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GEOMETRICAL INTUITION AND THE LEARNING AND TEACHING OF GEOMETRY

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Intuition is often regarded as essential in the learning of geometry, but how such skills might be effectively developed in students remains an open question. This paper reviews the role and importance of geometrical intuition and suggests it involves the skills to create and manipulate geometrical figures in the mind, to see geometrical properties, to relate images to concepts and theorems in geometry, and decide where and how to start when solving problems in geometry. Based on these theoretical considerations, we illustrate a range of student tasks that we argue should contribute to developing students' geometrical intuition.

INTRODUCTION

Across the world there is considerable variation in the design of the geometry curriculum. Some countries favour an approach using congruence as a central element, while other use similarity and transformations (Hoyles, Foxman and Küchemann, 2002; Jones, 2000a; 2002). Yet, as Hoyles *et al* found, "there is evidence of a state of flux in the geometry curriculum, with most countries looking to change" (*op cit* p121). This means that there remain considerable opportunities to improve the specification of the curriculum for geometry. In this paper we argue that 'geometrical intuition' should have a crucial role. While we admit that it is not straightforward to define 'intuition' clearly, we can illustrate that geometrical intuition is a kind of skill to imagine, create and manipulate geometrical figures in the mind when solving problems in geometry.

Although mathematicians such as Poincaré and Hilbert wrote very positively about the role of geometrical intuition, and while it is generally regarded as an essential component of mathematical thinking by many professional mathematicians (e.g. Atiyah, 2001), much additional research is needed on the relations between intuitive, inductive and deductive approaches to geometrical objects (Fischbein, 1994, p244). In contemplating a geometry curriculum that includes tasks for developing students' geometrical intuition, one problem is that such tasks are often neglected in current specifications (Jones and Mooney, 2003; Fujita and Jones, 2003b). The purpose of this paper is to examine the kinds of tasks that might support the development of geometrical intuition. In what follows we provide a theoretical background focusing on the roles of intuition, visualisations and mental images in geometry, and then examine tasks that can be found in various resources, from suggestions from pioneering geometry educators to the impact of the capabilities of dynamic geometry software.

THEORETICAL BACKGROUND

Mathematicians and educators have long expressed various views about the role and nature of geometrical intuition, and how it might help or hinder the learning of geometry (and other areas of mathematics). For example, Herbart (1776-1841), one of great contributors to education in the 19th Century, observed that when we solve problems in geometry, it is necessary to exercise not only what he called 'the reasoning skill [das

Schulssvermogen]' but also 'the imagination skill [die Einbildungskraft]'. This influenced the teaching of geometry in the early 20th Century so that in Germany, for example, Treutlien designed *Geometrical Intuitive Instruction* (*Der geometrische Anschauungsunterricht als Unterstufe eines zweistufigen geometrischen Unterrichtes an unseren hohen Schulen*) in 1911 with the intention of developing students' imaginative skills in geometry. Godfrey, a leading reformer in England at the start of the 20th Century, argued that mathematics is not undertaken solely by logic but that another important power is necessary. He called this 'geometrical power', describing it as "the power we exercise when we solve a rider [a difficult geometrical problem or proof]" (Godfrey, 1910, p197). To develop this power, Godfrey argued, it is essential to train student's "geometrical eye", something he defined as "the power of seeing geometrical properties detach themselves from a figure" (*ibid*). The experimental exercises in the textbooks by Godfrey (and Siddons) were carefully chosen and designed so that they would develop the students' 'geometrical eye' (see Fujita and Jones, 2003a).

More recently, a number of educators have contributed to clarifying the roles of intuition, visualisations and mental images in the teaching and learning of geometry (see Jones, 1998). Fischbein (1993), for example, proposed the notion of 'figural concept' such that, while a geometrical figure such as a square can be described as having intrinsic conceptual properties (in that it is controlled by geometrical theory), it is not solely a concept, it is an image too. (*ibid*, p141). Accordingly, geometrical reasoning is characterised *by the interaction between these two aspects, the figural and the conceptual*. Fischbein also suggested that "the *process of building figural concepts in the students' mind should not be considered a spontaneous effect of usual geometry courses*" (Fischbein, 1993, p156, emphasis in the original). Thus they need deliberate teaching. Mason (1991) suggests that diagrams can be thought of as a means, first and foremost, for awakening mental imagery and, secondly, as ways of augmenting, extending and strengthening mental imagery and hence mathematical thinking (*ibid* p84). Goldenberg, Cuoco, and Mark (1998, p7) argue that visualisations in geometry are very important when we solve problems in geometry and suggest that a prerequisite is "the ability to take a figure apart in the mind, see the individual elements, and make sufficiently good conjectures about their relationships to guide the choice of further experimental and analytic tools". This has echoes of Godfrey's 'geometrical eye'. Finally, the notion of 'mathematisation', taken as the ability to perceive mathematical relationships and to idealise them into purely mental material (Wheeler, 1976), is pertinent.

In summary, our theoretical position builds on Fischbein's notion that geometrical figures represent mental constructs that simultaneously possess conceptual and figural properties such that successful reasoning in geometry may be related to the harmony between figural and conceptual constrains. Thus, to be a successful problem solver in geometry, we argue that one must exercise skill in:

- Creating and manipulating geometrical figures in the mind,
- Perceiving geometrical properties,
- Relating images to concepts and theorems in geometry, and
- Deciding where and how to start when solving problems in geometry.

We call this 'geometrical intuition'. As exercising 'geometrical intuition' effectively does not necessarily develop spontaneous, it needs to be nurtured intentionally through appropriate tasks in a geometry curriculum. Thus, the issue becomes how to design tasks that effectively develop ‘geometrical intuition’ in various contexts. In the next section we illustrate how this might be done with a range of examples beginning with examples from historical resources. We chose such resources to illustrate that much can be learnt from the practice of great geometry teachers.

TASKS FOR THE DEVELOPMENT OF GEOMETRICAL INTUITION

Creating and manipulating geometrical images

In his book *Geometrical Intuitive Instruction* (1911), Treutlien introduced various tasks to develop what he called ‘spatial intuitive skills (das raumliche Anschauungsvermogen)’ (see Yamamoto, 2001). One of his examples is ‘Making new figures (bilden neuer Formen)’ in which students are required to make new figures by using triangles (a similar task is introduced in Goldenberg *et al*, 1998, p10.).

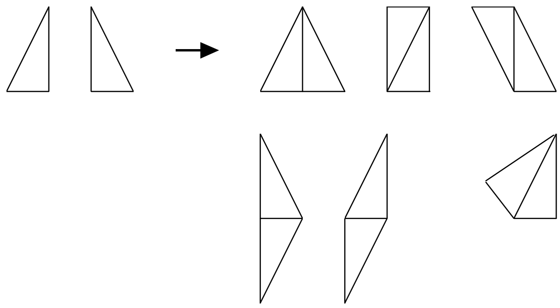

Figure 1. Example of ‘Making new figures’

The intention of such tasks is to stimulate students’ spatial imagination. Through the activity, students may be able to imagine figures or all combinations, or manipulate such activities in the mind, which Treutlein considered as important elements of spatial imagination.

Seeing geometrical properties

The UK reformer Godfrey also considered intuitive skills as important in geometry, and proposed the notion of the ‘geometrical eye’ (see Fujita and Jones, 2003a). Godfrey designed tasks with the intention of developing what he called the students’ ‘geometrical eye’. The tasks were mainly drawing and measuring figures, which directly related to definitions, theorems and riders. For example, the exercises before the theorem ‘the area of a triangle is measured by half the product of the base and the altitude’ were as follows:

> Draw an acute-angled triangle and draw the three altitudes. Repeat for a right-angled triangle. Repeat for an obtuse-angled triangle. In what case are two of the altitudes of a triangle equal?

Such exercises would make students pay attention to the height of triangles, which would be important in understanding the theorem. Another example is that the following exercises were studied before the theorem that ‘a straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord’:

> Draw a circle of about 3 in. radius, draw freehand a set of parallel chords (about 6), bisect each chord by eye. What is the locus of the mid-points of the chord?

Draw a circle and a diameter. This is an axis of symmetry. Mark four pairs of corresponding points. Is there any case in which a pair of corresponding points coincide?

What axes of symmetry has (i) a sector, (ii) a segment, (iii) an arc, of a circle?

These exercises are designed to help students become aware of the symmetry of the circle as well as leading them to discover the theorem. Also, to prove this theorem, it is necessary to show that triangles OAD and OBD (in Figure 2) are congruent, and the exercises would help students to see the congruency of the triangles. This illustrates how Godfrey and Siddons use experimental tasks to help develop students' geometrical eye.

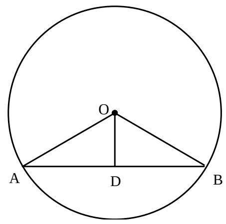

Figure 2. Circle and triangle

Imagining geometrical figures and problem solving

A question such as 'Find the number of squares in 3x3 geo-board' is probably good practice in imagining various squares on the board, but this question becomes more interesting when we ask 'Find the least number of pins which you have to take from the 3 x 3 geo-board so that you cannot make a square anymore' (this question is posed by Akai, 1997).

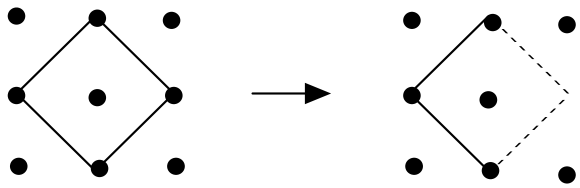

Figure 3. Squares on a 3x3 geoboard

To attack this problem, we might imagine various combinations of pins that form a square before taking pins from the geo-board, and verifying the decisions mentally. We can, of course, extend this problem to the cases of 4x4 and 5x5 geo-boards, and we have to exercise our imaginative skills more actively, since the combinations become more complex in these cases (a mathematical bonus to this question is that the pattern is 1, 3, 6, 10, …, triangular numbers!). Interestingly, Akai reported that children (aged at 12-13) could attempt to imagine various combinations on the geo-boards before taking pins, i.e. they started to exercises their imagination skills (Akai, 1997, p236). Also, this question gives an opportunity to think deductively, for example, we also have to justify that the following combination makes a square, which might be useful to build a figural concept of squares.

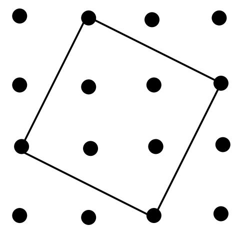

Figure 4. Squares on a 4x4 geoboard

IMPLEMENTATION AND RESOURCES

As the UK Royal Society notes in a comprehensive report on the teaching and learning of geometry "we need to develop a completely new pedagogy in geometry" (Royal Society 2001, p11). As we have argued in this paper, we think that a useful focus would be on developing pedagogical methods that mean that a deductive and an intuitive approach are mutually reinforcing when solving geometrical problems. In this it is well known that the use of tools (of all sorts) influences what geometry can be learnt and how it is learnt. For example, interesting challenges are possible by focusing on the limitations of tools, say in performing certain constructions with just compasses, or just the computer application *Logo*. Using software such as dynamic geometry packages means that learners interact with geometrical theory as they tackle problems using the software tool. This makes these computer environments, potentially, very powerful learning tools. Yet such interaction with geometric theory is not without problems. For example, it is not always clear what interpretations the learners are gaining of geometrical objects they encounter in this way (see, for example, Jones, 2000b). What is clear, however, is that with appropriate tasks, students can use a mixture of a deductive approach (by, for instance, knowing from geometrical theory that they need to construct, say, a perpendicular through a point) and an empirical intuitive approach provided for by software by being able, say, to 'drag' a second perpendicular to a given line into place to see if this provokes them into seeing *why* this enables them to solve the problem *and provide a suitable proof* (see Jones, 1998, for an elaboration on this). This illustrates how a deductive and an intuitive approach can prove to be mutually reinforcing when solving geometrical problems and lead to powerful learning.

CONCLUDING COMMENT

One of problems in geometry education is related to students' intuitive skills in that some students appear to be unable to 'see' geometrical properties, or decide where to start, when they solve proof exercises in geometry (for example, see Sinclair, 2003, p295). In this paper, we argue for the importance of 'geometrical intuition', and suggest that a major component of an innovative geometry pedagogy would be to improve on general appeals to develop 'geometrical intuition'. We define 'geometrical intuition' as a skill to create and manipulate geometrical figures in the mind, to see geometrical properties, to relate images to concepts and theorems in geometry, and decide where to start when solving problems in geometry. We suggest that tasks that require students to imagine and manipulate geometrical figures can link geometrical intuition more directly with geometrical theory, and involve active use of imagination skills. In this paper, we have just considered a few examples and have been able only to touch on the use of dynamic geometry software, so an obvious next task is to consider how we include such examples in current geometry curricula. Another undertaking that is needed is to examine practically how such tasks can be incorporated in actual teaching practices. Such research could make an important contribution to providing a firmer theoretical and practical basis for formulating new curricular and pedagogic models for geometry.

Author's note: All the authors of this paper contributed equally. The order of authorship is strictly alphabetical.

REFERENCES

Akai, T.: 2000, Mathematical activities on geo-boards. *Practical Studies in Elementary School Mathematics Education of Japan in 1990's*. Tokyo: Japan Society of Mathematical Education. 15-18.

Atiyah, M.: 2001, Mathematics in the 20th Century, *Mathematics Today*, **37**(2), 46-53.

Fischbein, E.: 1993, The Theory of Figural Concepts. *Educational Studies in Mathematics*, **24**(2), 139-162.

Fischbein, E: 1994, The Interaction between the Formal, the Algorithmic and the Intuitive Components in a Mathematical Activity, *In*: R. Biehler *et al* (eds), *Didactics of Mathematics as a Scientific Discipline*. Dordrecht: Reidel, 231-245.

Fujita, T. and Jones, K.: 2003a, The Place of Experimental Tasks in Geometry Teaching: learning from the textbooks design of the early 20th Century. *Research in Mathematics Education*, **5**, 47-62.

Fujita, T. and Jones, K.: 2003b, *Interpretations of National Curricula: the case of geometry in Japan and the UK*, British Educational Research Association 2003 Annual Conference, Heriot-Watt University, Sept 2003.

Godfrey, C.: 1910, The Board of Education Circular on the Teaching of Geometry, *Mathematical Gazette.* Vol. V, 195-200.

Goldenberg, E. P., Cuoco, A. A. and Mark, J.: 1998, A Role for Geometry in General Education. *In*: R. Lehrer and D. Chazan, D. (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space*. Mahwah, NJ: Lawrence Erlbaum Associates

Hoyles, C., Foxman, D. and Küchemann, D.: 2002, *A Comparative Study of Geometry Curricula*. London: QCA.

Jones, K.: 1998, Deductive and intuitive approaches to solving geometrical problems. *In*: C. Mammana and V. Villani (Eds.), *Perspective on the Teaching of Geometry for the 21st Century*, Dordrecht: Kluwer.

Jones, K.: 2000a, Critical Issues in the Design of the Geometry Curriculum. *In*: B. Barton (Ed), *Readings in Mathematics Education*. Auckland, New Zealand: University of Auckland. 75-91.

Jones, K.: 2000b, Providing a Foundation for Deductive Reasoning. *Educational Studies in Mathematics*, **44**(1&2), 55-85.

Jones, K.: 2002, Issues in the Teaching and Learning of Geometry. *In*: L. Haggarty (Ed), *Aspects of Teaching Secondary Mathematics*. London: RoutledgeFalmer. 121-139.

Jones, K. and Mooney, C.: 2003, Making Space for Geometry in Primary Mathematics. *In*: I. Thompson (Ed), *Enhancing Primary Mathematics Teaching*. London: Open University Press. 3-15.

Mason, J. 1991, Questions about Geometry, *In*: D. Pimm and E. Love, (Eds.), *Teaching and Learning School Mathematics*, Hodder and Stoughton, London, pp. 77-90.

Royal Society: 2001, *Teaching and Learning Geometry 11-19*. London: Royal Society.

Sinclair, M: 2003, Some Implications of the Results of a Case Study for the Design of Pre-constructed, Dynamic Geometry Sketches and Accompanying Materials, *Educational Studies in Mathematics,* **52**, 289-317.

Wheeler, D.: 1976/2001, Mathematisation as a Pedagogical Tool, *For the Learning of Mathematics,* **21**(2), 50-53.

Yamamoto, S.: 2001, The Meaning of Spatial Imagination in Geometrical Intuitive Instruction of P. Treutlein, *Journal of JASME: Research in Mathematics Education*, **7**, 105-116. [in Japanese]