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Developing Geometrical Reasoning

Margaret Brown
Kings College, University of London
<margaret.brown@kcl.ac.uk>

Keith Jones
University of Southampton, England
<D.K.Jones@soton.ac.uk>

Ron Taylor
Mathematics Inspector for Hampshire, England
<RonTGImath@aol.com>

Ann Hirst
University of Southampton, England
<A.E.Hirst@soton.ac.uk>

This paper summarises a report (Brown, Jones & Taylor, 2003) to the UK Qualifications and Curriculum Authority of the work of one geometry group. The group was charged with *developing and reporting on teaching ideas that focus on the development of geometrical reasoning at the secondary school level*. The group was encouraged to explore what is possible both within and beyond the current requirements of the UK National Curriculum and the Key Stage 3 strategy, and to consider the whole ability range.

In 2000, following a remit from UK Government ministers, the Qualifications and Curriculum Authority (QCA) embarked on a three-year project to consider whether the algebra and geometry components of the National Curriculum for mathematics needed strengthening at key stages 3 and 4 (secondary school levels). The group based at Southampton included teachers and a local authority officer from Hampshire Local Education Authority, together with mathematicians and mathematics educators from the University of Southampton, University College Chichester and King's College London. The group members were:

Jill Barton, Cams Hill School, Fareham
Margaret Brown, King's College London/SMP (member of QCA Advisory Group)
Ann Hirst, University of Southampton
Keith Hirst, University of Southampton (to March, 2003)
Keith Jones, University of Southampton
Carol Knights, Applemore College, Dibden Purlieu, Southampton
Jo Lees, Crestwood Community School, Eastleigh
Paul Morris, Brune Park Community School, Gosport
Adrian Oldknow, University College Chichester (member of QCA Advisory Group)
Peter Ransom, The Mountbatten School, Romsey
Ron Taylor, Mathematics Inspector for Hampshire (member of QCA Advisory Group)

Geometrical reasoning in the secondary classroom

Aims and Rationale

Important objectives in teaching mathematics at the secondary level include developing a knowledge and understanding of, and the ability to use, geometrical properties and theorems and encouraging the development and use of conjecture, deductive reasoning and proof. While such objectives are relatively easy to state, developing a suitable geometry curriculum that can be taught successfully to the majority of pupils remains an elusive goal (see, Jones, 2001). As the writers of one major curriculum development project observed:

> Of all the decisions one must make in a curriculum development project with respect to choice of content, usually the most controversial and the least defensible is the decision about geometry.

(Chicago School Mathematics Project staff 1971, p281)

A recent comparative study of geometry curricula found considerable variation in current approaches to the design of the school geometry curriculum (Hoyles, Foxman, & Küchemann, 2002). For example, a "realistic" or practical approach is apparent in Holland, while a theoretical approach is evident in France and Japan. Most countries, although not all, include elements of proof and proving in their curricula specifications for geometry. Here, too, there are variations, with some countries favoring an approach with congruence as a central element, while other use similarity and transformations. What is agreed is the central importance of geometry in mathematics. As the renowned mathematician Sir Michael Atiyah writes:

> … spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool...
>
> (Atiyah, 2001, p50)

The report examined what might be promising ways of developing classroom approaches that provide a suitable foundation for the development of deductive reasoning and proof in geometry at the secondary school level. The next section provides a rationale for the development of classroom teaching ideas that are reported on in the main part of this report.

Classroom approaches

Hiebert *et al* (1977) offers a useful framework for considering different "dimensions" of the classroom and the links between them. The key five dimensions noted by the authors are:

- The nature of classroom tasks
- The role of the teacher
- The social culture of the classroom
- Mathematical tools as learning supports
- Equity and accessibility.

The Royal Society/Joint Mathematical Council (2001) report on the teaching and learning of geometry for pupils aged 11-19 makes a number of recommendations about suitable approaches to the teaching of deductive reasoning in geometry. Such an approach, the report suggests, should mean that all pupils come to understand that deductive reasoning is more than simply stating a belief or checking that the result is valid in a number of specific cases. However, the report observes, "we accept that it is not an easy matter to determine how to achieve this with each pupil and each result and that a careful choice of approach will be needed" (p. 9).

The use of *local deduction*, an activity in which new theorems are deduced from accepted theorems, is an important aim of a mathematics curriculum. However deductive geometrical reasoning can be more widely interpreted to also include:

- Deriving a specific value of a variable (*e.g.*, the size of an angle) using both known theorems and known properties of shapes.
- Deducing a specific result in relation to a figure with given properties which does not have the generality or status of a theorem (*e.g.*, proving that two sides are equal

of a quadrilateral with a particular set of properties). This type of problem used to be known as a ‘rider’.

- Considering alternative definitions of geometrical shapes, deciding which of these are necessary, sufficient and minimal, and becoming familiar with the differences between the meanings of these terms.

Some classroom activities use *inductive reasoning* (empirical generalisation or verification) in geometrical contexts, such as measuring angles of several figures in order to generalise or to verify a result. Inductive reasoning is an important mathematical process and often leads to interesting conjectures.

Pupil understanding of proof and proving in the UK has been the subject of a major research study, carried out prior to the implementation of the current version of the National Curriculum, (*e.g.* Healy, & Hoyles, 1998, Küchemann, & Hoyles, 2002). Drawing from these considerations, in developing suitable teaching materials for mathematical reasoning, the following are suggested as guiding principles:

- The geometrical situations selected should be chosen, as far as possible, to be useful, interesting and/or surprising to pupils.
- Activities should expect pupils to explain, justify or reason and provide opportunities for pupils to be critical of their own, and their peers’, explanations.
- Activities should provide opportunities for pupils to develop problem-solving skills and to engage in problem posing.
- The forms of reasoning expected should be examples of *local* deduction, where pupils can utilise *any* geometrical properties that they know to deduce or explain other facts or results.
- To build on pupils’ prior experience, activities should involve the properties of 2D and 3D shapes, aspects of position and direction, and the use of transformation-based arguments that are about the geometrical situation being studied (rather than being about transformations *per se*).
- The generating of data or the use of measurements, while playing important parts in mathematics, and sometimes assisting with the building of conjectures, should not be an end point to pupils’ mathematical activity.

Language, communicating and reasoning

Enabling pupils to communicate their mathematics to their peers and others is an essential aim within the mathematics classroom. Moving from informal discussions, with imprecise oral descriptions and explanations, to precise, unambiguous and concise communication (including that involving written explanations and, for example, the invention of new symbols and diagrams) is one of the most challenging tasks for a teacher.

Giving pupils the opportunity to discuss their mathematics with their peers or teacher, rather than engaging solely in short ‘question and answer’ interactions, helps them to clarify their thinking and improve their understanding. It also helps them to organise their thoughts in preparation to presenting a cogent argument or line of reasoning. Communication is particularly important in deductive reasoning.

Activities developed and trialled

A large number of activities were designed, selected and developed during and between the group meetings, and tried out in schools by the five teachers and other colleagues. We only include details of three of the eight recorded in the full report. These include:

- Shape properties,
- diagonals (folded paper and using Geometer's Sketchpad),
- two-piece tangrams.

The classroom trialling was the most useful part of the work of the group. Members reported in several ways, using video, students' written work, computer recorded work and verbal reports. What is apparent from all the reports is that the materials by themselves will always be insufficient to develop geometrical reasoning to the extent that is desirable. It is clearly important that teachers use materials in ways that prompt pupils to engage in reasoning activities. The reports of the trials suggest several strategies, including:

- the questions which are posed in the materials themselves;
- teacher intervention during classroom activity to elicit reasons for pupils' conclusions;
- structured student recording;
- group work where students are required to give explanations to each other, and
- presentations by one or more students to the remainder of the class.

Classroom reports: reasoning about properties and definitions of shapes

Introduction

One aim of the 11-16 geometry curriculum is to help children to progress in sophistication in their ways of thinking about shapes and their properties. The van Hieles (1986) have described a progression through different levels in geometrical reasoning that might underlie a geometry curriculum. A considerable amount of international research has been carried out on this proposed progression; much of this research has broadly validated its usefulness but some has shown that it is a limited tool. However it seems helpful to refer to it in discussing different types of reasoning which occurred in some of the activities used in classrooms in this study.

Shape property activity

Jo gave the task, to Year 7 pupils, of writing down all they knew about two shapes represented by pictures - an equilateral triangle and a square. This activity, in addition to raising issues of logic and reasoning, was particularly useful in demonstrating the problems of written communication among some pupils (especially boys in set 4 out of 5). Some examples which include issues in both mathematics and language skills are given below; the spelling and punctuation is the student's own in each case.

- Jason - triangle. 3-sides. No angeles.
- Dale - triangle. 3 corners and 3 sides the same.
- Tom - equarltral. 3 equale sides. 3 ponits 60º each.
- Scott - squir. 4 side all the same. 4 right angles the same length.
- Daryl - It is a square with 4 equal sides, all the angles are 90º. They are called right angles. It is a quadrilateral.

These answers potentially provide the basis for some interesting discussions that highlight alternative definitions. So, rather than just marking and returning the answers, Jo decided to use a form of 'peer assessment' which involved students commenting on and comparing each other's work. Jo extracted some of the properties referred to by different children without indicating their origin, and asked which gave the best description.

- It is called a square.

- That is what a square is.
- A square is a quadrilateral.
- It has 4 sides and 4 corners.
- It has 4 right angles.
- It has 2 pairs of parallel lines.
- All the sides are the same length.

Note that the first two of these quotations, each attached to a rough drawing of a square, could be used to illustrate van Hiele level 1 thinking which does not go beyond matching shapes with names. The same task was given to the equivalent set (4 out of 5) in Year 8. It was clear that the Year 8 pupils had better vocabulary and some were able to use more precise language.

Investigating diagonal properties of quadrilaterals

This task, used by Carole, and built on an idea from Fielker (1981,1983), was trialled with a Year 9 top set. The lesson started with a review of the angle and line properties of quadrilaterals. A quadrilateral was drawn step by step on the board. Students had indicate when they were certain that they had enough information to identify the quadrilateral. This encouraged argument and counter-argument, and set the tone for the lesson. With a weaker group it may be more appropriate to focus on symmetry of quadrilaterals. The first part of the main task was explained:

1. *Fold a piece of A4 paper in half vertically and horizontally*
2. *Put a point on each half of each of the folds.*
3. *Join the 4 points to form a quadrilateral.*
4. *Investigate which quadrilaterals can be made and which ones can't.*

In some ways this is similar to the previous activity, but is the inverse in that students are asked to identify shapes with particular diagonal properties, giving reasons. Students then moved on to the second part of the task - a pair of non-perpendicular diagonals.

> Fold a second piece of paper in a different way and repeat the task to investigate which quadrilaterals can now be made. Explain why certain quadrilaterals can be made.

Again students were very competent in carrying out the task and correctly identifying the quadrilaterals constructed.

The final part of the task was to try and explain why certain quadrilaterals were not possible with each pair of diagonals. Some students had a really strong feel for the task while others were quite frustrated that they could visualise why certain things weren't possible but struggled to explain verbally. The students' frustration provided an excellent opportunity to illustrate the usefulness of geometric proof. Errors in relation to trapezia arose perhaps because of a limited idea of the shape as necessarily including 'two long parallel lines', or thinking that the two non-parallel lines should make equal angles with the parallel lines (a regression to van Hiele level 1?).

The task thus was successful in revealing some misconceptions and provided plenty of discussion points. However it proved to be challenging for this Year 9 set 1 to reason why some shapes could or couldn't be generated. In order to try to make the task a bit more accessible Carol decided to do a parallel version using the Geometers' Sketch Pad program to generate a more structured set of activities. These were designed to guide the students through the investigation at their own pace, with an exercise book and pencil to hand. On the screen different coloured text was used for instructions, for information which pupils could copy into their books if required, and questions that the pupils are required to discuss and record an answer to.

To enable the students to access this task, they needed to be able to use basic tools in Geometers' Sketchpad and understand 3-letter angle notation. The students were given the option to make notes in their exercise books or to answer the questions on screen and then save the document to their area. This helped to increase their motivation. Students were able to make connections between the shapes that were possible with each set of diagonals. They easily determined that to turn a rectangle into a square they needed to make the diagonals cross at right angles. As a plenary with Year 9 set 1, two students explained the properties of the diagonals and the shapes that were feasible with each.

This task did enhance students' understanding of properties of quadrilaterals and did encourage them to think about their reasoning. The use of Geometers' Sketch Pad enabled students to become familiar with the diagonal properties of quadrilaterals, and while the original exercise had proved rather difficult for some, the Geometer's Sketch Pad version, being more structured, was more successful with the majority students.

Reasoning using the 2-piece tangram activity

This activity has been developed by Peter and was first trialled by him and by colleagues in his school, at many different levels. The two piece jigsaw is made from a square with a cut being made from one vertex to the midpoint of a side. Peter explained,

> I give them a square piece of card from which they produce the two pieces. This means that they all work with a standard size square (with right angles!). Students are then asked how many shapes they can make by matching sides of equal length, what the shapes are, and how they can be sure that the pieces make these shapes.

A video-recording was made of the first session with each of two classes (Year 9, set 4 out of 5, Year 8 set 5 out of 5). This was viewed afterwards by two separate Heads of Mathematics groups in Hampshire as well as by the geometrical reasoning group. There was a great deal of very productive activity in each lesson. What surprised Peter was the mathematical vocabulary that was used by the pupils when explaining their results to each other both in pairs and as a larger group. There was much kinaesthetic work taking place – pupils explained with their hands as well as their voices. Pupils enjoyed using overhead projectors and explaining to their peers. In the Year 8 class he was delighted to see two of the lowest attainers, who normally lack confidence, volunteer to present their work to the class. The plenary session involved two groups explaining the reasoning to the others. It was noticeable that although the overhead projector transparency did not include much written reasoning, the pupils spent most time explaining their reasoning to the class. This was pleasing since that was the key objective of the lesson.

Their subsequent written work did not reflect the high level of geometrical understanding used in the class. Peter found that pupils are unwilling to write down much of what they explain verbally and with their hands. He felt that the year 9 pupils were working at the Van Hiele level 4 in the class, though this was not obvious in their written work. Peter also found that the work helped him to identify some of the misconceptions that pupils had. (*The transcript of a discussion revealed that one pupil thought that parallel lines had to be equal in length.*) Without this type of activity, and the discussion generated, he thought it unlikely that this misconception would have come to light.

When the video was shown to the group responsible for the report, it was felt that what both Year 8 and Year 9 classes achieved was impressive, especially in terms of articulate presentation and verbal reasoning. One girl in particular was using "because" as a natural part of her discourse about the shapes being discussed, and seemed to be operating towards van Hiele level 4. The strategy of getting the students working in groups with the goal of preparing an overhead transparency to present their findings to the class gave them a purpose for engaging in written communication, and helped them to codify their verbal reasoning. The video also made it clear that for an activity such as this to lead to geometrical reasoning as an outcome the teacher has to pose questions that elicit that reasoning, and constantly go beyond simply acknowledging empirical observations. The teacher's own knowledge of the logical structure of the area of geometry being investigated is clearly very important here.

The video also raised many interesting language issues, for example where the use of the phrase "which of the sides are parallel" appeared to elicit a different response to "which two sides are parallel". One boy accompanied oral description by signalling (hand gestures or pointing with a pen). The corresponding aspect in a written account would be static labelling, using letters to label points and signs to denote equal angles or parallel lines for example. It became more obvious that the two modes of communication (verbal and written) have significantly different characteristics, and that the transition between them needs careful management. Pupils need to be clear about the purpose of communication (in this case communication to the rest of the class).

Conclusions

In designing and trialling classroom material, the group found that the issue of how much structure to provide in a task is an important factor in maximising the opportunity for geometric reasoning to take place. The group also found that the role of the teacher is vital in helping pupils to progress beyond straightforward descriptions of geometrical observations to encompass the reasoning that justifies those observations. Teacher knowledge in the area of geometry is therefore important. The group found that pupils benefit form working collaboratively in groups with the kind of discussion and argumentation that has to be used to articulate geometrical reasoning. This form of organisation creates both the need and the forum for argumentation that can lead to mathematical explanation.

Whilst pupils can demonstrate their reasoning ability orally, either as part of a group discussion or through presentation of group work to a class, the transition to individual recording of reasoned argument causes significant problems. Several methods have been used successfully in this study to support this transition, but more research is needed into ways of helping written communication of geometrical reasoning to develop. Given the finding of the project that many pupils know more about geometrical reasoning than they can demonstrate in writing, the emphasis in assessment on individual written response does not capture the reasoning skills which pupils are able to develop and exercise. Sufficient time is needed for pupils to engage in reasoning through a variety of activities; skills of reasoning and communication are unlikely to be absorbed quickly by many students.

The study suggests that it is appropriate for all teachers to aim to develop the geometrical reasoning of all pupils, but equally that this is a non-trivial task. Obstacles that need to be overcome are likely to include uncertainty about the nature of mathematical reasoning and about what is expected to be taught in this area among many teachers, lack of exemplars of good practice (although we have tried to address this in this report), lack of time and freedom in the curriculum to properly develop work in this area, an assessment system which does not recognise students' oral powers of reasoning, and a lack of appreciation of the value of geometry as a vehicle for broadening the curriculum for high attainers, as well as developing reasoning and communication skills for all students.

References

Atiyah, M. (2001), Mathematics in the 20th Century: Geometry versus Algebra, *Mathematics Today*, **37**(2), 46-53.

Brown, M., Jones, K. & Taylor, R. (2003), *Developing geometrical reasoning in the secondary school: outcomes of trialling activities in classrooms*, Report, Qualifications and Curriculum Authority. ISBN: 0854328092

Chicago School Mathematics Project staff, (1971), The CSMP Development in Geometry, *Educational Studies in Mathematics*, **3**(3-4), 281-285.

Fielker, D.S. (1981). Removing the shackles of Euclid. *Mathematics Teaching* **95**, 16-20; **96**, 24-28; **97**, 321-327. [Reproduced as a booklet by ATM , 1983]

Healy L. and Hoyles C. (1998). *Justifying and Proving in School Mathematics: Summary of the results from a survey of the proof conceptions of students in the UK*. Research Report, Mathematical Sciences, Institute of Education, University of London.

Hiebert, J *et al* (1997). *Making Sense: Teaching and learning Mathematics With Understanding*. Portsmouth, NH: Heinemann.

Jones, K. (2000), Critical Issues in the Design of the Geometry Curriculum. *In*: Bill Barton (Ed), *Readings in Mathematics Education*. Auckland: University of Auckland, pp 75-91.

Küchemann, D. & Hoyles, C. (2002). The Quality of Students' Reasons for the Steps in a Geometric Calculation, *Proceedings of the British Society for Research into Learning Mathematics*, **22**(2), 43-48.

Royal Society (2001) *Teaching and learning geometry 11-19: Report of a Royal Society / Joint Mathematical Council working group.* London: Royal Society.

van Hiele, P.M. (1986). *Structure and Insight: a theory of mathematics education*. Orlando FL: Academic Press.