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Critical issues in the design of the school geometry curriculum[1]

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The fundamental problem in the design of the geometry component of the mathematics curriculum is simply that there is too much interesting geometry, more than can be reasonably included in the mathematics curriculum. The question that taxes curriculum designer is what to include and what to omit. This paper does not seek to resolve the disagreements over the geometry curriculum as, given the nature of the problem, such an endeavour is unlikely to be successful. Rather, the aim is to identifying and review critical issues concerning the design of the geometry curriculum. These issues include the nature of geometry, the aims of geometry teaching, how geometry is learnt, the relative merits of different approaches to geometry, and what aspects of proof and proving to accentuate.

Keywords: curriculum, geometry, teaching

Introduction

> Of all the decisions one must make in a curriculum development project with respect to choice of content, usually the most controversial and the least defensible is the decision about geometry.
>
> (The Chicago School Mathematics Project staff 1971, p281)

Designing a suitable geometry curriculum is probably the most difficult task for those who are charged with constructing mathematics curricula. It is also probably the most enduring dilemma in mathematics curricula design and has been probably been the subject of more inquiries and commentaries than any other area of the mathematics curriculum. Reports and commentaries on the geometry curriculum range from historical accounts, such as Stamper (1909) or Quast (1968) to national or regional inquiries, for instance, Mathematical Association (1923) or Willson (1977) and international studies, such as Morris (1986) or Mammana and Villani (1998).

The development of new mathematics curricula in the 1960s added a particular complexity to decisions about geometry as curricula were revised in order to base much more of school mathematics on the idea of function and to aim more at the mathematics that would lead to calculus and linear algebra. To accommodate these changes, all parts of the mathematics curriculum were reformulated. In terms of the geometry curriculum the practical effect was more or less to remove solid geometry from the curriculum and to convert the trigonometry component into part of a course about functions. Thus the impact

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was to reduce the overall amount of geometry while, at the same time, increasing the emphasis on co-ordinate geometry and introducing some elements of vector geometry, transformation geometry (including matrices) and topology. More recently the squeeze on the curriculum time devoted to geometry has been exacerbated by a substantial increase in the coverage of statistics and, especially of late, in a number of countries, a major focus on numeracy. Both of these recent developments have tended to deflect yet more attention away from geometry. In contrast to this reduction in the coverage of geometry at school (and university) level, the amount of geometry that is known has grown considerably since the end of the 19th century. Such is the extent of geometry that it is now possible to classify more than 50 geometries (see, for instance, Malkevitch, 1992).

These changes have left many unanswered questions for curriculum designers. In 1969, for example, Allendoerfer wrote (regarding the situation in the US at the time), “The mathematics curriculum in our elementary and secondary schools faces a serious dilemma when it comes to geometry. It is easy to find fault with the traditional course in geometry, but sound advice on how to remedy these difficulties is hard to come by” (Allendoerfer, 1969 p165). Such problems are shared across a range of countries and, as time has passed, have remained largely unresolved. In 1977 for example, in the UK, Willson commented that, “Among textbooks and teachers at present we find very wide differences of opinion about what is appropriate subject matter for school geometry and about how to approach it” (Willson 1977, p19). In the 1980s, the impetus for the Unesco study (Morris 1986) was that “There is no consensus on the content of geometry in schools” (*ibid* p9). In the 1990s the International Commission on Mathematical Instruction (ICMI) embarked on a study of the teaching and learning of geometry (reported in Mammana and Villani, 1998). The discussion document, written to inform the study, observed that, “Among mathematicians and mathematics educators there is a widespread agreement that, due to the manifold aspects of geometry, the teaching of geometry should start at an early age, and continue in appropriate forms throughout the whole mathematics curriculum. However, as soon as one tries to enter into details, opinions diverge on how to accomplish the task. There have been in the past (and there persist even now) strong disagreements about the aims, contents and methods for the teaching of geometry at various levels, from primary school to university” (International Commission on Mathematical Instruction, 1994 p345). The ICMI study concluded that, “It is improper to claim that it is possible to elaborate a geometry curriculum having universal validity” (Villani, 1998 p321).

The purpose of this paper is not to attempt to resolve the range of disagreements about the geometry curriculum. Given the range of issues, such an endeavour is unlikely to be successful. The aim is more modest (and hopefully achievable). It is to identify and review some of the critical issues in the design of the geometry curriculum, primarily at the school level. The focus is mainly on the *intended* curriculum - that set out in curricula statements and/or in textbooks - rather than the *experienced* or *learned* curriculum, the curriculum as experienced or learnt by students. The *intended* and the *experienced* curriculum can be very different. In the case of the experienced or learned curriculum, it can also be difficult to identify with any certainty, as there are a multitude of variables.

The paper begins with a brief consideration of the nature of geometry and the aims for teaching geometry. This provides the necessary background for the identification of critical issues in the design of the geometry curriculum.

The nature of geometry

Geometry is one of the longest-established branches of mathematics and its origins can be traced back through a wide range of cultures and civilisations. During the nineteenth and twentieth centuries, geometry, like most areas of mathematics, went through a period of growth that was near cataclysmic in proportion. As a consequence, the content of geometry and its internal diversity increased almost beyond recognition. The geometry of the ancient world, codified in the books of Euclid, rapidly become no more than a subspecies of the vast family of mathematical theories of space. The contemporary classification of more than 50 geometries (see Malkevitch, *ibid*) illustrates the richness of modern geometrical theory.

Much of the development of geometry during the twentieth century was inspired by the work of Felix Klein (1849-1925), who, at his inaugural lecture as professor of mathematics at the University of Erlangen in 1872, proposed that geometry be viewed as *the study of the properties of a space that are invariant under a given group of transformations*. This synthesis and (re)definition of geometry came to be known as the Erlanger Programme and profoundly influenced much subsequent mathematical development. With this definition it became possible to classify the various geometries into related 'families', ranging from topology as the most general, through projective and affine geometries, to Euclidean geometry which has the most restricted congruences (because more properties are invariant). This way of viewing geometry, and subsequent developments, spurred the delineation of many more geometries.

Geometry, in all its variety, is also rich in application. Here, briefly, are a few illustrative examples of current applications as suggested by Whitely (1999):

- Computer aided design (CAD) and geometric modeling (including designing, modifying, and manufacturing cars, planes, buildings, manufactured components, etc)
- Robotics
- Medical imaging (which has led to some substantial new results in fields like geometric tomography)
- Computer animation and visual presentations

Further areas where geometric problems arise are in chemistry (computational chemistry and the shapes of molecules), material physics (modeling various forms of glass and aggregate materials), biology (modeling of proteins, 'docking' of drugs on other molecules, etc), Geographic Information Systems (GIS), and most fields of engineering.

In recent times, the nature of geometry has continued to expand. A number of contemporary developments in mathematics are predominantly geometrical. These developments include work on dynamical systems (a major mathematical discipline closely intertwined with all main areas of mathematics) mathematical visualisation (the art of transforming the symbolic into the geometric), and geometric algebra (a representational and computational system for geometry that is entirely distinct from algebraic geometry). Some implications of these developments in geometry for the teaching of algebra are given in Jones (2001a). For other examples of the curricular implications of contemporary geometry, see Crowe and Thompson (1987) or Malkevitch (1998).

Thus geometry is continuing to evolve and now encompasses the understanding of diverse visual phenomena. A useful contemporary definition of geometry is that attributed to the highly-respected British mathematician, Sir Christopher Zeeman: "geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight" (Royal Society, 2001).

Having considered the nature of geometry it is now useful to give some consideration to the aims of geometry education.

The aims of geometry education

Deciding on the aims for geometry education involves considering both the nature of geometry and the range of its applications. Thus consideration must be given to spatial thinking, to visualisation, and, of course, to proof (for further consideration of these points, see, for example, Hoffer, 1981 and Usiskin, 1987). The Royal Society report on the teaching and learning of geometry (*op cit*) suggests that the contemporary aims of teaching geometry can be summarised as follows:

a) to develop spatial awareness, geometrical intuition and the ability to visualise;
b) to provide a breadth of geometrical experiences in 2 and 3 dimensions;
c) to develop knowledge and understanding of and the ability to use geometrical properties and theorems;
d) to encourage the development and use of conjecture, deductive reasoning and proof;
e) to develop skills of applying geometry through modelling and problem solving in real world contexts;
f) to develop useful ICT skills in specifically geometrical contexts;
g) to engender a positive attitude to mathematics; and
h) to develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry.

The breadth of knowledge that is contemporary geometry, and the range of aims that must be addressed in order to provide a full experience of geometry, are indicative of the issues that make designing a suitable geometry curriculum such a difficult task. The next section seeks to identify some of the critical issues in the design of the geometry curriculum.

Some critical issues in designing the geometry curriculum

The critical issues listed below have been identified through reviewing the range of writing about the geometry curriculum, many of which have already been refered to in this paper. Where possible some commentary is given on the various considerations that can be applied to the issue. In a number of cases, however, all that can be done is to raise questions for which the answers are, as yet, either unclear or unknown. For the most part this is because of lack of good evidence on which to base the decision. Much about the geometry curriculum remains un-researched or under-researched.

Any consideration of the content of the mathematics curriculum must consider both what is to be learnt and whether and in what order it can be learnt. In the case of the geometry curriculum, this means attending both to the structure of geometry and to what is known about how geometry can be learnt. Already this raises a major problem. Neither what is seen as the structure of geometry nor what is known about how geometry can be learnt are stable entities over time. As described above, the structure of geometry was radically reorganised by Klein's Erlanger programme. There is no way of knowing whether (or when) another radical reorganisation of geometry may be proposed or be found to be advantageous or necessary. Similarly, at least two major ways in which geometry is learnt have been proposed. As discussed in more detail below, both have met with some support and some criticism. Neither appears to be a wholly reliable basis on which to build a geometry curriculum. Thus in the design of the geometry curriculum we must be cautious about relying overly on current geometrical logic or on psychological models as both may be subject to change. In any case, the words of Stamper are well worth remembering, "A sequence that is logically best is not necessarily pedagogically the best.... Teachers of mathematics, above all others, have been slow in recognising these distinctions" Stamper (*opcit* p141).

Nevertheless, we must needs start somewhere and so, in this paper, we start with how geometry can be learnt.

How geometry can be learnt

It is no good having a geometry curriculum that cannot be learnt. Thus knowing something about how geometry can be learnt is essential. Of the range of theoretical work concerned with geometrical ideas, that of Piaget (and colleagues) and of the van Hiele's are probably the most well-known. The Piagetian work has two major themes (see Piaget, Inhelder and Szeminska, 1960, and Piaget and Inhelder, 1968). The first theme is that a learner's mental representation of space is not a perceptual 'reading off' of what is around them. Rather, as learners, we build up from our mental representation of our world through progressively reorganising our prior active manipulation of that environment. This theme remains reasonably well-supported by research. The second Piagetian theme is that the progressive organisation of geometric ideas follows a definite order and that this order is more experiential (and possibly more mathematically logical, depending on your mathematical perspective) than it is a re-enactment of the historical development of geometry. That is, initially topological relations, such as connectedness, enclosure, and continuity, are constructed by the learner, followed by projective (rectilinearity) and Euclidean (angularity, parallelism, and distance) relations. This second hypothesis suggests a learning sequence for geometry beginning with some topological ideas and gradually moving through affine and projective geometry to the geometry of metric spaces. Unfortunately such a model has received, at best, only mixed support. The available evidence suggests that all types of geometric ideas appear to develop over time, becoming increasingly integrated and synthesised (Clements & Battista, 1992). This does not mean that a geometry curriculum of the form suggested by the Piagetian model may not work just as well as existing geometry curricula, and possibly even better. It is the case that there has been no well-researched study of such a curriculum being used.

The van Hiele model also suggests that learners advance through levels of thought in geometry (van Hiele 1985). Van Hiele characterised these levels as visual, descriptive, abstract/relational, and formal deduction. At the first level, students identify shapes and figures according to their concrete examples. At the second level, students identify shapes according to their properties, and here a student might think of a rhombus as a figure with four equal sides. At the third level, students can identify relationships between classes of figures (for example, that a square is a special form of rectangle) and can discover properties of classes of figures by simple logical deduction. At the fourth level, students can produce a short sequence of statements to logically justify a conclusion and can understand that deduction is the method of establishing geometric truth. According to this model, progress from one of Van Hiele's levels to the next is more dependent upon teaching method than upon age.

While research is generally supportive of the van Hiele levels as useful in describing students' geometric concept development (in the absence of anything better), it remains uncertain how well the theory reflects children's mental representations of geometric concepts (Clements 2001). Various problems have been identified with the specification of the levels. For example, the labelling of the lowest level as 'visual' when visualisation is demanded at all levels, and the fact that learners appear to show signs of thinking from more than one level in the same or different tasks, in different contexts. It is important to remember that the model was developed in the 1950s at a time when the geometry curriculum was predominantly plane geometry in the Euclidean tradition. The model thus naturally reflects such origins. Its usefulness with respect to other approaches to plane geometry (such as via vectors or transformations) and to other geometries (such as spherical geometry) are not clear. As a consequence of these various factors, the van Hiele model appears to be of only limited use in determining the geometry curriculum and how this should be sequenced for teaching.

There is, of course, a host of other research on children's learning of aspects of geometry (including work on topics such as shape, angle, congruence, co-ordinates, transformations, vectors, and so on) and research on themes such as proving in geometry, visualisation, spatial thinking, etc. It is impossible to adequately summarise such work in the space available in this article and, in any case, much of it has not been directly related to the design of the geometry curriculum. It would be useful if it were, but that will have to be the subject of in-depth analysis that is beyond the scope of the current paper.

Bearing in mind the words of Stamper, quoted above, any decision about the design of the geometry curriculum has to take into account what is known about how geometry can be learnt (because the aim of teaching is for learning to take place). Nevertheless, there is also a range of other issues to consider that are more practical in nature. These are to do with emphasis, balance and scope and are related not only to how geometry can be learnt but also to the curriculum time available, both for mathematics in general, and for the proportion that is allocated to geometry in particular. These issues range from what geometry to include to how to devise a teaching sequence. They take in questions about the relative emphasis given to spatial thinking, visualisation and proof, the role and use of definitions, and when and how to specify the use of tools, including computer tools such as dynamic geometry software. These issues are considered below, beginning with what geometry to include.

What geometry to include

One of the major problems in the design of the geometry curriculum, and one that pains all those who love geometry, is that there is just too much interesting geometry around. This means that, in practice, some decision has to be made as to what to put in the curriculum and what to leave out. As Sawyer (1977 p12) puts it, in his elegant essay on geometry, "In the subject matter of geometry we suffer from an embarrassment of riches. We have so many tools for the discussion of geometric problems - Euclid, transformations, coordinates, matrices, calculus. However, it is noticeable that no one of these is the magic key that unlocks all doors".

Thus Sawyer highlights a fundamental issue that is also echoed by Fletcher (1971). That, in practice, geometry "makes use of a wide variety of techniques, no one of which is powerful enough to make the rest superfluous. Every branch has a selection of problems which are most easily solved by using its own methods" (*ibid* p397). Thus each approach to geometry has its own surprising and interesting theorems and results, its own interesting problems, indeed its own fundamentals. What is 'fundamental' in one geometry is not necessarily 'fundamental' in another. For example, 'fundamental' Euclidean results are valid in a Euclidean approach, but they are not necessarily 'fundamental' in a transformation approach. Each approach has its own set of theorems, which are key results to *that particular approach*. This is very different to saying that there exist *universal* 'fundamental' results.

This is the case even in plane geometry, something that currently makes up the bulk of the school geometry curriculum in all parts of the world. As Willson (1977), Barbeau (1988) and Nissen (2000) demonstrate, in theoretical terms there is nothing to choose between methods based on, say, congruent triangles (in the Euclidean tradition) and those based on transformations. Taking an isometry as a transformation that preserves congruence, any proof by congruence can be translated into a proof by transformations, and *vice versa*. One version may be neater or shorter than the other, but, in practice, neither approach is the sole purveyor of elegant proofs. The same goes for co-ordinate geometry. It is always possible to give an analytical proof of a theorem given in, say, the Euclidean tradition and a reasonable proportion of such proofs are as elegant as one would wish. The position of proofs using vectors is somewhat different so that, while there are some neat proofs using vectors, the proofs of many 'elementary' theorems are clumsy and lengthy. Of course, this goes back to the discussion above about what is 'fundamental'. What is 'basic' or 'elementary' to vector geometry may not be basic or elementary to other approaches. Whatever is the position with vectors, it is clearly the case that attempting to use the notion of 'elegant proofs' in choosing what geometry to include in the curriculum and what to omit is not that helpful.

Of course, geometry is not solely about proofs (see Hoffer 1981), it is also about spatial thinking and visualisation. As the UK Association of Teachers of Mathematics noted in 1964, "The problem for the schools is so to conduct the discussion of fundamental geometrical configurations that (i) the pupil's spatial imagination is stimulated and developed, and (ii) he (*sic*) learns to think in terms and in modes that will support, and not conflict with, his (*sic*) later mathematical activity" (Association of Teachers of Mathematics, 1964). This remains a central issue that is not solely about teaching. It is also about what geometry is to be taught and the relative emphasis given to each component.

Given that there are no easy answers to the general question of what geometry to include (and, hence, what to omit), there remain a whole host of questions at the level of detail. These include:

> Do 2D concepts and techniques hold the key to the exact analysis of both 2D and 3D problems or does the general neglect of 3D problems leave students in difficulties when they confront problems in both 3D *and* 2D?
>
> Are triangles (and, to some extent, parallelograms) the key to geometric analysis (and should they therefore occupy a large slice of the curriculum time) or is it impossible to say anything interesting about them without having to go through so many dull and uninteresting theorems that most pupils never get to anything surprising?
>
> Does mastery of relevant 2D geometrical techniques depend on recognising and mastering a hierarchy of 'elementary' notions (such as angle, length, triangles, circles, congruence, similarity, and so on) or are these notions not 'elementary' at all but are rather complex ideas that grow more sophisticated as the learner experiences more geometry?
>
> How important are trigonometry and co-ordinate geometry and when in the curriculum should they be introduced?
>
> What emphasis should be placed on geometric modelling?
>
> What is the place of measuring?
>
> Should ruler and compass constructions be included?

It is clear that different countries have made different decisions in relation to these questions. It is also clear that concentrating on one geometry means denying access to other geometries. Perhaps the issue of specifying the geometry curriculum can be approached from a different angle, by considering the key ideas or concepts in geometry.

Specifying the key ideas or concepts in geometry

This is somewhat different to the discussion about 'fundamentals'. This is not about what *results* can be called 'key' results across different geometries, but whether there are key ideas that underlie or are in some way central to all (or most) geometry. This is not about congruence versus transformations but about underlying geometrical ideas. A list of these might include ideas such as dimension, position, incidence, locus, invariance, symmetry and so on.

Let us consider a couple of these ideas. As noted above, the mathematician Felix Klein revolutionised geometry by defining it as the study of the properties of a configuration that are invariant under a set of transformations. As Schuster (1971 p82) declares, "Invariance is one of the most important ideas in all of mathematics, and geometry is unquestionably the most natural subject for the demonstration and use of this idea".

Another key idea throughout mathematics is symmetry, yet it is in geometry that it achieves its most immediacy. Technically, a symmetry can be thought of as a transformation of a mathematical object which leaves some property invariant. In practice, symmetry is frequently used to make arguments simpler, and usually more powerful. An example from plane geometry is that all of the essential properties of a parallelogram can be derived from the fact that a parallelogram has half-turn symmetry around the point of intersection of the diagonals. Symmetry is also a key organising principle in mathematics.

For example, probably the best way of defining quadrilaterals (*except* for the general trapezium, which is not an essential quadrilateral in any case, since there are no interesting theorems involving the trapezium that do not also hold for general quadrilaterals), is via their symmetries.

Given these brief considerations of invariance and symmetry, it is obvious that transformation is also a key idea in geometry. Indeed, transformation is the means by which the formal definitions of congruence, similarity and symmetry can be related to learners' previous intuitive ideas.

In this way it may be possible to specify a geometry curriculum around a number of key ideas or concepts, the sorts of ideas that often fall between the spaces of a curriculum that is defined in terms of techniques or results. In practical terms, however, this is yet to be done so there is no empirical evidence on which to base a sound curriculum. The idea is, nonetheless, worth pursuing as it may give a coherence to what might be experienced by learners of geometry (as presented in typical curricula) as a relatively incoherent 'bazaar'(to adopt the phrase of Hansen, 1998 p 238).

What aspects of proof and proving to accentuate

Proof and proving are, of course, central to geometry, just as they are central to mathematics. Yet it is clear from all the attempts to revise the geometry curriculum over the years, that constructing a geometry curriculum that provides learners with a meaningful experience of proof and proving is far from straightforward. Research studies (such as Williams, 1980 or Senk, 1985, conducted at a time when learners did little else but proof in geometry) have invariably shown that students fail to see a need for proof because all too often they are asked to prove things that are obvious to them. They are also unable to distinguish between different forms of mathematical reasoning such as explanation, argument, verification and proof because the emphasis in the curriculum is on the result (or on the format of the result) and learners are not exposed to the wider reasons for and form of proof. Such were the failures of attempts to teach such proof-dominated geometry curricula that in 1980 Usiskin, a well-known and highly-respected curriculum developer, famously wrote, "If proof were a new idea with which we were experimenting, too few would experience success to make the idea last" (Usiskin, 1980 p427).

This does not mean that it is impossible for students to learn deductive reasoning. It does mean that a good deal of well-researched curriculum development needs to take place before there can be any confidence that it would work any better than when Usiskin (*ibid*) called it "the failed experiment". A range of work, such as that by de Villiers (1999) and by Hanna (1998), is suggesting that increasing the emphasis on one of the major functions of proof, that of *explanation,* is central to learners' success in learning to prove. Giving explanation a higher profile, it is claimed, should help teachers connect with students' reasoning and guard against the students experiencing learning to prove as no more than a ritual determined by the teacher.

The availability of new tools, especially computer tools such as dynamic geometry packages, is also likely to impact not only on how geometry is learnt but also on *what* geometry is learnt. This applies to proof and proving as much as is does to geometric facts. While this is considered in more detail below, the general point is that the specification of the geometry curriculum needs to say something about proof and proving. The current available evidence is that giving learners experience of the various forms of proof, especially of proof as explanation, is essential.

Role and use of definitions

Involving learners in deciding on what is acceptable as a ‘proof’ was central to Fawcett’s approach to proof and proving (Fawcett, 1938). The same can be said about the role and use of definitions in geometry. As Blandford wrote in 1908 (quoted in Griffiths and Howson, 1974 pp216-217),

> “*To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions, is both interesting and highly educational*”.

Allowing learners to experience the consequences of using their own definitions means that they see that definitions are important yet hard to agree. The suggestion is that this should surmount some of the problems associated with knowing where to start in geometry proofs and make it easier for learners to accept hierarchical classifications, say of quadrilaterals (see de Villiers, 1998).

This discussion of the role and use of definitions, and the previous and subsequent items in this paper, may appear to have more to do with teaching than with the design of the curriculum. Indeed they do. Yet this relates to the earlier discussion about what is known about how geometry can be learnt. Just as technological tools, such as computer packages, can impact not only on how geometry is learnt but also on *what* geometry is learnt, so can teaching methods. New pedagogical techniques, or learning lessons from previously used techniques, can influence what is in the curriculum for geometry and when it is taught. This leads us to the final few issues that there is space to consider in this paper.

How to link geometrical intuition and deductive thinking

The need for the curriculum to be specified in a way that enables learners to link their geometrical intuition to the demands of deductive thinking is probably the most crucial issue in the design of the contemporary geometry curriculum. It means linking learners’ developing spatial awareness, and their ability to visualise, to their developing knowledge and understanding of, and ability to use, geometrical properties and theorems. Of course, there is a crucial role for the teacher here, but the point is that the curriculum should support this and not make it more difficult than it needs to be (it is difficult enough already).

How to deal with pupils’ ideas of geometric objects

Some of the problems that students have in learning geometry relate to the prototype phenomenon (in which a particular illustration of an isosceles triangle might, for example, cause students to over-generalise and assume that only triangles sitting on their base qualify as isosceles). Other problems arise from the specificities of diagrams (a property may appear to be true but is only so for a particular configuration and not in general). Both these sources of problems are well established (for a review, see Hershkowitz, 1990). Such problems have to be acknowledged by the curriculum so that teachers are not expected to teach topics with which students will have insurmountable problems.

When and how to specify the use of tools, including computer tools

As mentioned above, the use of tools, of all sorts, in geometry education influences what geometry can be learnt and, thereby influence what geometry can be specified in the curriculum. In general the curriculum needs to enable pupils to know what particular tools can do, and, above all, know when it is appropriate to use them.

Just as it is appropriate to see what tools can do, interesting challenges are possible by focusing on the limitations of tools. For example, in performing certain given constructions it is possible to try to use just a straight edge, or just compasses, or just Logo (a computer language with a graphics facility that can be used to explore various geometries).

In terms of computer software, a range of packages is available. The key feature of such software is their interactive nature. Especially with the general geometry packages (such as dynamic geometry software and Logo), learners interact with geometrical theory as they tackle problems using the software tool. This makes these computer environments, potentially, very powerful learning tools. Dynamic geometry software is perhaps the most versatile, being suitable for a range of geometry (synthetic, transformation, co-ordinate, vectors, hyperbolic, and so on). In addition, various measurements can be taken and areas calculated. With Logo it is also possible to tackle a range of geometry, including differential and fractal geometry.

In all these cases, use that is *integrated* in the teaching of geometry is what is needed so that it is the teaching of geometry that is central rather than the teaching of the software tool. The appropriate use of such software can enable learners to generate and manipulate geometrical diagrams quickly and explore geometric relationships using a range of examples. Yet such interaction with geometric theory is not without problems. For example, it is not always clear what interpretations the learners are gaining of geometrical objects they encounter in this way (see, for example, Jones, 1999). There is also the possibility that the opportunity afforded by the software of testing a myriad of diagrams through use of the 'drag' function, or of confirming conjectures through measurements (that also adjust as the figure is dragged), may *reduce* the perceived need for deductive proof (Hoyles and Jones, 1998).

A further dilemma for curriculum designers is that specifying particular tools has resource implications for schools that must adopt the curriculum. Yet ignoring tools may cost more in the long run as learners miss opportunities to learn geometry most effectively.

How to devise a teaching sequence

In making decisions about what to include in the geometry curriculum, someone has to decide on a teaching sequence. This involves addressing general questions about what topics to put in what order, and how to link together the various aspects and topics. It also involves deciding on specific questions such as should Pythagoras' theorem only be used after it has been proved and, if so, which proof of the theorem is most appropriate (out of probably in excess of 350).

Neither this issue of deciding a teaching sequence, nor any of the above critical issues, are easy to solve. This is testimony to the continuing problem of adequately specifying the geometry curriculum. Even if some of the problems, such as the issues concerning the approach to proof and proving, were adequately resolved, there are still further issues.

Final issues

One over-riding issue is how to know whether what is specified in the curriculum is teachable and learnable. Another is how to ensure there are teachers who can teach it creatively and effectively. It is worth noting the words of A. L. O'Toole, written in 1941 in the context of college geometry, "It is a shame that higher institutions that prepare secondary school teachers to send them back to the high schools knowing no more about the subject [geometry] they are to teach than when they left high school. In fact, they know less, for they have several years to forget what they once knew" (O'Toole, 1941). The reason that this is a special problem for geometry teaching is that, if anything, the amount of geometry *decreases* as one moves into the mathematics curriculum for 16-19 years olds and can disappear almost completely when one examines the mathematics offered at University level. This leads to a major problem that teachers are expected to teach geometry when they are likely to have done little geometry themselves since they were in secondary school, and possible little even then (for further discussion of this, see Jones, 2000, and Jones, 2001b).

Concluding comments

The over-riding problem for geometry is, in some sense, a good problem to have. No-one should complain that there is simply too much interesting geometry around. In other senses it is not such a good situation to be in. There are those who seek to make political capital over disagreements about mathematics education.

In terms of the geometry curriculum, for those who like to reduce complex arguments to dichotomies, the question could be put like this. Is it better that learners grasp in depth one approach to geometry (and, if so, which one) or experience a wide range of approaches to geometry. For some, the answer to this question (both parts) is obvious. It means a reinstatement of geometry in the euclidean tradition as the dominant or even sole form of geometry in schools (for example, see Wu, 1996).

Others would not wish to deny learners access to the richness that is geometry in all its facets. Indeed, to do so would be counterproductive. As Meserve (1967 p 11) wrote, "acceptance of the existence of many geometries is a necessary step in the obtaining of sufficient understanding to apply geometrical concepts effectively to mathematical problems".

The critical questions remain:

- What criteria should be applied to decide on the content and structure of the geometry curriculum?

- How can a judicious mix of elements from different geometries be arranged to fully realise the aims of geometry teaching?
- How can teachers be assisted in gaining the skills, knowledge and confidence to teach such a curriculum successfully?

To paraphrase Villani (1998, p321), answering these questions requires all those involved in, or in a position to influence, decisions about the geometry curriculum, to have a deep knowledge and understanding of the real needs and expectations of the specific community for which the curriculum has to be designed.

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