A boundary element model for nonlinear viscoelasticity

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Abstract. The boundary element methodology is applied to the analysis of non-linear viscoelastic solids. The adopted non-linear model uses the same relaxation moduli as the respective linear relations but with a time shift depending on the volumetric strain. Nonlinearity introduces an irreducible domain integral into the original integral equation derived for linear viscoelastic solids. This necessitates the evaluation of domain strains, which relies on proper differentiation of an integral with a strong kernel singularity. A time domain formulation is implemented through a numerical integration algorithm. The effectiveness of the developed numerical tool is demonstrated through the analysis of a plate with a central crack. The results are compared with respective predictions by the finite element method.

Introduction

Many polymers exhibit highly nonlinear viscoelastic behaviour in areas of stress or strain concentrations such as those arising from the presence of cracks. Material non-linearity manifests itself as considerable strain softening near the crack tip. The development of numerical techniques for the implementation of relevant constitutive models describing such behaviour has been an important research objective. Non-linear viscoelastic solutions, based on the finite element method (FEM), have been formulated and tested for efficiency and stability [1].

The boundary element method (BEM) has been extensively and very effectively used in modelling linear viscoelastic behaviour [2]. It has, in particular, been found a reliable tool for the analysis of viscoelastic fracture mechanics problems [3]. It seems, however, that there has not been any previous attempt to extend such formulations to modelling the nonlinear behaviour of polymers.

Various constitutive models have been proposed for representing nonlinear viscoelasticity in polymers [4]. Schapery [5] proposed a quite general and frequently applied model, which includes the principle of time-stress superposition. The latter is accounted for through the definition of 'reduced time', a concept originally introduced to account for temperature variation [6]. Based on experimental studies, Knauss and Emri [7, 8] linked the time-stress superposition model to the concept of free volume. This constitutive model has been applied to various problems [9, 10] and found to be a very effective analysis tool for assessing the effect of nonlinearity on the behaviour of polymer materials.

The non-linear visco-elastic model employed in the present BEM formulation is based on the reduced time concept, which is, in turn, considered as a function of mechanical free-volume changes. The relaxation moduli of linear visco-elasticity are thus employed in the Boltzmann constitutive equations with a time shift depending on the volumetric strain. The difference between the actual stress tensor and its linear counterpart generates an irreducible domain integral into the original integral equation derived for linear viscoelastic solids. Domain strains are obtained by differentiation of a domain integral with a strong kernel singularity resulting in a singular integral and a regular free term. A time domain formulation is implemented through a numerical integration algorithm. The effectiveness of the developed numerical tool is demonstrated through the analysis of a plate with a central crack subjected to remote tension. The results are compared with respective predictions by the finite element method.

Background theory

The linear viscoelastic model adopted in earlier BEM formulations [11] is, in accordance with Boltzmann's principle, of hereditary integral type

$$\sigma_{ij} = G_{ijkl}(t)\varepsilon_{kl}(0) + \int_0^t G_{ijkl}(t-\tau)\frac{\partial \varepsilon_{kl}(\tau)}{\partial \tau}\mathrm{d}\tau \tag{1}$$

where σ_{ij}, ε_{ij} are the stress and small strain tensors, respectively, and $G_{ijkl}(t)$ the relaxation moduli in the general case of an anisotropic medium. The problem is described relative to a Cartesian frame of reference x_i, i =1,2,3, adopting the summation convention for repeated indices. Introducing the notation for the Stieltjes convolution of two functions [12], eq (1) can be more concisely written as

$$\sigma_{ij} = G_{ijkl} * \mathrm{d}\varepsilon_{kl} \tag{2}$$

The nonlinear constitutive equations adopted here are [9]

$$\sigma_{ij} = \int_{-\infty}^t G_{ijkl}[\zeta(t)-\zeta(\tau)]\frac{\partial \varepsilon_{kl}(\tau)}{\partial \tau}\mathrm{d}\tau = G_{ijkl}[\zeta(t)]\varepsilon_{kl}(0) + \int_0^t G_{ijkl}[\zeta(t)-\zeta(\tau)]\frac{\partial \varepsilon_{kl}(\tau)}{\partial \tau}\mathrm{d}\tau \tag{3}$$

where $\zeta(t)$ is the reduced or intrinsic time, which may account for the effect of temperature [6], moisture and pressure variations on the relaxation moduli. A general definition of $\zeta(t)$ is

$$\zeta(t) = \int_0^t \frac{\mathrm{d}\tau}{\phi[v(\tau)]} \tag{4}$$

and ϕ is a shift factor, which depends on the fractional free volume v. Here, only the influence of mechanically induced aging is considered, thus v is expressed only in terms of volumetric strain by

$$v = v_0 + C\varepsilon_{kk} \tag{5}$$

where v_0 is the fractional free volume at some reference state and C is a material parameter which, in many cases, may be taken equal to unity. A possible expression for ϕ is [7]

$$\phi = \exp\left[b\left(\frac{1}{v} - \frac{1}{v_0}\right)\right] \tag{6}$$

where b is another material parameter.

Formulation

The derivation of an integral equation for non-linear viscoelastic problems begins with the reciprocal theorem of linear viscoelasticity [12]. Given two linear viscoelastic states (ε_{ij}, σ_{ij}) and ($\tilde{\varepsilon}_{ij}, \tilde{\sigma}_{ij}$), satisfying the constitutive eq (2) in the viscoelastic domain Ω, then

$$\int_\Omega \tilde{\varepsilon}_{ij} * \mathrm{d}\sigma_{ij}\mathrm{d}\Omega = \int_\Omega \varepsilon_{ij} * \mathrm{d}\tilde{\sigma}_{ij}\mathrm{d}\Omega \tag{7}$$

In a non-linearly deformed viscoelastic material, it is possible to define the notional stress field σ_{ij}^L related to the actual strain components by

$$\sigma_{ij}^L = G_{ijkl} * \mathrm{d}\varepsilon_{kl} \tag{8}$$

Then, the actual stress developing in the non-linear material can be written as

$$\sigma_{ij} = \sigma_{ij}^{L} + \sigma_{ij}^{NL} \tag{9}$$

where σ_{ij}^{NL} represents the effect of material non-linearity on stress, that is, the stress difference resulting from using constitutive eq (3) rather than eq (2). The reciprocity relation (7) is only valid for σ_{ij}^{L}; hence, for the non-linear problem, it should take the form

$$\int_{\Omega} \tilde{\varepsilon}_{ij} * \mathrm{d}(\sigma_{ij} - \sigma_{ij}^{NL})\mathrm{d}\Omega = \int_{\Omega} \varepsilon_{ij} * \mathrm{d}\tilde{\sigma}_{ij}\mathrm{d}\Omega \tag{10}$$

Substitution of the small strains-displacement relations into eq (10), integration by parts and the application of divergence theorem, gives

$$\int_{\Gamma} \tilde{u}_i * \mathrm{d}p_i \mathrm{d}\Gamma + \int_{\Omega} \tilde{u}_i * \mathrm{d}f_i \mathrm{d}\Omega = \int_{\Gamma} u_i * \mathrm{d}\tilde{p}_i \mathrm{d}\Gamma + \int_{\Omega} u_i * \mathrm{d}\tilde{f}_i \mathrm{d}\Omega + \int_{\Omega} \tilde{\varepsilon}_{ij} * \mathrm{d}\sigma_{ij}^{NL}\mathrm{d}\Omega \tag{11}$$

where Γ is the boundary of Ω while u_i, p_i and f_i are, respectively, the components of the displacement, traction and body force corresponding to the actual, nonlinear problem while $\tilde{u}_i, \tilde{p}_i$ and $\tilde{f}_i$ are the respective quantities for a second hypothetical linear field. The latter is assumed due to the body force

$$\tilde{f}_{ki} = \delta_{ki}\delta(\mathbf{x}–\boldsymbol{\xi})H(t) \tag{12}$$

acting on an infinite isotropic linearly viscoelastic domain, where δ_{ki} is the Kronecker delta, $\delta(\mathbf{x}–\boldsymbol{\xi})$ the delta function and $H(t)$ the Heaviside step function. Then, eq (11) is transformed to

$$\kappa_{ij}u_j(t) = \int_{\Gamma} (\tilde{u}_{ij} * \mathrm{d}p_j - \tilde{p}_{ij} * \mathrm{d}u_j)\mathrm{d}\Gamma + \int_{\Omega} \tilde{u}_{ij} * \mathrm{d}f_j \mathrm{d}\Omega - \int_{\Omega} \tilde{\varepsilon}_{ijk} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Omega \tag{13}$$

where $\tilde{u}_{ki}(\mathbf{x}–\boldsymbol{\xi},t)$ is the time-dependent fundamental solution while $\tilde{\varepsilon}_{ijk}$ and $\tilde{p}_{ij}$ are the corresponding strain and edge traction components. In eq (13), $\kappa_{ij} = \delta_{ij}$ for interior source points and $\kappa_{ij} = (1/2)\delta_{ij}$ for points on a smooth boundary.

The Laplace transform of $\tilde{u}_{ij}$ can be derived from the fundamental solution of the respective elastic problem via the correspondence principle. Inversion from the transform to the real time domain leads to the general form [11]

$$\tilde{u}_{ij}(\mathbf{x},\boldsymbol{\xi},t) = A(t)g_{ij}(\mathbf{x}-\boldsymbol{\xi}) + B(t)h_{ij}(\mathbf{x}-\boldsymbol{\xi}) \tag{14}$$

where the time functions $A(t)$ and $B(t)$ also depend implicitly on the relaxation moduli of the material while the spatial functions $g_{ij}(\mathbf{x} - \boldsymbol{\xi})$ and $h_{ij}(\mathbf{x} - \boldsymbol{\xi})$ also depend on the dimensionality of the problem.

Eq (13) is not a true boundary integral equation because of the presence of an irreducible domain integral dependent on material non-linearity. An iterative scheme accounting for that integral complements an existing time-stepping boundary element formulation [11] for solving the linear part of eq (13). The scheme relies on the evaluation of σ_{ij}^{NL} at internal points using eq (9) and the constitutive relations (1) and (3). The strains are given in terms of the displacement gradients, which are obtained from eq (13) with $\kappa_{ij} = \delta_{ij}$:

$$\frac{\partial u_i(\boldsymbol{\xi},t)}{\partial \xi_l} = \int_{\Gamma} \left[\frac{\partial \tilde{u}_{ij}}{\partial \xi_l} * \mathrm{d}p_j - \frac{\partial \tilde{p}_{ij}}{\partial \xi_l} * \mathrm{d}u_j \right] \mathrm{d}\Gamma + \int_{\Omega} \frac{\partial \tilde{u}_{ij}}{\partial \xi_l} * \mathrm{d}f_j \mathrm{d}\Omega - \frac{\partial}{\partial \xi_l} \int_{\Omega} \tilde{\varepsilon}_{ijk} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Omega \tag{15}$$

Differentiation of the singular domain integral. The strong singularity of the kernel $\partial\tilde{\varepsilon}_{ijk} / \partial\xi_l$, whose behaviour is of order O(r^{-2}), does not allow differentiation under the domain integral sign in the third term on the right-hand side of eq (15). The correct expression for that gradient is derived using a method proposed by Bui [13]. Thus, the irreducible domain integral is separated into two parts,

$$\frac{\partial}{\partial\xi_l}\int_{\Omega}\tilde{\varepsilon}_{ijk} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Omega = \lim_{R\to 0}\left[\frac{\partial}{\partial\xi_l}\int_{\Omega-\Omega_R}\tilde{\varepsilon}_{ijk} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Omega + \frac{\partial}{\partial\xi_l}\int_{\Omega_R}\tilde{\varepsilon}_{ijk} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Omega\right] \tag{16}$$

where Ω_R is a small circle of radius R, centred at the source point $\boldsymbol{\xi}$.

It can be shown that the second volume integral on the right-hand side of eq (16) is of the order O(R^2). The proof requires that σ_{ij}^{NL} as well its first and second derivatives to be continuous functions of $\mathbf{x}$ in the neighbourhood of $\boldsymbol{\xi}$. Then, a Taylor's series expansion of σ_{ij}^{NL} around $\boldsymbol{\xi}$ leads to

$$\int_{\Omega_R}\tilde{\varepsilon}_{ijk}(\mathbf{x},\boldsymbol{\xi},t) * \mathrm{d}\sigma_{jk}^{NL}(\mathbf{x},t)\mathrm{d}\Omega = R^2 D_{ijkm}(t) * \mathrm{d}\sigma_{jk}^{NL},_m(\boldsymbol{\xi},t) \tag{17}$$

where $D_{ijkm}(t)$ are linear combinations of $A(t)$ and $B(t)$, independent of $\boldsymbol{\xi}$. Hence the gradient of the right-hand side of eq (17) involves second derivatives of σ_{ij}^{NL} and therefore vanishes as $R \to 0$.

Since Ω_R depends on $\boldsymbol{\xi}$, differentiation of the first domain integral on the right-hand side of eq (16) produces an additional convective term. Thus

$$\frac{\partial}{\partial\xi_l}\int_{\Omega-\Omega_R}\tilde{\varepsilon}_{ijk} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Omega = \int_{\Omega-\Omega_R}\frac{\partial\tilde{\varepsilon}_{ijk}}{\partial\xi_l} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Omega - \int_{\Gamma_R} n_l\tilde{\varepsilon}_{ijk} * \mathrm{d}\sigma_{jk}^{NL}\mathrm{d}\Gamma \tag{18}$$

where Γ_R is the periphery of the circle with radius R and $\mathbf{n}$ is the outward unit normal to that circle. Using the formulas [14]

$$\int_{\Gamma_R} r,_i\, r,_j\, \mathrm{d}\Gamma = \pi R\delta_{ij}, \quad \int_{\Gamma_R} r,_i\, r,_j\, r,_k\, r,_l\, \mathrm{d}\Gamma = \frac{\pi R}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{19}$$

where $r = |\mathbf{x} - \boldsymbol{\xi}|$, the last term on the right-hand side of eq (18) is reduced to a sum of simple convolution integrals.

As $R \to 0$, the first integral on the right-hand side of eq (18) becomes the Cauchy principal value of the singular integral, whose existence has been proved and the methods to evaluate is presented below. The strain components corresponding to $\tilde{u}_{ij}$ and their gradients are obtained by successive differentiation of Eq. (14).

Evaluation of the Cauchy principal value. The domain integral on the right-hand side of eq (18) is evaluated by dividing the domain into cells, that is, two-dimensional subdomains Ω_c bounded by contours Γ_c. The integration is performed over each cell using an approximate model for the unknown σ_{ij}^{NL}. The radial integration method [14] was used to evaluate the Cauchy principal value of that integral over the cell containing the source point. The integration over all other cells was performed using numerical quadrature.

The domain integral on the right-hand side of eq (18) can be expressed as

$$\int_{\Omega_c}\frac{\partial\tilde{\varepsilon}_{ijk}(\mathbf{x},\boldsymbol{\xi},t)}{\partial\xi_l} * \mathrm{d}\sigma_{jk}^{NL}(\mathbf{x},t)\mathrm{d}\Omega = \left[\int_{\Omega_c}\frac{\partial\tilde{\varepsilon}_{ijk}(\mathbf{x},\boldsymbol{\xi},t)}{\partial\xi_l}\mathrm{d}\Omega\right] * \mathrm{d}\sigma_{jk}^{NL}(\boldsymbol{\xi},t)$$

$$+\int_{\Omega_c} \frac{\partial \tilde{\varepsilon}_{ijk}(\mathbf{x},\xi,t)}{\partial \xi_l} * \left[\mathrm{d}\sigma_{jk}^{NL}(\mathbf{x},t) - \mathrm{d}\sigma_{jk}^{NL}(\xi,t) \right] \mathrm{d}\Omega \tag{20}$$

The second integral on the right-hand side of eq (20) can be shown to be regular and therefore evaluated by standard numerical schemes; the strong singularity remains in the first integral. A polar coordinate system (r, θ) is defined with the origin at the source point $\boldsymbol{\xi}$. It can be shown that, relative to this system, the singular integral on the right-hand side of eq (20) can be transformed to

$$\int_{\Omega_c} \frac{\partial \tilde{\varepsilon}_{ijk}(\mathbf{x},\xi,t)}{\partial \xi_l} \mathrm{d}\Omega = \int_{\Gamma_c} \left[\frac{1}{r} \frac{\partial r}{\partial n} \right]_{\Gamma} \psi_{ijkl}(\theta,t) \int_0^{r(\Gamma)} \frac{\mathrm{d}r}{r} \mathrm{d}\Gamma \tag{21}$$

where

$$\frac{\partial \tilde{\varepsilon}_{ijk}}{\partial \xi_l} = \frac{1}{r^2} \psi_{ijkl}(\theta,t)$$

Since the integration is carried out in the Cauchy principal value sense, a small circle of radius R around the singular point $\boldsymbol{\xi}$ can be cut off. Thus, eq (21) becomes,

$$\int_{\Omega_c-\Omega_R} \frac{\partial \tilde{\varepsilon}_{ijk}(\mathbf{x},\xi,t)}{\partial \xi_l} \mathrm{d}\Omega = \int_{\Gamma_c} \left[\frac{\ln r}{r} \frac{\partial r}{\partial n} \right]_{\Gamma} \psi_{ijkl}(\theta,t) \mathrm{d}\Gamma + \ln R \int_0^{2\pi} \psi_{ijkl}(\theta,t) \mathrm{d}\theta \tag{22}$$

where $\partial r/\partial n = -1$ has been used in the second integral along the circle Γ_R. Using relations (19), it can be shown that the last integral on the right-hand side of eq (22) is identical to zero; this is an intrinsic property of ψ_{ijkl}. Hence, as $R \to 0$, eq (22) becomes

$$\int_{\Omega_c} \frac{\partial \tilde{\varepsilon}_{ijk}(\mathbf{x},\xi,t)}{\partial \xi_l} \mathrm{d}\Omega = \int_{\Gamma_c} \left[\frac{\ln r}{r} \frac{\partial r}{\partial n} \right]_{\Gamma} \psi_{ijkl}(\theta,t) \mathrm{d}\Gamma \tag{23}$$

Now the strongly singular domain integral has been transformed into a boundary integral. Since the source point is located inside the domain, no singularity occurs and standard Gaussian quadrature formulas can be used to calculate this integral.

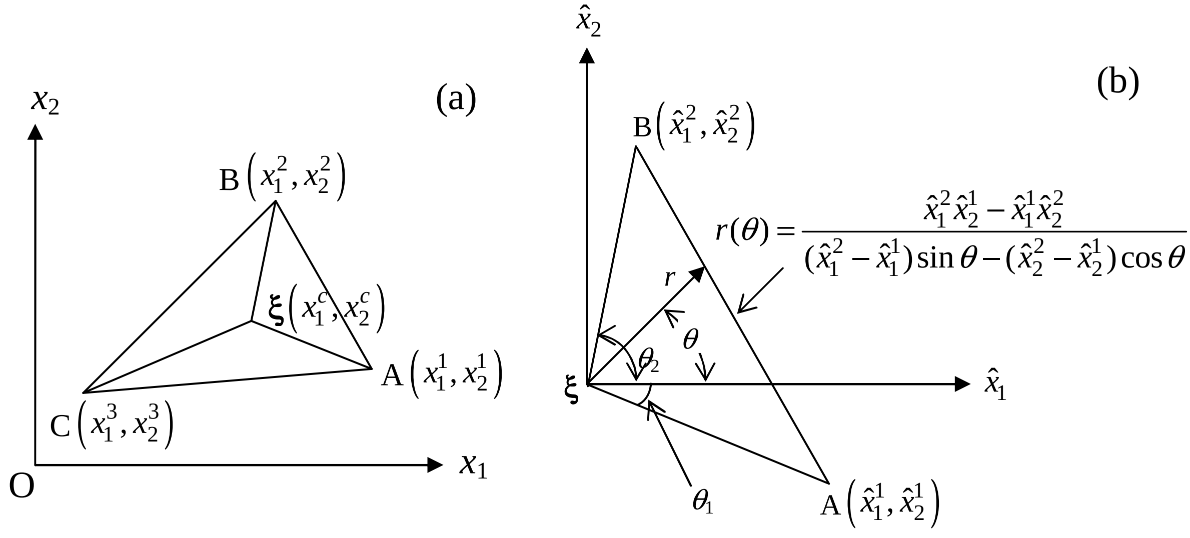

Figure 1 Triangular cell (a) and polar coordinate system with origin at the singular point $\boldsymbol{\xi}$ (b)

The domain was divided into small triangular cells, such as the one shown in Figure 1(a), with the source point $\boldsymbol{\xi}$ at the centre of the triangle. Relative to a polar frame of reference with origin at the centre of the triangle, the equation of side A($\hat{x}_i^1$)-B($\hat{x}_i^2$) of the triangle, shown in Fig. 1(b), can be expressed in terms of the local corner co-ordinates $\hat{x}_i^j = x_i^j - \xi_i$, where x_i^j are the co-ordinates of corner j.

Thus, in this case, the contour integral on the right-hand side of eq (23) is evaluated along each side of the triangle. Adopting a 'constant' cell model for σ_{jk}^{NL} , the stress difference in eq (20) vanishes and therefore

$$\int_{\Omega_c} \frac{\partial \tilde{\varepsilon}_{ijk}(\mathbf{x},\boldsymbol{\xi},t)}{\partial \xi_l} * \mathrm{d}\sigma_{jk}^{NL}(\mathbf{x},t)\mathrm{d}\Omega = \left[\sum \int_{\theta_m}^{\theta_n} \ln[r(\theta)]\psi_{ijkl}(\theta,t)\mathrm{d}\theta \right] * \mathrm{d}\sigma_{jk}^{NL}(\boldsymbol{\xi},t) \tag{24}$$

The integrals on the right-hand side of eq (24) can be calculated by the standard Gaussian quadrature formulas.

Numerical algorithm

Constant boundary elements were used in the present numerical implementation of BEM formulation based on integral eq (13), which also requires modelling in the time dimension. If the boundary surface Γ is discretised in E elements Γ_e, the following representation can be adopted,

$$u_j(\mathbf{x},t) = u_j^e(t)\,, \quad p_j(\mathbf{x},t) = p_j^e(t) \tag{25}$$

where $u_j^e(t)$ and $p_j^e(t)$ are the time dependent nodal values of displacement and traction, respectively. Over a cell, strain was modelled as uniform and, as a consequence of eqs (1), (3), (5) and (6), the shift factor as well as both linear and non-linear contributions to the stress are also constant within each cell.

It was assumed that the boundary variables $u_i(x,t)$ and $p_i(x,t)$ as well as the nonlinear part of total stress σ_{jk}^{NL} ($\mathbf{x}$,t) in the domain are linear with respect to time t within a small time step $\Delta t_\kappa = t_\kappa - t_{\kappa-1}$. With the viscoelastic fundamental solutions in the general form:

$$\tilde{u}_{ij} = b_{ij}^0 + \sum_{n=1}^{N} b_{ij}^n \mathrm{e}^{-\beta_n t}\;,\;\; \tilde{p}_{ij} = a_{ij}^0 + \sum_{m=1}^{M} a_{ij}^m \mathrm{e}^{-\alpha_m t}\;,\;\; \tilde{\varepsilon}_{ijk} = c_{ijk}^0 + \sum_{q=1}^{Q} c_{ij}^q \mathrm{e}^{-\gamma_q t} \tag{26}$$

the discretised form of eq (13) was obtained as

$$\begin{aligned}
\kappa_{ij}(\boldsymbol{\xi})u_i^{(K)}(\boldsymbol{\xi}) &= \sum_{n=0}^{N} B_j^{n(K)}(\boldsymbol{\xi}) + \sum_{\kappa=1}^{K}\sum_{n=1}^{N} [B_j^{n(\kappa-1)}(\boldsymbol{\xi})\mathrm{e}^{-\beta_n(t_K - t_\kappa)}(\mathrm{e}^{-\beta_n \Delta t_\kappa} - 1)] \\
&- \sum_{m=0}^{M} A_j^{m(K)}(\boldsymbol{\xi}) - \sum_{\kappa=1}^{K}\sum_{m=1}^{M} [A_j^{m(\kappa-1)}(\boldsymbol{\xi})\mathrm{e}^{-\alpha_m(t_K - t_\kappa)}(\mathrm{e}^{-\alpha_m \Delta t_\kappa} - 1)] \\
&- \sum_{q=0}^{Q} C_j^{q(K)}(\boldsymbol{\xi}) - \sum_{\kappa=1}^{K}\sum_{q=1}^{Q} [C_j^{q(\kappa-1)}(\boldsymbol{\xi})\mathrm{e}^{-\gamma_q(t_K - t_\kappa)}(\mathrm{e}^{-\gamma_q \Delta t_\kappa} - 1)]
\end{aligned} \tag{27}$$

where, for simplicity, the body force was assumed to be zero and

$$A_i^{m(\kappa)}(\boldsymbol{\xi}) = \int_\Gamma a_{ij}^m u_j^{(\kappa)} \mathrm{d}\Gamma\,,\;\; B_i^{n(\kappa)}(\boldsymbol{\xi}) = \int_\Gamma b_{ij}^n p_j^{(\kappa)} \mathrm{d}\Gamma\,,\;\; C_i^{q(\kappa)}(\boldsymbol{\xi}) = \int_\Omega c_{ijk}^q \sigma_{jk}^{NL(\kappa)} \mathrm{d}\Omega$$

$$u_i^{(\kappa)}(\mathbf{x}) = u_i(\mathbf{x},t_\kappa)\,,\;\; p_i^{(\kappa)}(\mathbf{x}) = p_i(\mathbf{x},t_\kappa)\,,\;\; \sigma_{jk}^{NL(\kappa)}(\mathbf{x}) = \sigma_{jk}^{NL}(\mathbf{x},t_\kappa)$$

An iterative scheme is proposed for solving eq (27) since the current values of the stresses σ_{jk}^{NL} are not known at the beginning of a time step. At the first iteration, the boundary displacements and tractions are determined at time $t = t_K$ assuming $\sigma_{jk}^{NL}(t_\kappa) = \sigma_{jk}^{NL}(t_{\kappa-1})$. Then displacement gradients are determined from eq (15) leading to initial estimates of domain strains and subsequently of stresses through constitutive eqs (1) and (3). The value of the domain integral can thus be revised and the procedure repeated until results from two successive iterations agree within an acceptable tolerance. Convergence of boundary displacements was the adopted criterion for terminating the iteration. It should be noted that, at $t = 0$ all unknown boundary values can be calculated when the integral eq (13) governs only the initial elastic response due to any non-zero initial values of the boundary or loading conditions. At the following time $t = t_1$ (step $\kappa = 1$), the respective unknown boundary values can be obtained from eq (27) with the current boundary conditions and the additional terms depending on the solution at the initial step as well as the non-linear contribution of the current step. The solution progresses to the next time step $\kappa = 2$ in a similar manner and a step-wise procedure is thus established which advances the solution until the final time step is reached. A suite of FORTRAN programs was developed for implementing this formulation.

Numerical results

Specimen geometry and material model. As a numerical test, the developed non-linear analysis was applied to a plate with a central crack under constant tension. The input data are approximately the same as those used by Moran and Knauss [9] who solved this problem using FEM. Due to symmetry relative to two orthogonal axes, only a quarter of the plate was modelled. The plate half-width was 13.44 mm, half-height 12 mm, and the crack half-length $a = 1$ mm. The material behaviour was represented by a standard linear solid model in shear

$$\mu(t) = \mu_0[\lambda + (1 - \lambda)e^{-\eta t}]$$

with $\mu_0 = 4800$ MPa, $\lambda = 0.1$, $\eta = 0.4$ and a constant Poisson's ratio $\nu = 1/3$. In order to simplify the evaluation of the volumetric strain, plane strain conditions were applied.

In order to calculate ϕ using eq (6), v_0 was assumed to be 0.01, and b was chosen equal to 0.05 [9]. A remote tension $\sigma_0 = 0.001E(0)$ was applied, where $E(0)$ is the initial value of the tensile relaxation modulus.

Boundary and domain meshing. 'Constant' boundary elements with variable element length were adopted. The two smallest elements, located on either side of the crack tip, were 0.005 mm long, the largest element at the loading edge of the plate was 3 mm long. In conformity with the boundary mesh, the domain mesh was arranged to be much denser near the crack tip, where the stress concentration and high nonlinearity occur.

Nonlinear stress field. Fig. 2 shows the normalized nonlinear stress field near the crack tip. For a linear viscoelastic plate with a constant Poisson's ratio under constant tensile loading, the stress field is constant over the time history. For nonlinear viscoelastic problems, the material undergoes considerable strain softening around the crack tip, where the high stress and strain occur. From Fig. 2 it is very clear that the stress field drops with time due to the strain softening, and this drop slows down with time becoming less significant as the strain itself changes more slowly. The stresses far from the crack tip were increased in order that overall equilibrium is satisfied. This response is similar to that predicted by FEM [9].

The program was also run with the parameter λ changed to 0.001 implying a more pronounced material time-dependace than originally assumed. From the respective results, it was clear that the effect of nonlinearity was higher than previously under the same loading conditions, though not as significant as predicted by FEM [9]. One possible reason for this is that an initial region of K-dominance is assumed in their paper, inside this region the strains are assumed to be infinite and the time-shift factor reduces to a constant value.

Concluding remarks

An initial attempt was made to validate the developed formulation and the resulting software through their application to a fracture problem. Although the cell size distribution around the crack tip was not ideal for capturing the local stress concentration, this mesh was considered acceptable for an initial assessment of the performance of the proposed method. The numerical results obtained confirmed the expected effect of non-linearity on the stress time history, which is highest in the neighbourhood of the crack tip. Compared however with those reported in a previous FEM study [9], significant discrepancy was noted. There was also a degree of inconsistency between the calculated boundary tractions and domain stresses. Further numerical test are required to assess the sensitivity of the solution to mesh refinements as well as other input and control parameters.

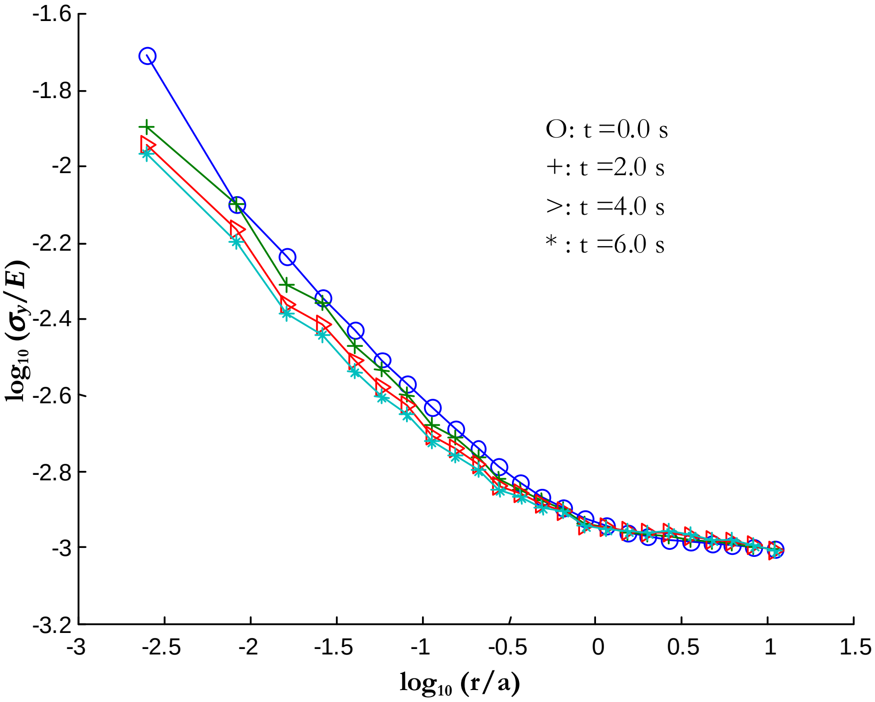

Figure 2. Normalized nonlinear stress field near the crack tip ($\lambda = 0.1$)

References

[1] J. G. J. Beijer and J. L. Spoormaker *Computers & Structures*, **80**, 1213-1229 (2002).

[2] S. Syngellakis *Engineering Analysis with Boundary Elements*, **27**, 125-135 (2002).

[3] S. Syngellakis and H. W. Wu *Engineering Fracture Mechanics*, **75**, 1251-1265 (2008).

[4] P. J. Dooling, C. P. Buckley and S. Hinduja *Polymer Engineering and Science*, **38**, 892-904 (1998).

[5] R. A. Schapery *Polymer Engineering and Science*, **9**, 295-310 (1969).

[6] L. W. Morland and E. H. Lee *Transactions of the Society of Rheology*, **4**, 233-263 (1960).

[7] W. G. Knauss and I. J. Emri *Computers & Structures*, **13**, 123-128 (1981).

[8] W. G. Knauss and I. Emri *Polymer Engineering and Science*, **27**, 86-100 (1987).

[9] B. Moran and W. G. Knauss *Journal of Applied Mechanics-Transactions of the ASME*, **59**, 95-101 (1992).

[10] A. Wineman and R. Kolberg *International Journal of Non-Linear Mechanics*, **32**, 863-883 (1997).

[11] S. Syngellakis and J. W. Wu *Engineering Analysis with Boundary Elements*, **28**, 733-745 (2004).

[12] M. E. Gurtin and E. Sternberg *Archive for Rational Mechanics and Analysis*, **11**, 291-356 (1962).

[13] H. D. Bui *International Journal of Solids and Structures*, **14**, 935-939 (1978).

[14] X. W. Gao and T. G. Davis *International Journal of Solids and Structures*, **37**, 4987-5008 (2000).