Jones, K. (2002), Issues in the Teaching and Learning of Geometry. *In*: Linda Haggarty (Ed), *Aspects of Teaching Secondary Mathematics: perspectives on practice*. London: RoutledgeFalmer. Chapter 8, pp 121-139. ISBN: 0-415-26641-6

8 Issues in the teaching and learning of geometry

Keith Jones

Preamble

In the Sherlock Holmes story, *The Adventure of the Priory School* (written by the British writer Arthur Conan Doyle and published in 1904), the great detective and his assistant, Dr Watson are examining some bicycle tracks[1]. The following conversation takes place:

Sherlock Holmes — This track, as you perceive, was made by a rider who was going from the direction of the school.

Dr Watson — Or towards it?

Sherlock Holmes — No, no, my dear Watson. The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school.

Holmes' insight of examining the depth of the tracks made by the bicycle is undoubtedly the most straightforward thing to do. However, by using some knowledge of geometry, Holmes could have found that it is not necessary to have information on the depth of the tracks to discover the direction in which the bicycle had travelled. See if you can work out which way the bicycle went in task 8.1, below.

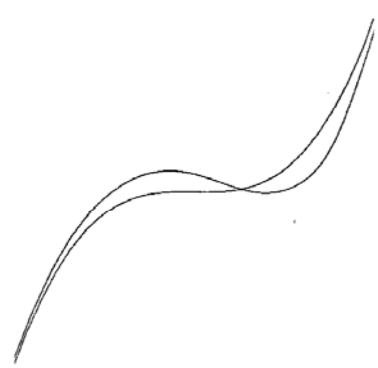
Task 8.1

These tracks either fit a bicycle moving forward *up* the page or *down* the page (but not both). Which is it?

Hints:

- Think about which of the two paths will fit the rear wheel and which will fit the front wheel.
- Think about how bicycle wheels sit along the tracks at any time.

An enlargement of the illustration may be helpful.



Which way did the bicycle go?

Introduction

Geometry is a wonderful area of mathematics to teach. It is full of interesting problems and surprising theorems. It is open to many different approaches. It has a long history, intimately connected with the development of mathematics. It is an integral part of our cultural experience being a vital component of numerous aspects of life from architecture to design (in all its manifestations). What is more, geometry appeals to our visual, aesthetic and intuitive senses. As a result it can be a topic that captures the interest of learners, often those learners who may find other areas of mathematics, such as number and algebra, a source of bewilderment and failure rather than excitement and creativity. Teaching geometry well can mean enabling more students to find success in mathematics.

These aspects and considerations also tend to make geometry a demanding topic to teach well. Teaching geometry well involves knowing how to recognise interesting geometrical problems and theorems, appreciating the history and cultural context of geometry, and understanding the many and varied uses to which geometry is put. It means appreciating what a full and rich geometry education can offer to learners when the mathematics curriculum is often dominated by other considerations (the demands of numeracy and algebra in particular). It means being able to put over all these things to learners in a way that is stimulating and engaging, and leads to understanding, and success in mathematics assessments.

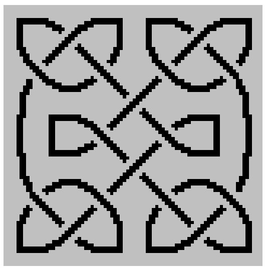
The aim of this chapter is introduce some of the special features of geometry and its teaching and learning. The chapter examines the nature of geometry, the reasons for it being included in the school mathematics curriculum, and how it can be best taught and learnt. The chapter contains a number of tasks that you might like to tackle, either at the point they occur in the chapter or at some later, convenient, time. Hopefully, you might be inspired to try some of them out with learners or with professional colleagues. At the end of the chapter there are commentaries and/or hints on many of the tasks.

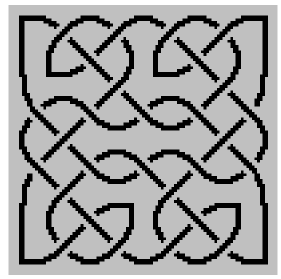
What is geometry?

The word ‘geometry’ comes from two ancient Greek words, one meaning earth and the other meaning to measure. These Greek words, as well as the word ‘geometry’, may themselves be derived from the Sanskrit word ‘Jyamiti ’ (in Sanskrit, ‘Jy a’ means an arc or curve and ‘Miti’ means correct perception or measurement). The origins of geometry are very ancient (it is probably the oldest branch of mathematics) with several ancient cultures (including Indian, Babylonian, Egyptian, and Chinese, as well as Greek) developing a form of geometry suited to the relationships between lengths, areas, and volumes of physical objects. In these ancient times, geometry was used in the measure of land (or, as we would say today, surveying) and in the construction of religious and cultural artefacts. Examples include the Hindu Vedas, thought to have been composed between 4000 BCE[2] to 3100 BCE, the ancient Egyptian pyramids, Celtic knots (see task 8.2), and many more examples. Sources of further information on the geometry of some religious and cultural artefacts are given in the commentary to task 8.2.

Task 8.2

The Celts were a dominant force in Europe during the 4th and 5th centuries CE. Intricately–designed jewellery, and illustrations in texts such as the Book of Kells, survive as examples of Celtic knot patterns.





- Work out how to construct these relatively simple Celtic knot patterns.
- Find other examples of the use of geometry in cultural and religious artefacts.

Around 300 BCE much of the accumulated knowledge of geometry was codified in a text that became known as Euclid's *Elements*. In the 13 books that comprise the *Elements*, and on the basis of 10 axioms and postulates, several hundred theorems were proved by deductive logic. The *Elements* came to epitomise the axiomatic-deductive method for many centuries. It is likely that no other works, except perhaps the Christian Bible and the Muslim Koran, have been more widely used, edited, or studied, and probably no other work has exercised a greater influence on scientific thinking. While some parchments do exist from the 9th century, it is said that over a thousand editions of Euclid's *Element*s have appeared since the first printed edition in 1482, and for more than two millennia this work dominated all aspects of geometry, including its teaching.

In the nineteenth century, geometry, like most academic disciplines, went through a period of growth that was near cataclysmic in proportion. Since then the content of geometry and its internal diversity has increased almost beyond recognition. The geometry of Euclid became no more than to a subspecies of the vast family of mathematical theories of space. If you do a search for geometry using the web version of the *Encyclopedia Britannica* (http://www.britannica.com/), you get the following message: did you mean differential geometry, hyperbolic geometry, Lobachevskian geometry, projective geometry, elliptic geometry, algebraic geometry, Euclidean geometry, analytic geometry, plane geometry, Riemannian geometry, or co-ordinate geometry? It is possible today to classify more than 50 geometries (see: Malkevitch 1991). This illustrates the richness of modern geometry but, at the same time, creates a fundamental problem for curriculum designers: what geometry should be included in the mathematics curriculum?

Task 8.3

Find out something about the contribution of some (or all) of these mathematicians to the development of geometry:

René Descartes	Isaac Newton	Leonhard Euler
Max Noether	Victor Poncelet	Bernhard Riemann
Lobachevsky and Bolyai	Felix Klein	Sophus Lie
Luitzen Brouwer		

The question of what geometry to include can be applied to the curriculum at any level, from pre-school to (University) graduate school. In order to approach this problem it is worth returning to the question of what is geometry and also to consider the aims of teaching geometry.

A useful contemporary definition of geometry is that attributed to the highly-respected British mathematician, Sir Christopher Zeeman: "geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight" (Royal Society/JMC 2001). These are transferable skills that are needed for (but not taught by) all other branches of mathematics (and science)". The Royal Society/JMC report suggests that the aims of teaching geometry can be summarised as follows:

- to develop spatial awareness, geometrical intuition and the ability to visualise;
- to provide a breadth of geometrical experiences in 2 and 3 dimensions;
- to develop knowledge and understanding of and the ability to use geometrical properties and theorems;
- to encourage the development and use of conjecture, deductive reasoning and proof;
- to develop skills of applying geometry through modelling and problem solving in real world contexts;
- to develop useful ICT skills in specifically geometrical contexts;
- to engender a positive attitude to mathematics; and
- to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

Given the above definition of geometry, and a consideration of the aims of teaching geometry, it is possible to say why it should be included in the school mathematics curriculum.

Why include geometry in the school mathematics curriculum?

The study of geometry contributes to helping students develop the skills of visualisation, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proof. Geometric representations can be used to help students make sense of other areas of mathematics: fractions and multiplication in arithmetic, the relationships between the graphs of functions (of both two and three variables), and graphical representations of data in statistics. Spatial reasoning is important in other curriculum areas as well as mathematics: science, geography, art, design and technology. Working with practical equipment can also help develop fine motor skills.

Geometry provides a culturally and historically rich context within which to do mathematics. There are many interesting, sometimes surprising or counter-intuitive results in geometry that can stimulate students to want to know more and to understand why. Presenting geometry in a way that stimulates curiosity and encourages exploration can enhance student's learning and their attitudes towards mathematics. By encouraging students to discuss problems in geometry, articulate their ideas and develop clearly structured arguments to support their intuitions can lead to enhanced communication skills and recognition of the importance of proof. The contribution of mathematics to student's spiritual, moral, social and cultural development can be effectively realised through geometry. As mentioned above, some ideas for using geometry to support spiritual and cultural development can be found in the references included in the commentary to task 8.2. Useful sources of material for supporting moral and social development can be found in the publications of the Stapleford Centre in Nottingham (for example, the Charis Mathematics resources for key stages 3 and 4), the "Summing up the World" series from Development Education in Dorset, and the Maths and Human Rights Resources Book published by Amnesty International. See appendix 1 for more details of other resources to support to the teaching and learning of geometry.

Geometry is a rich source of opportunities for developing notions of proof. While more is said about this in a later section, it is worth emphasising that visual images, particularly those, which can be manipulated on the computer screen, invite students to observe and conjecture generalisations. Proving conjectures requires students to understand how the observed images are related to one another and are linked to fundamental 'building blocks'. In dynamic geometry software (see task 8.6 and appendix 1) understanding observed images means working with points, circles, and parallel and perpendicular lines. In the programming language Logo, it involves understanding the way in which the 'turtle' moves (see task 8.7 and appendix 1).

We live on a solid planet in a 3D world and, as much of our experience is through visual stimulus, this means that the ability to interpret visual information is fundamental to human existence. To develop an understanding of how spatial phenomena are related and to apply that understanding with confidence to solve problems and make sense of novel situations has to be part of the educational experience of all students. Geometry offers a rich way of developing visualisation skills. Visualisation allows students a way of exploring mathematical and other problems without the need to produce accurate diagrams or use symbolic representations. Manipulating images in the head can inspire confidence and develop intuitive understanding of spatial situations. Sharing personal visual images can help to develop communication skills as well as enabling students to see that there are often many ways of interpreting an image or a written or spoken description.

Much of our cultural life is visual. Aesthetic appreciation of art, architecture, music and many cultural artefacts involves geometric principles – symmetry, perspective, scale, orientation, and so on. Understanding many scientific principles and technological phenomena also requires geometric awareness, as do navigation, orienteering and map reading. Recognition of the familiar and the unfamiliar requires an ability to characterise and note key features.

Numerous current applications of mathematics have a strong geometric component. In many cases, the problem includes getting 'geometric' information into a computer in a useful format, solving geometric problems, and outputting this solution as a visual or spatial form, as a design to be built, as an action to be executed, or as an image to entertain. Solving these problems requires substantial geometric knowledge. Here, briefly, are a few illustrative examples as suggested by Whitely (1999):

Computer Aided Design and Geometric Modeling: A basic problem is to describe, design, modify, or manufacture the shapes we want: cars, planes, buildings, manufactured components, etc. using computers. The descriptions need to be accurate enough to directly control the manufacturing and to permit simulation and testing of the objects, usually prior to making any physical models. Indeed, for example, the most recent Boeing aeroplane was entirely designed using computers, without the use of any physical models.

Robotics: To use a robot, we must input (using cameras, sensors, etc) a geometric model of the environment. The whole issue of what geometric vocabulary is used (e.g. solid modeling, polyhedral approximations, etc.), and how the information is structured is a major area of research in a field called 'computational geometry'.

Medical Imaging: Generating non-intrusive measurements (usually some form of picture) requires the construction of an adequate three-dimensional image of parts of the body. This can involve, for example, a series of projections or images from ultra-sound, or magnetic resonance imaging (MRI) from several directions or points. This raises questions about how many measurements are needed to construct the full three-dimensional image and what algorithm can be used to reconstruct the full image from the pieces. Such questions have led to some substantial new results in fields like geometric tomography.

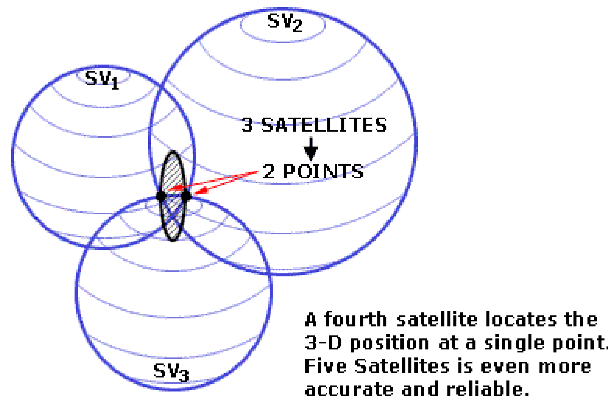
Computer Animation and Visual Presentations: How can the computer generate sufficiently rich images to fool our human perceptions of the static form and the moving objects? One of the computer scientists/ geometers who worked on the full-length animation video 'A Bug's Life' described it as an "exercise in handling texture and modeling clothing with new levels of mathematics". New mathematics with a geometric base, such as fractals, is a part of this work. So is geometric modeling.

Further areas where geometric problems arise are in chemistry (computational chemistry and the shapes of molecules), material physics (modeling various forms of glass and aggregate materials), biology (modeling of proteins, 'docking' of drugs on other molecules), Geographic Information Systems (GIS), and most fields of engineering. Task 8.4 provides an opportunity to explore one of these developments.

Task8. 4

GPS, or global positioning system, is a satellite navigation system operated by the US Department of Defense.

- Find out how GPS works.
- Investigate another modern application of geometry.



As shown above, there are a whole host of reasons why geometry should be a major part of the learning experience of mathematics at all levels. There is also a whole host of geometry to consider. The question remains: what geometry should be included in the mathematics curriculum? We examine this question more closely in the next section.

The geometry curriculum

Mention geometry and school in the same sentence to anyone who completed secondary school before 1970, and this meant only one thing: geometry in the Euclidean tradition. In fact, up to around the turn of the 20th century, for those who were able to attend school, it meant Euclid as written by Euclid, perhaps in an English translation but possibly as translated into Latin (probably dating from the 15th or 16th century), or even in the original Greek. Given that the books of Euclid were primarily an orderly compilation of what was known about geometry and arithmetic at the time they were written (around 300 BCE), and not a teaching programme as we know it today, the use of Euclid's *Elements* as a school textbook was not without problems. Indeed, the forerunner of the UK Mathematical Association was formed in 1871 as the Association for the Improvement of Geometry Teaching. A major issue of the time was whether or not any required proof *had* to be reproduced by students *exactly* in the form given in Euclid (including in the order the proof occurred in Euclid). For very many pupils their experience of geometry was far from positive.

As Euclidean geometry lost its status as the only geometry, following the work on other geometries, it became, by the middle of the 20th century, of little more than historic interest at University research level. Other geometries became the object of research. Then, in the wake of the launch of the Sputnik by the Soviets in 1957, a major revision of school mathematics (and science) was begun in most western countries. One of the reform ideas was to base much more of school mathematics on the idea of function and to aim more at the mathematics that would lead to calculus and linear algebra. The room for this innovation was made by reformulating all parts of the mathematics curriculum, but the practical effect seemed to be to remove solid geometry and to convert the trigonometry component into part of a course about functions. The impact of these changes was to reduce the amount of geometry while, at the same time, increasing the emphasis on co-ordinate geometry and introducing some elements of transformation geometry and topology. As a result, the amount of geometry taught in the Euclidean fashion probably became much less.

For students completing secondary school in the UK since around 1970, their geometrical experience is likely to be quite varied. While it is difficult to be precise about the form of geometry teaching practiced at that time (due to a lack of good evidence), a review of geometry in the mathematics curriculum published in 1977 (Willson 1977 p.*vi*) set out "to show that the introduction of modern transformation geometry does not rule out the teaching of more traditional Euclidean type proofs, and indicates some of the many fruitful points of contact between the two areas". Thus it is likely that geometry in the Euclidean fashion persisted in some places, while in others approaches based on transformations dominated. Popular textbooks of the time, including those produced by the School Mathematics Project, favoured the transformations approach.

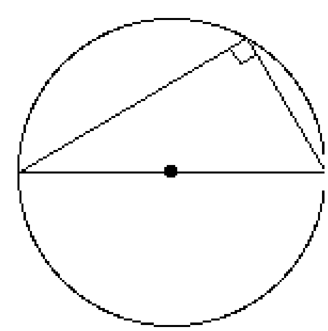
To illustrate that various approaches can be used to prove theorems in plane geometry, you might like to tackle task 8.5. The aim of the task is to see how many different, but valid, proofs you can find for the theorem that the angle in a semicircle is invariant and that it is equal to 90^o. The theorem is sometimes referred to as Thales' theorem, after Thales of Miletus in Greek Ionia who died in about 546 BCE and is credited with the first known proof in mathematics (but note that there are other theorems that can be known as Thales' theorem).

Task 8.5

One of the surprising results in plane geometry is that the angle in a semicircle is invariant and that it is equal to 90^o.

- See how many proofs you can find for this theorem.

Your proofs could be Euclidean, use transformation geometry, analytic (co-ordinate) geometry, complex numbers, or vectors.



The introduction of the National Curriculum for school mathematics in the UK in 1988 reflected the developing practice in the teaching of geometry. The specification focused primarily on plane geometry, covering aspects of the Euclidean, transformation and co-ordinate approaches to geometrical objects and problems. There was also a little bit of topology. Reflecting this wide view of geometry, the specification was called “Shape and Space” (and measures).

Since 1988, the curriculum has been revised a number of times. If one wanted to be particularly unkind one could conclude that the 2000 version of the curriculum for shape, space and measure for England is a strange mixture of smatterings of antique geometric construction (using a ruler and protractor is required by law!), 19th century arithmetic, 1950s grammar school geometry, and 1970s SMP-style transformation geometry. Co-ordinate geometry, by contrast, is negligible as is explicit development of spatial awareness and skills in visualisation. A more generous perspective might be to observe that, given the rich diversity that is geometry (as mentioned above, it is possible to categorise more than 50 geometries within mathematics), the 11-16 age-range curriculum attempts to introduce pupils to a range of important approaches to the study of the Cartesian plane.

The situation for students in the 16-19 age-range is such that there is even less geometry than in the 11-16 mathematics curriculum. The core curriculum specification for 2000 onwards has a section entitled co-ordinate geometry but the specification is actually about the algebraic formulae for straight lines and circles. Similarly, while there is a section headed vectors, the specification is predominantly algebraic (with “geometrical interpretation”). Information is less readily available about the amount of geometry taught in higher education. Some University mathematicians have called for more geometry to be taught at University level (and by geometers rather than logicians or historians of mathematics). It is unclear how much success they have had (for example, see Mathematical Association, 1993). It is certainly unhelpful to expect teachers to teach geometry competently given that they are likely to have studied little geometry since the age of 16 (and may be not much even then).

There is little doubt that the driving force behind curricular decisions in high school mathematics is the goal of preparing students for the study of calculus. A great deal of the manipulative skill in algebra, trigonometry, and analytic geometry is clearly prized because of its usefulness in calculus, or at least calculus as it has been traditionally conceived. Geometry, in its many guises, gets neglected, with spatial intuition and visualisation being particularly so. An emphasis on numeracy, through initiatives such as the National Numeracy Strategy, may only serve to reduce the coverage of geometry in schools just at the time when geometrical education has so much to offer the education of students.

Given the curriculum, we now turn to how best to teach it.

Teaching and learning geometry

There is a considerable amount of research in mathematics education that concerns the teaching and learning of geometry. It is neither sensible nor feasible to attempt to summarise it all (for a comprehensive review, see Clements 2001). Instead, a selection of issues is addressed below covering theories of geometric thinking, learning, and teaching.

Theories of learning geometry

Of the range of theoretical work concerned with geometrical ideas, that of Piaget (and colleagues) and of the van Hiele's are probably the most well-known. The Piagetian work has two major themes. The first theme is that our mental representation of space is not a perceptual "reading off" of what is around us. Rather, we build up from our mental representation of our world through progressively reorganising our prior active manipulation of that environment. Second, the progressive organisation of geometric ideas follows a definite order and this order is more experiential (and possibly more mathematically logical) than it is historical. That is, initially topological relations, such as connectedness, enclosure, and continuity, are constructed, followed by projective (rectilinearity) and Euclidean (angularity, parallelism, and distance) relations. The first of these Piagetian themes, concerning the process of the formation of spatial representations, remains reasonably well-supported by research. The second hypothesis has received, at best, mixed support. The available evidence suggests that all types of geometric ideas appear to develop over time, becoming increasingly integrated and synthesised.

The van Hiele model also suggests that learners advance through levels of thought in geometry. Van Hiele characterised these levels as visual, descriptive, abstract/relational, and formal deduction. At the first level, students identify shapes and figures according to their concrete examples. At the second level, students identify shapes according to their properties, and here a student might think of a rhombus as a figure with four equal sides. At the third level, students can identify relationships between classes of figures (for example, that a square is a special form of rectangle) and can discover properties of classes of figures by simple logical deduction. At the fourth level, students can produce a short sequence of statements to logically justify a conclusion and can understand that deduction is the method of establishing geometric truth. According to this model, progress from one of Van Hiele's levels to the next is more dependent upon teaching method than on age. Given traditional teaching methods, research suggests that most lower secondary students perform at levels one or two with almost 40% of students completing secondary school below level two. The explanation for this, according to the van Hiele model, is that teachers are asked to teach a curriculum that is at a higher level than the students.

According to the van Hiele model it is not possible for learners to bypass a level. They cannot see what the teacher sees in a geometric situation and therefore do not gain from such teaching. While research is generally supportive of the van Hiele levels as useful in describing students' geometric concept development (in the absence of anything better), it remain uncertain how well the theory reflects children's mental representations of geometric concepts. Various problems have been identified with the specification of the levels. For example that the labelling of the lowest level as "visual" when visualisation is demanded at all the levels, and the fact that learners appear to show signs of thinking from more than one level in the same or different tasks, in different contexts. An integral component of the van Hiele model is a specified teaching approach involving four phases. There is little research on this aspect of the model and hence little idea if it is successful.

Key ideas in teaching and learning geometry

In order to teach geometry most effectively, and give some coherence to classroom tasks, it is helpful if, in your preparation and teaching you keep in mind, and highlight where appropriate, key ideas in geometry. These include:

Invariance: in 1872, the mathematician Felix Klein revolutionised geometry by defining it as the study of the properties of a configuration that are invariant under a set of transformations. Examples of invariance proposition are all the plane angle theorems (such as Thales' theorem in task 8.5), and the theorems involving triangles (such as the sum of the angles of a plane triangle is 180^0). Pupils do not always find it straightforward to determine which particular properties are invariant. The use of dynamic geometry software (see task 8.6) can be very useful in this respect.

Symmetry: symmetry, of course, is not only a key idea in geometry but throughout mathematics, yet it is geometry that it achieves its most immediacy. Technically, a symmetry can be thought of as a transformation of a mathematical object which leaves some property invariant. Symmetry is frequently used to make arguments simpler, and usually more powerful. An example from plane geometry is that all of the essential properties of a parallelogram can be derived from the fact that a parallelogram has half-turn symmetry around the point of intersection of the diagonals. Symmetry is also a key organising principle in mathematics. For example, probably the best way of defining quadrilaterals (*except* for the general trapezium, which is not an essential quadrilateral in any case, since there are no interesting theorems involving the trapezium that do not also hold for general quadrilaterals), is via their symmetries.

Transformation: transformation permits students to develop broad concepts of congruence and similarity and apply them to all figures. For example, congruent figures are always related either by a reflection, rotation, slide, or glide reflection. Studying transformations can enable students to realise that photographs are geometric objects, that all parabolas are similar because they can be mapped onto each other, that the graphs of y = cos x and y = sin x are congruent, that matrices have powerful geometric applications, and so on. Transformations also play a major role in artwork of many cultures - for example, they appear in pottery patterns, tilings, and friezes.

The teaching and learning of proof in geometry

While the deductive method is central to mathematics and intimately involved in the development of geometry, providing a meaningful experience of deductive reasoning for students at school appears to be difficult. Research invariably shows that students fail to see a need for proof and are unable to distinguish between different forms of mathematical reasoning such as explanation, argument, verification and proof. For example, a large-scale survey in the US found that only about 30% of students completing full-year geometry courses that taught proof reached a 75% mastery level in proof writing. Even high-achieving students have been found to get little meaningful mathematics out of the traditional, proof-oriented high school geometry course.

Corresponding difficulties with proof have also been found with mathematics graduates. A number of reasons have been put forward for these student difficulties with proof. Amongst these reasons are that learning to prove requires the co-ordination of a range of competencies each of which is, individually, far from trivial, that teaching approaches tend to concentrate on verification and devalue or omit exploration and explanation, and that learning to prove involves students making the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures. Further reasons are that students are asked to prove using concepts to which they have just been introduced and to prove things that appear to be so obvious that they cannot distinguish by intuition the given from what is to be proved.

Nevertheless, despite the sheer complexity of learning to prove and the wealth of evidence suggesting how difficult it can be for students, there are a few studies that show that students can learn to argue mathematically. One promising approach is that being developed by de Villiers (see de Villiers 1999). De Villiers points out that, in addition to explanation, proof has a range of functions, including communication, discovery, intellectual challenge, verification, systematisation, and so on. These various functions, de Villiers argues, have to be communicated to students in an effective way if proof and proving are to be meaningful activities for them. In fact, de Villiers suggests that it is likely to be meaningful to introduce the various functions of proof to students more or less in the sequence shown in figure 1.

Figure 8.1: *the likely learning sequence of functions of mathematical proof*

Focusing on explanation, de Villiers argues, should counteract students becoming accustomed to seeing geometry as just an accumulation of empirically discovered facts in which explanation plays no role.

Resources for the teaching and learning of geometry

There can be a tendency to teach geometry by informing students of the properties associated with plane or solid shapes, requiring them to learn the properties and then to complete exercises which show that they have learned the facts. Such an approach can mean that little attempt is made to encourage students to make logical connections and explain their reasoning. Whilst it is important that students have a good knowledge of geometrical facts, if they are to develop their spatial thinking and geometrical intuition, a variety of approaches are beneficial. For example, some facts can be introduced informally, others developed deductively or found through exploration.

To teach geometry effectively to students of any age or ability, it is important to ensure that students understand the concepts they are learning and the steps that are involved in particular processes rather than the students solely learning rules. More effective teaching approaches encourage students to recognise connections between different ways of representing geometric ideas and between geometry and other areas of mathematics. The evidence suggests that this is likely to help students to retain knowledge and skills and enable them to approach new geometrical problems with some confidence.

When planning approaches to teaching and learning geometry, it is important to ensure that the provision in the early years of secondary school encourages students to develop an enthusiasm for the subject by providing opportunities to investigate spatial ideas and solve real life problems. There is also a need to ensure that there is a good understanding of the basic concepts and language of geometry in order to provide foundations for future work and to enable students to consider geometrical problems and communicate ideas. Students should be encouraged to use descriptions, demonstrations and justifications in order to develop the reasoning skills and confidence needed to underpin the development of an ability to follow and construct geometrical proofs.

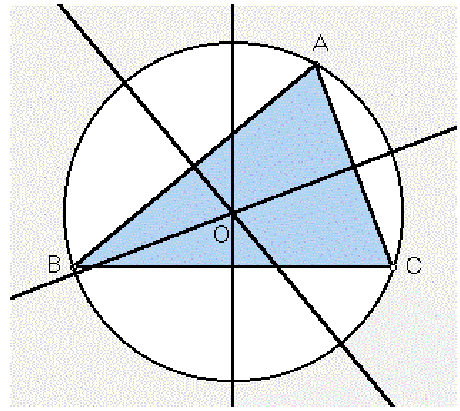
At Key Stage 4, for many students, the teaching of geometry requires similar teaching approaches to those used in earlier years. A formal, deductive approach to learning geometry needs to be treated with great care if it is to be appropriate for all pupils. With more able students it is possible to encourage a greater understanding of the need for definitions and of the laws of deductive logic. This can include notions of the place of axioms, an appreciation of the importance of proof, understanding of some proofs and the ability to construct simple proofs themselves. For all students, there should be an emphasis on problem solving involving real life applications of geometrical skills. See appendix 1 for some sources of materials on geometrical problem solving.

The geometry in the National Curriculum for Key Stages 3 and 4 can be taught making little use of practical resources but this is not necessarily the best way of doing so. It is useful to consider geometry as a practical subject and provide opportunities for students to use a range of resources to explore and investigate properties of shapes and geometrical facts. Particular consideration should be given to ways in which the ICT resources, which are increasingly available in schools, can be used to enhance the teaching and learning of geometry. The use of dynamic geometry enables the teacher or individual students to generate and manipulate geometrical diagrams quickly and explore relationships using a range of examples. Task 8.6 involves using dynamic geometry software to explore the various “centres” of a triangle. While use of such software can enliven geometry teaching it should be noted that it is not always clear what interpretations students make of geometrical objects encountered in this way (see, Jones 1999). There is also the possibility that the opportunity afforded by the software of testing a myriad of diagrams through use of the ‘drag’ function, or of confirming conjectures through measurements (that also adjust as the figure is dragged), may *reduce* the perceived need for deductive proof (Hoyles and Jones 1998).

Task 8.6

Use dynamic geometry software to construct the circumcircle of a triangle such that the figure is invariant when any object used in its construction is dragged (using the mouse).

- What other "centres" of the triangle can you find?

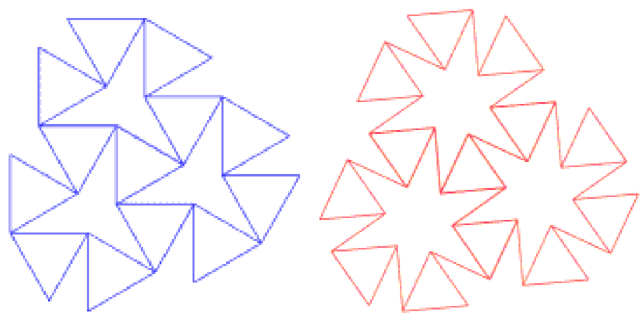


Other software can also be invaluable. Logo can be used to describe journeys and investigate properties of shapes (see task 8.7). Graph drawing packages are useful for co-ordinate and transformation geometry. All of these resources can be used individually by students or for whole class lessons using an interactive whiteboard.

Task 8.7

Use logo to construct designs such as these.

- What geometry as you doing?



Concluding Comments

Geometry is the part of the mathematics curriculum where it is possible to have the most fun. It is visual, intuitive, creative, and demanding. Use your imagination and tap into that of your pupils. Create striking classroom displays, suspend geometrical models from the ceiling of your classroom, involve your pupils in making things and imagining things, get them to decide on definitions and then explore the logical consequences. Too many students vote with their feet and give up studying mathematics. Geometry can help to keep them engaged.

New developments in computing technology mean that the 21st century will be one where spatial thinking and visualisation are vital. Geometry is where those all-important skills are nurtured. Engage with geometry yourself and get your pupils thinking geometrically.

Postscript: a few geometrical jokes

Here are a few jokes with a geometric flavour. They may amuse you and you might find them useful at some point (although there is no guarantee that anyone else will find them funny).

> What do you say when you see an empty parrot cage?
> *Polygon*
>
> What do you call a crushed angle?
> *A Rectangle*
>
> What do you call an angle that is adorable?
> *acute angle*
>
> What do you call a man who spent all summer at the beach?
> *Tangent*
>
> What do you call people who are in favour of tractors?
> *Protractors*

Commentaries and/or Hints on Tasks

*Task 8.1: Which way did the bicycle go?*If you have tried the hints given with this task and are still stuck then you are likely to be aware that this is not that easy a problem. Indeed, the idea is to challenge you and get you interested in such problems. Hopefully you have had the opportunity to discuss the problems with others. A further (big) hint is to think about the geometry of the bicycle frame. Bicycle wheels are, at all times, tangent to the curve they make as a track. The frame of the bike is a direct extension of the tangent vector from the back wheel. So, it is a straight-line segment extending out from the back wheel to the point just above where the front wheel is tangent to the other curve.

Thus, given any position of the rear wheel, and a selected direction of travel, you know where the front wheel is at the same time. (Extend the tangent vector until it crosses the other curve). If you move this rear wheel along one curve you see a series of measurements for the length of the part of the bicycle frame called the “top tube”. Are the measurements constant? If they are *not* constant, you are looking at an impossible path for the bike.

For more on the mathematics of bicycles see: Whitt, Frank Rowland, Wilson, David Gordon, and McCullagh, James C., *Bicycling Science*, Cambridge, Mass: The MIT Press, 1984.

Task 8.2: Celtic knots Probably the best thing to do is consult a resource which details how to construct Celtic knot patterns, such as *Celtic Knotwork*, by Iain Bain, Constable and Company, 1986 (although you could start with a grid with an even number of squares in both directions and place dots at alternating vertices, making sure to miss the corners).

The Book of Kells is housed in the Library of Trinity College Dublin and is one of the most sumptuously illustrated manuscripts to have survived from the early Middle Ages in Europe. It was completed in 800 CE and contains transcriptions of the four Gospels, lavishly illustrated and ornamented.

Other examples of the use of geometry in cultural and religious artefacts include:

- Moorish and Islamic architecture and design (perhaps the most famous example being the Alhambra Palace in Granada in southern Spain). See, for example, *Geometrical Concepts in Islamic Art*, El-Said, Issam, and Ayse Parman. Palo Alto, Calif: Dale Seymour Publications, 1987.
- the tradition of sand drawings in Africa south of the Equator (see the work of Paulus Gerdes, for example, Gerdes, P. (1998). On possible uses of traditional Angolan sand drawings in the mathematics classroom. *Educational Studies in Mathematics*. **19**, 3-22)
- weaving patterns from many cultures - for example, Knight, Gordon, (1984), The Geometry of Maori Art: weaving patterns. *New Zealand Mathematics Magazine*, **21**(3), 80-86.

Many more examples can be found in publications such as *Sacred Geometry: philosophy and practice*, Lawlor, Robert. London: Thames and Hudson, 1992, and in the chapter on native American geometry in *Native American Mathematics* edited by Michael Closs, Austin, Texas: University of Texas Press, 1986, pp. 387-407. Some fascinating (and difficult) geometrical problems have been found in Japanese Temples written on wooden tablets. The oldest one in existence dates from 1683, see: H. Fukagawa and D. Pedoe (1989), *Japanese Temple Geometry Problems*. Winnipeg, Canada: Charles Babbage Research Foundation.

Task 8.3: the development of geometry Among the developments you should be able to find out about are:

- the development of analytic geometry, in which algebraic notation and procedures are used for the description of geometric objects
- the study and classification of conic sections and other families of plane curves and the solution of problems involving them
- how the problem of perspective (in painting and other areas) became the basis of projective geometry
- how the field of differential geometry was initiated (by providing analytic expressions for the length of arc and the curvature of plane curves and, subsequently, surfaces) and how these ideas were generalised to spaces of any number of dimensions
- the development of non-Euclidean geometry
- the classification of geometries
- the development of topology

Task 8.4: What is GPS (and how does it work)? The GPS system relies on 24 satellites that orbit the earth twice each day. The distribution of these near circular orbits is even, such that they provide a uniform net around the entire surface of the earth. At any location, at any point in time, up to ten of these satellites may be 'visible' to a receiver of GPS signals. The satellites orbit the earth at a speed of about 4 km/s at a height of a little over 20 000 km. They are orbiting radio transmitters and the user's GPS instrument is a receiver. The signals from a number of satellites are received and processed by the GPS to provide position, height and time information. The process involves calculating the distance to each of the satellites in range and the intersection of circles produced using those calculated radii (as shown in the diagram accompanying this task). For more information, see GPSCO (1998), *Exploring GPS. A GPS Users Guide*: The Global Positioning System Consortium (GPSCO).

For information on some other modern application of geometry (and other mathematical topics) try: *For All Practical Purposes: mathematical literacy in today's world* by Sandra Savage, Daniel Reich, John Emert, and published by W H Freeman & Co 5th edition, 1999.

Task 8.5: The angle in a semicircle There are a whole host of proofs of the theorem that that angle in a semicircle is 90^o. These can be of the Euclidean (synthetic geometry) form, utilise transformation geometry, analytic (co-ordinate) geometry, complex numbers, or vectors. As long as it is a valid proof, no one method is *a priori* superior to any other. For examples of such proofs see: Willson, 1977, or Barbeau, E J E .J. (1988), Which Method is Best, *Mathematics Teacher*, February 1988: 87-90.

For more than 350 different proofs of Pythagoras' theorem see: E S Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, 1968.

Task 8.6: A dynamic circumcircle At its simplest, dynamic geometry software allows the user to create and then manipulate points, lines and circles on a computer screen. Some points and lines may be created so that they are freely moveable. These are referred to as the "basic" objects. Other objects can be created using geometric relationships with these basic objects, such as the midpoint or perpendicular bisector of a line, or the bisector of an angle. Any such relationships created in this way are maintained consistently when any basic object is dragged using the mouse. See appendix 1 for some details on different versions of dynamic geometry software.

There are many defined "centres" of a triangle. For a comprehensive discussion, see the Encyclopedia of Triangle Centers at: http://cedar.evansville.edu/~ck6/encyclopedia/

Task8. 7: designs on Logo Logo is the term used to describe a range of programs that in various ways provide the user with the means of controlling the movement of an object on the screen (the turtle). As Seymour Papert (credited as the inventor of logo) described it: powerful ideas in mind-sized bites. The controls consist of some simple instructions like: fd 40, bk 30, lt 90, rt 60. See appendix 1 for some details on different versions of logo.

Acknowledgments

I would like to record my appreciation to the following people with whom I've had the pleasure of sharing many ideas on the teaching and learning of geometry:

Members of the Geometry Working Group of the British Society for Research into Learning Mathematics (BSRLM). Reports of the working group are available in the proceedings of the BSRLM, see http://www.bsrlm.org.uk

Colleagues I have had the privilege of working with on the Royal Society/ Joint Mathematical Council inquiry into the teaching and learning of geometry.

A group of prospective secondary mathematics teaches I had the pleasure of working with on aspects of the teaching and learning of geometry while on secondment to the mathematics education unit in the Department of Mathematics at the University of Auckland, New Zealand during the second half of 2000.

Notes

1. This use of this example of geometry was inspired by a course on “'Geometry and the Imagination', led by some of the greatest mathematicians of our time, John Conway, Peter Doyle, Jane Gilman and Bill Thurston at the Geometry Center in Minneapolis, USA, June 17-28, 1991.
2. BCE is the acronym for “before the common era”. BCE and CE (common era) are increasingly being used to denote years as used across the world, in place of the solely Christian BC (before Christ) and AD (*anno Domini*, Latin for the year of [the] Lord).

Appendix

Some resources for teaching geometry

The information below is designed to support some of the ideas in this chapter. Space does not allow for a comprehensive list. Some further resources are given with the commentaries of some of the tasks in the chapter (see above).

Tarquin Publications (Stradbroke, Diss, Norfolk IP21 5JP) are a good source of materials for teaching geometry. Examples include:

Mathematics in Three Dimensions
3D Geoshapes and Polydron
Escher, Illusions & Perception
Geometrical Pattern Making
Tilings & Tessellations
Paper Engineering & Popups
DIME 3-D Visualising & Thinking
Tangrams,Pentacubes & Pentominoes

Dale Seymour Publications (P.O.Box 10888, Palo Alto, California, USA) publish some useful materials, including:

Mathematical Investigations, Books 1-3 (problem solving tasks covering a range of mathematics, including geometry)
Logic Geometry Problems
Blueprint for Geometry (designing and building a scale model of a house)
Designing Playgrounds
By Nature’s Design (geometry in nature)
Structures: The Way Things Are Built
Designing Environments

The Mind's Eye: Imagery in Everyday Life

Real-Life Math Problem Solving by Mark Illingworth (Scholastic, 1996) contains a number of problems involving geometrical ideas.

Dynamic Geometry Software

Cabri is available from Texas Instruments.
SketchPad is available from Key Curriculum Press.

Logo

WinLogo and *SuperLogo*, a new version for Windows, are available from Longman Logotron.
LogoWriter and *MicroWorlds logo* are produced by Logo Computer Systems Inc.
StarLogo is produced by the Media Lab at MIT, Boston, USA.

Note: all websites mentioned in this chapter were correct at time of going to print.

References

Clements. D. (2001), Teaching and Learning Geometry. In J. Kilparick (Ed), *Research Companion to the NCTM Standards for Mathematics.* Reston, VA: NCTM.

de Villiers, M (1999), *Rethinking Proof with Geometer's Sketchpad*. Key Curriculum Press, CA.

Hoyles, C. and Jones, K. (1998), Proof in dynamic geometry contexts, in C. Mammana and V. Villani (eds.), *Perspectives on the Teaching of Geometry for the 21st Century*. Dordrecht: Kluwer.

Jones K. (1999), Students interpretations of a dynamic geometry environment, in I. Schwank (ed.), *European Research in Mathematics Education.* Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Malkevitch, J (Ed) (1991), *Geometry's Future*. Arlington, MA: COMAP.

Mathematical Association (1993), *Geometry in the Undergraduate Syllabus: Report from Group III, Mathematical Content at University*. Leicester: Mathematical Association.

Royal Society/ Joint Mathematical Council (2001), *Teaching and Learning Geometry 11-19*. London: Royal Society/Joint Mathematical Council.

Whitely, W. (1999), The Decline and Rise of Geometry in 20th Century North America. *Proceedings of the 1999 Conference of the Mathematics Education Study Group of Canada.* St. Catharines, Ontario: Brock University.

Willson, W. W. (1977), *The Mathematics Curriculum: geometry*. Glasgow: Blackie/Schools Council.