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# $K^+ \rightarrow \pi^+ \pi^0$ Decays on Finite Volumes and at Next-to-Leading Order in the Chiral Expansion

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## Abstract:

We present the ingredients necessary for the determination of physical  $K \rightarrow \pi\pi$  decay amplitudes for  $\Delta I = 3/2$  transitions, from lattice simulations at unphysical kinematics and the use of chiral perturbation theory at next-to-leading order. In particular we derive the expressions for the matrix elements  $_{I=2} \langle \pi\pi | \mathcal{O}_W | K \rangle$ , where  $\mathcal{O}_W$  is one of the operators appearing in the  $\Delta S = 1$  weak Hamiltonian, in terms of low-energy constants at next-to-leading order in the chiral expansion. The one-loop chiral corrections are evaluated for arbitrary masses and momenta, both in full QCD and in the quenched approximation. We also investigate the finite-volume effects in this procedure.

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# 1 Introduction

The need for a quantitative control of strong-interaction effects in  $K \rightarrow \pi\pi$  decays is underlined by the recent measurement of a non-zero value of  $\varepsilon'/\varepsilon$  [1] (demonstrating the existence of direct CP-violation) and the long-standing  $\Delta I = 1/2$  rule. A precise evaluation of non-perturbative QCD effects in nonleptonic kaon decays by lattice simulations, although possible in principle, is a truly extraordinary challenge, with a number of major theoretical and computational obstacles to be overcome. Ultimately, of course, we would wish to perform the simulations in full QCD, with physical quark masses on a lattice which is both large enough and sufficiently fine-grained for finite-volume effects and lattice artefacts to be negligible (or at least under control). We are some way from being able to do this, and therefore need rely on approximations. In this paper we explain our strategy, first presented in ref. [2], for improving one of the key approximations, the use of Chiral Perturbation Theory ( $\chi$ PT). Specifically we propose to perform the calculations at next-to-leading order (NLO) in the chiral expansion. In this paper we present the necessary formulae for  $\Delta I = 3/2$  decays and calculate the chiral logarithms for both full QCD and for the quenched theory. The chiral logarithms have been evaluated for arbitrary meson masses and momenta.

In our calculations we incorporate the recent developments in our understanding of finite-volume effects in nonleptonic kaon decays [3, 4, 5]. At NLO in  $\chi$ PT, for  $\Delta I = 3/2$  transitions, the corrections which vanish only as powers of the volume can be eliminated using the techniques presented in these papers, even in the quenched approximation (note however, that the generalization of the techniques in refs. [3, 4, 5] to the case when the two-pions have non-zero total momentum has yet to be performed). This point is explained in some detail in sec. 4 below. This is not the case however, when applying quenched  $\chi$ PT to  $\Delta I = 1/2$  transitions. For these decays, the lack of unitarity in the quenched theory leads to a number of subtleties and major difficulties which need to be overcome in order to understand the volume dependence and to extract the amplitudes [6, 7, 8]. We expect similar difficulties for  $\Delta I = 3/2$  transitions beyond one-loop order in chiral perturbation theory in quenched QCD.

In this paper we do not discuss the *ultra-violet* problem, i.e. the construction of finite matrix elements of renormalized operators constructed from the bare lattice ones. For  $\Delta I = 1/2$  transitions this involves the subtraction of terms which diverge as inverse powers of the lattice spacing (and for fermion actions which explicitly break chiral symmetry, such as the Wilson action, there is also mixing with other operators of dimension 6). The subtraction of power divergences is unavoidable, and different lattice formulations of QCD, including ones which preserve chiral symmetry, are being used in attempts to perform these subtractions effectively and to determine the matrix elements as precisely as possible. The strategy discussed in this paper for  $\Delta I = 3/2$  decays can be applied to any of these formulations of lattice fermions.

The non-perturbative QCD effects in  $\Delta I = 3/2$   $K \rightarrow \pi\pi$  decays are contained in the matrix elements of the following three operators which appear in the  $\Delta S = 1$  effective

Hamiltonian:

$$\mathcal{O}_4 = (\bar{s}_\alpha d_\alpha)_L (\bar{u}_\beta u_\beta - \bar{d}_\beta d_\beta)_L + (\bar{s}_\alpha u_\alpha)_L (\bar{u}_\beta d_\beta)_L; \quad (1)$$

$$\mathcal{O}_7 = \frac{3}{2} (\bar{s}_\alpha d_\alpha)_L \sum_{q=u,d,s,c} e_q (\bar{q}_\beta q_\beta)_R; \quad (2)$$

$$\mathcal{O}_8 = \frac{3}{2} (\bar{s}_\alpha d_\beta)_L \sum_{q=u,d,s,c} e_q (\bar{q}_\beta q_\alpha)_R, \quad (3)$$

where  $\alpha, \beta$  are colour indices and  $e_q$  is the charge of quark  $q$ . The subscripts  $L, R$  signify “left” and “right” so that  $(\bar{q}_1 q_2)_{L,R} = \bar{q}_1 \gamma_\mu (1 \mp \gamma_5) q_2$ .  $\mathcal{O}_4$  transforms as the (27,1) representation of the  $SU(3)_L \times SU(3)_R$  chiral symmetry and the electroweak penguin (EWP) operators  $\mathcal{O}_{7,8}$  transform as the (8,8) representation.  $\mathcal{O}_4$  is a  $\Delta I = 3/2$  operator, whereas  $\mathcal{O}_{7,8}$  have both  $\Delta I = 1/2$  and  $3/2$  components. The definition of the EWP operators  $\mathcal{O}_{7,8}$  coincides with that used in refs. [9, 10] (where they were denoted by  $Q_{7,8}$ ). Our definition of  $\mathcal{O}_4$  is that introduced in ref. [11]. In terms of the operators defined in refs. [9, 10]  $\mathcal{O}_4$  is equal to the  $\Delta I = 3/2$  component of  $Q_1 + Q_2$ . In this paper we study the  $K \rightarrow \pi\pi$  matrix elements of these operators, with the two pions in an  $I = 2$  state, so that only the  $\Delta I = 3/2$  components of  $\mathcal{O}_{7,8}$  contribute.

In this project we are evaluating  $K \rightarrow \pi\pi$  matrix elements directly. All but one of the previous lattice studies evaluated  $K \rightarrow \pi$  matrix elements of the form  $\langle \pi | \mathcal{O}_W | K \rangle$  and used lowest order  $\chi$ PT to relate them to matrix elements  $\langle \pi\pi | \mathcal{O}_W | K \rangle$ , following the original proposal in ref. [12] (see also refs. [13, 14]). ( $\mathcal{O}_W$  represents any of the  $\Delta S = 1$  operators which appear in the effective weak Hamiltonian.) The exception is the quenched study of the CP-conserving  $K \rightarrow \pi\pi$   $\Delta I = 3/2$  decays in ref. [15], where the two pions were at rest. The previous studies are therefore not sensitive to final-state interactions.

In ref. [2] it was shown that it is possible to obtain all the low-energy constants necessary to determine the physical  $\Delta I = 3/2$  decay amplitudes at NLO in the chiral expansion by performing simulations with a simple range of momenta for the mesons<sup>1</sup>. Specifically the kaon and one of the final-state pions are at rest and the second pion has an arbitrary momentum (the energy of this second pion will be referred to as  $E_\pi$  in the following). In order to ensure that the final state has isospin equal to 2, we have to symmetrise the final state over the two pions. Since  $K \rightarrow \pi\pi$  matrix elements with this set of momenta are being computed by the SPQCD collaboration [16], we refer to it as SPQR kinematics. Of course, it is also possible to determine the low-energy constants which appear at NLO in the chiral expansion from simulations using other kinematical configurations<sup>2</sup>. For this reason in the following we will present the results with arbitrary kaon and pion masses and momenta.

Among the previous lattice studies are two recent determinations of  $K \rightarrow \pi\pi$  matrix elements from computations of  $K \rightarrow \pi$  matrix elements with quenched domain-wall fermions

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<sup>1</sup>The generalization of the formalism to  $\Delta I = 1/2$  decays is currently under investigation

<sup>2</sup>In ref. [17] it is suggested that an alternative set of kaon matrix elements be used to determine the low energy constants. We comment on this proposal in sec. 6.

and using  $\chi$ PT at lowest order [18, 19]. The very interesting results for the octet enhancement,  $\text{Re}\mathcal{A}_0/\text{Re}\mathcal{A}_2 = 9-12$  [18] and  $24-27$  [19] (compared to the experimental value of 22.2) and for  $\varepsilon'/\varepsilon = \{(-7)-(-2)\}10^{-4}$  [18] and  $\{(-8)-(-4)\}10^{-4}$  [19] (compared to the experimental value of  $+(17.2 \pm 1.8)10^{-4}$ ), will serve as important benchmarks for future calculations. ( $\mathcal{A}_I$  is the  $K \rightarrow \pi\pi$  amplitude for decays into two-pion states with isospin  $I$ .) These papers also contain detailed discussions of the systematic errors and the difficulties in estimating and controlling them. The importance of a reliable error analysis is underlined by a comparison of the results of the two collaborations and with the experimental values. For example, the cancellations which occur between different contributions to  $\varepsilon'/\varepsilon$  (in particular between the matrix elements of operators conventionally called  $\mathcal{O}_6$  and  $\mathcal{O}_8$ ) imply that each component must be determined with good precision. In this paper we tackle one of the main sources of systematic error, the use of  $\chi$ PT at lowest order.

The remainder of the paper is organized as follows. In the next section we explain our strategy in detail. We express the matrix elements in terms of the low-energy constants and demonstrate that it is possible to determine the physical matrix elements at NLO in the chiral expansion from numerical simulations which are currently feasible. We present a brief introduction to quenched and unquenched  $\chi$ PT in sec. 3. Sec. 4 is devoted to a study of finite volume effects in  $\chi$ PT. The NLO results are presented in sec. 5 for *physical* kinematics, i.e. with the momenta of the mesons satisfying  $p_{K^+} = p_{\pi^+} + p_{\pi^0}$ , but with the masses being arbitrary. In this section we also present the results for the special cases  $m_K = 2m_\pi$  and  $m_K = m_\pi$ , in both cases with each of the pions at rest, and compare our results with those of previous papers where this is possible. The expressions for the matrix elements at NLO in the chiral expansion for the SPQR kinematics are presented in appendices B.1 and C.1 for full QCD and the quenched approximation respectively. The results for general kinematics, i.e. with arbitrary masses and momenta, are very lengthy and we present them on the web site [23], from which interested users can import them into their programs. For full QCD we also present the results for general kinematics in appendix B.2. Sec. 6 contains our conclusions. There is one other appendix which contains integrals used in the study of finite-volume effects.

## 2 The Strategy

In this section we present our strategy for the determination of  $K \rightarrow \pi\pi$  matrix elements using chiral perturbation theory at next-to-leading order. We envisage computing the matrix elements in numerical simulations for a variety of (unphysical) masses and momenta. The generic form for the behaviour of the matrix element of a weak operator  $\mathcal{O}_W$  with masses and momenta is

$$\langle \pi^+ \pi^0 | \mathcal{O}_W | K^+ \rangle = A \rho \left( 1 + \text{chiral logs} \right) + \sum_i \lambda_i B_i, \quad (4)$$

where  $A$  and the  $B_i$  are kinematic factors and  $\rho$  and the  $\lambda_i$  are low-energy constants. (In the applications below  $\rho$  corresponds to  $\alpha^{(27,1)}$  or  $\gamma^{(8,8)}$  and the  $\lambda_i$ 's correspond to the  $\beta_i$ 's

or the  $\delta_i$ 's.)  $A$  is  $O(p^2)$  when  $\mathcal{O} = \mathcal{O}_4$  and  $O(p^0)$  when  $\mathcal{O} = \mathcal{O}_{7,8}$  and the  $B_i$  are  $O(p^4)$  and  $O(p^2)$  when  $\mathcal{O} = \mathcal{O}_4$  and  $\mathcal{O} = \mathcal{O}_{7,8}$  respectively. In both cases there is only one low-energy constant at lowest order of  $\chi$ PT and several at next-to-leading order. For the EWP operators  $\mathcal{O}_{7,8}$  the low energy constants  $\rho$  and  $\lambda_i$  include a factor of the electromagnetic coupling  $e^2$ . The expression “chiral logs” on the right-hand side of eq. (4) is shorthand for the full one-loop contribution, which includes both terms containing logarithms and also others without logarithms.

Of course we wish to determine the matrix element for a physical kaon decaying into two physical pions. Our strategy for doing this at NLO in the chiral expansion is as follows [2]:

1. Calculate the chiral logarithms at one-loop order in  $\chi$ PT for general masses and momenta. In this paper we perform this calculation for the  $\Delta I = 3/2$  operators  $\mathcal{O}_{4,7,8}$  in both full QCD and in the quenched approximation. Once the logarithms have been computed, we have the formula for the behaviour of each matrix element in terms of the low-energy constants  $\rho$  and  $\lambda_i$ .
2. Determine the matrix elements in eq. (4) using lattice simulations for a variety of masses and momenta. The kinematic range must be sufficient to enable the low-energy constants to be obtained from a fit to the behaviour of the matrix elements with the masses and momenta.
3. Use the low-energy constants determined in item 2 and the expression in eq. (4) with  $m_K$  and  $m_\pi$  equal to their physical values and zero energy-momentum transfer at the operator (i.e. with  $p_{K^+} = p_{\pi^+} + p_{\pi^0}$ ) to determine the physical matrix elements. In extracting the physical amplitudes we assume that the higher order terms in the chiral expansion can be safely neglected in the region of the extrapolation.

For compactness of notation, in the following we represent the low-energy constants by the same symbols in both full QCD and in the quenched theory. It must be stressed however, that the low-energy constants are not expected to be the same in the two cases. Indeed their dependence on the renormalization scale is in general different.

In the remainder of this section we give the expressions for the matrix elements of  $\mathcal{O}_{4,7,8}$  in terms of the low energy constants.

## 2.1 Counterterms for $\mathcal{O}_4$ up to $O(p^4)$

The general expression for the matrix element of  $\mathcal{O}_4$ , which transforms as a  $(27,1)$  under the  $SU(3) \times SU(3)$  chiral symmetry, for arbitrary meson masses and momenta is

$$\begin{aligned} \langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle &= -\frac{\sqrt{2}}{f_K f_\pi^2} \left\{ \alpha^{(27,1)} \left( 8p_{\pi^0} \cdot p_K + 6p_{\pi^0} \cdot p_{\pi^+} - 2p_{\pi^+} \cdot p_K + \text{chiral logs} \right) \right. \\ &\quad \left. + \beta_2 \left( -24m_\pi^2 m_K^2 + 24m_\pi^4 \right) \right\} \end{aligned}$$

$$\begin{aligned}
& +\beta_4 \left( 16m_\pi^2 (p_{\pi^0} \cdot p_K) + 16m_K^2 (p_{\pi^0} \cdot p_K) + 24m_\pi^2 (p_{\pi^0} \cdot p_{\pi^+}) \right. \\
& \quad \left. - 4m_\pi^2 (p_{\pi^+} \cdot p_K) - 4m_K^2 (p_{\pi^+} \cdot p_K) \right) \\
& +\beta_5 \left( -16m_\pi^2 (p_{\pi^0} \cdot p_K) - 12m_K^2 (p_{\pi^0} \cdot p_{\pi^+}) + 4m_\pi^2 (p_{\pi^+} \cdot p_K) \right) \\
& +\beta_7 \left( 16m_\pi^2 (p_{\pi^0} \cdot p_K) + 32m_K^2 (p_{\pi^0} \cdot p_K) + 12m_\pi^2 (p_{\pi^0} \cdot p_{\pi^+}) \right. \\
& \quad \left. + 24m_K^2 (p_{\pi^0} \cdot p_{\pi^+}) - 4m_\pi^2 (p_{\pi^+} \cdot p_K) - 8m_K^2 (p_{\pi^+} \cdot p_K) \right) \\
& +\beta_{22} \left( 24(p_K \cdot p_{\pi^0}) (p_K \cdot p_{\pi^+}) - 8(p_K \cdot p_{\pi^0}) (p_{\pi^0} \cdot p_{\pi^+}) + 32(p_K \cdot p_{\pi^+}) (p_{\pi^0} \cdot p_{\pi^+}) \right) \\
& \left. +\beta_{24} \left( 128(p_K \cdot p_{\pi^0})^2 - 96(p_{\pi^0} \cdot p_{\pi^+})^2 - 32(p_K \cdot p_{\pi^+})^2 \right) \right\}. \tag{5}
\end{aligned}$$

$f_\pi$  and  $f_K$  are the physical pion and kaon decay constants. At NLO in the chiral expansion they are given in terms of the parameter  $f$  of the chiral Lagrangian (see sec. 3 below) and the Gasser-Leutwyler coefficients in ref. [20] for full QCD and in ref. [21] for quenched QCD.  $\alpha^{(27,1)}$  is the leading order ( $O(p^2)$ ) low-energy constant whereas the  $\beta_i$  ( $i = 2, 4, 5, 7, 22, 24$ ) are the  $O(p^4)$  counterterms, where the subscript corresponds to the numbering of the operators in ref. [22]. These operators are listed explicitly in eq. (13) of sec. 3. Thus for a set of simulations, with a range of quark masses and momenta in the chiral regime, the mass and momentum dependence is given by eq. (5). We envisage determining the low-energy constants  $\alpha^{(27,1)}$  and  $\beta_i$  in lattice simulations (or at least a sufficient subset of these constants). We have calculated the chiral logarithms for arbitrary meson masses and momenta, for full QCD and in the quenched approximation and we present the results on the web site [23]. For full QCD we also present the results in appendix B.2.

It is possible to determine the physical matrix elements of  $\mathcal{O}_4$  by performing simulations with the SPQR kinematics. This will become clear in the following subsections where we rewrite eq. (5) for physical kinematics (subsection 2.1.1) and for SPQR kinematics (subsection 2.1.2). The corresponding one-loop expressions with the chiral logarithms included are given in:

- eqs. (79) – (82) for the physical kinematics in full QCD;
- eqs. (91) – (94) for the physical kinematics in quenched QCD;
- eqs. (108) – (116) for the SPQR kinematics in full QCD;
- eqs. (133) – (136) for the SPQR kinematics in quenched QCD.

### 2.1.1 The Matrix Element of $\mathcal{O}_4$ for Physical Kinematics

For the physical matrix element, when there is no momentum insertion at the weak operator  $\mathcal{O}_4$ , eq. (5) reduces to

$$\langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle_{\text{phys}} = -\frac{6\sqrt{2}}{f_K f_\pi^2} \left\{ \alpha^{(27,1)} \left( m_K^2 - m_\pi^2 + \text{chiral logs} \right) \right\}$$

$$\begin{aligned}
& +(\beta_4 - \beta_5 + 4\beta_7 + 2\beta_{22})m_K^4 + (4\beta_2 - 4\beta_4 - 2\beta_7 - 16\beta_{24})m_\pi^4 \\
& +(-4\beta_2 + 3\beta_4 + \beta_5 - 2\beta_7 - 2\beta_{22} + 16\beta_{24})m_K^2 m_\pi^2 \Big\}. \tag{6}
\end{aligned}$$

Of course it must be remembered that the chiral logarithms depend on the kinematics, however, as they are calculable we do not exhibit this dependence explicitly here. In principle, by varying the masses and momenta of the pions it is possible to determine  $\alpha^{(27,1)}$  and the  $\beta_i$ 's but in practice this is not the optimal strategy. For each set of pion masses and momenta one would have to tune the mass of the strange quark in order to ensure energy conservation, and in addition one would have to extract the matrix elements of non-leading (excited) energy levels. In the following subsection we demonstrate that by performing the simulations with the SPQR kinematics it is possible to determine the combinations of the low-energy constants which appear in eq. (6).

### 2.1.2 The Matrix Element of $\mathcal{O}_4$ for the SPQR Kinematics

For the SPQR kinematics the matrix element in eq. (5) reduces to

$$\begin{aligned}
\langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle_{\text{SPQR}} = & -\frac{6\sqrt{2}}{f_K f_\pi^2} \Big\{ \alpha^{(27,1)} \left( E_\pi m_\pi + \frac{1}{2} m_K (E_\pi + m_\pi) + \text{chiral logs} \right) \\
& + 4\beta_2 m_\pi^4 + (4\beta_4 + 2\beta_7) E_\pi m_\pi^3 + (\beta_4 - \beta_5 + \beta_7) m_\pi^3 m_K \\
& + (\beta_4 - \beta_5 + \beta_7 + 2\beta_{22}) E_\pi m_\pi^2 m_K + (-4\beta_2 + 8\beta_{24}) m_\pi^2 m_K^2 + (\beta_4 + 2\beta_7) E_\pi m_K^3 \\
& + (-2\beta_5 + 4\beta_7 + 4\beta_{22}) E_\pi m_\pi m_K^2 + (\beta_4 + 2\beta_7) m_\pi m_K^3 \\
& + (-16\beta_{24}) E_\pi^2 m_\pi^2 + 2\beta_{22} E_\pi^2 m_\pi m_K + 8\beta_{24} E_\pi^2 m_K^2 \Big\}. \tag{7}
\end{aligned}$$

It can readily be seen that by fitting the lattice results for the matrix elements as a function of the masses and  $E_\pi$ , the combinations of low-energy constants needed to determine the physical matrix elements in eq. (6) can be obtained.

## 2.2 Counterterms for $\mathcal{O}_{7,8}$ up to $\mathcal{O}(p^2)$

We now present the corresponding formulae for the  $\Delta I = 3/2$  EWP operators  $\mathcal{O}_{7,8}$ . In this case the matrix elements start at  $\mathcal{O}(p^0)$  and at NLO we have to consider all possible terms of  $\mathcal{O}(p^2)$ . The general expression for these matrix elements up to this order is

$$\begin{aligned}
\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = & \frac{2\sqrt{2}}{f_K f_\pi^2} \Big\{ \gamma^{(8,8)} (1 + \text{chiral logs}) + \\
& \delta_1 \left( \frac{1}{3} (p_{\pi^0} \cdot p_K) - (p_{\pi^0} \cdot p_{\pi^+}) + \frac{2}{3} (p_{\pi^+} \cdot p_K) \right) - \delta_2 ((p_{\pi^0} \cdot p_K) + (p_{\pi^0} \cdot p_{\pi^+})) \\
& - \delta_3 \left( \frac{4}{3} (p_{\pi^0} \cdot p_K) + \frac{2}{3} (p_{\pi^+} \cdot p_K) \right) + (\delta_4 + \delta_5) (2m_K^2 + 4m_\pi^2) + \delta_6 (4m_K^2 + 2m_\pi^2) \Big\}. \tag{8}
\end{aligned}$$

The  $O(p^2)$  counterterms  $\delta_i$ ,  $i = 1 - 6$ , correspond to the operators with the same numbers in ref. [24]. We give these operators explicitly in eq. (21) of sec. 3. A complete list of the  $O(p^2)$  independent counterterms can be found in ref. [25].

Since  $\mathcal{O}_7$  and  $\mathcal{O}_8$  both transform under the same representation, the generic form of the chiral expansion is the same but the values of the  $\gamma^{(8,8)}$  and the  $\delta_i$  on the right-hand side of eq. (8) depends on the operator. We leave this dependence implicit.

We present the results for the chiral logarithms for arbitrary meson masses and momenta, together with those for  $\mathcal{O}_4$ , on the web site [23]. For full QCD we also present the results in appendix B.2.

We rewrite eq. (5) for physical and SPQR kinematics in sec. 2.2.1 and 2.2.2 respectively. The one-loop expressions with the chiral logarithms included are given in:

- eqs. (87) and (88) for the physical kinematics in full QCD;
- eqs. (95) and (96) for the physical kinematics in quenched QCD;
- eqs. (117) – (120) for the SPQR kinematics in full QCD;
- eqs. (137) – (140) for the SPQR kinematics in quenched QCD.

### 2.2.1 The Matrix Element of $\mathcal{O}_{7,8}$ for Physical Kinematics

For the physical kinematics eq. (8) reduces to

$$\begin{aligned} \langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle_{\text{phys}} &= \frac{2\sqrt{2}}{f_K f_\pi^2} \left\{ \gamma^{(8,8)} (1 + \text{chiral logs}) + \right. \\ &\quad \left. (-\delta_2 - \delta_3 + 2(\delta_4 + \delta_5) + 4\delta_6) m_K^2 + (\delta_1 + \delta_2 + 4(\delta_4 + \delta_5) + 2\delta_6) m_\pi^2 \right\}. \end{aligned} \quad (9)$$

### 2.2.2 The Matrix Element of $\mathcal{O}_{7,8}$ for the SPQR Kinematics

Following the same strategy as for  $\mathcal{O}_4$ , we envisage determining the  $\gamma^{(8,8)}$ 's and  $\delta_i$ 's by performing the simulations with the SPQR kinematics. In this case eq. (8) reduces to

$$\begin{aligned} \langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle_{\text{SPQR}} &= \frac{2\sqrt{2}}{f_K f_\pi^2} \left\{ \gamma^{(8,8)} (1 + \text{chiral logs}) + 2(\delta_4 + \delta_5 + 2\delta_6) m_K^2 + \right. \\ &\quad \left. \frac{1}{2}(\delta_1 - \delta_2 - 2\delta_3)(m_\pi + E_\pi)m_K + 2(2(\delta_4 + \delta_5) + \delta_6)m_\pi^2 - (\delta_1 + \delta_2)m_\pi E_\pi \right\}. \end{aligned} \quad (10)$$

It is straightforward to verify that if one determines  $\gamma^{(8,8)}$  and the combinations of the  $\delta_i$ 's which appear in eq. (10) by fitting lattice data for SPQR kinematics, then one has all the necessary ingredients to determine the physical matrix elements using eq. (9).



### 3 Quenched and Unquenched $\chi$ PT for $K \rightarrow \pi\pi$ Decays

In this section we outline the main ingredients of the one-loop  $\chi$ PT calculations. The generic form of the (27,1) component of the weak Lagrangian for  $K \rightarrow \pi\pi$  up to  $O(p^4)$  is [22]

$$\mathcal{L}_{\text{eff}}^{(27,1)} = -\alpha^{(27,1)} \mathcal{O}_2^{(27,1)} - \sum_{i=2,4,5,7,22,24} \beta_i \mathcal{O}_{4,i}^{(27,1)} + \text{h.c.} + O(p^6) \quad (11)$$

where  $\mathcal{O}_2^{(27,1)}$  is the leading order ( $O(p^2)$ ) operator

$$\mathcal{O}_2^{(27,1)} = T_{kl}^{ij} (L_\mu)^k{}_i (L^\mu)^l{}_j, \quad (12)$$

and  $\mathcal{O}_{4,i}^{(27,1)}$  are the operators which appear at NLO ( $O(p^4)$ )

$$\begin{aligned} \mathcal{O}_{4,2}^{(27,1)} &= -T_{kl}^{ij} (P)^k{}_i (P)^l{}_j, \\ \mathcal{O}_{4,4}^{(27,1)} &= T_{kl}^{ij} (L_\mu)^k{}_i \{L^\mu, S\}^l{}_j, \\ \mathcal{O}_{4,5}^{(27,1)} &= i T_{kl}^{ij} (L_\mu)^k{}_i [P, L^\mu]^l{}_j, \\ \mathcal{O}_{4,7}^{(27,1)} &= T_{kl}^{ij} (L_\mu)^k{}_i (L^\mu)^l{}_j \text{tr}[S], \\ \mathcal{O}_{4,22}^{(27,1)} &= i T_{kl}^{ij} (L_\mu)^k{}_i [L_\nu, W^{\mu\nu}]^l{}_j, \\ \mathcal{O}_{4,24}^{(27,1)} &= T_{kl}^{ij} (W_{\mu\nu})^k{}_i (W^{\mu\nu})^l{}_j. \end{aligned} \quad (13)$$

The tensor  $T_{kl}^{ij}$  is defined as

$$\begin{aligned} T_{12}^{13} &= T_{12}^{31} = T_{21}^{13} = T_{21}^{31} = \frac{1}{2}, \\ T_{22}^{23} &= T_{22}^{32} = -\frac{1}{2}, \end{aligned} \quad (14)$$

with all other components zero.  $L_\mu$  and  $W_{\mu\nu}$  are defined by

$$\begin{aligned} L_\mu &= i \Sigma^\dagger \partial_\mu \Sigma, \\ W_{\mu\nu} &= 2 (\partial_\mu L_\nu - \partial_\nu L_\mu), \end{aligned} \quad (15)$$

and  $S$  and  $P$  by

$$S = \Sigma^\dagger \chi + \chi^\dagger \Sigma \quad \text{and} \quad P = i (\Sigma^\dagger \chi - \chi^\dagger \Sigma) \quad (16)$$

with

$$\chi = \frac{-2\langle 0 | \bar{u}u + \bar{d}d | 0 \rangle}{f^2} \mathcal{M}. \quad (17)$$

$\Sigma$  is the conventional exponential representation of the pseudo-Goldstone boson field

$$\Sigma = \exp\left(\frac{2i\Phi}{f}\right) \quad \text{with} \quad \Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}, \quad (18)$$

and  $\mathcal{M}$  is the quark mass-matrix  $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ .

For the (8,8) component of the  $\Delta S = 1$  weak Lagrangian the generic form is [24, 25]:

$$\mathcal{L}_{\text{eff}}^{(8,8)} = -\gamma^{(8,8)} \mathcal{O}_0^{(8,8)} - \sum_{i=1}^6 \delta_i \mathcal{O}_{2,i}^{(8,8)} + O(p^4), \quad (19)$$

where  $\mathcal{O}_0^{(8,8)}$  is the leading order ( $O(p^0)$ ) operator

$$\mathcal{O}_0^{(8,8)} = \text{tr} [\lambda_6 \Sigma^\dagger Q \Sigma], \quad (20)$$

and  $\mathcal{O}_{2,i}^{(8,8)}$  are the NLO ( $O(p^2)$ ) operators

$$\begin{aligned} \mathcal{O}_{2,1}^{(8,8)} &= \text{tr} [\lambda_6 L_\mu \Sigma^\dagger Q \Sigma L^\mu], \\ \mathcal{O}_{2,2}^{(8,8)} &= \text{tr} [\lambda_6 L_\mu] \text{tr} [\Sigma^\dagger Q \Sigma L^\mu], \\ \mathcal{O}_{2,3}^{(8,8)} &= \text{tr} [\lambda_6 \{\Sigma^\dagger Q \Sigma, L_\mu L^\mu\}], \\ \mathcal{O}_{2,4}^{(8,8)} &= \text{tr} [\lambda_6 \{\Sigma^\dagger Q \Sigma, S\}], \\ \mathcal{O}_{2,5}^{(8,8)} &= i \text{tr} [\lambda_6 [\Sigma^\dagger Q \Sigma, P]], \\ \mathcal{O}_{2,6}^{(8,8)} &= \text{tr} [\lambda_6 \Sigma^\dagger Q \Sigma] \text{tr} [S]. \end{aligned} \quad (21)$$

$Q$  and  $\lambda_6$  are the charge and Gell-Mann matrices respectively:

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \quad \text{and} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (22)$$

For the strong interactions we only need the leading order chiral Lagrangian

$$\mathcal{L}_{\text{strong}} = \frac{f^2}{8} \text{tr} [(\partial_\mu \Sigma^\dagger) (\partial^\mu \Sigma) + \Sigma^\dagger \chi + \chi^\dagger \Sigma]. \quad (23)$$

The NLO terms in  $\mathcal{L}_{\text{strong}}$ , which are proportional to the Gasser-Leutwyler  $L_i$  coefficients, contribute to the wave function renormalization. These effects are reabsorbed in the renormalization of the decay constant leading to the replacement of the factor  $1/f^3$  by  $1/(f_K f_\pi^2)$  in all the  $K \rightarrow \pi\pi$  matrix elements. We therefore do not present the NLO terms in the strong chiral Lagrangian.

### 3.1 Quenched $\chi$ PT

For the calculation in quenched  $\chi$ PT we use the strong Lagrangian introduced in ref. [21]

$$\mathcal{L}_{\text{strong}} = \frac{f^2}{8} \text{str} [(\partial_\mu \Sigma^\dagger) (\partial^\mu \Sigma) + \Sigma^\dagger \chi + \chi^\dagger \Sigma] - m_0^2 \Phi_0^2 + \alpha (\partial_\mu \Phi_0) (\partial^\mu \Phi_0). \quad (24)$$

In eq. (24) the trace over the chiral group indices in eq. (23) has been replaced by the supertrace over the indices of the graded group  $SU(3|3)_L \times SU(3|3)_R$ . The fields  $\Sigma$  and  $\chi$  are now graded extensions of  $\Sigma$  and  $\chi$  defined in eqs. (18) and (17). Since in the quenched approximation the singlet field remains degenerate with the pion and has to be treated as dynamical, the graded extension of the matrix  $\Phi$  is not supertraceless. We define the singlet field  $\Phi_0 = \text{str}[\Phi]/\sqrt{6}$  proportional to the graded extension of the  $\eta'$ .

In the weak operators each field in eqs. (12), (13), (20) and (21) is replaced by its graded extension, and every trace by the corresponding supertrace. Although we use the following extension for the charge matrix in the quenched approximation

$$Q = \text{diag}(2/3, -1/3, -1/3, 2/3, -1/3, -1/3), \quad (25)$$

the results for  $\Delta I = 3/2$  transitions do not depend on the choice of extension in the  $SU(2)$  (isospin) limit. This is because the EWP operators are always inserted on external quark lines, and hence the matrix elements are independent of the charges assigned to the ghost particles. For the same reason, for the present calculation we can simply extend  $\lambda_6$  to a six-by-six block diagonal matrix:

$$\lambda_6 \rightarrow \begin{pmatrix} \lambda_6 & 0 \\ 0 & 0 \end{pmatrix}. \quad (26)$$

## 4 $\chi$ PT on a Finite Volume

In this section we study, at one-loop order in  $\chi$ PT, the Euclidean correlation functions relevant for the finite-volume lattice determination of  $K \rightarrow \pi\pi$  decay amplitudes. As argued in the section 2, such a study is a necessary step to extract, by extrapolation, the physical amplitudes from correlation functions which at present can only be computed at unphysical values of masses and momenta [2]. Although the explicit discussion is presented for full QCD, it also applies to quenched (and partially quenched) QCD for  $\Delta I = 3/2$  decays (for  $\Delta I = 1/2$  decays in quenched QCD this is not the case [8]).

A significant step towards enabling the determination of the physical matrix element of  $\mathcal{O}_4$  from matrix elements computed in lattice simulations at unphysical kinematics, has been the evaluation, at one-loop order in perturbation theory in finite volumes, of  $K \rightarrow \pi\pi$  amplitudes with  $m_K = m_\pi$  and  $m_K = 2m_\pi$  and with all the mesons at rest [26, 27, 28]. (For a discussion of the advantages of using these kinematic points in the evaluation of  $\Delta I = 1/2$  decay amplitudes at leading order in the chiral expansion see refs. [29, 30].) The one-loop generating functional in the presence of weak interactions, also in the quenched approximation, was considered in ref. [31].

In discussing the behaviour of the correlation functions and matrix elements with the volume, we envisage varying the volume at fixed physics, i.e. with the masses, energies and momenta constant in physical units. From our explicit computations we see that all the one-loop corrections can be classified into five categories:

1. The infinite-volume chiral corrections to the matrix elements of the weak operators. These diverge in  $\chi$ PT, and are renormalized by operators of higher dimension [22], see sec. 5;
2. The finite-volume shift to the two-pion energy,  $\Delta W = W - 2E_\pi$ , which is independent of the weak operators, and only depends on the isospin of the two-pion state. We illustrate this point by showing explicitly that our calculations give the same shift  $\Delta W$  for the operators  $\mathcal{O}_4$ ,  $\mathcal{O}_7^{3/2}$  and  $\mathcal{O}_8^{3/2}$  (where the superscript 3/2 indicates the  $\Delta I = 3/2$  component of the operators);
3. The corrections which shift the energy argument ( $E$ ) of the amplitude  $\mathcal{A}(E)$  from its infinite volume value to the finite volume one, i.e.  $\mathcal{A}(2E_\pi) \rightarrow \mathcal{A}(W)$ . We show that in earlier studies the shift of the argument of  $\mathcal{A}(E)$  was wrongly interpreted as a finite-volume correction to the amplitude [26, 27];
4. The finite volume corrections to the sink (source) which is used to annihilate (create) the two pions. These corrections obviously depend on the sink/source and can be eliminated non-perturbatively by studying the source-sink correlation functions. In the following we consider two classes of correlation functions. In the first the two pions are annihilated at the same time (as used, for example, in ref. [3]) and in the second one pion is annihilated much later than the other (as used, for example, in refs. [4, 5, 32]).
5. Finally we have the Lellouch-Lüscher (LL) finite-volume corrections to the matrix elements themselves.

A very important result is that the finite-volume correction to the amplitude, in one-loop  $\chi$ PT and in the center of mass frame, is the universal LL factor with both types of sources mentioned in item 4 above. This will be explained in sec. 4.1 below. The generalization of the LL factor to the case where the two pions have a non-zero total three-momentum is being studied (see sec. 4.1.2).

The starting point for our studies is the Euclidean correlation function

$$\mathcal{C}_{\vec{q}_1 \vec{q}_2}(t_K, t_1, t_2) = \langle 0 | \pi_{\vec{q}_1}(t_1) \pi_{\vec{q}_2}(t_2) \mathcal{O}_W(0) K_0^\dagger(t_K) | 0 \rangle, \quad (27)$$

where

$$\pi_{\vec{q}}(t) = \int d^3x \, \pi(\vec{x}, t) e^{i\vec{q}\vec{x}} \quad (28)$$

is the Fourier transform of the one-pion interpolating field (with a similar definition for the kaon) and  $t_K < 0 < t_2 \leq t_1$ .  $\mathcal{O}_W$  is one of the  $\Delta I = 3/2$  weak operators, and from the correlation functions in eq. (27) we wish to determine the matrix element  $\langle \pi\pi | \mathcal{O}_W | K \rangle$ . In the following subsections we explain, at one-loop order in  $\chi$ PT, what the finite-volume effects in the correlation function are, and how one recovers the LL factor in the matrix elements.

In order to simplify the discussion of the different corrections occurring at one loop, while maintaining all the essential ingredients, we begin in sec. 4.1 with a study of the correlation functions in a  $\lambda\Phi^4$  theory (for the strong interactions) and later, in sec. 4.2 and 4.3 generalise the discussion to  $\chi$ PT .

#### 4.1 $K^+ \rightarrow \pi^+\pi^0$ Decays on a Finite Volume in a $\lambda\Phi^4$ Model

In this section we illustrate the structure of the relations between correlation functions in finite and infinite volumes, by considering  $K \rightarrow \pi\pi$  decays in a simple model in which the interactions between the pions are described by a  $\lambda\Phi^4$  theory. The bare  $K \rightarrow \pi\pi$  vertex is taken to be a constant ( $\lambda_{K\pi\pi}$ ). This model contains all the essential ingredients necessary to illustrate the key features of finite-volume effects. In the following section we generalise this discussion to  $\chi$ PT .

At tree-level, the correlation function in eq. (27) is given by

$$\begin{aligned} \mathcal{C}_{\vec{q}_1\vec{q}_2}^{(0)}(t_K, t_1, t_2) &\equiv \mathcal{A}^{(0)}(E_T) \times \mathcal{G}^{(0)}(t_K, t_1, t_2, E_T) \\ &= \lambda_{K\pi\pi} \times \frac{1}{2m_K 2E_{\vec{q}_1} 2E_{\vec{q}_2}} e^{-m_K|t_K|} e^{-E_{\vec{q}_1}(t_1-t_2)} e^{-E_T t_2}, \end{aligned} \quad (29)$$

where  $E_{\vec{q}_i}^2 = |\vec{q}_i|^2 + m_\pi^2$  and the total two-pion energy is given by  $E_T = E_{\vec{q}_1} + E_{\vec{q}_2}$ .  $\mathcal{A}^{(0)}(E_T)$  ( $= \lambda_{K\pi\pi}$ ) is the tree-level component of the  $K \rightarrow \pi\pi$  decay amplitude.

At one-loop order, we insert an interaction vertex ( $\mathcal{O}_s = -\lambda |\pi^0(x_s)|^2 |\pi^+(x_s)|^2 / 4$ , where  $\pi$  represents the pion field) into the correlation function, and its contribution is given by the expression

$$\mathcal{C}_{\vec{q}_1\vec{q}_2}^{(1)}(t_K, t_1, t_2) = \langle 0 | T[\pi_{\vec{q}_1}(t_1) \pi_{\vec{q}_2}(t_2) \mathcal{O}_W(0) \left[ \int d^4x_s \mathcal{O}_s(x_s) \right] K_0^\dagger(t_K)] | 0 \rangle \quad (30)$$

corresponding to the Feynman diagram in fig. 1(b). In this case the weak operator is simply given by the triple scalar vertex,  $\mathcal{O}_W(0) = \lambda_{K\pi\pi} K(0) \pi^\dagger(0) \pi^\dagger(0)$ .

In  $\chi$ PT , of the four diagrams shown in fig. 1 (and the graphs which renormalize the external meson fields) it is only diagram (b) which contains finite-volume effects which decrease as inverse powers of the volume. For the remaining diagrams the finite volume effects decrease exponentially and will be neglected. It is for this reason that the evaluation of this graph in the  $\lambda\Phi^4$  model illustrates the key properties of finite volume effects in  $\chi$ PT . Note however, that the evaluation of the graph in fig. 1(b) is different in this model and  $\chi$ PT (for example both the weak and strong vertices in  $\chi$ PT depend in general on the momenta).

The contribution from the diagram of fig. 1 (b) to the correlation function is:

$$-\lambda_{K\pi\pi} \lambda \frac{e^{-m_K|t_K|}}{2m_K} \frac{1}{L^3} \int dt_s \frac{e^{-E_{\vec{q}_1}|t_1-t_s|}}{2E_{\vec{q}_1}} \frac{e^{-E_{\vec{q}_2}|t_2-t_s|}}{2E_{\vec{q}_2}} \sum_{\vec{k}, \vec{l}} \int \frac{dk_4}{2\pi} \frac{dl_4}{2\pi} \frac{e^{-i(k_4+l_4)t_s}}{(k_4^2 + w_1^2)(l_4^2 + w_2^2)} \delta_{\vec{k}+\vec{l}, \vec{q}}$$

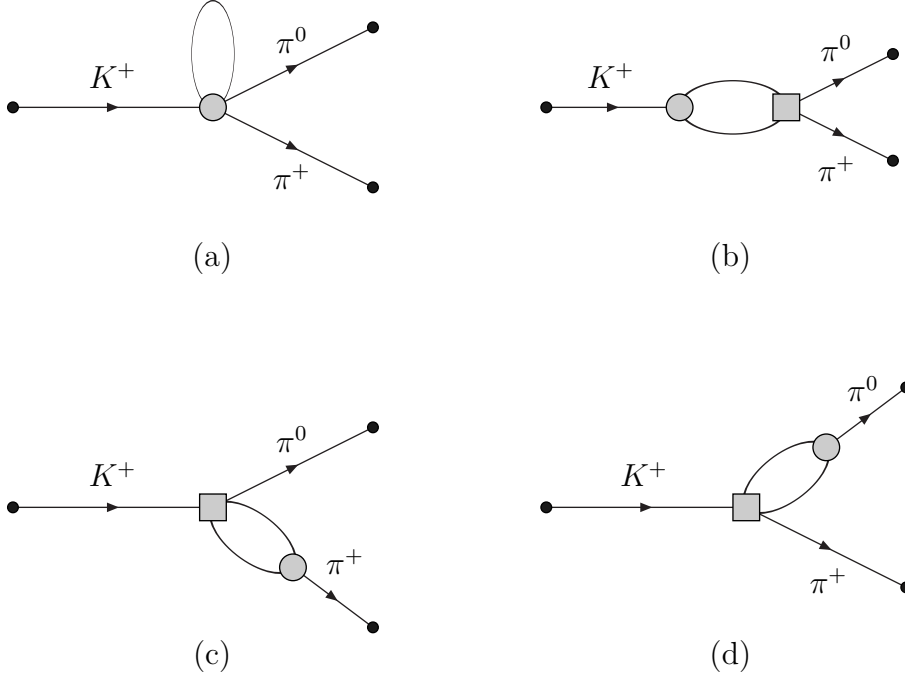


Figure 1: One-Loop Chiral Perturbation Theory Diagrams for the  $K^+ \rightarrow \pi^0 \pi^+$  decay. The grey circle (square) represents weak (strong) vertices. In addition one must also calculate the contributions from the diagrams which renormalize the wave functions of the external mesons.

$$\begin{aligned}
&= -\lambda_{K\pi\pi} \lambda \times \frac{e^{-m_K|t_K|}}{2m_K} \frac{1}{L^3} \sum_{\vec{k}, \vec{l}} \delta_{\vec{k}+\vec{l}, \vec{q}}^3 \int dt_s \frac{e^{-E_{\vec{q}_1}|t_1-t_s|}}{2E_{\vec{q}_1}} \frac{e^{-E_{\vec{q}_2}|t_2-t_s|}}{2E_{\vec{q}_2}} \frac{e^{-w_1|t_s|}}{2w_1} \frac{e^{-w_2|t_s|}}{2w_2} \\
&= -\lambda_{K\pi\pi} \lambda \times \frac{e^{-m_K|t_K|}}{2m_K} \frac{1}{L^3} \frac{1}{16E_{\vec{q}_1}E_{\vec{q}_2}} \sum_{\vec{k}, \vec{l}} \delta_{\vec{k}+\vec{l}, \vec{q}}^3 \frac{1}{w_1w_2} \{T_1 + T_2 + T_3 + T_4\}, \quad (31)
\end{aligned}$$

where

$$w_1^2 = m_1^2 + |\vec{k}|^2, \quad w_2^2 = m_2^2 + |\vec{l}|^2 \quad (32)$$

and  $\vec{q} = \vec{q}_1 + \vec{q}_2$ .  $T_1 - T_4$  are the contributions to the integrations over  $t_s$  from the intervals  $(-\infty, 0)$ ,  $(0, t_2)$ ,  $(t_2, t_1)$  and  $(t_1, \infty)$  respectively and are given by

$$\begin{aligned}
T_1 &= \frac{e^{-E_{\vec{q}_1}t_1 - E_{\vec{q}_2}t_2}}{E_{\vec{q}_1} + E_{\vec{q}_2} + w_1 + w_2} \\
T_2 &= \frac{e^{-E_{\vec{q}_1}(t_1-t_2) - (w_1+w_2)t_2} - e^{-E_{\vec{q}_2}t_2 - E_{\vec{q}_1}t_1}}{E_{\vec{q}_1} + E_{\vec{q}_2} - (w_1 + w_2)} \\
T_3 &= \frac{e^{-E_{\vec{q}_2}(t_1-t_2) - (w_1+w_2)t_1} - e^{-E_{\vec{q}_1}(t_1-t_2) - (w_1+w_2)t_2}}{E_{\vec{q}_1} - E_{\vec{q}_2} - w_1 - w_2} \quad (33)
\end{aligned}$$

$$T_4 = \frac{e^{-E_{\vec{q}_2}(t_1-t_2)-(w_1+w_2)t_1}}{E_{\vec{q}_1} + E_{\vec{q}_2} + w_1 + w_2}.$$

For simplicity we choose  $|\vec{q}_1| = |\vec{q}_2|$  and take  $m_1 = m_2$ , so that  $\vec{q} = 0$ ,  $w_1 = w_2 \equiv w$  and  $E_{\vec{q}_1} = E_{\vec{q}_2} \equiv E$ . The generalization to the SPQR kinematics is straightforward in perturbation theory, but as described in sec. 4.1.2, the generalization of the LL-factor to the case where the two pions have a non-zero total momentum has not been performed yet. Factorizing the tree level correlation function, we obtain

$$\begin{aligned} \mathcal{C}_{\vec{q}_1\vec{q}_2}^{(1)}(t_K, t_1, t_2) &= -\mathcal{G}^{(0)}(t_K, t_1, t_2, 2E) \times \lambda \times \\ &\frac{1}{L^3} \sum_{\vec{k}} \frac{\mathcal{A}^{(0)}(2w)}{(2w)^2} \left\{ \frac{1}{2(E+w)} + \frac{e^{2(E-w)t_2} - 1}{2(E-w)} + \frac{e^{2Et_2-2wt_1} - e^{2(E-w)t_2}}{-2w} + \frac{e^{2Et_2-2wt_1}}{2(E+w)} \right\}. \end{aligned} \quad (34)$$

Note that in this specific example the weak amplitude is independent of the energy and thus  $\mathcal{A}^{(0)}(2w) = \lambda_{K\pi\pi}$ . The four terms in braces in eq. (34) are  $T_1 - T_4$  respectively. For illustration we now consider the two cases  $0 < t_1 = t_2 \equiv t$  [3] and  $0 < t_2 \ll t_1$  [4, 5, 32]. In the first case,  $T_3 = 0$  and we have a factor

$$\frac{1}{L^3} \sum_{\vec{k}} \frac{1}{(2w)^2} \left\{ \frac{e^{2(E-w)t}}{2(E+w)} + \frac{e^{2(E-w)t} - 1}{2(E-w)} + \frac{1}{2(E+w)} \right\}. \quad (35)$$

In the second case, since  $t_1 \gg t_2$  we can neglect  $T_4$  since it is exponentially suppressed and the corresponding factor is

$$\frac{1}{L^3} \sum_{\vec{k}} \frac{1}{(2w)^2} \left\{ \frac{e^{2(E-w)t_2}}{2w} + \frac{e^{2(E-w)t_2} - 1}{2(E-w)} + \frac{1}{2(E+w)} \right\}. \quad (36)$$

Here we are considering finite volume corrections to correlation functions. As anticipated above, we will show that the finite volume corrections to the amplitudes, extracted after dividing by the appropriate source factor, are identical in the two cases, in spite of the differences of the expressions in eqs. (35) and (36).

We now consider the sum over the momentum  $\vec{k}$  and separate the contribution corresponding to  $w = E$ , with the two pions in the loop on the mass-shell, from the remaining terms. For the two cases we are studying, the contribution with  $w = E$  is given by

$$\frac{1}{L^3} \frac{\nu}{(2E)^2} \left\{ \frac{1}{4E} + t + \frac{1}{4E} \right\} \quad (t_1 = t_2 = t) \quad (37)$$

and

$$\frac{1}{L^3} \frac{\nu}{(2E)^2} \left\{ \frac{1}{2E} + t_2 + \frac{1}{4E} \right\} \quad (t_1 \gg t_2), \quad (38)$$

where  $\nu = \sum_{\vec{k}:w=E}$ . The term linear in time in eqs. (37) and (38), after resummation, corresponds to a shift of the two-pion energy due to the  $\lambda\Phi^4$  interactions:

$$W = E_T + \Delta W = 2E + \frac{\lambda\nu}{4L^3E^2}. \quad (39)$$

It is straightforward to check that the energy shift in eq. (39) corresponds to that obtained from the  $O(\lambda)$  term in the perturbative expansion of the Lüscher quantization condition [33, 34] using the one-loop expression for the  $s$ -wave phase-shift.

Of the remaining terms in each of eqs. (37) and (38), the first one comes from final-state pion rescattering and will be cancelled by dividing by the matrix element of the source (this will be demonstrated explicitly below). These final-state interactions are different in the two cases  $t_1 = t_2$  and  $t_2 \ll t_1$ . The third term in each of eqs. (37) and (38) is part of the (finite volume) one-loop correction to the “weak” matrix element and, as expected, is the same in both cases,  $t_1 = t_2$  and  $t_2 \ll t_1$ .

It is convenient to group the terms with  $w \neq E$  as follows:

$$\frac{1}{L^3} \sum_{\vec{k}: w \neq E} \frac{1}{(2w)^2} \left\{ \frac{E}{(E^2 - w^2)} e^{2(E-w)t} + \frac{w}{w^2 - E^2} \right\} \quad (t_1 = t_2 = t) \quad (40)$$

$$\frac{1}{L^3} \sum_{\vec{k}: w \neq E} \frac{1}{(2w)^2} \left\{ \frac{E}{2w(E-w)} e^{2(E-w)t_2} + \frac{w}{w^2 - E^2} \right\} \quad (t_1 \gg t_2). \quad (41)$$

In the limit  $t_2 \rightarrow \infty$ , the terms proportional to  $e^{2(E-w)t_2}$  with  $w > E$  are exponentially suppressed and only a finite number of contributions with  $w < E$  survive. As pointed out by Maiani and Testa, these terms dominate the correlation function over the contribution one is trying to isolate, i.e. the one with  $w = E$ . We denote these terms as *Maiani-Testa terms*. They can be eliminated by using the behaviour of the correlator as a function of  $t_2$  (at large  $t_2$ ) and separating the different exponentials, corresponding to the different energies [3, 4, 35]. We assume that this can be done in the numerical simulations and do not consider these contributions anymore. Up to exponentially small terms, the final term in eqs. (40) and (41) can be written as [34]

$$\frac{1}{L^3} \sum_{\vec{k}: w \neq E} \frac{1}{(2w)^2} \frac{w}{w^2 - E^2} = f(E) + \frac{z_n}{4(2\pi)^2 EL} + \frac{\nu}{8E^3 L^3}, \quad (42)$$

where  $\vec{n}$  is defined by  $\vec{q}_1 = \frac{2\pi}{L} \vec{n}$ ,

$$z_n \equiv \sum_{\substack{|\vec{i}| \neq |\vec{n}| \\ \vec{i} \in Z^3}} \frac{1}{(\vec{i}^2 - \vec{n}^2)}, \quad \text{and} \quad f(E) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3} \frac{1}{4w} \frac{1}{w^2 - E^2}. \quad (43)$$

It is instructive to rewrite  $f(E)$  in a form which makes clear the connection to the one-loop contribution to the physical  $K \rightarrow \pi\pi$  amplitude. We write  $f(E) = \text{Re}\{-if_M(E)\}$  where

$$f_M(E) \equiv \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\varepsilon)((q-k)^2 - m^2 + i\varepsilon)} \quad (44)$$

is the relevant integral in Minkowski space (hence the subscript  $M$ ). The relation  $f(E) = \text{Re}\{-if_M(E)\}$  can be readily verified using contour integration in  $k_0$ -space. The real part



of the  $K \rightarrow \pi\pi$  amplitude in Minkowski space is proportional to  $\text{Re}\{1 + i\lambda f_M(E)\} = 1 - \lambda f(E)$ , so that one correctly obtains the real part of the Minkowski amplitude from the Euclidean correlation function, in agreement with the general arguments of ref. [32]. (At this order in perturbation theory the modulus of the amplitude is equal to its real part.) We can also rewrite  $f(E)$  as a one-loop four dimensional integral in Euclidean space,  $f(E) = \text{Re}\{f_E(iE)\}$ , where

$$\text{Re} f_E(iE) = \text{Re} \left\{ \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + m^2)((q - k)^2 + m^2)} \right\}_{q^2 = -4E^2}. \quad (45)$$

After the integration has been performed in Euclidean space, the substitution  $q^2 = -4E^2$  is made.

From the above discussion we conclude that the correlation function, at one-loop order in perturbation theory and keeping only the terms with the required dependence on  $t_2$ , can be written in the form (for the two cases being considered):

$$\begin{aligned} \mathcal{C}_{\bar{q}_1 - \bar{q}_1}(t_K, t, t) &= \mathcal{A}^{(0)}(W) \times \mathcal{G}^{(0)}(t_K, t, t, W) \times \\ &\left\{ 1 - \lambda \left[ \text{Re} f_E(iW) + \frac{z_n}{4(2\pi)^2 EL} + \frac{\nu}{L^3} \frac{3}{16E^3} + \frac{\nu}{L^3} \frac{1}{16E^3} \right] \right\} + \mathcal{O}(\lambda^2) \quad (t_1 = t_2 = t), \end{aligned} \quad (46)$$

and

$$\begin{aligned} \mathcal{C}_{\bar{q}_1 - \bar{q}_1}(t_K, t_1, t_2) &= \mathcal{A}^{(0)}(W) \times \mathcal{G}^{(0)}(t_K, t_1, t_2, W) \times \\ &\left\{ 1 - \lambda \left[ \text{Re} f_E(iW) + \frac{z_n}{4(2\pi)^2 EL} + \frac{\nu}{L^3} \frac{3}{16E^3} + \frac{\nu}{L^3} \frac{1}{8E^3} \right] \right\} + \mathcal{O}(\lambda^2) \quad (t_1 \gg t_2). \end{aligned} \quad (47)$$

Although, in this case, the tree-level amplitude does not depend on the energy, we have nevertheless written it as  $\mathcal{A}^{(0)}(W)$  in eqs. (46) and (47), in order to be able to generalise the discussion to  $\chi$ PT in the following sections.

In order to determine the  $K \rightarrow \pi\pi$  decay amplitudes we need to divide the correlation functions by the appropriate factors associated with the source (kaon) and the sink (two pions). For single particle states this is straightforward, and there are no finite volume power corrections,

$$\langle 0 | K_{\vec{0}}(0) K_{\vec{0}}^\dagger(t_K) | 0 \rangle = L^3 Z_K^2 \frac{e^{-m_K |t_K|}}{2m_K}, \quad (48)$$

where the subscript  $\vec{0}$  indicates that the Fourier transform of the kaon field has been performed at zero momentum.

The determination of the factors  $\langle 0 | \pi\pi | n \rangle$  associated with the two-pion intermediate states  $|n\rangle$  is more subtle. These matrix elements can be obtained by evaluating four-pion correlation functions. However, one has to be careful to project out the matrix element corresponding to the required state  $|n\rangle$ , which in the present case is a two-pion, cubically invariant state which contains an s-wave component. Although one can, in principle, try to use the behaviour of the correlation function with  $t_2$  to project out the state with the

correct energy, this may be difficult in practice if this state has an energy close to one with no s-wave component. For this reason, it is useful to project out the contribution of the required state directly, by replacing exponential factors of the form  $\exp(i\vec{k} \cdot \vec{x})$  in Fourier transforms by their s-wave projections,  $\sin(kr)/kr$  ( $k = |\vec{k}|$  and  $r = |\vec{x}|$ ). For our purposes it will be sufficient instead, to average the correlation function  $\mathcal{C}_{\vec{q}_1 - \vec{q}_1}$  in eqs. (46) and (47) over all  $\vec{q}_1$ 's which can be transformed into each other by elements of the cubic group (in the formulae below we actually choose to average over the  $\nu$  momenta with the same modulus  $|\vec{q}_1|$ ). Such an average has a projection on the s-wave, but no projection onto  $l = 1, 2$  and 3 components of the states. Since at one-loop order, both in the  $\lambda\Phi^4$  theory and in  $\chi$ PT,  $l = 4$  components (and higher partial waves) cannot be generated, such an average over cubically equivalent states is sufficient. We now carry out this procedure explicitly.

We start by evaluating the averaged correlation function (see eq. (27))

$$\begin{aligned} \sum_{\vec{q}_1} C_{\vec{q}_1 - \vec{q}_1}(t_K, t_1, t_2) &= \sum_{\vec{q}_1} \sum_{n, m} \langle 0 | \pi_{\vec{q}_1}(t_1) \pi_{-\vec{q}_1}(t_2) | n \rangle \langle n | \mathcal{O}_W(0) | m \rangle \langle m | K^\dagger(t_K) | 0 \rangle \\ &= \frac{Z_K}{2m_K} e^{-m_K |t_K|} e^{-W t_2} \sum_{\vec{q}_1} \langle 0 | \pi_{\vec{q}_1}(t_1 - t_2) \pi_{-\vec{q}_1}(0) | W \rangle \langle W | \mathcal{O}_W(0) | K \rangle + \dots, \end{aligned} \quad (49)$$

where  $|W\rangle$  represents the cubically invariant two-pion state with energy  $W$  which contains an s-wave component. As explained above, the sum over  $\vec{q}_1$ , runs over all the three-momenta with the same modulus,  $|\vec{q}_1|$ . The ellipses in eq. (49) represent terms with a different behaviour in  $t_2$ , and as always, we assume that we can use the time dependence to isolate the first term on the left hand side.

In order to obtain the  $K \rightarrow \pi\pi$  matrix element from the correlation function in eq. (49), we need to divide by the matrix element

$$\sum_{\vec{q}_1} \langle 0 | \pi_{\vec{q}_1}(t_1 - t_2) \pi_{-\vec{q}_1}(0) | W \rangle. \quad (50)$$

We can determine this matrix element from the four-pion correlation function:

$$\begin{aligned} &\sum_{\vec{q}_1} \sum_{\vec{p}} \langle 0 | \pi_{\vec{q}_1}(t_1) \pi_{-\vec{q}_1}(t_2) \pi_{-\vec{p}}^\dagger(-t_2) \pi_{\vec{p}}^\dagger(-t_1) | 0 \rangle \\ &= e^{-2W t_2} \sum_{\vec{q}_1} \sum_{\vec{p}} \langle 0 | \pi_{\vec{q}_1}(t_1 - t_2) \pi_{-\vec{q}_1}(0) | W \rangle \langle W | \pi_{-\vec{p}}^\dagger(0) \pi_{\vec{p}}^\dagger(-(t_1 - t_2)) | 0 \rangle + \dots, \end{aligned} \quad (51)$$

where the sum over  $\vec{p}$  is over the same range as the one over  $\vec{q}_1$ . Evaluating the four-pion correlation function in perturbation theory we find:

$$\sum_{\vec{p}, \vec{q}_1} \langle 0 | \pi_{\vec{q}_1}(t_1) \pi_{-\vec{q}_1}(t_2) \pi_{-\vec{p}}^\dagger(-t_2) \pi_{\vec{p}}^\dagger(-t_1) | 0 \rangle = \begin{cases} \nu \frac{e^{-2W t_2}}{(2E)^2} L^6 \left(1 - \frac{\lambda\nu}{8E^3 L^3}\right) & (t_1 = t_2) \\ \text{and} & \\ \nu \frac{e^{-2W t_2 - 2E(t_1 - t_2)}}{(2E)^2} L^6 \left(1 - \frac{\lambda\nu}{4E^3 L^3}\right) & (t_1 \gg t_2) \end{cases} \quad (52)$$

Combining the results in eq. (52) with those for the  $K \rightarrow \pi\pi$  correlation functions in eqs. (46) and (47) we find that the one-loop contribution to the amplitude in both cases is

$$\frac{\langle W | \mathcal{O}_W(0) | K \rangle}{\sqrt{\nu}} = \mathcal{A}^{(0)}(W) \left\{ 1 - \lambda \left[ \text{Re } f_E(iW) + \frac{z_n}{4(2\pi)^2 E L} + \frac{\nu}{L^3} \frac{3}{16E^3} \right] \right\}. \quad (53)$$

The finite volume corrections in eq. (53) are precisely those given by the Lellouch-Lüscher factor (see subsec. 4.1.1). We stress that in the perturbative example studied in this section we use the conventional covariant normalization of states. The remaining diagrams in fig. 1 also contribute to the infinite volume result, but they do not have corrections which decrease as inverse powers of the volume.

The above argument is a simple example of the discussion presented in ref. [5], in which it was shown that from the correlation functions eq. (27), after dividing by the matrix elements corresponding to the source and the sink, one obtains the modulus of the amplitude, up to finite-volume corrections given by the Lellouch-Lüscher factor. The case  $t_2 \ll t_1$  was analysed in ref. [4] where it was shown that the correlation function eq. (27) gives the real part of the decay amplitude up to power corrections in the volume which are not those described by the Lellouch-Lüscher factor. This is before dividing by the two-pion matrix element eq. (50). Division by this matrix element removes a factor of  $\cos(\delta)$  (where  $\delta$  is the s-wave phase-shift), turning the real part of the amplitude into its modulus, and modifies the finite-volume corrections into the Lellouch-Lüscher factor [5]. The above exercise is an explicit one-loop demonstration of this effect, although at this order of perturbation theory the real part and modulus of the amplitude are equivalent ( $\cos(\delta) = 1$ ).

The remaining diagrams in fig. 1 do not have singularities in the range of integration and hence only have exponential corrections in the volume (i.e. terms which decrease like, for example,  $\exp(-mL)$ ). To illustrate this feature consider the complete contribution from  $T_1$  in eq. (31), which is proportional to

$$\frac{1}{L^3} \sum_{\vec{q}} \frac{1}{(2w)^2} \frac{1}{2(w+E)} = \int \frac{d^3q}{(2\pi)^3} \frac{1}{(2w)^2} \frac{1}{2(w+E)} + \mathcal{O}(e^{-mL}). \quad (54)$$

#### 4.1.1 Evaluation of the Decay Amplitude

The one-loop toy-model study in sec. 4.1 is an explicit illustration of the general procedure for evaluating infinite-volume amplitudes from correlation functions computed in a finite volume. Before proceeding to develop the discussion of sec. 4.1 in the context of chiral perturbation theory, we briefly summarise this general procedure for the determination of the decay amplitudes. For illustration we consider the case where the kaon is at rest and the total momentum of the two-pions is also zero (see however sec. 4.1.2). The procedure is as follows:

1. We evaluate the correlation function  $\mathcal{C}_{\vec{q}_1-\vec{q}_1}(t_K, t_1, t_2)$  (defined in eq. (27)) averaged over all  $\vec{q}_1$  with the same modulus. We envisage evaluating the matrix element with the two-pion energy equal to  $W = 2E_{q_1}$  + finite-volume corrections. Let  $\mathcal{C}_{\vec{q}_1-\vec{q}_1, W}(t_K, t_1, t_2)$  be the component of  $\mathcal{C}_{\vec{q}_1-\vec{q}_1}(t_K, t_1, t_2)$  whose behaviour with  $t_2$  is given by  $\exp(-Wt_2)$ .
2. We evaluate the four-pion correlation function (see eq. (51))

$$\mathcal{C}_{\vec{q}_1\vec{q}_2}^{4\pi}(t_1, t_2) = \langle 0 | \pi_{\vec{q}_1}(t_1) \pi_{-\vec{q}_1}(t_2) \pi_{-\vec{q}_2}^\dagger(-t_2) \pi_{\vec{q}_2}^\dagger(-t_1) | 0 \rangle, \quad (55)$$

where again we average over all  $\vec{q}_1$  and  $\vec{q}_2$  with modulus equal to  $|\vec{q}_1|$ . As above, let  $\mathcal{C}_{\vec{q}_1\vec{q}_2,W}^{4\pi}(t_1, t_2)$  be the component of  $\mathcal{C}_{\vec{q}_1\vec{q}_2}^{4\pi}(t_1, t_2)$  whose behaviour with  $t_2$  is given by  $\exp(-2Wt_2)$ .

3. Finally we calculate the kaon propagator,  $\mathcal{C}_0^K(t_K)$

$$\mathcal{C}_0^K(t_K) \equiv \langle 0 | K_0(-t_K) K_0^\dagger(t_K) | 0 \rangle \quad (56)$$

for sufficiently large  $|t_K|$  so that the correlation function is dominated by the kaon-state.

4. The finite-volume matrix elements are given by:

$$|_V \langle \pi\pi | \mathcal{O}_W(0) | K \rangle_V| = \frac{\mathcal{C}_{\vec{q}_1-\vec{q}_1,W}(t_K, t_1, t_2)}{\sqrt{\mathcal{C}_{\vec{q}_1\vec{q}_2,W}^{4\pi}(t_1, t_2) \mathcal{C}_0^K(t_K)}}, \quad (57)$$

where the subscripts  $V$  on the matrix element denotes that it is the finite-volume matrix element (with the finite-volume normalization). Note that the  $K \rightarrow \pi\pi$  matrix elements discussed in sec. 4.1 in general and eq. (53) in particular, which were presented using the covariant normalization, should be divided by  $\sqrt{8m_K E^2 V^3}$  to translate them into the finite-volume normalization in eq. (57).

5. To obtain the infinite-volume matrix elements  $\langle \pi\pi | \mathcal{O}_W(0) | K \rangle$  (with the conventional covariant normalization) from the finite-volume ones, we multiply the latter by the LL factor [3, 4]<sup>3</sup>

$$|\langle \pi\pi | \mathcal{O}_W(0) | K \rangle|^2 = 4\pi V^2 \{q\phi'(q) + k\delta'(k)\} \left(\frac{m_K}{k}\right)^3 |_V \langle \pi\pi | \mathcal{O}_W(0) | K \rangle_V|^2, \quad (58)$$

where  $k = \sqrt{W^2/4 - m_\pi^2}$ ,  $q = kL/(2\pi)$ ,  $\delta(k)$  is the s-wave (and in our case  $I = 2$ )  $\pi$ - $\pi$  phase-shift and the geometrical function  $\phi(q)$  is given by

$$\tan(\phi(q)) = -\frac{2\pi^2 q}{\sum_{\vec{i} \in Z^3} 1/(\vec{i}^2 - q^2)}. \quad (59)$$

This procedure holds for the matrix elements discussed in this paper in full QCD and also in the quenched approximation (this is not the case in the quenched approximation for  $\Delta I = 1/2$  transitions [8]). The example in sec. 4.1 is a model one-loop demonstration of the validity of the procedure.

At present the LL factor has not been generalized to the situation in which the two-pion final state has a non-zero total three-momentum, and so we are unable to carry out step 5. We now discuss this point in a little more detail.

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<sup>3</sup>In  $K^+ \rightarrow \pi^+\pi^0$  decays the final state particles are distinguishable and so the factor in eq. (58) differs by a factor of 2 compared with the corresponding one presented for indistinguishable final state particles [3, 4].

### 4.1.2 Finite-Volume Corrections in a Moving Frame

In order to implement our strategy, which requires the computation of matrix elements at unphysical kinematics in general and SPQR kinematics in particular, simulations have to be performed with the two pions having non-zero total three-momentum. Thus, if the calculations are to be precise up to exponential corrections in the volume, we need the generalization of the Lüscher quantization condition and the LL factor away from the centre-of-mass frame. The quantization condition in a moving frame has been presented in ref. [36], whereas the generalization of the LL-factor is being studied and we will report the results in a future publication. We anticipate that the generalization of the LL-factor will be obtained from the quantization condition in an analogous way to the derivation in the centre-of-mass frame given in ref. [4].

At this point we are able to check whether the energy shift obtained in perturbation theory following the procedure of sec. 4.1 generalized to a moving frame, agrees with the expansion of the quantization condition presented in ref. [36]. This is indeed the case. Repeating steps (34) – (39) for two particles with energies  $E_1$  and  $E_2$  we find that the energy shift is equal to  $\lambda\nu^d/(4E_1E_2L^3)$ , where  $\nu^d$  is the number of terms in the momentum sum with energy equal to  $E_1 + E_2$ . This agrees with the result obtained by expanding the quantization condition in ref. [36] and using the one-loop expression for the phase-shift <sup>4</sup>.

We postpone the general discussion of finite-volume corrections at non-zero total momentum until the corresponding LL-factors have been derived and now proceed to study one-loop chiral perturbation theory.

## 4.2 $K \rightarrow \pi\pi$ Decays in $\chi$ PT on a Finite Volume

In this section we generalise the previous discussion to  $\chi$ PT. In this case the weak amplitude in general depends on the energy, and hence other finite-volume correction terms appear in the correlation function. To illustrate this, consider the term in eq. (31) which contains  $T_2$ , but now generalized to allow for the energy dependence in the lowest order weak and strong amplitudes (the power corrections arise from singular terms as explained in the previous section):

$$\mathcal{C} = \mathcal{G}^{(0)}(t_K, t_1, t_2, 2E) \left\{ \mathcal{A}(2E) - \frac{1}{L^3} \sum_{\vec{k}} \frac{e^{2(E-w)t_2} - 1}{2(E-w)} \mathcal{A}(2w) \mathcal{N}(w) \right\} \quad (60)$$

where  $\mathcal{A}(2E)$  is the tree level amplitude and  $\mathcal{N}(w)$  is the strong interaction amplitude (in the model of sec. 4.1  $\mathcal{N}(w) = \lambda/4w^2$ ). Separating the contribution with  $w = E$  from the rest, we find

$$\mathcal{C} = \mathcal{G}^{(0)}(t_K, t_1, t_2, 2E) \left\{ \mathcal{A}(2E) \left[ 1 - \frac{\nu \mathcal{N}(E)}{L^3} t_2 \right] - \frac{1}{L^3} \sum_{\vec{k}: w \neq E} \frac{e^{2(E-w)t_2} - 1}{2(E-w)} \mathcal{A}(2w) \mathcal{N}(w) \right\}$$

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<sup>4</sup>We thank Kari Rummukainen for helping us to clarify this point.

$$= \mathcal{G}^{(0)}(t_K, t_1, t_2, W) \left\{ \mathcal{A}(2E) - \frac{1}{L^3} \sum_{\vec{k}: w \neq E} \frac{g(w)}{2(w^2 - E^2)} + \dots \right\} \quad (61)$$

where

$$g(w) = (w + E)\mathcal{A}(2w)\mathcal{N}(w), \quad (62)$$

and the ellipses represent terms which have a different  $t_2$  behaviour (as above, we assume that we can isolate the terms with the required behaviour in  $t_2$ ).

We now use the summation formula (for smooth functions  $F$ ),

$$\frac{1}{L^3} \sum'_{\vec{k}} \frac{F(k^2)}{K^2 - k^2} = \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \frac{F(k^2)}{K^2 - k^2} - \frac{z_K}{4\pi^2 L} F(K^2) + \frac{\nu}{L^3} F'(K^2), \quad (63)$$

where the prime on the summation indicates the omission of the  $\nu$  terms with  $k^2 = K^2$  and  $F'(K^2)$  denotes the derivative of  $F$  with respect to  $k^2$ , evaluated at  $k^2 = K^2$ . Eq. (42) is an example of this summation formula with  $F(k^2) = -1/(4w)$  and  $w^2 = k^2 + m^2$ . Applying eq. (63) to eq. (61) we find

$$\begin{aligned} \mathcal{C} = & \mathcal{G}^{(0)}(t_K, t_1, t_2, W) \left\{ \mathcal{A}(2E) (1 - h(E)) + \frac{\nu \mathcal{N}(E)}{2L^3} \frac{\partial \mathcal{A}}{\partial w} \Big|_{w=E} \right\} \\ & - \mathcal{G}^{(0)}(t_K, t_1, t_2, W) \mathcal{A}(2E) \left\{ E \mathcal{N}(E) \frac{z_n}{4\pi^2 L} - \frac{\nu}{4EL^3} \frac{\partial \{(w + E)\mathcal{N}(w)\}}{\partial w} \Big|_{w=E} \right\}, \end{aligned} \quad (64)$$

where  $-h(E)\mathcal{A}(2E)$  is the one-loop correction to the infinite-volume amplitude with

$$h(E) = \frac{1}{2} \mathcal{P} \int \frac{d^3k}{(2\pi)^3} \frac{g(w)}{w^2 - E^2}. \quad (65)$$

The one-loop results for the  $\Delta I = 3/2$  matrix elements of  $\mathcal{O}_{4,7,8}$  are presented in detail in Appendices B and C and on the web-site [23]. We now look at the finite-volume corrections on the right-hand side of eq. (64). The term proportional to  $\partial \mathcal{A} / \partial w$  has the rôle of shifting the argument of the amplitude from  $2E$  to  $W = 2E + \nu \mathcal{N}(E)/L^3$ . In an earlier study this term had been misinterpreted as a genuine finite-volume correction, whereas it is present to shift the argument of  $\mathcal{A}$ . For this reason the authors of ref. [27] concluded that there are different finite-volume corrections for the two cases when  $m_K = m_\pi$  and  $m_K = 2m_\pi$ , with the two pions at rest. These corrections must instead be the same since they only depend on the final state. Our explicit calculations, presented in sec. 4.3 below, demonstrate that this is indeed the case.

The second line on the right-hand side of eq. (64) is a universal correction which depends only on strong interactions and gives the Lellouch-Lüscher factor. This concludes our demonstration of finite-volume effects in one-loop  $\chi$ PT.

### 4.3 $K^+ \rightarrow \pi^+\pi^0$ Decays in $\chi$ PT at Finite Volume

We now briefly summarize the results obtained in full QCD in finite volume for the physical amplitudes, i.e. for a kaon at rest decaying into two pions with the insertion of a weak operator at zero four-momentum transfer. Of course, our approach requires us to perform lattice simulations (quenched and unquenched), with the insertion of non-zero momentum at the weak operator and the corresponding results will be presented below in sec. 5.2.

For the calculation of the correlation functions needed to determine the  $K^+ \rightarrow \pi^+\pi^0$  decay amplitude at one-loop in  $\chi$ PT we need the following propagators:

$$\begin{aligned} \int d^3x e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \langle \phi(\vec{x}, t_x) \phi^\dagger(\vec{y}, t_y) \rangle &= \frac{e^{-w_{\vec{q}}|t_x-t_y|}}{2w_{\vec{q}}} \\ \int d^3x e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \partial_\mu^y \langle \phi(\vec{x}, t_x) \phi^\dagger(\vec{y}, t_y) \rangle &= \frac{e^{-w_{\vec{q}}|t_x-t_y|}}{2w_{\vec{q}}} (iq_j \delta_{j\mu} + \delta_{\mu 4} w_{\vec{q}} \varepsilon(t_x - t_y)) \end{aligned} \quad (66)$$

where  $w_{\vec{q}}^2 = m_\pi^2 + |\vec{q}|^2$ ,  $\varepsilon$  represents the sign function, and the index  $j$  runs over the spatial components 1, 2 and 3.

For the evaluation of the tadpole diagrams we need the relations:

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} &= \frac{1}{L^3} \sum_{\vec{q}} \frac{1}{2w_{\vec{q}}} \quad \text{and} \\ \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{k^2 + m^2} &= \frac{1}{L^3} \sum_{\vec{q}} \left( \int \frac{dk_4}{2\pi} - \frac{m^2}{2w_{\vec{q}}} \right), \end{aligned} \quad (67)$$

which are valid up to terms which are exponentially small in the volume<sup>5</sup>.

When evaluating the diagrams (b), (c) and (d) we follow similar steps to those in eq. (31). In  $\chi$ PT a number of integrals appear with different numerators. The complete set of these integrals and the corresponding results, obtained upon integration over  $k_4$  and  $l_4$ , are given in Appendix A.

The result for the matrix element of  $\mathcal{O}_4$ , extracted from the correlation function with the kaon at rest and with  $W = m_K$ , where  $W$  is the finite-volume two-pion energy, is given by the expression:

$$\begin{aligned} \langle \pi^+\pi^0 | \mathcal{O}_4 | K^+ \rangle &= -\frac{6\sqrt{2}\alpha^{(27,1)}}{f_K f_\pi^2} \left[ (m_K^2 - m_\pi^2) \left[ 1 + \frac{m_K^2}{(4\pi f)^2} (I_{zf} + F(\sqrt{s}L)) \right] \right. \\ &\quad \left. + \frac{m_K^4}{(4\pi)^2 f^2} (I_a + I_b^E + I_{c+d}) \right] + O(m_{K,\pi}^4) \text{ counterterms}, \end{aligned} \quad (68)$$

where  $\sqrt{s}$  is the invariant mass of the two-pion final state. In this case  $\sqrt{s} = m_K$ , however since the finite-volume corrections depend only on the energy of the two-pion state, we

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<sup>5</sup>The divergent terms proportional to  $\int dk_4/(2\pi)$  cancel when all contributions to the amplitude are summed.

write the answer in terms of  $\sqrt{s}$ . The functions  $I_{zf,a,b,c+d}$  are defined in eqs. (80) and (81) below and  $I_b^E = \text{Re } I_b$ .  $\alpha^{(27,1)}$  is the leading order low-energy constant, and new constants appear at order  $m_{\pi,K}^4$ . As explained in the introduction, in our approach it is envisaged that all the low energy constants up this order will be determined by studying the behaviour of the matrix element as a function of masses and momenta.  $F(\sqrt{s}L)$  is the centre-of-mass finite volume correction

$$F(\sqrt{s}L) = -\frac{2z_n}{\sqrt{s}L} \left(1 - 2\frac{m_\pi^2}{s}\right) - \frac{8\pi^2\nu}{s^{3/2}L^3} \left(1 - 6\frac{m_\pi^2}{s}\right). \quad (69)$$

The shift in the energy of the two-pion state due to the finite volume is given by

$$\Delta W = W - 2E = \frac{\nu \left(1 - 2\frac{m_\pi^2}{s}\right)}{f^2 L^3}. \quad (70)$$

From eq. (70) with the pions at rest, using  $\Delta W = -4\pi a_0^{I=2}/(m_\pi L^3)$  [34], we can obtain the infinite-volume  $I = 2$ ,  $\pi - \pi$  scattering length

$$a_0^{I=2} = -\frac{m_\pi}{8\pi f^2}. \quad (71)$$

We remain now with the limit  $m_K \rightarrow 2m_\pi$ , corresponding to the two pions at rest, and compare our results with those of ref. [27]. In this case  $2z_n \rightarrow 2z_0 = -17.82726584$  and  $\nu = 1$  so that the finite volume correction becomes

$$F(m_K L) = \frac{17.827}{4m_\pi L} + \frac{\pi^2}{2m_\pi^3 L^3}. \quad (72)$$

This result is in agreement with the general LL formula, whereas it disagrees with the result in ref. [27] (see eq.(3.3) in [27]), where the corresponding correction was found to be:

$$\frac{17.827}{4m_\pi L} + \frac{5\pi^2}{2m_\pi^3 L^3}. \quad (73)$$

We have explicitly verified that the discrepancy is due to the fact that in ref. [27] it was not recognized that part of the  $1/L^3$  finite-volume correction is proportional to  $\partial\mathcal{A}/\partial w$  and is absorbed by the shift of the argument of the amplitude,  $\mathcal{A}(2E) \rightarrow \mathcal{A}(W)$  (see the discussion in section 4.2 above). This is also the origin for the erroneous conclusion of these authors that the finite-volume effects are different for the two-pions at rest with  $m_K = 2m_\pi$  and  $m_K = m_\pi$  [26, 27].

For the electroweak penguin operators  $\mathcal{O}_7$  and  $\mathcal{O}_8$  we obtain

$$\begin{aligned} \langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle &= \frac{2\sqrt{2}\gamma^{(8,8)}}{f_K f_\pi^2} \left[ 1 + \frac{m_K^2}{(4\pi f)^2} (I_{zf} + F(\sqrt{s}L)) + \frac{m_K^2}{(4\pi)^2 f^2} (J_a^E + J_b^E + J_{c+d}^E) \right] \\ &\quad + O(m_{K,\pi}^2) \text{ counterterms}, \end{aligned} \quad (74)$$



where in this case  $\sqrt{s} = m_K$ ,  $\gamma^{(8,8)}$  are the leading-order low-energy constants (for  $\mathcal{O}_7$  and  $\mathcal{O}_8$ ) and the functions  $J_i^E$  are the real part of the functions  $J_i$  defined in eq. (87). As expected, the finite volume corrections encoded in  $F(\sqrt{s}L)$  are the same as for  $\mathcal{O}_4$ .

For  $K \rightarrow \pi\pi$  matrix elements for which the two pions have zero total momentum, the finite-volume corrections take the universal form

$$\langle \pi^+ \pi^0 | \mathcal{O}_W | K^+ \rangle = \text{LO} \left( 1 + \frac{s}{(4\pi f)^2} F(\sqrt{s}L) + \dots \right), \quad (75)$$

where LO is the lowest order contribution and the ellipses represent NLO infinite-volume terms. For  $\Delta I = 3/2$  transitions this is also true for the quenched approximation, since the relevant one-loop contributions are as above. For  $\Delta I = 1/2$  transitions this is not the case [7, 8]. In the following section and in appendices B and C we present the results also for unphysical kinematics at one-loop order in  $\chi$ PT without explicitly exhibiting the finite-volume effects. As discussed in sec. 4.1.2 the general theory of finite-volume corrections to matrix elements with the two-pions at non-zero total momentum has not yet been developed. We are currently investigating this question, and will present the perturbative finite-volume corrections within the general framework.

## 5 One-loop corrections in $\chi$ PT

In the preceding section, we have demonstrated that, at one-loop order in chiral perturbation theory, the finite-volume corrections to the amplitudes extracted from correlators in which the pions are annihilated at the same time or at different times are exactly the same and correspond to the general LL formula. As explicit examples, we have given the one-loop amplitudes for the  $\Delta I = 3/2$  operators,  $\mathcal{O}_4$ ,  $\mathcal{O}_7$  and  $\mathcal{O}_8$ , in the physical case (i.e. with no momentum insertion at the weak operator), for a kaon at rest decaying into two pions in full QCD. In order to implement the strategy described in sec. 2 however, we also need the amplitudes in other kinematical situations for which there is an insertion of momentum at the weak operator. In this section we discuss the evaluation of the one-loop corrections in the chiral expansion, both in full QCD and in the quenched approximation for arbitrary meson masses and momenta. The general results are very long and complicated: for full QCD they are presented in appendix B.2 (and also on the web site [23]) and for the quenched approximation on the web site [23]. In presenting our results we do not exhibit explicitly all the NLO counterterms, i.e. the terms proportional to the  $\beta_i$ 's in sec. 2.1 and the  $\delta_i$ 's in sec. 2.2. Of course these terms must be included in any comparison of lattice data with  $\chi$ PT.

We now consider separately the following special cases, which are useful for current numerical simulations:

1. *physical kinematics* with the kaon at rest and with the mesons having arbitrary masses  $m_K$  and  $m_\pi$ . By “physical” here we mean that there is no insertion of momentum at

the weak vertex, so that the two pions each move with momentum  $k = \sqrt{m_K^2/4 - m_\pi^2}$ . The results for this case are presented fully in sec. 5.1 below;

2. *SPQR kinematics* with the kaon and one of the final-state pions at rest and the other one moving with arbitrary energy  $E_\pi$ . The final state is symmetrized to ensure that it has  $I = 2$ . The results are presented in appendix B.1 for full QCD and appendix C.1 for the quenched theory;
3. choices of the quark masses such that  $m_K = m_\pi$  and  $m_K = 2m_\pi$ , in both cases with all the particles at rest. These are presented in sections 5.2.1 – 5.2.4.

Below we follow the notation introduced in the preceding sections and assume  $SU(2)$  isospin symmetry ( $m_s \neq m_u = m_d$ ), together with the following relation for the  $\eta$  mass:

$$m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2. \quad (76)$$

Ultra-violet divergences are regulated using dimensional regularization and, following the conventions of ref. [20], renormalized by performing the subtraction of the term

$$\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + 1. \quad (77)$$

## 5.1 Matrix Elements for the Decay $K^+ \rightarrow \pi^+\pi^0$ with Physical Kinematics

In this section we describe the calculation of the matrix elements for the decay  $K^+ \rightarrow \pi^+\pi^0$ , in both full QCD (sec. 5.1.1) and the quenched theory (sec. 5.1.2). We underline again that by physical we mean that there is no insertion of momentum at the weak operator. The masses of the mesons are arbitrary.

### 5.1.1 Calculations in Full QCD

We start with the discussion in Minkowski space. At lowest order the matrix element of  $\mathcal{O}_4$ , obtained using the chiral operator defined in eq. (12), is given by

$$\langle \pi^+\pi^0 | \mathcal{O}_4 | K^+ \rangle = -\frac{6\sqrt{2}\alpha^{(27,1)}}{f_K f_\pi^2} (m_K^2 - m_\pi^2). \quad (78)$$

At one-loop order we have to compute the diagrams of fig. 1. In the tadpole diagram (a), all pseudoscalar mesons propagate in the loop. In the remaining diagrams, because of the symmetries of the Lagrangian, it is the combinations indicated in tab. 1 which contribute.

Our results for this case agree with those of ref. [37] and with those of Golterman and Leung [26], except for the finite terms which depend on the regularization/renormalization

Diagram	Particles
(b)	$\pi^+\pi^0$
(c)	$K^+\pi^0, K^0\pi^+, K^+\eta$
(d)	$K^0\pi^0, K^+\pi^-, K^0\eta$

Table 1: *Combinations of mesons looping in the different diagrams.*

(and the sign of the imaginary part of diagram (b) in fig. 1). The matrix element is given by

$$\langle \pi^+\pi^0 | \mathcal{O}_4 | K^+ \rangle = -\frac{6\sqrt{2}\alpha^{(27,1)}}{f_K f_\pi^2} \left[ (m_K^2 - m_\pi^2) \left[ 1 + \frac{m_K^2}{(4\pi f)^2} I_{zf} \right] + \frac{m_K^4}{(4\pi)^2 f^2} (I_a + I_b + I_{c+d}) \right], \quad (79)$$

plus the  $O(p^4)$  counterterms (i.e. the terms proportional to the  $\beta_i$ 's in eq. (6)). The integrals in eq. (79) are given by

$$I_{zf} = \frac{7}{3} \log(y) - \left[ \frac{2}{3} - \frac{1}{6}y \right] \log\left(\frac{m_\eta^2}{m_\pi^2}\right) - 3[1+y] \log\left(\frac{m_\pi^2}{\mu^2}\right), \quad (80)$$

$$I_a = \left( \frac{20}{3}y^2 - y - \frac{17}{3} \right) \log\left(\frac{m_\pi^2}{\mu^2}\right) - \left( \frac{1}{6}y^2 - y + \frac{4}{3} \right) \log\left(\frac{m_\eta^2}{m_\pi^2}\right) + \left( \frac{13}{3} - 4y \right) \log(y),$$

$$I_b = \left( 1 - \frac{13}{3}y + \frac{14}{3}y^2 \right) \log\left(\frac{m_\pi^2}{\mu^2}\right) + (1 - 3y + 2y^2) \left[ A(y) - i\pi\sqrt{1-4y} - 1 \right],$$

$$\begin{aligned} I_{c+d} &= \frac{1}{3} (5 - 8y - y^2) \log\left(\frac{m_\pi^2}{\mu^2}\right) + \left( \frac{10}{3} - \frac{4}{3y} - 2y \right) - A(y) \left( 1 + \frac{5}{8y^2} - \frac{13}{8y} \right) \\ &- \frac{1}{72} \left( \frac{1}{y^2} - \frac{1}{y} \right) B(y) + \left( \frac{1}{3}y^2 - \frac{19}{12}y + \frac{19}{18} - \frac{23}{72y} + \frac{1}{72y^2} \right) \log\left(\frac{m_\eta^2}{m_\pi^2}\right) \\ &- \left( \frac{11}{18y^2} - \frac{23}{9y} + \frac{65}{18} - \frac{7}{3}y \right) \log(y), \end{aligned} \quad (81)$$

where  $y = m_\pi^2/m_K^2$ ,  $\mu$  is the renormalization scale,

$$\begin{aligned} A(y) &= \sqrt{1-4y} \log\left(\frac{1+\sqrt{1-4y}}{1-\sqrt{1-4y}}\right) \quad \text{and,} \\ B(y) &= \sqrt{1-44y+16y^2} \log\left(\frac{7-4y+\sqrt{1-44y+16y^2}}{7-4y-\sqrt{1-44y+16y^2}}\right) \\ &= -2\sqrt{44y-1-16y^2} \arctan\left(\frac{\sqrt{44y-1-16y^2}}{7-4y}\right). \end{aligned} \quad (82)$$

In deriving the above expressions we have used eq. (76) and the fact that, with physical meson masses, the argument of the square root in eq. (82),  $1-44y+16y^2$ , is negative.

From eq. (79), we can extract the phase of the amplitude,  $\delta$ , defined by

$$\mathcal{A} = |\mathcal{A}|e^{i\delta}. \quad (83)$$

We find

$$\delta(k = \sqrt{m_K^2/4 - m_\pi^2}) = -\frac{m_K^2}{16\pi f^2}(1-2y)\sqrt{1-4y}. \quad (84)$$

Rewriting the right-hand side of eq. (84) in terms of  $k$  we recover the expression for the  $I = 2$  strong interaction  $\pi$ - $\pi$  scattering phase,

$$\delta(k) = -\frac{(2k^2 + m_\pi^2)}{8\pi f^2} \frac{k}{E}, \quad (85)$$

as required by Watson's theorem. The  $I = 2$  scattering length is given by

$$a_0^{I=2} = \lim_{k \rightarrow 0} \frac{\delta(k)}{k} = -\frac{m_\pi}{8\pi f^2}. \quad (86)$$

For the electroweak penguin operators  $\mathcal{O}_7$  and  $\mathcal{O}_8$  the corresponding formulae are:

$$\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = \frac{2\sqrt{2}\gamma^{(8,8)}}{f_K f_\pi^2} \left[ 1 + \frac{m_K^2}{(4\pi f)^2} I_{zf} + \frac{m_K^2}{(4\pi)^2 f^2} (J_a + J_b + J_{c+d}) \right], \quad (87)$$

where  $I_{zf}$  is given in eq. (80) plus the  $O(p^2)$  counterterms (i.e. the terms proportional to the  $\delta'_i$ s in eq. (9)). The integrals appearing on the right-hand side of eq. (87) are

$$\begin{aligned} J_a &= -\left(\frac{10}{3}y + \frac{11}{3}\right) \log\left(\frac{m_\pi^2}{\mu^2}\right) - \left(\frac{2}{3} - \frac{1}{6}y\right) \log\left(\frac{m_\eta^2}{m_\pi^2}\right) + 3\log(y), \\ J_b &= \left(1 - \frac{8}{3}y\right) \log\left(\frac{m_\pi^2}{\mu^2}\right) + (1-2y) \left[A(y) - i\pi\sqrt{1-4y} - 1\right], \\ J_{c+d} &= \left(\frac{13}{6} - y\right) \log\left(\frac{m_\pi^2}{\mu^2}\right) + \left(2 - \frac{3}{2y}\right) - A(y) \left(\frac{5}{8y^2} - \frac{1}{y}\right) \\ &\quad - \frac{1}{24y^2} B(y) + \left(-\frac{1}{3}y + \frac{19}{12} - \frac{11}{12y} + \frac{1}{24y^2}\right) \log\left(\frac{m_\eta^2}{m_\pi^2}\right) + \\ &\quad - \left(\frac{7}{12y^2} - \frac{4}{3y} + \frac{4}{3}\right) \log(y). \end{aligned} \quad (88)$$

$A(y)$  and  $B(y)$  have been defined in eq. (82). The strong interaction phase is of course, the same as for  $\mathcal{O}_4$ , since it only depends on the isospin of the two-pion state. For this case our results agree with those of ref. [38]. In ref. [24] the one-loop contributions to the matrix elements are presented numerically and we agree with these results.

**Calculations in Euclidean Space** The calculation of the Euclidean matrix elements in terms of the low-energy constants at NLO in  $\chi$ PT proceeds along similar lines. In the Euclidean theory the imaginary part is zero and the arguments of the logarithms correspond to the absolute values of the arguments in the Minkowski case. For a generic (Euclidean) kaon momentum  $p_K$  (which is also therefore the momentum of the two-pion state) the logarithms take the form

$$\log \left( \frac{\left| 1 + \sqrt{1 + 4 \frac{m_\pi^2}{p_K^2}} \right|}{\left| 1 - \sqrt{1 + 4 \frac{m_\pi^2}{p_K^2}} \right|} \right). \quad (89)$$

Setting  $p_K^2 \rightarrow -m_K^2$  we recover the Minkowski amplitude without the phase

$$\langle \pi^+ \pi^0 | \mathcal{O}_4^E | K^+ \rangle_E = -\frac{6\sqrt{2}\alpha^{(27,1)}}{f_K f_\pi^2} \left[ (m_K^2 - m_\pi^2) \left[ 1 + \frac{m_K^2}{(4\pi f)^2} I_{zf} \right] + \frac{m_K^4}{(4\pi)^2 f^2} (I_a + I_b^E + I_{c+d}) \right]. \quad (90)$$

The functions  $I_x$  are those defined in eqs. (80) and (81) and  $I_b^E = \text{Re } I_b$ .

### 5.1.2 Calculations in Quenched QCD

In this section we present the results for the matrix elements in quenched QCD. As above we do not exhibit the NLO counterterms given in eqs. (6) and (9) explicitly, but of course they must be included in any comparison with lattice data.

The matrix element of the operator  $\mathcal{O}_4$  takes the form:

$$\langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle = -\frac{6\sqrt{2}\alpha^{(27,1)}}{f_\pi^2 f_K} \left[ (m_K^2 - m_\pi^2) \left[ 1 + \frac{m_K^2}{(4\pi f)^2} I_{zf}^q \right] + \frac{m_K^4}{(4\pi)^2 f_q^2} (I_a^q + I_b^q + I_{c+d}^q) \right], \quad (91)$$

where the integrals are given by

$$\begin{aligned} I_{zf}^q &= -\frac{2m_0^2}{9m_K^2} \left[ 1 + \frac{1}{2(1-y)} \log \left( \frac{y}{2-y} \right) \right] + \frac{2\alpha}{9} \left[ 1 + \frac{y(2-y)}{2(1-y)} \log \left( \frac{y}{2-y} \right) \right] \\ I_a^q &= -\left( \frac{2}{3} + y - \frac{5}{3}y^2 \right) \log \left( \frac{m_\pi^2}{\mu^2} \right) + \frac{2}{3} (1-y) \log(y) + \\ &\quad + \frac{m_0^2}{9m_K^2} \left[ -2(2-y) + \frac{2-y}{1-y} \log \left( \frac{2-y}{y} \right) \right] + \\ &\quad + \frac{\alpha}{9} \left[ 2(2-y) - \frac{y(2-y)^2}{(1-y)} \log \left( \frac{2-y}{y} \right) \right], \\ I_b^q &= \left( 1 - \frac{13}{3}y + \frac{14}{3}y^2 \right) \log \left( \frac{m_\pi^2}{\mu^2} \right) + (1-3y+2y^2) \left[ A(y) - i\pi\sqrt{1-4y} - 1 \right], \\ I_{c+d}^q &= -\frac{5+y+2y^2}{6} \log \left( \frac{m_\pi^2}{\mu^2} \right) + \left( \frac{3}{2} - \frac{1}{2y} - y \right) \end{aligned} \quad (92)$$

$$\begin{aligned}
& - \left( \frac{1}{2} + \frac{1}{4y^2} - \frac{3}{4y} \right) A(y) + \left( -\frac{2}{3} - \frac{1}{4y^2} + \frac{5}{4y} + \frac{2}{3}y \right) \log(y) \\
& + \frac{m_0^2}{m_K^2} \left[ \left( \frac{1}{6y^2} - \frac{1}{3y} \right) A(y) - \frac{1-8y+18y^2-8y^3}{6y^2(1-8y+4y^2)} C(y) - \frac{4}{9}y \right. \\
& \quad \left. + \frac{3-15y+24y^2-8y^3}{18y^2(1-y)} \log\left(\frac{2-y}{y}\right) + \left( 1 + \frac{1}{3y^2} - \frac{4}{3y} \right) \log(y) \right] + \\
& + \alpha \left[ \left( \frac{2}{3} + \frac{1}{12y^2} - \frac{7}{12y} \right) A(y) + \left( \frac{5}{12y^2} - \frac{15}{4y} + \frac{59}{6} - 8y + 2y^2 \right) \frac{C(y)}{1-8y+4y^2} + \right. \\
& \quad \left. + \frac{1}{1-y} \left( -\frac{65}{12} - \frac{5}{12y^2} + \frac{5}{2y} + \frac{9}{2}y - \frac{29}{18}y^2 + \frac{2}{9}y^3 \right) \log\left(\frac{2-y}{y}\right) \right. \\
& \quad \left. - \frac{(1-y)^2(1-2y)}{3y^2} \log(y) + \left( \frac{1}{y} - 4 + \frac{31}{9}y \right) \right]. \tag{93}
\end{aligned}$$

with

$$C(y) = \sqrt{1-8y+4y^2} \log\left(\frac{3-2y+\sqrt{1-8y+4y^2}}{3-2y-\sqrt{1-8y+4y^2}}\right). \tag{94}$$

For the EWP operators the corresponding results are

$$\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = \frac{2\sqrt{2}\gamma^{(8,8)}}{f_\pi^2 f_K} \left[ 1 + \frac{m_K^2}{(4\pi f)^2} I_{zf}^q + \frac{m_K^2}{(4\pi)^2 f^2} (J_a^q + J_b^q + J_{c+d}^q) \right] \tag{95}$$

where  $I_{zf}^q$  is defined in eq. (92) and the remaining integrals are given by

$$\begin{aligned}
J_a^q &= -\frac{1-y}{3} \log\left(\frac{m_\pi^2}{\mu^2}\right) + \frac{1}{3} \log(y) + \frac{m_0^2}{9m_K^2} \left[ -2 + \frac{1}{1-y} \log\left(\frac{2-y}{y}\right) \right] + \\
& \quad + \frac{\alpha}{9} \left[ 2 - \frac{y(2-y)}{1-y} \log\left(\frac{2-y}{y}\right) \right], \\
J_b^q &= \left( 1 - \frac{8}{3}y \right) \log\left(\frac{m_\pi^2}{\mu^2}\right) + (1-2y) \left[ A(y) - i\pi\sqrt{1-4y} - 1 \right], \\
J_{c+d}^q &= \left( \frac{4}{3} + \frac{y}{3} \right) \log\left(\frac{m_\pi^2}{\mu^2}\right) - 1 - \frac{1}{2y} A(y) - \left( \frac{1}{3} + \frac{1}{2y} \right) \log(y) \\
& \quad + \frac{m_0^2}{m_K^2} \left[ -\frac{A(y)}{12y^2(1-y)} + \frac{-1+6y-16y^2+8y^3}{12y^2(1-y)(1-8y+4y^2)} C(y) \right. \\
& \quad \left. + \frac{4}{9} - \frac{1}{3y} + \frac{3-6y-2y^2}{36y^2(1-y)} \log\left(\frac{2-y}{y}\right) \right] + \\
& \quad + \alpha \left[ -\left( \frac{1-6y+4y^2}{12y^2(1-y)} \right) A(y) + \left( \frac{1}{3y} - \frac{4}{9} \right) \right. \\
& \quad \left. + \left( \frac{17}{3} + \frac{1}{4y^2} - \frac{2}{y} - 5y + \frac{4}{3}y^2 \right) \frac{C(y)}{(1-y)(1-8y+4y^2)} + \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{9-36y+42y^2-28y^3+8y^4}{36y^2(1-y)}\log\left(\frac{2-y}{y}\right) \\
& -\frac{(1-y)(1-3y)}{3y^2}\log(y)\Big].
\end{aligned} \tag{96}$$

## 5.2 $K^+ \rightarrow \pi^+\pi^0$ Matrix Elements with the SPQR Kinematics

As explained in the Introduction and section 2, our aim is to extract the physical  $K \rightarrow \pi\pi$  matrix elements from simulations at unphysical kinematics using NLO  $\chi$ PT. This requires the calculation of the one-loop chiral corrections to the  $K^+ \rightarrow \pi^+\pi^0$  amplitudes with the insertion of weak operators carrying four-momentum. In this section we consider the matrix elements for the SPQR kinematics, in which the kaon and one of the final-state pions are at rest and the second pion has an energy  $E_\pi$ . We have seen in section 2 that the computation of these matrix elements are sufficient to determine all the required low-energy constants at NLO in the chiral expansion.

In order to ensure that the final state has  $I = 2$ , it is necessary to symmetrise over the momenta of the two pions. Note also that the only diagram which contributes to the rescattering phase is diagram (b).

The results for full QCD are presented in appendix B.1 and for the quenched theory in appendix C.1. Here we present the results for the two choices of quark masses corresponding to  $m_K = 2m_\pi$  or  $m_K = m_\pi$  in each case with both pions at rest.

### 5.2.1 Matrix Elements with $m_K = 2m_\pi$ and both Pions at Rest in Full QCD

Consider first the operator  $\mathcal{O}_4$ . In the limit where both pions are at rest ( $\omega = 1$ ) and  $m_K = 2m_\pi$  ( $z = 1/2$ ) from eqs.(108) – (111) we can readily recover the corresponding result ( $y = 1/4$ ) in eqs. (79) – (81):

$$\begin{aligned}
\langle \pi^+\pi^0 | \mathcal{O}_4 | K^+ \rangle = & -\frac{18\sqrt{2}\alpha^{(27,1)}m_\pi^2}{f_K f_\pi^2} \left[ 1 + \frac{m_\pi^2}{4\pi^2 f^2} \left( -\frac{19}{2} \log\left(\frac{m_\pi^2}{\mu^2}\right) - \frac{23}{6} \right. \right. \\
& \left. \left. + \frac{4}{3} \arctan \frac{1}{2} - \frac{10}{3} \log(4) - \frac{31}{12} \log(5) \right) \right].
\end{aligned} \tag{97}$$

The corresponding result for the EWP operators is:

$$\langle \pi^+\pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = \frac{2\sqrt{2}}{f_K f_\pi^2} \gamma^{(8,8)} \left[ 1 + \frac{m_\pi^2}{4\pi^2 f^2} \left( -6 \log\left(\frac{m_\pi^2}{\mu^2}\right) - \frac{9}{2} + 4 \arctan \frac{1}{2} - \frac{11}{4} \log(5) \right) \right]. \tag{98}$$

### 5.2.2 Matrix Elements in the $SU(3)$ limit, $m_K = m_\pi$ , and with both Pions at Rest in Full QCD

In the  $SU(3)$  limit  $m_s = m_d = m_u$ , for which  $m_K = m_\pi$ , with the two pions both at rest eqs.(108) – (111) reduce to

$$\langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle = -\frac{12\sqrt{2}\alpha^{(27,1)}m_\pi^2}{f_K f_\pi^2} \left[ 1 + \frac{m_\pi^2}{(4\pi)^2 f^2} \left( -15 \log\left(\frac{m_\pi^2}{\mu^2}\right) - 3 \right) \right], \quad (99)$$

in agreement with the result given in eq. (82) of ref. [26], up to scheme-dependent finite terms.

For  $\mathcal{O}_{7,8}$  the corresponding result is:

$$\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = \frac{2\sqrt{2}}{f_K f_\pi^2} \gamma^{(8,8)} \left[ 1 + \frac{m_\pi^2}{(4\pi f)^2} \left( -8 \log\left(\frac{m_\pi^2}{\mu^2}\right) \right) \right]. \quad (100)$$

### 5.2.3 Matrix Elements with $m_K = 2m_\pi$ and both Pions at Rest in Quenched QCD

The results for the matrix elements with the SPQR kinematics in the quenched theory are presented in detail in appendix C.1. In this and the following subsection we present the results for the special cases  $m_K = 2m_\pi$  and  $m_K = m_\pi$  respectively, in each case with both the pions at rest.

For the matrix element of  $\mathcal{O}_4$  in the quenched theory with  $m_K = 2m_\pi$  We find

$$\begin{aligned} \langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle = & -\frac{18\sqrt{2}\alpha^{(27,1)}m_\pi^2}{f_\pi^2 f_K} \left\{ 1 + \frac{m_\pi^2}{4\pi^2 f^2} \left[ -2 \log\left(\frac{m_\pi^2}{\mu^2}\right) - \frac{3}{2} - \frac{4}{3} \log(4) + \right. \right. \\ & + \frac{m_0^2}{4m_\pi^2} \left( -\frac{8}{9} + \frac{40}{27} \log(7) - \frac{4}{3} \log(4) \right) + \\ & \left. \left. + \alpha \left( \frac{17}{9} + 2 \log(4) - \frac{113}{54} \log(7) - \frac{2}{\sqrt{3}} \arctan \frac{\sqrt{3}}{5} \right) \right] \right\}. \quad (101) \end{aligned}$$

We are grateful to M. Golterman for revisiting this calculation and confirming his agreement with the expression in eq. (101) [39] (the result in eq. (101) is in disagreement with that presented in eqs. (3.1) and (3.2) of ref. [27]). The corresponding result for the EWP operators is:

$$\begin{aligned} \langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = & \frac{2\sqrt{2}}{f_\pi^2 f_K} \gamma^{(8,8)} \left\{ 1 + \frac{m_\pi^2}{4\pi^2 f^2} \left[ \frac{3}{2} \log\left(\frac{m_\pi^2}{\mu^2}\right) - \frac{3}{2} + 2 \log(4) + \right. \right. \\ & + \frac{m_0^2}{4m_\pi^2} \left( -\frac{4}{3} + \frac{10}{9} \log(7) - \frac{8\sqrt{3}}{9} \arctan \frac{\sqrt{3}}{5} \right) + \\ & \left. \left. + \alpha \left( \frac{4}{3} + \log(4) - \frac{13}{9} \log(7) + \frac{8\sqrt{3}}{9} \arctan \frac{\sqrt{3}}{5} \right) \right] \right\}. \quad (102) \end{aligned}$$



### 5.2.4 Matrix Elements in the $SU(3)$ limit, $m_K = m_\pi$ , and with both Pions at Rest in Quenched QCD

Finally we present the results with  $m_K = m_\pi$  in the quenched theory with the pions at rest. In this case for the matrix element of  $\mathcal{O}_4$  we do agree with the result in ref. [26] finding

$$\langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle = -\frac{12\sqrt{2}\alpha^{(27,1)}m_\pi^2}{f_K f_\pi^2} \left[ 1 + \frac{m_\pi^2}{(4\pi)^2 f^2} \left( -3 \log \left( \frac{m_\pi^2}{\mu^2} \right) - 3 \right) \right]. \quad (103)$$

For the matrix elements of  $\mathcal{O}_{7,8}$  the corresponding result is:

$$\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = \frac{2\sqrt{2}\gamma^{(8,8)}}{f_\pi^2 f_K} \left\{ 1 + \frac{m_\pi^2}{(4\pi f)^2} 4 \log \left( \frac{m_\pi^2}{\mu^2} \right) \right\}. \quad (104)$$

## 6 Conclusions

The precise evaluation of  $K \rightarrow \pi\pi$  matrix elements in lattice simulations is a major long-term challenge. In this paper we have explained our strategy for the reduction of an important source of systematic error in current studies, viz. the use of  $\chi$ PT at leading order in which final state interactions are neglected. We propose to use lattice computations of  $K \rightarrow \pi\pi$  matrix elements with meson masses and momenta which are unphysical (but accessible to lattice simulations) to determine all the low energy constants necessary to obtain the physical amplitudes at NLO in  $\chi$ PT. We have presented all the necessary ingredients to implement this strategy for  $\Delta I = 3/2$  transitions. In particular we have evaluated all the chiral logarithms at one-loop order for arbitrary masses and momenta.

We have shown in sec. 2 that it is possible to determine all the necessary low-energy constants from matrix elements with the SPQR kinematics, i.e. with the kaon and one of the pions at rest and with the second pion having an arbitrary momentum. We present the results for this choice of kinematics separately, but stress that of course this is not the only choice for the determination of the low-energy constants (for this reason we present the very lengthy results for general kinematics).

We incorporate into our work the recent progress in understanding finite-volume effects in  $K \rightarrow \pi\pi$  decays. For  $\Delta I = 3/2$  transitions we find that these effects are given by the Lellouch-Lüscher factor (for zero total three-momentum), even in the quenched approximation. This is not the case for  $\Delta I = 1/2$  transitions, where the absence of unitarity in the quenched theory leads to major subtleties and difficulties, such as the dependence of the final state interactions on the choice of  $\Delta I = 1/2$  weak operator and non-standard behaviour of the correlation functions with time and volume. This is the subject of a paper under preparation. We are also investigating the generalization of the LL-factor to the case in which the total three-momentum of the two-pions is not equal to zero.

In ref. [17] a particular implementation of our proposed strategy is discussed. The conclusion of ref. [17] is that it is possible to determine all the low-energy constants necessary to

obtain the physical matrix elements for (27,1) and (8,1) operators from the knowledge of:

- i)  $K \rightarrow \pi\pi$  matrix elements with both the pions at rest for  $m_K = 2m_\pi$  and  $m_K = m_\pi$ ;
- ii)  $K \rightarrow \pi$  matrix elements at non-zero momentum;
- iii) the matrix elements for  $K^0\text{--}\bar{K}^0$  mixing.

We agree with this conclusion for the  $\Delta I = 3/2$ , (27,1) operators studied above. Notice however that our complete basis for the  $\chi$ PT weak operators at order  $p^4$  differs in the replacement of the operator  $O_{20}$  with our operator  $O_{22}$ <sup>6</sup>. The two choices turn out to be equivalent also in the most general case with unphysical kinematics (i.e. with an injection of momentum by the weak operator).

For  $\Delta I = 3/2$  decays our strategy can also be implemented for partially quenched lattice QCD, and we will present the corresponding one-loop results in a future paper. (For the case  $m_K = m_\pi$ , with all the mesons at rest and with degenerate sea-quarks, the matrix element of  $\mathcal{O}_4$  has been evaluated in partially quenched QCD in ref. [28].)

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## A Integrals which appear in the evaluation of the correlation functions

In this appendix we list the energy integrals which appear in the evaluation of the contributions of diagrams (b), (c) and (d) of Fig.1 and present the corresponding results. We write each integral in the generic form:

$$I(\mathcal{N}) = \int \frac{dk_4 dl_4}{(2\pi)^2} \frac{\mathcal{N} e^{-i(k_4+l_4)t}}{(k_4^2 + w_1^2)(l_4^2 + w_2^2)} \quad (105)$$

---

<sup>6</sup>The numbering of operators follow ref. [22] in both cases.

where  $\mathcal{N}$  is a polynomial in  $k_4$  and  $l_4$ ,  $w_1^2 = m_1^2 + \vec{k}^2$  and  $w_2^2 = m_2^2 + \vec{l}^2$ .

Using the identity

$$\int \frac{dx}{2\pi} \frac{x^n e^{-ixt}}{x^2 + w^2} = \left(i \frac{\partial}{\partial t}\right)^n \int \frac{dx}{2\pi} \frac{e^{-ixt}}{x^2 + w^2} \quad (106)$$

we obtain

$$\begin{aligned} I(1) &= \frac{1}{(2w_1)(2w_2)} e^{-(w_1+w_2)|t|} \\ I(k_4) &= -i \varepsilon(t) \frac{1}{4w_2} e^{-(w_1+w_2)|t|} \\ I(l_4) &= -i \varepsilon(t) \frac{1}{4w_1} e^{-(w_1+w_2)|t|} \\ I(k_4 l_4) &= -\frac{1}{4} e^{-(w_1+w_2)|t|} \\ I(k_4^2) &= \frac{\delta(t)}{2w_2} - \frac{w_1}{4w_2} e^{-(w_1+w_2)|t|} \\ I(l_4^2) &= \frac{\delta(t)}{2w_1} - \frac{w_2}{4w_1} e^{-(w_1+w_2)|t|} \\ I(k_4^2 l_4) &= i \varepsilon(t) \frac{w_1}{4} e^{-(w_1+w_2)|t|} \\ I(k_4 l_4^2) &= i \varepsilon(t) \frac{w_2}{4} e^{-(w_1+w_2)|t|} \\ I(k_4^2 l_4^2) &= \delta(t) \int \frac{dk_4}{2\pi} - \frac{w_1 + w_2}{2} \delta(t) + \frac{w_1 w_2}{4} e^{-(w_1+w_2)|t|} \end{aligned}$$

where  $\varepsilon$  represents the sign function.

## B Results in Full QCD

In this appendix we present the results for matrix elements in full QCD. The results for general kinematics, i.e. for arbitrary quark masses and momenta, with the two pions in an I=2 state, are given in subsection B.2. We start however, with subsection B.1 which contains the results for SPQR kinematics.

The integral  $I_{zf}$ , which corresponds to the one-loop contributions from the renormalization of the mesonic wave-functions and from the replacement of the factor  $1/f^3$  which appears at lowest-order in the chiral expansion, by  $1/(f_K f_\pi^2)$  is given by ( $y = m_\pi^2/m_K^2$ ):

$$I_{zf} = -3(1+y) \log\left(\frac{m_\pi^2}{\mu^2}\right) + \frac{7}{3} \log(y) + \frac{1}{6} (y-4) \log\left(\frac{m_\eta^2}{m_\pi^2}\right). \quad (107)$$

The results are given in terms of the variables,  $y = m_\pi^2/m_K^2$ ,  $z = m_\pi/m_K$  and  $\omega = E_\pi/m_\pi$ , where  $E_\pi$  is the energy of the pion whose momentum is (in general) not equal to zero.

## B.1 $K^+ \rightarrow \pi^+\pi^0$ Decays with SPQR Kinematics in Full QCD

### B.1.1 Results for the matrix elements of $\mathcal{O}_4$ in full QCD with SPQR kinematics

The matrix elements of  $\mathcal{O}_4$  with SPQR kinematics is given by:

$$\begin{aligned} \langle \pi^+\pi^0 | \mathcal{O}_4 | K^+ \rangle &= \frac{-6\sqrt{2} m_K^2}{f_K f_\pi^2} \alpha^{(27,1)} \left\{ \frac{z(1+\omega+2z)}{2\omega} \left( 1 + \frac{m_K^2}{16\pi^2 f^2} (I_{zf}) \right) \right. \\ &\quad \left. + \frac{m_K^2}{16\pi^2 f^2} (I_a + I_b + I_{c+d}) \right\}, \end{aligned} \quad (108)$$

where  $I_{zf}$  is given in eq. (107) and  $I_{a,b,c+d}$  are as follows:

$$\begin{aligned} I_a &= \left( \frac{1}{12\omega} \right) \left\{ 4 \left( 12z^2 - \omega + 8z(1+\omega) \right) \log(z^2) \right. \\ &\quad - \left( 56z^2 - 4\omega + 44z(1+\omega) + 26z^3(1+\omega) + 4z^4(21+\omega) \right) \log\left( \frac{m_\pi^2}{\mu^2} \right) \\ &\quad \left. + z \left( -4 + z^2 \right) (3 + 2z + 3\omega) \log\left( \frac{m_\pi^2}{\mu^2} \right) \right\}, \end{aligned} \quad (109)$$

$$\begin{aligned} I_b &= \left( \frac{-z^3}{3\omega^2} \right) \left\{ 3(1+2z+\omega) + 3\sqrt{\frac{1-\omega}{1+\omega}} (1+2z+\omega) \log\left( \frac{-1+\sqrt{\frac{1-\omega}{1+\omega}}}{1+\sqrt{\frac{1-\omega}{1+\omega}}} \right) \right. \\ &\quad \left. + [-3 - 2\omega + \omega^2 - 2z(3 - 3\omega + \omega^2)] \log\left( \frac{m_\pi^2}{\mu^2} \right) \right\}, \end{aligned} \quad (110)$$

and

$$I_{c+d} = I_{c+d}^1 + I_{c+d}^2 \log\left( \frac{m_\pi^2}{\mu^2} \right) + I_{c+d}^3 \log(z^2) + I_{c+d}^4 \log\left( \frac{m_\pi^2}{\mu^2} \right) + I_{c+d}^5. \quad (111)$$

The separate components in eq. (111) are as follows:

$$\begin{aligned} I_{c+d}^1 &= \left( \frac{z^2}{9\omega^2(-2z+\omega+z^2\omega)^2} \right) \left\{ 2z^5\omega^2(-16-15\omega+\omega^2) + 3\omega^2(3-26\omega+3\omega^2) \right. \\ &\quad + 2z\omega(-18+140\omega-33\omega^2+\omega^3) + z^4\omega(43+129\omega-\omega^2+9\omega^3) \\ &\quad \left. + z^2(36-269\omega+174\omega^2-79\omega^3+18\omega^4) - 2z^3(11+78\omega-107\omega^2+48\omega^3+30\omega^4) \right\}, \end{aligned} \quad (112)$$

$$\begin{aligned} I_{c+d}^2 &= \left( \frac{1}{6\omega^2} \right) \left\{ -2\omega^2 - 2z^4\omega^2 + 4z\omega(1+\omega) + z^2(-6+28\omega-6\omega^2) \right. \\ &\quad \left. - z^3(-1+\omega^2) \right\}, \end{aligned} \quad (113)$$

$$\begin{aligned}
I_{c+d}^3 = & \left( \frac{1}{54 (-1+z) \omega^2 (-2z+\omega+z^2\omega)^3} \right) \times \\
& \left\{ -18\omega^5 + 18z\omega^4 (7+2\omega) + 4z^{10}\omega^3 (-7-3\omega+4\omega^2) \right. \\
& -6z^2\omega^3 (63-11\omega+21\omega^2) - 2z^9\omega^2 (-135-50\omega+12\omega^2+8\omega^3) \\
& +5z^3\omega^2 (126-179\omega+129\omega^2+8\omega^3) + 3z^8\omega (-90-156\omega-265\omega^2+71\omega^3) \\
& +z^4\omega (-576+1926\omega-1373\omega^2+75\omega^3-202\omega^4) \\
& +z^7 (36+582\omega+1116\omega^2-357\omega^3+1047\omega^4-36\omega^5) \\
& +6z^5 (36-179\omega+222\omega^2-210\omega^3+197\omega^4+56\omega^5) \\
& \left. -6z^6 (42+85\omega-195\omega^2+309\omega^3+29\omega^4+77\omega^5) \right\}, \tag{114}
\end{aligned}$$

$$\begin{aligned}
I_{c+d}^4 = & \left( \frac{z}{108 (-1+z) \omega^2 (-2z+\omega+z^2\omega)^3} \right) \\
& \times \left\{ -36z^{10}\omega^4 - 36\omega^4 (1+\omega) + z^9\omega^3 (187+3\omega-4\omega^2) \right. \\
& +12z\omega^3 (18+29\omega+3\omega^2) + z^8\omega^2 (-378+11\omega+141\omega^2+4\omega^3) \\
& -z^2\omega^2 (432+1052\omega+303\omega^2+163\omega^3) \\
& -3z^7\omega (-114+6\omega+188\omega^2+29\omega^3+21\omega^4) \\
& -3z^4\omega (230-30\omega+363\omega^2+151\omega^3+28\omega^4) \\
& +z^3\omega (288+1314\omega+566\omega^2+771\omega^3+163\omega^4) \\
& +3z^6 (-36-26\omega+288\omega^2+14\omega^3+89\omega^4+21\omega^5) \\
& \left. +3z^5 (36-178\omega+192\omega^2-111\omega^3+19\omega^4+28\omega^5) \right\} \tag{115}
\end{aligned}$$

and

$$\begin{aligned}
I_{c+d}^5 = & \left( \frac{z^3}{6\omega^2 (-2z+\omega+z^2\omega)^3} \right) \sqrt{1-\omega^2} \log \left( \frac{1-\sqrt{1-\omega^2}}{1+\sqrt{1-\omega^2}} \right) \times \\
& \left\{ 18z^5\omega^2 + 6\omega^2 (-1+4\omega) - 3z^4\omega (5+2\omega+11\omega^2) - 3z\omega (-8+21\omega+5\omega^3) \right. \\
& \left. +3z^2 (-8+17\omega-4\omega^2+7\omega^3) + z^3 (-2+24\omega-11\omega^2+25\omega^4) \right\} \\
& + \left( \frac{2z^2 (1+z)}{9(-1+z)^2\omega} \right) \sqrt{(-1+z)^2 (-2-z+z^2)} \times \\
& \log \left( \frac{2+3z-2z^2-2\sqrt{(-1+z)^2 (-2-z+z^2)}}{2+3z-2z^2+2\sqrt{(-1+z)^2 (-2-z+z^2)}} \right) + \tag{116}
\end{aligned}$$

$$\left( \frac{z^2}{54\omega^2(-2z+\omega+z^2\omega)^3} \right) \sqrt{12z\omega-12z^3\omega-8\omega^2+4z^4\omega^2+z^2(9-5\omega^2)} \times \\ \left\{ 6\omega^3+3z^4\omega(-11+5\omega^2)+z\omega^2(-31+13\omega^2)+z^2(33\omega-21\omega^3) \right. \\ \left. +z^5(14\omega^2-8\omega^4)+z^3(18-19\omega^2+13\omega^4) \right\} \\ \log \left( \frac{3z+2\omega-2z^2\omega-\sqrt{12z\omega-12z^3\omega-8\omega^2+4z^4\omega^2+z^2(9-5\omega^2)}}{3z+2\omega-2z^2\omega+\sqrt{12z\omega-12z^3\omega-8\omega^2+4z^4\omega^2+z^2(9-5\omega^2)}} \right).$$

### B.1.2 Results for the matrix elements of $\mathcal{O}_{7,8}$ in full QCD with SPQR kinematics

The matrix elements of the EWP operators are given by:

$$\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = \frac{2\sqrt{2}}{f_K f_\pi^2} \gamma^{(8,8)} \left[ 1 + \frac{m_K^2}{16\pi^2 f^2} (I_{zf} + J_a + J_b + J_{c+d}) \right], \quad (117)$$

where  $I_{zf}$ , the one-loop contribution from the renormalization of the mesons' wavefunctions and the replacement of  $1/f^3$  by  $1/f_K f_\pi^2$  is given in eq. (107) and the  $J_{a,b,c+d}$  are as follows:

$$J_a = \frac{1}{6} \left\{ 18 \log(z^2) - 2(11 + 10z^2) \log\left(\frac{m_\pi^2}{\mu^2}\right) + (-4 + z^2) \log\left(\frac{m_\eta^2}{m_\pi^2}\right) \right\} \quad (118)$$

$$J_b = \frac{-2z^2}{3\omega} \left\{ 3 + 3\sqrt{\frac{1-\omega}{1+\omega}} \log\left(\frac{-1+\sqrt{\frac{1-\omega}{1+\omega}}}{1+\sqrt{\frac{1-\omega}{1+\omega}}}\right) + (-3+\omega) \log\left(\frac{m_\pi^2}{\mu^2}\right) \right\}, \quad (119)$$

and

$$J_{c+d} = \frac{-(1+\omega)z(4\omega+4z^2\omega-z(5+3\omega))}{\omega(-2z+\omega+z^2\omega)} + \left( \frac{-2\omega-6z^2\omega+15z(1+\omega)}{6\omega} \right) \log\left(\frac{m_\pi^2}{\mu^2}\right) \\ + \left( \frac{1}{6(-1+z)\omega(-2z+\omega+z^2\omega)^2} \right) \left\{ 2\omega^3 - 4z^6\omega^2(1+\omega) + 3z\omega^2(3+5\omega) + \right. \\ \left. 2z^5\omega(15+10\omega+7\omega^2) - z^2\omega(34+77\omega+13\omega^2) - z^4(30+46\omega+93\omega^2+5\omega^3) \right. \\ \left. + z^3(30+102\omega+41\omega^2+43\omega^3) \right\} \log(z^2) + \\ \left( \frac{1}{12(-1+z)\omega(-2z+\omega+z^2\omega)^2} \right) \left\{ 8\omega^3 - 4z^7\omega^3 - 4z\omega^2(7+\omega) + z^6\omega^2(17+5\omega) + \right. \\ \left. z^5\omega(-28-21\omega+\omega^2) + 2z^2\omega(16+6\omega+5\omega^2) - z^3(18+16\omega+25\omega^2+\omega^3) + \right. \\ \left. z^4(18+32\omega+5\omega^2+5\omega^3) \right\} \log\left(\frac{m_\eta^2}{m_\pi^2}\right) \quad (120)$$

$$\begin{aligned}
& + z^2 \left( \frac{6\omega + 6z^2\omega - z(7 + 5\omega^2)}{2\omega(-2z + \omega + z^2\omega)^2} \right) \sqrt{1 - \omega^2} \log \left( \frac{1 - \sqrt{1 - \omega^2}}{1 + \sqrt{1 - \omega^2}} \right) + \\
& \left( \frac{2z}{3(-1 + z)^2} \right) \sqrt{(-1 + z)^2(-2 - z + z^2)} \log \left( \frac{2 + 3z - 2z^2 - 2\sqrt{(-1 + z)^2(-2 - z + z^2)}}{2 + 3z - 2z^2 + 2\sqrt{(-1 + z)^2(-2 - z + z^2)}} \right) \\
& + z \left( \frac{2\omega + 2z^2\omega - z(3 + \omega^2)}{6\omega(-2z + \omega + z^2\omega)^2} \right) \times \sqrt{12z\omega - 12z^3\omega - 8\omega^2 + 4z^4\omega^2 + z^2(9 - 5\omega^2)} \\
& \times \log \left( \frac{3z + 2\omega - 2z^2\omega - \sqrt{12z\omega - 12z^3\omega - 8\omega^2 + 4z^4\omega^2 + z^2(9 - 5\omega^2)}}{3z + 2\omega - 2z^2\omega + \sqrt{12z\omega - 12z^3\omega - 8\omega^2 + 4z^4\omega^2 + z^2(9 - 5\omega^2)}} \right).
\end{aligned}$$

## B.2 Results for the Matrix Elements in Full QCD for General Kinematics

In this section we present the results for the matrix elements in full QCD for general kinematics, i.e. with arbitrary masses and momenta for the mesons. The results are rather long and complicated so we also present them on the web site [23]. The results are presented in terms of the following functions:

$$\begin{aligned}
d(p) &= \frac{1}{p^2}, \\
\text{div}(p) &= -1 - \log \left( \frac{p^2}{\mu^2} \right), \\
B(p, m_1, m_2) &= \sqrt{(1 - m_1^2 d(p) + m_2^2 d(p))^2 - 4m_2^2 d(p)}, \\
L1(p, m) &= -\log \left( \frac{B(p, m, m) - 1}{B(p, m, m) + 1} \right), \\
LL(p, m_1, m_2) &= \log \left( \frac{1 - (m_1^2 + m_2^2)d(p) + B(p, m_1, m_2)}{1 - (m_1^2 + m_2^2)d(p) - B(p, m_1, m_2)} \right).
\end{aligned} \tag{121}$$

### B.2.1 Result for the matrix elements of $\mathcal{O}_4$ in Full QCD with General kinematics:

We start with the matrix elements of  $\mathcal{O}_4$  which we write in the form

$$\langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle = O_4^{\text{tree}} \left[ 1 + \frac{m_K^2}{16\pi^2 f^2} I_{zf} \right] + O_4^a + O_4^b + O_4^{c+d} + \text{counterterms}, \tag{122}$$

where the counterterms are given in eq. (5). The two-pion state is symmetrised over the two-momenta  $p_1$  and  $p_2$  to ensure that it has isospin 2.  $I_{zf}$  is given in eq. (107) and we now present the remaining ingredients of the right-hand side of eq. (122).

$$O_4^{\text{tree}} = - \frac{\alpha^{(27,1)} \times \left\{ 3\sqrt{2} (p_K \cdot p_1 + p_K \cdot p_2 + 2p_1 \cdot p_2) \right\}}{f_\pi^2 f_K}, \tag{123}$$

$$\begin{aligned}
O_4^a = & \frac{\alpha^{(27,1)} \log\left(\frac{m_K^2}{\mu^2}\right) m_K^2 (12p_1 \cdot p_2 + 8p_K \cdot p_1 + 8p_K \cdot p_2 - m_K^2)}{4\sqrt{2}f^2 f_\pi^2 f_K \pi^2} \\
& + \frac{\alpha^{(27,1)} \log\left(\frac{m_\eta^2}{\mu^2}\right) (4m_K^2 - m_\pi^2) \left(2p_1 \cdot p_2 + 3(p_K \cdot p_1 + p_K \cdot p_2)\right)}{16\sqrt{2}f^2 f_\pi^2 f_K \pi^2} \\
& + \frac{\alpha^{(27,1)} \log\left(\frac{m_\pi^2}{\mu^2}\right) m_\pi^2 (86p_1 \cdot p_2 + 29p_K \cdot p_1 + 29p_K \cdot p_2 + 4m_\pi^2)}{16\sqrt{2}f^2 f_\pi^2 f_K \pi^2}, \quad (124)
\end{aligned}$$

$$\begin{aligned}
O_4^b = & \frac{\alpha^{(27,1)}}{4\sqrt{2}f^2 f_\pi^2 f_K \pi^2} \times \left[ \left( \text{div}(p_1 + p_2) - \log(d(p_1 + p_2)m_\pi^2) \right) \right. \\
& \times \left( 6(p_1 \cdot p_2)^2 + 3(p_1 \cdot p_2) (p_K \cdot p_1 + p_K \cdot p_2 - 2m_\pi^2) \right. \\
& \left. \left. + m_\pi^2 (-p_K \cdot p_1 - p_K \cdot p_2 + 2m_\pi^2) \right) \right] \\
& + \frac{\alpha^{(27,1)}}{12\sqrt{2}f^2 f_\pi^2 f_K \pi^2} \times \left\{ 36(p_1 \cdot p_2)^2 + (p_1 \cdot p_2) \left( 18p_K \cdot p_1 + 18p_K \cdot p_2 - 50m_\pi^2 \right) \right. \\
& + m_\pi^2 (-9p_K \cdot p_1 - 9p_K \cdot p_2 + 22m_\pi^2) \\
& + 4d(p_1 + p_2)m_\pi^2 \\
& \times \left[ 8(p_1 \cdot p_2)^2 (2 + 3d(p_1 + p_2)m_\pi^2) \right. \\
& + m_\pi^2 (3p_K \cdot p_1 + 3p_K \cdot p_2 + 4m_\pi^2 (-5 + 6d(p_1 + p_2)m_\pi^2)) \\
& \left. \left. + (p_1 \cdot p_2) (3p_K \cdot p_1 + 3p_K \cdot p_2 + 4m_\pi^2 (-1 + 12d(p_1 + p_2)m_\pi^2)) \right) \right] \Big\} \\
& + \frac{\alpha^{(27,1)}}{8f^2 f_\pi^2 f_K \pi^2} \times \left\{ L1(p_1 + p_2, m_\pi) \sqrt{2 - 8d(p_1 + p_2)m_\pi^2} \right. \\
& \times \left[ m_\pi^2 (-1 + 2d(p_1 + p_2)m_\pi^2) (p_K \cdot p_1 + p_K \cdot p_2 - 2m_\pi^2 + 8d(p_1 + p_2)m_\pi^4) \right. \\
& + 2(p_1 \cdot p_2)^2 (3 + 6d(p_1 + p_2)m_\pi^2 + 8d(p_1 + p_2)^2 m_\pi^4) \\
& + (p_1 \cdot p_2) (3p_K \cdot p_2 - 6m_\pi^2 + 2d(p_1 + p_2)p_K \cdot p_2 m_\pi^2 + 32d(p_1 + p_2)^2 m_\pi^6 \\
& \left. \left. + (p_K \cdot p_1) (3 + 2d(p_1 + p_2)m_\pi^2) \right) \right] \Big\}, \quad (125)
\end{aligned}$$

and finally the monster

$$\begin{aligned}
O_4^{c+d} = & \frac{\alpha^{(27,1)}}{864\sqrt{2}f^2 f_\pi^2 f_K \pi^2} \\
& \times \left\{ 3B(-p_1 + p_K, m_\eta, m_K) \times LL(-p_1 + p_K, m_\eta, m_K) \right\}
\end{aligned}$$



$$\begin{aligned}
& \times \left[ (p_1 \cdot p_2) \left[ -9 \left( 3p_K \cdot p_1 + 8m_K^2 - 2m_\pi^2 \right) \right. \right. \\
& \quad + 2d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2)^3 \left( -2p_K \cdot p_1 + m_K^2 + m_\pi^2 \right) \\
& \quad - d(-p_1 + p_K)^2 (m_K^2 - m_\pi^2) \\
& \quad \times \left( 41m_K^4 + 44m_K^2 m_\pi^2 - 13m_\pi^4 + 18(p_K \cdot p_1) \left( -5m_K^2 + m_\pi^2 \right) \right) \\
& \quad \left. \left. - 3d(-p_1 + p_K) \left( -5m_K^4 - 14m_K^2 m_\pi^2 + m_\pi^4 + 3(p_K \cdot p_1) \left( 5m_K^2 + m_\pi^2 \right) \right) \right] \right. \\
& + (p_K \cdot p_2) \left[ -9 \left( 3p_K \cdot p_1 + 10m_K^2 - 4m_\pi^2 \right) \right. \\
& \quad - 2d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2)^3 \left( -2p_K \cdot p_1 + m_K^2 + m_\pi^2 \right) \\
& \quad + 3d(-p_1 + p_K) \left( 19m_K^4 - 2m_K^2 m_\pi^2 + m_\pi^4 + 9(p_K \cdot p_1) \left( -3m_K^2 + m_\pi^2 \right) \right) \\
& \quad + d(-p_1 + p_K)^2 (m_K^2 - m_\pi^2) \\
& \quad \times \left( 35m_K^4 + 44m_K^2 m_\pi^2 - 7m_\pi^4 + (p_K \cdot p_1) \left( -78m_K^2 + 6m_\pi^2 \right) \right) \left. \right] \\
& + 2 \log \left( d(-p_1 + p_K) m_K^2 \right) \\
& \times \left[ 108d(-p_1 + p_K)^5 (m_K^2 - m_\pi^2)^5 \left( -2p_K \cdot p_1 + m_K^2 + m_\pi^2 \right)^2 \right. \\
& \quad - 2d(-p_1 + p_K)^4 \left( 2p_K \cdot p_1 - m_K^2 - m_\pi^2 \right) (m_K^2 - m_\pi^2)^3 \\
& \quad \times \left( 121(p_1 \cdot p_2) m_K^2 - 216m_K^4 - 121(p_1 \cdot p_2) m_\pi^2 - 108m_K^2 m_\pi^2 - 216m_\pi^4 \right. \\
& \quad \left. \left. - 121p_K \cdot p_2 (m_K^2 - m_\pi^2) + 270p_K \cdot p_1 (m_K^2 + m_\pi^2) \right) \right. \\
& \quad + 27 \left( 6(p_K \cdot p_1)^2 - 7(p_1 \cdot p_2) m_K^2 + m_K^4 + m_\pi^4 - (p_K \cdot p_2) \left( 7m_K^2 + 2m_\pi^2 \right) \right. \\
& \quad \left. \left. + (p_K \cdot p_1) \left( -p_1 \cdot p_2 - 7p_K \cdot p_2 + 3m_K^2 + 3m_\pi^2 \right) \right) \right. \\
& \quad + 9d(-p_1 + p_K) \\
& \quad \times \left[ - \left( (p_1 \cdot p_2) m_K^4 \right) + 21m_K^6 - 16(p_1 \cdot p_2) m_K^2 m_\pi^2 + 3m_K^4 m_\pi^2 - (p_1 \cdot p_2) m_\pi^4 - 3m_K^2 m_\pi^4 \right. \\
& \quad - 21m_\pi^6 + (p_K \cdot p_2) \left( -23m_K^4 - 8m_K^2 m_\pi^2 + 49m_\pi^4 \right) \\
& \quad \left. + 18(p_K \cdot p_1) \left( m_K^4 - m_\pi^2 \left( 2p_1 \cdot p_2 - 2p_K \cdot p_2 + m_\pi^2 \right) \right) \right] \\
& - 3d(-p_1 + p_K)^2 \\
& \quad \times \left[ -133(p_1 \cdot p_2) m_K^6 + 171m_K^8 + 138(p_1 \cdot p_2) m_K^4 m_\pi^2 - 111(p_1 \cdot p_2) m_K^2 m_\pi^4 \right. \\
& \quad - 110(p_1 \cdot p_2) m_\pi^6 - 171m_\pi^8 - 90(p_K \cdot p_1)^2 (m_K^4 - m_\pi^4) \\
& \quad + (p_K \cdot p_2) \left( 31m_K^6 - 12m_K^4 m_\pi^2 - 51m_K^2 m_\pi^4 + 248m_\pi^6 \right) \\
& \quad \left. + 3(p_K \cdot p_1) \left( -51m_K^6 + 11(p_1 \cdot p_2) m_\pi^4 + 51m_\pi^6 - m_K^4 \left( 19p_1 \cdot p_2 + 15m_\pi^2 \right) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + (p_K \cdot p_2) \left( 43m_K^4 - 56m_K^2 m_\pi^2 - 59m_\pi^4 \right) \\
& + 5m_K^2 \left( 16(p_1 \cdot p_2)m_\pi^2 + 3m_\pi^4 \right) ] \\
& + d(-p_1 + p_K)^3 \left( m_K^2 - m_\pi^2 \right) \\
& \times [ -593(p_1 \cdot p_2)m_K^6 + 675m_K^8 - 123(p_1 \cdot p_2)m_K^4 m_\pi^2 + 324m_K^6 m_\pi^2 \\
& + 153(p_1 \cdot p_2)m_K^2 m_\pi^4 + 162m_K^4 m_\pi^4 + 563(p_1 \cdot p_2)m_\pi^6 + 324m_K^2 m_\pi^6 + 675m_\pi^8 \\
& + 540(p_K \cdot p_1)^2 (m_K^2 + m_\pi^2)^2 \\
& + (p_K \cdot p_2) \left( 461m_K^6 + 255m_K^4 m_\pi^2 - 21m_K^2 m_\pi^4 - 695m_\pi^6 \right) \\
& - (6p_K \cdot p_1) \left( 252m_K^6 + 151(p_1 \cdot p_2)m_\pi^4 + 252m_\pi^6 + m_K^4 (-161p_1 \cdot p_2 + 108m_\pi^2) \right. \\
& \quad \left. + 39(p_K \cdot p_2) (3m_K^4 + 2m_K^2 m_\pi^2 - 5m_\pi^4) \right. \\
& \quad \left. + 2m_K^2 (5p_1 \cdot p_2 m_\pi^2 + 54m_\pi^4) \right) ] ] \\
& + \log \left( d(-p_1 + p_K)m_\eta^2 \right) \\
& \times \left[ - (p_K \cdot p_2) \right. \\
& \quad \times [ 27 \left( 3p_K \cdot p_1 + 10m_K^2 - 4m_\pi^2 \right) \\
& \quad + 2d(-p_1 + p_K)^4 (m_K^2 - m_\pi^2)^4 \left( -2p_K \cdot p_1 + m_K^2 + m_\pi^2 \right) \\
& \quad - 9d(-p_1 + p_K) \left( 25m_K^4 + 16m_K^2 m_\pi^2 - 5m_\pi^4 + 6(p_K \cdot p_1) (7m_K^2 - m_\pi^2) \right) \\
& \quad - d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2)^2 \left( 77m_K^4 + 80m_K^2 m_\pi^2 - 13m_\pi^4 \right. \\
& \quad \quad \left. - 18(p_K \cdot p_1) (9m_K^2 - m_\pi^2) \right) \\
& \quad + 3d(-p_1 + p_K)^2 \left( 82m_K^6 + 186m_K^4 m_\pi^2 - 60m_K^2 m_\pi^4 + 8m_\pi^6 \right. \\
& \quad \quad \left. + 3(p_K \cdot p_1) (-77m_K^4 + 4m_K^2 m_\pi^2 + m_\pi^4) \right) ] \\
& + (p_1 \cdot p_2) \\
& \quad \times [ -27 \left( 3p_K \cdot p_1 + 8m_K^2 - 2m_\pi^2 \right) \\
& \quad + 2d(-p_1 + p_K)^4 (m_K^2 - m_\pi^2)^4 \left( -2p_K \cdot p_1 + m_K^2 + m_\pi^2 \right) \\
& \quad - d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2)^2 \left( 83m_K^4 + 80m_K^2 m_\pi^2 - 19m_\pi^4 \right. \\
& \quad \quad \left. - 6(p_K \cdot p_1) (29m_K^2 - 5m_\pi^2) \right) \\
& \quad - 9d(-p_1 + p_K) \left( 67m_K^4 - 32m_K^2 m_\pi^2 + m_\pi^4 + 6(p_K \cdot p_1) (7m_K^2 - m_\pi^2) \right) ]
\end{aligned}$$

$$\begin{aligned}
& + 3d(-p_1 + p_K)^2 \left( -3(p_K \cdot p_1) \left( 119m_K^4 - 52m_K^2 m_\pi^2 + 5m_\pi^4 \right) \right. \\
& \quad \left. + 2 \left( 74m_K^6 + 84m_K^4 m_\pi^2 - 57m_K^2 m_\pi^4 + 7m_\pi^6 \right) \right) ] \\
& - 27 \log(d(-p_1 + p_K)m_\pi^2) \\
& \times \left[ -12(p_K \cdot p_1)^2 + 11(p_K \cdot p_1)(p_K \cdot p_2) - 6(p_K \cdot p_1)m_K^2 + 4(p_K \cdot p_2)m_K^2 - 2m_K^4 \right. \\
& \quad - 6(p_K \cdot p_1)m_\pi^2 + 8(p_K \cdot p_2)m_\pi^2 - 2m_\pi^4 \\
& \quad + 8d(-p_1 + p_K)^5 (m_K^2 - m_\pi^2)^5 (-2p_K \cdot p_1 + m_K^2 + m_\pi^2)^2 \\
& \quad + (p_1 \cdot p_2) (-p_K \cdot p_1 + 6m_K^2 + 2m_\pi^2) \\
& \quad - 2d(-p_1 + p_K)^4 \times (2p_K \cdot p_1 - m_K^2 - m_\pi^2) \times (m_K^2 - m_\pi^2)^3 \\
& \quad \times [9(p_1 \cdot p_2)m_K^2 - 16m_K^4 - 9(p_1 \cdot p_2)m_\pi^2 - 8m_K^2 m_\pi^2 - 16m_\pi^4 \\
& \quad - 9(p_K \cdot p_2) (m_K^2 - m_\pi^2) + 20(p_K \cdot p_1) (m_K^2 + m_\pi^2) ] \\
& + d(-p_1 + p_K) \\
& \quad \times [ -23(p_1 \cdot p_2)m_K^4 + 14m_K^6 + 2m_K^4 m_\pi^2 - (p_1 \cdot p_2)m_\pi^4 - 2m_K^2 m_\pi^4 \\
& \quad - 14m_\pi^6 + (p_K \cdot p_2) (-7m_K^4 + 31m_\pi^4) \\
& \quad + 2(p_K \cdot p_1) \left( -7(p_1 \cdot p_2)m_K^2 + 6m_K^4 - 11(p_1 \cdot p_2)m_\pi^2 - 6m_\pi^4 \right. \\
& \quad \left. + (p_K \cdot p_2) (7m_K^2 + 11m_\pi^2) \right) ] \\
& + d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2) \\
& \quad \times [ -47(p_1 \cdot p_2)m_K^6 + 50m_K^8 - 9(p_1 \cdot p_2)m_K^4 m_\pi^2 + 24m_K^6 m_\pi^2 + 15(p_1 \cdot p_2)m_K^2 m_\pi^4 \\
& \quad + 12m_K^4 m_\pi^4 + 41(p_1 \cdot p_2)m_\pi^6 + 24m_K^2 m_\pi^6 + 50m_\pi^8 + 40(p_K \cdot p_1)^2 (m_K^2 + m_\pi^2)^2 \\
& \quad + (p_K \cdot p_2) (37m_K^6 + 19m_K^4 m_\pi^2 - 5m_K^2 m_\pi^4 - 51m_\pi^6) \\
& \quad - 2(p_K \cdot p_1) \left( 56m_K^6 + 33p_1 \cdot p_2 m_\pi^4 + 56m_\pi^6 + 3m_K^4 (-13p_1 \cdot p_2 + 8m_\pi^2) \right. \\
& \quad + (p_K \cdot p_2) (29m_K^4 + 14m_K^2 m_\pi^2 - 43m_\pi^4) \\
& \quad \left. + 6m_K^2 ((p_1 \cdot p_2)m_\pi^2 + 4m_\pi^4) \right) ] \\
& - d(-p_1 + p_K)^2 \\
& \quad \times [ -20(p_K \cdot p_1)^2 (m_K^4 - m_\pi^4) \\
& \quad + (p_K \cdot p_1) \left( -34m_K^6 + 9p_1 \cdot p_2 m_\pi^4 + 34m_\pi^6 + m_K^4 (27p_1 \cdot p_2 - 10m_\pi^2) \right. \\
& \quad + 3(p_K \cdot p_2) (m_K^4 - 12m_K^2 m_\pi^2 - 13m_\pi^4) \\
& \quad \left. + 2m_K^2 ((18p_1 \cdot p_2)m_\pi^2 + 5m_\pi^4) \right) ]
\end{aligned}$$

$$\begin{aligned}
& + 2 \times \left( -23(p_1 \cdot p_2)m_K^6 + 19m_K^8 + 6(p_1 \cdot p_2)m_K^4m_\pi^2 - 6(p_1 \cdot p_2)m_K^2m_\pi^4 \right. \\
& \quad - 13(p_1 \cdot p_2)m_\pi^6 - 19m_\pi^8 \\
& \quad \left. + (p_K \cdot p_2) \left( 8m_K^6 + 9m_K^4m_\pi^2 - 9m_K^2m_\pi^4 + 28m_\pi^6 \right) \right) ] \\
& + 27B(-p_1 + p_K, m_\pi, m_K) \times LL(-p_1 + p_K, m_\pi, m_K) \\
& \times \left[ 12(p_K \cdot p_1)^2 - 11(p_K \cdot p_1)(p_K \cdot p_2) + 6(p_K \cdot p_1)m_K^2 - 4(p_K \cdot p_2)m_K^2 + 2m_K^4 \right. \\
& \quad + 6(p_K \cdot p_1)m_\pi^2 - 8(p_K \cdot p_2)m_\pi^2 + 2m_\pi^4 \\
& \quad + 8d(-p_1 + p_K)^4(m_K^2 - m_\pi^2)^4(-2p_K \cdot p_1 + m_K^2 + m_\pi^2)^2 \\
& \quad - 6d(-p_1 + p_K)^3(2p_K \cdot p_1 - m_K^2 - m_\pi^2)(m_K^2 - m_\pi^2)^2 \\
& \quad \times \left( 3(p_1 \cdot p_2)m_K^2 - 4m_K^4 - 3(p_1 \cdot p_2)m_\pi^2 - 4m_\pi^4 \right. \\
& \quad \left. - 3(p_K \cdot p_2)(m_K^2 - m_\pi^2) + 4(p_K \cdot p_1)(m_K^2 + m_\pi^2) \right) \\
& \quad \left. + (p_1 \cdot p_2)(p_K \cdot p_1 - 2(3m_K^2 + m_\pi^2)) \right) \\
& + d(-p_1 + p_K)^2 \\
& \times [ -29(p_1 \cdot p_2)m_K^6 + 26m_K^8 + 9(p_1 \cdot p_2)m_K^4m_\pi^2 - 8m_K^6m_\pi^2 - 3(p_1 \cdot p_2)m_K^2m_\pi^4 \\
& \quad - 4m_K^4m_\pi^4 + 23(p_1 \cdot p_2)m_\pi^6 - 8m_K^2m_\pi^6 + 26m_\pi^8 \\
& \quad - 8(p_K \cdot p_1)^2(m_K^4 - 6m_K^2m_\pi^2 + m_\pi^4) \\
& \quad + (p_K \cdot p_2)(19m_K^6 + m_K^4m_\pi^2 + 13m_K^2m_\pi^4 - 33m_\pi^6) \\
& \quad - 2(p_K \cdot p_1)(20m_K^6 + 5m_\pi^4(3p_1 \cdot p_2 + 4m_\pi^2) - m_K^4(21p_1 \cdot p_2 + 4m_\pi^2) \\
& \quad + (p_K \cdot p_2)(11m_K^4 + 14m_K^2m_\pi^2 - 25m_\pi^4) \\
& \quad + m_K^2(6(p_1 \cdot p_2)m_\pi^2 - 4m_\pi^4)) ] \\
& + d(-p_1 + p_K) \\
& \times [ 17(p_1 \cdot p_2)m_K^4 - 12m_K^6 - 8(p_1 \cdot p_2)m_K^2m_\pi^2 - 3(p_1 \cdot p_2)m_\pi^4 - 12m_\pi^6 \\
& \quad + 12(p_K \cdot p_1)^2(m_K^2 + m_\pi^2) + (p_K \cdot p_2)(3m_K^4 - 12m_K^2m_\pi^2 + 23m_\pi^4) \\
& \quad + (p_K \cdot p_1)((p_K \cdot p_2)(-25m_K^2 + 11m_\pi^2) \\
& \quad - 3(2m_K^4 + 7(p_1 \cdot p_2)m_\pi^2 + 2m_\pi^4 - m_K^2(5p_1 \cdot p_2 + 4m_\pi^2))) ] \\
& + (p_1 \leftrightarrow p_2) \Big\} \\
& + \frac{\alpha^{(27,1)}}{144\sqrt{2}f^2f_\pi^2f_K\pi^2} \times \left[ -72d(-p_1 + p_K)^4(m_K^2 - m_\pi^2)^4(-2p_K \cdot p_1 + m_K^2 + m_\pi^2)^2 \right. \\
& \quad \left. - 72d(-p_2 + p_K)^4(m_K^2 - m_\pi^2)^4(-2p_K \cdot p_2 + m_K^2 + m_\pi^2)^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + 4d(-p_1 + p_K)^3 \left( 2p_K \cdot p_1 - m_K^2 - m_\pi^2 \right) \left( m_K^2 - m_\pi^2 \right)^2 \\
& \quad \times \left( 41(p_1 \cdot p_2)m_K^2 - 63m_K^4 - 41(p_1 \cdot p_2)m_\pi^2 - 18m_K^2m_\pi^2 \right. \\
& \quad \left. - 63m_\pi^4 - 41(p_K \cdot p_2) \left( m_K^2 - m_\pi^2 \right) + 72(p_K \cdot p_1) \left( m_K^2 + m_\pi^2 \right) \right) \\
& + 4d(-p_2 + p_K)^3 \left( 2p_K \cdot p_2 - m_K^2 - m_\pi^2 \right) \left( m_K^2 - m_\pi^2 \right)^2 \\
& \quad \times \left( 41(p_1 \cdot p_2)m_K^2 - 63m_K^4 - 41(p_1 \cdot p_2)m_\pi^2 - 18m_K^2m_\pi^2 \right. \\
& \quad \left. - 63m_\pi^4 - 41(p_K \cdot p_1) \left( m_K^2 - m_\pi^2 \right) + 72(p_K \cdot p_2) \left( m_K^2 + m_\pi^2 \right) \right) \\
& - 18\text{div}(-p_2 + p_K) \left( 6(p_K \cdot p_2)^2 - 7p_1 \cdot p_2m_K^2 + m_K^4 + m_\pi^4 \right. \\
& \quad \left. - p_K \cdot p_1 \left( 7p_K \cdot p_2 + 7m_K^2 + 2m_\pi^2 \right) + p_K \cdot p_2 \left( -p_1 \cdot p_2 + 3m_K^2 + 3m_\pi^2 \right) \right) \\
& - 3 \times [ 36(2 + \text{div}(-p_1 + p_K)) (p_K \cdot p_1)^2 + 72(p_K \cdot p_2)^2 - 278(p_1 \cdot p_2)m_K^2 \\
& \quad - 42\text{div}(-p_1 + p_K)(p_1 \cdot p_2)m_K^2 + 42m_K^4 + 6\text{div}(-p_1 + p_K)m_K^4 \\
& \quad + (10p_1 \cdot p_2)m_\pi^2 + 4m_K^2m_\pi^2 + 42m_\pi^4 + 6\text{div}(-p_1 + p_K)m_\pi^4 \\
& \quad + (p_K \cdot p_2) \left( -28p_1 \cdot p_2 - (71 + 42\text{div}(-p_1 + p_K))m_K^2 \right. \\
& \quad \left. + (1 - 12\text{div}(-p_1 + p_K))m_\pi^2 \right) \\
& \quad + (p_K \cdot p_1) \left( -28p_1 \cdot p_2 - 6\text{div}(-p_1 + p_K)(p_1 \cdot p_2) \right. \\
& \quad \left. - 2(100 + 21\text{div}(-p_1 + p_K))(p_K \cdot p_2) \right. \\
& \quad \left. + (-71 + 18\text{div}(-p_1 + p_K))m_K^2 + m_\pi^2 + 18\text{div}(-p_1 + p_K)m_\pi^2 \right) ] \\
& + 3d(-p_2 + p_K) \\
& \quad \times [ -89(p_1 \cdot p_2)m_K^4 + 65m_K^6 - 24(p_1 \cdot p_2)m_K^2m_\pi^2 + 15m_K^4m_\pi^2 + 41(p_1 \cdot p_2)m_\pi^4 \\
& \quad + 15m_K^2m_\pi^4 + 65m_\pi^6 - 64(p_K \cdot p_2)^2 \left( m_K^2 + m_\pi^2 \right) \\
& \quad - 2(p_K \cdot p_2) \left( 7m_K^4 - 16p_1 \cdot p_2m_\pi^2 + 7m_\pi^4 + 2m_K^2 \left( -10p_1 \cdot p_2 + m_\pi^2 \right) \right) \\
& \quad + (p_K \cdot p_1) \left( -31m_K^4 + 24m_K^2m_\pi^2 - 113m_\pi^4 + 8p_K \cdot p_2 \left( 13m_K^2 + 2m_\pi^2 \right) \right) ] \\
& + 3d(-p_1 + p_K) \\
& \quad \times [ -89(p_1 \cdot p_2)m_K^4 + 65m_K^6 - 24(p_1 \cdot p_2)m_K^2m_\pi^2 + 15m_K^4m_\pi^2 \\
& \quad + 41(p_1 \cdot p_2)m_\pi^4 + 15m_K^2m_\pi^4 + 65m_\pi^6 - 64(p_K \cdot p_1)^2 \left( m_K^2 + m_\pi^2 \right) \\
& \quad + (p_K \cdot p_2) \left( -31m_K^4 + 24m_K^2m_\pi^2 - 113m_\pi^4 \right) \\
& \quad + (2p_K \cdot p_1) \left( -7m_K^4 + 16p_1 \cdot p_2m_\pi^2 - 7m_\pi^4 - 2m_K^2 \left( -10p_1 \cdot p_2 + m_\pi^2 \right) \right. \\
& \quad \left. + (p_K \cdot p_2) \left( 52m_K^2 + 8m_\pi^2 \right) \right) ]
\end{aligned}$$

$$\begin{aligned}
& -2d(-p_2 + p_K)^2 \\
& \times [ -202(p_1 \cdot p_2)m_K^6 + 165m_K^8 + 66(p_1 \cdot p_2)m_K^2m_\pi^4 - 42m_K^4m_\pi^4 \\
& + 136(p_1 \cdot p_2)m_\pi^6 + 165m_\pi^8 + 48(p_K \cdot p_2)^2 (m_K^4 + 4m_K^2m_\pi^2 + m_\pi^4) \\
& + 2(p_K \cdot p_1) (m_K^2 - m_\pi^2) (77m_K^4 + 101m_K^2m_\pi^2 + 92m_\pi^4 - 30p_K \cdot p_2 (4m_K^2 + 5m_\pi^2)) \\
& - 6(p_K \cdot p_2) (53m_K^6 + 34(p_1 \cdot p_2)m_\pi^4 + 53m_\pi^6 - m_K^4 (56p_1 \cdot p_2 + 5m_\pi^2) \\
& + m_K^2 (22p_1 \cdot p_2m_\pi^2 - 5m_\pi^4)) ] \\
& -2d(-p_1 + p_K)^2 \\
& \times [ -202(p_1 \cdot p_2)m_K^6 + 165m_K^8 + 66(p_1 \cdot p_2)m_K^2m_\pi^4 - 42m_K^4m_\pi^4 \\
& + 136(p_1 \cdot p_2)m_\pi^6 + 165m_\pi^8 + 48(p_K \cdot p_1)^2 (m_K^4 + 4m_K^2m_\pi^2 + m_\pi^4) \\
& + 2(p_K \cdot p_2) (77m_K^6 + 24m_K^4m_\pi^2 - 9m_K^2m_\pi^4 - 92m_\pi^6) \\
& - 6(p_K \cdot p_1) (53m_K^6 + 34p_1 \cdot p_2m_\pi^4 + 53m_\pi^6 - m_K^4 (56p_1 \cdot p_2 + 5m_\pi^2) \\
& + m_K^2 ((22p_1 \cdot p_2)m_\pi^2 - 5m_\pi^4)) \\
& + 10(p_K \cdot p_2) (4m_K^4 + m_K^2m_\pi^2 - 5m_\pi^4) ] ] . \tag{126}
\end{aligned}$$

### B.2.2 Result for the matrix elements of $\mathcal{O}_{7,8}$ in Full QCD with General Kinematics:

For the EWP operators the corresponding expressions are:

$$\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = O_{7,8}^{tree} [1 + \frac{m_K^2}{16\pi^2 f^2} I_{zf}] + O_{7,8}^a + O_{7,8}^b + O_{7,8}^{c+d} + \text{counterterms}, \tag{127}$$

where the counterterms are given in eq. (122),  $I_{zf}$  is given in eq. (107),

$$O_{7,8}^{tree} = \frac{2\sqrt{2}\gamma^{(8,8)}}{f_\pi^2 f_K}, \tag{128}$$

$$O_{7,8}^a = -\frac{3\gamma^{(8,8)} \log\left(\frac{m_K^2}{\mu^2}\right) m_K^2}{4\sqrt{2}f^2 f_\pi^2 f_K \pi^2} - \frac{7\gamma^{(8,8)} \log\left(\frac{m_\pi^2}{\mu^2}\right) m_\pi^2}{8\sqrt{2}f^2 f_\pi^2 f_K \pi^2} + \frac{\gamma^{(8,8)} \log\left(\frac{m_\pi^2}{\mu^2}\right) (-4m_K^2 + m_\pi^2)}{24\sqrt{2}f^2 f_\pi^2 f_K \pi^2}, \tag{129}$$

$$\begin{aligned}
O_{7,8}^b = & -\frac{\gamma^{(8,8)} (\text{div}(p_1 + p_2) - \log(d(p_1 + p_2)m_\pi^2)) (3p_1 \cdot p_2 - m_\pi^2)}{6\sqrt{2}f^2 f_\pi^2 f_K \pi^2} \\
& - \frac{\gamma^{(8,8)} [m_\pi^2 (-3 + 4d(p_1 + p_2)m_\pi^2) + (p_1 \cdot p_2) (6 + 4d(p_1 + p_2)m_\pi^2)]}{6\sqrt{2}f^2 f_\pi^2 f_K \pi^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma^{(8,8)} L1(p_1 + p_2, m_\pi) \sqrt{2 - 8d(p_1 + p_2)m_\pi^2}}{12f^2 f_\pi^2 f_K \pi^2} \\
& \times \left[ m_\pi^2 \left( -1 + 2d(p_1 + p_2)m_\pi^2 \right) + (p_1 \cdot p_2) \left( 3 + 2d(p_1 + p_2)m_\pi^2 \right) \right], \tag{130}
\end{aligned}$$

and

$$\begin{aligned}
O_{7,8}^{c+d} &= \frac{\gamma^{(8,8)}}{864\sqrt{2}f^2 f_\pi^2 f_K \pi^2} \\
&\times \left\{ -3 B(-p_1 + p_K, m_\eta, m_K) \times LL(-p_1 + p_K, m_\eta, m_K) \right. \\
&\quad \times \left[ -27 p_K \cdot p_1 - 6 m_K^2 + 6 m_\pi^2 \right. \\
&\quad + d(-p_1 + p_K)^2 (m_K^2 - m_\pi^2)^2 (-2 p_K \cdot p_1 + m_K^2 + m_\pi^2) \\
&\quad + d(-p_1 + p_K) (-7 m_K^4 - 16 m_K^2 m_\pi^2 + 5 m_\pi^4 + 3 p_K \cdot p_1 (7 m_K^2 - m_\pi^2)) \left. \right] \\
&+ 3 B(-p_2 + p_K, m_\eta, m_K) \times LL(-p_2 + p_K, m_\eta, m_K) \\
&\quad \times \left[ -27 p_K \cdot p_2 - 6 m_K^2 + 6 m_\pi^2 \right. \\
&\quad + d(-p_2 + p_K)^2 (m_K^2 - m_\pi^2)^2 (-2 p_K \cdot p_2 + m_K^2 + m_\pi^2) \\
&\quad + d(-p_2 + p_K) (-7 m_K^4 - 16 m_K^2 m_\pi^2 + 5 m_\pi^4 + 3 p_K \cdot p_2 (7 m_K^2 - m_\pi^2)) \left. \right] \\
&+ \log(d(-p_1 + p_K)m_\eta^2) \\
&\quad \times \left[ 9 (9 p_K \cdot p_1 + 2 m_K^2 - 2 m_\pi^2) \right. \\
&\quad - 9 d(-p_1 + p_K) (m_K^2 - m_\pi^2) (-2 p_K \cdot p_1 + 7 m_K^2 - m_\pi^2) \\
&\quad + d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2)^3 (-2 p_K \cdot p_1 + m_K^2 + m_\pi^2) \\
&\quad - d(-p_1 + p_K)^2 (m_K^2 - m_\pi^2) (28 m_K^4 + 34 m_K^2 m_\pi^2 - 8 m_\pi^4 + 9 p_K \cdot p_1 (-7 m_K^2 + m_\pi^2)) \left. \right] \\
&+ \log(d(-p_2 + p_K)m_\eta^2) \\
&\quad \times \left[ 9 (9 p_K \cdot p_2 + 2 m_K^2 - 2 m_\pi^2) \right. \\
&\quad - 9 d(-p_2 + p_K) (m_K^2 - m_\pi^2) (-2 p_K \cdot p_2 + 7 m_K^2 - m_\pi^2) \\
&\quad + d(-p_2 + p_K)^3 (m_K^2 - m_\pi^2)^3 (-2 p_K \cdot p_2 + m_K^2 + m_\pi^2) \\
&\quad - d(-p_2 + p_K)^2 (m_K^2 - m_\pi^2) (28 m_K^4 + 34 m_K^2 m_\pi^2 - 8 m_\pi^4 + 9 p_K \cdot p_2 (-7 m_K^2 + m_\pi^2)) \left. \right] \\
&+ 9 B(-p_1 + p_K, m_\eta, m_K) \times LL(-p_1 + p_K, m_\eta, m_K) \\
&\quad \times \left[ 21 p_K \cdot p_1 - 4 (m_K^2 + m_\pi^2) \right. \\
&\quad + 7 d(-p_1 + p_K)^2 (m_K^2 - m_\pi^2)^2 (-2 p_K \cdot p_1 + m_K^2 + m_\pi^2)
\end{aligned}$$

$$\begin{aligned}
& + d(-p_1 + p_K) \left( -3 m_K^4 - 8 m_K^2 m_\pi^2 - 3 m_\pi^4 + 7 p_K \cdot p_1 (m_K^2 + m_\pi^2) \right) \Big] \\
& + 9 B(-p_2 + p_K, m_\eta, m_K) \times LL(-p_2 + p_K, m_\eta, m_K) \\
& \times \left[ 21 p_K \cdot p_2 - 4 (m_K^2 + m_\pi^2) \right. \\
& + 7 d(-p_2 + p_K)^2 (m_K^2 - m_\pi^2)^2 (-2 p_K \cdot p_2 + m_K^2 + m_\pi^2) \\
& \left. + d(-p_2 + p_K) \left( -3 m_K^4 - 8 m_K^2 m_\pi^2 - 3 m_\pi^4 + 7 p_K \cdot p_2 (m_K^2 + m_\pi^2) \right) \right] \\
& + 6 \times \left[ -22 d(-p_1 + p_K)^2 (m_K^2 - m_\pi^2)^2 (-2 p_K \cdot p_1 + m_K^2 + m_\pi^2) \right. \\
& - 22 d(-p_2 + p_K)^2 (m_K^2 - m_\pi^2)^2 (-2 p_K \cdot p_2 + m_K^2 + m_\pi^2) \\
& + 6 [-15 (2 + \text{div}(-p_1 + p_K)) p_K \cdot p_1 - 15 (2 + \text{div}(-p_2 + p_K)) p_K \cdot p_2 \\
& - 3 m_K^2 + \text{div}(-p_1 + p_K) m_K^2 + \text{div}(-p_2 + p_K) m_K^2 + 9 m_\pi^2 \\
& + 3 \text{div}(-p_1 + p_K) m_\pi^2 + 3 \text{div}(-p_2 + p_K) m_\pi^2] \\
& + d(-p_1 + p_K) (37 m_K^4 + 70 m_K^2 m_\pi^2 + 13 m_\pi^4 - 12 p_K \cdot p_1 (7 m_K^2 + 3 m_\pi^2)) \\
& + d(-p_2 + p_K) (37 m_K^4 + 70 m_K^2 m_\pi^2 + 13 m_\pi^4 - 12 p_K \cdot p_2 (7 m_K^2 + 3 m_\pi^2)) \Big] \\
& + 2 \log(d(-p_1 + p_K) m_K^2) \\
& \times \left[ 9 d(-p_1 + p_K) (m_K^2 - m_\pi^2) (6 p_K \cdot p_1 + 3 m_K^2 - m_\pi^2) \right. \\
& + 31 d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2)^3 (-2 p_K \cdot p_1 + m_K^2 + m_\pi^2) \\
& - 9 (-15 p_K \cdot p_1 + m_K^2 + 3 m_\pi^2) \\
& \left. - d(-p_1 + p_K)^2 (m_K^2 - m_\pi^2) (31 m_K^4 + 82 m_K^2 m_\pi^2 + 49 m_\pi^4 - 9 p_K \cdot p_1 (7 m_K^2 + 11 m_\pi^2)) \right] \\
& + 2 \log(d(-p_2 + p_K) m_K^2) \\
& \times \left[ 9 d(-p_2 + p_K) (m_K^2 - m_\pi^2) (6 p_K \cdot p_2 + 3 m_K^2 - m_\pi^2) \right. \\
& + 31 d(-p_2 + p_K)^3 (m_K^2 - m_\pi^2)^3 (-2 p_K \cdot p_2 + m_K^2 + m_\pi^2) \\
& - 9 (-15 p_K \cdot p_2 + m_K^2 + 3 m_\pi^2) \\
& \left. - d(-p_2 + p_K)^2 (m_K^2 - m_\pi^2) (31 m_K^4 + 82 m_K^2 m_\pi^2 + 49 m_\pi^4 - 9 p_K \cdot p_2 (7 m_K^2 + 11 m_\pi^2)) \right] \\
& - 9 \log(d(-p_1 + p_K) m_\pi^2) \\
& \times \left[ -21 p_K \cdot p_1 + 4 (m_K^2 + m_\pi^2) \right. \\
& - d(-p_1 + p_K) (m_K^2 - m_\pi^2) (-14 p_K \cdot p_1 + m_K^2 + m_\pi^2) \\
& \left. + 7 d(-p_1 + p_K)^3 (m_K^2 - m_\pi^2)^3 (-2 p_K \cdot p_1 + m_K^2 + m_\pi^2) \right]
\end{aligned}$$



$$\begin{aligned}
& +d(-p_1 + p_K)^2 \left( 21 p_K \cdot p_1 \left( m_K^4 - m_\pi^4 \right) - 2 \left( 5 m_K^6 + 6 m_K^4 m_\pi^2 - 6 m_K^2 m_\pi^4 - 5 m_\pi^6 \right) \right) \\
& -9 \log \left( d(-p_2 + p_K) m_\pi^2 \right) \\
& \times \left[ -21 p_K \cdot p_2 + 4 \left( m_K^2 + m_\pi^2 \right) \right. \\
& - d(-p_2 + p_K) \left( m_K^2 - m_\pi^2 \right) \left( -14 p_K \cdot p_2 + m_K^2 + m_\pi^2 \right) \\
& + 7 d(-p_2 + p_K)^3 \left( m_K^2 - m_\pi^2 \right)^3 \left( -2 p_K \cdot p_2 + m_K^2 + m_\pi^2 \right) \\
& \left. + d(-p_2 + p_K)^2 \left( 21 p_K \cdot p_2 \left( m_K^4 - m_\pi^4 \right) - 2 \left( 5 m_K^6 + 6 m_K^4 m_\pi^2 - 6 m_K^2 m_\pi^4 - 5 m_\pi^6 \right) \right) \right] \Bigg\}. \quad (131)
\end{aligned}$$

## C Results in Quenched QCD

Finally we present the results for matrix elements in quenched QCD. The results for general kinematics, i.e. for arbitrary quark masses and momenta are too lengthy to be exhibited in this paper; we present them on the web site [23]. In the following subsection we present the results for the matrix elements with the SPQR kinematics.

In the quenched approximation, the integral  $I_{zf}^q$ <sup>7</sup>, which corresponds to the one-loop contributions from the renormalization of the mesonic wave-functions and from the replacement of the factor  $1/f^3$  which appears at lowest-order in the chiral expansion, by  $1/(f_K f_\pi^2)$  is given by ( $y = m_\pi^2/m_K^2$ ):

$$I_{zf}^q = -\frac{2m_0^2}{9m_K^2} \left[ 1 + \frac{1}{2(1-y)} \log \left( \frac{y}{2-y} \right) \right] + \frac{2\alpha}{9} \left[ 1 + \frac{y(2-y)}{2(1-y)} \log \left( \frac{y}{2-y} \right) \right]. \quad (132)$$

As for full QCD, the results in this appendix are defined in terms of the variables,  $y = m_\pi^2/m_K^2$ ,  $z = m_\pi/m_K$  and  $\omega = E_\pi/m_\pi$ , where  $E_\pi$  is the energy of the pion whose momentum is (in general) not equal to zero.

### C.1 $K \rightarrow \pi^+ \pi^0$ Decays with SPQR Kinematics in Quenched QCD

#### C.1.1 Result for the matrix elements of $\mathcal{O}_4$ in Quenched QCD with SPQR kinematics

The matrix elements of  $\mathcal{O}_4$  with the SPQR kinematics in the quenched approximation is given by

$$\begin{aligned}
\langle \pi^+ \pi^0 | \mathcal{O}_4 | K^+ \rangle &= \frac{-6\sqrt{2} m_K^2}{f_K f_\pi^2} \alpha^{(27,1)} \left\{ \frac{z(1+\omega+2z)}{2\omega} \left( 1 + \frac{m_K^2}{16\pi^2 f^2} \left( I_{zf}^q \right) \right) \right. \\
&\quad \left. + \frac{m_K^2}{16\pi^2 f^2} \left( I_a^q + I_b^q + I_{c+d}^q \right) \right\}, \quad (133)
\end{aligned}$$

---

<sup>7</sup>We introduce a superscript  $q$  in this appendix to denote the quenched approximation.

where  $I_{zf}^q$  is given in eq. (132) and the  $I_{a,b,c+d}^q$  are as follows:

$$\begin{aligned}
I_a^q &= \left( \frac{-1}{3\omega} \right) \left\{ \left( -2z^2 + \omega - 2z(1 + \omega) \right) \log(z^2) \right. \\
&+ \left. \left( 2z^2 - \omega + 2z(1 + \omega) + 2z^3(1 + \omega) + z^4(6 + \omega) \right) \log \left( \frac{m_\pi^2}{\mu^2} \right) \right\} \\
&- \frac{\alpha(3 + 2z + 3\omega) \left( 2 - 2z^2 + z^2(-2 + z^2) \log \left( -1 + \frac{2}{z^2} \right) \right)}{18z^3(-1 + z^2)\omega} \\
&- \frac{m_0^2(3 + 2z + 3\omega) \left( -2 + 2z^2 + \log \left( -1 + \frac{2}{z^2} \right) \right)}{18z(-1 + z^2)\omega m_\pi^2}, \tag{134}
\end{aligned}$$

$$\begin{aligned}
I_b^q &= \left( \frac{-z^3}{3\omega^2} \right) \left\{ 3(1 + 2z + \omega) + 3\sqrt{\frac{1-\omega}{1+\omega}}(1 + 2z + \omega) \log \left( \frac{-1 + \sqrt{\frac{1-\omega}{1+\omega}}}{1 + \sqrt{\frac{1-\omega}{1+\omega}}} \right) \right. \\
&+ \left. \left[ -3 - 2\omega + \omega^2 - 2z(3 - 3\omega + \omega^2) \right] \log \left( \frac{m_\pi^2}{\mu^2} \right) \right\}, \tag{135}
\end{aligned}$$

and

$$\begin{aligned}
I_{c+d}^q &= \left( \frac{-1}{18(-1 + z)\omega^2(-2z + \omega + z^2\omega)^3} \right) \times \\
&\left\{ -3 \left[ 10z\omega^4 - 2\omega^5 + 2z^9\omega^2(3 + \omega^2) - 2z^2\omega^3(9 - 6\omega + 5\omega^2) \right. \right. \\
&- z^3\omega^2(-22 + 56\omega - 41\omega^2 + \omega^3) - z^8\omega^2(12 + 42\omega - 9\omega^2 + \omega^3) \\
&+ z^4\omega(-32 + 90\omega - 64\omega^2 + 15\omega^3 - 17\omega^4) \\
&+ z^7(-8 + 24\omega + 56\omega^2 - 18\omega^3 + 55\omega^4 - 3\omega^5) \\
&+ 2z^5(12 - 18\omega + 28\omega^2 - 33\omega^3 + 36\omega^4 + 6\omega^5) \\
&\left. \left. - 2z^6(8 + 14\omega - 23\omega^2 + 48\omega^3 + 13\omega^5) \right] \log(z^2) \right. \\
&- 6(-1 + z)z^3\sqrt{1 - \omega^2} \left[ 3z^5\omega^2 + 3\omega^2(-1 + 2\omega) - 3z^4\omega^2(1 + 2\omega) \right. \\
&- 3z\omega(-4 + 6\omega + \omega^3) + 6z^2(-2 + 3\omega - \omega^2 + \omega^3) \\
&\left. \left. + z^3(-4 + 12\omega - 7\omega^2 + 5\omega^4) \right] \log \left( \frac{1 - \sqrt{1 - \omega^2}}{1 + \sqrt{1 - \omega^2}} \right) \right. \\
&+ (-1 + z)(-2z + \omega + z^2\omega) \left[ 2z^2(-9\omega^2(1 - 4\omega + \omega^2) + z^5\omega^2(7 + 9\omega + 2\omega^2) \right. \\
&+ z\omega(36 - 128\omega + 45\omega^2 - 7\omega^3) - z^4\omega(-2 + 45\omega + 2\omega^2 + 9\omega^3) \\
&\left. \left. - 2z^2(18 - 64\omega + 45\omega^2 - 26\omega^3 + 9\omega^4) + z^3(-8 + 72\omega - 101\omega^2 + 54\omega^3 + 19\omega^4) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& +3 \left( -2z + \omega + z^2 \omega \right)^2 \left( 2\omega^2 + 2z^4 \omega^2 + 2z\omega(1 + \omega) + z^3(2 + 3\omega + \omega^2) \right. \\
& \left. + 2z^2(3 - 5\omega + 3\omega^2) \right) \log \left( \frac{m_\pi^2}{\mu^2} \right) \Big] \Big\} \\
+ & \alpha \left( \frac{-z^2}{18(-1+z)^5(1+z)^2\omega^2(-2z+\omega+z^2\omega)^3} \right) \times \\
& \left( \frac{1}{2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2)} \right) \sqrt{(-1+z)^3(1+z)} \times \\
& \left\{ \frac{-1}{z} \sqrt{(-1+z)^3(1+z)} \left[ 2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2) \right] \right. \\
& \times \left[ z[-12\omega^4 + 4z^{11}\omega^4 + z^{10}\omega^3(-6 + 9\omega - 5\omega^2) - 2z\omega^3(-23 + 8\omega + \omega^2) \right. \\
& + 2z^2\omega^2(-45 + 61\omega + 9\omega^2 + \omega^3) + z^9\omega^2(18 - 72\omega - 3\omega^2 + 5\omega^3) \\
& - 2z^3\omega(-27 + 123\omega - 60\omega^2 + 54\omega^3 + 14\omega^4) \\
& - 2z^7\omega(42 - 92\omega + 15\omega^2 - 16\omega^3 + 17\omega^4) \\
& + 2z^8\omega(-10 + 69\omega - 45\omega^2 + 18\omega^3 + 17\omega^4) \\
& - z^6\omega(78 + 100\omega + 42\omega^2 + 39\omega^3 + 63\omega^4) \\
& + 2z^4(-2 + 85\omega - 108\omega^2 + 126\omega^3 - 54\omega^4 + 14\omega^5) \\
& + z^5(4 + 126\omega - 192\omega^2 + 204\omega^3 + 19\omega^4 + 63\omega^5)] \log \left( -1 + \frac{2}{z^2} \right) \\
& - 2(1+z)\omega \left( (-1+z) \times \right. \\
& \left[ -(\omega^3(1+\omega)) + 3z^9\omega^2(-4 - 3\omega + \omega^2) + z\omega^2(6 + 27\omega + \omega^2) \right. \\
& + z^8\omega(24 + 66\omega + 25\omega^2 - 3\omega^3) - z^2\omega(12 + 107\omega + 17\omega^2 + 22\omega^3) \\
& - z^7\omega(132 + 79\omega + 39\omega^2 + 22\omega^3) - 4z^5(3 - 8\omega - 8\omega^2 + 14\omega^3 + \omega^4) \\
& + 2z^4(-46 - 8\omega - 64\omega^2 - 35\omega^3 + 2\omega^4) \\
& + z^6(72 + 16\omega + 109\omega^2 + 67\omega^3 + 22\omega^4) \\
& \left. + z^3(8 + 160\omega + 41\omega^2 + 97\omega^3 + 22\omega^4) \right] \\
& + z(1+z) \left[ 6\omega^3 + 3z^8\omega^2(-3 - 2\omega + \omega^2) + z\omega^2(-23 - 3\omega + 2\omega^2) \right. \\
& + 3z^7\omega(6 + 21\omega + 4\omega^2 - 3\omega^3) + z^6\omega(-126 - 89\omega - 33\omega^2 + 2\omega^3) \\
& - 3z^2\omega(-14 + 7\omega + 3\omega^2 + 2\omega^3) + 3z^5(16 + 40\omega + 47\omega^2 + 19\omega^3 + 6\omega^4) \\
& \left. + z^3(-24 + 54\omega + 11\omega^2 + 78\omega^3 + 13\omega^4) \right]
\end{aligned}$$

$$\begin{aligned}
& - z^4 \left( 48 + 36 \omega + 145 \omega^2 + 78 \omega^3 + 23 \omega^4 \right) \Big] \log(z^2) \Big) \Big] \\
+ & 3 (1+z)^2 \left( -4 + 10 z - 3 z^2 - 7 z^3 + 4 z^4 \right) \omega \left( -2 z + \omega + z^2 \omega \right)^3 \times \\
& \left[ 2 z \omega - 2 z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 \left( -1 + \omega^2 \right) \right] \times \\
& \log \left( \frac{1 + z - z^2 - \sqrt{(-1+z)^3 (1+z)}}{1 + z - z^2 + \sqrt{(-1+z)^3 (1+z)}} \right) \\
- & (-1+z) z^2 \sqrt{(-1+z)^3 (1+z)} \times \\
& \left[ \sqrt{1-\omega^2} \left[ -6 \omega^4 - 6 z^{10} \omega^4 + 27 z^9 \omega^5 + 9 \omega^6 - 9 z \omega^3 \left( -2 + 3 \omega^2 \right) \right. \right. \\
& - 3 z^7 \omega \left( 4 - 3 \omega^2 + 7 \omega^4 \right) \\
& - 3 z^8 \omega^2 \left( -6 + 10 \omega^2 + 7 \omega^4 \right) + z^6 \omega^2 \left( -38 + 67 \omega^2 + 55 \omega^4 \right) \\
& + z^3 \omega \left( 2 - 67 \omega^2 + 113 \omega^4 \right) + z^5 \left( 10 \omega + 40 \omega^3 - 92 \omega^5 \right) \\
& + z^2 \left( -10 \omega^2 + 29 \omega^4 - 25 \omega^6 \right) \\
& \left. \left. + z^4 \left( 4 + 18 \omega^2 - 42 \omega^4 - 22 \omega^6 \right) \right] \log \left( \frac{1 - \sqrt{1-\omega^2}}{1 + \sqrt{1-\omega^2}} \right) \right. \\
+ & \frac{1}{z^2} \sqrt{2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2)} \times \\
& \left[ -12 \omega^5 + 6 z^{11} \omega^4 \left( -3 + \omega^2 \right) + 22 z \omega^4 \left( 2 + \omega^2 \right) - z^2 \omega^3 \left( 38 + 73 \omega^2 \right) \right. \\
& - z^8 \omega \left( -12 + 107 \omega^2 + \omega^4 \right) - 2 z^3 \omega^2 \left( -8 + 15 \omega^2 + 5 \omega^4 \right) \\
& + 2 z^7 \omega^2 \left( 48 + 55 \omega^2 + 35 \omega^4 \right) \\
& - 2 z^6 \omega \left( 7 + 50 \omega^4 \right) + z^4 \omega \left( 2 + 97 \omega^2 + 177 \omega^4 \right) + z^{10} \left( 48 \omega^3 + 9 \omega^5 \right) \\
& \left. \left. + z^9 \left( -42 \omega^2 + 11 \omega^4 - 35 \omega^6 \right) - z^5 \left( 4 + 58 \omega^2 + 129 \omega^4 + 49 \omega^6 \right) \right] \right. \\
& \left. \log \left( \frac{z + \omega - z^2 \omega - \sqrt{2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2)}}{z + \omega - z^2 \omega + \sqrt{2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2)}} \right) \right] \Big\} \\
+ & m_0^2 \times \left( \frac{-z^4}{18(-1+z)^5(1+z)^2\omega^2(-2z+\omega+z^2\omega)^3m_\pi^2} \right) \times \\
& \left( \frac{1}{2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2)} \right) \sqrt{(-1+z)^3(1+z)} \times \\
& \left\{ \frac{-1}{z} \sqrt{(-1+z)^3(1+z)} \left[ 2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2) \right] \times \right. \\
& \left. \left[ - \left( \omega^4 (1+\omega) \right) - 3 z^9 \omega^3 \left( -3 - 2 \omega + \omega^2 \right) + z \omega^3 \left( 6 + 15 \omega + \omega^2 \right) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + z^8 \omega^2 (-12 - 45\omega + 4\omega^2 + 3\omega^3) - z^2 \omega^2 (12 + 49\omega + 2\omega^2 + 5\omega^3) \\
& + z^3 \omega (8 + 74\omega - 35\omega^2 + 18\omega^3 + 5\omega^4) \\
& + z^4 \omega (-42 + 106\omega - 81\omega^2 + 33\omega^3 + 12\omega^4) \\
& + z^7 \omega (6 + 84\omega - 75\omega^2 + 30\omega^3 + 13\omega^4) \\
& + z^5 (4 - 86\omega + 110\omega^2 - 141\omega^3 + 27\omega^4 - 12\omega^5) \\
& - z^6 (4 + 54\omega - 154\omega^2 + 93\omega^3 - 38\omega^4 + 13\omega^5) \Big] \times \log \left( -1 + \frac{2}{z^2} \right) \\
& + 2(1+z)\omega \Big[ (-1+z) (-2z+\omega+z^2\omega)^2 \times \\
& (4z^4\omega - \omega(1+\omega) - z^3(3+\omega^2) + z^2(-5-2\omega+\omega^2) + z(2+5\omega+\omega^2)) \\
& - z(1+z) (-6\omega^3 - 3z^5\omega^2(-10-3\omega+\omega^2) + z^6\omega^2(-4-3\omega+\omega^2) \\
& + z\omega^2(20+9\omega+7\omega^2) - 2z^4\omega(18+7\omega+18\omega^2+2\omega^3) \\
& - 3z^2\omega(12+2\omega+9\omega^2+7\omega^3) + z^3(24+46\omega^2+30\omega^3+20\omega^4)) \log(z^2) \Big] \\
& + 3(1+z)^2 (1-3z+2z^2) \omega (-2z+\omega+z^2\omega)^3 \times \\
& (2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2)) \log \left( \frac{1+z-z^2-\sqrt{(-1+z)^3(1+z)}}{1+z-z^2+\sqrt{(-1+z)^3(1+z)}} \right) \\
& - (-1+z)\sqrt{(-1+z)^3(1+z)} \times \\
& \left[ \sqrt{1-\omega^2} [6z^{10}\omega^4 + 9z\omega^5 - 3\omega^6 + 3z^2\omega^4(-3+5\omega^2) - 3z^9\omega^3(2+7\omega^2) \right. \\
& + 2z^4\omega^2(-1+17\omega^2+5\omega^4) + z^8\omega^2(-10+32\omega^2+11\omega^4) \\
& + z^7\omega(14-13\omega^2+23\omega^4) + 2z^5\omega(-7+2\omega^2+26\omega^4) + z^3(15\omega^3-63\omega^5) \\
& \left. - z^6(4-24\omega^2+75\omega^4+29\omega^6)] \log \left( \frac{1-\sqrt{1-\omega^2}}{1+\sqrt{1-\omega^2}} \right) \right. \\
& + \sqrt{2z\omega - 2z^3\omega - \omega^2 + z^4\omega^2 - z^2(-1+\omega^2)} \times \\
& [3\omega^5 + 2z^9\omega^4(-4+\omega^2) - 2z\omega^4(4+5\omega^2) + z^8\omega^3(14+13\omega^2) \\
& - z^7\omega^4(23+13\omega^2) + z^2\omega^3(-1+37\omega^2) - z^3\omega^2(4+3\omega^2+5\omega^4) \\
& - 2z^4\omega(-5+8\omega^2+24\omega^4) + z^6(-10\omega+3\omega^3-5\omega^5) \\
& \left. + z^5(4-8\omega^2+54\omega^4+22\omega^6)] \right. \\
& \left. \times \log \left( \frac{z+\omega-z^2\omega-\sqrt{2z\omega-2z^3\omega-\omega^2+z^4\omega^2-z^2(-1+\omega^2)}}{z+\omega-z^2\omega+\sqrt{2z\omega-2z^3\omega-\omega^2+z^4\omega^2-z^2(-1+\omega^2)}} \right) \right] \Big\}. \tag{136}
\end{aligned}$$

### C.1.2 Results for the matrix elements of $\mathcal{O}_{7,8}$ in quenched QCD with SPQR kinematics

The matrix elements of the EWP operators are given by:

$$\langle \pi^+ \pi^0 | \mathcal{O}_{7,8} | K^+ \rangle = \frac{2\sqrt{2}}{f_K f_\pi^2} \gamma^{(8,8)} \left[ 1 + \frac{m_K^2}{16\pi^2 f^2} (I_{zf}^q + J_a^q + J_b^q + J_{c+d}^q) \right] \quad (137)$$

where  $I_{zf}$ , the one-loop contribution from the renormalization of the mesons' wavefunctions and the replacement of  $f^3$  by  $f_K f_\pi^2$  is given in eq. (132) and the  $J_{a,b,c+d}$  are as follows:

$$J_a^q = \frac{(\log(z^2) + (-1 + z^2)\log(\frac{m_\pi^2}{\mu^2}))}{3} - \frac{\alpha(2 - 2z^2 + z^2(-2 + z^2)\log(-1 + \frac{2}{z^2}))}{9(-1 + z^2)} - \frac{m_0^2 z^2 (-2 + 2z^2 + \log(-1 + \frac{2}{z^2}))}{9(-1 + z^2)m_\pi^2}, \quad (138)$$

$$J_b^q = \frac{-2z^2 \left( 3 + 3\sqrt{\frac{1-\omega}{1+\omega}} \log\left(\frac{-1+\sqrt{\frac{1-\omega}{1+\omega}}}{1+\sqrt{\frac{1-\omega}{1+\omega}}}\right) + (-3 + \omega)\log(\frac{m_\pi^2}{\mu^2}) \right)}{3\omega} \quad (139)$$

and

$$\begin{aligned} J_{c+d}^q &= \left( \frac{1}{3(-1+z)\omega(-2z+\omega+z^2\omega)} \times \right. \\ &\quad \left\{ [\omega^2 + z\omega(1+2\omega) + z^2(-3-7\omega+\omega^2) + z^3(3+2\omega^2)] \log(z^2) \right. \\ &\quad \left. + 3(-1+z)z^2\sqrt{1-\omega^2} \log\left(\frac{1-\sqrt{1-\omega^2}}{1+\sqrt{1-\omega^2}}\right) \right. \\ &\quad \left. + (-1+z)(-2z+\omega+z^2\omega) \left[ -3z(1+\omega) + (\omega+z^2\omega+3z(1+\omega)) \log\left(\frac{m_\pi^2}{\mu^2}\right) \right] \right\} \\ &+ \alpha \left( \frac{z^2}{18(-1+z)^5(1+z)^2\omega(-2z+\omega+z^2\omega)^2} \right) \times \\ &\quad \left( \frac{1}{2z\omega-2z^3\omega-\omega^2+z^4\omega^2-z^2(-1+\omega^2)} \right) \sqrt{(-1+z)^3(1+z)} \times \\ &\quad \left\{ \frac{2}{z^2} \sqrt{(-1+z)^3(1+z)} \left[ 2z\omega-2z^3\omega-\omega^2+z^4\omega^2-z^2(-1+\omega^2) \right] \times \right. \\ &\quad \left[ z[z^6(5-7\omega)\omega+z^7(-5+\omega)\omega^2+2z^8\omega^3-3\omega^2(1+\omega)+z\omega^2(15+\omega) \right. \right. \\ &\quad \left. \left. + z^5\omega(7-10\omega+3\omega^2)+2z^4\omega(4+5\omega+3\omega^2) \right. \right. \\ &\quad \left. \left. + z^2(3-12\omega+17\omega^2-10\omega^3)-z^3(3+20\omega-7\omega^2+12\omega^3) \right] \log\left(-1+\frac{2}{z^2}\right) \right. \\ &\quad \left. + (-1+z^2)\omega[-(z(-5+\omega)\omega)-2\omega^2-4z^5\omega^2+z^4\omega(13+\omega)] \right\} \end{aligned}$$

$$\begin{aligned}
& - z^2 (2 - 7\omega + \omega^2) - z^3 (10 + \omega + 5\omega^2) \\
& + 3z (1 + z) \left( \omega (1 + \omega) + z^4 \omega (1 + \omega) - 2z\omega (3 + \omega) \right. \\
& \quad \left. - 2z^3 \omega (3 + \omega) + 2z^2 (2 + \omega + 3\omega^2) \right) \log(z^2) \Big] \\
& - \left( 2 + z - 4z^2 - z^3 + 2z^4 \right) \omega \left( -2z + \omega + z^2 \omega \right)^2 \times \\
& \quad \left[ 2z\omega - 2z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2) \right] \times \\
& \quad \log \left( \frac{1 + z - z^2 - \sqrt{(-1 + z)^3 (1 + z)}}{1 + z - z^2 + \sqrt{(-1 + z)^3 (1 + z)}} \right) \\
& - 3(-1 + z) \sqrt{(-1 + z)^3 (1 + z)} \times \\
& \left[ z \sqrt{1 - \omega^2} [-2z^7 \omega^3 + z^2 (2 - 5\omega^2) + \omega^2 (-2 + 3\omega^2) + z^6 \omega^2 (4 + 3\omega^2) + 2z^3 \omega (-2 + 7\omega^2) \right. \\
& \quad + z (4\omega - 6\omega^3) - 2z^5 (\omega + 2\omega^3) + z^4 (\omega^2 - 6\omega^4)] \log \left( \frac{1 - \sqrt{1 - \omega^2}}{1 + \sqrt{1 - \omega^2}} \right) \\
& \quad + \sqrt{2z\omega - 2z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2)} \\
& \quad \times [2z^7 \omega^3 - 2\omega^4 + z^5 \omega (2 + 3\omega^2) - z^6 \omega^2 (4 + 3\omega^2) \\
& \quad + z\omega (-2 + 7\omega^2) + z^3 (2\omega - 14\omega^3) + z^2 (-2 + 5\omega^2 - 2\omega^4) \\
& \quad + z^4 (\omega^2 + 7\omega^4)] \times \\
& \quad \left. \log \left( \frac{z + \omega - z^2 \omega - \sqrt{2z\omega - 2z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2)}}{z + \omega - z^2 \omega + \sqrt{2z\omega - 2z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2)}} \right) \right] \Big\} \\
& + m_0^2 \left( \frac{z^2}{18(-1 + z)^5 (1 + z)^2 \omega (-2z + \omega + z^2 \omega)^2 m_\pi^2} \right) \times \\
& \left( \frac{1}{2z\omega - 2z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2)} \right) z \sqrt{(-1 + z)^3 (1 + z)} \times \\
& \left\{ \frac{1}{z} \sqrt{(-1 + z)^3 (1 + z)} \left[ 2z\omega - 2z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2) \right] \times \right. \\
& \left[ 2(-1 + z^2) \omega [z(-5 + \omega) \omega + 2\omega^2 + 4z^5 \omega^2 - z^4 \omega (13 + \omega) \right. \\
& \quad + z^2 (2 - 7\omega + \omega^2) + z^3 (10 + \omega + 5\omega^2)] \\
& \quad + [z (-5 + \omega) \omega^2 + 2\omega^3 + 3z^6 \omega^2 (1 + \omega) + z^5 \omega (-6 - 15\omega + \omega^2) \\
& \quad + z^2 \omega (8 - 7\omega + \omega^2) + 2z^3 (-3 + 2\omega - 7\omega^2 + 4\omega^3) \\
& \quad + 2z^4 (3 + 9\omega - 5\omega^2 + 4\omega^3)] \log \left( -1 + \frac{2}{z^2} \right) \Big] \\
& \quad \left. - 3(-1 + z^2) \omega (-2z + \omega + z^2 \omega)^2 \times \right.
\end{aligned}$$

$$\begin{aligned}
& \left[ 2 z \omega - 2 z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2) \right] \times \\
& \log \left( \frac{1 + z - z^2 - \sqrt{(-1 + z)^3 (1 + z)}}{1 + z - z^2 + \sqrt{(-1 + z)^3 (1 + z)}} \right) \\
& - 3(-1 + z) \sqrt{(-1 + z)^3 (1 + z)} \times \\
& \left[ \sqrt{1 - \omega^2} [6 z^5 \omega + 3 z^2 \omega^2 + 2 z \omega^3 + 2 z^7 \omega^3 - \omega^4 - z^6 \omega^2 (6 + \omega^2) \right. \\
& \left. - 2 z^3 \omega (2 + 3 \omega^2) + z^4 (-2 + 5 \omega^2 + 2 \omega^4)] \log \left( \frac{1 - \sqrt{1 - \omega^2}}{1 + \sqrt{1 - \omega^2}} \right) \right. \\
& \left. + \sqrt{2 z \omega - 2 z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2)} \times \right. \\
& \left. [2 z^2 \omega - 3 z \omega^2 + \omega^3 + z^4 \omega (-4 + \omega^2) \right. \\
& \left. - z^5 \omega^2 (-2 + \omega^2) + z^3 (2 - \omega^2 + \omega^4)] \times \right. \\
& \left. \log \left( \frac{z + \omega - z^2 \omega - \sqrt{2 z \omega - 2 z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2)}}{z + \omega - z^2 \omega + \sqrt{2 z \omega - 2 z^3 \omega - \omega^2 + z^4 \omega^2 - z^2 (-1 + \omega^2)}} \right) \right] \Bigg\}. \tag{140}
\end{aligned}$$

## References

- [1] NA48 Collaboration, V.Fanti et al., *Phys. Lett.* **B465**, 335 (1999);  
KTEV Collaboration, A. Alavi-Harati et al., *Phys. Rev. Lett.* **83**, 22 (1999)
- [2] SPQCDR Collaboration, Ph.Boucaud et al.,  
*Nucl. Phys. Proc.Suppl.* **106**, 329 (2002) (hep-lat/0110206)
- [3] L. Lellouch and M. Lüscher, *Commun. Math. Phys.* **219**, 31 (2001)
- [4] C.-J.D. Lin, G. Martinelli, C.T. Sachrajda, M. Testa, *Nucl. Phys.* **B619**, 467 (2001)
- [5] C.-J.D. Lin, G. Martinelli, C.T. Sachrajda, M. Testa, *Nucl. Phys. Proc.Suppl.* **109**, 218 (2002)
- [6] C. Bernard and M.F.L. Golterman *Phys. Rev.* **D53**, 476 (1996)
- [7] M.F.L. Golterman and E. Pallante *Nucl. Phys. Proc.Suppl.* **83**, 250 (2000)
- [8] C.-J.D. Lin, G. Martinelli, E. Pallante, C.T. Sachrajda and G. Villadoro, in preparation
- [9] M. Ciuchini, E. Franco, G. Martinelli, L. Reina and L. Silvestrini *Z. Phys.* **C68**, 239 (1995)
- [10] G. Buchalla, A.J. Buras and M.E. Lautenbacher, *Rev.Mod.Phys.* **68**, 1125 (1996)



- [11] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B120**, 316 (1977)
- [12] C. Bernard, T. Draper, A. Soni, H.D. Politzer and M.B. Wise, *Phys. Rev.* **D32**, 2343 (1985)
- [13] R.C. Brower, G. Maturana, M.B. Gavela and R. Gupta, *Phys. Rev. Lett.* **53**, 1318 (1984)
- [14] N. Cabibbo, G. Martinelli and R. Petronzio, *Nucl. Phys.* **B244**, 381 (1984)
- [15] JLQCD Collaboration, S. Aoki et al., *Phys. Rev.* **D58**, 054503 (1998)
- [16] SPQCDR Collaboration, Ph.Boucaud et al.,  
*Nucl. Phys. Proc.Suppl.***106**, 323 (2002) (hep-lat/0110169)
- [17] J. Laiho and A. Soni hep-ph/0203106
- [18] CP-PACS Collaboration, J. Noaki et al., hep-lat/0108013 (2001)
- [19] RBC Collaboration, T. Blum et al., hep-lat/0110075 (2001)
- [20] J. Gasser and H. Leutwyler, *Nucl. Phys.* **B250**, 465 (1985)
- [21] C. Bernard and M. Golterman, *Phys. Rev.* **D46**, 853 (1992)
- [22] J. Kambor, J. Missimer and D. Wyler, *Nucl. Phys.* **B346**, 17 (1990)
- [23] C.-J.D. Lin et al., <http://www.hep.phys.soton.ac.uk/kpipi/>
- [24] V. Cirigliano and E. Golowich, *Phys. Lett.* **B475**, 351 (2000)
- [25] G. Ecker et al., *Nucl. Phys.* **B591**, 419 (2000)
- [26] M.F.L. Golterman and K.-C. Leung, *Phys. Rev.* **D56**, 2950 (1997)
- [27] M.F.L. Golterman and K.-C. Leung, *Phys. Rev.* **D58**, 097503 (1998)
- [28] M.F.L. Golterman and K.C. Leung, *Phys. Rev.* **D57**, 5703 (1998)
- [29] C. Bernard, T. Draper, G. Hockney and A. Soni, *Nucl. Phys. Proc.Suppl.* **4**, 483 (1988)
- [30] C. Dawson et al., *Nucl. Phys.* **B514**, 313 (1998)
- [31] E. Pallante, *JHEP* **9901**, 012 (1999)
- [32] L. Maiani and M. Testa *Phys. Lett.* **B245**, 585 (1990)
- [33] M. Lüscher, *Commun. Math. Phys.* **104**, 177 (1986); *Nucl. Phys.* **B354**, 531 (1991);  
*Nucl. Phys.* **B364**, 237 (1991)

- [34] M. Lüscher, *Commun. Math. Phys.* **105**, 153 (1986)
- [35] M. Ciuchini, E. Franco, G. Martinelli and L. Silvestrini, *Phys. Lett.* **B380**, 353 (1996)
- [36] K. Rummukainen and S. Gottlieb, *Nucl. Phys.* **B450**, 397 (1995)
- [37] J. Bijnens, E. Pallante and J. Prades, *Nucl. Phys.* **B521**, 305 (1998)
- [38] E. Pallante, A. Pich and I. Scimemi, *Nucl. Phys.* **B617**, 441 (2001)
- [39] M.F.L. Golterman, Private Communication