I.O.S.

ESTIMATING RETURN VALUES OF WAVE HEIGHT

by

D.J.T. CARTER

&

P.G. CHALLENOR

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INSTITUTE OF OCEANOGRAPHIC SCIENCES

INSTITUTE OF OCEANOGRAPHIC SCIENCES

Wormley, Godalming,
Surrey, GU8 5UB.
(0428 - 79 - 4141)

(Director: Dr. A.S. Laughton)

Bidston Observatory,
Birkenhead,
Merseyside, L43 7RA.
(051 - 653 - 8633)

(Assistant Director: Dr. D.E. Cartwright)

Crossway,
Taunton,
Somerset, TA1 2DW.
(0823 - 86211)

(Assistant Director: M.J. Tucker)

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Institute of Oceanographic Sciences
Wormley, Godalming,
Surrey, GU8 5UB
United Kingdom

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ESTIMATING RETURN VALUES OF WAVE HEIGHT

1. INTRODUCTION

The N-year return value of wave height is a statistic which is frequently used by design engineers concerned with coastal and offshore structures as a measure of severe wave climate - generally with N equal to 50 or 100. Extremes of other environmental parameters are also frequently described by return values; for example 50-year return value of wind speed and 100-year return value of sea level.

Estimates of 50-year return values in waters around the UK have been provided to the Department of Energy, since 1972 by the Institute of Oceanographic Sciences, for inclusion in the Department's guidance notes on the design and construction of offshore installations.

These estimates have been amended over the years as wave data at different sites have become available and to take account of changes in the estimates, by the Meteorological Office, of 50-year return values of wind speed over waters around the UK - which are also included in the Department of Energy's guidance notes.

The method of estimating the 50-year return value of wave height was devised to produce results from very little data - a few series of measurements each obtained for about a year - and this method has been altered very little since its introduction. With the passage of years, longer series of wave data are becoming available and other methods can be considered - methods which do not depend so much upon unjustified assumptions as that in present use, and which give some estimates of confidence limits for the 50-year return values.

This report discusses the present method of estimation and the problems associated with it, and suggests other methods which might be used.

There are other aspects of extreme waves, of comparable importance to the offshore industry as their height, which require further research, such as wave period and steepness, and the distribution and structure of breaking waves. However, only extreme wave height is dealt with in this report.

The methods of analysing wave height discussed in this report assume that the underlying distribution of wave height does not vary from year to year. Little research has been carried out into the effect upon extreme wave height estimation of long-term, or secular, variations of climate. However, Rye(1976) estimates, using visual wave height data from 'Famita', that the 100-year return value of wave height in the northern North Sea is increasing by 3% - 4% per annum; and concludes that "future research should be strongly intensified on this subject".

The methods of analysis in this report also assume that the very large waves, which occur once in 50 years or once in 100 years, are from the same statistical distribution as other, lower waves; so may be estimated by extrapolation from a statistical analysis of lower wave heights, and no new physical process is involved. If this assumption is incorrect, then none of the methods is valid. However, there is no known reason why the physics of very high waves should be different from that of lower waves, provided that the water depth is sufficient.

2. DEFINITIONS

2.1 Return Value

The N-year return value is that which is exceeded on average once in N years.

This definition of return value is in terms of events, so any formula has to include the number of events which occur in a year, and any reference to a return value should include the interval between events - 3 hourly, or hourly etc. Suppose M values of wave height are obtained each year from a population with a cumulative probability distribution function P(h), so that:

$$\text{Prob}\ (H < h) = P(h)$$

Then the N-year return value, h_N, is given by:

$$P(h_N) = 1 - 1/MN \qquad (2.1)$$

For example, if wave height is recorded at three-hourly intervals, then assuming the heights are independent and taking an average year as 365.25 days, the 50-year return value of three hourly events is given by:

$$P(h_{50}) = 1 - 1/(8\text{x}365.25\text{x}50) = 0.9999932$$

Sometimes the N-year return value is defined slightly differently, as that value which is exceeded on average during one year in N. Thus multiple occasions of exceeding h_N during a year are not taken into account. It is statistically very unlikely that the 50-year or 100-year return value will be exceeded more than once during any particular year, so in practice the difference between the two definitions is generally unimportant. For example, estimates of 50-year return value of significant wave height might differ by about 10 cm. But as N becomes smaller, so the difference between the two definitions becomes larger, and 2-year or 3-year return values could be quite different.

If the events defining the N-year return value are not independent, then the definition of the N-year return value as the height exceeded on average once in N years is still valid, but these exceedances will no longer be random, and will therefore not have a Poisson distribution; and the alternative definition described above as that value which is exceeded on average during one year in N will be lower.

The N-year return value of wave height is sometimes defined in terms of maxima occurring during storms (see Nolte 1974, pp 6-8). This avoids the problem of dependence of successive waves, but the estimated return value can depend upon the criterion used to denote a storm. Such a definition of return value seems well-suited to the analysis of waves caused by tropical storms, but not so useful for mid- and high-latitude storms which are often not so clearly separated.

2.2 Wave Height

Generally, 50-year return values of wave height are specified in terms either of significant wave height, h_s, or of the most likely maximum wave height (crest to trough of a zero-up-crossing wave) in a specified period such as three hours, $h_{max,3hr}$.

Significant wave height is defined as

$$h_s = 4\sqrt{m_0} \tag{2.2}$$

where m_0 is the area under the energy-frequency spectrum, i.e. the variance of the sea surface elevation. Originally significant wave height was defined as the mean height of the highest 1/3 zero-up-crossing waves, because this value appeared to be close to visual estimates of the wave height. The two definitions are almost identical for narrow band waves, and in general appear to be in good agreement. Note that the present definition is in terms of surface elevation and not of crest to trough heights.

The most likely highest zero-up-crossing wave during a specified time depends upon the number of waves and upon the distribution of these waves. Assuming that the distribution is Rayleigh with root mean square, rms, of $h_s/\sqrt{2}$, and that the surface is statistically stationary during the specified period, p, then the value of $h_{max,p}$ is given by

$$h_{max,p} \simeq h_s(\ln N_z/2)^{1/2} \tag{2.3}$$

where N_z is the number of zero-up-crossing waves during the period p (i.e. $N_z = p/T_z$ where T_z is the zero-up-crossing wave period).

It should be remembered that $h_{max,p}$ is not the largest wave height that will occur, only the most likely highest. For example, in a three-hour period, with $T_z = 10$ seconds, then from the above equation

$$h_{max,3hr} = 1.87\ h_s$$

But there is a 10% probability of a wave height exceeding 2.15 h_s during any three-hour period. The method of calculating this and similar values is given in Section 3, together with a more detailed discussion of equation 2.3.

The design of offshore structures requires return values of highest wave heights, which is why "Offshore Installations: Guidance on design and construction" by the Department of Energy (1977) gives estimates of 50-year return values of most likely highest wave height. However, significant wave height is the more fundamental property. The value of $h_{max,p}$ can be determined from equation 2.3 for any p during which the waves are stationary, provided that the zero-up-crossing period and hence N_z can be estimated - and because of the form of equation 2.3, only an approximate estimate is required.

3. PRESENT METHOD OF ESTIMATING RETURN VALUES - SHORT TERM STATISTICS

The method consists of two stages: first estimating, from wave measurements, the significant wave height, h_s, and hence the most likely maximum wave height in three hours, $h_{max,3hr}$, using what might be called 'short-term statistics of waves'; then analysing these values of h_s or of $h_{max,3hr}$ to estimate 50-year return values - 'long term statistics', which are discussed in Section 4.

3.1 Fundamental Theory

The first stage is based upon the results obtained by Cartwright and Longuet-Higgins (1956) and Cartwright (1958), for the distribution of crest elevation in a record of a broad-band sea and the distribution of maximum crest elevation. These distributions are in terms of the spectral width parameter ϵ, defined by

$$\epsilon^2 = 1 - m_2{}^2/m_0 m_4 \qquad (3.1)$$

where

$$m_i = \int_0^\infty f^i E(f)\,df$$

and E is the surface elevation spectrum.

For a narrow band sea, i.e. $\epsilon = 0$, the crest elevations have a Rayleigh distribution given by

$$\text{Prob}(X<x) = 1 - \exp(-x^2/2)$$

where x = crest elevation $/m_0^{1/2}$

Cartwright and Longuet-Higgins (1956) show that, for $\epsilon < 1$, the higher crests in a record are approximately as if from a Rayleigh distribution, suggesting that the distribution of the maximum of N crests can be approximated by a Rayleigh to the power N. Cartwright (1958) shows graphically that this suggestion is justified for ϵ less than about 0.9, and that a close approximation to the distribution of the maximum crest elevation of N mutually independent crests is given for large N by

$$P_N(x) \simeq \exp[-N(1-\epsilon^2)^{1/2}\exp(-x^2/2)]$$

An analytical proof for all ϵ is given by Walden and Prescott (1980).

But $N(1 - \epsilon^2)^{1/2}$ is the number of zero-up-crossings, N_z, in the record containing N crests, so

$$P_N(x) \simeq \exp[-N_z\exp(-x2/2)] \quad (3.2)$$

Thus $P_N(x)$ may be described in terms of N_z (or the zero-up-crossing period $T_z = (m_0/m_2)^{1/2}$) rather than in terms of N (or the crest period $T_c = (m_2/m_4)^{1/2}$). This is fortunate because it is very difficult to measure m_4 which is strongly affected by instrumental high frequency cut-off (see Rye 1977). Similarly the number of crests in an analogue record is dependent upon instrument response.

The modal value of the distribution (3.2) is given by

$$x_{mode} = 2\ln N_z \tag{3.3}$$

(See for example Longuet-Higgins, 1952, pp 259-260.)

Note that equation 3.2 does not imply that zero-up-crossing elevations have a Rayleigh distribution. Zero-up-crossing elevations are a sub-set of crest elevations, generally including independent, higher values of the set; so an assumption that the distribution of zero-up-crossing crest elevations may be approximated by a Rayleigh distribution is not unreasonable - but it has not been proved, except for $\epsilon = 0$.

Expressions are derived by Cartwright (1958) for the moments of this distribution, P_N; in particular the first moment or expected value is given by

$$M_1 = (2\theta)^{1/2}\left(1 + \frac{1}{2}A_1\theta^{-1} - \frac{1}{8}A_2\theta^{-2} + \frac{1}{16}A_3\theta^{-3} - \frac{5}{128}A_4\theta^{-4} + \frac{7}{256}A_5\theta^{-5} - \ldots\right) \tag{3.4}$$

where

$\theta = \ln N_z$

$A_1 = \gamma = 0.5772\ldots$

$A_2 = 1.9781$

$A_3 = 5.4449$

$A_4 = 23.5615$

$A_5 = 68.067$

Thus given a wave record with N_z zero-up-crossings and a maximum crest elevation η, and equating $\eta/m_0^{1/2}$ with M_1 in equation 3.4 leads to an estimate of m_0.

Cartwright (1958) also derives the distribution of the second highest value of x (where x is crest elevation/$m_0^{1/2}$) and the mean, M_1', of this distribution in terms of N_z, which may also be used to estimate m_0 given a wave record. Cartwright (1958 para 4) examines the effect of correlation between crest elevations and shows that unless N is very small or the spectrum particularly narrow the effect is slight, provided that the second highest crest height is taken to mean the highest except for the highest crest itself and m waves on either side of it, where the crests have m-dependence. In practice this condition is approximated by selecting the second highest crest from all crests outside the zero-up-crossings wave containing the highest crest.

3.2 Present IOS technique of estimating significant wave height

This technique, called the 'Tucker/Draper' method, for estimating m_0 and hence significant wave height is based upon the above consideration of crest elevation distribution applied to an analogue wave record covering about 15 minutes. (See Tucker 1961, and Draper 1966, or Tann 1976 for a more detailed description.) This method assumes that the analogue wave trace is statistically symmetrical about the mean water surface, and it uses the average of the highest crest and deepest trough - assumed independent - as an estimate of the expected highest crest in the given record length from which to calculate m_0 using equation 3.4, and hence to make an estimate of h_s ($=4\sqrt{m_0}$). A second estimate of h_s is obtained from the average of the second highest crest and second deepest trough in the record (outside the zero-up-crossing wave containing the highest and deepest values): and the significant wave height is taken to be the mean of the two estimates.

The Tucker/Draper method of estimating significant wave height from the two highest and two lowest values on a wave trace was devised to cope with analogue wave records. With the introduction of digitally recording instruments, more efficient methods will replace it.

3.3 Standard error of the Tucker/Draper estimate of significant wave height

An advantage of the Tucker/Draper method is that the mean water surface does not have to be determined. A disadvantage is that the standard error of the estimated significant wave height has not been derived, because of the difficulty of determining analytically the covariance of the highest and second highest crest elevations; but an estimate can be obtained using simulated data. (Standard errors of estimates from one or two highest values or from one or two second highest values can be derived theoretically, examples are given in Table 3.1.)

Suppose that in a record with N_z zero-up-crossing waves, A and C are the maximum crest elevation and deepest trough depth, and B and D are the second highest and second deepest values - all measured from the mean water surface and normalised using $m_0^{1/2}$. Then the Tucker/Draper estimate of significant wave height, h_s, is given by

$$\hat{h}_s/h_s = [(A+C)/M_1 + (B+D)/M_1']/4 \tag{3.5}$$

where M_1 and M_1' are the expected values of the highest and second highest crests respectively, and are functions of the given N_z. Therefore

$$S.E.(\hat{h}_s/h_s) = [(\mathrm{var}A+\mathrm{var}C)/M_1^2 + (\mathrm{var}B+\mathrm{var}D)/M_1'^2 + 2\mathrm{cov}(AB)/M_1M_1' + 2\mathrm{cov}(CD)/M_1M_1']^{1/2} \tag{3.6}$$

The covariances (cov) of the other combination of A, B, C and D are zero. The variances (var) are given by Cartwright (1958), the values of cov (AB) and cov (CD) are equal and can be estimated from values A, B, C and D obtained from simulated Rayleigh distributed crest heights, and hence an estimate of $S.E.(\hat{h}_s/h_s)$ can be derived.

TABLE 3.1

Method of estimation	Number of zero-up-crossing waves 32	64	128	256
from A	0.152	0.130	0.114	0.101
from A+C	0.107	0.0922	0.0807	0.0710
from B	0.126	0.104	0.0883	0.0770
from B+D	0.0893	0.0735	0.0624	0.0544

Standard error of
estimated significant wave ht./significant wave ht.
(i.e. $S.E.(h_s/h_s)$)

where A: maximum crest elevation in the record
B: elevation of second highest crest
C: deepest trough depth
D: depth of second deepest trough

simulated values and equation 3.5. The efficiency of the Tucker/Draper method can be obtained from the estimate of standard error (see Appendix A). Results from a simulation of two sets of 8192 values are given in Table 3.2, the covariance and correlation between the highest and second highest crests are given in Table 3.3.

The rather large differences in covariances estimated from the two data sets, shown in Table 3.3, suggest that insufficient numbers were simulated to give a good estimate of covariance and correlation, but the covariance terms in the expression for the standard error are only about half the variance terms, so the results in table 3.2 can be expected to be more accurate. Table 3.2 shows that the standard error of $\hat{h}_s/h_s$ from the Tucker/Draper method is about 0.06, depending upon N_z, and is very close to the standard error obtained using only the two second highest excursions (i.e. the mean of the second highest crest and second deepest trough) which is given in Table 3.1.

This analysis assumes N_z is constant; in fact it is a random variable. However, Tables 3.1-3.3 show that large changes in N_z give quite small changes in the standard error of estimates, and the assumption would seem to be justified.

A standard error of about 6% of h_s seems generally acceptable if these estimates of h_s are to be combined, for example into a histogram of annual distribution, but when considering individual estimates it is rather large, So the Institute of Oceanographic Sciences, when processing a year's records, usually takes the few records with the highest Tucker/Draper estimate of h_s and recalculates h_s by directly estimating the surface variance.

TABLE 3.2

Method of Estimation	Number of zero-up-crossing waves 32	64	128	256
from simulated data using (3.5)	0.0876	0.0719	0.0610	0.0533
from (3.6) with cov. from simulated data	0.0878	0.0732	0.0630	0.0553
Efficiency (%)	45	34	23	15

Standard error of Tucker/Draper estimates of $\hat{h}_s/h_s$

TABLE 3.3

	Number of zero-up-crossing waves (N_z) 32	64	128	256
Covariance				
from simulated data set 1	0.0845	0.0645	0.0605	0.0491
" " " set 2	0.0722	0.0609	0.0454	0.0430
mean of two sets	0.0783	0.0627	0.0530	0.0461
Correlation coefficient (mean of two sets)	0.63	0.66	0.65	0.74

Covariance and correlation between highest and second highest crest elevation in N_z zero-up-crossing waves, estimated from simulated data.

3.4 Wave height distribution

The analysis of wave records discussed so far, including the estimation of significant wave height, has only been concerned with the distributions of surface elevation and crest elevation about the mean level. The distribution of wave height (crest to trough) poses a more difficult problem. The distribution of crest elevation is derived from a consideration at a random instant of zero first derivative and negative second derivative of surface elevation: wave height requires the introduction of a time scale or a period distribution.

For example Cramer and Leadbetter (1967) derive the following expression for the mean crest-to-trough height ($\bar{h}_c$):

$$\bar{h}_c = (2\pi m_2{}^2/m_4)^{1/2}$$

Taking zero-up-crossing wave period $T_z = (m_0/m_2)^{1/2}$ and crest-to-trough wave period $T_c = (m_2/m_4)^{1/2}$ gives

$$\bar{h}_c = (2\pi)^{1/2} h_s/4 T_c T_z$$

or in terms of the spectral width parameter $\epsilon = (1-m_2{}^2/m_0 m_4)^{1/2}$:

$$\bar{h}_c = [2\pi(1-\epsilon^2)]^{1/2} h_s/4 \quad (3.7)$$

Lindgren (1972) gives an expression for the distribution of h_c, but it is not in a closed form, including integrals that can only be solved numerically: this distribution is not Rayleigh unless $\epsilon = 0$.

The distribution of zero-up-crossing wave height (h_z) has not been determined. It is generally assumed that h_z has a Rayleigh distribution. There is no theoretical justification for this assumption, except for the case of $\epsilon = 0$; but analysis of wave records usually seem to give results in reasonable agreement with it. For example: Longuet-Higgins (1952 para 5) compares the distribution of waves from records with the Rayleigh distribution and finds 'surprisingly close' agreement; Goda (1970) analyses wave records simulated from various spectral formulae and finds that zero-up-crossing wave heights follow a Rayleigh distribution for ϵ in the range 0.03 - 0.86 irrespective of the spectral shape and cut-off frequency; Borgmann (1973) analyses the highest 19 waves in each 20 minute record obtained during Hurricane Carla in the Gulf of Mexico, and finds these high waves fit a Rayleigh distribution.

So assuming a Rayleigh for zero-up-crossing wave height h_z:

$$\mathrm{Prob}(h_z < h) = 1 - \exp[-(h/h_{rms})^2] \qquad (3.8)$$

where h_{rms} is the root mean square value of h_z.

Longuet-Higgins (1952) shows that for a narrow-band sea ($\epsilon = 0$) the mean height of the highest one third zero-up-crossing waves ($h_{1/3}$) is given by

$$h_{1/3} = 1.416..\ h_{rms} \simeq \sqrt{2}\ h_{rms}$$

(Longuet-Higgins (op cit) works in terms of crest height, but for $\epsilon = 0$ wave height is twice crest height). Equating $h_{1/3}$ to $h_s = 4(m_0)^{1/2}$ leads to:

$$h_{rms} \simeq h_s/\sqrt{2} = 2\sqrt{2m_0} \qquad (3.9)$$

i.e.

$$\text{Prob}(h_z<h) = 1 - \exp[-2(h/h_s)^2] \qquad (3.10)$$

Recently Forristall (1978) and others have found that wave data from storms - notably Hurricane Camille - do not appear to fit the Rayleigh distribution given by equation 3.10, and that using this equation over-estimates wave heights in the upper tail of the distribution. Forristall suggests using a two-parameter Weibull distribution; but Longuet-Higgins (1980) finds that the Rayleigh distribution with a root mean square value of 0.925 $h_s/\sqrt{2}$ appears to fit about as well as the two-parameter Weibull. Normalised values of h_z are analysed in these papers, but judging by the source of the wave data, the discrepancy from the generally-used Rayleigh distribution might only occur with high values of h_z and of h_s. However, Longuet-Higgins (1980) shows that this reduction in rms value is not due to finite wave steepness.

3.5 Estimating maximum wave height assuming stationarity

Assuming the distribution of individual zero-up-crossing wave height h_z is given by equation 3.10, then the distribution of the highest in N_z waves is given by

$$P = \text{Prob}(h_{max}<h) = \{1 - \exp[-2(h/h_s)^2]\}^{N_z} \qquad (3.11)$$

Using the approximation for x/k small:

$$(1 - x/k)^k \simeq \exp(-x)$$

then if $\exp[-2(h/h_s)^2]$ is small and N_z is large

$$\text{Prob}(h_{max}<h) \simeq \exp\{-N_z \exp[-2(h/h_s)^2]\} \tag{3.12}$$

From equation 3.11

$$h_{max}/h_s = \{-0.5 \ln(1 - P^{1/N})\}^{1/2} \tag{3.13}$$

from which the median, percentiles etc. can be calculated.

Longuet-Higgins (1952) gives the mean value of the maximum for large N_z as

$$\bar{h}_{max}/h_s \simeq [(\ln N_z)/2]^{1/2} + \gamma/\{4[(\ln N_z)/2]^{1/2}\} \tag{3.14}$$

where $\gamma = 0.5772 \ldots$

A more accurate solution for the expected maximum of multiple samples from a Rayleigh distribution is derived by Cartwright (1958), and given in equation 3.4

The mode of the distribution of h_{max} is shown by Longuet-Higgins (1952) to be given by

$$h_{mode}/h_s = [(\ln N_z)/2]^{1/2} + O[(\ln N_z)/2]^{-3/2} \tag{3.15}$$

For large N_z it can be shown (see Annex D) that

$$h_{mode}/h_s \simeq [(\ln N_z)/2]^{1/2} + [(\ln N_z)/2]^{-3/2}/16 \qquad (3.16)$$

Thus the differences between mean and mode of the highest wave is given, from equations 3.4 and 3.16 by

$$(\bar{h}_{max} - h_{mode})/h_s = [(\ln N_z)/2]^{1/2} . \{0.1443 - 0.1243[(\ln N_z)/2]^{-1}\} \qquad (3.17)$$

The value of this expression varies from 0.055 to 0.060 for N_z between 50 and 8 000.

Note that the distribution of h_{max} (equation 3.12) and of maximum crest height (equation 3.2) are similar, with the most likely values given by equation 3.4 and .15, i.e.

$$(\text{crest max})_{mode} \simeq m_0^{1/2}(2\ln N_z)^{1/2}$$

$$\simeq h_s(\ln N_z)^{1/2}/2\sqrt{2}$$

and

$$(h_{max})_{mode} \simeq h_s(\ln N_z)^{1/2}/\sqrt{2}$$

Thus the assumption of equation 3.10 for the distribution of h_z leads to the most likely wave height being twice the most likely crest height. Similarly the expected maximum h_z is twice the expected crest height. This implies a very high correlation between a crest height and the following trough depth, which suggests that the distribution (3.10) over-estimates maximum wave heights - supporting the contention of Longuet-Higgins (1980) that the Rayleigh distribution for h_z has a root mean square value less than the $h_s/\sqrt{2}$ implied by (3.10). Longuet-Higgins (1980) estimates a value corresponding to $0.925\ h_s/\sqrt{2}$ from hurricane data. If generally correct this implies - from equation 3.17 - that the usually quoted most likely highest wave height is in fact closer to the expected value.

Generally the most likely maximum wave height during a three-hour period, $h_{max,3hr}$, is estimated - from equation 2.3 - and used for further, long-term analysis. The method of deriving $h_{max,3hr}$ assumes that h_s is constant for the three hours. A method introducing a varying h_s has been proposed by Nolte (1974). He assumes that h_s varies linearly between observations - but ignores any change in wave period. This method should give better results when the variation in h_s during each three hour period is within the bounds of the three hourly observations. Assuming independence of hourly observations within a three hour period, on average 17% of the three hour periods would have h_s for the inner two hours within the bounds of the outer two values. Correlation between the observations would increase this percentage. Taking hourly data from a Shipborne Wave Recorder on the Morecambe Bay Light Vessel, from 5th January to the 22nd February 1957, the proportion was found to be 30%. So Nolte's method would show some improvement with roughly a third of the data, but the extra effort is probably not worthwhile. Moreover, the present method, assuming h_s is constant, leads to conservative - i.e. higher - estimates.

3.6 Conclusions

Significant wave height, h_s, is defined in terms of surface elevation (equation 2.2), and not in terms of crest-to-trough wave heights. Thus enabling an estimate of h_s to be determined from a 15-20 minute record using the theory developed by Cartwright and Longuet-Higgins for the distribution of the highest and second highest crest elevation.

This 'Tucker/Draper' method, employed by the Institute of Oceanographic Sciences, was developed to facilitate the manual processing of analogue wave records. A simulation exercise indicates that the standard error of the estimate of h_s is about 6% and shows that the method has a rather poor statistical efficiency (see Table 3.2). It is being replaced by more accurate methods with the introduction of digitally recording instruments.

The distribution of crest-to-trough wave height is less tractable than the distribution of crest elevations. Lindgren (1972) has solved this problem, but the solution is not readily applicable; even he has not tackled the theoretical distribution of zero-up-crossing wave heights.

Generally zero-up-crossing wave heights, h_z, are assumed to have a Rayleigh distribution with a root mean square value of $h_s/\sqrt{2}$. During the last few years some doubt has been cast upon this assumption, with data showing that this distribution over-estimates wave heights in the upper tail. Forristall (1978) suggests that a two-parameter Weibull distribution should be used, instead of a

Rayleigh, but Longuet-Higgins (1980) shows that a Rayleigh distribution with a rms of $0.925h_s/\sqrt{2}$ fits Forristall's data as closely as a Weibull distribution.

The principal use of the Rayleigh distribution for h_z seems to be to derive from h_s the most likely highest wave in three hours, $h_{max,3hr}$. This modal value is about 6% less than the expected highest wave, which is comparable to the reduction of wave height resulting from Longuet-Higgins suggested value of rms. Thus if this suggestion is correct, then the values derived for return values of $h_{max,3hr}$ are the expected highest and not the most likely highest, the difference for engineering purposes is perhaps not important.

4. PRESENT METHOD OF ESTIMATING RETURN VALUES - LONG TERM STATISTICS

4.1 Basic Method

The Institute of Oceanographic Sciences normally records 15-20 minute wave traces at three hourly intervals, from which are estimated the significant wave height (h_s) and the most likely highest wave in a three hour period ($h_{max,3hr}$), obtained using the method described in Section 3. Given these estimates of h_s and $h_{max,3hr}$ for a year or more, then N-year return values are determined by fitting a distribution to the data, and extrapolating into the tail of the distribution to the height with a cumulative probability corresponding to once in N years, from equation 2.1.

Sometimes the return value of $h_{max,3hr}$ is obtained by first estimating the return value of h_s and its associated value of zero-up-crossing wave period, T_z, from which the number of waves in three hours is obtained, and then using equation 2.3. The value of T_z is determined either visually from a 'scatter plot' of h_s: T_z or by assuming a wave steepness; because of the form of equation 2.3, the choice of T_z is not critical.

The method was first described by Draper (1963) when a log-normal distribution was proposed for the long-term distribution of wave height. Since then, a three-parameter Weibull distribution has generally been found to give a better fit, but Saetre (1974) decides that a Fisher-Tippett Type I distribution is better than a Weibull distribution for data from Famita (in the northern North Sea at 57.5°N 3°E), while Fortnum and Tann (1977) find a Fisher-Tippett Type III distribution fits values of $h_{max,3hr}$ from the Seven Stones Light Vessel (near 50°N,6°W). These distributions are defined in Appendix C. There is no theoretical justification for choosing any of these distributions; the one which appears to give the best straight line when the data are plotted on appropriately scaled paper is used. Thus the choice of distributions has been limited to those for which plotting paper can be constructed.

The present method does not plot all the data. The cumulative probabilities of data in half metre sets are evaluated - e.g. 0.79 less than 6 m, 0.82 less than 6.5 m... - and these values are plotted. The straight line is fitted either by 'least squares' or by eye - often the points corresponding to the lower wave heights do not appear to fall on a straight line, and a line is then fitted through only the higher points.

4.2 Discussion of method

This method has a number of practical advantages: it is simple to apply and it includes a visual representation of the data. However, for various reasons it is unsatisfactory. The major drawback is the lack of justification for the choice of distribution; how well the data appears to fit the chosen distribution, is no criterion for how good a fit can be expected into the tail of the distribution. Goodness-of-fit tests are too weak to be of any practical value. There can be no statistical justification for extrapolating from one year's data to the fifty-year return value (as is often done). However, if the result seems reasonable, compared for example with an estimate derived from the fifty-year return value of wind speed, then it suggests that the data offer no reason why the estimate is wrong.

There are other statistical errors in the present method - but they seem unimportant compared with the choice of distribution. They include using the 0.5 metre cumulative values rather than the full data set, and using a least squares fit when the data are neither independent nor normally distributed - so confidence limits for the regression line are also inappropriate. There is a further problem of plotting position: Gümbel (1958 para 1.2.7) advocates that the m^{th} ranked value of n values should be fitted at a probability of $m/(n + 1)$; and that for grouped data the mean value of the m^{th} to $(m + k)^{th}$ observations should be plotted at $m(m + k)/(n + 1)$. The method used in the present method of plotting at fixed height increments (of 0.5 m) gives undue weight to the higher values, even resulting in a plotted value for a 0.5 m step with no observations.

4.3 Effect of correlation of three-hourly estimates of h_s

Another statistical problem is raised by the possible correlation of successive three-hourly values of significant wave height. Even if they are correlated, the method of estimating the N-year return value from the cumulative distribution function using a probability from equation 2.1 is still applicable, i.e. the value exceeded on average once in N years will be obtained, but the distribution of exceedances about this mean value of N-years will be different. However, if the N-year return value is defined in terms of that height which is exceeded on average during one storm or 1 year in N, then the effect of correlation is to change the N-year return value.

Suppose, for sake of argument, that the larger values of significant wave height always occur in pairs, then the 50-year return value of 3-hourly h_s is the 100-year return value for pairs, the 50-year return value for pairs would be less. Similarly any positive correlation between waves, causing bunching of the high waves, will reduce the N-year return value of wave height in a bunch. To derive the precise return value of wave height in a bunch or in a storm requires a knowledge of the correlation of three-hourly values of h_s , and this is not known - it is not even known whether the correlation is a function of wave height. Other methods such as analysis of extremes (described in Section 7) or a 'peaks over threshold' model (see for example NERC, 1975, para 2.7) might be used.

Tann (1976) examines the effect of grouping of three hourly values of significant wave height during a 50-year storm upon the estimate from those wave heights of the most likely highest wave during the storm. Assuming that storms always last for D hours, and that significant wave height is constant throughout, then the 50-year return value of significant wave height in a storm, h_D, is given by extrapolating the cumulative distribution of significant wave height to a probability given by

$$P = 1 - Dhr/50yr \tag{4.1}$$

The value of h_D decreases as D increases, but the number of waves in D hours increases; and these two effects tend to compensate when $h_{max,Dhr}$ is estimated from equation 2.3 - assuming zero-up-crossing period is constant during the storm. In Tann's example - using Famita data fitted to a 3-parameter Weibull - he finds $h_{max,Dhr}$ slowly decreases with increasing D, with $h_{max,15hr}$ 3% - or 1.1 m - less than $h_{max,3hr}$. However, the assumptions that storms always last for D hours and that significant wave height and wave period are constant throughout seem too narrow to draw any general conclusions concerning grouping, let alone correlation effects.

4.4 Correcting for between-year variations

Because the estimates of N-year return wave heights are based upon so little data - usually only for one year - the present method includes an attempt to adjust the results, taking into account any climatic variation during the year of recording which can be detected from any nearby long series of wind measurements. The mean wind speed for the year is often used. For example in Draper (1976), because the average of the hourly mean wind speed at the Dowsing Light Vessel is estimated to be 7% lower during the year of wave measurements than the long-term annual average, it is suggested that recorded wave heights would have been about 10% lower than average - the relationship between wind speed and wave height is that proposed by Darbyshire (1963). Sometimes the number of hours or days of gales is used as a measure of climatic storminess.

An analysis of 50-year return values of $h_{max,3hr}$ for five years of data from Seven Stones Light Vessel by Fortnum and Tann (1977) shows a range in the estimates from each year's data of 24.4 m to 28.8 m. Table 4.1 gives the five values together with estimates from Fortnum and Tann of the corresponding mean wind speed and the maximum wind speed each year at the nearby Isles of Scilly during the years of recording (which did not all commence on 1st January). No correlation is evident between return wave height and either mean wind speed or maximum wind speed.

TABLE 4.1

Year	Seven Stones L.V. 50-yr return value $h_{max,3hr}$ (m)	Isle of Scilly mean wind speed* (m/s)	Isle of Scilly max wind speed* (m/s)
1968	24.4	5.6	19.1
1969	24.4	5.5	23.7
1971/72	27.5	5.6	19.1
1972/73	24.7	5.9	21.8
1973/74	28.8	5.4	23.2

Comparison of wind speed and wave height
(Values from Fortnum and Tann (1977))
*three hourly synoptic values (excluding winds from 050^o to 100^o) except for 1972/73 when adjusted one -hourly anemograph values were used.

Painting (1980) investigates the connection between wind speed and high wave heights at Ocean Weather Station India. He compares 50-year return value of wind speed and of wave height for the years 1957-74 (obtained from three-parameter Weibull analyses of three years data, running through the series from 1956 to 1975: wave heights from visual observations). Painting finds "little correlation" -as evidenced in Figures 4.1 and 4.2 which show plots of 50-year wave height against 50-year wind speed and against annual mean wind speed, using values taken from diagrams in his paper.

Therefore, whilst accepting the need to adjust estimates of return wave heights to allow for climatic variations during the years ot recording, the present method of doing this seems unsatisfactory: and further research is needed to try to improve upon it.

4.5 Lack of Confidence Limits

The present method does not include any estimate of confidence limits, and it seems most unlikely that any could be produced for this method of fitting cumulative half metre probability values by least squares. However, assuming that the chosen distribution is correct, then limits could be estimated if the distribution were fitted to all the data, using maximum likelihood estimators.

FIGURE 4.1

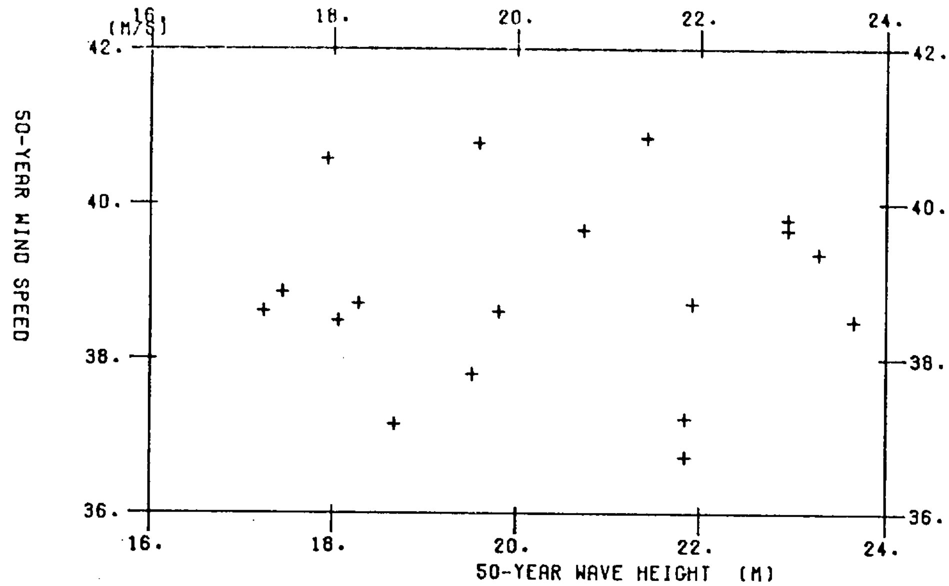
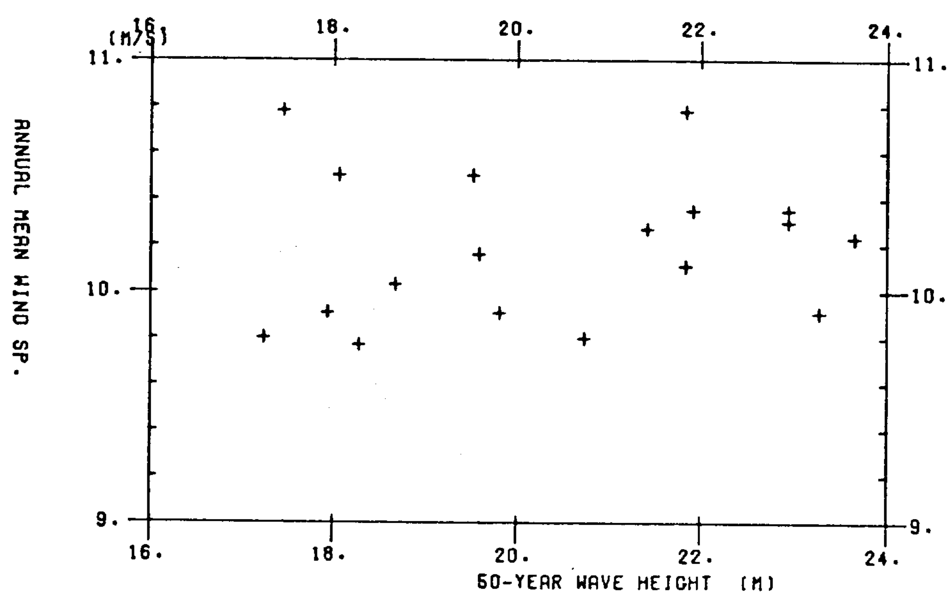


FIGURE 4.2



Figures 4.1 & 4.2: From analysis of wind and visual wave data at OWS 'I' for 1956-1975 by Painting(1980). 50-year return values: 3-parameter Weibull, fitted to data for each 3-years running.

4.6 Conclusions

The method of estimating low-probability return values of wave height by fitting one or two years' data to a distribution is unsatisfactory. Mainly because there is no theoretical justification for the choice of distribution so that considerable error can occur when extrapolating far into the tail of the distribution to determine the 50-year or 100-year return value.

Confidence limits are not produced for these estimates of return value. These could be derived, with slight modification to the method of fitting the distribution, but they would be conditional upon the choice of distribution being correct, so would not take into account the largest source of error.

If it is necessary to derive estimates of return values from only one year's data then it would appear sensible to attempt to relate the wave condition during that year with wind conditions that year and thence with the wind climate. However, the data available - such as that in Table 4.1- indicates that the present method of working with mean wind speed is unsatisfactory. The connection between wind speed and wave height is far too complex for such a simplistic approach.

5. ESTIMATING RETURN VALUES OF WAVE HEIGHT FROM WIND SPEED

Because of the connection between wind speed and wave height and because there are many more observations of wind speed than of wave height, several methods have been developed for deriving return values of wave height from wind speed data. There are various formulae in the literature for estimating wave height given wind speed together with its fetch and duration, e.g. Darbyshire and Draper (1963); and a method frequently used to estimate return value of wave height is to apply one of these formulae to an estimated return value of wind speed. Another method, being developed by N. Hogben at the National Maritime Institute, is to use the few joint observations of wind speed and wave height that are available to obtain a relationship between the distribution of these parameters thus enabling the wave height distribution - and hence return values - to be derived from any set of wind data. A third approach is to use a wave model with meteorological storm data to generate hindcast wave heights which are then analysed to give return values.

5.1 Estimating return values of wind speed

There are numerous sites in the U.K. at which anemograph records have been obtained regularly for many years - often for more than 20 years, sometimes for 50 years; and various analyses have been carried out to determine return values of wind speeds, notably analyses of annual maxima by Hardman et al (1973) - using a Fisher-Tippett Type I distribution with parameters estimated by the method of moments. (Extreme value analysis is described in Section 7.) The number of years of records available at U.K. stations and the return values estimated from them are given in Tables 3.8 and 3.10 of Shellard (1976). There are problems in analysing wind speeds: anemometers are at different heights above the ground, instruments have sometimes been changed or moved over the years, the effects of topography can be important but are difficult to quantify, the method of analysing annual extremes ignores seasonal variations and possible differences in wind speed distributions from different directions. However, because of the many years of data and because some of the inconsistencies in the data can be resolved by mapping the estimates of return values, the accuracy of these estimates is undoubtedly better than those for waves - although no confidence limits have been produced.

Determining return values of wind speed over the sea is a more dubious process. There are a few series of records from Ocean Weather Stations and from Light Vessels, and in recent years from off-shore platforms, but the bulk of wind data are observations from ships of opportunity. The U.K. Meteorological Office has used these data to prepare a chart of the 50-year return value of hourly mean wind speed over the North Sea and U.K. waters, published by the Department of Energy (1977) - a revised version is due to be produced shortly. The method of analysing the wind data to obtain this chart has not been published, but the method used to produce the revised version is based, according to D.J. Painting (private communication), upon fitting a three parameter Weibull to obtain the 5-year return value, then using the ratio between 50-year and the 5-year return values determined from coastal sites data.

The UK Meteorological Office finds, from anemograph records, that there is a constant ratio during storms between return values of the hourly mean wind and return values meaned over other periods; the ratios are given, for periods up to 24 hours, in Department of Energy (1977, Table 2.3).

5.2 Estimating wave height given return values of wind speed

The problem of calculating the return value of wave height from the corresponding wind speed is two-fold: an estimate of fetch or duration is required and a formula giving wave height from wind speed has to be selected. The choice of formula can lead to quite different answers for the high wind speeds associated with the 50-year or 100-year return value.

For example, Table 5.1 shows estimates of 50-year return value of significant wave height deduced from a 50-year return value of hourly mean wind speed of 35 ms^{-1} and a fetch of 120 km (appropriate for example to deep water off Douglas, Isle of Man where the maximum fetch is from the E.S.E.). Three formulae are used to estimate wave heights in this table: that for "coastal waters" given by Darbyshire and Draper (1963), one by Bretschneider (1973), and that derived from the JONSWAP results by Hasselmann et al (1976) and given by Carter (1981). The 50-year return value of significant wave height is 9.2 m, 7.1 m or 5.6 m, depending upon the choice of formula.

Results are given in Table 5.2 for a duration-limited wave with a 50-year return value of hourly mean wind speed of 42 ms^{-1} (appropriate to O.W.S. India at 59° N, 19° W). Wind speeds meaned beyond 24 hours were obtained by extrapolation of the figures given in Table 2.3 of Department of Energy (1977). Table 5.2 indicates a 50-year return value of significant wave height of about 18.4 m using the formula for "oceanic waters" given by Darbyshire and Draper (1963) and 16.4 m using Bretschneider, and 18.6 m using the JONSWAP results - but the extrapolation required to obtain the mean wind speeds for such long durations and the corresponding fetches - of 1000 - 2000 km - for the Bretschneider and JONSWAP values makes them questionable. A striking difference is seen in Table 5.2 between wave heights from the three formulae for the same duration. In the open ocean a wind of 42 ms^{-1} would not suddenly get up over a flat calm sea, and those differences for short durations must reflect the allowance in the formulae for both the sea state prevailing at the site at the initial time and the swell waves in the general vicinity. The JONSWAP results give a significant wave height, for a wind of 42 ms^{-1} for one hour, initially over a calm sea, of only 1.8 m (A measure of extrapolation is involved in the derivation of this value, because the wind speed during the JONSWAP experiment never rose above 20 ms^{-1}.)

TABLE 5.1

50-year mean wind speed (m/s)	duration of mean wind (hr)	Significant wave height (m)		
		Darbyshire & Draper(1963)+	Bretschneider (1973)	JONSWAP (Carter,1981)
35.0	1	3.2*	3.2*	1.4*
33.6	3	8.8*	5.5*	2.9*
32.6	6	9.2	7.1	4.6*
31.4	9	8.7	6.8	5.6
30.5	12	8.3	6.6	5.4

+Formula for "coastal waters"
*Duration-limited, otherwise fetch-limited with fetch of 120 km

Estimates of 50-year wave height off Douglas,IOM.

TABLE 5.2

50-year mean wind speed (m/s)	duration of mean wind (hr)	Significant wave height (m)		
		Darbyshire & Draper(1963)+	Bretschneider (1973)	JONSWAP (Carter,1981)
42.0	1	9.6	4.1	1.8
40.3	3	17.4	7.1	3.7
39.1	6	18.4	9.9	5.6
37.7	9	17.4	11.5	7.5
36.5	12	16.8	12.7	8.8
35.0	18	15.7	14.5	11.1
33.6	24	14.6	15.6	13.0
32.3	30	13.5	16.1	14.5
31.2	36	12.6	16.4	15.7
30.1	42	11.8	16.4	16.8
29.1	48	11.0	16.3	17.7
28.3	54	10.5	16.2	18.6
27.4	60	9.8	15.8	18.0

+Formula for "oceanic waters"

Estimates of 50-year wave height at OWS India

5.3 Method proposed by Hogben

Hogben and Miller (1980) outline a method of estimating wave height distribution from wind data. The first step is to find expressions - from available joint measurements of wave height and wind speed - for the mean wave height $\bar{h}$ (either significant wave height or visually observed values) and the variance of wave height σ^2 in terms of wind speed w. Wave height is separated into sea and swell components (h_{sea} and h_{swell}). They propose the following numerical relationships:

$$\bar{h}_{sea} = aw^n$$

$$\bar{h}^2 = (\bar{h}_{sea})^2 + (\bar{h}_{swell})^2$$

i.e.

$$\bar{h} = [(\bar{h}_{swell})^2 + (aw^n)^2]^{1/2} \qquad (5.1)$$

and

$$\sigma = \bar{h}_{swell}(b + cw) \qquad (5.2)$$

where $\bar{h}_{swell}$, a, b, c and n are empirical coefficients

Values for a, b, c, and n are given in Hogben (1979, Fig. 8b & c) for (i) open ocean sites, from analysis of data from OWS India and Seven Stones Light Vessel, and (ii) more sheltered sites, from analysis of data from Shambles, Varne, Owers, and Mersey Bar Light Vessels. Hogben and Miller (1980) suggest using a value for $\bar{h}_{swell}$

of 2 m and 1 m, for (i) and (ii) respectively. These values for the empirical coefficients - for wind speed w in knots, wave heights and standard deviations in m - are given in Table 5.3.

TABLE 5.3

	(i) Ocean Sites	(ii) More sheltered Sites
$\bar{h}_{swell}$(m)	2.0	1.0
a	0.033	0.023
b	0.5	0.75
c	0.0125	0.01875
n	1.46	1.38

Values for coefficients in equations 5.1 and 5.2 with wind speed w in knots and wave height and standard deviation in metres.

The second step is to assume that the conditional distribution of wave height -given wind speed - is a gamma distribution, and to determine the two parameters of this distribution from the mean and variance ($\bar{h}$ and σ from equations 5.1 and 5.2).

The gamma distribution is given by:

$$p(x) = c^{b+1} x^{b} \exp[-cx] / \Gamma(b+1)$$

with the mean and variance given by:

$$\text{mean}(x) = (b + 1)/c$$

$$\text{var}(x) = (b + 1)/c^{2}$$

Then, given any histogram of wind speed distribution, the distribution of wave height for each wind speed bracket is summed to give a marginal wave height histogram.

Finally, this wave height histogram can be fitted to a distribution, for example by plotting on Weibull paper, and extrapolated to determine the 50-year return value. Given regular wind observations (e.g. three-hourly) then the usual formulae can be used to determine plotting positions and the probability of the 50-year return value. If wind data are not at regular intervals and possibly with spatial variation, such as observations from ships of opportunity in a 1^{o} square, then the plotting positions and the return probability cannot be accurately defined; and an estimate would have to be made assuming a knowledge of temporal and spatial dependence of wave height. Hogben suggests that it is common practice to assume uniform density of observations in time and space, and to use for the return probability the reciprocal of the estimated number of observations during the return period. This requires the observations to be independent. More information is needed about the data set to judge whether this is satisfactory.

This method has been developed by Hogben using a considerable quantity of measured wind and wave data. His results in general seem to justify the use of equations 5.1 and 5.2 and of the gamma distribution. However, there is little evidence that these assumptions are valid at very high wind speeds and wave heights, so the estimates derived of fifty-year return wave heights are of questionable validity. Hogben and Miller (1980, para 4) admit that "the reliability of such 'extrapolation' methods ... can only be established by validation against high quality data based on measurements at fixed stations". As indicated in Hogben(1980,para 3.1.2), investigations aimed at establishing such validation are being undertaken by NMI in collaboration with the Meteorological Office.

5.4 Model methods

The use of formulae connecting significant wave height with wind speed, fetch, and duration has been extended into the formulation of significant wave models; but in recent years more emphasis has been given to the development of spectral models, estimating the wave energy in frequency bands: an early example was that by Darbyshire (1961). With increasing sophistication of the models, the demands upon the computer became too great, and led, in particular, to the development of a parametric wave model for the northern North Sea.

This North Sea Wave Model (NORSWAM) is described by Ewing et al (1979). The wind fields over the North Sea and adjacent waters were determined from synoptic weather charts during 42 storms, chosen as representative of all storms in the North Sea between 1966 and 1976. (Ref: Harding and Binding, 1978.) Wind waves were determined at grid points using a parametric wave model derived by Hasselmann et al (1976), while swell waves were advected along straight rays, using energy transfer between sea and swell given by Gunther et al (1979). The maximum values of significant wave heights thus determined at each grid point during each of the 42 storms were analysed by Ewing et al (1979), using a Fisher-Tippett Type I distribution, to derive estimates of 50-year return values. Results, shown in Figure 5.1, are about 1-2 m higher than significant wave heights corresponding to the maximum wave heights given in Department of Energy (1977).

NORSWAM was a lengthy and expensive project, although not so lengthy or nearly as expensive as obtaining real wave data throughout the area for ten years! Some uncertainty was introduced into the results by the selection of about one third of the storms that occurred during the period 1966-1976. Further work with the model could lead to improved results if the grid size for wind input outside the North Sea were reduced from 300 km to the 100 km used inside the North Sea; and the model could be extended to cover the southern North Sea if refraction of swell waves and dissipation of energy through bottom friction were included.

Considering the completely different approaches used to estimate the 50-year return values given in Department of Energy (1977) and by

NORSWAM, and the limitations of both methods, there is a surprisingly good measure of agreement in the results for the northern North Sea.

Further estimates could be extracted from the NORSWAM results, for example return values from specific directions.

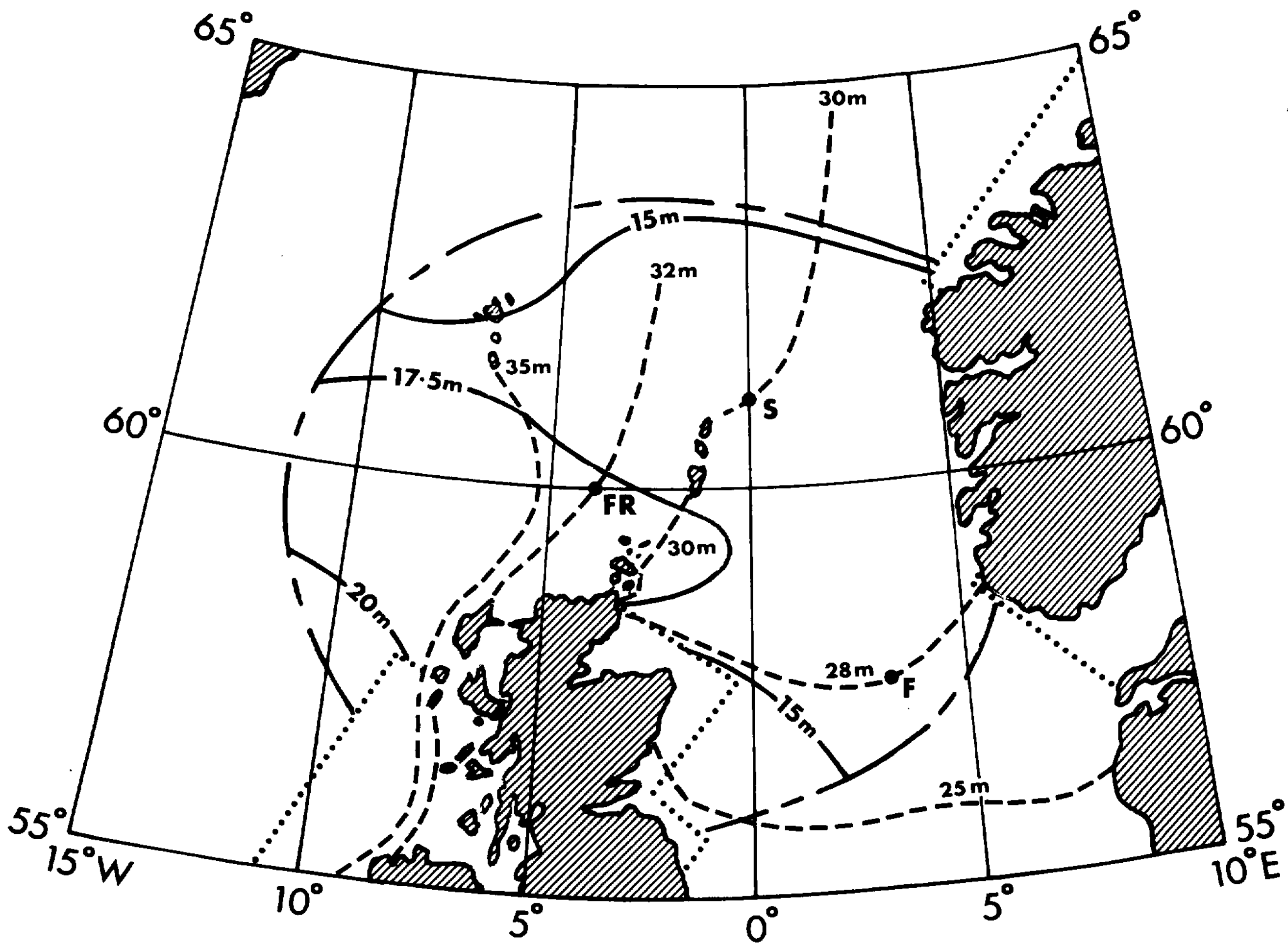


Figure 5.1 50-year return value of wave height

—— significant wave height, from NORSWAM study (Ewing, 1979)

- - - most likely highest wave in a 12-hr storm (D.En., 1977)

(Copy of Ewing, 1979, Fig.8)

Other wind/wave models might be used. A comparison is being made by J.A. Ewing (personal communication) between the NORSWAM model and that used by the Meteorological Office (described by Golding, 1978), using wave data measured during March 1980 at a number of stations.

A review of models, including a discussion of the relative merits of spectral models and parametric methods, is given by Ewing(1980). He concludes that estimates of significant wave height from models are generally adequate for engineering purposes, but that, since both approaches are based on linear or weakly non-linear assumptions, further work is necessary on the applicability of the results to extreme sea conditions.

5.5 Conclusions

There is so much wind data available compared with the amount of wave height data that estimating return wave heights from the wind data would seem to be a reasonable method. However, in practice the benefit of the quantity of wind data seems to be balanced out by complexity of the wind speed/wave height relationship, with rather small errors in wind speed estimations giving large errors in wave height. As expressed by Overland (1979): "The importance of accurate winds on the final wave calculations is demonstrated by a casual investigation of significant wave height tables. An increase in wind speed from 10 to 13 ms^{-1} (i.e. 5 knots) results in an increase in significant wave height from 2.5 to 4.0 m".

The 50-year return wave height estimated from a 50-year return value of wind speed depends critically upon the choice of formula - none of which was derived from observations with such high wind speeds. Moreover the estimates of 50-year return value of wind speed over the open ocean are themselves questionable - although it seems reasonable to use the spatial distribution of these estimated wind speeds as a guide to relative severity of storm conditions.

The method proposed by Hogben, involving the determination of wave height distribution from wind speed data, is being evaluated at the National Maritime Institute, with the co-operation of the Meteorological Office. Hogben acknowledges that validation tests against high quality data are required. It seems possible that the method will be more useful for describing the general wave climate than for estimating extremes.

The analysis of hindcast data from wave models - which incorporates the physics of wind/wave and wave/wave interaction - would seem to offer the best method - although the cost and effort involved are considerable.

6. RETURN VALUES OF ZERO-UP-CROSSING WAVE HEIGHT

The distribution of the maximum zero-up-crossing wave height, which was discussed in Sections 3 and 4 - with modal value given by equation 2.3 - assumes stationarity of the sea surface during the time p, i.e. it is the distribution of $h_{max,p}$ given significant wave height h_s. The present technique of recording wave data at three-hourly intervals assumes that the sea surface may be regarded as stationary for about three hours. The highest wave that might occur during a longer period, for example during a year, is not necessarily given by equation 2.3 with h_s equal to the largest value of significant wave height during the year. Not only is there a distinct probability of a larger wave occurring during the three hours with this value of h_s, but higher individual waves could also occur at other times, with lower values of h_s.

Assuming that three-hourly value of h_s are independent and that the highest waves within a 3-hourly period have a Rayleigh distribution, then the distribution of maximum wave height is given by

$$\text{Prob}(h_{max}<h) = \prod_i \{1 - \exp[-2(h/h_{si})^2]\}^{N_i} \qquad (6.1)$$

where N_i is the number of waves during the period when the significant wave height is h_{si} and may be estimated as $3hr/T_{zj}$ where T_{zj} is the zero-up-crossing wave period. Thus, given a 'scatter plot' showing the joint distribution of h_s and T_z during a year, then the median or other quantiles can be determined. Note that no allowance is made for the distribution of h_s during the years, so these results only apply to the time at which the measurements were made, and cannot be used to estimate N-year return values.

6.1 Battjes' technique

Battjes (1970) derives the marginal distribution of individual zero-up-crossing wave heights, h_z, from the joint probability distribution of significant wave height and zero-up-crossing period. This marginal distribution is given by

$$P(h_z<h) = \int \mathrm{Prob}(h_z<h|h_s).p(h_s).dh_s$$

where $p(h_s)$ is the p.d.f. of h_s associated with a random wave - not at a random time.

In practice, the integration is replaced by a summation, $\mathrm{Prob}(h_z<h|h_s)$ is taken to be a Rayleigh distribution, and $p(h_s)$ is estimated from the scatter plot. An estimate of the number of waves during a three-hour period is $3hr/T_z$, so the number of waves associated with a particular value of h_{si} and period T_{zj} in the scatter plot is $n_{ij}.3hr/T_{zj}$ where n_{ij} is the number of occurrences of h_{si} and T_{zj} from the scatter plot so

$$p(h_s) = (\sum_j n_{ij}T_{zj}^{-1})/(\sum_i \sum_j n_{ij}T_{zj}^{-1})$$

and

$$P(h_z<h) = \frac{\sum_i \sum_j \{1 - \exp[-2(h/h_{si})^2]\}^{n_{ij}T_{zj}^{-1}}}{\sum_i \sum_j n_{ij}T_{zj}^{-1}} \qquad (6.2)$$

This is the probability distribution of individual wave heights compared with equation 6.1 which gives the distribution of maximum wave height. It cannot be used directly to estimate 50-year return values, any more than equation 6.1, because it ignores the variation of h_s that would be observed over the years - and if, for example, there was one isolated very high value of h_s in the observed year this would essentially determine the tail of the distribution given in equation 6.2.

Battjes' method for deriving the 50-year return value is to compute values for $P(h_z<h)$ from equation 6.2 for increments of h (he uses increments of 4 ft) up to about twice the maximum value of h_s (which is roughly the 1-year return wave height). A Weibull distribution is then fitted to these values of (P,h) by plotting them on appropriately scaled probability paper and fitting a straight line which is extrapolated to give the 50-year return value. The relevant probability is obtained from an estimate of the total number of waves in fifty years, N_{50}. The formula for this depends upon the precise nature of the scatter plot. If this plot is derived from three hourly observations but expressed in parts per thousand, so the total of the numbers on the plot plus the number of calms per 1000 observations add up to 1000, then - assuming 365.25 days per year

$$N_{50} = 50 \times 2.922 \sum_i \sum_j n_{ij} T_{zj}^{-1}$$
$$= 146.1 \sum_i \sum_j n_{ij} T_{zj}^{-1}$$

and

$$P_{50} = 1-1/N_{50}$$

Battjes (1970) applied his method to data from Ocean Weather Stations India and Juliet and from five Light Vessels around the U.K. He fits a two-parameter Weibull distribution to these data by eye, and gets values for the shape parameter close to 1.0 - between 0.93 and 1.06 except for Morecambe Bay data for which he gets 0.85. "Thus implying that in general the long-term distribution of individual wave heights is nearly exponential (Battjes, op. cit.)

The three-parameter Weibull is given by

$$\mathrm{Prob}(h_z<h) = \begin{cases} 0 & h<A \\ 1-\exp[-((h-A)/B)^C] & h>A \end{cases} \qquad (6.3)$$

where B,C >0

The two-parameter Weibull is obtained from this with A = 0.

If the shape parameter C = 1, then 6.1 reduces to the negative exponential distribution:

$$\mathrm{Prob}(h_z<h) = \begin{cases} 0 & h<A \\ 1-\exp[-(h-A)/B] & h>A \end{cases} \qquad (6.4)$$

Carter and Draper (1979) apply Battjes' method to wave data from Ocean Weather Station Alpha (62°N, 33°W), but fit a three-parameter Weibull distribution and obtain a value for the shape parameter C of

0.998 (a two-parameter Weibull gives 0.90). Parameter A and B are 1.65 ft and 4.69 ft respectively. Clearly the negative exponential distribution cannot fit wave heights less than A, and must become increasingly inaccurate as wave height decreases towards A; but a negative exponential distribution appears to fit extremely well wave heights above about 4 feet at O.W.S. Alpha.

The Morecambe Bay records have recently been re-analysed by Draper and Carter (1981). A particular problem with this data set is that observations were obtained hourly during daylight only, thus

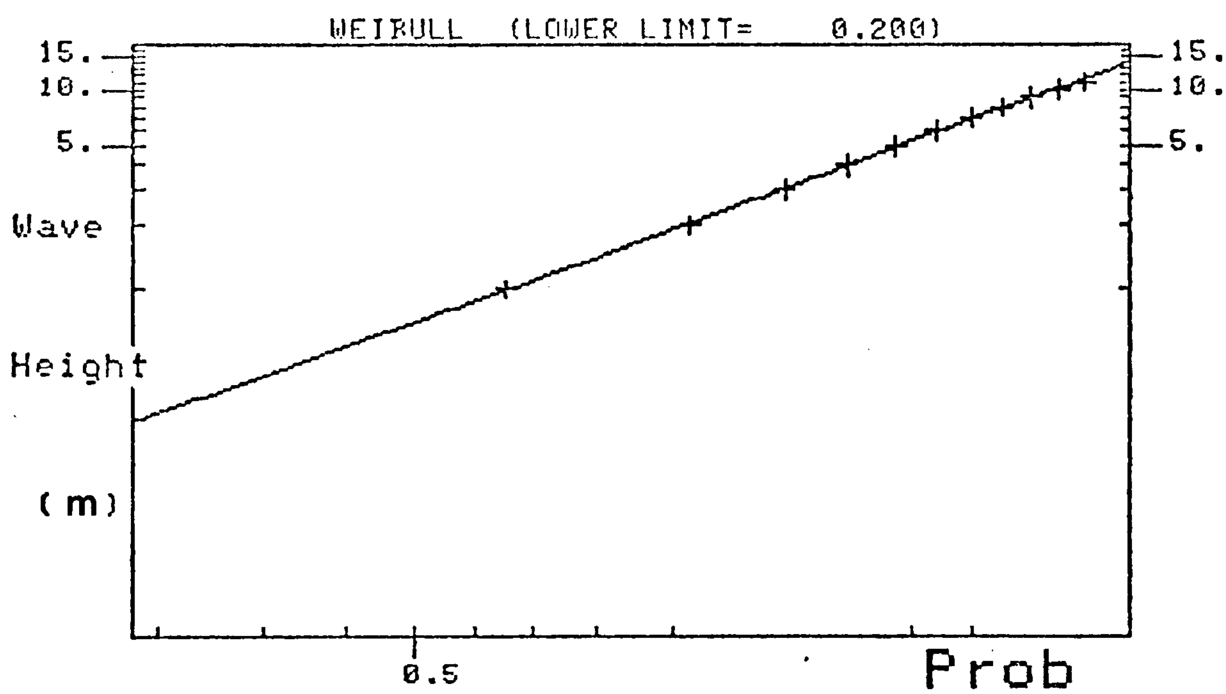


Figure 6.1 Distribution of zero-up-crossing wave height (Weibull scale) from Morecambe Bay Light Vessel data, 1957.

giving a highly correlated data set. The result of applying Battjes' method to the three-hourly measurements (at 0900, 1200, 1500 and 1800z) are shown in Figure 6.1, with cumulative probability plotted up to a wave height of 11 m on a Weibull scale. The value for parameter A of 0.2 m was determined by maximising the correlation coefficient; parameters B and C, found by linear regression, are 0.7470 and 1.0093 respectively. (Fitting a two-parameter Weibull gave C = 1.0858.)

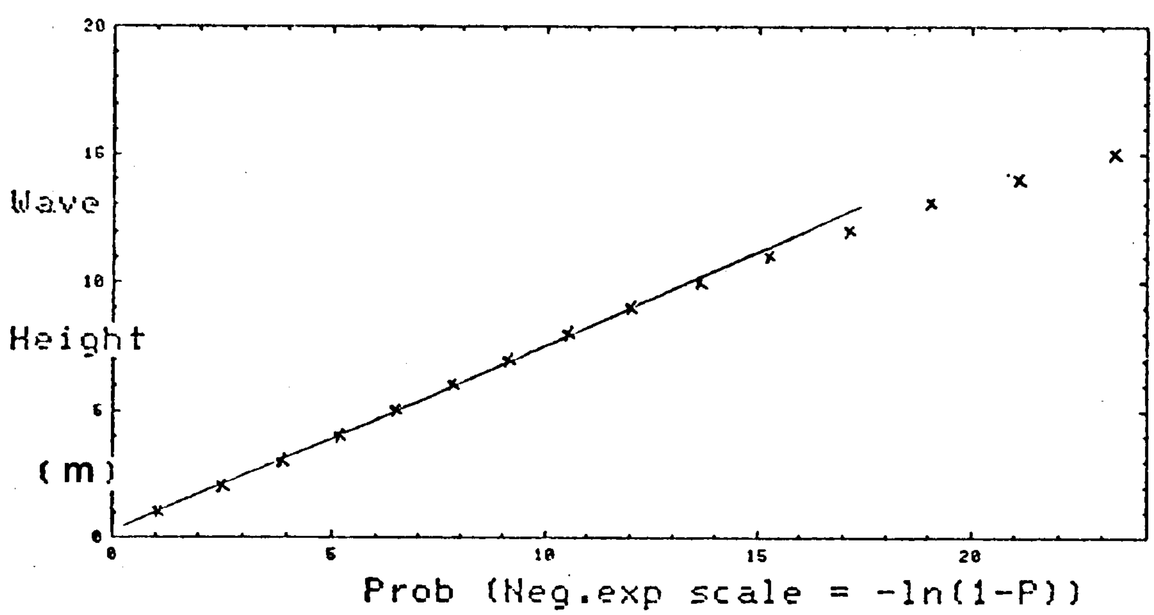


Figure 6.2 Distribution of zero-up-crossing wave height (negative exponential scale) from Morecambe Bay Light Vessel data, 1957.

Using the scatter plot in Draper and Carter (1981) derived from hourly data gave quite similar results, with A = 0.2 m, B = 0.7454 m and C = 1.0152.

Using the scatter plot in Draper (1968) - i.e. that analysed by Battjes (1970) - gives a value for C of 0.79 with a two-parameter Weibull (compared with Battjes' value of 0.85 determined by fitting a line by eye) and of 0.72 with a three- parameter Weibull. Therefore, the large difference found for C by Battjes with that from the more recent analysis seems to be due to inadequacies in the original analysis of the Morecambe Bay Wave records, and not to Battjes choice of a two-parameter Weibull.

The result from a scatter plot of three-hourly Morecambe Bay data plotted on negative exponential scales is shown in Figure 6.2. The fit up to 10 m appears very good. (The maximum value of significant wave height in the data was 5.5 m.) The effect of limiting the height of the data for this analysis is shown in Table 6.1.

TABLE 6.1

Limiting height(m)	50-year return value(m)
5	14.4
6	14.5
7	14.5
8	14.5
9	14.4
10	14.3
11	14.1
12	13.9
13	13.6
14	13.4
15	13.2

Variations in estimates of 50-year return value of zero-up-crossing wave height with upper limit of regression

Pickands (1975) shows that the upper tail of many distributions tend to a negative exponential distribution (see Section 8 of this report). But the distribution of individual waves is derived in a rather complicated way from two other distributions: the Rayleigh distribution and the unknown distribution of significant wave height. Whether Pickands' theory is applicable requires further research. The many thousands of waves involved ensure that the data extend well into the upper tail of the distribution, so an asymptotic limit would seem to be appropriate. However, in practice, even with Battjes' 'cut-off' at twice the maximum observed significant wave height, there is a tendency for the values in the extreme tail to be dominated by this one observed maximum; thus giving a Rayleigh distribution for individual wave heights - which would eventually tend to its own negative exponential distribution.

The negative exponential distribution fits the wave data so well that it might be supposed that any distribution combined with a Rayleigh distribution must have this shape, but this is disproved by the Morecambe Bay data used by Battjes.

6.2 Conclusions

Estimating return values of extreme wave using the technique proposed by Battjes (1970) makes several assumptions, notably that three-hourly estimates of significant wave height are independent and persist for three hours, and that, for a given significant wave height (or surface elevation variance), zero-up-crossing wave heights have a Rayleigh distribution; the effects of these assumptions have not been quantified. Battjes fits a Weibull distribution to the wave heights, but analysis of various sets of Shipborne Wave Recorder records indicates that a negative exponential distribution could be used. It is possible that the work of Pickands (1975) on asymptotic distributions of upper tails could be extended to justify theoretically the use of the negative exponential.

In practice there is the problem of deciding the height limit of data to use when fitting the distribution; Battjes' choice of twice the maximum observed significant wave height seems to be about right; but results from Morecambe Bay data indicate that using a higher value could seriously underestimate the 50-year return value.

7. THE ANALYSIS OF EXTREMES

Extreme value theory is frequently used to estimate return values of environmental parameters, such as wind speed, rainfall, and sea level. Usually one of the three asymptotic extreme value distributions of Fisher and Tippett is fitted to annual maxima. An advantage of the method is that the distribution of the population from which the maxima arise does not have to be known - but several years' data are required. In recent years sufficient wave data for this method have become available at a few sites around the U.K.

7.1 The Fisher-Tippett Limits

Consider an independent, identically distributed sample, X_i, size n. If the distribution of X ($=P(X<x)$) is $F(x)$ then the distribution of the largest in this sample is $F^n(x)$. Fisher and Tippett (1928) show that if there is a limiting distribution for the extreme values in a sample (as the sample size tends to infinity) then it must satisfy the following stability postulate:

$$F^n(a_m x+b_m) \ = F(a_{mn} x+b_{mn})$$

where a_m and b_m are constants depending upon m but not x, i.e. the distribution of the largest in a sample size mn must be the same as the distribution of the largest in n samples size m.

(The variable $a_m x+b_m$ is called the reduced variate).

Fisher and Tippett (1928) further show that there are only three possible solutions to this equation.

These solutions are:

The Fisher-Tippett Type I distribution:

$$P(X<x) = F(x) = \exp - \exp(-(x-A)/B) \qquad B>0 \qquad (7.1)$$

The Fisher-Tippett Type II distribution:

$$P(X<x) = F(x) = 0 \qquad x<A$$

$$= \exp - ((x-A)/B)^{-k} \qquad x>A \qquad (7.2)$$

The Fisher-Tippett Type III distribution:

$$P(X<x) = F(x) = \exp - ((A-x)/B)^{k} \qquad x<A \qquad (7.3)$$

$$= 1 \qquad x>A$$

These distributions are known under a wide variety of names. The Fisher-Tippett Type I (FT-I) is also known as the Gumbel distribution, the double exponential distribution, the extreme value distribution and the extreme value type I distribution. It is the most common of the three extreme value distributions and the latter half of this section will be concerned almost entirely with inferences based upon it.

The FT-II is also known as the Frechet distribution. It is not as widely used as the FT-I, although it has been applied to wave heights by Thom (1971). The transformation log (X-A) transforms an FT-II variable into an FT-I, if A is known. Similarly the transformation - log (A-X) will produce an FT-I from an FT-III variable. The FT-III is closely related to the Weibull distribution.

When choosing which extreme value distribution to use in practice it is useful to know which, theoretically, is the limit of which distribution. Technically any distribution whose limit is a particular extreme value distribution is said to be in the 'domain of attraction' of that distribution. A lot of work has been done on the domains of attraction of extreme distributions and various conditions have been produced for them (for details see Johnson and Kotz (1970) or Galambos (1978)). However, for our purposes the following description should be sufficient. Distributions that are in the domain of attraction of the FT-II are all unbounded above and have 'heavy tails', the best example (apart from the FT-II itself) is the Cauchy distribution. All distributions in the domain of attraction of the FT-III are bounded above, but this is not a sufficient condition, there are many distributions that are bounded above that have the FT-I as limit. The domain of attraction of the FT-I contains distributions with and without upper limits, it is by far the largest and contains all the 'well known' distributions such as the Weibull, the log normal, the normal etc. For this reason all the discussion in the remainder of this section will refer to the FT-I alone. One of the more useful criteria for determining to which domain of attraction a distribution belongs is in terms of 'positive moments' (these are similar to ordinary moments but defined only for positive x). If all the positive moments exist then the distribution has the FT-I as its limit, otherwise it belongs to the domain of attraction of the FT-II or FT-III according to whether or not it is bounded above.

7.2 The Fisher-Tippett Type I Distribution

As given above the FT-I distribution is

$$F(x) = \exp(-\exp(-(x-A)/B)$$

The probability density function is given by

$$f(x) = B^{-1} \exp[-((x-A)/B + \exp(-(x-A)/B))]$$

A plot of this pdf with A = 0, B = 1 is shown in Figure 7.1 The distribution is unbounded and unimodal with the mode at x = A; the median (= the 2 year return value if used with annual data) is given by

$$X_{(50)} = A\text{-}B\log\log 2$$

The 100pth percentile is given by

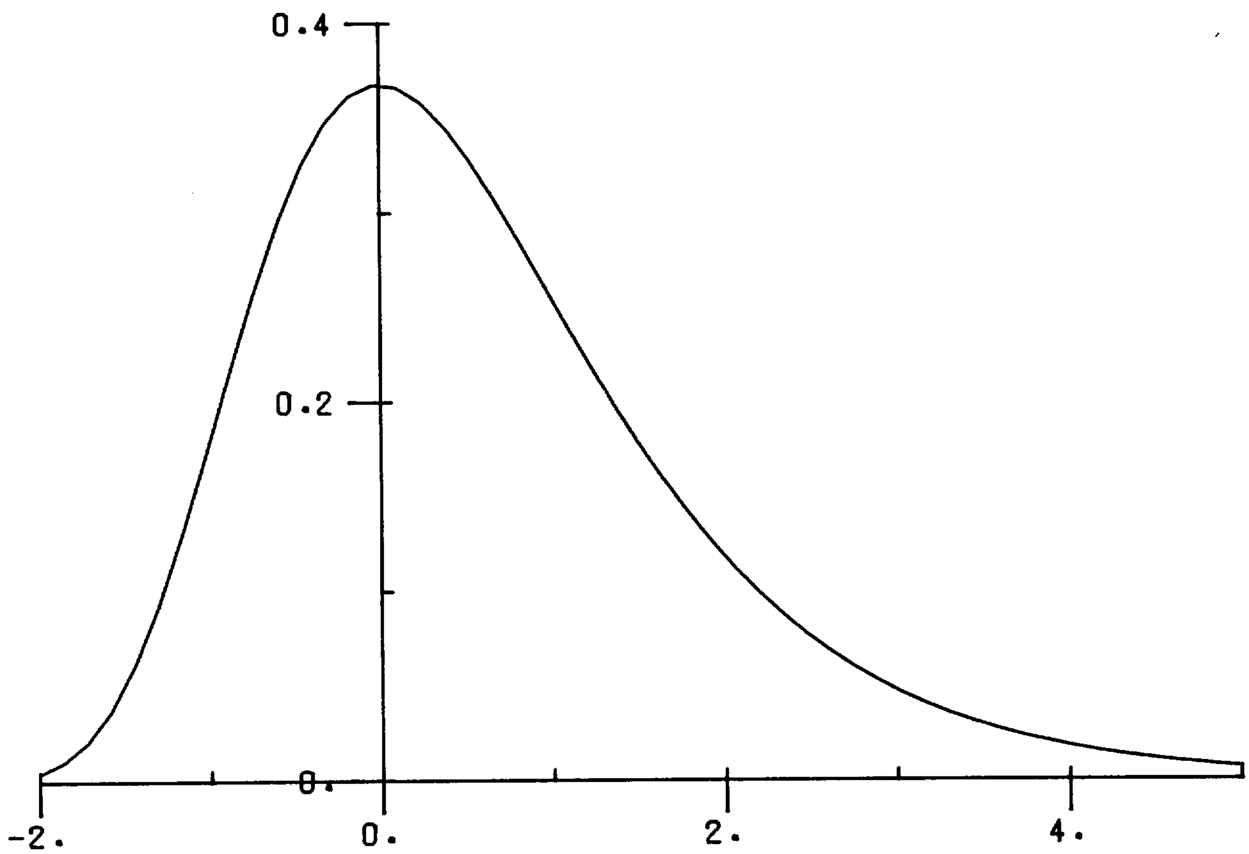
$$X_{(100p)} = A\text{-}B\log\log(1/p) \qquad (7.3)$$

All moments exist and the mean and variance are

$$\mu = A + \gamma B \qquad (\gamma \text{ is Euler's constant} = 0.5772)$$

$$\sigma^2 = \pi^2 B^2/6$$

The parameters, A and B, are location and scale parameters respectively. This means that whatever the values of A and B the shape of the distribution remains constant, varying A causing a location shift and B a scale change along the X-axis.



FISHER-TIPPET TYPE 1

FIGURE 7.1

7.3 Methods of Estimation

If p is P(X<x) the distribution function can be written

$$p = \exp(-\exp(-(x-A)/B))$$

Rearranging this gives

$$x = A-B \log-\log(1/p)$$

Thus FT-I probability paper can be constructed and estimates of A and B obtained. (Plotting positions are discussed in Appendix A).

The likelihood equations are

$$\sum_{j=1}^{n} \exp(-(x_j-A)/B) = n$$

$$\sum_{j=1}^{n} (x_j-A) \; [1-\exp(-(x_j-A)/B)] = nB$$

These cannot be solved analytically but an iterative scheme is easily produced (for details see Johnson and Kotz (1970)).

The maximum likelihood estimates are biased. However, simulations have shown that this bias is small and the mean square error is smaller than that for estimates derived from probability paper. (For details of these calculations see Carter and Challenor (1978)). It is therefore recommended that the maximum likelihood estimators are used for practical applications, possibly with a correction for bias. Alternative estimators are the moments estimators and while these are useful for quick initial estimates they are not entirely satisfactory.

Once the parameters A and B have been estimated it is a simple matter to insert the estimates into equation (7.3) and obtain estimates of return values. If maximum likelihood estimators are used then these estimators will also be biased (with Seven Stones data Carter and Challenor (1978) found that the bias in H_{50} was of the order of 4%).

In a series of papers Lawless (1972,1973a,1973b,1974,1978) considers confidence intervals for the FT-I. Most of these papers are concerned with confidence limits that are conditional on certain 'ancillary statistics' (i.e. on some functions of the data). However, in one paper (Lawless, 1974) he gives an approximation for unconditional intervals. This approximation has been tabulated by Challenor (1979). Tables are given there of confidence limits for A, B, $X_{(50)}$, $X_{(98)}$ and $X_{(99)}$ (where $X_{(p)}$ is the p^{th} percentile) and prediction limits of X_2, X_{10}, X_{50} and X_{100} (where X_n is the largest in a future sample size n).

7.4 Seasonal Variation

Fisher and Tippett (1928) obtain their stability postulate by assuming that the original data are independently and identically distributed. In practice, however, the data are neither independent nor identically distributed. Galambos (1978) considers the theory of both non-independent and non-identically distributed data. This theory, however, requires that the distribution of the original data is known, which is rarely the case in practice.

The problems caused by dependence do not seem to have a very great effect and if samples are taken at large enough intervals the dependence should not be important. However, it should always be borne in mind that the data are not independent and that this may have an effect, even if it is believed to be minor.

The problems of the data being non-identically distributed have been considered empirically by Carter and Challenor (1978, 1981). They split the year up into months and assuming the data are identically distributed within these months they analyse them separately. An 'annual' distribution is then found by combining the monthly distributions thus

$$P(X<x) = \prod_{i=1}^{12} F_i(x) \tag{7.4}$$

where $F_i(x) = P(X<x)$ in month i.

This equation assumes that the monthly extremes are independent, this may be called in question, but as there is little evidence either way it is not an unreasonable assumption to make. The return values obtained in this way are consistently greater than those calculated from annual maxima. (However, Dickinson (1977) seems to get contrary results, see Carter and Challenor (1981) for possible reasons for this.)

Why does analysing the data as monthly rather than annual maxima raise the return values? Partly this is due to the theoretical difference given in Carter and Challenor (1981) but also - as explained in that paper - it is because of extreme value theory: if the distributions fitted to the monthly extremes are one of the

Fisher-Tippett distributions then the distribution of annual maxima derived from (7.4) cannot be one. Although this explains why we cannot get the same answers it does not tell us whether the results obtained from monthly extremes should be greater than or less than those from annual extremes. To investigate this problem and either confirm or reject the hypothesis obtained from analysing data, a Monte Carlo experiment was performed. (N.B. In the above discussion the year was referred to as broken down into months, this need not of course be the case, any interval which is short enough for wave height to be considered stationary and long enough to ensure convergence to the Fisher-Tippett limits and independence of maxima could be used. There is no need for these intervals even to be of the same size, as long as the components in (7.4) are weighted accordingly.)

The simulation experiment was performed with the original data having an FT-I distribution so as to avoid any problem with the asymptotic limits not being reached (the maxima from samples of any size are distributed FT-I if the original data is FT-I). The experiment was in two parts. In the first part all the data generated were from the same distribution and the maxima for the whole sample and for each half of the sample were taken. This represents the case where the data are distributed identically throughout the year and are erroneously assumed to come from 'monthly' distributions, (the number of 'months' was taken to be two here simply to reduce the cost). The second part was identical, except that the data were produced from two distributions. This represents the case where the data have a different distribution in each 'month'. In each part samples size 200 were produced 6, 12 and 24 times, estimates produced and the whole process repeated one hundred times. The results are given in Tables 7.1 and 7.2.

Table 7.1 shows little difference between the bias, variance, and mean square error of the estimate of the 50-year return value (h_{50}) obtained by analysing the simulated values (from one distribution) as a single sample or as if from two separate distributions. Indicating that if data were identically distributed throughout the year, then no systematic error would be introduced into the estimate of h_{50} if the data were wrongly assumed to be from several distributions.

Table 7.2 shows that assuming an identically distributed population, when there were two distinct populations, led to considerably greater (negative) bias in the estimate of h_{50}. So assuming an identically distributed population would be expected to give a lower estimate for h_{50} than obtained by analysing the data from each population separately. The variance of the single sample estimates of h_{50} are less,but the mean square errors are greater.

TABLE 7.1

Sample Size		Single Sample	Separate Halves
6			
	Bias	-0.343	-0.163
	Variance	0.906	0.883
	MSE	1.024	0.909
12			
	Bias	-0.252	-0.183
	Variance	0.452	0.357
	MSE	0.515	0.391
24			
	Bias	-0.245	-0.287
	Variance	0.087	0.097
	MSE	0.147	0.179

Estimates of 50-year Return Value - Single Distribution
(A= 9.0, B= 0.75; h_{50}= 11.9265)

TABLE 7.2

Sample Size		Single Sample	Separate Halves
6			
	Bias	-1.816	-0.734
	Variance	2.774	4.647
	MSE	6.07	5.19
12			
	Bias	-1.757	-0.722
	Variance	1.058	1.979
	MSE	4.14	2.50
24			
	Bias	-1.752	-0.665
	Variance	0.438	1.094
	MSE	3.51	1.54

Estimates of 50-year Return Value - Two Distributions
(A= 8.0,B= 0.65 ; A= 6.5,B= 1.8 ;h_{50}= 13.542)

7.5 Conclusions

The statistical theory of extreme values enables return values to be calculated from a firm theoretical basis. There is no need for an arbitrary choice of distribution to fit the data, the choice is restricted to three, with some theoretical guidelines as to which should be used. It is necessary to have several years' wave data, which are presently available from a few sites, and even there the amount of data is so limited that confidence intervals are very broad - but an added advantage of extreme value analysis is that a method has been derived for estimating confidence intervals.

Problems arise, in common with other methods, concerning the dependence of observations and variation in the distribution of wave height throughout the year. Nevertheless, extreme value analysis would seem to offer the soundest statistical method of estimating return wave heights, if sufficient data are available.

8. FITTING THE TAIL OF WAVE HEIGHT DISTRIBUTIONS

In Section 7 the three limiting distributions for extreme values were described, similar distributions exist as limiting approximations to the upper tails of wave height distributions. If one of these, chosen by some means, is fitted to the data then estimates of return values can be obtained. Difficulties arise in choosing which 'tail distribution' to use and in estimating where the 'tail' of the data begins.

8.1 The Limiting Tail Distributions

Pickands (1975) shows that there are three limiting shapes for the upper tails of distribution functions of reduced variables (the reduced variables are similar to those considered for extreme values in Section 7), if such a limit exists. These limits are analogous to, and by using the results of Resnick (1971) can be shown to be equivalent to, the limiting distributions for extreme values described above. These distributions are:

$$F(x) = 1-e^{-x/a} \qquad x,a>0$$

$$F(x) = 1-(1+cx/a)^{-1/c} \qquad x,a,c>0$$

$$F(x) = 1-(1-|c|x/a)^{1/|c|} \qquad c<0,a>0, \quad 0<x<a/|c|$$

$$= 1 \qquad x>a/|c|$$

These distributions correspond to the FT-I , II and III respectively. The Type I tail distribution is, of course, the negative exponential distribution. Types II and III are examples of 'generalised' Pareto distributions (type I is a limiting case as c tends to zero).

8.2 Methods of Estimation

8.2.1 Problems of Estimation

When analysing extreme values the problem of estimation can be broken down into two parts: estimating which distribution is appropriate and estimating the parameters of that distribution. Here the problem is more complicated for, in addition to deciding which of the three tail distributions to use and estimating its parameters, a decision has to be made as to where in the 'tail' of the data the fit is to be made. This latter problem is not trivial and so far has not been solved completely. Several proposed solutions will now be given.

8.2.2. Pickands' Solution

Pickands (1975) proposes the following solution.

Starting at the largest data point move backwards through the ordered data fitting a generalised Pareto distribution at each point. The Kolmogorov-Smirnov distance is calculated at each point and the distribution used to calculate return values is the one fitted at the point that gives the smallest Kolmogorov-Smirnov distance.

(The Kolmogorov-Smirnov distance is defined as

$$Dn = \sup|S_n - F(x_n)|$$

where S_n is the empirical distribution function and F(.) is the distribution being fitted, evaulated at x).

The method used by Pickands to fit the generalised Pareto distribution is based upon quartiles.

Consider the mth largest value, Z_m (Z_i, i=1,n, represents the data in descending order) then the parameters are estimated by

$$c = (\log 2)^{-1} \log [(Z_m - Z_{2m})/(Z_{2m} - Z_{4m})]$$

and

$$a = (Z_{2m} - Z_{4m}) / \int_0^{\log 2} \exp(cu)\,du$$

$$= c(Z_{2m} - Z_{4m})/(2^c - 1) \qquad \text{if } c \neq 0$$

$$= (Z_{2m} - Z_{4m})/\log 2 \qquad \text{if } c = 0$$

(N.B. the tail is fitted up to Z_{4m} not Z_m).

The above procedure does not give good results in practice. The method of, possibly, choosing a different form of the tail distribution at each data point seems highly unstable and can give wildly varying estimates of return values. Possibly an improved method of estimating the parameters would help, but by using similar arguments to those advanced in Section ·7 the generalised Pareto distribution can be dispensed with and the negative exponential considered on its own.

8.2.3 Fitting the Negative Exponential Tail

The simplest way to fit an exponential tail is to follow Pickands′ method above only estimating a by its maximum likelihood estimator

$$a = \sum_{i=1}^{m} z_i \ /m$$

This method appears to work quite well, only there is little justification for using the minimum Kolmogorov-Smirnov distance as the goodness-of-fit test for the tail distribution. In addition to this it would be nice to be able to put some confidence level on the fit. For these reasons it is suggested that an exact goodness-of-fit test for the negative exponential distribution, of which there are several available, should replace the Kolmogorov-Smirnov distance. A good description of exact tests and tables of percentage points are given in Stephens (1974), no specific goodness-of-fit test is recommended here as any reasonably powerful test could be used. The procedure is similar to Pickands′ except that instead of choosing the fit that gives the smallest Kolmogorov-Smirnov distance, the goodness-of-fit test is performed until, for a certain size of tail, it is failed. The tail size chosen is the one used immediately before failing the test. So far it has not been possible to put any confidence limits on estimates produced by this method.

8.3 Conclusions

At first sight the method described above holds certain advantages over the other methods descibed in this report. The distributions used have a firm theoretical basis and unlike the method of extremes one year's data is sufficient. However the problems of estimation, in particular from where the tail distribution should be fitted, have not been finally solved. The proposed solution given above has not yet been adequately tested and only experience will tell if it gives reliable estimates. The method of fitting tail distributions must be regarded,at present, as experimental and in need of further research.

9. EXAMPLES

9.1 Seven Stones Light Vessel

A Shipborne Wave Recorder was first fitted in the Seven Stones Light Vessel (near 50^{o}N, 6^{o}W) in January 1962. A description of the data obtained during 1962 is given by Draper and Fricker (1965), but no estimate is made of return wave height. However, Battjes (1970) gives the result of fitting a three-parameter Weibull to the 1962 data, and this can be used to obtain a 50-year return value.

The analysis of the records from 1963 to 1967 is now considered to be unsatisfactory, and they are being re-analysed. Data for 1968 to 1974 have been reported on by Fortnum and Tann (1977), they derive 50-year return values by fitting various distributions to the data and by the analysis of annual maxima. An analysis of monthly maxima for the seven years of data obtained between 1968 and 1977 is given by Carter and Challenor (1978).

Various estimates of the 50-year return value of significant wave height from these papers are shown in Table 9.1, together with that from the Department of Energy(1977) and estimates for March obtained by the tail-fitting methods described in Section 8 - see also Table 4.1. Table 9.1 shows the range of acceptable values from 12.3 m to 15.2 m; corresponding values of $\hat{h}_{max,3hr}$ are approximately 23.4m and 28.9 m.

TABLE 9.1

Source	Height(m)	Comment
Battjes (1970)	13.5	Weibull to 1962 data
Dept. of Energy (1977)	13.2*	
Fortnum & Tann (1977)	12.3*	FT-III to all data 1968-74 (the best fit according to Fortnum & Tann)
"	12.5*	Weibull to all data 1968-74
"	13.2*	FT-I to all " " "
"	12.8*-15.2*	FT-I to each of 5 year's data
"	13.7*	Mean of FT-I to each of 5 year's data
"	12.9*	FT-III to 5 annual maxima
Carter & Challenor (1978)	12.4	FT-I to 7 annual max. (not recommended by Carter and Challenor)
"	14.2	FT-I to 7 March maxima (highest result for an individual month)
"	14.8	Product of monthly maxima analysis (value recommended by Carter and Challenor)
This report (Section 8)	8.6	Pickands method to March '73 data - max poss. 8.8m (clearly unacceptable) Neg. exponential fit to tail of March '73 data:
"	14.3	a) Cramer-von Mises goodness-of-fit test
"	13.8	b) Anderson-Darling g-o-f test

Various estimates of 50-year return value of significant wave height at Seven Stones L.V. (*derived from estimates of h_{max} by dividing by 1.9)

9.2 Famita

A Shipborne Wave Recorder was fitted in the Norwegian M.V. Famita in 1969, and wave data have been obtained from the ship's station at 57^{o} 30'N, 3^{o} E during the winter months since then - but with some considerable gaps when Famita was off-station replenishing or on other duties, so that about a quarter of the potential data is missing.

Draper and Driver (1971) give an analysis of the first winter's data. Saetre (1974) analyses three winters' data, and decides that an FT-I distribution fits the data better than a Weibull distribution.

Fortnum (1978) analyses six winters' data, fitting Weibull, FT-I, FT-III and log-normal distributions to the data to obtain estimates of 50-year return value of $h_{max,3hr}$. Neither the Weibull nor log-normal distribution gives a good fit over all the data, but the FT-I and FT-III appear to fit well. (The parameter limiting the height of the FT-III distribution is found to be very high, 193 m, so there is little difference between the FT-I and FT-III.)

Ewing et al (1979) use the results of the NORSWAM project - described in Section 5.4 - to obtain a 50-year return value of significant wave height at a grid point close to Famita's position.

Carter and Challenor (1978) derive 50-year return values by extreme value analysis of monthly maxima for six winters; and have since calculated the results including a seventh winter (1980, unpublished). Carter and Challenor (1981) give estimates from extreme value analysis of each month's data in the NORSWAM data set, and from extreme value analysis of data for each storm type in the NORSWAM data set, for the grid point close to Famita's position.

All these estimates of return value are given in Table 9.2, together with that from Department of Energy (1977). Values range from 14.4 m to 18.1 m, corresponding to $h_{max,3hr}$ of about 27.4 m to 34.4 m.

TABLE 9.2

Source	height(m)	Comment
Draper & Driver (1971)	14.4*	log-normal fitted to 1969/70 data
Saetre (1974)	16.3	FT-I fitted to 3 year's data
"	15.2	Weibull " " " "
"	14.6	Storm model with FT-I
Dept. of Energy (1977)	14.7*	
Fortnum (1978)	14.8*	FT-I to 6 winters data
"	14.5*	FT-III to " " "
"	14.9*	Weibull " " " (a poor fit)
"	16.1*	log-normal " " (" " ")
Ewing et al (1979)	16.2	NORSWAM result
Carter & Challenor (1981)	17.6	Product of monthly maximum analyses for 6 years
"	14.5	FT-I to winter maxima (Not recommended by Carter & Challenor)
"	18.1	Product of analyses of each month's data from NORSWAM
"	17.6	Product of analyses of each storm from NORSWAM
Carter & Challenor (1980 Unpublished Manuscript)	16.7	Product of monthly maximum analyses for 7 years (inc 1976/77)

Various estimates of 50-year return value of significant wave height at Famita (*derived from estimates of h_{max} by dividing by 1.9)

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10. GENERAL CONCLUSIONS

The present method used by the Institute of Oceanographic Sciences - and other organisations - to estimate the 50-year return value of wave height from a few years' wave data is in some ways unsatisfactory. The method involves fitting a distribution for which there is no theoretical justification, and moreover the amount of data available is insufficient to provide any evidence of goodness-of-fit around the probability of the 50-year return value.

Of the methods described in this report, the most satisfactory is the analysis of extreme values. However, it is necessary to have at least 5 years of data, which are available at very few sites; and 10-15 years of data are required to reduce the confidence limits to reasonable proportions. The method is theoretically sound only if the wave height values from which the maxima are derived are independent and identically distributed - and they are neither; but the lack of independence is probably not significant and the within-year variation of wave height distribution can be allowed for by analysing the maxima from each calendar month separately.

Two other methods which might be applied to only one year's data are described in this report, but neither has yet been developed to a level at which it could be recommended. These are a modification of the method of Battjes (Section 6) - fitting a negative exponential to zero-up-crossing wave heights, and the tail-fitting method described in Section 8. However, because of the between-year variation in wave distribution, it is highly doubtful if either of these methods will ever give a satisfactory estimate of 50-year wave height from only one year's data.

N. Hogben uses an empirical connection between the distributions of wind speed and wave height to derive a wave height distribution from wind data. His method - described in Section 5.3 - is being developed at the National Maritime Institute with the co-operation of the Meteorological Office. Careful validation will be required before it is possible to decide whether the method gives satisfactory estimates of return value. Because the connection between the distributions of wave height and wind speed is empirical, validation will have to extend well into the tail of the distributions. If successful, the method offers a possible means of investigating the effects of climatic variation upon return wave height.

The best way of obtaining values for the 50-year return wave height would be to make measurements for many years at numerous sites. The nearest approach to this ideal is to analyse hindcast wave heights from wind/wave models. Considerable progress has been made in recent years, both in our understanding of wave generation and in the development of models, but much remains to be done before these results can be used confidently to estimate extreme wave heights. For validation purposes, at least one long-term measuring station is necessary.

The principal conclusion of this report is that there is no method of estimating low probability return values of wave height that can be applied satisfactorily at present except at the few sites with some years of data. If it were not for the pressing needs of the offshore industry, one would not attempt to estimate these values. It is important for offshore engineers to appreciate the limitations of the estimates provided to them.

The range of significant wave heights shown in Tables 9.1 and 9.2, as estimates of 50-year return values, of about 3 m (corresponding to a range in $h_{max,3hr}$ of about 6 m) gives an optimistic rough indication of the accuracy of these estimates, because of the lack of justification for the various methods. A more realistic idea of the accuracy is given by the range of 90% confidence limits of the 50-year return value each month estimated at Seven Stones by Carter and Challenor (1978) using extreme value analysis; the average range for the twelve months is 6 m, with errors skewed towards the upper level.

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APPENDIX A

STATISTICAL METHODS OF ESTIMATION

A1 Properties of Estimators

Before describing the most common methods of point estimation a brief description will be given of those properties that it is thought desirable, or undesirable, for an estimator to possess.

Because an estimator is a statistic, i.e. a function of the sample values, it has a distribution and therefore possesses moments etc. like any other random variable. The expectation of the estimator need not be the true value of the parameter; N.B. this does not rule out the estimator as a good estimator we could just as well stipulate that the median or mode of the estimator should be the true value as the expectation. The difference between the expected value of the estimator and the true value of the parameter is defined as the bias in the estimator. If the parameter is denoted by θ and its estimator by $\hat{\theta}$ then

$$\text{bias} = E\ (\hat{\theta}) - \theta$$

A good example of a biased estimator is the maximum likelihood estimator of σ^2 for a normal distribution (μ, σ^2) with μ unknown. The M.L. estimator is

$$\frac{1}{n} \sum_{i=1}^{n} (X_i - X)^2$$

It is not only the expectation of an estimator that is important, its variance must also be considered. There are theoretical lower limits on the variance of an unbiased estimator (the Cramer-Rao lower bound). Few estimators achieve this bound, but the ratio of the lower bound to the variance of an estimator given as a percentage is called its efficiency. An estimator with no bias and large variance is obviously unsatisfactory as is one with large bias and small variance. A measure of combined bias and variance is the mean square error (MSE). This is defined as

$$MSE = E[(\hat{\theta} - \theta)^2]$$

It is easily shown that this is the same as

$$MSE = V(\hat{\theta}) + (\text{bias }(\hat{\theta}))^2$$

Usually estimators are compared in terms of their MSE and minimum mean square error (MMSE) estimators are sought. Occasionally only unbiased estimators are considered and minimum variance unbiased estimators (MVUE) are required.

One thing that is essential for any estimator is that it should become 'better' as the sample size increases. This property is referred to as consistency, i.e. if an estimator is mean square error consistent then its MSE tends to zero as the sample size tends to infinity.

A final property that is desirable for estimators to possess is that they should be functions of the sufficient statistic(s). A sufficient statistic is one that contains all the information in the sample concerning the parameters of the distribution. For example in the case of the normal distribution with both parameters unknown the sufficient statistics are $\sum X_i$ and $\sum X_i^2$. Estimators that are not functions of the sufficient statistics do not extract the full quantity of information from the data.

A2 Methods of Estimation

The methods of estimation considered here are the method of moments (briefly), maximum likelihood, and the use of probability paper. There are other estimators, e.g. best linear unbiased estimators (BLUE) or minimum variance unbiased estimators (MVUE), details of which may be found in statistical textbooks. Throughout this discussion examples will be given from the Rayleigh distribution:

$$f(x) = 2x\,\exp(-x^2/\theta^2)/\theta^2 \qquad x>0$$

$$=0 \qquad \text{otherwise}$$

A2.1 The Method of Moments

The method of moments consists simply of equating sample and population moments until there are as many equations as parameters and then solving these equations. The method only works of course if the population moments exist, for example the method cannot be used for the Cauchy distribution. Because of the nature of the method the estimators are always functions of the sample moments, indeed for a single parameter distribution the estimator will be a function of the sample mean. Therefore in most cases the moments estimator is not a function of a sufficient statistic.

In the case of the Rayleigh distribution the population mean is given by

$$E(x) = \theta/2$$

equating this to the sample mean gives

$$\theta/2 = \sum_{i=1}^{n} x_i/n$$

therefore

$$\hat{\theta} = 2\sum_{i=1}^{n} x_i/n$$

(Note if θ is the unknown parameter $\hat{\theta}$ is normally used to represent its estimator, however this usage is sometimes restricted to maximum likelihood estimators.)

A2.2 Maximum Likelihood

Consider a sample $x_1,\ldots,x_n$ from a distribution with pdf $f(x)$, then the probability of obtaining this sample is

$$\prod_{i=1}^{n} f(x_i)\,dx_i$$

since $f(x)dx$ is the probability that X lies in the interval x, x+dx. If the distribution contains one or more unknown parameters $\underline{\theta}$ it can be denoted by $f(x \cdot \underline{\theta})$. The likelihood is defined as

$$\prod_{i=1}^{n} f(x_i : \hat{\underline{\theta}})$$

i.e. the probability of the sample expressed as a function of $\underline{\theta}$, ignoring any multiplicative constants. By maximising the likelihood w.r. to $\underline{\theta}$ an estimator is obtained which is 'most likely'.

Because most p.d.f.'s contain exponential functions it is usually easier to work with the log likelihood

$$L = \sum_{i=1}^{n} \log f(x_i;\hat{\theta})$$

The log transformation is monotonic and therefore the value of $\underline{\theta}$ giving the maximum log likelihood is equal to that giving the maximum likelihood.

Maximum likelihood estimators (MLE) possess certain useful properties: the MLE of $g(\underline{\theta})$ is $g(MLE(\underline{\theta}))$, they are mean square error consistent (and hence asymptotically unbiased) and if a sufficient estimator exists then it is a function of the maximum likelihood estimator. Under certain regularity conditions MLE's are asymptotically normally distributed with mean equal to the parameter being estimated and variance equal to the Cramer-Rao lower bound. For large sample sizes they are therefore 100% efficient.

Returning to the example of the Rayleigh distribution; the likelihood is

$$\prod_{i=1}^{n} 2\, x_i \theta^{-2} \exp(-x_i^{\,2}/\theta^2).$$

$$\text{i.e.}\quad L = n \log 2 - 2n \log\theta + \sum\{\log(x_i) - x_i^{\,2}/\theta^2\}$$

differentiating w.r. to θ and setting the derivative to zero gives

$$-2n\hat{\theta}^{-1} + 2\sum x_i^2 \hat{\theta}^{-3} = 0$$

i.e. $$n\hat{\theta}^2 = \sum x_i^2$$

$$\hat{\theta} = \sqrt{\sum x_i^2 / n}$$

A2.3 The Use of Probability Paper

Consider the distribution function of a random variable X. If this depends on not more than two parameters (θ_1 and θ_2) then it may be possible to express the relationship between F(x) (=P(X<x)) and x as a linear function of θ_1 and θ_2. By using some estimate of F(x) this relationship can be plotted. This plot has two uses; firstly, the fit of a straight line to the data can be used as a measure of 'goodness-of-fit'; secondly by taking the slope and intercept of the straight line estimates of θ_1 and θ_2 can be formed. An example will make this clearer. Consider the Rayleigh distribution

$$f(x) = 2x\theta^{-2} \exp(-x^2/\theta^2)$$

$$P(X<x)=F(x) = 2\theta^{-2} \int_0^x t \exp(-t^2/\theta^2)\,dt$$

$$= 1-\exp(-x^2/\theta^2)$$

Put F(x)=p

i.e. $$p = 1-\exp(-x^2/\theta^2)$$

then

$$x^2 = \theta^2 \log[1/(1-p)]$$

So that if x^2 is plotted against $\log[1/(1-p)]$ then data from a Rayleigh distribution should lie on a straight line which will pass through the origin and have slope equal to θ^2.

There are certain problems associated with the above procedure. One is the estimation of F(x), these estimates are normally called plotting positions and are dealt with below. Another is the narrow range of application of the method. Most distributions are such that their distribution functions cannot be forced into a linear function of the parameters and so suitable probability paper cannot be produced. All distributions with three or more parameters, of course, are in this set. It is possible to estimate three parameters by using probability plotting: plot as if the third parameter were known, and then vary this 'known' parameter until the 'best fit' is achieved. If this is done the use of the plots for goodness-of-fit testing is, of course,invalid.

A2.4 Plotting Positions

When plotting on probability paper, consideration has to be given to the 'best' position in which to plot the data, i.e. the estimate of F(x) to be used. The optimal plotting positions in general vary with both the distribution being plotted and the use to which the plot is to be put - whether a visual representation of the data is required or a means of estimating parameters. Good general descriptions of the problems and some suggested solutions are given in Barnett (1975) and Gerson (1975).

The use of probability plotting was first suggested by Hazen (1914), for the normal distribution. The plotting position for the i^{th} point he suggested was i/n. This has the great disadvantage that the largest value has to be omitted (F(x)=1 being plotted at infinity). Since that time numerous plotting positions have been suggested. Simple, and robust, alternatives are: $i/(n+1)$ (Weibull, 1939), and $(i-1/2)/n$ (Hazen, 1930)

In the case of plotting to test the goodness-of-fit of a statistical model, it seems sensible to plot x_i against the cumulative probability associated with the expected value of x_i i.e. $F(E(x_i))$ - where x_i is the i^{th} ordered value of n from a density $f(x)$ and cumulative distribution $F(x)$.

The distribution of $x_i=y$, g_i say, is given by

$$g_i(y) = i^nC_i[F(y)]^{i-1}[1-F(y)]^{n-i}f(y) \qquad \text{(A1)}$$

(For proof see for example Gibbons, 1971 para 2.3)

$$\text{So} \quad E(y) = \int y\, g_i(y)dy \qquad \text{(A2)}$$

For example if $F(x)$ is the uniform distribution, $U(0,1)$ then it may be shown that

$$E(x_i) = i/(n+1) \qquad \text{(A3)}$$

i.e. the plotting position suggested by Weibull (1939). There is, in general no analytical solution to equation A2, - although equation A3 is an approximate solution for any distribution, and is recommended by Gumbel (1958, para 1.2.7). Numerous other approximations have been suggested for various forms of $F(x)$.

A good approximation for the normal distribution, according to Blom (1958), is $(i-3/8)/(n+1/4)$. Bernard and Bas-Levenbach (1953) suggest using the median rather than the expectation and give $(i-0.3)/(n+0.4)$.

In the case of the FT-1, Gringorten (1963) recommends $(i-0.44)/(n+0.12)$ if $n>20$, and shows that equation A3 is not satisfactory for this distribution. He deduces this plotting position assuming an expression of the form $(i-a)/(n+1-2a)$ and chooses the value of a so as to "place the largest observed value as close as possible to the ideal straight line". NERC(1975, para 1.3.2) advocates using Gringorten's plotting position. However, for the FT-1, the exact values can be derived, using the moment generating function of g_i given by Gumbel (1958 para 6.1.4). This leads to the expected value of $y_m=(x_m-a)/b$ where y_m is the reduced variate of the m^{th} value from above $(m=n+1-i)$:

$$z_m=E(y_m) = \sum_{k=0}^{m-1} \{ {}^nC_{m-1} \,.\, {}^{m-1}C_k \,.\, (-1)^k \,.\, (n-m+1)(n-m+k+1)^{-1} \, [\gamma+\ln(n-m+k+1)] \}$$

and the corresponding cumulative frequency

$$F(E(y_m)) = \exp(-\exp(-z_m))$$

For example, for the largest value, m=1:

$$E(Y_1) = \gamma + \ln n$$

$$F(E(Y_1)) = \exp(-\exp(-\gamma - \ln n))$$

$$\approx 1/(1.753)^{1/n}$$

The criteria for 'optimal' plotting positions that are to be used in estimation are rather different. The theory is somewhat complex; details are given in Chernoff and Lieberman (1954, 1956), Barnett (1975) and Cran (1975). For example, Barnett (1975) gives plotting positions which might be used in conjunction with linear regression to obtain unbiassed estimates, that are 'optimal' in some sense, for both the normal and FT-1 distributions. In our experience, those for the FT-1 give similar results to maximum likelihood, however we have not examined the MSE's of these estimators.

APPENDIX B

CONFIDENCE AND PREDICTION LIMITS

Confidence intervals are an alternative means of estimating parameters to the usual point estimator. Instead of giving a single number as the estimate an interval is given, which has associated with it a probability. If both point and interval estimates are given the latter can be used as a measure of confidence in the former. The probability associated with a confidence interval is sometimes supposed to give the probability that the true value of the parameter lies within the given interval. This is incorrect. What the associated probability does give us is the proportion of intervals, calculated from repeated sampling, that will cover the true value of the parameter. It should always be borne in mind that it is the ends of the interval that are random variables and the true value of the parameter that is fixed.

An infinitude of, say, 90% confidence intervals can be produced for any parameter. The definition does not state how the interval should be aligned with the point estimate, it does not even have to be continuous. However in practice the intervals given are usually one of two types. The first type is centred upon the point estimator, $\hat{\theta}$, and are usually given as $\hat{\theta} \pm t$ where t is half the length of the interval. The second, and more common, type is where the interval is positioned such that there is an equal chance (p/2 for a 100 (1-p)% interval) that the confidence interval is either entirely above or entirely below the true value of the parameter.

For symmetric distributions these two types are of course identical. All confidence intervals given in this report are of the latter type.

Prediction intervals are confidence intervals on a predicted future observation, i.e. they give an indication of the precision of predictions made from the data. They differ from confidence intervals in that here both the ends of the interval and the point, the future observation, are random variables.

APPENDIX C

SUMMARY OF USEFUL DISTRIBUTIONS

For further details of these distributions, see either NERC(1975) or Johnson & Kotz(1970).

C1. Normal distribution

pdf: $f(x) = (\sqrt{2\pi}\,\sigma)^{-1}\exp[-(x-\mu)^2/\sigma^2]$ $\qquad -\infty<x<\infty$

mean $= \mu$

variance $= \sigma^2$

C2. Log Normal distribution

pdf: $f(x) = [(x-\theta)\sqrt{2\pi}\,\sigma]^{-1}\exp[-\{\log(x-\theta) - \zeta\}^2/\sigma^2]$ $\qquad (x>\theta)$

[$Z=\log(X-\theta)$ is normally distributed.]

mean $= \exp(\zeta + \sigma^2/2) + \theta$

variance $= \exp(2\zeta + \sigma^2)\ [\exp(\sigma^2) - 1]$

The two-parameter version is given by $\theta = 0$

C3. Gamma distribution

pdf: $f(x) = (x-\theta)^{\lambda-1}\exp[-(x-\theta)/\alpha]/\alpha^{\lambda}\,\Gamma(\lambda)$ $\qquad (x>\theta)$

mean $= \alpha\lambda + \theta$

variance $= \lambda\alpha^2$

The two-parameter gamma distribution is given by $\theta = 0$

C4. Cauchy distribution

pdf: $f(x) = 1/\{\pi\lambda\,[1 + ((x-\theta)/\lambda)^2]\}$ $\quad -\infty<x<\infty$

This distribution has no moments of order ≥ 1, so it does not possess a mean or variance.

median = mode = θ

C5. Weibull distribution

pdf: $f(x) = \lambda\,\alpha^{-1}\{(x-\theta)/\alpha\}^{\lambda-1}\exp[-\{(x-\theta)/\alpha\}^{\lambda}]$ $\quad x>\theta$

$P(X<x) = F(x) = 1 - \exp[-\{(x-\theta)/\alpha\}^{\lambda}]$

mean $=\alpha\,\Gamma(\lambda^{-1} + 1) + \theta$

variance $= \alpha^2\{\Gamma(2\lambda^{-1} + 1) - [\Gamma(\lambda^{-1} + 1)]^2\}$

The two-parameter version is given by $\theta = 0$

C6. Exponential distribution

pdf: $f(x) = \alpha^{-1}\exp[-(x-\theta)/\alpha]$ $\quad x>\theta$

$P(X<x) = F(x) = 1 - \exp[-(x-\theta)/\alpha]$

mean $= \theta + \alpha$

variance $= \alpha^2$

The exponential distribution is a special case of the gamma distribution (λ =1) and of the Weibull distribution (λ =1).

C7. Rayleigh distribution

pdf: $f(x) = 2\,\alpha^{-2}(x-\theta)\exp\{-(x-\theta)^2/\alpha^2\}$ $x>\theta$

$P(X<x) = F(x) = 1 - \exp\{-(x-\theta)^2/\alpha^2\}$

mean $= \frac{\sqrt{\pi}}{2}\,\alpha + \theta$

variance $= (1 - \frac{\pi}{4})\,\alpha^2$

The Rayleigh distribution is a special case of the Weibull distribution ($\lambda=2$).

C8. Fisher-Tippett Type I or Gumbel distribution

pdf: $f(x) = \alpha^{-1}\exp\{-(x-\theta)/\alpha - \exp[-(x-\theta)/\alpha]\}$ $-\infty<x<\infty$

$P(X<x) = F(x) = \exp\{-\exp[-(x-\theta)/\alpha]\}$

mean $= \theta + \gamma\alpha$ ($\gamma = 0.5772\ldots$)

variance $= \frac{\pi^2\alpha^2}{6}$

Further details are given in Section 7.2

C9. Fisher-Tippett Type II or Frechet distribution

pdf: $f(x) = \alpha^{1/\lambda}\,\lambda^{-1}(x-\theta)^{-(1+1/\lambda)}\exp[-\{(x-\theta)/\alpha\}^{-1/\lambda}]$

$x>\theta,\ \lambda>0$

$P(X<x) = F(x) = \exp[-\{(x-\theta)/\alpha\}^{-1/\lambda}]$

mean $= \alpha\Gamma(1-\lambda) + \theta$

$$\text{variance} = \alpha^2\{\Gamma(1-2\lambda) - [\Gamma(1-\lambda)]^2\}$$

Therefore the mean does not exist if $\lambda > 1$ and the variance does not exist if $\lambda > 1/2$

$Y=\log(X-\theta)$ is distributed FT-I

C10. Fisher-Tippett Type III distribution

$$\text{pdf:} \quad f(x) = \lambda\alpha^{-1}\{(\theta-x)/\alpha\}^{\lambda-1}\exp[-\{(\theta-x)/\alpha\}^{\lambda}] \qquad x<\theta$$

$$P(X<x) = F(x) = \exp[-\{(\theta-x)/\alpha\}^{\lambda}]$$

$$\text{mean} = \theta - \alpha\Gamma(\lambda^{-1}+1)$$

$$\text{variance} = \alpha^2[\Gamma(2\lambda^{-1}+1) - \{\Gamma(\lambda^{-1}+1)\}^2]$$

$Y=-X$ is distributed Weibull: $Z=-\log(\theta-X)$ is distributed FT-I.

APPENDIX D

MOST LIKELY MAXIMUM OF n VALUES FROM A RAYLEIGH DISTRIBUTION

The cumulative distribution of n values from a Rayleigh distribution is

$$\mathrm{Prob}(x_{max}<x) = F(x) = [1 - \exp(-x^2/2)]^n \qquad \text{D1}$$

The most likely value of x_{max} is given by

$$d^2F/dx^2 = 0$$

which reduces to:

$$1 - x^2 + (nx^2 - 1)\exp[-(x^2/2)] = 0 \qquad \text{D2}$$

i.e.

$$x^2/2 = \ln[(nx^2 - 1)/(x^2 - 1)]$$

So

$$x^2/2 = \ln n + \ln[(nx^2-1)/(nx^2-n)] \qquad \text{D3}$$

Putting $x^2 = 2\theta$

then equation D3 reduces to:

$$\theta = \ln n + \ln\{[(4n\theta-n-1)/(n-1)+1]/[(4n\theta-n-1)/(n-1)-1]\}$$

For $|z|>1$

$$\ln\{(z+1)/(z-1)\} = 2\{(1/z) + (1/z)^3/3 + (1/z)^5/5 + \ldots\}$$

Therefore

$$\theta = \ln n + 2\{[(n-1)/(4n\theta-n-1)] + [(n-1)/(4n\theta-n-1)]^3 + \ldots\} \quad \text{D4}$$

For large n

$$(n-1)/(4n\theta-n-1) \simeq n/(4n\theta-n)$$

$$\simeq 1/(4\theta-1)$$

But Longuet-Higgins(1952,equation 67) shows that

$$\theta = \ln n + O\{(\ln n)^{-1}\}$$

i.e.

$$\theta \simeq \ln n \quad \text{D5}$$

Therefore

$$1/[4\theta-1] = 1/[4(\ln n) - 1] + O\{(\ln n)^{-1}\}$$

Therefore, from equation D4

$$\theta \simeq \ln n + 2/[4(\ln n) - 1]$$

$$\simeq (\ln n)\{1 + 1/[2(\ln n)^2]\}$$

Comparing equations 3.10 and D1

$$\theta = x^2/2 = 2(h/h_s)^2$$

Therefore the most likely maximum value of h is given, for large n, by

$$2(h/h_s)^2 \simeq \ln n\{1 + 1/[2(\ln n)^2]\}$$

i.e. $$h/h_s \simeq ((\ln n)/2)^{1/2}\{1 + 1/[4(\ln n)^2]\}$$

Therefore

$$h/h_s \simeq ((\ln n)/2)^{1/2}\{1 + 1/[16((\ln n)/2)^2]\} \qquad \text{D6}$$

Although equation D6 is only true for large n, a comparison of exact values of h (from a numerical solution of equation D2) with those from equation D6 shows an error of less than 2% for n>4. (Values are given in Table D1, which also gives values of h/h_s estimated from equation D5 showing very much larger errors, particularly for small n.)

TABLE D1

n	exact solution	equation D5	% error	equation D6	% error
2	0.728	0.589	19.13	0.895	22.91
3	0.840	0.741	11.78	0.895	6.48
4	0.912	0.833	8.76	0.941	3.18
5	0.966	0.897	7.10	0.984	1.83
7	1.043	0.986	5.36	1.052	0.88
10	1.119	1.073	4.17	1.124	0.38
15	1.201	1.164	3.12	1.203	0.18
20	1.257	1.224	2.64	1.258	0.07
100	1.536	1.517	1.20	1.535	0.02

Comparison of approximate solutions for the most likely maximum wave height of n values from a Rayleigh distribution with the exact solution from equation D2.

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