IS WHAT YOU SEE WHAT YOU GET? REPRESENTATIONS, METAPHORS AND TOOLS IN MATHEMATICS DIDACTICS

Janet Ainley[1], Bill Barton[2], Keith Jones[3], Maxine Pfannkuch[2], Mike Thomas[2]

[1]University of Warwick, United Kingdom

[2]University of Auckland, New Zealand,

[3]University of Southampton, United Kingdom

dkj@southampton.ac.uk

Abstract: *This paper is exploratory in character. The aim is to investigate ways in which it is possible to use the theoretical concepts of representations, tools and metaphors to try to understand what learners of mathematics 'see' during classroom interactions (in their widest sense) and what they might get from such interactions. Through an analysis of a brief classroom episode, the suggestion is made that what learners see may not be the same as what they get. From each of several theoretical perspectives utilised in this paper, what learners 'get' appears to be something extra. According to our analysis, this something 'extra' is likely to depend on the form of technology being used and the representations and metaphors that are available to both teacher and learner.*

Introduction

"What you see is what you get" (WYSIWYG) was a catchphrase on the 1960s US TV show, *Rowan and Martin's Laugh-In*. In the 1980s it became a byword in computer-based desktop publishing, referring to any technology enabling the user to see images on-screen exactly as they appear when printed. The development of graphical user interfaces (for forms of software that are proving useful in mathematics didactics, such as Logo, spreadsheets, dynamic geometry, graph plotting and statistical modelling software) has raised questions about how learners' interactions with these interfaces mediate their understanding of mathematical ideas. This paper seeks to open up discussion about how the theoretical resources of representations, metaphors and tools can assist in an examination of what learners in classroom interactions with technology 'see', what understanding they might get from such interactions, and the implications for the theoretical ideas.

Following an overview of some of the main theoretical ideas, a brief extract from a piece of classroom research is considered from several different perspectives. The aim is to see in what ways these perspectives might both illuminate and constrain interpretation of the classroom incident. The paper concludes with some commentary on the relationship between the various perspectives and what they might mean in terms of practical classroom responses.

Theoretical Preamble

There is extensive literature on representations, metaphors and tools in mathematics education yet this literature often focuses on one of these theoretical ideas and, in the main, does not seek to examine the relationship between them. Indeed, each of the terms is reasonably complex in itself, with discussion continuing as to what each one is and how the idea might be useful. In this section we identify some of the main features of each theoretical idea, as a precursor to using them in an analysis of a classroom interaction.

In the literature on representation, a distinction is often made between internal and external representation. An internal representation is a hypothesised mental construct; an external representation is a material notation of some kind (such as a graph, a table or an equation). As Kaput (1998) observes, such a distinction is cognitivist in essence and does not necessarily take account of other perspectives on thinking and learning. These other perspectives, including socio-cultural viewpoints, for example, raise questions about whether learner interactions with screen images are usefully described in terms of internal and external representations, or whether such images constitute a new class of representation.

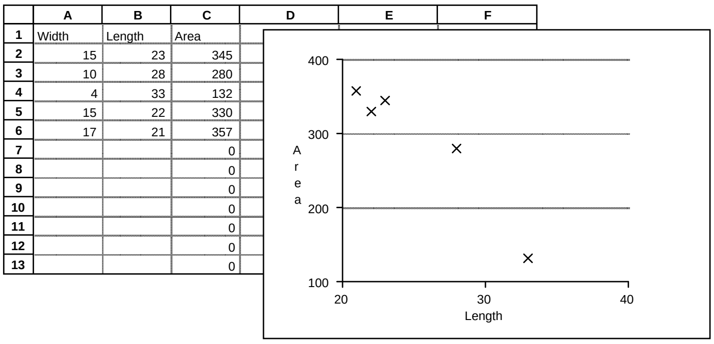
The literature on representation currently makes little reference to notions of metaphor even though work on the latter (for example, Lakoff and Nuñes, 2000) shows how it is possible to point to deep metaphors which are implicitly embedded in our language and which are therefore part of the way we talk/think, whether we like it or not. Metaphor, in the general sense, characterises the substitution of one similar concept for another one. Metaphor is a widely-used idea in software design (examples being the desk-top, menus, windows, etc) and in human computer interaction. In mathematics, metaphor occurs as translation of structure from one domain to another and has been posited as crucial for our sense of understanding mathematical ideas. For their part, images (such as screen images) are posited as part and parcel of the metaphorical mappings that bring new mathematical concepts into existence (see, for example, Sfard, 1997). This raises the question of the relationship between ideas of metaphor and representation, about which we hope to stimulate discussion through this paper.

The notion of tools is widely used in mathematics education. At its most straight-forward, the term refers to physical implements. But the use of the term has expanded to include not only physical implements but also technical procedures (like the algorithms of arithmetic), symbolic resources (such as those of natural languages and mathematical and musical notation), and, most recently, cognitive processes. Such use of the word tool can be considered as metaphorical, as a way of understanding the use of technical procedures, symbolic resources and cognitive processes. Computer environments (such as microworlds) and electronic calculators are frequently referred to as both technological and cognitive tools. These tools, as well as being physical artefacts, encompass technical and symbolising capabilities and become objects to think with. It is widely recognised that tools change the way that activities are carried out and can shape the conceptions of the user (Gutiérrez, Laborde, Noss and Rakov, 1999; Lajoie, 1993).

The wide use of the terms representation, metaphor, and tool in mathematics education highlights the complexity of trying to understand and describe what may be happening when learners (and their teachers) interact with mathematics when using computer software, calculators, or other technology. In an attempt to begin to try to clarify the relationships between these terms, and perhaps their interactions, a segment of classroom interaction has been taken as a catalyst for producing various perspectives on the role of the technological imagery in learning.

An Interaction from the Mathematics Classroom

The task described below was set within what Ainley, Nardi and Pratt (2000) call an *Active Graphing* approach. With this approach, children are encouraged to make a scattergraph as soon as they have a few pieces of data. The children are then expected to discuss the graph, perhaps with their teacher, and make conjectures about any patterns that emerge before deciding what data to collect next to test these conjectures. The data extract in Figure 1 is taken from the work of two 9-year-old children, Laura and Daniel (both pseudonyms). The children were working on a task (introduced verbally by the teacher) in which each group was given a 75 cm length of ribbon, and challenged to make a rectangular frame which had the largest possible area. The children collected initial data by pinning a length of ribbon on to a display board to make the frame, then measuring the length and width of the frame. They entered these results on a spreadsheet, and, with help, set up a third column with a formula to calculate the area of the frame.



	A	B	C	D	E	F
1	Width	Length	Area			
2	15	23	345			
3	10	28	280			
4	4	33	132			
5	15	22	330			
6	17	21	357			
7			0			
8			0			
9			0			
10			0			
11			0			
12			0			
13			0			

Figure 1. Children's spreadsheet work on the rectangle task

The data extract below is taken from their discussion of their first graph (see Figure 1).

Researcher: Okay, so what is this graph saying?
Laura: It's a hill.
Daniel: It's like a mountain there.
Laura: I think it's going to come down again.
Daniel: and go back to nought.

The first thing to note is that the spreadsheet technology available to the children, while supporting the rapid display of their data, did place constraints on aspects of their interaction. For example, the children had to organize their data in a particular way, since the software only allows graphs to be made from adjacent columns. Further, while the graphing facility within the particular software (*ClarisWorks)* creates a window containing a graph that can be dragged to different sizes and proportions, such dragging changes both the appearance of the graph and the scaling on the axes. By default, the software selects scales and ranges of values on the axes which display the data points centrally in the window (as in the example shown in Figure 1). Thus the axes may not start from zero with the consequence that the full range of possible values of a given variable may not be visible. The recognition of this shaping of student activity by the technology is an important prerequisite for a theoretical discussion of their learning (see, for example, Jones, 1999).

Analysis

In this section, the above data extract is subject to analysis from four different perspectives, including modelling, multiple representations, co-construction, linguistic, etc. in order to examine the convergence, or otherwise, of these viewpoints and to see what they reveal about the roles of representation, metaphor, and tool.

§1 Representation as tool and symbol

At first sight it seems that the children do not respond directly to the researcher's invitation to read information from the graph. Their first two comments suggest that they are seeing the graph as a picture. The fact that they do see a picture, rather than a series of separate points, is likely to be significant. They seem to be looking *through* the individual points of the graph to construct a coherent image that takes in the whole of the data set.

The latter two comments are even more interesting. Laura's comment suggests that she is extrapolating to imagine data which has not yet been collected, and what is more she is doing this 'backwards' to a part of the graph (to the left of the current position of the y axis) which does not yet exist on the screen. This suggests that she is using the graph as *a tool* that she can (mentally) manipulate to make conjectures about the outcome of the experiment.

Laura's use of the words 'come down again' may also link to the overall purpose of the task. The children are trying to find a maximum area, and the 'hill' for Laura seems to contain the idea of a value increasing and decreasing. The experimental data the children have collected shows that they started with a width of 15, tried smaller widths, tried 15 again (though without seeming to notice that they get a different area!), and then tried a larger width. This sequence reflects a sense of 'going up and down' that links closely with the hill metaphor.

The graph *as a symbol* has as its referent the tabulated data, and the rectangular frames which have been created. Nemirovsky and Monk (2000) talk about symbolizing as *the creation of a space in which the absent is made present and ready to hand*. This seems quite a useful way of seeing what is happening for Laura and Daniel as they look at the graph. The graph symbol allows them to hold all the data in one space, so that they can see something about the overall pattern of what is happening. The symbol contains all the complexity in a more manageable way than the data, and so allows them to talk about how 'it' is changing. Notice that when Laura says she thinks 'it' is going to go down, 'it' might be any or all of:

- the trend in the graph,
- the value of the area in the data,
- the size of the space within the frames they are making.

The tabular representation can also be viewed as a symbol in the sense that it is a counterpart for the conceptual object of function (actually of several functions in this case), via the input-output process. That is, it stands for function in a metaphorical sense but may not name or point to it. This representation-symbol contains a number of other symbols, including those for specific numbers, and the symbolisation of variables in the table using natural language, namely ‘width’, ‘length’ and ‘area’. For the learner, a tabular representation can assist in the construction of understanding of properties of a function, such as its one-to-one nature (a problem with the data in this table!). However, if a student decides, for example, to use the table values to interpolate or extrapolate *other* values for the function, or to calculate the perimeter of the rectangle by adding a further column with its associated symbols, then such activity has moved them beyond *looking through* to a stage of *acting on* the representation.

Once the table is complete the focus shifts to using the software as a graphing tool, for drawing the graph of area against length. In this case the students have taken the route the tool directs them in and have drawn a discrete set of points to represent the functional relationship with each point symbolised by a little cross. One can see this graphical picture either as a counterpart symbol, or a representation of a function, comprising other symbolic objects. These include the counterpart symbols which are described as axes and the language symbols ‘Length’ and ‘Area’, both of which stand for the independent and dependent variables. The little crosses are also symbols pointing to ordered pairs in the function, etc. A student can pay attention to this representation and construct some properties of a function as a process or an object (Tall *et al*, 2000) as with a table, but it becomes more interesting in some sense when the student interacts with it and uses it as a tool. The comments such as those of Daniel who says of the graph that “it’s like a mountain there.” and of Laura who describes it as a hill appear to require a global modelling strategy. They may or may not have seen a continuous model of the function in their mind’s eye when making these statements, but their interaction with the representation has comprised more than *looking through* it. They see the graph as an entity, an ‘it’. They are imposing a global model on the graph and *construing* properties of the process or an object underlying the model. Later Daniel again identifies a local property of the graph, namely that it appears to head “back to nought”. This may have been inspired by Laura’s comment about the trend of the values saying “…it’s going to come down again”, again paying attention to a local property. This brief encounter with the activity demonstrates, when modelling functions in a computer environment for building understanding, the students’ interaction with the tabular and graphical representation as tools is crucial.

§3 Multiple representations of a real situation

Tools can be said to both aid and initiate thinking. Before the data are entered, the table is a tool to organise the data collected and the graph is a tool to organise the table data, whereas once the data are entered the table and graph are both representations (models) of the real situation. When the students are confronted with a representation, a dialogue with, or interrogation of, the representation is operationalised. In the vignette above, that three representations (models) can be found infers that students had to think in order to change from one representation to another. Wild & Pfannkuch (1999) call the thinking that is required to move between representations, or to change representations to engender understanding, *transnumeration.* Overall, using the approach of Wild & Pfannkuch would mean characterising the dialogue in terms of five fundamental elements – recognition of need for data, *transnumeration*, consideration of variation, reasoning with statistical models, and integrating the contextual and statistical.

The first question confronting the students is what measures should be captured from the real system. The children must think how to capture the notion of area so they decide to make a rectangle and measure the width and length to the nearest cm (*transnumeration*). These are determined to be the relevant measures for the problem. They then represent these measures in a table of data as a way of systemising their thoughts (*transnumeration*). They calculate the area using a spreadsheet tool much as they would use a calculator or pencil-and-paper. The table-of-data representation has no order that easily allows the students to interact with the relationship between the variables. Whatever was noticed or not noticed by the students, the table of data must be changed in some way to convey new or increased meaning. The students have to think that perhaps a graphical representation will allow them more insight into the data. What variables should they graph? What graphical tool should they choose? When they obtain a graphical representation (*transnumeration*), a dialogue between them and the data ensues.

In this episode only two elements of statistical thinking – reasoning with a statistical model and recognition of the need for data – are activated. What features can the students see in the data? First they see a hill. The students perhaps do not have the language to discuss trends and therefore use the metaphor of a hill to describe the pattern they are ‘seeing’. When they further describe the pattern they imagine what the representation might look like if there were more data. The statistical-system tools allow multiple representations of the real situation to be seen so that students can engage in a dialogue with the data in a search for meaning and ultimately understanding about the real situation.

§4 Co-construction of representational relationships

A critique of some of the work in mathematics education on multiple representations (generally numerical, graphical, and symbolic) is that the various representations can be no more than (external) representations of each other with no grounding for the learner in any experience. For the learner, they are not 'representations' since they are not representing anything known to the learner. The data extract above illustrates that this does not necessarily have to be the case. Here the representations are linked to concrete, experientially-real data; in this particular example the construction of rectangular picture frames from a fixed length of ribbon. So it could be said that the phenomena of making picture frames is at the centre of the activity, and the representations are means of understanding and reasoning about the phenomenon.

The mathematical relationships hidden in the spreadsheet formula used to calculate the area of each rectangular frame of ribbon, and, indeed, in the model of space that is the Euclidean plane that controls the phenomenon' are also models of the phenomenon. So there is a two-way (at least) representational relationship. It is a form of co-construction. The forms of representation available to, or, more particularly, used by, the learners control, or influence, the exploration of the phenomenon just as the phenomenon influences, or controls, each representation.

The forms of representation permitted in the software environment are given, or, perhaps more accurately, proscribed, as the learners are not free to create their own representations but can only make use of those representations provided by the software. The representations that are available, in turn, generate imagery which is intimately connected with the metaphor of the cognitive tool. The representations available to the learners are not static. The *active graphing* approach exploits the potential of computer-based environments for the *active exploration of phenomena*. The children in this example have experience of this approach and, in the last two lines of dialogue, are making predictions based on their interpretation of the graph. They seem to be using the representations, particularly the graphical representation, as a means of building up a sense of the quadratic relationship that models the phenomenon they are exploring.

As Leont'ev (1981, pp. 55-6) argues, "the tool mediates activity and thus connects humans not only with the world of objects but also with other people". Thus the process by which learners create meaning is embedded within the setting or context and is mediated by the forms of interaction and by the tools being used. Here the argument is that the learners create representations, albeit limited by the forms of representation available via the tool, and the (available) representations create the learner's ideas of those representations.

Discussion on Tools, Representations and Metaphors

In terms of tools, and taking the spreadsheet in its totality as a tool, in this example of classroom interaction the tool is not used here to its full potential. For example, for these children in this case, 'length' is not taken to be dependent upon 'width'. Tools are like this –their full potential is rarely used. In addition, tools are not mathematical in themselves. They are only *used* mathematically.

In terms of representations, it could be said that there are three representations in this example: a table, a graph, and a dialogue. Yet the word representation carries an implication that *a thing* is being represented. Here the representations are not just different aspects of a mathematical relationship, the relationships are different kinds of thing in each case. For example, in the classroom example analysed above, the table of data has no 'shape', no sense of increasing or decreasing. It is raw, unordered data. Yet it implicitly contains the formula relationship between the sides of a rectangle and its area. In contrast, the graph does not contain that information: the points on the graph cannot be read to the accuracy of the formula. What it does contain is an ordering of the data. Finally, the pupil dialogue is about a trend, not about the formula, nor about the data as individual points.

These three things, the table, the graph, and the pupil dialogue, are different things in kind. In fact, it could be said that there is not *a thing* being represented at all. When we talk about representations, we talk *as if* there were something to present. This is a metaphor. The metaphor is that mathematics is *like a thing*. Nominalising in mathematics is a metaphor whereby mathematics is likened to objects in the world. There are other options, for example we could talk about mathematical ideas as actions.

If talking of mathematical relations as objects is a metaphor, what is it a metaphor of? To ask that question is to fall into the same trap: it implies that there has to be a metaphor of anything. We talk in metaphors because there is no other way of doing it. Mathematics is created by talking about a relationship, tabulating it, graphing it, describing it. These are all (Wittgenstein, in Shanker, 1987) *normative activities*: the communication lays down the ways it makes sense to talk about, describe, or illustrate these ideas. The benefit of having many re-presentations is that this mathematical idea has a lot of different aspects – no one representation embodies the entirety of the idea.

In the above analysis, the terms representation, metaphor and tool were each given a variety of roles. For example, 'tool' was interpreted as a function of a representation (§2); 'representation' was used as a model (§3), and as a mode of description (§4); 'metaphor' was used as a linguistic feature (§1 and §3), as a relationship between a function and its representation (§2). The lesson here is that we must be careful not to assume the same functionality for our use of such terms as well as not assuming consonant interpretations.

Concluding Comments

Perhaps there is the basis of productive discussion arising from people's different uses of the same theoretical concepts. In terms of our actions as researchers, we are reminded that it is essential to be very clear, very early in any writing or discussion, that the meaning of our conceptual constructs is evident. The other side of this coin is that, when reading the work of others, we should not to jump to conclusions about what these constructs mean when used by other authors.

But mostly we are reminded that constructs are just that. They are constructed by us, and are therefore useful or not useful. They are not true or false. There is no unequivocal thing that can take the name 'representation'. The consequence of this is that constructs must be judged for how they speak to the readers or listeners. Do they help teachers understand learning experiences or teaching behaviour? Do they help researchers frame useful questions? Do they add to the analytical tools available to mathematics educators? And so on.

This paper is titled "Is What You See What You Get?" because we want to focus on our ability to understand the relationship between technology (particularly visual technology) and mathematics learning. Can we make use of the ideas of representation, tool and metaphor to discuss what is "seen" and what is "got"? At the risk of being glib, the various perspectives used above return the following different answers to these two critical questions.

§1 What is seen is a picture, much of which is able to be inferred through the use of technological tools. What the learner gets are symbols that can be given meaning (added value?) through metaphors.

§2 What is seen are representations generated through the use of a tool. The representation is a metaphor of the mathematical relation. What the learner gets are properties of these representations that have been construed from them as objects.

§3 What is seen is data transformed in different ways. What the learner gets is the power to ask questions and to reason.

§4 What is seen are representations generated from real experiences. What the learner gets is the ability to co- (and/or re-) construct the representations in response to the questions they raise.

In total, what are seen are tool-generated representations of different, yet related, things. What the learner gets is a way of communicating mathematics. In none of the above is what the learner sees the same as what the learner gets. In every case, the learner gets something extra. Perhaps that is the power of (all) technology.

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