APPLICATION OF FUNCTIONAL ANALYSIS TO THE SOUND FIELD RECONSTRUCTION

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1 INTRODUCTION

The problem of sound field reconstruction is a subject of relevance in many branches of acoustics. In the audio industry, considerable research has been dedicated to the study of recording and reproduction systems that allow an accurate rendering of the spatial information that is considered to be an important component of the sound scene. This research activity is also encouraged by the increasing diffusion of multi-channel recording and reproduction systems and multi-channel audio formats. Loudspeaker systems using the 5.1 format are now widely accepted in the consumer market and systems with an increasingly large number of loudspeakers are now being proposed.

The rendering of the spatial information describing the sound scene can be attempted in a number of different ways. One of the possibilities is to attempt the physical reconstruction or the synthesis of the desired sound field using an array of loudspeakers. Theories like Wave Field Synthesis [1] and High Order Ambisonics [2] have been proposed and applied in order to realize this task, and other approaches to the same problem have been proposed more recently [3], [4]. All these theories are based on the physical description of the sound field using rigorous mathematical models. The aim of this paper is to propose the theory of an alternative approach to sound field reconstruction, that is based on functional analysis. The latter has been widely used in other branches of physics, such as quantum mechanics, which share with acoustics some important mathematical models. This suggests that a similar approach could be useful in order to produce a greater insight into the sound field reconstruction problem. An approach based on a mathematical background analogous to that presented in this paper has already been used for practical inverse problems of engineering interest, such as those described, for example, in [5] and [6]. An extensive and rigorous mathematical introduction to functional analysis applied to acoustic and electromagnetic scattering problems can be found in [7]. Concepts such as Hilbert spaces, compact operators and self-adjoint operators are introduced in the course of this paper, and the reader can refer to [8], [9] and [10] for an extensive discussion of these mathematical tools.

The target of the system described in this paper is the reconstruction of a sound field over a region of space Ω that does not contain acoustic sources or scattering objects, which means that the sound field in that region can be described by the homogeneous Helmholtz equation. The loudspeaker array is assumed to be an ideally infinite distribution of sources continuously arranged on a three dimensional surface S, that contains the region of space Ω over which the sound field reconstruction is attempted. This is represented in Figure 1. It is also assumed that the information on the desired sound field is represented by the knowledge of the acoustic pressure $p(\mathbf{x},t)$ on the boundary $\partial\Omega$ of the reconstruction area. This implies that either the original sound field was measured using an ideally infinite number of omnidirectional microphones continuously arranged over $\partial\Omega$, or that $p(\mathbf{x},t)$ was defined using an analytical model of the desired sound field. The advantage of this approach is that the sound field reconstruction problem can be modeled by an integral equation of the first kind, and many useful results that functional analysis provides with respect to integral operators can be used in order to give an important insight into this engineering problem. Since the derived integral equation defines an ill-posed problem, a brief discussion is presented in the final section of this paper of ill-conditioning of inverse problems and regularization methods. The reader is referred to [6], [7], [10] and [11] for a more detailed introduction to this topic.

2 THEORY OF SOUND FIELD RECONSTRUCTION SYSTEM

2.1 Definition of the problem

Let the reconstruction area $\Omega \subset \mathbb{R}^3$ be a region of space limited by a smooth, bounded and simply connected boundary $\partial\Omega$. Assume that the acoustic pressure $\psi(\mathbf{x},t)$ of a sound field is defined over this region, satisfying the homogeneous wave equation

$$\nabla^2\psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2\psi(\mathbf{x},t)}{\partial t^2} = 0 \qquad \mathbf{x} \in \bar{\Omega} \tag{1}$$

where c is the speed of sound, considered to be uniform over Ω, and the symbol $\bar{\Omega}$ represents the closure of Ω, that is $\bar{\Omega} = \Omega \oplus \partial\Omega$. For a monochromatic sound field with angular frequency ω, equation (1) can be reduced to the homogeneous Helmholtz equation

$$\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0 \qquad \mathbf{x} \in \bar{\Omega} \tag{2}$$

where $k = \omega / c$ is the wave number, and the harmonic time dependence $e^{j\omega t}$ is implicitly assumed. Let $p(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$ be the continuous function that represents the value of $\psi(\mathbf{x},t)$ on the boundary $\partial\Omega$. Assume then that the loudspeaker array that is used for the reconstruction corresponds to an ideally continuous monopole source layer arranged over a smooth, bounded and simply connected surface S, as showed in Figure 1. It is also assumed that Ω is contained in S.

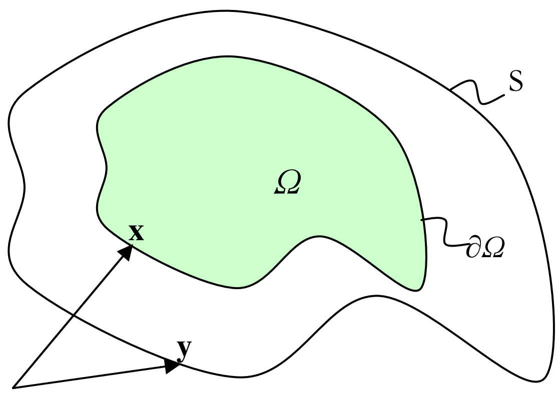

Figure 1: Cross-section of the reconstruction volume Ω and of the 3D surface S

2.2 The Dirichlet Problem

It is important to note at this point that, under certain conditions, the knowledge of the acoustic pressure on the boundary $\partial\Omega$ is sufficient in order to completely define the sound field inside Ω. This is equivalent to proving the uniqueness of the solution of the interior Dirichlet problem

$$\begin{cases} \nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0 & \mathbf{x} \in \bar{\Omega} \\ \psi(\mathbf{x}) = p(\mathbf{x}) & \mathbf{x} \in \partial\Omega \end{cases} \tag{3}$$

where the second equation represents the Dirichlet boundary condition. It is worth saying that there is no need to discuss the existence of the solution and to impose any condition on the smoothness of $p(\mathbf{x})$ as long as it is assumed that the boundary condition $p(\mathbf{x})$ is not chosen arbitrarily but corresponds to that of a real sound field. Therefore the solution must exist, but there is no evidence that it is unique.

If $p(\mathbf{x}) = 0 \ \ \forall \mathbf{x} \in \partial\Omega$, then the Dirichlet boundary condition is said to be homogeneous. For a certain geometry of $\partial\Omega$, a non-trivial solution to the Dirichlet problem in the interior of Ω with homogeneous boundary conditions on $\partial\Omega$ is possible only for a countable and infinite set of wave numbers $\{k_n\}$, related mathematically to the eigenvalues of the negative Laplacian operator. These eigenvalues are physically associated with the resonances of the cavity having the shape of Ω and pressure release boundaries. Provided the wave number k in equation (3) with inhomogeneous boundary conditions does not correspond to any of the eigenvalues $\{k_n\}$, then the homogeneous Dirichlet problem in Ω has only the trivial solution and the problem (3) with inhomogeneous boundary conditions has a unique solution. This is shown in [7, p.108] and [12, Chapter 7]. This means that, under the above-mentioned condition, the measurement of the acoustic pressure on $\partial\Omega$ is sufficient in order to define the sound field in Ω. Furthermore, the reconstruction of the acoustic pressure on $\partial\Omega$ ensures that the reconstruction is achieved also in the interior region Ω. This is an important result that implies that, if the frequency in question is not one of the resonances of the pressure release cavity, the measurement and reconstruction effort can be limited to the boundary $\partial\Omega$ of the reconstruction area.

2.3 Formulation of the reconstruction problem as an integral equation

Let $G(\mathbf{y} \,|\, \mathbf{x})$ be the free field Green function solution to the free field inhomogeneous wave equation

$$\begin{aligned} &\nabla^2 G(\mathbf{y} \,|\, \mathbf{x}) + k^2 G(\mathbf{y} \,|\, \mathbf{x}) = \delta(\mathbf{y} - \mathbf{x}) \\ &\mathbf{y} \in S, \quad \mathbf{x} \in \Omega \end{aligned} \tag{4}$$

and assume that this function can be a good model, at a given frequency, of the electro-acoustic transfer function between each loudspeaker, represented by a point source located at $\mathbf{y}$, and any point $\mathbf{x}$ in Ω. It is now possible to write an expression of the acoustic pressure $\hat{\psi}(\mathbf{x})$ of the reconstructed sound field as the linear superposition of the infinite number of point sources arranged on S. That is

$$\hat{\psi}(\mathbf{x}) = \int_S G(\mathbf{y} \,|\, \mathbf{x}) a(\mathbf{y}) ds(\mathbf{y}) \qquad \mathbf{x} \in \Omega \tag{5}$$

where $a(\mathbf{y})$ is a complex function representing the driving signal (monopole strength) of each loudspeaker. In view of the uniqueness of the Dirichlet problem when the problem does not involve one of the resonance frequencies, the reconstructed sound field $\hat{\psi}(\mathbf{x})$ equals the desired sound field $\psi(\mathbf{x})$ if the acoustic pressure is correctly reconstructed on the boundary $\partial\Omega$. That is to say, provided the loudspeaker driving function $a(\mathbf{y})$ is such that

$$p(\mathbf{x}) = \int_S G(\mathbf{y} \,|\, \mathbf{x}) a(\mathbf{y}) ds(\mathbf{y}) \qquad \mathbf{x} \in \partial\Omega \tag{6}$$

Obviously, if the desired boundary condition $p(\mathbf{x})$ is imposed, then $a(\mathbf{y})$ is the unknown of the problem. Equation (6) is a Fredholm integral equation of the first kind, and $G(\mathbf{y} \,|\, \mathbf{x})$ is the kernel of the integral. As will be shown later, this equation represents an inverse problem that is, in general, ill-posed. For the definition of an ill-posed problem the reader can refer to [6] or [11]. It is possible to rewrite equation (6) using an operational notation as

$$(Ha)(\mathbf{x}) = \int_S G(\mathbf{y} \mid \mathbf{x}) a(\mathbf{y}) ds(\mathbf{y}) \qquad \mathbf{x} \in \partial\Omega \tag{7}$$

where H is an operator that acts on the function $a(\mathbf{y})$ defined over S, and transforms it into a function defined over $\partial\Omega$. It may be useful to mention that the function $a(\mathbf{y})$ belongs to a Hilbert space Y of dimension M. Explaining the concept of Hilbert spaces is beyond the aim of this paper and the reader can refer to [8], [9] and [10] for a detailed explanation. For the case under consideration it suffices to provide an intuitive idea of Hilbert space. The space can be described as an infinite set of functions defined over a certain domain (S in the case of Y), over which is possible to define an inner product between two elements of the set as

$$\langle a(\mathbf{y}) | b(\mathbf{y}) \rangle = \int_S \overline{a(\mathbf{y})} b(\mathbf{y}) ds(\mathbf{y}) \tag{8}$$

and the norm of an element and the distance between two elements, respectively as

$$\begin{aligned} \|a(\mathbf{y})\| &:= \sqrt{\langle a(\mathbf{y}) | a(\mathbf{y}) \rangle} = \sqrt{\int_S \overline{a(\mathbf{y})} a(\mathbf{y}) ds(\mathbf{y})} \\ d\left(a(\mathbf{y}), b(\mathbf{y})\right) &:= \sqrt{\int_S \overline{\left(a(\mathbf{y}) - b(\mathbf{y})\right)} \left(a(\mathbf{y}) - b(\mathbf{y})\right) ds(\mathbf{y})} \end{aligned} \tag{9}$$

The symbol $\bar{\square}$ represents the complex conjugate of $\square$. Two functions $a(\mathbf{y}), b(\mathbf{y}) \in Y$ are said to be orthogonal if their inner product $\langle a(\mathbf{y}) | b(\mathbf{y}) \rangle = 0$. The Hilbert space has dimension M if each element of Y can be expressed as a linear combination of M mutually orthogonal functions belonging to Y, in the same way that any vector of a Euclidean vector space of dimension M can be expressed as a linear combination of M vectors constituting an orthogonal basis for that space. Hilbert spaces can have both finite and infinite dimension. In the same way as $a(\mathbf{y})$ belongs to the Hilbert space Y, the function $p(\mathbf{x})$ defined over $\partial\Omega$ belongs to the Hilbert space X, for which the inner product, the norm and distance are defined as for Y (but the domain of integration is $\partial\Omega$ instead of S).

It is now useful to introduce the concept of an adjoint operator. The adjoint operator H^+ of H is such that

$$\langle (Ha)(\mathbf{x}) | p(\mathbf{x}) \rangle = \langle a(\mathbf{y}) | (H^+ p)(\mathbf{y}) \rangle \tag{10}$$

It is important to point out that if the operator H transforms a function defined over a domain S into a function defined over $\partial\Omega$, the adjoint operator acts on a function defined over $\partial\Omega$ generating a function defined over S. Under the proper assumptions of smoothness of the kernel $G(\mathbf{y} \mid \mathbf{x})$, the operator H defined in (7) is compact [8, p. 454]. The rigorous definition of compactness of an operator is beyond the scope of this paper, and the interested reader can refer to [8] and [10], but it is important to state that the operator H is compact in order to use the properties of compact operators. As a consequence of the compactness of H, the adjoint operator H^+ exists and is compact [8, p.416]. It can be observed that H^+ has the form [10]

$$(H^+ p)(\mathbf{y}) = \int_{\partial\Omega} \overline{G(\mathbf{y} \mid \mathbf{x})} p(\mathbf{x}) ds(\mathbf{x}) \qquad \mathbf{y} \in S \tag{11}$$

and can be understood as a “time reversed” acoustic propagation of an infinite distribution of monopole sources on $\partial\Omega$ to a point $\mathbf{y} \in S$. It is now possible to define the operator H^+H, which maps a function of Y to another function of Y. It has the analytical form

$$(H^{+}Ha)(\mathbf{y}) = \int_{\partial\Omega} \overline{G(\mathbf{y} \mid \mathbf{x})} \int_{S} G(\boldsymbol{\xi} \mid \mathbf{x}) a(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) ds(\mathbf{x}) \qquad \mathbf{y} \in S \, . \tag{12}$$

This operator is a product of two compact operators, and is therefore a compact operator [7, p.89], [8, p.422]. Furthermore, $H^{+}H$ is self-adjoint, that is to say

$$\left(H^{+}H\right)^{+} = H^{+}H \tag{13}$$

and it is therefore possible to use all the proprieties of compact, self-adjoint operators. Of special relevance is the spectral theorem of self-adjoint operators. Consider the eigenvalue problem

$$\left(H^{+}Ha_n\right)(\mathbf{y}) = \lambda_n a_n(\mathbf{y}) \tag{14}$$

The eigenfunctions $\{a_n\}$, henceforth also called modes, can be chosen to be of unitary norm ($\|a_n(\mathbf{y})\| = 1$). The non-negative square roots μ_n of the non negative eigenvalues of $H^{+}H$, that is $\mu_n = \sqrt{\lambda_n}$, are real and are called the singular values of H. It is useful to order them with decreasing magnitude, from the largest to the smallest. Let $N(H)$ be the null-space of the operator H, defined as the set of functions $\tilde{a}(\mathbf{y})$ such that

$$N(H) = \left\{\tilde{a}(\mathbf{y}) : (H\tilde{a})(\mathbf{x}) = 0\right\}$$

The null space of an operator, as explained in [11], can be understood for the case under consideration as the set of loudspeaker driving functions for which, at the considered frequency, the reconstructed acoustic pressure profile $\hat{p}(x) = 0$.

2.4 Spectral Theorem and Singular Value Decomposition

The spectral theorem for compact, self-adjoint operators states that each function $a(\mathbf{y})$ of Y can be expressed as a linear combination of the eigenfunctions of $H^{+}H$, plus a function belonging to $N(H)$:

$$a(\mathbf{y}) = \sum_{n=1}^{N} \langle a_n(\mathbf{y}) \,|\, a(\mathbf{y}) \rangle a_n(\mathbf{y}) + (Qa)(\mathbf{y}) \, . \tag{15}$$

The operator Q represents the orthogonal projection of $a(\mathbf{y})$ on $N(H)$ and N is the number of nonzero eigenvalues of $H^{+}H$ (that can also be infinite). It is possible to generate a set of orthogonal functions $\{p_n(\mathbf{x})\} \subset X$ (also called modes) by letting H act on the eigenfunctions $\{a_n(\mathbf{y})\}$

$$(Ha_n)(\mathbf{x}) = \mu_n p_n(\mathbf{x}) \, . \tag{16}$$

The functions $\{p_n(\mathbf{x})\}$ are mutually orthogonal because, considering equations (10), (14), (16), the orthogonality of $\{a_n(\mathbf{y})\}$ and the fact that $\|a_n(\mathbf{y})\| = 1 \;\; \forall n$ leads to

$$\begin{aligned} \langle p_n(\mathbf{x}) \,|\, p_m(\mathbf{x}) \rangle &= \left\langle \frac{(Ha_n)(\mathbf{x})}{\mu_n} \,\middle|\, \frac{(Ha_m)(\mathbf{x})}{\mu_m} \right\rangle = \left\langle \frac{a_n(\mathbf{y})}{\mu_n} \,\middle|\, \frac{(H^{+}Ha_m)(\mathbf{y})}{\mu_m} \right\rangle = \\ &= \frac{\lambda_m^2}{\mu_n \mu_m} \langle a_n(\mathbf{y}) \,|\, a_m(\mathbf{y}) \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \end{aligned} \tag{17}$$

The set of functions $\{a_n(\mathbf{y})\}$ and $\{p_n(\mathbf{x})\}$ can be interpreted respectively as the loudspeaker array modes and microphone array modes described in [11]. Combining the two latter equations it is possible to express the action of H on a function $a(\mathbf{y})$ as

$$(Ha)(\mathbf{x}) = \sum_{n=1}^{N} \mu_n \langle a_n(\mathbf{y}) | a(\mathbf{y}) \rangle p_n(\mathbf{x}) . \tag{18}$$

This powerful representation of the integral operator defined in (7) is called the singular value decomposition of the compact operator H . The function $p(\mathbf{x})$ can be therefore represented as

$$p(\mathbf{x}) = \sum_{n=1}^{N} \mu_n \langle a_n(\mathbf{y}) | a(\mathbf{y}) \rangle p_n(\mathbf{x}) + (Rp)(\mathbf{x}) \tag{19}$$

where the operator R is the orthogonal projection on the null-space of the adjoint operator $N(H^+)$. This can be understood as the set of functions $\{\tilde{p}(\mathbf{x})\} \subset X$ that can not be generated by the operator H [11].

In practical terms, $N(H^+)$ represents the set of acoustic pressure profiles $\{\tilde{p}(\mathbf{x})\} \subset X$ that can not be reconstructed by the continuous distribution of sources on S . Any acoustic pressure profile $p(\mathbf{x})$ corresponding to a physical case can be expressed as the sum of an acoustic pressure profile that can not be reconstructed (the orthogonal projection $(Rp)(\mathbf{x})$ of $p(\mathbf{x})$ on $N(H^+)$) plus the linear superposition of different orthogonal modes $p_n(\mathbf{x})$ that can be reconstructed by the monopole source distribution on S .

It is now possible to seek a solution, when this exists. Multiplying both sides of (19) by the complex conjugate of $p_m(\mathbf{x})$ and integrating over $\partial\Omega$ one obtains, because of the orthogonality of the functions $\{p_n(\mathbf{x})\}$,

$$\begin{aligned} &\int_{\partial\Omega} \overline{p_m(\mathbf{x})} p(\mathbf{x}) ds(\mathbf{x}) = \langle p_m(\mathbf{x}) | p(\mathbf{x}) \rangle = \\ &= \int_{\partial\Omega} \overline{p_m(\mathbf{x})} \left(\sum_{n=1}^{N} \mu_n \langle a_n(\mathbf{y}) | a(\mathbf{y}) \rangle p_n(\mathbf{x}) + (Rp)(\mathbf{x}) \right) ds(\mathbf{x}) = \\ &= \mu_m \langle a_m(\mathbf{y}) | a(\mathbf{y}) \rangle \end{aligned} \tag{20}$$

Hence, if the solution to equation (6) exists, it is given by

$$a(\mathbf{y}) = \sum_{n=1}^{N} \langle a_n(\mathbf{y}) | a(\mathbf{y}) \rangle a_n(\mathbf{y}) = \sum_{n=1}^{N} \frac{1}{\mu_n} \langle p_n(\mathbf{x}) | p(\mathbf{x}) \rangle a_n(\mathbf{y}) \tag{21}$$

An exact solution is possible if and only if the desired acoustic pressure profile is described by a function $p(\mathbf{x})$ that has zero orthogonal projection on $N(H^+)$, since any function belonging to $N(H^+)$ cannot be reconstructed by the distribution of monopole sources on S . This shows that a solution of equation (6) is not possible for all pressure profiles $p(\mathbf{x})$ and the inverse problem is therefore ill-posed. However, even if a solution to the inverse problem (6) does not exist, that is if $p(\mathbf{x})$ has a nonzero orthogonal projection on $N(H^+)$, the approximation of $p(\mathbf{x})$ expressed by

$$\hat{p}(\mathbf{x}) = \int_S G(\mathbf{y} \,|\, \mathbf{x}) \sum_{n=1}^{N} \frac{1}{\mu_n} \langle p_n(\mathbf{x}) | p(\mathbf{x}) \rangle a_n(\mathbf{y}) ds(\mathbf{y}) \tag{22}$$

is the approximation that can be generated by the continuous distribution of sources on S which is the closest to $p(\mathbf{x})$ in relation to the distance defined by (9). This is the approximation that minimizes the root mean square error.

2.5 Ill-conditioning of the inverse problem

Even if a solution exists, the inverse problem (6) can be ill-conditioned and its solution can be unstable. This can be easily seen considering the fact that, in the general case, the eigenvalues $\{\lambda_n\}$ of the compact, self-adjoined operator $H^{+}H$ can accumulate at zero, and hence the singular values $\{\mu_n\}$ of H will monotonically decrease, possibly approaching zero. Observing equation (21) it can be noticed that if the desired acoustic pressure profile $p(\mathbf{x})$ has a nonzero inner product $\langle p_n(\mathbf{x}) | p(\mathbf{x}) \rangle$ with a mode $p_n(\mathbf{x})$ related to a very small singular value μ_n, then the related mode $a_n(\mathbf{y})$ will become very large due to the inverse of μ_n in equation (21). Furthermore, if the acoustic pressure profile $p(\mathbf{x})$ is perturbed such that

$$p^{\delta}(\mathbf{x}) = p(\mathbf{x}) + \delta p_n(\mathbf{x}) \tag{23}$$

then the effect of this perturbation is amplified by a factor $1/\mu_n$ in the loudspeaker driving function $a(\mathbf{y})$, obtaining a perturbed solution

$$a^{\delta}(\mathbf{y}) = a(\mathbf{y}) + \frac{\delta a_n(\mathbf{y})}{\mu_n}.$$

In more detail, it holds that [7, p.91]

$$\frac{\left\| a^{\delta}(\mathbf{y}) - a(\mathbf{y}) \right\|}{\left\| p^{\delta}(\mathbf{x}) - p(\mathbf{x}) \right\|} = \frac{1}{\mu_n}. \tag{24}$$

This error amplification factor can become very large because of the roll-off of the singular values. In a practical sense, this means that the presence of small singular values implies that a small error in the measurement of the desired acoustic pressure profile $p(\mathbf{x})$ or the inaccuracy in the positioning of the loudspeakers can have a devastating effect on the reconstructed sound field, The reconstructed field might then differ largely from that desired because of the presence of small errors in the data.

There are many different ways to “regularize” the solution to the ill-posed problem (6) and to compute an approximate solution $\hat{a}(\mathbf{y})$ that does not generate an exact sound field reconstruction, but is more robust than the solution given by (21). Many regularization methods are described in the literature such as, for example, spectral damping and Tikhonov regularization. They will not be discussed in this paper and the reader is invited to refer to [6] or [7].

3 CONCLUSIONS

The outline has been presented of a theory for the reconstruction of a sound field over a source free area. The system is constituted by an ideally continuous distribution of monopole sources over a three dimensional smooth surface that contains the reconstruction volume. The uniqueness of the related Dirichlet problem has been discussed and it has been shown that, provided the operating frequency does not correspond to one of those related to the eigenvalues of the homogeneous

Dirichlet problem, knowledge of the acoustic pressure profile on the boundary of the reconstruction volume is sufficient in order to completely define the sound field over all the reconstruction volume. Using the same argument, it has been shown that under the same conditions the accurate reconstruction of the target pressure profile on the boundary of the reconstruction volume implies the reconstruction of the desired sound field over the entire reconstruction volume. The reconstruction problem has been formulated analytically as a Fredholm integral equation of the first kind, its solution providing the driving function for the continuous distribution of point sources. The ill-posedness of the inverse problem has been discussed and the singular value decomposition of the compact operator involved in the integral equation has been presented in order to seek a solution to the inverse problem. As the problem is ill-posed, the solution might be not exact but can be approximated. Finally, the robustness of the solution to errors in the data has been discussed and some regularization methods have been mentioned.

The practical realization of a system based on this theory obviously involves the reformulation of the latter when a limited number of point sources is considered. This subject is currently part of the research activity of the authors. Further work might also involve the study of the problem when the wave number considered is one of the eigenvalues of the homogeneous Dirichlet problem. Another important aspect is the study of the null-space of the adjoint operator $N(H^{+})$ and of the spread of the singular values of the operator H in relation to the shape of the surface S over which the point sources used for the reconstruction are arranged. Finally, it could be interesting to attempt to reformulate the problem by removing the free field assumption and choosing a different Green function $G(\mathbf{y}\,|\,\mathbf{x})$ that can model the reflections of the room in which the reconstruction is attempted.

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