British Society for Research into Learning Mathematics

Geometry Working Group

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GEOMETRY AND PROOF

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Is Euclidean geometry the most suitable part of the school mathematics curriculum to act as a context for work on mathematical proof? This paper examines some of the issues regarding the teaching and learning of proof and proving specifically in relation to Euclidean geometry.

INTRODUCTION

Up to the late 1950s, Euclidean geometry was regarded as the place in the school mathematics curriculum where students learnt proofs and were introduced to the axiomatic structure of mathematics. The other parts of the mathematics curriculum (number and algebra, at that time) were less concerned with such matters.

In the wake of the launch of the Sputnik by the Soviets in 1957, a major revision of school mathematics (and science) was begun in most western countries (see Moon, 1986, for a general account and Howson, 2000 or Jones, 2001 for the impact on geometry). One of the reform ideas was to base much more of school mathematics on a set theoretic foundation, reflecting the emphasis in university mathematics at that time: maths was to be ‘modern’. From this base, it was natural to introduce functions in school mathematics which lead to calculus and linear algebra. In the UK these major changes were in the ‘O’ level (or equivalent) syllabuses, catering for about 25% of school pupils at age 16. The room for this innovation was made by reformulating all parts of the mathematics curriculum, but the practical effect was to reduce the study of solid geometry and to convert the trigonometry component into part of a course about functions. The impact of these changes was to reduce the overall amount of geometry while, at the same time, increasing the emphasis on co-ordinate geometry and introducing some elements of transformation geometry and topology.

One consequence of these changes was that geometric proof was allocated proportionally much less curriculum time than in previous eras. Subsequently, curricula were developed that required proof and proving to permeate the whole mathematics curricula (examples of the latest versions of such curricula include the *National Curriculum for England: Mathematics* (DfEE, 1999) and the Standards for School Mathematics published by the US National Council of Teachers of Mathematics (NCTM, 2000)). In the UK, with the advent of GCSE in 1986, curriculum reformers were cautious about requiring deductive-style proof from the majority of pupils and the emphasis moved to the communication of mathematical reasoning rather than production of Euclidean-style proofs.

With the introduction, in England and Wales, of the National Curriculum in 1989 the issue of mathematical reasoning and justification became channelled into the attainment targets ‘Using and applying mathematics’. Explicit requirements to prove, in a traditional mathematical sense, were specified for the very highest attainers, but it is possible that these did not demand a priority in teaching. A consequence was that the main experience of proof for new mathematics undergraduates was during their A-level studies prior to University and, in the main, this was probably fairly limited. This provoked reactions from those who teach mathematics to undergraduates. In the UK, a publication spearheaded by the London Mathematical Society (LMS, 1995), a major professional association in the UK for pure mathematicians, complained about the lack of emphasis on proof in the curriculum, (as well as other things), despite the school curriculum specification that mathematical reasoning should be taught. In the US the debate (about school mathematics in general) has been so heated and, at times, acrimonious, that it has become known as the “math wars” (see, for example, Schoen *et al*, 1999).

In terms of the geometry curriculum, there have been several calls for Euclidean-style geometry to be reinstated as a major component of school mathematics and for this to be where students gain their experience of proof and proving (see, for example, McClure, 2000, or Wu, 1996). The purpose of this paper is to review these arguments for the reinstatement of Euclidean geometry and examine some of the issues regarding the teaching and learning of proof and proving.

PROOF IN EUCLIDEAN GEOMETRY

McClure (2000) considers Euclidean-style geometry to be the best place to begin “a student’s serious mathematical training” (p45) because:

- It involves familiar objects that can be thought about both visually and verbally
- The statements it makes about these objects are readily intelligible and frequently dramatic
- The logical methods involved tend to be less subtle than those in other introductory parts of mathematics; for example, they involve fewer quantifiers
- It is possible to do serious mathematical learning in this subject without having a perfect understanding of what axiomatic systems are and what the rules are for working with them

McClure’s interest is mathematics undergraduates, rather then school students, and he goes on to contrast geometry in the Euclidean tradition as a place to learn about proof and proving with other areas of the undergraduate mathematics curriculum, including analysis and linear algebra. In both cases he suggests that geometry in the Euclidean tradition is superior in this context. In analysis he maintains that the mathematical objects (such as limit) can only be approached by means of subtle definitions that involve the careful use of quantifiers. In linear algebra he suggests that the proofs are too sophisticated and deal with unfamiliar objects (such as a subspace) that rely on formal definitions.

Wu (1996) is similarly convinced that geometry in the Euclidean tradition is a place to learn about proof and proving but this time the focus is high school mathematics. The argument is that Euclidean-style proofs are often short, only require a few concepts (angle, line segment, etc.), are supported by a visual prop, and are quite formal in structure. Wu maintains that the only other topic available for this purpose is the real number system but claims that "anyone who has ever gone through a development of the real number system, starting from the Peano axioms or the axioms of a complete ordered field, would know that this alternative is fraught with perils. …. Moreover, the discussion would soon be dominated by continuity considerations and they are definitely out of the reach of the 10^{th} and 11^{th} graders " (p227-228).

DIFFICULTIES WITH PROOF IN EUCLIDEAN GEOMETRY

Both McClure and Wu are University mathematicians. While McClure restricts himself to University mathematics (and outlines how Euclidean-style proofs can provide a good 'bridging' course for first year mathematics undergraduates), Wu ventures into school mathematics. Neither Wu nor McClure discuss the fact that a school geometry curriculum dominated by proofs in the Euclidean tradition has been tried in the past and been found to be wanting. As Howson (2000) writes, ""Euclid-style" geometry [was] found extremely difficult (and often uninteresting) by most [school] students". He quotes Tammadge reporting on his experiences as an examiner of the top 20% or so of English students: "Only a small percentage of candidates attempted questions on this topic and they normally regurgitated the theorem and collapsed when it came to the rider [i.e. a request to prove a corollary to the theorem]". In similar terms, past research studies by Williams (1980) or by Senk (1985) provide evidence across a wide range of schools of how little those pupils who followed such a geometry curriculum could do at the end of their course. Such were the failures of attempts to teach such a proof-dominated geometry curricula that in 1980 Usiskin, a well-known and highly-respected curriculum developer, famously wrote, "If proof were a new idea with which we were experimenting, too few would experience success to make the idea last" (Usiskin, 1980 p427).

The reasons for this lack of success in teaching proof are numerous (for a recent review, see Dreyfus, 1999). Research studies have invariably shown that students fail to see a need for proof because all too often they are asked to prove things that are obvious to them. Students also fail to distinguish between different forms of mathematical reasoning such as heuristic or argument, explanation or proof. A major gap in the research literature is how little is known about how children can be supported in shifting from "because it looks right" or "because it works in these cases" to convincing arguments which work in general.

TEACHING AND LEARNING PROOF

There is some current research that may indicate positive ways forward. A range of work, such as that by de Villiers (1999) and by Hanna (1998), is suggesting that increasing the emphasis on one of the major functions of proof, that of explanation, is central to learners' success in learning to prove. Giving explanation a higher profile, it is claimed, should help teachers connect with students' reasoning and guard against the students experiencing learning to prove as no more than a ritual determined by the teacher. However, mathematical proof is more structurally specific than a general explanation. In particular, learning to prove involves learners taking on this precise form of reasoning as their own (Rodd, 2000: 236) such that they tend to require proof-like explanations in order to become convinced.

In addition, the availability of new tools, especially computer tools such as dynamic geometry software, also has implications for the way proof and proving can be taught and learnt (see, for example, Mogetta *et al*, 1999, but see Hoyles and Jones 1998 for some cautionary remarks). Proof and proving can also be met in other parts of the mathematics curriculum (see Tall, 2000, or Rodd and Monaghan, 2001).

In the current version of the National Curriculum for England (DfEE 1999), the programmes of study include *geometrical reasoning* as well as a *reasoning* component of the 'Using and applying' attainment target. Even at the 'foundation' level at Key Stage 4 (for the lower attaining 14-16 year olds), students are to be taught to distinguish between practical demonstrations and proofs and to show step-by-step deduction in solving a geometrical problem (DfEE 1999:78). For teachers teaching this, or a similar, curriculum, the challenge is to develop teaching methods which do not turn pupils off or get them solely to learn by rote (as appears to have been the case in the past). This will certainly require new pedagogical approaches which are likely to involve technology like dynamic geometry, as well as discursive methods of engagement and methods of assessment which reduce the pressure to rote learn.

CONCLUDING COMMENTS

Proof and proving are, of course, central to all mathematics. In terms of school mathematics, the NCTM standards (*ibid*) state that "Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied". They go on to suggest that "in mathematically productive classroom environments, students should expect to explain and justify their conclusions. When questions such as, What are you doing? or Why does that make sense? are the norm in a mathematics classroom, students are able to clarify their thinking, to learn new ways to look at and think about situations, and to develop standards for high-quality mathematical reasoning". This sort of language is consonant with what is known about how to teach mathematical reasoning in general, and proof and proving in particular. It suggests that an atmosphere of collective classroom enquiry is important generally.

The NCTM standards also observe that, "Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by 'doing proofs' in geometry". Perhaps students at the tertiary level find proof so difficult because their previous experience is limited. If mathematical reasoning was a consistent part of students' mathematical experience throughout the school years, then students might become accustomed to this way of thinking. Both in the UK and in the US mathematical reasoning – not just mathematical techniques or results – is considered important for all students. Nevertheless, unless teaching methods can be developed to engage all, there is a real danger of returning to the situation of non-comprehension to which Howson (*ibid*) refers.

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BSRLM Geometry Working Group

The BSRLM geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. The aim of the group is to share perspectives on a range of research questions that could become the basis for further collaborative work. Suggestions of topics for discussion are always welcome. The group is open to all.

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