noted that 'the profile and teaching of shape and space, and the proficiency of pupils in this area on entry to Year 7, were raised [by Key Stage 3 teachers] as particular concerns'.

Indicators of the profile of geometry within the NNS can be gleaned through examining the relative number of *key* teaching objectives devoted to geometry and by noting the number of hours allocated to geometry in the suggested Yearly Teaching Programme. For example, in the *Framework* only around 10 per cent of the Key Objectives are devoted to geometrical/spatial thinking and reasoning. As a result, between four and eight times as much number/calculation work (depending on the age of the pupils) is recommended as compared with geometrical/spatial work. In terms of the percentage of mathematics teaching time that should be devoted to geometry, the RS/JMC report (2001: 13) recommends that for secondary age pupils, 'geometry should occupy 25%–30% of the teaching time, and hence a similar proportion of the assessment weighting'. Even being generous with what can be counted as geometrical in the NNS *Framework*, the amount of geometry suggested at the primary level falls very much short of this, being about 12 per cent of the teaching time across the whole of the primary years, and in Year 2 falling to just 7 per cent (just at the time pupils take their first national tests).

This lack of priority for geometry within the NNS is also evidenced in the structure of the three-part daily mathematics lesson. The first part of the suggested lesson structure is entitled 'Oral work and mental *calculation*' (emphasis added) (DfEE 1999b: 13), with each of the examples of an oral and mental starter given within Section 1 being based either on recall of number facts or of number calculations. The implication of this concentration on calculation is that mental geometry is neither feasible nor, indeed, desirable while, given the importance of spatial and visual thinking, just the opposite is the case – see Figure 1.4 for a suggestion of a useful mental geometry activity. The emphasis in the NNS on oral and mental *calculation* is further highlighted within the objectives given in the Yearly Teaching Programmes (DfEE 1999b: Section 3). Strategies for the development of rapid recall of number facts and mental calculation are clearly identified in discrete sections. Mental geometry is not given this status. Despite the modification of 'oral and mental *calculation*' to 'oral and mental *starter*' in some of the NNS training material, within the NNS documentation as a whole almost the only guidance on geometry is given within the supplementary training document *Shape and Space Activities*, yet even here it is revealing that a number of the suggested pupil activities are, in fact, number and/or algebra activities simply contextualized within geometry.

A particularly influential way in which pupils experience the curriculum is through the commercial mathematics scheme, or set of textbooks, used in

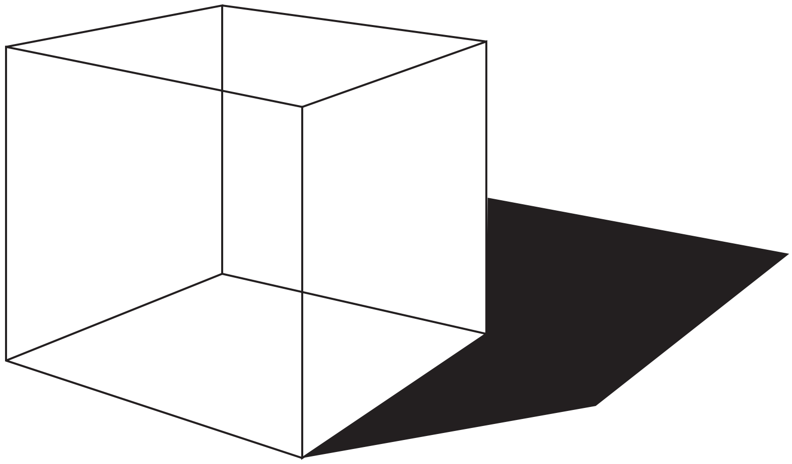

Figure 1.4 Mental geometry: what shadows can a cube cast?

their classrooms. A small-scale survey of such schemes carried out in preparation for this chapter suggests that the majority of those used within schools in England reflect the priorities of the NNS in giving far greater prominence to work on number as compared to work on geometry. For instance, many commercially produced schemes have multiple books devoted to developing number, compared with maybe only one containing the whole of geometry and measures (and frequently data handling too).

Overall, all the above evidence suggests that the advice to pupils should be 'do not be ill at primary school or you might easily miss whole sections of the geometry curriculum'! While this lack of emphasis on geometry at the primary level is replicated in many countries, it is important to re-examine the treatment of geometry, beginning with what can be gleaned from the research that has been carried out.

Evidence from research

There is a considerable amount of research on the teaching and learning of geometry. It is neither feasible nor sensible to attempt to summarize it all. Rather, major themes are identified below with a view to examining the aspects of geometry that the NNS has omitted or to which it has paid little attention (for details of references to this research, see Appendix 14 of the RS/JMC report 2001).

Piaget argued that the progressive organization of geometric ideas in children follows a definite order and this order is more experiential (and possibly more mathematically logical) than it is a reflection of the historical development of geometry. For example, although topology is a recently developed area of mathematics, Piagetian research suggests that, for learners, topological relations, such as connectedness, enclosure and continuity, are formed first. After this come the ideas of rectilinearity (such as the outline of objects) associated with projective geometry. Finally, the child is ready to acquire Euclidean notions of angularity, parallelism and distance. At best, this suggested learning sequence has received mixed support from research. The available evidence indicates that all types of geometric ideas appear to develop over time, becoming increasingly integrated and synthesized as children progress.

Another learning sequence, suggested by van Hiele (1986), puts identifying shapes and figures according to their concrete examples at the first ('visual') level for learners. At the second ('descriptive') level, learners identify shapes according to their properties (e.g. that a rhombus is a figure with four equal sides). At the third ('abstract') level, students can identify relationships between classes of figures (e.g. that a square is a special form of rectangle) and can discover properties of classes of figures by simple logical deduction. At the fourth ('formal') level, students can produce a short sequence of statements to logically justify a conclusion and can understand that deduction is the method of establishing geometric truth. Available research, while generally supportive of this model, has identified various problems with the specification of the levels. Particular problems include the labelling of the lowest level as 'visual', when visualization is demanded at all the levels, and the fact that learners appear to show signs of thinking from more than one level in the same or different tasks, in different contexts.

Neither the Piagetian nor the van Hiele models are strongly evident in the *Framework*. This is especially true of the Piagetian model as the *Framework*, reflecting the National Curriculum, makes no mention of topology or of projective geometry even though such ideas can be accessible at the primary level.[1] In terms of the van Hiele model, the *Framework* could be said to include the progression from description to classification but the heavy emphasis on descriptive language, at the expense of *geometrical* problem solving, which is given relatively little attention in the NNS, is likely to mean that children's progression will be somewhat limited.

Overall, research on the teaching and learning of geometry indicates that physical experience, especially the physical manipulation of shapes, is important at all ages, that a wide variety of geometrical experiences are necessary in order for pupils to gain a firm understanding of geometrical relationships and that computer packages such as *Logo* and dynamic geometry software, while not being a panacea, have much to offer. While all these things are present in the NNS to some degree, the lack of time apportioned to geometry is likely to mean that none of them are fully realized. Given the importance of geometry within mathematics, the next section offers some idea of what it might mean to give proper space to geometry within the primary mathematics curriculum.

Making space for geometry in the NNS

When describing what they see as the impoverished nature of school geometry at the primary level in the USA, Battista and Clements (1988: 11) note that the poor performance of primary age pupils in geometry 'is due, in part, to the current elementary school geometry curriculum, which focuses on recognizing and naming geometric shapes and learning to write the proper symbolism for simple geometric concepts'. To remedy this situation they argue that geometry at the primary level should be 'the study of objects, motions, and relationships in a spatial environment'. This means that pupils' first experiences with geometry should emphasize the informal study of physical shapes and their properties and have as their primary goal the development of students' intuition and knowledge about their spatial environment. Subsequent experiences should involve analysing and abstracting geometric concepts and relationships in increasingly formal settings.

While the NNS could be said to attempt to provide children with an appropriate grounding in geometrical ideas, it is limited by the amount of curricular time it allocates to geometry. As a result, opportunities are missed which could mean that children do not make the kind of progress envisaged. For example, the NNS could be seen to provide a strengthening of the treatment of transformation geometry as, by the end of their primary schooling, children should be confident with rotations, reflections and translations. Yet the link between symmetry and the various transformations is not always made explicit. In the *Framework*, for instance, rotation appears to be considered solely as a transformation and the opportunity is missed to extend this to include rotational symmetry, even though the latter is specified in the statutory National Curriculum.

Other ways in which opportunities are missed occur when so few of the *Framework*'s 'supplements of examples' encompass geometrical problem solving (most of the examples use terms like 'recognize', 'identify', 'describe') and when connections are not made between geometrical ideas. For example, the *Framework* appears to make no mention of tessellations, yet this area of geometry can provide an intuitive visual foundation for a variety of geometric content that can be treated more formally in a deductive manner at a later stage.

Studying tessellations, as Figure 1.5 illustrates, can be a springboard into the angle properties of regular and irregular polygons and because each tile has to be identical and can be made to fit onto any other tile exactly (by means of translations, rotations or reflections), pupils can be introduced, intuitively, to the concept of *congruency*. Pupils can also be encouraged to look for larger figures with the same shape, thus intuitively introducing them to the concept of *similarity*. Of course, the *Framework*, in its supplement of examples, does provide illustrations of how, for instance, as an outcome of 'recognizing translations', pupils might 'make patterns by repeatedly translating a shape' (DfEE 1999b: 106). Yet not only does the *Framework* fail to mention that the outcome of repeatedly translating a shape can be a tessellation, it also signally fails to develop the example so that it becomes possible to make the sorts of links

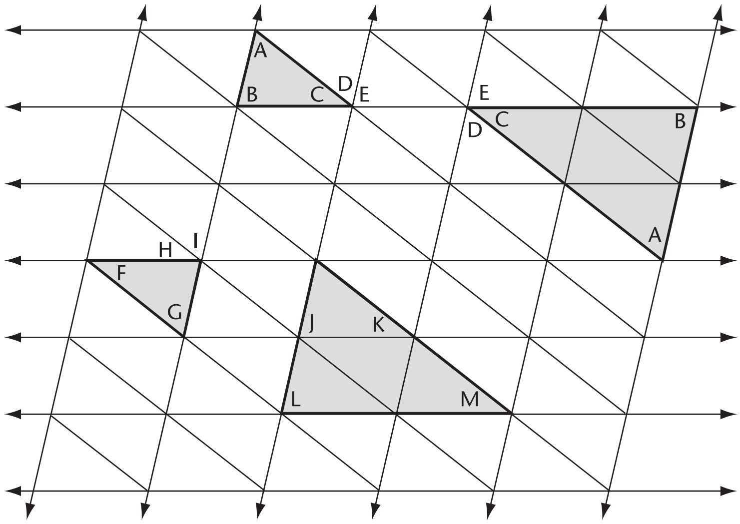

Figure 1.5 Analysing a tessellation

Make a building out of ten cubes by looking at the three pictures of it below.

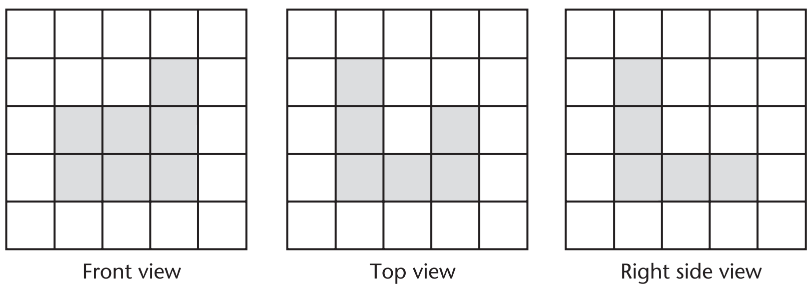

Figure 1.6 An example of a 3D problem

between geometrical ideas, such as those indicated above, that are essential if pupils are going to progress.

Other geometrical topics where omissions are likely to limit pupil development include the limited treatment of 3D geometry, where the emphasis appears to be on recognizing and describing rather than on solving problems like the example in Figure 1.6, and where some topics, such as initial ideas about planes of symmetry, and about polyhedra, could have been included. Further examples of omissions are the lack of some experience of early topology and the failure to include suitable applications of geometry (such as in various mechanisms).

Conclusion

No doubt the development of the NNS was heavily influenced by the work of the Numeracy Task Force (established by the UK government in May 1997), with its explicit focus on the teaching of number skills. Yet in comparing the statutory requirements for geometry, as laid out in the National Curriculum for mathematics, with the suggested time allocations recommended within the NNS *Framework*, it is clear that it is extremely difficult to adequately address all the geometry requirements within the time allowed (146 hours of geometry across the whole of the primary years, out of 1200 hours of mathematics altogether). Given the vitally important role of geometry, and the emerging concerns voiced by Key Stage 3 teachers about the lack of geometrical skills of incoming secondary pupils (Barnes *et al.* 2003), it would be sensible to research the impact of the allocation of time given to geometry within the NNS to ensure that it is not failing to enable children to develop clear and lasting geometrical knowledge, skills and understanding. In addition, it would also be helpful if further studies of the development of spatial thinking and geometrical visualizing were carried out with a view to making the specification of mental geometry in the NNS documentation far more explicit, probably in ways similar to the prominence given to oral and mental calculation. On the available evidence it is likely that, until spatial and visual thinking is given greater status within the mental and oral segment of the daily mathematics lesson, and until more curriculum space is devoted to geometry, children may well continue to have insufficient opportunity to develop the fundamental visualization and spatial reasoning skills that are so important in an increasingly visual world.

The success of any revised NNS *Framework* that gives greater prominence to geometry would inevitably depend, to a large extent, on the expertise of the teachers who teach it and the teaching methods that they use. In respect of the latter, the RS/JMC report (2001: 11) affirms that 'in many respects, we need to develop a completely new pedagogy in geometry', something on which teachers and researchers can collaborate in devising and evaluating. In respect of the former, ongoing research is suggesting that geometry is the area of mathematics in which prospective primary teachers have the most to learn (Jones *et al.* 2002). The urgent need for professional development opportunities for teachers in the area of geometry is stressed by the RS/JMC report and supported by the recommendation from the Advisory Committee on Mathematics Education (2002: 2) that 'the Government should initiate urgently the process of developing and funding a long-term programme of CPD [continuing professional development] for teachers of mathematics that can meet their needs at various stages of their careers'. Such an initiative is needed to help ensure that any improvement in the treatment of geometry in primary mathematics is a success.

Note

1 The mental geometry activity suggested in Figure 1.4, for example, can be used to introduce, at the intuitive level, ideas of affine geometry (where parallelism is preserved) and projective geometry (where parallelism is not preserved), in that if the cube is held so that one face is parallel to a screen, the transformation from shape to screen (in the form of the shadow) is an affine transformation, while if a face is not held parallel, the transformation is projective.

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