Fundamental properties of Bragg gratings and their application to the design of advanced structures

by

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This thesis presents the analysis of the local properties of Bragg gratings and their application to the improvement of standard designs and advanced structures. The time spent by light inside each grating section is derived in terms of complex-valued quantities, and clear meaning is given to both the real and imaginary parts. Improved physical understanding of propagation and energy distributions inside periodic structures is obtained. Local properties also explain in a more intuitive way well understood features of different gratings, which improves intuition of new complex designs. The analysis of the effect of perturbations is immediate using this approach and has important practical applications. Independent confirmation of the theory is obtained, and experimental measurement of the imaginary part of the local time delay is given. Phase errors affect the grating writing techniques, and the related sensitivity is analysed in detail. The robustness of different designs is discussed with respect to such manufacturing errors. Fine tuning of standard or advanced grating designs by means of suitable error distributions is also proposed, and optimised characteristics either in the reflectivity or in the dispersive response are obtained. This method is integrated with inverse scattering designs to further boost their performances. Improved complex designs are also proposed in case losses affect propagation in the grating. Cladding mode losses are compensated using an iterative layer-peeling algorithm. The design of the first wide-band dispersion-compensating grating realised with a standard single mode fibre is shown. Background losses and UV-induced losses in gratings are also compensated using a modified layer-peeling method. The physical limitations related to grating design in lossy media are explained using the derived understanding of local properties. New advanced designs are also considered that fully exploit the theoretical potentialities and manufacturing capabilities of Bragg gratings. The performance of code-division multiple access systems based on superstructured gratings is improved by combining encoding, bandwidth filtering, and dispersion compensation in the same high reflectivity grating.
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DECLARATION OF AUTHORSHIP

I, Fabio Ghiringhelli, declare that the thesis entitled

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and the work presented in it are my own. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- I have published as part of Ph.D. work some of the research material contained within this thesis as journal and conference papers (see List of Publications).

Signed: .................................................................

Date: .................................................................
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Odi et amo. Quare id faciam fortasse requiris.
Nescio, sed fieri sentio et excrucior.

Catullus, Carmen LXXXV
# Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Equation</th>
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<tbody>
<tr>
<td>( A )</td>
<td>Coefficient matrix in ECWEs</td>
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</tr>
<tr>
<td>( D(x) )</td>
<td>Generic diagonal matrix</td>
<td></td>
</tr>
<tr>
<td>( M )</td>
<td>Matrix for grating optimisation via phase perturbation</td>
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<tr>
<td>( T )</td>
<td>Transfer matrix</td>
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</tr>
<tr>
<td>( T^\Delta )</td>
<td>Propagation matrix over a section of length ( \Delta )</td>
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<tr>
<td>( T^S )</td>
<td>Scattering matrix in the point-scattering approximation</td>
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<td>( \Delta \tau_R )</td>
<td>Matrix of the grating local time delay/sensitivity</td>
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<tr>
<td>( q )</td>
<td>Column vector of core-core and core-cladding complex coupling coefficients in a grating</td>
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<td>( \tilde{\delta} )</td>
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<td>( \mathcal{O}(N) )</td>
<td>Algorithmic complexity</td>
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<td>Amplitude of a backward-propagating cladding mode</td>
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<tr>
<td>( C )</td>
<td>Chirp parameter in \textit{sech} or Gaussian pulses</td>
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<tr>
<td>(</td>
<td>C</td>
<td>)</td>
</tr>
<tr>
<td>( CR )</td>
<td>Grating chirp rate</td>
<td></td>
</tr>
<tr>
<td>( c )</td>
<td>Velocity of light in vacuum</td>
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<td>( \delta(x) )</td>
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<td>( dW )</td>
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<td>( dW_e, dW_m )</td>
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<td>( dz )</td>
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<tr>
<td>( d\tau_0 )</td>
<td>Free-space traversal time through an infinitesimal layer</td>
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NOMENCLATURE

\( E, E_k \) Electric field (for a given mode \( k \))
\( E_+, E_- \) Forward-/backward-propagating electric fields
\( e_{\ell,k}(x,y) \) Electric field transverse profile for a given mode \( k \)
\( F, f \) Generically refer to reflected, transmitted, or total components
\( G \) Gain factor in lossy inverse scattering
\( g \) Photodetector gain
\( H \) Magnetic field
\( H_{\text{enc}}(f), H_{\text{dec}}(f) \) Spectral response of CDMA encoder/decoder gratings
\( h_{\text{chip}}(\xi) \) Impulse response of a single chip in CDMA gratings
\( h_{\text{enc}}(\xi), h_{\text{dec}}(\xi) \) Impulse response of CDMA encoder/decoder gratings
\( h_{\text{disp}}(\xi) \) Impulse response of the dispersion compensation contribution
\( I \) Intensity
\( IL \) Insertion Loss
\( \text{Im}\{\} \) Imaginary part
\( k \) Generic constant
\( k_0 \) Free-space plane wave propagation constant
\( L \) Length
\( L, L_{\text{eff}} \) CDMA code length/effective length
\( L_{\text{chip}} \) Chip length in CDMA systems
\( L_{\text{coh}} \) Coherence length of manufactured gratings
\( L_D \) Fibre dispersion length
\( L_{gr}, L_{\text{eff}} \) Grating length/effective length
\( L_{IS} \) Grating length used in IS simulations
\( L_{\text{link}} \) Length of a transmission span in a communication system
\( M \) Number of wavelengths
\( m \) Generic integer number
\( \text{max}\{I_{Q_i-Q_i^*}\} \) Maximum value of the out-of-peak auto-correlation signal in a CDMA system
\( \text{max}\{I_{Q_i-Q_j^*}\} \) Maximum value of the cross-correlation signal in a CDMA system
\( N \) Number of sections
\( N_R, N_T, N_{\text{tot}} \) Local number of passes in a grating section (reflection/transmission/total)
\( n \) Refractive index
\( n_{co}, n_{cl} \) Core/cladding refractive index in a step-index fibre
\( n_{\text{eff}} \) Fibre effective refractive index
\( P \) Probability
\( p \) Number of CDMA chips
\( \text{peak}\{I_{Q_i-Q_i^*}\} \) Peak of the auto-correlation signal in a CDMA system
\( Q_i \) Generic CDMA signature code
\( q(z) \) Grating complex cross-coupling coefficient
R Transmission data rate
\( R, r \) Generic reflectivity/reflection coefficient
\( \tilde{R}, \tilde{r} \) Perturbed grating reflectivity/reflection coefficient
\( R_1, R_2 \) Reflectivity of the sub-grating preceding/following a given layer
\( R_{\text{comp}}, r_{\text{comp}} \) Computed reflectivity/reflection coefficient in IS design with loss compensation
\( R_{gr}, r_{gr} \) Grating reflectivity/reflection coefficient
\( R_{\text{MAX}} \) Maximum grating reflectivity
\( R_{\text{Round–trip}} \) Round-trip reflectivity of an effective Fabry-Pérot cavity
\( R_{\text{target}}, R_{IS} \) Target/computed reflectivity in IS design
\( \text{Re}\{\} \) Real part
\( r_+, r_- \) Grating reflection coefficient (forward/backward direction)
\( r_1, r_2 \) Reflection coefficient of the sub-grating preceding/following a given layer
\( \tilde{S} \) Poynting vector
\( S_{IN} \) Poynting vector flux entering a grating
\( s \) Index referring to a generic grating layer
\( s_d(\xi) \) Electrical signal after photodetection
\( T, t \) Generic transmissivity/transmission coefficient
\( \tilde{T}, \tilde{t} \) Perturbed grating transmissivity/transmission coefficient
\( T_1, T_2 \) Transmissivity of the sub-grating preceding/following a given layer
\( T_{\text{chip}} \) Chip duration
\( T_{\text{FWHM}} \) Pulse duration at FWHM
\( T_{\text{loss}} \) Maximum possible transmissivity with cladding mode losses
\( T_{gr}, t_{gr} \) Grating transmissivity/transmission coefficient
\( T_{\text{min}} \) Minimum grating transmissivity
\( T_{\text{pulse}} \) Pulse duration (FWHM)
\( T_{\text{smooth}} \) Duration of a smoothing function (FWHM)
\( t_1, t_2 \) Transmission coefficient of the sub-grating preceding/following a given layer
\( U_0(f), V_0(f) \) Envelope of the forward-/backward-propagating core mode (frequency)
\( V_c(f) \) Envelope of backward-propagating cladding modes (frequency)
\( u(\xi), v(\xi) \) Envelope of the forward-/backward-propagating mode (time)
\( v \) Grating fringe visibility
\( v_D \) Dwell time-based velocity
\( v_e, v_g \) Energy/group velocity
\( W \) Total stored energy
\( w(\xi) \) Time windowing function
\( x, y \) Transverse spatial coordinates
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>z</td>
<td>Mode propagation direction</td>
</tr>
<tr>
<td>$a_0$</td>
<td>Propagating mode loss coefficient</td>
</tr>
<tr>
<td>$a_{dB}$</td>
<td>Power loss (dB)</td>
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<tr>
<td>$\beta_0, \beta_c$</td>
<td>Propagation constant of the core/cladding mode</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>Second-order dispersion/group delay slope (in $s^2/m$)</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Damping factor</td>
</tr>
<tr>
<td>$\Gamma_{core}$</td>
<td>Core power confinement factor</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Effective propagation constant</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Length of a grating section</td>
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<tr>
<td>$\Delta a_0$</td>
<td>Variation in the material absorption coefficient</td>
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<tr>
<td>$\Delta \xi$</td>
<td>Time duration</td>
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<td>$\Delta \xi_{tot}$</td>
<td>Maximum time delay compensation for a given grating length</td>
</tr>
<tr>
<td>$\Delta \xi_w$</td>
<td>Windowing function duration</td>
</tr>
<tr>
<td>$\Delta \delta, \Delta \lambda, \Delta f$</td>
<td>Detuning/wavelength/frequency bandwidth</td>
</tr>
<tr>
<td>$\Delta s_{chip}$</td>
<td>Chip physical length in a CDMA grating</td>
</tr>
<tr>
<td>$\Delta \Lambda$</td>
<td>Grating chirp function</td>
</tr>
<tr>
<td>$\Delta \lambda_{disp}, \Delta f_{disp}$</td>
<td>Dispersion compensated bandwidth (wavelength/frequency)</td>
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<tr>
<td>$\Delta n$</td>
<td>Variation in the refractive index</td>
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<tr>
<td>$\Delta R_{gr}, \Delta T_{gr}$</td>
<td>Grating reflectivity/transmissivity variation due to perturbation</td>
</tr>
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<td>$\Delta \tau_{target}$</td>
<td>Desired time delay variation via phase perturbation</td>
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<tr>
<td>$\delta$</td>
<td>Detuning from Bragg condition</td>
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<tr>
<td>$\delta \lambda$</td>
<td>Wavelength detuning</td>
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<tr>
<td>$\delta \lambda_{cm}$</td>
<td>Separation between core and cladding mode Bragg resonances</td>
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<td>$\delta n$</td>
<td>Refractive index modulation</td>
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<td>$\delta n_{IS}$</td>
<td>Refractive index profile resulting from IS design</td>
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<td>$\delta q(z)$</td>
<td>Perturbation of the grating complex coupling coefficient</td>
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<td>$\delta r$</td>
<td>Grating reflection coefficient variation due to perturbation</td>
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<td>$\epsilon_0$</td>
<td>Vacuum permittivity</td>
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<tr>
<td>$\zeta$</td>
<td>Light path inside a grating</td>
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<tr>
<td>$\eta_B$</td>
<td>Grating filling factor</td>
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<tr>
<td>$\theta_R, \theta_T$</td>
<td>Phase of the reflection/transmission coefficient</td>
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<td>$\Theta(x)$</td>
<td>Heaviside step function</td>
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<td>$\kappa$</td>
<td>Coupling coefficient of a grating</td>
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<td>$\kappa_0, \kappa_c$</td>
<td>Core-core and core-cladding coupling coefficients of a grating</td>
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<td>$\Lambda$</td>
<td>Grating pitch</td>
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<td>$\hat{\Lambda}$</td>
<td>Periodicity of a phase error distribution</td>
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<td>$\Lambda_{PM}$</td>
<td>Phase mask pitch</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wavelength</td>
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NOMENCLATURE

- $\lambda_{\text{Bragg}}$: Grating Bragg wavelength
- $\lambda_{\text{Bragg},\text{co}}$: Bragg wavelength for the core-core mode coupling
- $\lambda_{\text{Bragg},\text{cm}}$: Bragg wavelength for the core-cladding mode coupling
- $\lambda_{\text{UV}}$: Wavelength of the UV light used to write a FBG
- $\lambda_{\text{cut-off}}$: Effective cut-off wavelength in chirped gratings
- $\mu_0$: Vacuum permeability
- $\xi$: Time
- $\rho$: Round-trip coefficient of an effective Fabry-Pérot cavity
- $\varrho$: Discrete reflector coefficient
- $\sigma$: Self-coupling coefficient
- $\hat{\sigma}$: Effective detuning from Bragg condition
- $\hat{\tau}$: Traversal time operator
- $\tau_{\text{coh}}$: Coherence time
- $\tau_D$: Dwell time
- $\tau_{\text{enc}}, \tau_{\text{dec}}$: Encoded/decoded waveform durations
- $\tau_i$: Self-interference time
- $\tau_{\text{phase}}$: Classical definition of phase time delay/retardation
- $\tau_{\text{R}}, \tau_{\text{T}}$: Grating time delay in reflection/transmission
- $\tau_{\text{tot}}$: Total time delay in a grating section
- $\phi$: Phase of the grating modulation pattern
- $\phi_{\text{chip}}$: Phase of the impulse response of a single chip in CDMA gratings
- $\phi_{\text{Round-trip}}$: Round-trip phase shift of an effective Fabry-Pérot cavity
- $|\psi\rangle$: State of light propagating inside a grating (generic)
- $|\psi_{\text{IN}}\rangle$: State of light entering the grating
- $|\psi_{\text{R}}, \psi_{\text{T}}\rangle$: State of light exiting the grating (either reflected or transmitted)
- $\omega$: Angular frequency
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Definition</th>
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<tr>
<td>AC</td>
<td>Alternating Current</td>
</tr>
<tr>
<td>ASE</td>
<td>Amplified Spontaneous Emission</td>
</tr>
<tr>
<td>AWG</td>
<td>Arrayed-Waveguide Grating</td>
</tr>
<tr>
<td>BCH</td>
<td>Bose-Chaudhuri-Hochquenghem (code family)</td>
</tr>
<tr>
<td>BER</td>
<td>Bit Error Rate</td>
</tr>
<tr>
<td>CDMA</td>
<td>Code Division Multiple Access</td>
</tr>
<tr>
<td>CL</td>
<td>Constant Loss (lossy IS)</td>
</tr>
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<td>CLP</td>
<td>Continuous Layer-Peeling</td>
</tr>
<tr>
<td>CPM</td>
<td>Continuous Phase Modulation</td>
</tr>
<tr>
<td>CW</td>
<td>Continuous Wave</td>
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<tr>
<td>CWEs</td>
<td>Coupled-Wave Equations</td>
</tr>
<tr>
<td>DBR</td>
<td>Distributed Bragg Reflector</td>
</tr>
<tr>
<td>DC</td>
<td>Direct Current or Dispersion Compensating (grating)</td>
</tr>
<tr>
<td>DFB</td>
<td>Distributed FeedBack (laser)</td>
</tr>
<tr>
<td>DLP</td>
<td>Discrete Layer-Peeling</td>
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<tr>
<td>DS</td>
<td>Direct Scattering</td>
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<td>DS-CDMA</td>
<td>Direct-Sequence Code Division Multiple Access</td>
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<td>ECWEs</td>
<td>Extended Coupled-Wave Equations</td>
</tr>
<tr>
<td>EDFA</td>
<td>Erbium-Doped Fibre Amplifier</td>
</tr>
<tr>
<td>FEC</td>
<td>Forward Error Correction</td>
</tr>
<tr>
<td>FBG</td>
<td>Fibre Bragg Grating</td>
</tr>
<tr>
<td>FH-CDMA</td>
<td>Frequency-Hopping Code Division Multiple Access</td>
</tr>
<tr>
<td>FE-CDMA</td>
<td>Frequency-Encoded Code Division Multiple Access</td>
</tr>
<tr>
<td>FRED</td>
<td>FREquency Doubled (argon laser)</td>
</tr>
<tr>
<td>FT</td>
<td>Fourier Transform</td>
</tr>
<tr>
<td>FWHM</td>
<td>Full Width Half Maximum</td>
</tr>
<tr>
<td>GA</td>
<td>Genetic Algorithm</td>
</tr>
<tr>
<td>GLM</td>
<td>Gel’fand-Levitan-Marchenko (integral IS technique)</td>
</tr>
<tr>
<td>GDR</td>
<td>Group Delay Ripple</td>
</tr>
<tr>
<td>ILP</td>
<td>Integral Layer-Peeling</td>
</tr>
<tr>
<td>IR</td>
<td>InfraRed (radiation)</td>
</tr>
<tr>
<td>IS</td>
<td>Inverse Scattering</td>
</tr>
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<td>ACRONYMS</td>
<td>Definition</td>
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<td>-------------------------------------------------</td>
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<tr>
<td>ISI</td>
<td>Inter-Symbol Interference</td>
</tr>
<tr>
<td>LP</td>
<td>Layer-Peeling (differential IS technique)</td>
</tr>
<tr>
<td>LPG</td>
<td>Long Period Grating</td>
</tr>
<tr>
<td>MAI</td>
<td>Multiple Access Interference</td>
</tr>
<tr>
<td>NA</td>
<td>Numerical Aperture</td>
</tr>
<tr>
<td>OCDMA</td>
<td>Optical Code Division Multiple Access</td>
</tr>
<tr>
<td>OOC</td>
<td>Optical Orthogonal Code</td>
</tr>
<tr>
<td>OOK</td>
<td>On-Off Keying</td>
</tr>
<tr>
<td>PL</td>
<td>Proportional Loss (lossy IS)</td>
</tr>
<tr>
<td>PLC</td>
<td>Planar Lightwave Circuits</td>
</tr>
<tr>
<td>QAM</td>
<td>Quadrature Amplitude Modulation</td>
</tr>
<tr>
<td>RF</td>
<td>RadioFrequency</td>
</tr>
<tr>
<td>SMF</td>
<td>Single Mode Fibre</td>
</tr>
<tr>
<td>SONET-SDH</td>
<td>Fibre optics transmission protocols</td>
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<tr>
<td>SSFBG</td>
<td>SuperStructured Fibre Bragg Grating</td>
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<tr>
<td>SVEA</td>
<td>Slowly Varying Envelope Approximation</td>
</tr>
<tr>
<td>TCM</td>
<td>Trellis Code Modulation</td>
</tr>
<tr>
<td>TDM</td>
<td>Time Division Multiplexing</td>
</tr>
<tr>
<td>TEM</td>
<td>Transverse Electro-Magnetic (field)</td>
</tr>
<tr>
<td>TMM</td>
<td>Transfer Matrix Method</td>
</tr>
<tr>
<td>UV</td>
<td>UltraViolet (radiation)</td>
</tr>
<tr>
<td>WKB</td>
<td>Wentzel-Kramers-Brillouin (phase-integral method)</td>
</tr>
<tr>
<td>WDM</td>
<td>Wavelength Division Multiplexing</td>
</tr>
</tbody>
</table>
Chapter 1

Thesis overview

1.1 Historical background

Fibre Bragg gratings, holey fibres, photonic crystals: a large amount of the scientific research and industrial development of optical technologies in the last decade is related to these hot topics. All of them consider a single, fundamental concept of physics: Bragg scattering in periodic structures (also known as photonic band-gaps). Independently from the number of dimensions light is confined into, these structures present a periodic variation of the local propagation constant of light (wavenumber, or optical momentum), typically introduced by a periodic variation of the refractive index of the material light is guided through. An unrivalled control over the propagation of light is obtained, which spans from wavelength and spatial selectivity, to modification of the propagation velocity and of the dispersive characteristics of the medium. Light can be confined in space by Bragg scattering, but it can also be forced away from certain regions of space; it can be slowed down or quasi-stopped, as it can be sped up. Many interesting aspects of physics can be studied within these micro-engineered materials, given their unusual properties, and many can be directly applied to the production of novel devices with unparalleled features, ready for application to telecommunications, or for improving laser sources.

Momentum conservation is the simple principle ruling the properties of these structures. It results in the Bragg relation $\lambda_{\text{Bragg}} = 2n_{\text{eff}}\Lambda$, which connects the periodicity of the perturbation $\Lambda$ to the wavelength $\lambda_{\text{Bragg}}$ at which the interaction occurs (here represented for the simplest one-dimensional case). Nanometre precision is required for interactions involving optical waves, since $\lambda_{\text{Bragg}} = 1550$ nm implies $\Lambda \simeq 535$ nm if infrared radiation in silica glass is considered. The realisation of structures on this spatial scale is a tough technological challenge, especially if multi-dimensional arrangements are targeted. Groups all over the world tried to establish reliable ways to modify the refractive index of a material with the required precision, and one-dimensional structures (Bragg gratings) were demonstrated first. In the simplest configuration, a stack of different dielectric materials creates a photonic band-gap, but only a limited number of periods can be obtained, and a limited control over the refractive index exists. Multi-layer coatings and thin-film filters are produced with this technology, but their
design is limited, and manufacturing errors were a significant problem in the early stages (~1970).

The real breakthrough occurred in 1978, when Hill et al. [1] demonstrated the possibility of periodically perturbing in a permanent way the refractive index of an optical fibre by UV exposure. The fabrication of the “raw” material and the fabrication of the periodic structure are disentangled, which adds flexibility to both processes. Moreover, light can be used to indirectly control light thanks to photosensitivity, a first step towards more complicated self-assembling techniques. The importance of this result was actually highlighted only in 1989, when a writing technique that enables periodic patterns to be imprinted at a generic wavelength (the transverse holographic method) was proposed by Meltz et al. [2]. This technique is based on an interferometric set-up and gives maximum flexibility in the choice of the Bragg wavelength of the imprinted grating. Gratings were successfully manufactured in spectral regions of interest for fibre optic communications and fibre sensing, raising the general interest towards the practical applications of periodic structures. The introduction of the phase-mask technique [3] was the final step. Cheaper laser sources and inexpensive set-ups were needed for manufacturing fibre Bragg gratings (FBGs), and an increasing number of groups started to produce gratings and to study their applications.

However, as soon as manufacture was established, practical requirements were pushed forward. The possibility to ultimately control both the amplitude and phase characteristics of optical signals offers the potential for almost-ideal filter characteristics [4] and dispersion management [5], which made this first generation of FBGs an ideal component for commercial telecommunication systems. But simple, uniform designs are not suitable for these applications, as they do not take advantage of the possibility to tailor either the amplitude profile (apodisation) or the local periodicity (chirp) of the grating. The mathematical modelling was driven by this need, and techniques originally used in digital signal processing (windowing) were therefore re-applied to the design of more complex photonic band-gaps [4, 6, 7]. New methods were introduced for manufacture of apodised [8, 9, 10] and chirped gratings [10, 11, 12], and new characterisation techniques which allowed the obtained gratings to be measured were tested [13, 14, 15, 16]. These designs were far from optimal, and the modelling tool was not appropriate given the complexity of propagation in highly scattering structures. But they allowed gratings to keep pace with the growing demand, and establish themselves as valuable candidates for the development of next-generation optical systems with no-compromise high specification optical devices. Extended models were developed trying to understand the spectral features of manufactured FBGs, including fibre propagation issues such as cladding modes and radiation modes in normal and tilted gratings [17, 18, 19, 20], and the effect of grating non-uniformities [21, 22, 23]. Long-period gratings (LPGs) based on the coupling between core mode and co-propagating cladding modes were studied and manufactured [24, 25]. Simplified interpretations according to well accepted engineering models were also proposed [26, 27, 28], given the large number of people
getting involved in grating technology and coming from different backgrounds (electronics, microwaves,...). Physical understanding, mathematical modelling, and engineering capabilities were interacting with each other and driving the progress in the field.

At the same time, the use of the Fourier transform relationship that holds between the grating profile and grating spectrum in the low reflectivity limit (Born approximation [29]) drew attention to even more complicated grating profiles, such as sinc-apodised structures with ideal pass-band features [30], or Moiré-type structures able to replicate the same grating spectrum over multiple bands [31, 32]. Technology was again the bottleneck, since standard apodisation techniques based on amplitude masks were not adequate for such complicated profiles. Only the development of a continuous grating writing technique enabled progress in fibre Bragg gratings to proceed further [33, 34, 35, 36]. Both apodisation and chirp of the grating are accurately controlled on a local base by scanning the UV writing beam along the fibre length, so that (virtually) every profile can be practically manufactured. Despite the higher complexity of this technique, successful demonstrations of superstructured gratings with both unparallel complexity [30, 31, 32] and physical length [37] were given, setting new limits to the potentialities of this technology.

Now, the design tools were no longer up to date. Windowing techniques do not allow a generic target spectrum to be synthesised, and Fourier transform designs fail with high reflectivity. An exact synthesis tool was needed for taking full advantage of the intrinsic potentialities of gratings, and of the new manufacturing capabilities. The analysis of how this inverse scattering problem was solved in different areas of mathematics and physics allowed the development of new, powerful algorithms for the design of the grating profile starting from the desired spectral response. Exact inversion of either the integral (GLM) [38, 39, 40] or the differential (layer-peeling) [41, 42, 43] equations describing propagation in periodic structures was obtained, and genetic algorithm approaches to the problem were also shown [44, 45, 46], despite their worse computational efficiency. High reflectivity pass-band gratings and ultra-low sidelobe suppression [47, 48], complete control over the grating dispersive characteristics [49, 50, 51], multi-channel gratings [52], complex spectral profiles for signal shaping [53, 54] and processing [55, 56] were designed and experimentally demonstrated thanks to the quality of the manufacturing technology. No intuitive design of these profiles is possible whatsoever, since they typically present nonuniform, oscillating apodisation profiles, complex chirp, and distributed discrete phase shifts. The function of the various portions of an inverse scattered grating is often not understood even after the profile is available.

This framework brings us to the present status of research in the area of Bragg gratings. System demonstrations have shown the advantages of gratings over competitive optical technologies in telecommunications [50, 57, 58] and sensing. Always novel applications are being found despite the deflation of research funding in the area following the crisis of the telecommunication market. New manufacturing set-ups with the same flexibility as the continuous grating writing system have been proposed and tested in
the last few years [59, 60, 61, 62], and also different approaches to the production of complex gratings are under investigation (use of complex phase masks [63]). At the same time, inverse scattering techniques have been evolving, and evolving fast. GLM [64], layer-peeling [65, 66, 67], and genetic algorithm [68] methods have been applied to the design of co-propagating structures, such as LPGs or couplers. Recent publications show that reconstruction of lossy gratings is possible [69], as well as more efficient implementations of the lossless algorithm [70], which are expected to be even faster and more reliable in the ultra-high reflectivity limit. More flexible design techniques are also under study [71, 72], with the aim of optimising the obtained profiles when some of the spectral features can only be imprecisely defined.

This excitement in the grating modelling community has now to find practical application and obtain experimental demonstrations. Are the described second-order effects properly accounted for, or is more accurate modelling needed? Can new grating designs enable new applications in optics? The answer has not yet been given, and possibly the future of grating technology depends on a positive outcome in the next few years. Theory leads to technology, technology demands improved modelling and design techniques, and new profiles with novel features push manufacturing on. Also, the approaches used for Bragg gratings are a possible pathway for the upgrade to higher dimensionality, where, of course, more degrees of freedom exist and better results can be expected.

1.2 Motivations and main achievements

The previous historical discussion about grating technology shows that design and manufacturing techniques have both been pushed to the limit, and industrial application of the grating technology has already started. What else is needed in this area of research? Once “anything” can be done, the target is to make it better, to make it more reliable, to make it simpler given the limitations to be faced, and ultimately to reduce its cost. Also, of course, to find valuable applications where the advantages of the technology can be fully exploited. All these targets seem to refer to the engineering point of view, and to deal more with industrial development rather than basic research. But, surprisingly, a large room for improvement still exists in the analysis of gratings, and this improvement is driven by the need for solving practical problems.

If the quality of a grating is unsatisfactory due to errors in the manufacture, the obvious task is to improve the set-up and remove the source of the errors. However, this is possible only when such a source has been identified, which can be a serious problem due to size of the imperfections that are introduced (a few nanometres do make a huge difference in gratings). Direct characterisation is complex and often unreliable, while indirect techniques based on inverse scattering reconstruction suffer from noise limitations when strong gratings are considered. The alternative approach followed in this work is to identify the “spectral trace” left by errors which perturb an almost ideal grating. An understanding of the grating properties on a local (section-by-section)
basis is necessary in order to do it, and such an analysis has not been reported to date, despite the fact that the synthesis process is based on a layer-peeling (section-by-section) identification of the grating profile.

Another cause of poor quality in manufactured gratings is given by the presence of propagation losses in real waveguides (either fibres or planar structures, but also multi-layers suffer from the same limitation), which are not taken into account in the design process. They deform the obtained spectral response even if no errors are introduced, and often simple manufacturing solutions have to be discarded since the corresponding losses cannot be compensated for. The standard layer-peeling algorithm has to be extended, and this is carried out for different types of losses in this work (cladding mode losses, uniform propagation losses, losses proportional to the local strength of the grating). In particular, it is shown for the first time that a standard fibre can be used for writing large-band dispersion compensating gratings, despite the high distortion due to cladding mode coupling. Even in this case, knowledge of the local properties inside the grating is important, not for the implementation of the extended algorithms but rather for a deeper understanding of the way they work.

This analysis of the local properties of gratings gives an answer to questions such as “Where does light go when it enters a periodic structure?” or “How is power stored in the grating in order to get such complicated spectral responses?”, which have not been satisfied to date. Odd concepts will be introduced, such as “complex time delays”, whatever the notion of complex times physically means. However, they are not as abstract and “just for physicists” as they might look at a first glance. Knowing the whereabouts of photons inside the gratings also tells us how a nonuniform structure works, giving a better understanding of the features of complex designs. Non-intuitive profiles do make sense once what happens locally is highlighted, and the improved perception is then useful to tackle new problems without passively relying on the outcome of mathematical techniques such as inverse scattering algorithms. The manufacturing process takes advantage of this, as already highlighted, since these “complex times” are related to nothing else but the grating sensitivity. And the sensitivity is useful to analyse a grating profile even before it is actually manufactured, since it tells us how precise the production process has to be in order to match the target design, and where it is more critical to be precise. A new tool to evaluate the performance of almost equivalent, theoretical profiles is established in this way. A new tool that can also be used to introduce small corrections to a design that does not quite match the specifications, if necessary. This sounds odd in the era of perfect reconstruction and design via inverse scattering, but it is valuable if simple and inexpensive manufacturing techniques have to be used, and not-ideal (but still optimised) spectral characteristics are acceptable.

Finally, a more standard application of the layer-peeling design will be considered, i.e., the synthesis of complex superstructured gratings for Optical Code Division Multiple Access (OCDMA) telecommunication systems. Again, the need for improved designs comes from the manufacturing process, but it is a need for increased rather than reduced
complication in this case. Indeed, full advantage with respect to competitive technologies can be gained only if the potentialities of gratings are fully exploited, which means that profiles with apodisation, chirp and phase shifts at the limit of the manufacturing capability are required. Gratings with high reflectivity and multiple functionalities (coding, wavelength filtering, and dispersion compensation) will be designed within the limits of the present technology, and their performance discussed with respect to standard implementations.

1.3 Summary of content

The structure of this work is mainly divided into two subsections. Chapter 2 is still introductory, and it gives the mathematical background of grating theory. The different models that are generally accepted are highlighted, and the choice of the simple (but also approximate) coupled-wave theory is justified. The choice of the inverse scattering algorithm used in the following is also discussed. A more detailed review of the different manufacturing techniques is presented. The superior performance of the continuous grating writing technique when complex grating profiles are considered is shown, but the present technological limitations are also considered, in order to set reasonable boundaries to design space.

The first section deals with the analysis of local properties in Bragg gratings. The theory is developed in Chapter 3, is compared with other approaches used in physics, and it is shown to be coherent with well accepted results. A partial experimental demonstration is also given. Chapter 4 is focussed on the application of this theory to the analysis of the sensitivity of Bragg gratings to phase errors. Different grating types commonly used in practical applications are considered, and the advantages of the knowledge of the local properties are discussed. Better understanding of complex profiles is obtained, and a tool is derived for the correction of systematic errors in the manufacturing process, or the fine tuning of simple, non-optimised designs.

The second section is related to the extension and the application of layer-peeling, inverse scattering techniques. Chapter 5 first reviews the main characteristics and limitations of the discrete layer-peeling method that is applied in the rest of this work. Then cladding mode losses are addressed, with specific attention to the problem of writing dispersion compensating, chirped gratings in standard optical fibres. An iterative inverse scattering algorithm is proposed, and its effectiveness is demonstrated by designing an ultra-flat, linear time delay grating in a SMF. A new transfer matrix method for the direct computation of gratings with cladding modes is also presented, with a largely improved computational efficiency with respect to standard integration methods. Compensation of propagation losses related either to background losses in the waveguide or to UV-exposure related losses is considered next, and the obtained designs are explained by effectively using the theory developed in the first section. Chapter 6 considers the application of the standard algorithm to the design of highly complex gratings for
CDMA applications. High reflectivity gratings and multi-function, dispersion compensating gratings are analysed in detail, and the limitations due to the manufacturing process are also taken into account.

The final Chapter summarises the main achievements of this work and suggests possible routes for future development of the techniques used.
Chapter 2

Introduction to Bragg gratings

This Chapter provides a brief overview of the fundamentals of Bragg reflection in gratings and the main fabrication methods available to date. The different approaches to modelling of propagation inside periodic structures are reviewed, and the inverse problem of designing gratings given the required spectral features is addressed in Section 2.1. The background of the coupled-wave theory used in this thesis is presented in Section 2.2, together with the main properties of gratings derived from general considerations. A review of layer-peeling techniques used for inverse-engineering Bragg gratings is given in Section 2.3. Finally, the different manufacturing techniques used to produce fibre Bragg gratings are considered in Section 2.4, and the advantages and limitations of the continuous grating writing technique developed within the ORC are discussed. A general framework to understand practical issues and design requirements is set.

2.1 Choice of the analysis and synthesis techniques

2.1.1 Modelling of gratings

The problem of modelling propagation of light inside a periodically modulated structure such as a Bragg grating can be tackled using the different approaches that have been proposed during the last 30 years.

The correct method is to solve the Helmholtz equation for the given geometry by taking into account a general form of the field in terms of Bloch waves and of the Floquet theorem [73], i.e., considering a field distribution that must have the same periodicity as the modulated medium because of symmetry considerations. A characteristic equation is derived, and the dispersion diagram obtained by solving it provides all the information about propagation. This approach is typical in solid state physics and quantum physics, and it was first applied to electromagnetic propagation in stratified media by Yeh et al. [74, 75]. It gives the exact solution of the electromagnetic problem, since Bloch waves are the eigenmodes of periodic media in the same way as plane waves are the natural modes of free space propagation. This analysis is applicable to periodic structures with arbitrary large modulations, and provides an immediate understanding of the dispersion
and the microstructure of the field [76].

On the other hand, a rigorous derivation of the characteristic equation is complex and mathematically involved. Even in the case of a uniformly modulated medium of infinite length, the general expression of the fields (written as a summation over all the possible wavevectors that match the periodicity of the medium) is often approximated by considering only the two dominant counter-propagating components. Furthermore, generalisation of the theory to account for aperiodic gratings and finite length structures was proposed only after the advent of fibre Bragg gratings [64, 77, 78], and its outcome is still limited and immature. The analysis is approximate and counter-intuitive and, despite the claims, a direct understanding of the main issues inside complex structures is not possible with the present formalism. The corrections with respect to the simpler approach described in the following are limited to modulation strengths at the limit of what is achievable with the present FBG technology. Their practical advantage in this area is therefore not significant. Finally, no inverse design has ever been considered starting from the Bloch mode theory, and this is a major limitation to its practical use at the moment. Better knowledge of the fundamental physics behind gratings is worthless if it cannot be used to improve the corresponding designs.

All these limitations are possible areas for future investigation and theoretical improvement, and they might become highly attractive in case new technologies characterised by a larger modulation depths of the periodic structure are developed. Indeed, this applies especially for the multi-dimensional implementations of Bragg scattering cited previously, namely, holey fibres and photonic crystals. However, this model is clearly not suitable for the research topics that are of interest in this thesis.

Other methods based on fundamental physics-oriented approaches were applied to Bragg gratings. WKB analysis [26], application of the variational principle [79] or of Hamiltonian optics [80, 81] are available in the literature, but the formulation is always cumbersome, the results are still approximated, and negligible advantages in the understanding of these structures are obtained. Indeed, very limited interest was raised by these works.

Conversely, the most widely used approach to the analysis of Bragg gratings has been coupled-mode theory. Coupled-mode theory is a general tool developed in different areas of physics to model the interaction between fields in a periodically perturbed structure [82]. Rather than finding the real modes of the perturbed medium (which are orthogonal by definition and propagate unaltered in the grating), the fields in the unperturbed waveguide are considered and the effect of the perturbation is accounted for by assuming that these quasi-modes are now coupled together by the periodic modulation [83]. Power transfer between quasi-modes occurs on the basis of momentum conservation inside the system, so that all the processes are phase-matched [83]. This approach is inherently approximate, and is valid only when the amplitude of the modulation is small with respect to the unperturbed propagation constant (i.e., it has to be $\delta n \ll n$ for refractive index modulated gratings). But it is conceptually simple and intuitive, since it allows
the flow of energy to be followed as it propagates inside the structure. It also enables
the description of finite and nonuniform structures [29], so that apodisation, chirp, and
phase shifts can be accounted for within the same coupled-wave equations (CWEs)
by simply considering position dependent coupling coefficients. Moreover, the problem
of inverting the resulting set of equations (also known as Zakharov-Shabat system)
has already been studied extensively in connection with a class of nonlinear evolution
equations (soliton propagation) [84]. Layer-peeling and differential methods were also
analysed in geophysics, filter design, and voice synthesis [41].

For these reasons, this mathematical approach is chosen for the description of prop-
agation in Bragg gratings.

A closed-form solution of the coupled-wave equations is possible only for uniform
gratings, while aperiodic and finite structures require a numerical computation. Different
transfer matrix approaches to this problem are available and are typically faster than
the direct integration using numerical routines such as Runge-Kutta. Weller-Brophy
and Hall [85] presents an extension of the well known Rouard’s method employed in
thin-film design, where the coupled wave equations are solved within each period of
the grating and the effective reflection coefficient of an effective layer is found. The
grating is transformed into an equivalent multi-layer stack, very long and with a very
low index-contrast. This method is accurate and allows the computation of gratings
in which the base-period is not sinusoidal, but the problem is intractable when “long”
gratings with more than $10^5$ periods are considered (which typically means centimetre-
long structures only). A more practical, though less precise, transfer matrix method was
proposed by Yamada and Sakuda [86]. The nonuniform grating is divided into sections,
where the length of each one is much bigger than the biggest period of the corrugation
and where the modulation inside each one is such that they can be considered uniform,
even though each one has different parameters. Each section is described therefore by
a transfer matrix corresponding to a uniform grating [83], and the overall structure is
characterised by a global matrix obtained as the product of the individual matrices.
This approach is well suited for long gratings, such as those fabricated in fibres, but is
is inherently approximated due to the staircase representation of the grating. A careful
choice of the section length is required to obtain good results without needing a very
long computation time.

2.1.2 Synthesis of gratings

Different solutions have been proposed for the design of Bragg gratings. The synthesis
problem is complicated due to the nonlinearity of the coupled-wave equations, which
does not allow the refractive index profile to be easily obtained, given a desired reflection
spectrum $r(\lambda)$.

From a historical point of view, the problem was partially solved by considering the
Born approximation and the related Fourier transform relation between the coupling
coefficient and the reflection coefficient of the grating. This method is suitable for weak
gratings, but it has also been generally applied to strong gratings. However, the stronger the grating is, the larger the distortion of the actual spectral response is with respect to the ideal one. An extension of the Fourier transform technique was proposed by Winick and Roman [87], enabling the design of practical fibre Bragg grating filters at high reflectivities. However, this synthesis procedure is approximate in nature and is not reliable when designing very complex filters. Apodisation techniques developed in digital signal processing applications were also used to improve the spectral features of grating-based filters and dispersion compensators (Gaussian, raised cosine, sine, or Blackman windowing, see Erdogan [88]). However, an extensive analysis of the various apodisation possibilities is required in order to obtain an optimised (but still approximate) grating response [4, 89]. Only empirically-established trade-offs between bandwidth efficiency and sidelobe suppression (for filters) or between low and high frequency ripples in the time delay response (for dispersion compensators) are obtained using this method. Customised features cannot be synthesised, which makes this technique inappropriate for the design of novel devices with tight requirements.

An analytical tool for the automated synthesis of complex grating designs is therefore necessary, and it is obtained only by direct inversion of the coupled-wave equations. Different inverse scattering algorithms are available. A first group of inverse scattering methods relies on the exact solution of the problem in terms of integral equations. The coupled-wave equations are equivalent to the so called Gel’fand-Levitan-Marchenko (GLM) integral equations, for which exact solutions exist in the case where the design reflection coefficient is given by a rational function [38]. Iterative, numerical approaches were used in order to extend the applicability of this method and design generic fibre gratings [39, 40, 90]. First, the Born approximation is considered (a single scattering event occurs inside the grating), and then all the other multiple reflections are included step by step (3, 5, 7, ... scattering events also result in reflected light from the grating). This process requires a large number of iterations, especially when abrupt transitions occur in the grating profile or the grating reflectivity is high. Truncation is often required, which results in a reduced precision in the reconstruction. More importantly, it has a low algorithm efficiency, with a complexity that grows as $O(N^3)$, where $N$ is the number of sections in the grating. The described identification process provides valuable information about how light is trapped inside the structure, but its inefficiency makes it unsuitable for the design of complex structures. It has indeed found a very limited practical application in the last few years.

A second group of inverse scattering algorithms exists, called differential or direct methods [41]. Originally developed by geophysicists, they fully exploit the properties of the layered media by using causality arguments and identifying the medium recursively layer-by-layer. For this reason, they are often referred to as layer-peeling algorithms (LP). They are characterised by numerical stability and high efficiency, since the complexity grows only as $O(N^2)$. The process of propagation of light is mimicked, and a better understanding of local issues in gratings can be obtained. This technique was
first applied to fibre Bragg grating design by Feced et al. [42]. The inverse scattering problem was proven to be as simple as the direct computation [91], and clearer physical understanding is gained with the simplified approach proposed by Skaar et al. [43], which moreover has a faster numerical implementation. Layer-peeling algorithms are therefore preferred in this work because of the better intuition associated with the layer-peeling process and the superior speed and accuracy in the reconstruction of complex gratings. They will be described, used, and further extended throughout this thesis.

However, different classes of layer-peeling algorithms are still possible. Both continuous (CLP) [91] and discrete (DLP) [43] implementations are available, and differ in the way fields are propagated after the identification of the medium in a certain layer. CLP uses the coupled-wave equations, so that the model is not inherently discrete; nevertheless, some form of discretisation has to be introduced in order to numerically solve the equations and propagate the fields. DLP approximates the grating as a series of discrete, localised reflectors separated by free-space regions. The correct phase-dependence inside the grating cannot be precisely maintained over a wide band, unless very short sections are considered. This means that CLP offers advantages in terms of flexibility, but DLP is significantly faster and is often more stable, given the bandwidth used to present the target reflection coefficient [43]. In both cases, convergence problems are found in the reconstruction of ultra-strong gratings [92], due to the accumulation of the numerical errors along the grating and to the strong reduction of the signals propagating deep into the grating. The discrete implementation is therefore preferable and will be used in this work.

Very recently, a new integral-layer-peeling algorithm (ILP) has been proposed [70]. The idea of identifying the medium layer-by-layer is preserved, but each layer is now given by a nonuniform, weak subgrating that can be synthesised using the GLM integral method in a very precise way and with a limited number of iterations. Errors due to the limited bandwidth do not rapidly accumulate along the grating, resulting in a reduced total error and in an increased stability of the algorithm when very strong structures are considered (a uniform grating with $R_{gr} = 1 - 10^{-10}$ has been designed with this technique). Improvements in computational efficiency are also found when weaker gratings are designed. A new approach to the design of finite-length gratings via standard layer-peeling has also been investigated [93], based on the research of the necessary and sufficient conditions for the target response to be realisable as a grating of limited length $L_{gr}$. The potentialities of these new methods are very promising and indeed yet to be fully exploited.

A completely different approach to grating design is to reduce the synthesis problem to the minimisation of a nonlinear function of $N$ parameters (the coupling coefficients of the $N$ sections that the grating is divided into). This function is given by the difference between the target spectrum and the present spectrum, eventually weighted in order to give different importance to different requirements. This flexibility is useful in many grating designs, where some features are more important than others or some parameters
do not even need to be strictly specified, and it is an advantage with respect to layer-peeling techniques which are rigid, in the sense that the reflection coefficient \( r \) to be synthesised has to be unequivocally defined over the available bandwidth. The resulting degrees of freedom are used to impose requirements on the grating profile, such as its maximum length, or the lack of chirp, or a minimised maximum amplitude of the required refractive index modulation.

Genetic algorithms (GA) are well suited for this task. They are probabilistic parallel search algorithms based on natural selection. A population of “individuals” is created, each individual representing a possible solution to the problem, and ranked according to its “fitness”, which measures how well the solution complies with the requirements. New generations are made from this population by creating offspring from pairs of individuals, with more fit individuals having a higher chance of reproducing. The more promising areas of the search space are explored in this way, until a sufficiently optimised solution is found. Genetic (or evolutionary) algorithms have been applied to the design of fibre Bragg gratings [44, 45] and long-period gratings [68, 94]. Unfortunately, the GA approach is typically slow, given the large number of individuals and generations required to find a solution, and can face problems of convergence if the algorithm parameters (number of representatives in each generation, extent of the mutations, selection criteria) are not properly chosen. Their range of application has therefore been very limited, and they will be no longer discussed in this work.

Flexible approaches are nevertheless a valuable tool, and recently new iterative designs based on this principle have been proposed. Firstly, adaptive algorithms for the minimisation of a suitably defined error function have been considered [72]. Convergence towards the best solution is not stochastic as in GA implementations, but deterministic and it is based on the Levenberg-Marquardt algorithm [95]. Secondly, classical layer-peeling techniques have been extended [71]. The direct and inverse problems are iteratively solved while at the same time controlling some of the grating profile and spectral characteristics and allowing the remaining grating profile and spectral characteristics to evolve. Both the new approaches have shown remarkable results and are expected to find many applications in grating design in the future. However, neither of these methods have been considered in this work due to their novelty.

2.2 Analysis of gratings: formalism

2.2.1 Coupled-wave equations

As discussed in Section 2.1, the coupled-mode theory is applied in the following [83, 96]. According to this approach, the propagation of light in the perturbed structure is analysed by using the eigenmodes of the unperturbed guiding medium as a suitable modal base. This leads to the definition of quasi-modes which actually exchange power among them, while propagating along \( z \), via the distributed interaction with the perturbation.

A scalar approach is used throughout this work. Typical optical fibres can be
modelled using the weakly-guiding approximation, and support the propagation of almost TEM and linearly polarised eigenmodes (identified by the subscript $k$) for which $\vec{E}_k(x, y, z) = E_k(x, y, z) \vec{e}$. A vectorial solution of the wave equation is not needed. If a monochromatic excitation is considered in the unperturbed structure, the electric field is in the form

$$E_k(x, y, z) = e^{t_k(x, y)}e^{j(\beta_k z - \omega_0 \xi)},$$

where $z$ is the propagating direction and $\xi$ is time. The propagation constant $\beta_k$ and the transverse profile $e^{t_k(x, y)}$ of the considered mode are found by solving the scalar Helmholtz equation

$$\nabla^2 E_k + k_0^2 n^2 E_k = 0 \rightarrow \left[ \nabla^2 + \frac{\partial^2}{\partial z^2} + k_0^2 n^2(x, y) \right] e^{t_k(x, y)}e^{j(\beta_k z - \omega_0 \xi)} = 0,$$

$$\left[ \nabla^2 e^{t_k(x, y)} + (k_0^2 n^2(x, y) - \beta_k^2) e^{t_k(x, y)} \right] = 0,$$

where $k_0 = 2\pi/\lambda$ is the propagation constant in vacuum and $n(x, y)$ is the refractive index transverse profile of the unperturbed structure.

In general, a Bragg grating is characterised by a refractive index perturbation that can be expressed by

$$\delta n(x, y, z) = \bar{n}(x, y, z) \left\{ 1 + v(z) \cos \left[ \frac{2\pi}{\Lambda} z + \phi(z) \right] \right\},$$

where $\bar{n}(x, y, z)$ is the background index variation, $v(z)$ is the perturbation fringe visibility (related to the apodisation profile), $\Lambda$ is the grating period, and $\phi(z)$ describes the grating chirp. In fibre Bragg gratings, this is typically obtained by UV exposure of the fibre, and the induced variation extends only to the photosensitive region of the fibre. A transverse dependence $(x, y)$ of $\delta n$ is considered in general in Eq. (2.3). An ideally infinite number of forward and backward propagating modes $E_k^\pm$ are needed in the perturbed structure to solve the Helmholtz equation with $\hat{n} = \bar{n}(x, y, z) + \delta n(x, y, z)$:

$$E(x, y, z) = \sum_k \left[ A_k(z)e^{t_k(x, y)}e^{j(\beta_k z - \omega_0 \xi)} + B_k(z)e^{t_k(x, y)}e^{j(-\beta_k z - \omega_0 \xi)} \right],$$

where $A_k(z)$ and $B_k(z)$ are the forward and backward position-dependent slowly varying amplitudes of the $k$th mode, respectively.

Taking into account Eq. (2.2) for the unperturbed structure and integrating over the transverse profile of the fibre (different eigenmodes are orthogonal), the computation ends up with the following expression:

$$\frac{\partial A_j(z)}{\partial z} e^{j\beta_j z} - \frac{\partial B_j(z)}{\partial z} e^{-j\beta_j z} = j \sum_k K_{k,j}(z) \left[ A_k(z)e^{j\beta_k z} + B_k(z)e^{-j\beta_k z} \right],$$

where the coefficient $K_{k,j}(z)$ takes into account the transversally averaged coupling
between different modes $k$ and $j$ and can be written as

$$K_{k,j}(z) = \frac{\omega_0 \varepsilon_0}{4} \int \int \left[ \hat{n}^2(x, y, z) - n^2(x, y) \right] e_{t,k}(x, y) e^{*}_{t,j}(x, y) dxdy, \quad (2.6)$$

with $\varepsilon_0$ vacuum permittivity and normalising the modal transverse field amplitude in order to have unitary power [96]. If the refractive index perturbation is small, $\hat{n}^2 - n^2 \simeq 2n \delta n$ and Eq. (2.6) can more clearly be expressed as

$$K_{k,j}(z) = \sigma_{k,j}(z) + 2\kappa_{k,j}(z) \cos \left[ \frac{2\pi}{\Lambda} z + \phi(z) \right]. \quad (2.7)$$

$\sigma_{k,j}(z)$ is related to the average refractive index change, while $\kappa_{k,j}(z)$ takes into account the coupling related to the effective fringe visibility of the grating. The above approximation is always applicable in fibre Bragg gratings since typically $\delta n < 10^{-3}$, but it gives a quantitative idea about the limits above which coupled-mode theory fails and a correct Bloch mode analysis is required.

Equation (2.5) can be greatly simplified by considering the synchronous approximation [96]. If the $z$ phase dependence of the driving terms on the right side of (2.5) does not match the phase variation $\pm \beta_j$ of the driven field, no significant contribution to mode coupling is obtained since it is averaged out by integration upon the propagation direction $z$ (phase-matching conditions). This is nothing but an application of momentum conservation inside the system:

$$\beta_j + \frac{2\pi}{\Lambda} = -\beta_k \quad \rightarrow \quad \frac{2\pi n_{\text{eff},j}}{\lambda} + \frac{2\pi n_{\text{eff},k}}{\lambda} = -\frac{2\pi n_{\text{eff}}}{\lambda}, \quad (2.8)$$

where $n_{\text{eff}}$ is the effective refractive index and depends on the properties of the guiding structure. Only the fundamental modes $E_0^+ = A_0 e^{i(\beta z - \omega_0 t)}$ and $E_0^- = B_0 e^{i(\beta z - \omega_0 t)}$ have to be taken into account (where $\beta = \beta_0$ is set to simplify the notation), if the interaction close to the Bragg wavelength

$$\lambda_{\text{Bragg}} = 2n_{\text{eff}} \Lambda \quad (2.9)$$

is considered. The spatial evolution of their field envelopes, hereon simply labelled $A(z)$ and $B(z)$, is given by the following coupled wave equations (CWE):

$$\begin{align*}
\frac{dA(z)}{dz} &= j\sigma(z)A(z) + j\kappa(z)e^{i\phi(z)}B(z)e^{-j2\delta z}, \quad (2.10a) \\
\frac{dB(z)}{dz} &= -j\sigma(z)B(z) - j\kappa(z)e^{-j\phi(z)}A(z)e^{j2\delta z}, \quad (2.10b)
\end{align*}$$

where $\delta = \beta - \pi/\Lambda$ is the detuning from the Bragg wavelength (since $\delta = 0$ for $\lambda = \lambda_{\text{Bragg}}$). Eq. (2.10) can be derived under general conditions and without introducing the slowly varying envelope approximation (SVEA), as proven by [97, 98]. From Eq. (2.7), the self-coupling coefficient $\sigma$ and the cross-coupling coefficient $\kappa$ (usually called coupling coefficient of the grating since it is directly responsible for power trans-
fer between coupled modes) are
\[
\sigma(z) = \frac{\omega_0 \varepsilon_0}{2} \iint n(x, y) \Delta n(x, y, z) e_I(x, y) e^*_I(x, y) \, dx \, dy
\]
\[
= \frac{\omega_0 \varepsilon_0}{2} n \Delta n(z) \Gamma_{\text{core}}, \tag{2.11}
\]
\[
\kappa(z) = \frac{\nu}{2} \sigma(z). \tag{2.12}
\]

The parameter \( \Gamma_{\text{core}} \) accounts for the transverse dependence of the refractive index, of the UV-induced perturbation, and of the propagating modes. In the particular case of a step index profile and a photosensitive region limited to the fibre core, \( \Gamma_{\text{core}} \) is given by the core power confinement factor of the fundamental mode [88].

In order to simplify the solution of the system of differential equations, Eq. (2.10) is often written in a more concise way by introducing the effective forward- and backward-propagating fields \( U \) and \( V \):
\[
U(\delta, z) = A(z) e^{j\delta z} \quad \rightarrow \quad E_+(\delta, z) = U(\delta, z) e^{j\pi z / \Lambda_{\text{eff}}},
\]
\[
V(\delta, z) = B(z) e^{-j\delta z} \quad \rightarrow \quad E_-(\delta, z) = V(\delta, z) e^{-j\pi z / \Lambda_{\text{eff}}},
\]
where the dependence upon both detuning \( \delta \) and position \( z \) is shown. Therefore:
\[
\frac{\partial U(\delta, z)}{\partial z} = j\hat{\sigma}(\delta, z) U(\delta, z) + q(z) V(\delta, z), \tag{2.15a}
\]
\[
\frac{\partial V(\delta, z)}{\partial z} = -j\hat{\sigma}(\delta, z) V(\delta, z) + q^*(z) U(\delta, z). \tag{2.15b}
\]

In Eq. (2.15), \( \hat{\sigma} = \sigma + \delta \) is the effective detuning parameter inside the grating, and a complex coupling coefficient which takes into account both coupling and chirp has been introduced:
\[
q(z) = j\kappa(z) e^{j\phi(z)}. \tag{2.16}
\]

The modulus \( |q| = |\kappa| \) gives the strength of the modal coupling and is related to the amplitude of the perturbation. The accumulated phase shift along the grating length gives the optimum coupling condition related to phase matching. Actually, using a slightly different effective field substitution in Eq. (2.15) (see Erdogan [88] for details), it is apparent that optimum phase-matching is given by the combined effect of the grating chirp \( \phi(z) \), of the average effective index variation \( \delta n_{\text{eff}} = \delta n \Gamma_{\text{core}} \) of the mode, and of the wavelength detuning from the nominal Bragg wavelength \( \lambda_{\text{Bragg}} \). An effective grating periodicity can therefore be introduced:
\[
\Lambda_{\text{eff}}(z) = \Lambda \left[ 1 + \frac{\sigma(z) \Lambda}{\pi} - \frac{\Lambda}{2\pi} \frac{d\phi(z)}{dz} \right]^{-1}, \tag{2.17}
\]
and the effective Bragg condition at a certain position \( z \) inside the grating is given by
\[
\beta - \pi / \Lambda_{\text{eff}}(z) = 0.
\]

The CWEs (2.15) can be expressed in the time domain \( \xi \) by the Fourier transforma-
tion $\hat{\sigma} \leftrightarrow \xi$. The corresponding slowly varying impulse envelopes $u(\xi, z)$ and $v(\xi, z)$ for the forward- and backward-propagating fields, respectively, are

\[
\frac{\partial u(\xi, z)}{\partial z} = -\frac{\partial u(\xi, z)}{\partial \xi} + q(z)v(\xi, z), \tag{2.18a}
\]

\[
\frac{\partial v(\xi, z)}{\partial z} = \frac{\partial v(\xi, z)}{\partial \xi} + q^*(z)u(\xi, z), \tag{2.18b}
\]

where it is assumed, according to the sign conventions used as in Eq. (2.1), that

\[
u(\xi, z) = \frac{1}{2\pi} \int U(\hat{\sigma}, z)e^{-j\hat{\sigma}\xi}d\hat{\sigma}, \quad v(\xi, z) = \frac{1}{2\pi} \int V(\hat{\sigma}, z)e^{-j\hat{\sigma}\xi}d\hat{\sigma}. \tag{2.19}\]

The reflection coefficient $r = r(\delta)$ of the grating is directly derived once the above system of differential equations is solved under the boundary conditions $E_-(\delta,L_{\text{gr}}) = 0$ (no backward-propagating field after the grating) and $E_+(\delta,0) = 1$ (white light signal launched into the grating), which corresponds to an impulse grating excitation $u(\xi, 0) = \delta(0)$. $L_{\text{gr}}$ is the physical length of the considered grating, that is assumed to extend from $z = 0$ to $z = L_{\text{gr}}$. It can be expressed in either frequency ($\delta$) or time ($\xi$) domain by

\[
r(\delta) = \frac{E_-(\delta,0)}{E_+(\delta,0)} = V(\delta,0) = \int v(\xi, 0)e^{j\hat{\sigma}\xi}d\hat{\sigma}. \tag{2.20}\]

The above set of equations is extensively used in the analysis of mode coupling, in both optics and many other different physics applications. Simple extensions are possible in order to apply the same analytic and numeric tools developed in relation to the CWE to more general problems. In particular, the analysis of propagation in lossy scattering media simply requires the consideration of a corresponding complex propagation constant $\beta = \beta_0 + j\alpha_0$ in Eq. (2.10), where $\alpha_0$ is the attenuation expressed in m$^{-1}$. Moreover, coupling with higher order modes of the considered fibre (cladding modes) can be taken into account by properly applying the synchronous approximation to this case and writing an extended set of coupled differential equations (ECWEs, [99, 100]). This topic is extensively described in Section 5.2.3. Both the extensions previously described are used in conjunction with inverse scattering techniques for complex grating design in lossy media (see Chapter 5).

### 2.2.2 Solving the coupled-wave equations

In general, the CWEs (2.10) cannot be solved analytically for a generic choice of the chirp and apodisation profiles. In the first order Born approximation [29], valid only in the weak grating limit (i.e., when the probability of multiple scattering inside the grating is negligible and the overall reflectivity is less than $0.1 - 0.4$%), it can be shown from Eq. (2.10) that the grating reflection coefficient $r = \frac{E_-(0)}{E_+(0)}$ expressed in the detuning $\hat{\sigma}$ domain (corresponding to the more commonly used $\delta$ domain if no background index
variation is present, i.e., \( \sigma = 0 \) is

\[
r = r(\delta) = -\int_0^{L_{gr}} q^*(z)e^{j2\delta z}dz,
\]

(2.21)

where the integration extends over the whole length \([0, L_{gr}]\) of the grating. A simple Fourier transform relationship can be established between the grating spectral shape and the induced refractive index profile. This appealing result does not hold when higher reflectivities are considered, but the intuition deriving from the Fourier analysis is still valuable and was extensively used in the past for the design of apodised and chirped gratings (see Erdogan [88] for a detailed review).

An analytic solution exists only when a uniform grating is considered. If constant apodisation and chirp parameters are assumed, i.e., \( \sigma(z) = \sigma, \kappa(z) = \kappa, \) and \( \phi(z) = \phi, \) so that \( q(z) = q, \) the reflection and transmission coefficients \( r(\delta) \) and \( t(\delta) \) are given by

\[
r(\delta) = \frac{E_-(\delta,0)}{E_+(\delta,0)} = \frac{-q_2}{\gamma} \sinh(\gamma L_{gr})}{\cosh(\gamma L_{gr}) - j\frac{2}{\gamma} \sinh(\gamma L_{gr})},
\]

(2.22)

\[
t(\delta) = \frac{E_+(\delta,L_{gr})}{E_+(\delta,0)} = \frac{e^{j\pi L_{gr}/\Lambda}}{\cosh(\gamma L_{gr}) - j\frac{2}{\gamma} \sinh(\gamma L_{gr})},
\]

(2.23)

where \( \gamma^2 = |q|^2 - \hat{\sigma}^2 = \kappa^2 - \hat{\sigma}^2 \) is the propagation constant inside the scattering region. The band-gap region is defined by the value of \( \gamma \): inside the band gap \( \gamma \) is real and the propagation is evanescent, while outside the band gap \( \gamma \) is imaginary and normal propagation occurs. The peak reflectivity of a uniform grating occurs at the Bragg wavelength \( (\delta = 0) \) and is defined from Eq. (2.22) by

\[
R_{MAX} = |r(0)|^2 = \tanh^2(\kappa L_{gr}),
\]

(2.24)

where \( \gamma|_{\delta=0} = \kappa \) is typically used neglecting the self-coupling.

If a nonuniform structure of generic strength is considered, the reflection coefficient has to be computed numerically. A direct numerical integration of the CWEs (2.15) is possible, considering a local reflection coefficient \( \hat{r}(\delta,z) = V(\delta,z)/U(\delta,z) = E_-(\delta,z)/E_+(\delta,z)e^{j2\delta z}, \) expressed as a function of the effective propagating fields \( U \) and \( V, \) and deriving the corresponding differential equation in the form of the Riccati equation:

\[
\frac{\partial \hat{r}(\delta,z)}{\partial z} = -2j\tilde{\sigma}(\delta,z)\hat{r}(\delta,z) - q(z)\hat{r}^2(\delta,z) + q^*(z).
\]

(2.25)

Equation (2.25) can be integrated using the standard Runge-Kutta method under the boundary condition \( \hat{r}(L_{gr}) = 0. \) This technique is simple, but the numerical algorithm can require a very high number of steps to ensure convergence and is therefore quite slow.

A more efficient solution is represented by the use of a transfer matrix method (TMM) [86], as previously discussed. The grating is divided into \( N \) short, uniform gratings. Each
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subgrating has to be long enough so that the synchronous approximation used in the derivation of the CWEs still holds, but sufficiently short so that the variation of the grating parameters is negligible. The corresponding transfer matrix $T_i$ for a uniform grating labelled $i$ of length $\Delta_i$ can be computed from Eq. (2.10), and considering the local parameters $q_i, \gamma_i$, and $\hat{\sigma}_i$ results in

$$T_i = \begin{bmatrix} \cosh(\gamma_i \Delta_i) - j \frac{q_i^*}{\gamma_i} \sinh(\gamma_i \Delta_i) & -\frac{q_i^*}{\gamma_i} \sinh(\gamma_i \Delta_i) \\ -\frac{q_i^*}{\gamma_i} \sinh(\gamma_i \Delta_i) & \cosh(\gamma_i \Delta_i) + j \frac{q_i^*}{\gamma_i} \sinh(\gamma_i \Delta_i) \end{bmatrix} \begin{bmatrix} e^{-j\pi \Delta_i / \Lambda} \\ e^{j\pi \Delta_i / \Lambda} \end{bmatrix},$$  \hspace{1cm} (2.26)$$

where it is assumed, disregarding the explicit $\delta$ dependence of the electric fields and reflection coefficient in order to simplify the notation, that

$$\begin{bmatrix} E_+(z_i) \\ E_-(z_i) \end{bmatrix} = T_i \cdot \begin{bmatrix} E_+(z_i + \Delta_i) \\ E_-(z_i + \Delta_i) \end{bmatrix}. \hspace{1cm} (2.27)$$

The overall transfer matrix $T$ of the complex grating is given by the product of the matrices of the individual sections:

$$\begin{bmatrix} E_+(0) \\ E_-(0) \end{bmatrix} = T_1 \cdot T_2 \cdot \ldots \cdot T_N \cdot \begin{bmatrix} E_+(L_{gr}) \\ E_-(L_{gr}) \end{bmatrix} = T \begin{bmatrix} E_+(L_{gr}) \\ E_-(L_{gr}) \end{bmatrix},$$  \hspace{1cm} (2.28)$$

and can be efficiently computed numerically.

The last numerical approach that is applied to grating computation is a further approximation of the piecewise-uniform method just described. Every single subgrating is considered as a single, discrete reflector, and the whole structure is interpreted as a stack of such reflectors separated by free-space propagating regions (no scattering inside). Following Feced et al. [42], the transfer matrix $T_i$ is replaced by the product $T_i^S \cdot T_i^\Delta$, where

$$T_i^S = \frac{1}{\sqrt{1 - |\varrho_i|^2}} \begin{bmatrix} 1 & \varrho_i^* \\ \varrho_i & 1 \end{bmatrix} \hspace{1cm} (2.29)$$

is the discrete reflector matrix obtained from Eq. (2.26) by letting $\kappa_i \to \infty$ and holding $\kappa_i \Delta_i$ constant. The discrete reflector coefficient $\varrho_i$ is given by

$$\varrho_i = -\frac{q_i^*}{|q_i|} \tanh(|q_i| \Delta_i) \hspace{0.5cm} \text{with} \hspace{0.5cm} q_i = j \kappa_i e^{j \phi_i}. \hspace{1cm} (2.30)$$

The propagation matrix $T_i^\Delta$ is again derived from Eq. (2.26) by letting $\kappa_i \to 0$:

$$T_i^\Delta = \begin{bmatrix} e^{-j(\beta + \sigma_i)\Delta_i} & 0 \\ 0 & e^{j(\beta + \sigma_i)\Delta_i} \end{bmatrix} = \begin{bmatrix} e^{-j\hat{\beta}_i \Delta_i} & 0 \\ 0 & e^{j\hat{\beta}_i \Delta_i} \end{bmatrix},$$  \hspace{1cm} (2.31)$$

where $\hat{\beta}_i$ is the effective propagation constant in the perturbed medium and accounts for changes in the average refractive index. The propagation of the field amplitudes using
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The reflection coefficient \( r = r(0) \) of the grating is obtained by starting the recursion at \( z = L_{gr} \), where \( r(L_{gr}) = 0 \), and moving backward using Eq. (2.32). The beauty and strength of recursive relations like Eq. (2.32) is that they can be used with the same ease for both solving the direct (analysis) and the inverse (synthesis) problems. This approach is fast, since it involves the computation of \( O(N) \) hyperbolic functions instead of \( O(N^2) \) as when using Eq. (2.26). It even can provide better results when low-reflectivity gratings are simulated. Indeed, the “correct” method (2.26) represents the grating using a staircase approximation of the actual apodisation and chirp profiles, while the “discrete” method (2.32) simply samples the grating in fixed positions. According to the Fourier theory valid in the weak grating limit, sampling the grating profile with a step \( \Delta \) results in a periodic repetition of the original spectral response with period \( \Delta \delta = \frac{2\pi}{\delta} \) in the detuning domain. If this bandwidth is smaller than the bandwidth used to represent the grating spectrum, multiple replicas overlap and the spectral response is deformed (aliasing). If it is larger, no deformation occurs and the grating is perfectly represented. On the contrary, the staircase approximation is obtained by convolving the sampled profile just described with a rectangular function of length \( \Delta \) (sample-and-hold), which means that the spectrum is multiplied by a sinc-function with zeros in \( \delta = m\Delta \). The spectral response is therefore deformed. Unfortunately, this result cannot be extended to the more interesting case of strong gratings, since the multiple scattering occurring inside each subgrating is not taken into account in the point-reflection approach. Therefore, the grating characteristics are not modelled in an accurate way.

2.2.3 Properties of reflection and transmission spectra

The reflection and transmission coefficients of the grating are easily obtained once the transfer matrix \( \mathbf{T} \) (2.28) is known:

\[
\mathbf{T} = \begin{bmatrix} \frac{1}{t_r} & -\frac{r_+}{t_r} \\ \frac{r_+}{t_r} & t - \frac{r_+ r_-}{t_r} \end{bmatrix}.
\]  

Equation (2.33) is a direct consequence of the definition of transfer matrix, once the left and right reflection coefficients \( r_+ \) and \( r_- \), respectively, and the transmission coefficient \( t \) of the structure are properly identified:

\[
r_+ = |r_+| e^{j\theta_{R,+}} = \frac{E_-(0)}{E_+(0)} \bigg|_{E_-(L_{gr})=0},
\]  

\[
r_- = |r_-| e^{j\theta_{R,-}} = \frac{E_+(L_{gr})}{E_-(L_{gr})} \bigg|_{E_+(0)=0}.
\]
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\[ t = |t|e^{j\theta_T} = \left. \frac{E_+(L_{gr})}{E_+(0)} \right|_{E_-(L_{gr})=0} = \left. \frac{E_- (0)}{E_- (L_{gr})} \right|_{E_+(0)=0}. \]  

(2.34c)

Six different real parameters have to be identified in order to completely describe a grating, both the amplitude and phase of each coefficient in Eq. (2.34). Interesting properties of the transfer matrix coefficients are a consequence of the reciprocity of the medium [101]. In general, \( t_+ = t_- = t \), as is shown in Eq. (2.34c). If the scattering medium is lossless, further relations hold:

\[ |r_\pm|^2 + |t|^2 = 1, \]  

(2.35)

\[ \frac{r_+}{r_-} = -\frac{t}{t^*}. \]  

(2.36)

Equation (2.35) is the trivial power conservation and shows that \( |r_+| = |r_-| \), while Eq. (2.36) connects the phases of the two reflection coefficients and of the transmission coefficient:

\[ \theta_{R,+} + \theta_{R,-} = 2\theta_T + \pi. \]  

(2.37)

Using Eq. (2.36), the transfer matrix can simply be written in the lossless case as

\[ T = \begin{bmatrix} 1 & \frac{r_+}{r_-} \\ \frac{1}{r_+} & \frac{1}{r_-} \end{bmatrix}. \]  

(2.38)

Amplitude and phase of the transmission coefficient (2.34c) are also related in the lossless case. It was shown [13, 99] that \( t(\omega) \) is a minimum phase function since it has no zeros in the complex \( \omega \) plane, and therefore a logarithmic Hilbert transform uniquely relates \( |t(\omega)| \) and \( \theta_T(\omega) \):

\[ \theta_T(\omega) = \mathcal{H} \left[ \ln |t(\omega)| \right]. \]  

(2.39)

The knowledge of a single reflection coefficient (either \( r_+ \) or \( r_- \), i.e., only two real parameters) is a necessary and sufficient condition for a complete reconstruction of the grating characteristics in the lossless case. Indeed, \( |t| \) is obtained from Eq. (2.35), the corresponding phase \( \theta_T(\omega) \) from Eq. (2.39), and finally the reflection coefficient at the other grating side from Eq. (2.36).

Finally, the time delay experienced by reflected and transmitted light in the grating is classically related to the phase time definition, i.e., to the stationary phase principle [102]. Indeed, the first derivative of phase with respect to \( \omega \) is dimensionally a time, and it was shown to accurately describe the propagation of a pulse peak when negligible pulse distortion occurs [103]. From Eq. (2.37), the following relation holds in the lossless case:

\[ \tau = \frac{\partial \theta_T(\omega)}{\partial \omega} \quad \rightarrow \quad \tau_{R,+} + \tau_{R,-} = 2\tau_T. \]  

(2.40)

For grating applications in which the dispersive profile of the device is particularly important, the group delay ripple \( GDR \) rather than the time delay \( \tau \) of the grating is
typically used. GDR is defined as the deviation of the spectral response of the considered device with respect to the ideal characteristic. If a linear time delay is considered, as in dispersion compensators, the GDR is simply defined as

\[ GDR(\lambda) = \tau_R(\lambda) - \tau_{id}(\lambda) = \tau_R(\lambda) - D_{TOT} \lambda, \]  

(2.41)

where \( D_{TOT} \) is total dispersion to be compensated (in ps/nm). If this dispersion is generated by propagation through a link of telecommunication fibre, \( D_{TOT} = D_\lambda L_{link} \) depends on both the length \( L_{link} \) (in km) to be compensated and intrinsic dispersion of the fibre \( D_\lambda \) (in ps/nm/km)

\[ D_\lambda = -\frac{\lambda^2}{2\pi c} \beta_2 \]

with

\[ \beta_2 = \frac{\partial^2 \beta}{\partial \omega^2}, \]

(2.42)

the second-order dispersion term of the fibre with propagation constant \( \beta \), which produces a linear variation of the group delay (\( \beta_2 \) is proportional to the group delay slope).

### 2.3 Synthesis of gratings: the layer-peeling algorithm

The layer-peeling algorithm developed by Feced et al. [42] and Skaar et al. [43] allows the refractive index \( \delta n(z) \) and chirp \( \phi(z) \) profiles to be computed from a given reflectivity and time delay spectrum of the grating. This process has the same efficiency as the direct spectral computation starting from \( \delta n(z) \) and \( \phi(z) \). The described algorithm itself is exact, but, as will be clear later, the finite bandwidth and spectral resolution associated with the numerical implementation introduce a certain degree of approximation in the design process and limit the final agreement between the synthesised grating and the target.

#### 2.3.1 Description of the algorithm

Let us obtain an intuitive feeling of how the inverse scattering works. The core of the synthesis procedure is based on causality arguments, and simple physical understanding is obtained by considering the field propagation through the grating in the time domain and a multiple path picture (see Chapter 3 for further details). Considering Fig. 2.1, the grating is excited by a forward propagating impulse \( u(\xi, 0) = \delta(0) \), where \( \xi \) is time and \( z = 0 \) is the first layer in the grating that has to be identified. By causality, the impulse response \( v(\xi, 0) \) of the optical system at time \( \xi = 0 \) is related only to the grating coupling coefficient at \( z = 0 \). While an increasingly larger number of possible different paths are available for \( \xi > 0 \) due to multiple scattering, only the path \( z_{R,1}(\xi) \) (unambiguously related to the scattering coefficient at \( z = 0 \)) contributes to the reflected field at \( \xi = 0 \), since light has no time to propagate more deeply into the structure and to be scattered back. Therefore, given the target reflection coefficient \( r(\delta, 0) \) of the grating through the reflectivity \( R \) and the time delay \( T_R \), the impulse response is known since a Fourier transform \( r(\delta, 0) \xrightarrow{\mathcal{F}} v(\xi, 0) \) holds (see Eq. (2.20)), and the complex coupling coefficient
Figure 2.1: Multiple paths possible in a grating. \( u(\xi, z) \) is to the forward propagating field at time \( \xi \) and position \( z \), while \( v(\xi, z) \) is the backward propagating field. At layer \( z = 0 \), the grating is probed with a delta function and the corresponding impulse response, related to the reflection coefficient \( r(\delta, 0) \) by the Fourier transformation (2.20), is given by \( v(\xi, 0) \).

\( q(z) \) at \( z = 0 \) is uniquely determined.

The input fields can now be propagated through this first, known layer, as shown in Fig. 2.2 assuming an infinitesimal step \( dz \). This actually corresponds to the calculation of the local reflection coefficient \( r(\delta, z) = E_-(\delta, z)/E_+(\delta, z) \) of the target grating at \( z = dz \), but it can also be interpreted as the identification of a new target reflection coefficient to be synthesised. The effect of the first layer is “peeled off”, and the same situation as at the beginning is reached. Therefore, the new impulse response \( \tilde{v}(\xi, 0) \) is calculated and the coupling coefficient \( q(dz) \) is easily obtained. Iterating the procedure described until the target spectrum is reduced to zero, i.e., until no more photons have to be reflected in order to obtain the reflectivity desired at \( z = 0 \), the required grating coupling profile is reconstructed layer-by-layer along the whole grating length.

The intuitive algorithm just explained can be formalised either by assuming the correct CWEs (2.10) or via the simplified discrete reflector model described by Eqs. (2.29) and (2.31). The first approach leads to the “continuous layer-peeling” algorithm and it is extensively described by Poladian [91] and Skaar et al. [43]. Even though mathematically precise, the practical numerical implementation requires a finite step \( dz \rightarrow \Delta \) to be used and is intrinsically approximate even in this case. As discussed in Skaar et al. [43], an independent choice of layer thickness \( \Delta \), wavelength range \( \Delta \lambda \), and wavelength resolution \( d\lambda \) is possible, but the layer step required for convergence is shorter and the computation time significantly longer with respect to the discrete model implementation. Therefore, only the “discrete layer-peeling” synthesis method [42, 43] is considered in the following and extended to lossy gratings in the next Sections.

According to the formalism introduced in Section 2.2.2, the grating can be modelled by a stack of discrete, localised reflectors \( \rho_i \) located at positions \( z_i = i\Delta \) and separated
Figure 2.2: Propagation of the target reflection spectrum along the grating once the layer has been identified. Note that \( r(\delta, z) \) corresponds to the local reflection coefficient in the grating. An effective impulse response \( \tilde{v}(\xi, z) \) can be associated to \( r(\delta, z) \) and the coupling coefficient \( q(z) \) obtained from \( \tilde{v}(\xi, 0) \).

by free space propagation regions of length \( \Delta \):

\[
T^S_i = \frac{1}{\sqrt{1 - |q_i|^2}} \begin{bmatrix} 1 & q_i^* \\ q_i & 1 \end{bmatrix}, \quad T^\Delta_i = \begin{bmatrix} e^{-j\beta_i \Delta_i} & 0 \\ 0 & e^{j\beta_i \Delta_i} \end{bmatrix},
\]

where the propagation constant \( \beta_i \) is assumed to be in the unperturbed medium and

\[
\beta_i \to \hat{\beta}_i \text{ is considered in the free propagation matrix, i.e., the self-coupling coefficient } \sigma_i \text{ in Eq. (2.31) is neglected. This assumption is valid since the effect of a nonuniform background index can be included in the chirp profile, i.e., in the phase of the complex coupling constant } q_i \text{ [88]. Moreover, it must be stressed that the single scattering event has infinite bandwidth since it is related to a point defect. The wavelength selectivity of the grating is accounted for by the propagating matrices and the layer thickness } \Delta.
\]

Considering the identification of the scattering event at position \( z = z_i \), it is apparent from the point-scattering definition (2.43) and referring to Fig. 2.1 that

\[
\tilde{v}(0, z_i) = q_i \tilde{u}(0, z_i) = q_i \Rightarrow q_i = \frac{1}{2\pi} \int_{-\infty}^{+\infty} r(\delta, z_i) e^{-j\xi} d\delta \bigg|_{\xi=0},
\]

where \( \tilde{u}(\xi, z_i) = \delta(0) \) and \( \tilde{v}(\xi, z_i) \propto \mathcal{F}\{r(\delta, z_i)\} \) are the forward- and backward-propagating time responses of the new target grating at \( z = z_i \). Using Eq. (2.44), the grating coupling constant and chirp are obtained, and the propagation of the field amplitudes \( z_i \to z_{i+1} \) using the transfer matrices \( T^S_i \cdot T^\Delta_i \) can be performed in a computationally efficient way using the simple recursion

\[
r(\delta, z_i) = \frac{E_-(z_i)}{E_+(z_i)} \Rightarrow r(\delta, z_i + \Delta) = \frac{E_-(z_i + \Delta)}{E_+(z_i + \Delta)} = \frac{-q_i + r(\delta, z_i)}{1 - q_i^* r(\delta, z_i)} e^{-j2\beta_i \Delta}.
\]

The iterative application of Eqs. (2.45) and (2.46) leads to the complete characterisation of the grating along the length \( L_{gr} \), and ends when the target spectrum \( r(\delta, L_{gr}) \) is reduced to zero.
2.3.2 Choice of the parameters and limitations

The previous relations show how simple the discrete layer-peeling algorithm is theoretically. But they have to be numerically implemented using a discrete set of wavelengths, a finite bandwidth, and using discrete Fourier transforms. These practical issues have a major impact on the actual characteristics and approximations of the algorithm.

As the inverse scattering algorithm is based on the impulse response \( v(\xi, 0) \) of the grating, the target impulse response has to be physically realisable, i.e., it has to correspond to a physically possible grating. It has to be zero for negative times to ensure causality and also absolutely integrable to ensure stability [42]. Therefore, both a time shift by a \( \xi_0 \) interval and windowing with a suitable function \( w(\xi - \xi_0) \) of length \( \Delta \xi_w = 2L_{yr} \) are usually necessary before the actual algorithm can be applied. On the other hand, the windowing procedure described causes a spectral deformation of the target grating to be synthesised. In particular, the use of a non-smooth windowing profile can introduce abrupt transitions in the impulse response and, therefore, the typical Gibbs oscillations in the frequency domain, especially when squared frequency response and high reflectivity are required. This phenomenon is not only detrimental for the final synthesised filter shape (increase in in-band reflectivity and time delay ripple), but can introduce convergence problems in the layer-peeling algorithm if a target spectrum with \( R(\tilde{\delta}) > 1 \) is obtained in a certain bandwidth \( \tilde{\delta} \) after the windowing process. Indeed, a reflectivity \( R(\tilde{\delta}) > 1 \) is not physically possible in passive media, and this causes the algorithm to diverge.

The reason for obtaining a non-physically realisable starting impulse response is given by the interplay between the infinite impulse response of the grating, the finite bandwidth \( \Delta \delta \) of the target spectrum numerically implemented, and the aliasing related to the use of a discrete Fourier transform to compute \( v(\xi, 0) \) from \( r(\delta, 0) \). The presence of feedback terms in the CWEs is characteristic of systems containing poles in their frequency response. Therefore, the corresponding impulse response is not limited by the physical grating length, but theoretically extends to \( \xi \to +\infty \). This effect is physically related to the paths experiencing multiple scattering inside the structure and, therefore, both highly attenuated and highly delayed. Moreover, a frequency limited signal is known to be theoretically unlimited in time according to the “uncertainty” relation \( \Delta \delta \Delta \xi \approx 2\pi \). This means that the impulse response \( v(\xi, 0) \) related to a frequency limited version of the ideal target spectrum is characterised by both a leading edge (not present in the ideal filter) and a magnified trailing edge. The issues previously described become really important when the discrete nature of the numerical implementation of \( r(\delta, 0) \) and the use of discrete Fourier transforms is considered. A frequency discrete signal (with frequency resolution \( d\delta \)) is periodic in the time domain, with a fundamental periodicity \( \Delta \xi \) given by

\[
\Delta \xi = \frac{2\pi}{d\delta}.
\]  
(2.47)

Therefore, the leading and trailing edges of the target impulse response \( v(\xi, 0) \) are
“folded” on a $\Delta \xi$ timescale. Aliasing results in the numerical version $v(\xi_i, 0)$, preventing the correct identification of the grating starting point and deforming the desired response to be synthesised. Using a shorter sampling rate $d\delta$ in the frequency domain reduces the aliasing related to the discrete representation of signals. However, increased computation time is obtained in this case, and a trade-off has to be identified in the definition of the sampling period $d\delta \rightarrow d\lambda$ of the target spectrum used in the calculation.

In order to minimise these problems, a useful procedure is to overestimate the grating length $L_{gr}$ assumed in the windowing-apodising process with respect to the expected grating length $L_{exp}$. Either a further section of length $(L_{gr} - L_{exp})$ is added in the inverse scattering simulation after using a windowing function of length $\Delta \xi_w = 2L_{exp}$, or the length $\Delta \xi_w = 2L_{gr}$ of the applied windowing function $w(\xi - \xi_0)$ is “artificially” increased with respect to the expected grating length $L_{exp}$. The first solution does not improve the approximation of the original impulse response, but the added length $(L_{gr} - L_{exp})$ is useful for compensating the multiple reflections within the synthesised grating that are created by the finite windowing size. The second solution results in a reduced deformation of the impulse response and a better approximation of the target spectrum is obtained. Often the increased length does not contain any valuable information about the grating, since it simply takes into account the highly attenuated tails of the impulse response and gives refractive index variations at very low levels that have limited physical meaning (see Section 2.4.4). An improvement of the final synthesised grating is usually obtained even if the grating is truncated and the tails of the inverse scattered profile are disregarded in the direct simulation below a certain refractivity level.

The second method generally gives better results for a given total grating length $L_{gr}$, but it can produce worse time delay ripple if truncation is considered (see Section 4.3.3 for an intuitive understanding of this feature). Therefore, an optimised windowing-apodising technique must be found case by case if the algorithm potentialities have to be pushed to the limit. Anyway, the obtained differences are often below the experimental precision of the grating writing techniques, and have just a theoretical (limited) value.

As far as the simulation bandwidth $\Delta \delta$ is concerned, the total non-zero part of the grating target spectrum has to be included in order to avoid temporal aliasing. For a theoretically defined spectrum, this task is automatically achieved, but it can constitute a serious limitation when grating reconstruction is attempted starting from a measured spectrum (refer to Section 3.4.3). In particular, it results in blurred refractive index and chirp transitions, and in a length of the reconstructed grating that exceeds the actual grating length, as is clear from Fig. 3.16. Practically, measuring the $R(\delta) > -30$ dB part of the spectrum is usually sufficient in order to get a clear definition of the grating features. Moreover, as known from discrete Fourier transform theory, the overall bandwidth in the frequency domain limits the available time resolution in the time domain, which corresponds to the spatial resolution $\Delta$ in the grating reconstructed


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profile:

\[ d\xi = 2\Delta = \frac{2\pi}{\Delta\delta} \quad \Rightarrow \quad \Delta = \frac{\pi}{\Delta\delta}. \] (2.48)

Therefore, using band-limited versions of the target spectrum does not allow a complete spatial characterisation of the grating refractive index and chirp profiles. Indeed, as already discussed in Section 2.2.1, from a Fourier transform point of view the finite step of the spatial sampling function does not constitute a problem if the bandwidth requirements previously described are met (the spectral amplitude is negligible at the band-edge). However, the multiple scattering inside each \( \Delta \)-long section is not taken into account in case of strong coupling, and better spatial resolution improves the final agreement between the simplified point-scattering approach and the exact solution.

The considerations previously described lead to the choice of a small frequency sampling period \( d\delta \), a large total frequency range \( \Delta\delta \), and to an increased grating length \( L_{gr} \) in the simulation. The obvious drawback is represented by an increased inverse scattering computational time. The total number of samples used in the numerical representation of the target spectrum is \( M = \Delta\delta/d\delta \), and it corresponds to \( M \) computations of Eq. (2.46) for each grating layer. The total number of sections \( N = L_{gr}/\Delta \) grows both with \( L_{gr} \) and with the bandwidth \( \Delta\delta \), since better spatial definition is obtained in this case. The final algorithm complexity is therefore \( \mathcal{O}(MN) \), with \( M > N \) since the total impulse response \( \Delta\xi \propto M \) has to be longer than the windowed length \( \Delta\xi_w \propto N \) used in the inverse scattering simulation.

An improvement in the computation efficiency is obtained by running the inverse scattering algorithm in the time domain instead of in the frequency domain. The discrete scattering approximation of the CWEs (2.18) is easily derived from Eq. (2.43):

\[
\begin{bmatrix}
  u(\xi + \Delta, z_{i+1}) \\
  v(\xi - \Delta, z_{i+1})
\end{bmatrix} = \frac{1}{\sqrt{1 - |\varrho_i|^2}} \begin{bmatrix}
  1 & \varrho_i^* \\
  \varrho_i & 1
\end{bmatrix} \begin{bmatrix}
  u(\xi, z_i) \\
  v(\xi, z_i)
\end{bmatrix},
\] (2.49)

where \( u(\xi, z_i) \) and \( v(\xi, z_i) \) are the forward- and backward-propagating signals at position \( z_i = i\Delta \) and time \( \xi \) in the original grating (i.e., the origin is not shifted here, so that \( u \neq \tilde{u} \) and \( v \neq \tilde{v} \)). It is worth pointing out that \( u(\xi, z_i) = 0 \) for \( \xi < z_i \) because of causality. The synthesis relation (2.45) becomes

\[
v(\xi_i, z_i) = \varrho_i u(\xi_i, z_i) \quad \Rightarrow \quad \varrho_i = \frac{v(\xi_i, z_i)}{u(\xi_i, z_i)}\bigg|_{\xi=z_i=i\Delta}.
\] (2.50)

In this case, only the first \( N \) samples of the starting impulse response \( v(\xi, 0) \) are needed in order to reconstruct the complex coupling coefficient of \( N \) layers. Therefore, a very large number of frequency samples \( M \) is defined to minimise aliasing effects in the discrete Fourier transform \( r(\delta, 0) \xrightarrow{\mathcal{F}} v(\xi, 0) \). But, after \( v(\xi, 0) \) is computed, shifted, and windowed, only the first \( N \) samples are retained, reducing the algorithm complexity to \( \mathcal{O}(N^2) < \mathcal{O}(MN) \).
2.4 Manufacturing techniques

As already introduced in Section 1.1, a large number of different manufacturing technologies are available for the production of fibre Bragg gratings. The main characteristics of the most important fabrication methods are reviewed in this Section in order to understand which are the present practical limitations. This both sets the boundaries on the complexity of advanced designs that can be analysed, and shows where the problems are and where improved theoretical understanding is needed. The attention is focussed here on fibre-based implementations, despite the fact that most of the techniques developed in this work are applicable to generic periodic structures, such as UV-written gratings in planar waveguides or surface relief gratings. The high level of reliability demonstrated by fibre grating technologies, briefly summarised in Section 1.1, justifies this choice, compared to the relative immaturity of grating fabrication in different waveguides. However, no detailed description of the various theories about physical origin of photosensitivity is included in this thesis. The reader is referred to general reviews on the topic [104].

2.4.1 Interferometric fabrication techniques

The first practical technique for forming Bragg gratings in photosensitive fibres was the interferometric fabrication technique, demonstrated by Meltz et al. [2]. It utilises an interferometer that splits the incoming UV light into two beams and then recombines them to form an interference pattern on the photosensitive core of the fibre. Cylindrical lenses are normally placed in the interferometer to focus the interfering beams to a fine line matching the fibre core. A periodic refractive index modulation is thus permanently induced in the fibre. The Bragg grating period \( \Lambda \) depends on both the irradiation wavelength \( \lambda_{UV} \) and the half angle between the intersecting UV beams, so that Bragg gratings can be inscribed at any wavelength by simply changing the intersecting angles. Both fibre and pattern are stationary, which limits the grating length to the spot size of the beam. A further disadvantage of the interferometric technique is its susceptibility to mechanical vibration and, in general, to variations in the two separated optical paths. The former can cause the fringe pattern to drift, washing out the grating, while the latter may affect its stability. Moreover, good quality gratings require the use of an optical source with good spatial and temporal coherence and excellent output power stability, which makes high-energy excimer lasers unsuitable for this technique. Special laser configurations are necessary, but they are extremely costly. Frequency doubled, Argon lasers (FRED) are a better solution, but their output power is much lower and, besides increasing the grating writing time, the maximum refractive index change obtainable is often reduced.

2.4.2 Phase mask techniques

The introduction of the phase mask technique [3] overcomes the typical problems of the interferometric approach. A phase mask is a uni-dimensional surface-relief structure of
period $\Lambda_{PM}$ written on a UV-transparent silica plate by holographic methods or electron beam lithography. The profile of the periodic relief grating is chosen such that maximum diffraction efficiency for the $\pm 1$ orders is obtained at the UV writing wavelength, while the 0th order is highly suppressed. A near-field pattern with periodicity $\Lambda = \Lambda_{PM}/2$ is produced by a normal-incidence UV beam by interference of the $\pm 1$ orders. It can be imprinted in the core of the fibre by placing the fibre in close contact with the phase mask. The simplicity of using a single optical element provides a robust and inherently stable method for reproducing fibre Bragg gratings. Sensitivity to mechanical vibrations is minimised, and low temporal coherence of the laser source does not affect the writing capability due to the balanced geometry of the system. This allows excimer lasers to be used much more efficiently than is possible with an interferometer. A further advantage is represented by the possibility of scanning the UV-beam through the phase-mask and writing gratings whose length is not limited by the interfering pattern size [11].

The phase mask technique is compatible with the realisation of both apodised and chirped gratings, but with a significant complication in the fabrication techniques. Both a phase and an amplitude masks are necessary for apodised gratings [9], and a two step process is also required to avoid spurious chirp resulting from a non-uniform variation in the average refractive index. It is possible to use a simpler technique which requires no amplitude mask if a fixed apodisation profile ($\sin^2$) is sufficient, but a double exposure is still required [10]. For chirped gratings, a double exposure was originally used [105] by writing a uniform grating and then changing the background index of the fibre linearly. Fibre cladding etching was also proposed for the same reason, but an effective fabrication was possible only with the use of (step-)chirped phase masks [106]. These masks provide a reasonable, though non-optimal [107], approximation of a linear dispersion profile, but allow the complication to be removed from the grating writing process. Nevertheless, the quality of the grating depends on the quality of the phase mask.

Masks fabricated using e-beam technology are made of small regions stitched together, which makes them suitable for imprinting very complex structures directly in the mask. However, errors in the phase continuity of the mask (stitching errors) are directly transferred into glitches in the grating that ultimately limit the device quality, especially when long masks are considered [108]. Conversely, holographic masks have generally better quality [108], but suffer from poor characteristics towards the edge of the mask [109, Chapter 6]. Recently, improvements in both cited techniques have been reported [110, 111, 112], and it is nowadays possible to include very complex apodisation and chirp profiles also in holographic phase masks [63, 113]. This approach is highly attractive for mass production of relatively short gratings, but further improvement of the phase mask fabrication technology is required before it can compete (in terms of grating quality) with the continuous writing techniques discussed in the following Section.

Increase in the length of the grating beyond the physical dimension of the mask is possible if multiple gratings are written separately in a single piece of fibre [12]. The phase between each sub-section was controlled using UV post-processing of the
refractive index of the joining regions to maintain phase coherence inside the structure. Unfortunately, imperfections in the stitching procedure and different ageing of these regions can lead to severe performance limitations. More recently, a phase-sensitive alignment between the end of a previously written grating and the interference pattern of the subsequent section has been attempted [114, 115], so that the grating structure should be inherently coherent and no post-processing is needed. However, the quality of the obtained gratings is still questionable.

Another drawback of the phase mask technique is its inherent lack of flexibility. The Bragg wavelength of the grating is determined by the mask pitch $\Lambda_{PM}$ since there is no possibility of changing the angle of intersection of the two diffracted beams, and a limited tunability is obtained only by physically stretching the fibre either during or after the inscription. A possible solution is given by the “mask image projection” technique [116] with the use of tunable interferometers [60, 117]. The phase mask is used to generate two diffracted orders, which are then remotely combined via an optical arrangement that can vary their angle of intersection. Flexibility is increased as a simple two beam interference pattern rather than the complex near-field of the phase mask is used, and the phase mask does not have to be in close proximity to the fibre. The inevitable trade-off is the increased sensitivity to perturbations that can plague interferometric methods, even if this effect is typically minimised by using Sagnac-type interferometers.

In all the methods presented so far, the relative position between the fibre and the phase mask is fixed. At most, the optics are tuned to add wavelength flexibility, which is detrimental for the stability of the writing set-up. An alternative solution where no movements in the optics are required is to allow a relative movement of either the mask or the fibre [34]. Gratings detuned from the fundamental phase mask wavelength are obtained if a fixed velocity is used; chirped structures can be fabricated using a fixed acceleration; discrete phase shifts are introduced in the grating by a localised movement; apodisation is controlled by a rapid dithering of the fibre [35].

### 2.4.3 Continuous grating fabrication techniques

A more elegant solution to fabricating long and complex gratings is to synthesise the structure with many short exposures that are separated by one (or several) grating periods. In a first attempt [33], each grating plane was produced separately by a focussed single pulse of an excimer laser. Absolute flexibility in the grating design is possible given the point-by-point definition of the periodic structure. However, a relatively long process time is required, and errors in the grating spacing due to either thermal effects or strain variations are likely to occur. This still limits the grating to short lengths. Moreover, gratings for applications at 1550 nm were produced only using second- or third-order Bragg scattering due to the impossibility to focus the UV radiation down to a few hundred nanometres.

A more effective approach is to write short (beam diameter sized) subgratings that may overlap [36, 118]. The traditional idea of phase mask-based FBGs is turned upside-
down. The UV beam does not scan the mask and imprint its pattern into a stationary
fibre, but rather the beam and the mask are (nominally) stationary, while the fibre
is translated continuously through the interference pattern. In order to generate a
modulated pattern, rather than a uniform increase in the refractive index, a CW, UV
beam is modulated so that successive exposures are separated by a single grating fringe
(or a fraction of a fringe). Indeed, apodisation, chirp, and discrete phase shifts are
introduced simply by controlling the relative phases of such exposures on a local basis.
A large number of practical demonstrations have already been referenced in Section 1.1.

The continuous writing technique has several advantages with respect to other fab-
rication methods: gratings can be much longer than available phase masks, and are
limited only by the length of the translation stage used; their characteristics and designs
are not limited by the available phase masks, since a wide range of structures can be
made without the need for complicated optical set-ups; the grating quality is not limited
by the phase mask quality; gratings are continuous in nature, and are free from large
phase glitches as found in stitched gratings; a modulated CW source avoids any problem
of fibre (or mask) damage and intensity noise associated with high-energy pulsed lasers.
A further key point of this approach is its inherent error reduction in the fabrication
process. The location of the multiple exposures that define the grating is essential in
determining the quality of the final result, but it is at the mercy of quantisation and
noise on the measurement of position (since nanometre or sub-nanometre precision is
typically required). If the beam size $D_{\text{beam}}$ used in the grating writing comprises of
many grating fringes $\frac{D_{\text{beam}}}{\Lambda}$, the basic Central Limit Theorem in statistics assures that
the position uncertainty in the resulting grating structure is reduced by a factor $\sqrt{\frac{D_{\text{beam}}}{\Lambda}}$,
which means by more than one order of magnitude for beam sizes of 55 $\mu$m or more.

Controlling of the phase (position) of the different exposures is indeed essential in
this technique. Single phase jumps result in localised phase shifts; a wavelength change
requires a linear dephasing of successive exposures; linear chirp is introduced using a
quadratic phase change (the understanding is similar to the effect of a relative movement
between mask and fibre described in the previous Section). Apodisation is obtained
by considering the grating as formed by related pairs of single-exposure subgratings.
The overall index change induced by such a doublet is simply the vector addition of
the two exposures, so that maximum visibility occurs when they are in phase, and no
modulation when they are out-of-phase by $\pi$ ($\pm \frac{\pi}{2}$ with respect to the nominal position).
The same UV fluence is delivered on each section of fibre, so that a constant background
index variation is obtained and no self-chirping effects occur. Almost arbitrary complex
features are obtained by combining all these features together.

In the described implementations, an interferometric control of the relative position
of mask and fibre is used to trigger an acousto-optic modulator, used as an ON-OFF
shutter, when pre-determined positions are reached. A sort of digital coding of the
characteristics of the grating is used. A very similar technique was proposed by Brennan
and LaBrake [59]. However, a fibre spool rather than a linear translation stage was used,
extending the theoretical length limitation for FBGs to 10 m or more, and an analogue modulation of the UV writing beam was considered, similarly to standard magnetic tape recording. Given the (known) constant velocity of the fibre, the nominal frequency $\omega$ corresponding to the phase mask pitch $\Lambda_{PM}$ is found. Wavelength detuning corresponds to a frequency variation, while apodisation is obtained by changing the visibility of the control signal (and keeping the average value constant to prevent self-chirping). Despite the theoretical advantages of this approach, the quality of the gratings presented so far is not yet completely satisfactory [119, 120]. Analogue modulation is used also in the configuration proposed by Yoffe et al. [60], but the UV beam has constant intensity in this case and the features previously discussed are obtained by phase mask dithering (similarly to [34, 35]). This method is still used and provides good results if a careful design of the analogue waveform is considered [121], even though it has not the same level of maturity as the “standard” continuous grating writing described first.

Very recently, other continuous techniques have been demonstrated. Varming et al. [61] showed that splitting the two polarisations of the UV writing beam, shifting them by an effective $\pi$ phase shift, and controlling their relative intensity is a possible way of writing apodised gratings with $\pi$ shifts included by simple phase mask scanning. A single, uniform mask is needed, and no fibre or mask movements have to be controlled with nanometre precision. The maximum grating length is limited to the phase mask size, no chirped gratings can be manufactured, but the simplification of the set-up makes it appealing for “standard” FBG production. Finally, Knothe and Brinkmeyer [62] combined the continuous writing idea with phase mask projection using a Sagnac interferometer. Chirp is obtained by tuning the intersection angle of the two beams, while apodisation is controlled by a phase shifter inside the polarisation sensitive interferometer (details are reported in the paper). Arbitrary structures can be obtained, but the length of the grating is again fixed since no relative movement between fibre and mask is used. Due to their recent demonstrations, the potentialities of the latter two approaches for reliable manufacturing of very complex structures cannot be evaluated yet.

2.4.4 Practical limitations in continuous grating fabrication

According to the previous discussion, the continuous grating writing technique developed and used at the ORC is still the most reliable and powerful manufacturing method when very complex grating profiles have to be realised, such as the one produced by the inverse scattering algorithm. In this Section attention is focussed on the limitations of this technique, so that it is possible to discriminate between grating designs which are practically manufacturable, and designs which are mathematically correct, but of no practical interest. A detailed description of manufacturing issues and limitation is given in [109, Chapter 4].

A first aspect to be considered is how much wavelength detuning and chirp can be obtained by using the dephasing approach described in the previous Section. Complete constructive interference between exposures allows the maximum index change to be
obtained, while dephasing changes the periodicity of the modulation, but also reduces the fringe contrast (self-apodisation effect). Minimum fringe visibility is obtained when a $\pi$ phase change occurs between exposures at either extent of the beam. Interestingly, this limit corresponds to the first zeros in the reflectivity response of a weak, unapodised, unchirped grating of length $D_{\text{beam}}$. The spectral response of a single UV-exposure subgrating ultimately determines the range of reflection bands that can be realised with a uniform phase mask.

Maximum tunability is obtained by minimising the beam size used in the writing process. However, (1) smaller beams imply less statistical reduction of noise, and subsequently poor quality gratings, and (2) tight focussing reduces the depth (and therefore the contrast) of the UV interference pattern. If phase mask projection is not used, the minimum distance between the mask and the photosensitive core of the fibre is $\sim 65 \mu m$, which means $D_{\text{beam}} > 30 \mu m$ given the diffraction angle of the mask (typically $13^\circ$). But less than 10% of the UV power actually contributes to the interference pattern on the core in this case (the two diffracted beams are almost physically separated), while $D_{\text{beam}} > 100 \mu m$ is required to have at least 50% of used power. Using this value, the maximum detuning to get a dephasing of $\pi$ is $\sim 8$ nm, but it is reduced to only $\sim 2$ nm if a 1 dB degradation of the fringe visibility is tolerated.

This is not a fundamental limit of the continuous writing technique, but rather a limit on the degree of wavelength detuning that can be achieved without the need for complex optical arrangements or dedicated phase masks. 30 nm detuning range for full EDFA-bandwidth applications is obtained by using chirped masks whose period changes by $\sim 20$ nm across the mask length and by scanning the beam. The advantage of this method is that it maintains the optical stability of the phase mask approach while allowing a wavelength detuning much larger than otherwise available. The main disadvantage is represented by possible variations in the local chirp and quality of the mask, that directly affect the characteristics of the fabricated gratings.

The beam dimension $D_{\text{beam}}$ also determines the possibility of apodising the grating using pairs of dephased exposures and, in particular, the maximum rate of change of the apodisation. Indeed, a change in the level of apodisation is obtained by changing the $\pm$ dephasing of the pairs along the grating, i.e., introducing chirp in the two equivalent sequences of subgratings in the pair. Self-apodisation may occur if this chirp is too large, preventing an accurate control of the apodisation (for instance, it would be impossible to pass from zero to full fringe visibility). The practical (and reasonable) result of the analysis is that the beam should be less than the size of any feature in the grating structure that goes from a state of no apodisation to one of full apodisation (or vice-versa). Similar analysis is applicable to discrete phase shifts introduced in the structure. The phase shift in the subgratings is instantaneous, but the effect is blurred by the multiple exposures and finally extends over the full size of the beam.

As a result, the continuous grating writing is characterised by a limited spatial resolution of the longitudinal features that can be effectively imprinted in the fibre. Noise
averaging is fundamental in very complex gratings and cannot be sacrificed. Effective utilisation of the UV power is necessary if very large index modulations are required. These issues prevent the use of a very tightly focussed writing beam, and, as a best result to date, complex features on a 200 – 300 µm scale have been demonstrated [50] with full control over amplitude and phase inside the grating, very high refractive index modulation, and excellent adherence of the grating characteristics to the theoretical design. However, no simultaneous control of complex apodisation and chirp was required in the cited grating. If accurate definition of both profiles have to be achieved, the spatial resolution requirements need to be further relaxed (dephasing related to chirp would also affect apodisation and vice-versa, limiting the control over the local features). Features on a few-millimetre scale have typically been obtained [49] and represent a more reasonable benchmark for the writing set-up.

Another point to be addressed is the overall grating length $L_{gr}$ which can be obtained. As already discussed, the continuous grating writing set-up is theoretically limited only by the physical length of the linear stage used to translate the fibre. However, further problems occur in the practical grating fabrication. Accurate position measurement over the full stage movement is required; mechanical properties of long stages can affect the movement and introduce velocity noise (changes in the UV fluence result, i.e., spurious chirp). Sag in the fibre when mounted on the stage requires accurate tracking of the core position with respect to the UV beam to keep the UV fluence constant and prevent phase errors. Relaxation, during the writing process, of the tension under which the fibre is held between the grooves at either side of the stage (to also limit the sag) also introduces chirp; the process time is proportionally increased, with increased requirements on the stability of the UV source and of the optics.

Intrinsic fibre non-uniformities also play a significant role over very long lengths [109, Chapter 6]. Fluctuations in the fibre diameter typically occur due to the fibre drawing process, resulting in small changes in the effective refractive index of the fibre of the order of $\delta n \sim 10^{-6}$. Unpredictable variations in the strength and chirp of the grating result. The possibility of producing coherent grating structures of long length is therefore limited, so that long gratings have typically a poorer quality. A quantitative parameter is given by the coherence length $L_{coh}$, defined as the length over which phase coherence is maintained inside the grating despite noise and random fluctuations [109, 122]. Typically, reported values of $L_{coh}$ are of the order of tens of centimetres, and therefore they are a tighter limit than the actual translation stage length. Excellent potentialities for manufacture of up-to-1 m long chirped gratings have been shown [37], but it is not obvious that more complex structures, such as inverse scattering designed gratings, can be scaled to this physical length. Indeed, coherence control is fundamental for these applications, while linear dispersion compensators are not so demanding.

Packaging of these gratings is a further issue to be considered. It is not a fundamental limitation of the manufacturing technique, but it cannot be disregarded if practical application (and not laboratory demonstration) is targeted. Proper athermal packaging
is needed to stabilise gratings against environmental fluctuations, and typical packaging
lengths are again of the order of tens of centimetres. Recent industrial research [120]
has showed that different solutions suitable for several metre-long gratings are possible,
although not trivial or cheap.

Finally, the non-uniformity of the fibres and the fluctuations in the writing process
have another consequence on the complex designs that can be realised. Indeed, an in-
trinsic noise floor around $\delta n \sim 10^{-6}$ exists in the manufacture of gratings, so that any
theoretical feature at levels lower than or comparable to this value cannot be experimen-
tally reproduced. A large relative error is expected in this case, limiting the dynamic
range of the writing set-up.
Chapter 3

Time delay distributions in Bragg gratings

3.1 Motivations and background

The propagation of light in one-dimensional periodic or quasi-periodic media has been extensively studied mainly in the context of the fundamental properties of photonic band-gaps [74]. Multilayer mirrors, corrugated waveguides, and fibre Bragg gratings (FBGs) are among the most common implementations of such structures. These scattering structures are characterised by a high dispersion around the stop-band edges and show significant variation in the group velocity as the wavelength is varied across the stop-band [123]. Compared to free space propagation, the group velocity $v_g$ can be either increased for in-band propagation or decreased close to transmission resonances, and significant time delay variations (where $\tau = L/v_g$) can be achieved with the introduction of nonuniform perturbations or defects [88].

However, only the overall time delay characteristic of the entire scattering region has been analysed so far. Typically, a transfer matrix approach is used to calculate the reflection and transmission coefficients. It is based on coupled wave theory for fibre gratings [83, 86] or on the actual layer parameters for multilayers [124]. The time delay (phase time) is obtained by differentiation of the global phase retardation $\theta$ with respect to $\omega$ [88]. No knowledge of the local properties is gained. Questions such as “How is the time delay accumulated along the length?” and “Are there sections that are likely to affect the light propagation more than others?”, are important for both physical and technological reasons. Only a layer-by-layer analysis of reflected and transmitted light behaviour can give such insight.

The object of this Chapter is the derivation of analytical expressions for the local time delays $\tau_R(z)$ and $\tau_T(z)$ for reflected and transmitted light, respectively, and their application to some significant cases (uniform gratings, chirped gratings) [125]. A multilayer is considered in Zhu et al. [126], and a Fabry-Pérot-like picture is proposed for layer-by-layer characterisation. The structure is divided into two different sections preceding and
following the currently investigated layer, and multiple reflections are experienced inside this effective cavity before either transmission or reflection takes place. The probability of each path is calculated through the reflectivity $R$ and the transmissivity $T$, and the average time spent in the layer can be evaluated since each possible path has a well-defined associated transit time. The propagating field is treated as a classical particle in this analysis, and the scattering probabilities are calculated by wave theory. But light is best described by a waveform and its field (probability amplitude) evolution. As pointed out elsewhere [127], the fictitious particle picture (“photon”) can still be used, but only taking into account that it has to be associated with a probability amplitude $|\psi\rangle \rightarrow E$, i.e., a complex number with a phase that varies spatially. Moreover, “if a particle can reach a final state by two possible routes, the total amplitude for the process is the sum of the two routes considered separately” [128]. This means that the infinite series of different paths leading to either transmission or reflection can be independently identified and their interference effects taken into account, if the fields $E$ (phase information retained) and not intensities $I$ are considered in the Fabry-Pérot-like picture. The average time delay of both transmitted and reflected photons can be evaluated in each grating section since a precise traversal time is associated with every single path. This approach was first introduced by Wang and Zhang [129] and Japha and Kurizki [130]. It has recently been used by Lee et al. [131] to analyse a single Fabry-Pérot cavity, and it was shown to be an application of Feynman paths in the spirit of Sokolovski and Connor [132].

The derivation presented is structure independent and can be applied to any scattering medium, either uniform or nonuniform. The obtained local time delay is a complex number, in which the real part is directly related to the time spent by light in the considered section of the grating. Perfect agreement is found between the classical definition of phase time and the local time delays integrated along the grating length. But further proof that the presented analysis is correct is obtained by relating the local time delays to well-established operative definitions of time delay used throughout different areas of physics. The most common approach is to correlate the time spent by light in a certain region to the change in a physical quantity induced by an external perturbation, when a direct and linear relation can be inferred between such a change and the time of interaction. From a historical perspective, this approach originated from the problem of measuring the duration of quantum mechanical collisions and was first applied to the traversal time for particle tunnelling by Büttiker and Landauer [133]. Different external perturbations were proposed in quantum mechanics, such as oscillating barrier [133], time-modulated incident wave [134], or Larmor precession of spin [135]. Later, the same approach was transferred to optics, by either introducing lossy layers [136] or applying magnetic fields and exploiting the resulting Faraday rotation [137]. The effect of the introduction of an infinitesimal phase error in a localised position inside the grating is analysed in this work, in close similarity to the approach by Steinberg [136]. The obtained results are consistent with the general expressions derived for the local time
delay, but better physical understanding is obtained in the latter case. Moreover, precise physical meaning is given to the corresponding imaginary part using this method, and a framework for the experimental verification of the presented approach is obtained.

The obtained results are likely to be helpful in energy storage characterisation (important in active devices design, i.e., distributed Bragg reflector and distributed feedback lasers [138]) and in the design of devices with a particular dispersion (i.e., dispersion compensating fibre Bragg gratings [5]). The approach derived can also contribute to the broad discussion about “traversal time under a potential barrier”, which applies to both electromagnetic evanescent propagation and quantum particle tunnelling because of the analogy between the Helmholtz and the time-independent Schrödinger equations [139] (see Chiao and Steinberg [140] and references therein for an exhaustive review). But the main practical application is represented by the possibility to easily derive the sensitivity of the grating to external or intrinsic perturbations, independently of the physical significance of local time delays, or the meaning of complex times. This is important from an engineering point of view since it is the basis for a robustness analysis of each grating design. Intuitively, a device is more sensitive to imperfections where the light actually dwells the most. Analyses with both single and distributed perturbations are possible, and they will be extensively addressed in Chapter 4.

In this Chapter, the local time delay is defined in Section 3.2 and analytical expressions are presented. The actual derivations are given in Appendix A. The energy velocity is also considered and the relationship with previously defined time delays discussed. The derived expressions are applied in Section 3.3.1 to a uniform grating and the corresponding time delay distributions are described. The same approach is applied in Section 3.3.2 to chirped gratings to improve our understanding of characteristic features such as group delay ripple. The perturbation analysis of gratings is introduced in Section 3.4. The analytical formulas that connect the local time delay results to general parameters of the perturbed structure (time delay, reflectivity and transmissivity) are obtained, and the theoretical aspects related to the identification and measurement of local time delays are outlined. The close relationship with the weak measurement theory for tunnelling times developed by Steinberg [136] is clearly shown. Finally, details of the experimental measurement of the imaginary part of the local time delay are given in Section 3.4.3.

3.2 Derivation of time delay distributions

3.2.1 Approach to local time delay computation

The first step towards deriving a local measure of the time spent by light inside a certain section of the grating is to introduce a proper definition of traversal time. In particular, a definition which allows an independent characterisation of the traversal time of light that is either finally transmitted or finally reflected is appealing, since different properties are expected inside the scattering region depending on the final state \(|\psi_f\rangle\), where \(f = T\)
for transmitted light and \( f = R \) for reflected light. The mean traversal time of a certain section inside a scattering medium is computed here using the following definition \[130\]

\[
\tau_f(s) = \frac{\langle \psi_f | \hat{\tau}_f(s) | \psi_f \rangle}{\langle \psi_f | \psi_f \rangle} = \frac{\int \hat{\tau}_f(s) E_f^*(\xi) E_f(\xi) d\xi}{\int E_f^*(\xi) E_f(\xi) d\xi}. \tag{3.1}
\]

\(| \psi_f \rangle\) is the final state to be characterised, \( E_f \) is the associated electric field, \( s \) is the considered layer in the scatterer, and the integration is performed over the time \( \xi \). In a grating of length \( L_{gr} \), the final state is \( | \psi_T \rangle \rightarrow E_f = E_T(\xi, L_{gr}) \) for the transmission time delay and \( | \psi_R \rangle \rightarrow E_f = E_R(\xi, 0) \) for the reflection time delay, where the fields are evaluated at time \( \xi \) and in position \( z = L_{gr} \) or 0, respectively. The traversal time operator \( \hat{\tau}_f(s) \) through \( s \) can be defined once the actual \( m \)th path \( \zeta_{f,m} = \zeta_{f,m}(\xi) \) to reach the considered final state \( | \psi_f \rangle \) at time \( \xi \) is known:

\[
\hat{\tau}_{f,m}(s) = \int \Theta_s[\zeta_{f,m}(\xi)] d\xi \quad \text{where} \quad \Theta_s[\zeta_{f,m}] = \begin{cases} 1 & \text{if } \zeta_{f,m} \in s, \\ 0 & \text{if } \zeta_{f,m} \notin s, \end{cases} \tag{3.2}
\]

where \( \Theta_s \) selects only the portion of the path spent inside the investigated layer \( s \). Given a possible path \( \zeta_{f,m}(\xi) \), the related time delay is well defined and unique.

It is therefore necessary to identify all the paths \( \zeta_{f,m} \) possible in a typical grating. Fig. 3.1 shows a simple sketch of these paths for both transmitted (a) and reflected (b) light. The distributed scattering produces a continuum of possible different trajectories \( \{\zeta_{f,1}, \zeta_{f,2}, \ldots, \zeta_{f,m}, \ldots\} \) for each final state \( | \psi_f \rangle \), and therefore

\[
\hat{\tau}_f(s) = \int_m \hat{\tau}_{f,m}(s) \delta_{f,m} dm, \tag{3.3}
\]

where \( \delta_{f,m} \) means that the time delay operator \( \hat{\tau}_{f,m}(s) \) has to be associated only with the contribution \( | \psi_{f,m}(s) \rangle \) of the path \( \zeta_{f,m} \) to the final state \( | \psi_f \rangle \), i.e.

\[
\hat{\tau}_f(s) | \psi_f(s) \rangle = \int_m \hat{\tau}_{f,m}(s) \delta_{f,m} | \psi_f(s) \rangle dm = \int_m \tau_{f,m}(s) | \psi_{f,m}(s) \rangle dm \tag{3.4}
\]

with time delay eigenvalue \( \tau_{f,m}(s) \). Using (3.4), Eq. (3.1) may be written in a more concise way

\[
\tau_f(s) = \int_m \tau_{f,m}(s) \frac{| \psi_f \rangle \langle \psi_{f,m}(s) |}{\langle \psi_f | \psi_f \rangle} dm \\
= \int_m \tau_{f,m}(s) \frac{\int E_f^*(\xi) E_{f,m}(\xi) d\xi}{\int E_f^*(\xi) E_f(\xi) d\xi} dm = \int_m \tau_{f,m}(s) P_{f,m} dm, \tag{3.5}
\]

where \( \langle \psi_f | \psi_{f,m}(s) \rangle \) is the projection of each field component \( | \psi_{f,m}(s) \rangle \) onto the global field \( | \psi_f(s) \rangle \). This means that each \( \tau_{f,m}(s) \) has to be weighted by how much the corresponding path contributes to the global field. In Eq. (3.5), a generally complex probability \( P_{f,m} \) can be formally identified, since it can be directly shown that \( \sum_m P_{f,m} = 1 \).
Indeed, the $\hat{\tau}_f(s)$ operator so defined is non-Hermitian due to the nonorthogonality of the final state components $|\psi_{f,m}(s)\rangle$. Therefore the occurrence of complex probabilities $P_{f,m}$ and complex eigenvalues $\tau_f(s)$ is expected [136]. The related physical picture corresponds to the interference between different paths at either side of the structure when coherent fields are considered [130]. Introducing a complex-valued time delay may seem odd and unphysical, but $\text{Re}\{\tau_f(s)\}$ is shown in Section 3.3.1 to be associated with well established time delay definitions. A physical meaning will be also given to $\text{Im}\{\tau_f(s)\}$ in Section 3.4 [131].

The approach described leads to the computation of the “center of mass” arrival time if applied to pulse propagation and to the whole grating length [129]. The main advantage of the proposed formalism is the possibility to analyse the time delay characteristics of the structure layer by layer, provided that all the possible paths $|\psi_{IN}\rangle \rightarrow |\psi_f\rangle$ and the corresponding contributions $|\psi_{f,m}\rangle \rightarrow E_{f,m}(\xi)$ are identified.

Equation (3.5) looks different from expression (7) reported by Japha and Kurizki [130], where the same time delay definition is applied to propagation through a dielectric slab, i.e., a Fabry-Pérot cavity. However, it has been analytically verified that the same final expressions are obtained applying Eq. (3.5) to the same scattering structure, but with less involved computations and with a clearer physical understanding. For this reason, the formalism of Eq. (3.5) is used in the rest of the study.

### 3.2.2 Fields and time delay computation

In order to compute the local time delay using the previously introduced approach, an input field $|\psi_{IN}\rangle$ entering the structure as shown in Fig. 3.2 and a generic layer $s$ inside it are defined. The considered layer is treated in the following derivation as a free space propagation region, i.e., only straight trajectories $\zeta_{f,m}(\xi)$ are possible inside $s$. This approximation is valid as long as the layer length $dz$ is small compared to the inverse of the coupling constant $\kappa$ of the grating, which means that the probability of a scattering event inside is negligible. It is therefore natural to consider a layer of infinitesimal length $dz \rightarrow 0$ to keep the derivation as general as possible. In this case, the traversal time
\[ \Delta \tau_0(s) = n_{\text{eff}} \frac{dz}{c} = d\tau_0 \to 0, \]  

(3.6)

where \( n_{\text{eff}} \) is the effective refractive index of the medium when no modulation of the grating is present. A point-by-point definition of the properties of the grating is obtained in this way, so that the layer \( s \) can be simply identified with its position \( z \in [0, L_{gr}] \) inside the grating, and the concept of “local number of passes \( N_f(z) \)” rather than the actual local time delay \( \tau_f(s) \) is more effectively used. The simple relationship between number of passes and time delay is

\[ \tau_f(z) = d\tau_0 N_f(z), \]  

(3.7)

and the local time delay inside a layer \( s \) of finite length is directly obtained by integration

\[ \tau_f(s) = \frac{d\tau_0}{dz} \int_s N_f(z) dz = \frac{n_{\text{eff}}}{c} \int_s N_f(z) dz. \]  

(3.8)

With respect to the layer \( s \), the grating can be seen as a Fabry-Pérot cavity in which the left (labelled 1) and right (labelled 2) reflectors are fully characterised by their reflection and transmission coefficients \( r \) and \( t \), shown in Fig. 3.2. The complex coefficients \( r_{1+} \) and \( t_{1+} \) refer to forward propagating paths in the left reflector (left to right), and \( r_{1-} \) and \( t_{1-} \) to backward propagating paths (right to left). The general relation \( t_{1+} = t_{1-} \) holds in a reciprocal medium [101]. A monochromatic excitation is considered in the following analysis,

\[ |\psi_{\text{IN}}\rangle \longrightarrow E_{\text{IN}} = e^{j(\beta z - \omega_0 \xi)}, \]  

(3.9)

where \( \beta = \frac{\omega_0}{c} n_{\text{eff}} \) is the propagation constant in the unperturbed medium, and \( \omega_0 \) is the angular frequency related to the wavelength by \( \omega_0 = \frac{2\pi}{\lambda} \). Steady state conditions are therefore assumed, and only the lossless case is addressed, where the relation \( R + T = \)
\(|r|^2 + |t|^2 = 1\) holds. Note that the transmission and reflection coefficients are dispersive and different for every considered layer \(s\), but the notation \(t = t(\omega, s)\), \(r = r(\omega, s)\) is used later on for the sake of simplicity. It is also useful to introduce the round-trip reflection coefficient \(\rho\) of the effective Fabry-Pérot cavity associated with the layer \(s\),

\[
\rho = r_1 r_2 e^{j2\phi} \rightarrow r_1 - r_2,
\]

where \(\phi = \beta dz = \omega_0 \Delta \tau_0(s)\) is the phase delay for a single pass through the layer and can be neglected if a layer of infinitesimal length \(dz\) is considered.

A discrete set of possible paths \(\{\xi_{f,0}, \xi_{f,1}, \ldots, \xi_{f,m}, \ldots\}\) with a well defined time \(\tau_{f,m}(z) = d\tau_0 N_f(z)\) spent inside the layer \(s\) can be associated with both the transmitted \(|\psi_T\rangle\) and reflected \(|\psi_R\rangle\) final states, as shown in Figs. 3.3 (a) and 3.3 (b). The final state can be computed and expressed through its \(|\psi_{f,m}(z)\rangle\) components if the expressions of \(r\) and \(t\) for the given structure are known. This is possible, for instance, using the transfer matrix approach described in Section 2.2.2 \([86, 101, 124]\). In transmission,

\[
|\psi_T\rangle = |\psi_{IN}\rangle t_1 t_2 \sum_{m=0}^{\infty} \rho^m
\]

\[
= |\psi_{IN}\rangle \frac{t_1 t_2}{1 - \rho} = |\psi_{IN}\rangle t_{gr},
\]

where \(t_{gr}\) is the transmission coefficient of the whole grating and \(\phi\) has been neglected.

\[\text{Figure 3.3: Possible paths leading to light transmission (a) and reflection (b) according to multiple scattering. The number of passes } N(s) \text{ through the considered layer } s \text{ (shaded area) is easily identified for each path. Note that } |\psi_{R,0}(s)\rangle \text{ does not reach the layer } s \text{ in reflection.} \]
With similar analysis in reflection,

\[ |\psi_R\rangle = |\psi_{IN}\rangle \left[ r_{1+} + t_{1+} r_{2-} \sum_{m=0}^{\infty} \rho^m \right] \]  

(3.12a)

\[ = |\psi_{IN}\rangle \frac{r_{1+} + t_{1+} r_{2-}}{1 - \rho} = |\psi_{IN}\rangle r_{gr}, \]  

(3.12b)

where \( r_{gr} \) is the total grating reflection coefficient. The expressions obtained with this approach are consistent with transfer matrix results, and each path component \( |\psi_{f,m}(z)\rangle \) is clearly shown in Eqs. (3.11a) and (3.12a). This identification allows us to perform a weighted average over all possible interfering paths and to calculate the layer time delay according to Eq. (3.5).

The detailed derivation of both transmitted and reflected time delays is presented in Appendix A. The local time delay \( \tau_T(z) \) accumulated in layer \( s \) by transmitted light is found to be

\[ \tau_T(z) = \frac{\langle \psi_T | \hat{\tau}_T(z) | \psi_T \rangle}{\langle \psi_T | \psi_T \rangle} = d\tau_0 N_T(z) = d\tau_0 \cdot \frac{1 + \rho}{1 - \rho}. \]  

(3.13)

\( N_T(z) \) is evaluated by taking into account that each field component \( |\psi_{T,m}(z)\rangle \) in Fig. 3.3 (a) experiences \((2m-1)\) passes in the layer due to multiple scattering in the grating before being transmitted. An important feature of Eq. (3.13) is that the real part of the transmission time delay is always positive in every layer, as shown by Eq. (A.4a). Equation (3.13) is a generalisation of Eq. (10) in Lee et al. [131].

Starting from Eq. (3.12b), the reflection time delay \( \tau_R(z) \) becomes

\[ \tau_R(z) = \frac{\langle \psi_R | \hat{\tau}_R(z) | \psi_R \rangle}{\langle \psi_R | \psi_R \rangle} = d\tau_0 N_R(z) = d\tau_0 \cdot 2 \frac{T_1}{T_1 - \rho} \]  

(3.14)

where \( T_1 \) and \( R_1 \) are the left mirror transmissivity and reflectivity, respectively. With reference to Fig. 3.3 (b), it is worth noting that the time spent in layer \( s \) by \( |\psi_{R,0}(z)\rangle \) is \( \tau_{R,0}(z) = 0 \) since these photons are reflected before reaching the layer, while for all the other field components \( N_{R,m}(z) = 2m \). The real part of \( \tau_R(z) \) can be shown to be negative for certain wavelengths and in certain grating positions. A possible justification of this unintuitive result is given in Section 3.3.1, describing uniform grating calculations.

The overall time \( \tau_{tot}(z) \) spent in layer \( s \) by a photon, independently of its transmitted or reflected final state, can be derived using Eqs. (3.13) and (3.14). Since the final states are different, the two fields do not interfere and \( \tau_{tot}(z) \) is given by the average value weighted by each final state probability (given by the transmissivity \( T_{gr} \) and reflectivity \( R_{gr} \) of the whole grating) [128, 141]:

\[ \tau_{tot}(z) = \tau_T(z) T_{gr} + \tau_R(z) R_{gr} = d\tau_0 \cdot T_{gr} \frac{1 + R_2}{1 - R_2}, \]  

(3.15)

where \( R_2 \) is the right mirror reflectivity. It is important to note that the total time delay so computed is always real and positive, despite the fact that the two independent
components are complex. The physical explanation of this distinctive feature is given in Section 3.2.3, where $\tau_{tot}(z)$ is shown to be related to the energy density distribution inside the scatterer. Equation (3.15) corresponds to the result presented in Zhu et al. [126].

Since $\tau_{tot}(z)$ is real, Eq. (3.15) can be expanded as

$$
\tau_{tot}(z) = \text{Re}\{\tau_T\}T_{gr} + \text{Re}\{\tau_R\}R_{gr},
$$

(3.16a)

$$
0 = \text{Im}\{\tau_T\}T_{gr} + \text{Im}\{\tau_R\}R_{gr},
$$

(3.16b)

where the corresponding expressions for the real and imaginary parts of the time delay are given by Eqs. (A.4) and (A.9). Equation (3.16b) is a direct consequence of energy conservation, as is shown in Section 3.4.1.

From the previously calculated expressions and Eq. (3.8), it is possible to characterise the time delay of the entire grating by simply summing up all the individual layer contributions:

$$
\tau_f = \frac{n_{\text{eff}}}{c} \int_0^{L_{gr}} N_f(z) dz,
$$

(3.17)

where $f$ stands for $T$, $R$, or “tot”. The equivalence of the real part of Eq. (3.17) for reflected and transmitted light with the classic phase time delay [88] cannot be proved analytically for a general structure. But an analytic proof can be obtained in the uniform grating case and assuming a dispersionless effective refractive index $n_{\text{eff}}$, and is derived in Appendix B. In general, an almost perfect agreement is obtained even in nonuniform structures by numerical simulations.

As an example, a square-dispersionless grating is considered in Fig. 3.4. The grating is $L_{gr} = 140$ mm-long, with central wavelength $\lambda_{\text{Bragg}} = 1550$ nm and nominal bandwidth $\Delta \lambda = 0.2$ nm. The in-band reflectivity is $R_{\text{MAX}} \approx 0.99$ and the filling factor (defined as the ratio between the $-1$ dB and the $-30$ dB bandwidth) is $\eta_B = 0.84$. The corresponding reflectivity spectrum is shown in Fig. 3.4 (a), while the actual refractive index profile obtained with inverse-scattering algorithms [42, 43] is plotted in Fig. 3.4 (b) (see Chapter 5 for details). Fig. 3.4 (c) magnifies the reflection time delay inside the reflection band of the grating. The comparison between the standard time delay computation based on transfer matrix and phase derivative [86, 88] (blue line) and the local time delay approach through Eqs. (3.13) and (3.17) (red line) is highly satisfactory despite the complexity of the refractive index profile. Minor differences ($< 0.02$ ps) are present over the entire simulated spectrum, even for out-of-band wavelengths with $-40$ dB attenuation (not plotted here). They can be attributed to the limited, but still finite, length of each layer used in the simulation ($\Delta \approx 207 \mu$m in this case).

### 3.2.3 Energy distribution and dwell time

The analysis of local time delays can also be used to characterise the energy distribution inside the grating. The electromagnetic energy density distribution $dW$ and the Poynting...
vector $\vec{S}$, related to energy flux, can easily be derived knowing the field distribution in the various sections [75]:

$$dW(z) = dW_e(z) + dW_m(z) = \frac{i}{2}\epsilon_0 n_{\text{eff}}(z)^2 |\vec{E}(z)|^2 + \frac{1}{2}\mu_0 |\vec{H}(z)|^2,$$

(3.18)

$$\vec{S}(z) = \frac{i}{2}\text{Re}\left\{\vec{E}(z) \times \vec{H}^\ast(z)\right\},$$

(3.19)

where $\epsilon_0$ and $\mu_0$ are the vacuum permittivity and permeability and $n_{\text{eff}}$ is the effective refractive index, which is assumed to be non-dispersive.

Considering purely transverse fields, the electric field $E(z) = E_0 e^{j(\beta z - \omega t)}$ is scalar and can be calculated in every position inside the structure with the multiple scattering picture described above, which is equivalent to the transfer matrix method. The forward- and backward-propagating fields $E_+$ and $E_-$ in layer $s$ are

$$E_+(z) = E_{1+} \frac{t_1}{1-\rho},$$

(3.20a)

$$E_-(z) = E_+(z) r_2,$$

(3.20b)

and the total electric and magnetic fields in each layer are expressed in terms of these
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components as

\[ E(z) = E_+(z) + E_-(z), \]  
(3.21a)

\[ H(z) = \frac{\beta}{\omega_0 \mu_0} [E_+(z) - E_-(z)], \]  
(3.21b)

as shown by normal mode analysis [96, 142]. According to Eqs. (3.18), (3.20), and (3.21), the energy density distribution along the grating is given by

\[ dW(z) = \left( \frac{1}{2} \epsilon_0 n_{\text{eff},0}^2 |E_{\text{IN}}|^2 T_{gr} \right) \frac{1 + R_2}{1 - R_2} + \frac{1}{4} \epsilon_0 \left[ 2 n_{\text{eff},0} \delta n(z) + \delta n^2(z) \right] |E(z)|^2, \]  
(3.22)

where the effective refractive index is expressed as \( n_{\text{eff}}(z) = n_{\text{eff},0} + \delta n(z) \), i.e., the modulated contribution \( \delta n(z) \) has been separated from the average value \( n_{\text{eff},0} \). The electric and magnetic contributions to the total energy density can be shown to be equal if no modulation is present, and to result in the first term in Eq. (3.22). Comparing its expression to Eq. (3.15), it is apparent that this contribution can be directly related to the total time delay spent by light inside the grating

\[ \frac{1}{2} \epsilon_0 n_{\text{eff},0}^2 |E_{\text{IN}}|^2 T_{gr} \frac{1 + R_2}{1 - R_2} = \frac{1}{2} \epsilon_0 n_{\text{eff},0}^2 |E_{\text{IN}}|^2 \frac{\tau_{\text{tot}}(z)}{\tau_0} = S_{\text{IN}} \tau_{\text{tot}}(z), \]  
(3.23)

where \( S_{\text{IN}} = \frac{1}{2} n_{\text{eff}} \sqrt{\frac{\epsilon_0}{\mu_0}} |E_{\text{IN}}|^2 \) is the flux of the incident transverse wave.

Conversely, the second term is related to the local imbalance \( dW_e - dW_m \) between the electric and the magnetic components of the energy density. But this imbalance is insignificant for structures in which the refractive index contrast is small, such as fibre Bragg gratings. Indeed, the total energy \( W = \int_{z_0}^{L_{gr}} dW(z) dz \) stored in the grating and computed using Eqs. (3.22) and (3.23) is given by

\[ W = S_{\text{IN}} \tau_{\text{tot}} + \frac{1}{4} \epsilon_0 \int_0^{L_{gr}} \left[ 2 n_{\text{eff},0} \delta n(z) + \delta n^2(z) \right] |E(z)|^2 dz, \]  
(3.24)

while the corresponding result, independently obtained by D’Aguanno et al. [143, Eq. (2)] by using the boundary conditions of the gratings, is

\[ W = \frac{1}{2} \epsilon_0 \int_0^{L_{gr}} n_{\text{eff}}^2(z) |E(z)|^2 dz - |E_{\text{IN}}|^2 \frac{2 n_{\text{eff},0}}{2 \omega_0 \mu_0 c} \text{Im} \{ r_{gr} \}, \]  
(3.25)

where, again, the second term is the difference \( dW_e - dW_m \). Substituting this expression into Eq. (3.24) and taking into account the definition of the incoming flux \( S_{\text{IN}} \), it is finally obtained that

\[ W = S_{\text{IN}} \tau_{\text{tot}} - S_{\text{IN}} \frac{\text{Im} \{ r_{gr} \}}{\omega_0} = S_{\text{IN}} \tau_{\text{tot}} - S_{\text{IN}} \tau_i, \]  
(3.26)

where the newly defined quantity \( \tau_i \) is consistent with the definition of a self-interference time delay introduced by Winful [144]. \( \tau_i \) is there described as \( \text{“the time spent by} \)
the incident pulse dwelling in front of the barrier as it interferes with itself during the
tunnelling process”. But the maximum contribution of this self-interference term is
\( \tau_i < 1/\omega_0 < 1 \) fs even for strong gratings at \( \lambda \sim 1550 \) nm. It is readily found that this
contribution is negligible and can be practically discarded, if typical parameters valid for
FBGs are considered, with \( L_{gr} > 1 \) mm and \( \delta n \approx 10^{-4} \). This corresponds to considering
only the average refractive index \( n_{eff,0} \) in the energy computation. Conversely, the local
expression for \( n_{eff} = n_{eff}(z) \) has to be used in multilayer structures which are only
micrometres long and have \( \delta n > 10^{-1} \).

A further definition of time spent inside the structure can be introduced starting
from Eq. (3.26):

\[
\frac{W}{S_{IN}} = \tau_D = \tau_{tot} - \tau_i \simeq \tau_{tot}
\] (3.27)

According to the definition given by Smith [145], \( \tau_D = \frac{W}{S_{IN}} \) can be identified with the
dwell time inside the structure. The dwell time is generally considered as a measure
of the time spent by a wave packet in a given region of space. It has been applied to
electron quantum tunnelling by Büttiker [135] and extended to optical tunnelling by
Steinberg and Chiao [139]. The dwell time \( \tau_D \) is well described as “the ratio between
the total integrated particle density \( \mathcal{N} \) in the barrier region divided by the incident
current \( j \)” in [146]. In optics, the number of stored photons (i.e., the stored energy \( W \)) corresponds to \( \mathcal{N} \) and the incident photon flux (i.e., the Poynting vector) to \( j \), and
Eq. (3.27) is obtained. The correspondence of the dwell time \( \tau_D \) with the total time
delay \( \tau_{tot} \) computed by the multiple path approach in the small index-contrast limit is
an important result. It is an analytical proof of the validity of the proposed multiple
path approach in this approximation. It must also be noted that Eq. (3.27) shows that
the derivation obtained in [144] is generally valid only if the total phase time \( \tau_{tot} \) is
considered. The expressions presented in that paper for the transmission and reflection
components are valid only in a symmetric structure, in which all the different phase
times are identical, as shown in Appendix A.

Physically, the dwell time so-defined can be related to the time necessary to build
up the final photon density in the grating, which under steady state conditions also
corresponds to the time to empty the cavity and is related to the cavity Q factor. This
definition has only been applied to the analysis of the entire grating [135, 139, 144]. But
a local cavity can also be associated with each layer of length \( dz \) inside the grating. The
same physical meaning is also attributed, in Appendix C, to \( \tau_D(z) = \tau_{tot}(z) \), defined for
each layer by

\[
\tau_D(z) = \frac{dW(z)dz}{S_{IN}}
\] (3.28)

Therefore, \( \tau_D(z) \) can actually be interpreted as a local dwell time inside the grating in
the small index-contrast approximation.

Despite the formal parallelism between the electron wave function \( \Psi(z) \) and the elec-
tric field \( E(z) \), outlined by Steinberg and Chiao [139] and derived from the Schrödinger
and Maxwell equations analogy, it must be stressed that Eq. (3.27) is the correct extension of the dwell time to electromagnetism. Indeed, the energy of an electromagnetic wave is stored in both the electric and magnetic fields. Considering $E$ only (as suggested by Steinberg and Chiao [139]) would produce an extra term related to the self-interference between forward- and backward-propagating components in (3.27), and the agreement and physical insight obtained would be lost.

The idea of energy velocity $v_e$ can also be introduced once the energy distribution inside the grating and the net Poynting vector flux are known [147]. Indeed, the energy velocity inside a periodic structure was defined as $v_e(z) = \frac{S(z)}{dW(z)}$ in Yeh et al. [75]. Typically, energy velocity is associated with the speed at which energy is transferred inside the scattering medium, rather than with phase (or envelope) variations, as for phase velocity or group velocity $v_g$. Therefore, it is often considered as a more suitable definition of velocity when generic pulse shapes or complex transfer functions of the media (due to either a complex absorption spectrum or multiple scattering) are considered.

The net Poynting vector flux $S(z)$ inside the grating along the propagation direction $z$ is obtained from Eqs. (3.19), (3.20), and (3.21)

$$S(z) = \frac{\beta}{2\omega_0\mu_0} |E_{IN}|^2 T_{gr} = S_{IN} T_{gr}.$$  \hspace{2cm} (3.29)

It is found to be constant along the grating and related to the incoming flux $S_{IN}$ through the grating transmissivity $T_{gr}$, as expected because of energy conservation in passive, lossless media under steady state conditions. After substituting Eqs. (3.22) and (3.29), the energy velocity $v_e(z)$ becomes

$$v_e(z) = \frac{S(z)}{dW(z)} \simeq \frac{c}{n_{eff}} \frac{1 - R_2}{1 + R_2} \leq \frac{c}{n_{eff}},$$  \hspace{2cm} (3.30)

where the small-contrast approximation has been considered again. Equation (3.30) ensures that the energy velocity in a periodically perturbed medium is always subluminal, i.e., less that the corresponding velocity in an unperturbed medium $c/n_{eff}$. Eq. (3.30) has also interesting consequences when very strong gratings ($R_{gr} \simeq 1$) are considered. Indeed, $R_2 \simeq R_{gr} \simeq 1$ at the beginning of the grating if a uniform structure is considered (see Section 3.3.1), which means that the corresponding energy velocity is $v_e(0) \sim 0$. No energy transport occurs inside the grating, corresponding to the complete reflection of all the incoming power and to the formation of a standing wave. Energy is simply stored inside the structure and exchanged in time between the electric and magnetic fields. Similar considerations also apply to very strong linearly chirped gratings, but the region in which the formation of a full standing wave occurs extends from the beginning of the structure to the effective reflection point $z = z_{eff}(\lambda)$ in this case.

A different expression for the local velocity $v_D(z)$ can be introduced using (3.27) and
following D’Aguanno et al. [143],

\[
v_D(z) = \frac{dz}{\tau_D(z)} = \frac{S_{IN}}{dW(z)} = \frac{S(z)}{dW(z)T_{gr}} = \frac{v_e(z)}{T_{gr}}, \quad (3.31)
\]

which can be shown to be superluminal under certain propagation conditions, namely, for small values of \(T_{gr}\). But the dwell time \(\tau_D\) is by no means a time-of-flight and is not related to any well defined energy transport phenomena. It contains weighted averaged contributions from all photons present in layer \(s\), regardless of their final state. Therefore, it lacks directionality and any velocity associated with this time definition has no clear physical meaning. The concept of “delay” and “velocity” are distinct in this case, since velocity assumes that the phenomenon in question, namely a pulse peak, actually propagates, while a time delay by itself makes no assumption about the mechanism responsible for the delay (it could be absorption, re-emission, storage, reflection, not just propagation) [144]. Therefore, no violation of causality occurs since no real tunnelling can be associated with \(\tau_D\). This result is in agreement with [103, 148], where pulse propagation simulations show that the intensity of the transmitted pulse does not exceed, at any time, the incident intensity in the absence of the grating, i.e., energy is always propagating at subluminal velocity.

A more detailed definition of stored energy in linear dielectrics has recently been given by Ware et al. [149, 150], where both the usual field contribution (3.18) and an exchange contribution related to the accumulation of the energy transferred to the medium are considered. New insight in the relations between energy and group velocities is given by this approach, and the importance of the instantaneous spectrum experienced by each layer inside the medium to describe propagation and delays is highlighted. Unfortunately, no extension of this analysis to scattering media has been proposed to date, and it is a possible field for future work. The possibility to relate the instantaneous spectrum and the local time delays may further improve the understanding of propagation in a localised section of a periodic structure.

### 3.3 Time delay distributions in typical gratings

The previously described approach to the computation of local time delays inside a grating structure is now applied to a few significant cases. The excellent agreement between the proposed method and generally accepted definitions of phase time [88] is emphasised even though an analytical proof of such an equivalence is given in Appendix B for the special case of uniform gratings. Also, the characteristics of the local time delay distributions at different wavelengths and for the different final states of light (either reflected or transmitted) are discussed, showing some of the peculiarities found and giving intuitive explanations of the corresponding effects.
Figure 3.5: Reflectivity (a) and time delay (b) for a uniform grating with $L_{gr} = 10$ mm, $\delta n = 1.48 \times 10^{-4}$, $R_{MAX} \approx 0.99$, and Bragg wavelength $\lambda_{Bragg} = 1550$ nm. The time delay is computed using the phase derivative approach (blue) and using Eq. (3.17) (red). $\Re\{\tau_T\}$, $\Re\{\tau_R\}$, and $\tau_{tot}$ are identical and superimposed. The time delay evolution in the grating for the four marked wavelengths is described in Fig. 3.6.

### 3.3.1 Uniform gratings

The previously described theoretical picture has been applied to a uniform grating of length $L_{gr} = 10$ mm, refractive index modulation $\delta n = 1.48 \times 10^{-4}$, and maximum reflectivity $R_{MAX} \approx 0.99$. The effective refractive index is $n_{eff} = 1.45$, the bandwidth between the first zeros is $\Delta \lambda \approx 0.23$ nm, and the Bragg wavelength is set to $\lambda_{Bragg} = 1550$ nm. The grating is divided into 400 sections of length $\Delta = 50$ $\mu$m. Such a fine sampling enables the minimisation of the numerical error as described before.

Figure 3.5 shows the spectral shape of the grating (a) and the real part of the integrated time delay (b) computed using the local time delay approach described in Section 3.2. $\tau_{tot}$, $\Re\{\tau_T\}$, and $\Re\{\tau_R\}$ are identical and superimposed for a uniform grating, and they are plotted with a red line. The corresponding transfer matrix result [86] is essentially superimposed, as expected from the analytical equivalence proved in Appendix B. Minor differences are still present due to the finite length of each layer, but they are practically negligible ($< 0.1$ ps).

The main advantage of the approach described above is the possibility to analyse the contribution of every single section to the overall time delay. The longitudinal distributions of time delays have been computed in Fig. 3.6 for the four wavelengths indicated in Fig. 3.5, for both transmitted and reflected light, and for the dwell time in each section. For the sake of clarity, the effective number of passes $N_T(z)$, $N_T(z)$, and $N_{tot}(z)$ in the selected layer $s$ are shown in Fig. 3.6 using blue, red, and green lines, respectively. Again, it has to be stressed that the integrated $\tau_{tot}$, $\Re\{\tau_T\}$, and $\Re\{\tau_R\}$ are identical, despite their completely different spatial distributions. Indeed, the equivalence of the phase delays is related to the symmetry of a uniform grating and to general properties of gratings, as Eq. (2.40). But the way light is delayed inside the structure has to be different for transmitted and reflected light, since they are expected to penetrate the cavity in a different way depending on their final state. The number of passes at the grating ends ($z = 0$ and $L_{gr}$) is always fixed irrespective of the wavelength, since the
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Figure 3.6: Local time delay distributions in the uniform grating shown in Fig. 3.5: \(\text{Re}\{N_R(z)\}\), blue lines; \(\text{Re}\{N_T(z)\}\), red lines; \(N_{\text{tot}}(z)\), green lines. The effective number of passes \(N(z)\) is shown. The four wavelengths marked in Fig. 3.5 are considered: (a) \(\lambda_1\); (b) \(\lambda_2\); (c) \(\lambda_3\); (d) \(\lambda_4\).

Associated effective cavity has \(\rho = 0\) and no multiple reflections occur. Transmitted light (red) simply crosses these layers once during propagation towards the end of the grating, and \(N_T(0) = N_T(L_{gr}) = 1\). Reflected light (blue) always passes twice through the very beginning of the grating since it enters the grating and is finally bounced back, and \(N_R(0) = 2\). Conversely, it never reaches the very end of the structure since otherwise it would never be reflected, and \(N_R(L_{gr}) = 0\). These general relations can be derived analytically from the corresponding Eqs. (3.13)-(3.15) in the limits \(z \rightarrow 0, L_{gr}\).

The Bragg wavelength (\(\lambda_1\)) is presented in Fig. 3.6 (a), while (c) shows the evolution at a wavelength corresponding to the time delay maximum and close to the first transmission resonance (\(\lambda_3\)). As expected, \(N_{\text{tot}}(z) \simeq \text{Re}\{N_R(z)\}\) when \(R_{gr} \simeq 1\) (green and blue lines in (a)), and \(N_{\text{tot}}(z) \simeq \text{Re}\{N_T(z)\}\) when \(T_{gr} \simeq 1\) (green and red lines in (c)). At the Bragg wavelength, the dwell time decays exponentially along the structure in accordance with the evanescent propagation at this wavelength. Almost all the energy is stored in the first part of the grating. Instead, much higher energy storage is achieved at the transmission resonance. Light is trapped in the grating centre between two highly reflective mirrors, and constructive interference between different paths takes place, as is clear from Eq. (2.22) and practically shown by DFB laser properties (lasing at the band-gap edge in uniform structures [151, Fig. 3]). Indeed, lasing occurs only when the contributions of multiple cavity round-trips add up in phase. Figures 3.6 (b) and (d) refer to wavelengths \(\lambda_2\) and \(\lambda_4\) which have the same overall time delay \(\tau = 48\) ps.
However, the different scattering conditions produce completely different distributions of delay inside the structure and, therefore, different energy density distributions (green lines). Light is mainly confined at the beginning of the structure when propagation at the band-gap edge is considered (b), while two regions of larger energy storage are found on the first reflectivity sidelobe (d). They correspond to the creation of a partial standing wave inside the grating, given by the interference between the forward- and backward-propagating components. It is already apparent that each wavelength “sees” the grating in a different way. This special mapping technique can potentially be used to characterise the grating along its length, to design different devices, or to understand the spectral response differences under various perturbations and noise distributions (see Section 3.4 and Chapter 4.

Considering the transmission time delay (red lines), the light that passes through the grating is mainly delayed at the edges of the structure for propagation inside the band-gap, i.e., for $\lambda = \lambda_1$, Fig. 3.6 (a). At this wavelength, the multiple paths $|\psi_{T,m}(z)\rangle$ are interfering destructively since $T_{\text{gr}} \approx 0$. This is true for all the layers $s$, since the final transmissivity $T_{\text{gr}}(z) = T_{\text{gr}}$ is independent of the considered layer $s$, and similar interfering conditions are required along the whole grating. The destructive interference is almost complete around the grating centre due to sufficient contributions from either side of the layer. It actually gets more complete as the grating reflectivity increases, since the number of interfering paths increases, and the corresponding time delay distribution goes asymptotically to zero. However, the contributing paths are predominantly only from one side as the grating edges are approached, and destructive interference becomes gradually incomplete. The time delay distribution at the grating ends is therefore always finite. This picture is confirmed by comparing the time delays of gratings of different strengths, as shown in Fig. 3.7. The higher the reflectivity, the lower the transmission time delay in the structure centre, as shown in (a), so that the integrated traversal time decreases for increasing grating strength (b).

This phenomenon is typical of structures with tunnelling and/or evanescent wave propagation and it is known as the “Hartman effect” [102]. This effect produces a transmission time delay that is actually shorter than the one in vacuum in sufficiently strong gratings, as shown in (b) for $R \gtrsim 0.7$. Transmission under these conditions has been interpreted as “superluminal” [103, 129, 152]. When pulse transmission through periodic structures is considered, it has been well established that “superluminal” effects are associated with lack of sufficient destructive interference at the leading edge of the pulse and with strong destructive interference during the main duration of the pulse [129, 130]. On the other hand, the picture presented here shows that the Hartman effect and the associated superluminal effect under steady state conditions are due to strong and nearly complete destructive interference at the central part of the periodic structure and partial interference near its edges. A sort of dualism is therefore found between temporal position inside the pulse and spatial position inside the grating. It must be noted that a completely different interpretation of the physical origin of the
“Hartman” effect has been recently given by Winful [144]. The very idea of propagation is questioned, since no actual propagation occurs inside the grating for wavelengths inside the band-gap. Evanescent waves are indeed present, and any delay phenomenon is described as the effect of coupling to and from a resonator rather than as the effect of propagation [153, 154]. This theory is highly explicatory, but the results are applied only to uniform scatterers (photonic band-gap barriers in general) and to the whole grating region. A corresponding localised analysis would be interesting in order to get a different physical understanding of the results presented in this Section.

Considering the other wavelengths, a symmetric distribution along the grating length is always found, in accordance with Eq. (A.4a) when symmetric structures are analysed. Propagation of transmitted light is practically unperturbed at the band-gap edge, and \( N_T(z) \approx 1 \) in every position \( z \) as if no scattering structure were present, see Fig. 3.6 (b). This occurs despite the transmissivity at this wavelength being only \( T_{gr}(\lambda_2) \approx 0.2 \), and large scattering therefore occurs inside the grating. Similar comments apply to propagation on the sidelobe peaks, as in (d), where the integrated time delay \( \tau_T \) is equal to the one in free space propagation. A quasi-uniform distribution is obtained, with oscillations around \( N_T = 1 \) due to both the amplitude and the phase of the round-trip reflectivity \( \rho = r_1 r_2 \) of the effective cavity seen from each position \( z \). The limited effect is mainly due to the phase contribution, which is close to \( \pm \pi/2 \) all over the grating. No conditions for pure constructive or destructive interference are met, and large averaging occurs. Conversely, constructive interference is obtained close to transmission resonances (c), where light is trapped inside the grating and a large time delay results. The final effect is very similar (or, actually, dual) to the one just described for the Hartman effect, and stronger gratings present longer delays in the centre of the grating in this case. At the same time, the delay decreases by increasing the order of the transmission resonance. Indeed, the round-trip reflectivity of the effective cavity and the number of interfering paths are reduced, since the detuning from the Bragg condition is larger.
Conversely, destructive interference is never obtained for out-of-band propagation, and no light propagation at large effective velocity is found.

The corresponding reflection time delay is shown in blue in Fig. 3.6. A simple physical interpretation is possible for propagation inside the band-gap, as shown in (a) and (b). Light is effectively reflected by the periodic structure and is mainly confined at the beginning of the grating, as commented for the total time delay $\tau_{tot}$. $\text{Re}\{\tau_R(z)\}$ is found to be always positive in these cases. But counter-intuitive features are obtained as soon as the wavelength $\lambda$ moves outside the band-gap, as it can be shown that the number of passes becomes negative in certain positions inside the grating. In particular, it may assume very high positive and negative values close to reflectivity minima, as shown for $\lambda = \lambda_3$ in Fig. 3.6 (c). Extreme results are found in correspondence to the very transmissivity resonance (not shown here), where almost infinite time delay is formally obtained in localised regions of the scatterer. Indeed, the practical value of these results is limited, since they correspond to wavelengths at which reflection is negligible by all means. It is anyway remarkable that the time delay of the whole grating, integrated according to Eq. (3.17), is consistent with the phase time results even in this case.

This behaviour is again related to interference effects between the different possible paths. Considering Fig. 3.3 (b), it is clear that the reflected component $|\psi_{R,0}(z)\rangle$ does not contribute to time delay directly, but only through interference with the other contributing paths at the beginning of the grating. The real part of this interference contribution can be negative, since the field components $|\psi_{R,m}(z)\rangle$ are not orthogonal in general, and it can produce a negative reflection time delay in the layer. Despite the uneasiness of the idea of negative time-of-flight, an intuitive explanation of this steady state result is obtained if pulse propagation examples are considered [155]. Negative time delays are associated with the peak of the reflected pulse leaving the grating region before the incident peak actually enters it. This is possible if strong interfering conditions are met, as on the transmission resonances where constructive interference occurs. The first part of the pulse is reflected, since there are no other paths to interfere with. Conversely, large destructive interference occurs between the contributions coming from multiple round-trips in the effective cavity and the reflection $|\psi_{R,0}(z)\rangle$ from the preceding portion of the grating. As is clear by now, interference is more and more effective as the number of round-trips increases, and therefore occurs on the main lobe of the pulse rather than on the leading edge.

Unfortunately, this explanation based on an effective Fabry-Pérot cavity cannot give an intuitive idea of all the features of the reflection time delay. Indeed, the effect of the cavity has to be the same as for the transmission time delay previously discussed. But the flux which is finally transmitted is constant along the grating because of energy conservation in transmission, while the number of photons entering the layer and being finally reflected depends on the position inside the grating, and its contribution cannot be neglected. Moreover, the interference with the preceding part of the grating has to be accounted for, while no further interfering components are present in transmission.
Therefore, no simple qualitative analysis is possible.

### 3.3.2 Chirped gratings

An important application of local time delays is the analysis of the temporal response of chirped gratings used for second-order dispersion compensation, i.e., for compensation of a linear group delay [5]. A better knowledge of the time delay distribution along the grating is expected to provide a better understanding of the characteristic features and limitations of these devices, that use distributed Bragg reflection to obtain temporal dispersion. An unapodised, linearly chirped grating is analysed first, taking into account both the longitudinal time delay distribution and the spectral time delay ripple. In particular, it is known (and it is shown in Fig. 3.8 (b)) that the periodicity of this group delay ripple (GDR) is different for different wavelengths, and it gets shorter with an increased value of the time delay. The commonly accepted explanation of this effect relates these oscillations to the creation of a Fabry-Pérot cavity between the effective reflection point at each wavelength and the beginning of the grating [21, 156]. A longer cavity corresponds to a shorter free spectral range and to faster spectral oscillations. A discontinuity of the propagation constant $\beta$ is found in $z = 0$, as is clear if a rigorous Block mode theory is applied to these structures [76, 81]. Indeed, the eigenmodes outside and inside the grating region are different, and this mismatch causes spurious reflections which are sufficient to create a cavity, despite their small amplitude. The same concept is also visualised using an effective impedance approach to propagation in periodic media [27], in close analogy to typical applications in electromagnetism and microwaves. The local time delay approach analysed here confirms this general understanding, and it also gives a better idea about how the cited mismatch affects the properties inside the structure on a localised basis. A suitable apodisation profile is presented next, and its improved characteristics discussed.

The considered grating is shown in Fig. 3.8. Both the reflectivity $R_{gr}$ and the reflection time delay $\tau_R$ are plotted in (a) and (b), respectively, as a function of the detuning $\delta\lambda$ with respect to the Bragg wavelength $\lambda_{Bragg} = 1550$ nm. The grating’s...
length is $L_{gr} = 150$ mm, the (constant) refractive index modulation is $\delta n = 6 \times 10^{-5}$, and chirp rate is $CR = -0.065$ nm/cm. The average reflectivity of the grating is $R_{MAX} \approx 0.9$ and the bandwidth at -1 dB is $\Delta \lambda = 0.925$ nm. The nominal dispersion is $D_{TOT} = -1500$ ps/nm, sufficient for compensation of $\sim 80$ km of standard SMF fibre, but the maximum in-band ripple is as large as 100 ps, which makes this grating useless for any practical application.

Fig. 3.9(a) shows the comparison between the phase derivative approach (2.40) based on standard transfer matrix calculations and the local time delay computation (3.17). Only reflected light is considered in this case, since dispersion compensators are designed to work only in reflection configurations. The good accuracy of the method presented is apparent even in this case for reflected light, with an error limited to $< 1$ ps over the analysed bandwidth. Fig. 3.9 (b) compares the reflection time delay distributions for three different wavelengths that correspond to subsequent ripple maxima and minima, $\lambda_1$ (black), $\lambda_2$ (green), and $\lambda_3$ (magenta) in (a). Only the portion of the grating up to the local band-gap regions for the considered wavelength [26] is plotted in order to magnify the corresponding effect. The contribution to the reflection time delay from the region of the grating following the band-gap is expected to be negligible, given the limited power that leaks through the band-gap towards the transmission side (as seen in Fig. 3.6 (a) for a uniform grating). This approximation is more and more correct with increasing grating strength, given the fact than $N_{tot} \rightarrow N_R$ for $R_{gr} \rightarrow 1$, as shown by Eq. (3.15).

In all cases, light moving along the grating approaches the band-gap and gets slower or faster depending on the relative detunings, as in Fig. 3.5 for a uniform grating. In uniform gratings this is accomplished by changing the wavelength $\lambda$ for a fixed $\lambda_{Bragg}$; while $\lambda$ is fixed and the local $\lambda_{Bragg}(z)$ changes in chirped gratings. A parallelism between a chirped grating and an effective uniform grating was actually proposed, introducing an
effective grating length [6]

\[ L_{\text{eff}} = \sqrt{\frac{2\pi}{f_\gamma}} \quad \Rightarrow \quad L_{\text{eff}} = \sqrt{\frac{\lambda_0^2}{2n_{\text{eff}} |CR|}}, \quad (3.32) \]

where the former expression proposed by Hill [6] has been rewritten using grating parameters used in this work (central Bragg wavelength \( \lambda_0 \), effective refractive index \( n_{\text{eff}} \), and chirp rate \( CR \)). Good agreement is found in describing both pitch and amplitude of the light slowing-down close to the grating band-gap using Eq. (3.32) and taking into account the detuning in each layer, but unsatisfactory results are obtained for higher detunings. The effect of chirp cannot simply be approximated with an uniform grating, and, consequently, characterising the dispersive properties of chirped gratings by considering only undisturbed propagation and an effective reflection point is a limited and misleading picture.

An approximated but highly intuitive picture of the slowing-down of light approaching the band-gap was also given by Russell and Birks [81] using Hamiltonian optics. Potential and kinetic energy components are identified inside the periodic media, where the first refers to the creation of a standing wave inside the cavity due to the interference of forward- and backward-propagating components, while the second is related to the group velocity \( v_g \) of light moving in and back from the effective reflection point. The more energy is stored in the standing wave, i.e., the higher the local reflectivity, the slower it moves towards the band-gap. The results in Fig. 3.9 (b) agree with that analysis, in the sense that the value of \( N_R(z) \) averaged along \( z \) qualitatively follows the trend predicted by the Hamiltonian approach. But the method presented here is valid even close and inside the band-gap region, where the Hamiltonian method fails to give an accurate description of propagation since it does not take tunnelling effects into account. Two main differences are found in the local delay approach. Oscillations of the local time delay (and therefore of the velocity \( v_g \)) along \( z \) are present. Secondly, the averaged value of \( N_R(z) \) is different for wavelengths corresponding to maxima or minima of the GDR, as is clear comparing the distributions of \( \lambda_1 \) and \( \lambda_2 \) (black and green, respectively) in Fig. 3.9 (b). These peculiarities account for the creation of GDR in the uniform chirped grating, which is not explained by the results in [81]. Therefore, they are expected to be related to the reflections from the grating boundaries and the associated Fabry-Pérot interference fringes, which are indeed disregarded in [81], as explicitly discussed in the paper.

In order to test the previous hypothesis, the effective Fabry-Pérot approach described in Section 3.2.2 is used in the following. Propagation is analysed layer-by-layer by considering the effective cavity seen from each layer, i.e., the right and left reflectors shown in Fig. 3.2. The effective cavity is completely characterised by the reflectivity
and round-trip phase shift at each section $s$:

\[ R_{\text{Round-Trip}}(z) = R_1(z)R_2(z) = |\rho(z)|^2, \]  
\[ \phi_{\text{Round-Trip}}(z) = \phi_1(z) + \phi_2(z) + 2\phi(z) \approx \phi_1(z) + \phi_2(z), \]

where $R_1(z)$ and $R_2(z)$ are the left and right mirror reflectivities seen from layer $s$, $\phi(z)$ is the single pass phase shift inside the layer (which can be disregarded since $dz \to 0$), and $\phi_1(z)$ and $\phi_2(z)$ are the mirror phase delays, related to the effective length $L_{FP}(z)$ of the Fabry-Pérot cavity.

The same analysis carried out for a uniform grating in transmission can be applied to the chirped grating in reflection, if the region of the grating before the band-gap is considered. Indeed, the light flux that travels along the grating and is finally reflected is almost constant, since $\lambda$ is highly detuned from the local $\lambda_{\text{Bragg}}(z)$ and few scattering events occur. Therefore, the differences between the local time delays in different layers are only determined by interference between different paths. Moreover, the left reflector $R_1(z)$ is highly detuned from resonance, resulting in a weak mirror whose reflectivity oscillates as the layer gets closer to the local band-gap. By contrast, the right reflectivity $R_2(z)$ is quasi-constant along the structure, since it is mainly due to the band-gap contribution (in particular for gratings with high reflectivity). Therefore, only $R_1(z)$ is considered in the analysis.

The time delay distribution $N_R(z)$ along the grating is compared in Fig. 3.10 to the round-trip phase shift $\phi_{\text{Round-Trip}}(z)$ (red lines) for wavelengths corresponding to either ripple minima ($\lambda_1$, a, black) or maxima ($\lambda_2$, b, green). It is apparent from Figs. 3.10 (a) and (b) that the maximum delay is experienced in layers where the round-trip phase shift $\phi_{\text{Round-Trip}}(z)$ is close to $2m\pi$. Field trapping and energy build up are effective when multiple reflected photons add in phase and constructive interference takes place, as seen with reference to Fig. 3.6 (c). Conversely, destructive interference, i.e., $\phi_{\text{Round-Trip}}(z) \approx \pi + 2m\pi$, leads to cavity depopulation and light speeding up, as it is clear from the discussion about the Hartman effect. Figures 3.10 (c) and (d) show the corresponding left mirror reflectivities (blue lines) and compare them with the round-trip phases. For $\lambda_1$ (c), constructive interference is achieved where the cavity reflectivity is lower, while for $\lambda_2$ (d), the same condition takes place where $R_1(z)$ is higher. The interference is therefore less effective in the first case than in the second, due to the lack of sufficient interfering paths. It produces a shorter local time delay at $\lambda_1$ with respect to $\lambda_2$ in every layer $s$, so that the final time delay presents the large ripple shown in Fig. 3.8 (b). The increase in the reflectivity envelope also explains the increase in the fringe visibility of the local time delay approaching the band-gap ($z \approx 20$ mm in the examples), since all interference effects are enhanced by the higher cavity confinement, as seen in Fig. 3.7. At the same time, this intuitive picture also explains why the amplitude of the GDR is larger when stronger gratings are considered, as shown in [89, Fig. 7]. It is also clear that an effective way to damp this detrimental group delay ripple is to use modified structures in which the left reflectivity $R_1(z)$ grows smoothly with
z, without presenting an oscillatory shape as in Fig. 3.10 (c) and (d). This aspect is further discussed later in this Section.

The variation of the oscillation pitch along the grating may be related to cavity effective length $L_{FP}(s)$. Changing $s$, the right effective reflection point is almost constant inside the band-gap region. On the other hand, the left effective reflection point is always close to the considered layer, since the closest region of the left mirror is always the less detuned from the local $\lambda_{Bragg}(z)$, and it has the higher scattering probability. This means that the effective length decreases moving towards the band-gap, and a shorter cavity is characterised by a longer free spectral range and longer pitch.

The previous analysis also explains the ripple period dependence upon wavelength. Considering two different wavelengths corresponding to consequent ripple minima, say $\lambda_1$ (black line) and $\lambda_3$ (magenta line) in Fig. 3.9 (b), it is apparent that the local time delay distributions are practically identical and simply shifted one to respect to the other by one period of the time delay oscillation, as is clear from the analysis of Fig. 3.10. This feature is clarified by Fig. 3.11, which shows the band-gap region (shaded area) and the local Bragg wavelength (red dashed line) along the grating length [26], together with the two wavelengths previously defined. It is possible to define pairs of corresponding layers $s_3$ and $s_1$ associated with each wavelength, which have the same detuning $\delta \lambda$ from...
Figure 3.11: Schematic representation of the band-gap region (shaded area) and the local Bragg wavelength $\lambda_{\text{Bragg}}(z)$ along the grating. The two wavelengths described in the text are shown ($\lambda_1$ and $\lambda_3$) and layers having the same detuning $\Delta \lambda$ marked.

$\lambda_{\text{Bragg}}(z)$:

$s_1 : \lambda_1 - \lambda_{\text{Bragg}}(z_1) = \delta \lambda, \quad s_3 : \lambda_3 - \lambda_{\text{Bragg}}(z_3) = \delta \lambda.$

Each layer $s_3$ has a corresponding layer $s_1$ at wavelength $\lambda_1$, while the layers within the marked segment $a$ do not have any. These pairs of layers give equivalent contributions to the overall time delay. Indeed, the right reflectors in the associated Fabry-Pérot cavities are identical, since the two layers are at the same distance from the local Bragg reflection points (and since negligible light leaks through the band-gap to sample the last part of the structure). As long as the left reflectors are considered, they differ because of the segment $a$, but this is not important since it is the most detuned part of grating. This is particularly true if the pair $\hat{s}_1$ and $\hat{s}_3$ in Fig. 3.11 is considered (green dots), while larger differences are found for the pair $\tilde{s}_1$ and $\tilde{s}_3$ (blue dots), i.e., closer to the beginning of the grating. Nevertheless, the two cavities are almost equivalent and so are the related local time delays. The only significant difference between the two wavelengths is given by the unpaired layers within the segment $a$, which present the shorter pitch in the time delay distribution due to the length $L_{\text{FP}}(s)$ of the associated cavity. The pitch is shorter and shorter for wavelength reflected deeper into the grating, which also means that the frequency change between consecutive minima, i.e., the spectral ripple period, is shorter due to the linear relation existing between wavelength and spatial coordinate (well visualised in Fig. 3.11).

This analysis also explains why a suitable apodisation can considerably reduce the in-band time delay ripple despite the fact that it only affects the very end of the grating [89]. In order to check this, a linearly chirped grating with raised cosine apodisation has been simulated, and the corresponding time delay characteristics are shown in Fig. 3.12 (a). The grating’s length is $L_{\text{gr}} = 150$ mm as for the uniform grating discussed previously, the (constant) refractive index modulation is raised to $\delta n = 6.4 \times 10^{-5}$ to compensate for the added apodisation and keep the reflectivity to $R_{\text{MAX}} \simeq 0.9$.
Figure 3.12: Time delay in reflection (a) for chirped grating with \( L_{gr} = 150 \) mm, \( \delta n = 6.4 \times 10^{-5} \), \( CR = -0.065 \) nm/cm, and \( R_{MAX} = 0.9 \). A raised cosine apodisation extending for 20 mm at each side of the grating is used. The phase time (2.40) (blue line) and the layer-by-layer approach (3.17) (red line) are compared. Excellent agreement is found. \( \lambda_1 \) (black), \( \lambda_2 \) (green), and \( \lambda_3 \) (magenta) correspond to (residual) ripple maxima or minima. Their distributions of number of passes \( N_R(z) \) are shown in (b) using the same colour coding. Only the first part of the grating (\( z < 55 \) mm) is shown.

Figure 3.13: Distributions of (a) \( N_R(z) \) (black line) and of (b) the left reflectivity \( R_1(z) \) (blue line) compared to the round-trip phase \( \phi_{Round-Trip}(z) \) (red line) for the grating shown in Fig. 3.12 and at the wavelength \( \lambda_2 \). \( \phi_{Round-Trip}(z) = 0 \) and \( \phi_{Round-Trip}(z) = \pi \) correspond to peaks and minima in the time delay distribution in Fig. 3.12 (b), respectively.

long apodisation sections are considered at either end of the structure. The chirp rate is still \( CR = -0.065 \) nm/cm, giving a nominal dispersion of \( D_{TOT} = -1500 \) ps/nm. Fig. 3.12 (a) shows that the usual satisfactory agreement between phase time and local time delay approaches is found, and that the high frequency ripple is highly suppressed by apodisation (even though a significant low frequency distortion from linear dispersion compensation is introduced on the red side, see Section 4.4). The local time delay distributions along the grating are plotted in Fig. 3.12 (b) for the three wavelengths marked in (a), and corresponding to maxima and minima in the residual \( GDR \). Time delay ripple maxima and minima have almost the same distribution in this case, and the envelope of the oscillation goes to zero at the beginning of the grating where the apodisation is applied. Indeed, the refractive index modulation is really weak in this region, and light tends to propagate as in an unperturbed medium.

Fig. 3.13 shows the characteristics of the effective Fabry-Pérot cavity along \( z \) in this
apodised grating for \( \lambda = \lambda_2 \). It is apparent that the left mirror reflectivity (blue) is substantially different from the one in Fig. 3.10 (c) and (d). The reflectivity oscillations before the band-gap are almost suppressed, and the mutual relation between interference effects and cavity confinement no longer affects the local properties of the grating. Moreover, the absolute value of \( R_1(z) \) is also smaller due to the reduced refractive index modulation at the boundaries of the structure. This means that the amplitude of the time delay oscillations is also smaller with respect to the uniform case and to wavelengths corresponding to GDR peaks. Indeed, \( \max \{ N_R(z) \} \simeq 3.2 \) in the apodised case, while \( \max \{ N_R(z) \} \simeq 3.6 \) in the uniform one.

### 3.4 Local time delay and perturbed gratings

The analysis of complex local time delays developed in Section 3.2 provides insight into the way light propagates inside a scattering medium. But, so far, the only proof that the presented approach is correct is given by the agreement between phase times and local time delays integrated along the grating length. No evidence has been given to support the local identification of the number of passes \( N \) in each section inside the grating. This problem is directly addressed in this Section. Local time delays will be shown to be related to well established definitions of time delay used throughout different areas of physics, and to be a valuable tool for the analysis of gratings presenting localised or distributed errors. The first aspect is more interesting from a theoretical point of view, since it is the basis for a possible experimental verification of the presented approach. Better understanding of propagation issues in scattering media is derived by a direct measurement of local properties in real structures. The second aspect mainly concerns an engineering point of view as a direct application of theory, and it will be analysed in detail in Chapter 4.

#### 3.4.1 The local time delay approach to phase errors in gratings

Perturbed gratings are expected to be a valuable field of application of the local time delay approach from both a theoretical and a practical point of view. Different perturbations can be thought of for typical scattering structures, but phase errors will be extensively addressed here. This choice is compatible with the typical focus of this work on fibre (or planar) Bragg devices characterised by a limited modulation of the refractive index, such as the ones obtained by UV exposure. Phase errors are easily produced and controlled inside FBGs, which makes them suitable for possible experimental verifications of local time delay’s theory. They are also intrinsically related to the manufacturing process of these devices, given the complexity of precisely controlling the periodic pattern of the structure with the necessary nanometer resolution. Phase errors are due to positioning instabilities of the optics during the writing process, or stitching error in the phase mask used for manufacturing the grating, or velocity fluctuations of the writing stage or UV fluence fluctuations in the continuous grating writing set-up used within
the ORC. Therefore, the obtained results can find direct practical applicability. A more
detailed explanation of the advantages of this choice is given in the following, when the
theoretical background and the related aspects will be clearer.

If an infinitesimal phase perturbation $d\phi = \beta dz$ is introduced in a defined position $z$
inside the grating, the transmission and reflection coefficients $\tilde{t}_{gr}$ and $\tilde{r}_{gr}$ of the perturbed
grating are readily found using the formalism in Section 3.2.2 and Eq. (3.11b). The
transmission coefficient is given by

$$\tilde{t}_{gr}(z) = \frac{t_1 + t_2 e^{j\phi}}{1 - \rho e^{2j\phi}}, \quad (3.34a)$$

$$t_{gr} = \frac{t_1 + t_2}{1 - \rho} \quad (3.34b)$$

for the perturbed and unperturbed structures, respectively. $\rho = r_1 - r_2$ from Eq. (3.10),
and $d\phi = 0$ in the unperturbed grating. Using the Taylor expansions for $d\phi \to 0$ in the
previous expressions, it is possible to write, after a lengthy manipulation described in
Appendix D,

$$\tilde{t}_{gr}(z) \simeq t_{gr} e^{j\phi N_r(z)} = t_{gr} e^{j\phi N T(z)} \Delta \tau_0(z), \quad (3.35)$$

where $\rho e^{2j\phi} \simeq \rho$ for an infinitesimal perturbation and $\Delta \tau_0(z)$ is defined by Eq. (3.6).
The reflection coefficient computation is similar, but mathematically more involved.
From Eq. (3.12b) and using the same approximations as before,

$$\tilde{r}_{gr}(z) \simeq r_{gr} e^{j\phi N R(z)} = r_{gr} e^{j\phi N R(z)} \Delta \tau_0(z). \quad (3.36)$$

It is analytically confirmed that the effect of the perturbation is directly related to the
complex time delays (3.13) and (3.14). The previous relations are also consistent with
an intuitive understanding of propagation in multiple scattering structures. The more
times light crosses the layer $z$ where the perturbation is located, the bigger the phase
shift that is accumulated, so that the total phase shift is given by

$$\Delta \theta(z) = d\phi N(z), \quad (3.37)$$

where $N(z)$ is the effective number of passes in the layer $z$ and $\theta$ is the phase of the
reflection coefficient. Eq. (3.37) gives directly Eqs. (3.35) and (3.36) if $N$ is derived using
the local time delay approach. But the effect of the perturbation can be further analysed
taking into account that the derived time delays are complex-valued. Considering the
generic final transmissivity or reflectivity coefficient $F = \{T, R\}$, the effect of the phase
error introduced in the grating is

$$\tilde{F}_{gr}(z) = F_{gr}(z) \left| e^{j\phi N_F} \right|^2 = F_{gr}(z) e^{-2d\phi \text{Im}\{N_f(z)\}}.$$
If a small perturbation is assumed and $\tilde{F}_{gr}(z) \simeq F_{gr} + \Delta F_{gr}$,

$$-2d\phi \text{Im}\{N_f(z)\} = \ln \left( \frac{\tilde{F}_{gr}(z)}{F_{gr}} \right) = \ln \left( 1 + \frac{\Delta F_{gr}(z)}{F_{gr}} \right) \simeq \frac{\Delta F_{gr}(z)}{F_{gr}},$$

where the last equality is correct only if $\Delta F_{gr} \ll F_{gr}$ applies. Therefore, the phase and amplitude variations in the perturbed structure are simply written as

$$\Delta \theta_{F}(z) = d\phi \text{Re}\{N_{F}(z)\},$$  \hspace{0.5cm} (3.38)

$$\Delta F_{gr}(z) \simeq -2d\phi F_{gr} \text{Im}\{N_{F}(z)\}.$$  \hspace{0.5cm} (3.39)

Therefore, both the real and imaginary parts of the transmission and reflection time delays $\tau(z) = d\tau_0 N$ have a precise physical meaning. Introducing a phase perturbation results in a phase change related to the real part of the time delay, while the amplitude change is related to the corresponding imaginary part. They are directly related to measurable variations in the transmission and reflection coefficients. Using Eq. (3.39), it is also straightforward to show that Eq. (3.16b) results in $\Delta T_{gr} + \Delta R_{gr} = 0$. The total time delay $\tau_{TOT}$ being real is therefore a direct consequence of the principle of energy conservation, consistently with the results in Section 3.2.3.

Eqs. (3.38) and (3.39) are the starting point for the analysis of the sensitivity of a scattering structure with respect to the introduction of a single phase error, discussed in Chapter 4. They are simple and offer an immediate physical understanding, but they still are an approximation valid only when the amplitude of the perturbation is small and the unperturbed grating can be considered for the computation of the local time delay. Indeed, a very large phase error completely changes the interference conditions between the two parts that the starting grating is divided into, as is clear for the limiting case in which $d\phi = \pi$. The corresponding value of the local time delay is totally changed, and Eq. (3.36) fails to give a correct estimation of the perturbation of the reflection coefficient. Analytically, this is clear from the derivation in Appendix D. The Taylor expansions used throughout the derivation do not hold if $d\phi \rightarrow 0$, and the proposed approximations are not correct. Moreover, the time delay approach to phase error analysis developed so far is based on the assumption that a single error occurs inside the scattering structure. The form of Eq. (3.36) suggests that a straightforward generalisation to a known distribution of phase errors $d\Phi(z) = \sum_z d\phi(z)$ is possible and

$$\tilde{r}_{gr} \simeq r_{gr} \prod_z e^{j\phi(z)N_R(z)} = r_{gr} e^{j \sum_z d\phi(z)N_R(z)} = r_{gr} e^{j \int N_R(z) d\phi(z)}.$$  \hspace{0.5cm} (3.40)

An analogous expression is obtained for the transmission coefficient $\tilde{t}_{gr}$. The previous discussion shows that the validity of Eq. (3.40) is more limited than for the single error case. The accumulation of small errors alters the unperturbed local time delay distribution in an unpredictable way, while the previous expression assumes that the effect of a phase error $d\phi(\xi)$ does not affect the local characteristics $N_R(z)$ of the grating.
in a different position \( z \neq \bar{z} \). A large deviation from the predicted effect is therefore expected, with a limited applicability of Eq. (3.40) to practical cases.

3.4.2 Theoretical aspects

The analysis developed by Steinberg and Chiao [157] and Steinberg [136] gives a theoretical interpretation of the described effect, using the theory of “weak measurements” developed by Aharonov and Vaidman [158]. In a classical quantum measurement theory, it is not possible to get information about both the time spent by light in a certain region and the final transmitted or reflected state. The first measurement collapses the system status on the measured eigenstate and thus the system evolution is irreversibly altered. But if the measurement is sufficiently “gentle” (but therefore imprecise), the system does not collapse and both pieces of information (i.e., the weak value of the time delay) can be obtained by averaging a large set of such measurements. The final result is in general a complex number. The real part is related to the mean variation in the measured quantity (“pointer”) and gives the final result of the measurement. The imaginary part is shown to be associated with the mean shift in the pointer conjugate momentum, which corresponds to the back-action of the measurement on the system. It can be thought of as a measure of how much the system has been perturbed.

According to the above definition, the perturbation scheme proposed in this Section is a weak measurement. The optical phase of light is the measurement pointer, since the phase shift induced by the perturbed layer is used as a clock. The conjugate momentum is represented by the photon number, i.e., the transmissivity and reflectivity of the grating [136]. The variations \( \Delta T_{gr} \) and \( \Delta R_{gr} \) are interpreted as the effects of the perturbation introduced in the system by the measurement and are proportional to the time delay imaginary parts. This theoretical approach also shows that correct results are obtained as long as the state of the system remains to a large extent undisturbed by the measuring procedure. Therefore, good agreement between the complex time delay distributions calculated with Eqs. (3.13) and (3.14) and the effect of a localised phase defect is expected where \( \text{Im}\{N(z)\} \) is small, while differences are expected when the perturbation significantly affects the grating. This is the case for a relatively strong perturbation, or for either a weak or strong perturbation near the transmission resonances of a uniform grating, as in Fig. 3.5, for instance.

This theoretical framework and the results in Eqs. (3.38) and (3.39) completely justify the choice of phase perturbations from the point of view of an experimental measurement. The introduction of a localised phase error in a fibre (or planar) Bragg grating is simple and effective. The easiest way is to slightly change the background refractive index in a limited section of the structure, which is equivalent to introducing a chirp in the grating according to Eq. (2.17). Such a local variation can be induced by applying a localised heating [15] or strain [159], and experimental verification of the simulated time delay distributions is obtained by simply scanning the grating. This is possible since the perturbation induced by these techniques is reversible. Moreover, no excess losses are
introduced in the perturbed region, allowing an independent characterisation of the real and imaginary parts of the time delay by measuring the induced phase and amplitude variations, respectively, as is clear from Eqs. (3.38) and (3.39). UV trimming of the grating is a possible different solution, but its effect is not reversible and a small, but finite change in the local losses of the grating is introduced, as discussed in detail in Section 5.3 with reference to inverse scattering techniques applied to lossy materials. Independent identification of the effects of the phase and loss variations is no longer possible in this case.

A point defect is created inside a grating with a fairly simple set-up thanks to the intrinsic length of fibre gratings, and high spatial resolution of the measurement is obtained with few complications. This is a clear advantage compared to optical multilayers, which are characterised by a larger index contrast but very short lengths, typically in the $\mu$m range. These devices have been successfully used for optical time-of-flight measurements [157, 160, 161], but their dimensions and quasi-discrete structure makes them unsuitable for localised measurements, even though a possible scheme was actually proposed by Steinberg [136]. The length of FBGs offers several advantages even for “simple” time-of-flight measurements, and it has been recently exploited for elegant measurements of “superluminal” propagation in uniform scattering media [162]. Pulses in the sub-nanosecond scale rather than in the femtosecond scale can be used without losing the required sensitivity in the determination of the pulse peak. A clear simplification of both sources and detection schemes is obtained, since standard optoelectronics equipment is sufficient. At the same time, the strength of the barrier can be optimised (typically to $T_{gr} \sim -20$ dB) to permit detection of the transmitted pulse at a reasonable power level. Finally, experimental confirmations of more complex effects are easily available using fibre grating technology, such as “superluminal” pulse reflection in a double-Lorentzian photonic band-gap [163] or tunnelling in a double-barrier photonic band-gap [54].

A simple set-up is required for the direct measurement of the imaginary part of the local time delay, since only an amplitude variation in the transmitted or reflected signals has to be detected. The corresponding experimental verification is described in Section 3.4.3. A drawback of the phase perturbation as a clock is the relative complexity of the measurement of a (small) phase change at optical frequencies. An interferometric set-up based on a Michelson or Mach-Zehnder configurations is necessary, depending on the reflected or transmitted components are considered. The main problems to be tackled for such an experimental verification are: locking the interferometer at quadrature to obtain maximum visibility, active stabilisation of the interferometer against environmental changes, polarisation issues in optical fibres’ configurations, and finally simultaneous measurement of induced perturbation and fringe visibility (affected in turn by the transmissivity-reflectivity variation of the perturbed grating).

For other possible perturbation schemes, the theoretical interpretation of Steinberg allows generalisation of the results in Eqs. (3.38) and (3.39) to different possible “clocks”.

Chapter 3 Time delay distributions in Bragg gratings
A magnetic clock is proposed in the optical domain in [137]. The Faraday rotation experienced by light propagating in the perturbed structure is proportional to the real part of the complex time delay, while the conjugate momentum is represented by the degree of ellipticity of the outgoing radiation. But the application of an external magnetic field and the measurement of the induced Faraday effect is impractical and no experimental verification has been proposed to date. Changing locally the loss of the structure appears to be a more feasible technique [136]. The effect on the macroscopic parameters of the grating is dual with respect to the phase shifts, since amplitude variations are proportional to the time spent in the perturbed layer while phase variations are proportional to the corresponding imaginary part. A practical realisation can be obtained by using a grating written in an active fibre. If the fibre is doped with a suitable active medium (such as erbium or ytterbium) and a grating with Bragg wavelength within an absorption line of the medium is written, the propagation is lossy and the local loss coefficient can be changed by side pumping the fibre and inducing local transparency via population inversion of the active medium. The hypothesis of lossless propagation that has been used in Section 3.2 does not hold, but the derivation presented there can be extended to lossy or gain materials by considering complex propagation coefficients, as long as transmission and reflection time delays are considered. The simple detection of the variation of the grating transmissivity and reflectivity would give direct information about the real part of the local time delay, if expressions analogous to Eqs. (3.38) and (3.39) hold. But every variation in the material absorption $\Delta \alpha_0$ also produces a variation in the material refractive index $\Delta n$ due to Kramers-Kronig relations [164], and therefore local perturbations of both loss and phase are introduced in the grating. By duality with respect to the previously analysed case,

$$\tilde{f}_{gr}(z) = f_{gr}(z)e^{j\phi_F N_F} e^{-\Delta \alpha_0 dz N_F} = f_{gr}(z)e^{(-\Delta \alpha_0 + j \frac{\omega_0}{c} \Delta n) N_F dz}, \quad (3.41)$$

where $f = \{t, r\}$ is the transmission or reflection coefficient, respectively, $\omega_0$ is the angular frequency, and $d\phi = \omega_0 d\tau_0 = \frac{\omega_0}{c} \Delta n dz$. $dz$ is the (supposedly) short length the perturbation extends over. Eq. (3.41) does not produce decoupled relations for the amplitude and phase variations of the perturbed grating

$$\Delta F(z) = -2\Delta \alpha_0 dz \Re\{N_F(z)\} \left[1 + \frac{\omega_0}{c} \frac{\Delta n}{\Delta \alpha_0} \Im\{N_F(z)\}\right], \quad (3.42)$$

$$\Delta \theta_F(z) = -\Delta \alpha_0 dz \Im\{N_F(z)\} \left[1 - \frac{\omega_0}{c} \frac{\Delta n}{\Delta \alpha_0} \Re\{N_F(z)\}\right]. \quad (3.43)$$

If the typical values of atomic susceptibility in an Erbium-doped fibre [165] and practical fibre Bragg grating parameters ($\delta n = 10^{-4}$, $L_{gr} = 20$ mm) are considered, the two contributions to the amplitude variation related to $\Re\{N_F(z)\}$ and $\Im\{N_F(z)\}$ are comparable. Simultaneous detection of the phase and amplitude variations is necessary to measure either the real or the imaginary part of the local time delay. Therefore, this approach is not as practical as expected, and it is apparent that decoupling phase shift
and loss is necessary to keep the experimental set-up as simple as possible.

3.4.3 Measurement of the imaginary part of the local time delay

As was already pointed out in the previous Section, the measurement of the imaginary part of the local time delay $\text{Im}\{\tau(z)\}$ is a much simpler task compared to the measurement of the profile of $\text{Re}\{\tau(z)\}$ if a phase perturbation is used as a clock. Experimental verification of $\text{Im}\{\tau(z)\}$ has actually been performed [166]. Details of the experiment are presented here even though a more accurate analysis of the imaginary part of the local time delay will be considered in the following Chapter, discussing the sensitivity of gratings to external perturbation. This choice is because such a verification is primarily important from a physical point of view, since it gives a practical verification of the proposed approach to time delay characterisation, while the following Chapter is more deeply focussed on engineering aspects and applications of this method.

The experimental set-up used is a variation of the configuration described by Brinkmeyer et al. [15] for the characterisation of strength and chirp of a grating, and it is shown in Fig. 3.14. From a purely theoretical point of view, this set-up is the optical equivalent of the time-modulated barrier scheme proposed by Büttiker and Landauer [133] for measuring electron tunnelling. According to this approach, the external source of the phase perturbation (CO$_2$ laser) is modulated with an angular frequency $\omega_m$ by a gating signal, and a time-dependent phase shift is introduced in the grating. The induced phase variation has the form

$$\phi(\xi) = d\phi \cdot \frac{1 - \cos \omega_m \xi}{2}. \quad (3.44)$$

An optical signal launched into the grating is transmitted by the perturbed structure according to the result in Eq. (3.35). The corresponding optical signal arriving at the photodetector is given by

$$s(\xi) = \text{Re}\left\{A e^{i(-\omega_0 \xi + \phi)} t_{gr} e^{i\phi(\xi) N_\tau(z)}\right\}, \quad (3.45)$$

where $z$ is the position at which the phase error is introduced. The detected intensity
signal can be written under the assumption of small perturbation as

$$s_d(\xi) = g \cdot \frac{A^2 T_{gr}}{2} \cdot e^{2\phi(\xi) \cdot \text{Im}\{N_T(z)\}} \simeq g \cdot \frac{A^2 T_{gr}}{2} \cdot [1 + 2\phi(\xi) \cdot \text{Im}\{N_T(z)\}]$$

$$= g \cdot \frac{A^2 T_{gr}}{2} \cdot \left[1 + 2d\phi \cdot \frac{1 - \cos \omega_m \xi}{2} \cdot \text{Im}\{N_T(z)\}\right],$$

(3.46)

where the second term is the received optical power and $g$ is the gain of the photodetector. Finally, the lock-in amplifier acts as a narrow band filter centred at the reference signal frequency $\omega_m$. The obtained amplitude measurement is

$$s_{\text{output}} = g \cdot \frac{A^2 T_{gr}}{2} \cdot d\phi \cdot \text{Im}\{N_T(z)\} = g \cdot \frac{A^2 T_{gr}}{2} \cdot d\phi \cdot \frac{\text{Im}\{\tau_T(z)\}}{d\tau_0},$$

(3.47)

where $d\tau_0$ is the time-of-flight in the perturbed region when no perturbation is actually present. The measured signal is proportional to the desired value $\text{Im}\{\tau_T(z)\}$ and consistent with Eq. (3.39). Analogous analysis applies to the reflected signal, with the only difference given by the necessity to use an optical circulator in the set-up. Even considering a highly attenuated output (i.e., $-30$ dB), Eq. (3.47) shows that the proposed configuration is likely to be effective for measurements of reflectivity and transmissivity variations. The longitudinal profiles shown in Figs. 4.2 and 4.3 (left columns) can be experimentally verified, but the absolute value of $\text{Im}\{N_T(z)\}$ cannot be obtained since it also depends on $d\phi$, which is not directly measurable.

Describing the actual experiment in more detail, light from a tunable laser diode is launched into the fibre grating, which is mounted on a translation stage and perturbed by a nondestructive local heating induced by a $CO_2$ laser beam. The $CO_2$ power output ($P_{\text{out}} = 50$ mW) is gated with a 2 Hz modulation signal, and the corresponding disturbance signal is detected via a lock-in amplifier with time constant $\tau = 3$ s for different wavelengths, both inside and outside the band-gap. Such a low modulation frequency is selected in order to maximise the magnitude of the AC component of the perturbed signal with respect to the DC component. Both the transmitted and reflected signals are simultaneously detected, allowing a direct comparison irrespective of source wavelength fluctuations or environmental changes. No IR focusing lens was available, and a 2 mm $CO_2$ spot size is scanned along the grating. The stage scanning speed is set to 1 mm/s, giving an effective spatial resolution of 3 mm over the lock-in integration time constant. This resolution is anyway comparable with the dimensions of the perturbation. Therefore, a relatively long, low index-change grating is chosen to allow a sufficient spatial resolution of the grating features. The tested grating is uniform, 40 mm-long, and has an estimated refractive index modulation $\delta n \simeq 3.4 \times 10^{-5}$. The minimum transmissivity is $T_{\text{min}} \simeq -18$ dB and the -1 dB bandwidth is 0.043 nm.

Figure 3.15 shows the measured data (dotted line) and compares them with simulations obtained with Eqs. (3.13), (3.14), and (3.39). The grating is characterised at two different wavelengths $\lambda_A$ and $\lambda_B$ marked on the measured reflectivity spectrum in Fig. 3.16 (a), both inside ($\lambda_A$, (a) and (c)) and outside ($\lambda_B$, (b) and (d)) the band-
Chapter 3 Time delay distributions in Bragg gratings

Figure 3.15: Comparison between measured data (dotted line) and simulations, for both an ideal uniform grating (dashed line) and the inverse-scattered refractive index profile of the measured grating (solid line). The two wavelengths marked in Fig. 3.16 are shown: inside the band-gap ($\lambda_A \simeq \lambda_{\text{Bragg}}$, left column); close to the second sidelobe peak ($\lambda_B \simeq \lambda_{\text{Bragg}} + 50$ pm, right column). (Upper row) reflected signal; (lower row) transmitted signal. All the signals are plotted in normalised units. The actual grating extent is marked.

To test this hypothesis, the measured reflectivity and time delay of the actual grating has been fed into a layer-peeling inverse scattering algorithm [42, 43] to calculate the corresponding refractive index profile $\delta n(z)$. Part of the measured reflectivity spectrum is shown in Fig. 3.16 (a). An accurate description of the algorithm and its limitations is presented in Section 2.3 and it is not addressed here. Nevertheless, it has to be stressed that the limited bandwidth of the measured spectrum ($\lambda_{\text{max}} - \lambda_{\text{min}} = 0.9$ nm) and the windowing process, necessary to obtain a causal and physically realisable time response, limits the resolution of the reconstructing algorithm. The result of these limitations is given by the rather noisy refractive index modulation $\delta n(z)$ shown in Fig. 3.16 (b). It is clear that the reconstructed grating extends beyond the actual grating length, which is marked. Indeed, the length of the reconstructed grating is set...
Figure 3.16: (a) Reflectivity spectrum of the measured grating with $\delta \lambda = \lambda - \lambda_{\text{Bragg}}$. The measured wavelengths $\lambda_A$ (Fig. 3.15, left column) and $\lambda_B$ (Fig. 3.15, right column) are shown. (b) Corresponding inverse-scattered refractive index profile $\delta n(z)$. The actual extent of the grating is marked.

To $L_{1S} = 200$ mm to compensate for the limited spectral data available (see Section 2.3). The low sidelobe suppression, at the edges of the reconstructed spectrum, introduces large spurious features despite the low grating strength [70, 92]. It is apparent that the measured grating is far from ideally uniform. The expected local time delay distribution is recomputed using the profile obtained and it is represented (solid line) in Fig. 3.15. The agreement with experimental data is dramatically improved inside the grating region, and both the in-band peaked shape and the out-of-band lobe asymmetry are now well described.

These results are the first experimental confirmation of the validity of the perturbation approach in measuring the local time delay properties inside a tunnelling structure such as a fibre Bragg grating. Moreover, they prove the importance of the local time delay approach in evaluating the sensitivity of these structures to any intentionally or unintentionally induced local or distributed perturbation. For instance, Figs. 3.15 (a) and (c) show that the introduction of phase errors in the central region of the grating has a much larger effect on the reflectivity and transmissivity of wavelengths around the Bragg wavelength. This can intuitively be explained by considering the effective Fabry-Pérot cavity picture. The central sections are surrounded by almost equivalent mirrors, so that maximum fringe visibility and therefore sensitivity to external perturbations are expected. It must be noted that this is true despite the phase shift induced in the effective Fabry-Pérot cavity being smaller in the centre of the grating with respect to the sides, as shown by Fig. 3.6 (a), red line. In Figs. 3.15 (b) and (d), out-of-band propagation is considered and a quasi-periodic perturbation effect is found, consistent with the considerations in [15] and with the profiles shown in Chapter 4, Figs. 4.2 and 4.3. It has also been verified that the quasi-periodicity changes with the out-of-band wavelength (not shown here). This implies that localised phase shifts will affect different wavelengths to a different degree. If random phase error distributions are considered, the longitudinal dependence is averaged and all the wavelengths will be affected in a similar way, obtaining an out-of-band background level for the spectral response [122]. On the contrary, if the phase error distribution contains a dominant periodicity, maximum variation of the
amplitude response is expected at the wavelengths whose $\text{Im}\{N(z)\}$ periodicity matches the external perturbation. These features will be discussed in Chapter 4.

### 3.5 Conclusions

A method for the characterisation of the local time delay of periodic structures has been developed. Each layer divides the grating into two distinct reflectors, so that a multiple reflection approach is used to calculate all the possible classical paths for both transmitted and reflected light. This effective Fabry-Pérot analysis allows a field decomposition in terms of components whose traversal time inside the layer is well defined, and the average time spent in the layer by either transmitted or reflected light can be evaluated. A generally complex-valued time is obtained, but clear physical meaning is given to both its real and imaginary parts. The real part is related to the actual traversal time, while the imaginary part gives the extent of the influence of the measurement on the system. Finally, a total time spent in the structure is derived by appropriately weighting and summing up the two contributions. This time is analytically shown to be always real and positive, and to be also directly related to the power distribution inside the grating if small index-contrast gratings are considered. A brief discussion about different definitions of time delays has also been considered, with particular reference to the notion of energy velocity. The local time delay approach confirms that energy velocity is always subluminal inside a grating, despite the results obtained for the transmitted component (which can nominally be superluminal) and for the reflected component (which can even be negative).

The correctness of this approach has been proven in different ways. Analytical agreement between the reflection and transmission time delays derived with this method and phase times based on transfer matrix calculations is found for a uniform grating. A satisfactory numerical agreement is also found for more complex structures, such as chirped gratings and square dispersionless gratings designed with inverse scattering techniques. Finally, local time delays have been shown to be physically and mathematically related to the effect of the introduction of a localised perturbation inside the grating. If a phase error is considered, the corresponding phase and amplitude variations in the reflection and transmission coefficients are proven to be related to the (complex) time spent by light in the perturbed region. A more detailed discussion about the practical implication of this result is given in Chapter 4, together with a numerical confirmation of the validity of the proposed framework. Here, a (partial) experimental confirmation has been described. The change in the optical power transmitted and reflected by a quasi-uniform grating has been shown to be proportional to the imaginary part of the local time delay by scanning along the grating a small phase shift generated with $\text{CO}_2$ laser radiation.

A few examples in which local time delays give an increased physical understanding of propagation in scattering media have been considered. In particular, the Hartman effect has been analysed, which results in an apparent superluminal transmission of
wavelengths inside the band-gap of the grating. The central part of the structure is shown to give negligible contribution to the total traversal time, especially for highly reflective gratings. The group delay ripple and energy distribution in chirped gratings have also been discussed, and the mutual effects of light slow-down at the band-gap edges and of interference due to cavity confinement have been described. Nevertheless, other features of the obtained time delay distributions remain non-intuitive, especially when the reflected component is taken into account. Superluminality and negative time delays have been qualitatively explained by considering the corresponding results obtained for propagation of narrow-band pulses, but a complete understanding of how the derived local properties affect propagation and energy storage in this case has not been achieved yet.
Chapter 4

Applications of time delay distributions

4.1 Introduction

The derivation presented in the previous Chapter allows the definition and the computation of the local time delays experienced by transmitted and reflected light inside a scattering medium. Apart for the physical meaning of this identification, the local properties are useful in the analysis of these structures when external perturbations are introduced in the original grating, such as phase errors. From a physical point of view, perturbations are a simple and effective means for characterising the local properties of a grating. This aspect has been discussed extensively in Section 3.4. But the analysis of perturbations is even more important from a practical point of view, given the level of maturity of grating technology. Many different technologies are available nowadays for the production of high quality gratings (fibre Bragg gratings, planar lightwave circuits, thin-film filters), and these products are used in many commercial applications (or close to the commercial stage). Single channel or band filters [50, 52], dispersion compensation [49, 51, 119], signal manipulation and processing [55, 56] are a few typical examples. The possibility of characterising the sensitivity of a particular device to phase perturbations is very important in all these cases. Indeed, phase errors are typically introduced by the practical manufacturing process, given the tight requirements and the sub-micron scale of the periodicity of a grating.

The importance is twofold. Firstly, the robustness to manufacturing errors of different designs for the same device can be compared in the synthesis stage. A better identification of the trade-offs between complexity and manufacturing reliability is possible if a good understanding is obtained of “where” and “how much” a certain grating design is sensitive, especially if the typical errors introduced by the manufacturing process are known. In addition, an intuitive feeling about how each design is deformed by errors in a certain position (or by a distribution of small errors) is gained, and it constitutes a helpful guideline for a clever design of different structures. Secondly, man-
ufacturing imperfections can be identified and tackled more easily once it is possible to
determine where errors have been introduced from the analysis of the spectral deforma-
tions with respect to the ideal structure. The perturbation analysis is therefore a
means for characterising the manufacturing process when a direct measurement of the
spatial characteristics of the grating is not possible, or the corresponding results are
not sufficiently reliable. The combination of the different techniques proposed so far
(scanning of a perturbation along the grating’s length [15, 167], measurement of the
diffraction efficiency of the imprinted grating on a point-by-point basis [14, 16, 168],
using inverse scattering techniques starting from the measured spectral data [169, 170])
with the method presented here is expected to improve the reliability of the obtained
data.

The local time delay analysis covers all these features and possible applications, but
handling time-of-flight definitions and the concept of negative or complex delays is not
appealing from an engineering point of view. The improved understanding given by
the identification of suitable interaction times is a definite advantage of this method.
However, it is convenient to describe the approach derived in Chapter 3 simply in terms
of the sensitivity of the gratings, and to assign variations in the group delay response
or in the reflectivity to a generic sensitivity parameter, independently from the physical
significance of local time delays, or from the meaning of complex times.

Different approaches to the sensitivity to phase (or generic) perturbations were pro-
posed in the past, as a means of both the analysis and synthesis of periodic structures.
A perturbation method derived directly from the Helmholtz equation was developed by
Uno and Adachi [171]. It allows the computation of the variation in the spectral char-
acteristics once the unperturbed fields and the extent of the disturbance are known. A
similar method was used by Feced and Zervas [122], but starting from the well known
expression of the coupled-wave equations. In both cases, the result is accurate as long
as the phase errors do not significantly alter the field distribution inside the gratings,
and therefore they are valid only in a small perturbation regime. Another formulation
was used by Brinkmeyer et al. [15] for characterisation of gratings and reconstruction of
their longitudinal coupling coefficient profile. The validity of the Born approximation
is a further assumption in this case, which makes this approach suitable only for very
weak gratings or for wavelengths outside the main reflection peak. All these schemes
are reviewed in the following, and similarities and differences with the local time delay
are pointed out. Significantly, the formulas these methods are based on are not intu-
itively derived, which often does not allow a fast identification of the physical reasons
for the obtained results. However, they provide a better estimation of the real varia-
tions induced in the gratings in some conditions, and an optimised approach is therefore
proposed by combining different methods.

Finally, the proposed approach can also be used as a tool for the correction of exper-
imental errors by post-processing of the grating, extending the work by Sumetsky et al.
[172]. For instance, the profile of a phase error distribution, i.e., of an added chirp, that
allows compensation of non-ideal characteristics can be easily derived, provided that
the sensitivity of the (ideal) grating can be used, i.e., that the deviation from the ideal
shape is small. This method will be further developed by considering it as a design tool
to be used in conjunction with the inverse scattering algorithm. Indeed, length require-
ments and numerical issues often limit the possibility of perfect reconstruction of the
target spectrum. The resulting sensitivity can be used to correct these imperfections,
and further boost the accuracy of the design.

The definition of the sensitivity and the analysis of the various perturbation methods
is considered in Section 4.2, where an optimised combination of the different expressions
is also defined. Section 4.3 deals with the description of the properties of a few exam-
ple s of gratings which have practical applications. The sensitivity of uniform structures
is considered first, and the accuracy of the model is discussed in this case. Apodised
and square dispersionless gratings are analysed next as examples of filtering structures.
It is shown that the knowledge of the sensitivity of the grating provides a better un-
derstanding of complex apodisation profiles, allowing the identification of the part of
the grating which determines different spectral features. Finally, chirped gratings for
dispersion compensation are considered. The application of the presented approach to
the compensation of manufacturing errors or inverse scattering limitations is presented
in Section 4.4, and it is specifically applied to the correction of group delay ripple in
apodised linear dispersion compensators and those designed by inverse scattering.

4.2 Sensitivity to phase errors in gratings

The foundation for the analysis of the sensitivity of gratings to phase perturbations of
the nominal refractive index modulation profile has been given in Section 3.4, where the
connection between local time delays and variations in the reflection and transmission
spectra of the structure has been developed. Eqs. (3.38) and (3.39) provide the basic
results of the proposed analysis, and they are appealing since they are both simple and
intuitive.

However, a more practical definition of the sensitivity of the gratings parameters to
a phase perturbation has to be independent from the magnitude of the perturbation
itself. It is natural to take into account the derivative of the previous expressions with
respect to the phase shift $d\phi$, as in Furman and Tikhonravov [101, Section 1.4] for optical
multilayers. The resulting formulas for the grating’s sensitivity are

$$\frac{\partial (\Delta \theta_F)}{\partial d\phi} = \text{Re}\{N_F(z)\}, \quad (4.1)$$

$$\frac{\partial (\Delta F_{gr})}{\partial d\phi} \approx -2 F_{gr} \text{Im}\{N_F(z)\}, \quad (4.2)$$

where $F = \{R,T\}$ refers to either the reflectivity or the transmissivity of the grating.
The variation of the amplitude is expressed here as an absolute parameter, and it shows
where the perturbation is larger irrespective of the wavelength. It can be useful in
case the effects at different wavelengths have to be compared, and the overall spectral deformation has to be studied. It is obvious that even a definition normalised to the unperturbed reflectivity or transmissivity is possible

$$\frac{\partial \left( \frac{\Delta F_{gr}}{F_{gr}} \right)}{\partial d\phi} \simeq -2 \text{Im} \{ N_F(z) \}. \quad (4.3)$$

The sensitivities of both phase and amplitude are simple linear functions of the local number of passes inside the cavity in this case. This definition is a better tool for understanding how the perturbation affects each wavelength independently. It is also the natural parameter for checking the theoretical comments in Section 3.4.2 about the conditions for which the local time delay approach is correct. Indeed, a small variation in the reflectivity or transmissivity of the structure does not necessarily mean that a small perturbation is used and that a good agreement with theory will be found. A very small $\Delta F_{gr}$ can result if a wavelength close to a transmission resonance is used, but it can still be several orders of magnitude larger than the actual value of $F_{gr}$ in unperturbed conditions. Limited agreement has to be expected in this case.

Therefore, the time delay approach to phase error sensitivity proposed here has to be compared with different formulations of the same problem. On one hand, it gives an independent confirmation of the correctness of the proposed approach and of the extent of the approximations introduced. On the other, mixed solutions can be derived, trying to optimise the choice of each approach to different spectral regions of the grating or different strengths of the error introduced, if a large difference in the fit of the real perturbation is obtained. Indeed, a practical comparison between the different approaches is carried out in Section 4.3.1 using a uniform grating as a valuable example. From this simple case, it is empirically found that the optimum choice for a mixed approach is represented by using the time delay approach

$$\tilde{f}_{gr}(z) = f_{gr} e^{j d\phi N_f(z)}$$

for in-band propagation,

and using a linearised form of (4.4)

$$\tilde{f}_{gr}(z) = f_{gr} [1 + j d\phi N_f(z)]$$

for out-of-band propagation. \( (4.5) \)

In the previous expressions, $f_{gr}$ is the generic reflection or transmission coefficient in the unperturbed structure, $\tilde{f}_{gr}$ is the corresponding coefficient in the perturbed grating, and $N_f(z)$ is the number of passes in the position $z$ of the perturbation. Eq. (4.5) is also the outcome of alternative approaches to phase error analysis, as is clear in the following Section 4.2.1. The corresponding expressions for the phase and amplitude variations are
obtained with a simple manipulation and are

$$\Delta \theta_F(z) = \arctan \left\{ \frac{d\phi \Re\{N_F(z)\}}{1 - d\phi \Im\{N_F(z)\}} \right\},$$

(4.6)

$$\Delta F_{gr}(z) = -2d\phi F_{gr} \Im\{N_F(z)\} + d\phi^2 \Re\{N_F(z)\},$$

(4.7)

which are less intuitive than Eqs. (3.38) and (3.39), and are nonlinear with respect to the phase perturbation $d\phi$.

The mixed approach will be applied to all the analyses of different grating types considered in this Chapter. Despite a formal explanation of the reason why such an approach is effective has not been found, an intuitive understanding is given at the end of Section 4.3.1. Further confirmation of its applicability has also been given by a stochastic analysis of perturbed gratings following [122]. This work has not been included in this thesis and is currently unpublished.

### 4.2.1 Different approaches to phase errors

The most obvious approach to the analysis of gratings with errors is to develop a perturbation analysis directly from the Helmholtz equation or the coupled-wave equations. The variation of the reflection coefficient $\tilde{r}$ is first computed as a function of the fields in the perturbed structure, obtaining an expression which is formally correct, but of limited use since the characteristics of such a modified structure are initially unknown. If the effect of the perturbation is limited, the expression of the fields in the unperturbed grating can then be substituted without introducing a large error, and a first order approximation is found.

The corresponding derivation has been proposed by Uno and Adachi [171] in the context of inverse scattering techniques of inhomogeneous, layered media. If the corresponding analysis is derived using the formalism and notation introduced in Chapter 2 for gratings with generic apodisation and chirp profiles, the final expression obtained for the perturbed reflection coefficient $\tilde{r}$ is

$$\tilde{r} = r + j \int_0^{L_{gr}} 2 \Im \left\{ \delta q(z) e^{j \frac{2\pi z}{\Lambda}} \right\} E^2(z) dz,$$

(4.8)

where a generic perturbation $\delta q = \tilde{q} - q$ of the complex coupling coefficient of the grating (defined by Eq.(2.16)) is considered. $\Lambda$ is the nominal period of the modulation pattern, and $E(z) = E_+(z) + E_-(z)$ is the electric field distribution inside the unperturbed grating. As discussed before, the only approximation introduced in Eq. (4.8) is given by

$$E(z) \tilde{E}(z) \simeq E^2(z),$$

(4.9)

where $\tilde{E}$ is the electric field distribution in the perturbed structure. Eq. (4.8) can be written in a form that is more suitable for the analysis using standard CWEs formalism, i.e., using the forward- and backward-propagating fields $U$ and $V$ introduced by
Figure 4.1: Refractive index pattern $\tilde{n}_{\text{eff}}$ corresponding to a localised phase shift $d\phi_P = \frac{\pi}{2}$ (blue line). The green dotted line shows the unperturbed pattern, and $-|\Delta z|$ (shadowed area) the effective free propagation section equivalent to $d\phi_P$.

Eqs. (2.13) and (2.14):

$$\tilde{r} = r - \int_0^{L_{\text{gr}}} \left[ \delta q^*(z)U^2(z) - \delta q(z)V^2(z) \right] dz. \quad (4.10)$$

Eq. (4.10) is theoretically valid for an arbitrary complex modification of both phase and amplitude of the grating’s profile. In order to compare it with the result of the local time delay approach (3.36), a single phase error has to be considered

$$\tilde{r} = \kappa, \quad \tilde{\phi} = \phi + d\phi_P \Theta(z - \tilde{z}), \quad (4.11)$$

where $\Theta(z - \tilde{z}) = \int \delta(z - \tilde{z})dz$ is the Heaviside step function starting in $\tilde{z}$, and $d\phi_P$ is the shift in the periodic pattern of the grating. It must be noted that $d\phi_P \neq d\phi$ has been used so far, since the latter refers to a phase shift in the propagating field related to propagation in a free space region of length $dz$, as shown in Fig. 3.2. Fig. 4.1 gives a schematic idea of the corresponding modification in the periodic pattern when a phase shift $d\phi_P = \frac{\pi}{2}$ is considered. A jump in the modulation pattern (blue line) is found, and it is shown to correspond to the insertion of a free propagation layer (shadowed area) of negative length $-|\Delta z|$. Therefore, $d\phi_P = -2d\phi$ holds, and the corresponding perturbation of the complex coupling coefficient is given by

$$\delta q(z) = \tilde{q} - q = j\kappa e^{j\phi} \left( e^{j d\phi_P \Theta(z - \tilde{z})} - 1 \right) \simeq q \left[ jd\phi_P \Theta(z - \tilde{z}) \right]. \quad (4.12)$$

Finally, if the previous expression is plugged into Eq. (4.10) and the CWEs (2.15) are used, the expression for a single phase error, according to the formulation by Uno, is

$$\tilde{r} = r + j d\phi_P \frac{t_1^2 t_2^2}{(1 - p)^2} \left| \frac{r}{z = \tilde{z}} \right| = r + j d\phi_P r N_R(\tilde{z}). \quad (4.13)$$

The details of the computation and of the definition of $d\phi_P$ are reported in Appendix E, where it is also stressed that the comparison of the previous expression with Eq. (3.36)
shows that the two results are completely equivalent within the limit $d\phi N_R \to 0$, so that the exponential relation can be linearised. It must be noted that this position is also consistent with the analytical derivation presented in Eq. (D.6).

An alternative formulation of the perturbation problem directly applied to the analysis of phase shifted structures was proposed by Brinkmeyer et al. [15]. Similarly to the discussion in Section 3.4.2, the introduction of a localised phase shift via thermal perturbation of the grating was used as a tool for experimentally measuring the coupling coefficient profile $q(z)$ inside a grating. Under the assumption of weak scattering and, therefore, outside the main reflection band of the grating, the Born approximation applies and Eq. (2.21) gives a good estimation of the actual reflection coefficient. Using Eq. (4.12), it is easily found that

$$\tilde{r} = r + j d\phi p \int_{\tilde{z}}^{L_{gr}} q^*(z) e^{j2\sigma z} dz, \quad (4.14)$$

A Fourier transform relation exists between the section of the grating following the phase error and the perturbation of the reflection coefficient. This is intuitive since only the portion of light which actually samples the perturbed region of the grating can be affected, and it is a direct consequence of the linearity of the Born approximation. Eq. (4.14) looks very different from the previously considered expressions, but a complete agreement between the different formulations is found. Indeed, the forward and backward envelopes $U(z)$ and $V(z)$ have very simple expressions in the Born regime, since the back-reflected light is negligible and $V(z) \simeq 0$ can be assumed. From the CWEs (2.15), the forward-propagating field is

$$U(z) \simeq e^{j\sigma z}, \quad (4.15)$$

and the Fourier transform expression in (4.14) can be written as

$$\tilde{r} \simeq r + j d\phi p \int_{\tilde{z}}^{L_{gr}} q^*(z) U^2(z) dz$$

$$\simeq r + j d\phi p \int_{\tilde{z}}^{L_{gr}} [q^*(z) U^2(z) + q(z) V^2(z)] \, dz, \quad (4.16)$$

which is completely equivalent to the general formulation by Uno (see Eq. (4.13)) and, therefore, to the time delay approach. It has to be stressed that this equivalence applies only to weakly scattering gratings or to out-of-band wavelengths in strong gratings, since it is based on the assumption of an undepleted forward-propagating component.

A similar analysis was also proposed by Feced and Zervas [122], starting from a slightly different expression for the reflection coefficient $\tilde{r}$. A general amplitude of phase perturbation was considered in this case, but the results are only related to phase shifts here for compatibility with the previous derivations. The reflection coefficient is written
as

\[ r = - \int_0^{L_{gr}} [q^*(z)U(z)e^{-j\delta z}] e^{j2\delta z} dz, \tag{4.17} \]

and therefore as a function of the coupling profile \( q(z) \), of the effective detuning \( \delta = \sigma + \delta \), and of the actual distribution of the forward-propagating envelope \( U(z) \) inside the grating. Eq. (4.17) is absolutely general, since it is directly derived from the CWE (2.15b) under the assumption that \( U(z) \) is known. This is typically not the case when a real complex grating has to be analysed, which limits the practical applicability of Eq. (4.17). But it can be useful in a perturbation analysis if the perturbed coupling coefficient \( \tilde{q} = q + \delta q \) is considered together with the unperturbed envelope distribution \( \tilde{U} \simeq U \). This approach is similar to the general analysis by Uno, but the CWEs rather than the Helmholtz equation are used as a starting point. Again, the expression for the modified reflection coefficient \( \tilde{r} \) is given by

\[ \tilde{r} = r - \int_0^{L_{gr}} [\delta q^*(z)U(z)e^{-j\delta z}] e^{j2\delta z} dz, \tag{4.18} \]

and the corresponding formula in the case when a single phase error \( d\phi_P \) is considered is

\[ \tilde{r} = r + jd\phi_P \int_0^{L_{gr}} [q^*(z)U(z)e^{-j\delta z}] e^{j2\delta z} dz. \tag{4.19} \]

Taking into account Eq. (4.15), the previous expression is completely equivalent to the formulations of the problem previously described. It is clear that the use of the actual distribution of \( U(z) \), rather than its approximation in the Born regime, makes Eq. (4.19) more suitable for the analysis of perturbation inside the band-gap and in strong gratings.

All the described approaches confirm that the effect of a single phase perturbation inside a grating can be simply interpreted on the basis of the number of times light actually crosses the perturbed region of the grating. Different approximations are introduced in each different analysis, and therefore a different agreement is expected between these results and the true spectrum of the perturbed structure computed using the rigorous transfer matrix formalism. This is generally true for different grating structures and at different wavelengths. The main advantage of the approach proposed in Section 3.4.1 is the increased physical understanding it provides, which is lacking in all the other methods. Indeed, despite being physically well defined, they are cumbersome and no intuition can be gained from integral relations such as (E.8), (4.16), or (4.19).

A simplified interpretation of this physical phenomenon was also proposed by Weber and Wang [173] back in 1990, dealing with the effect of layer thickness variations on the phase of the reflected light from DBR structures. Layer variations are practically equivalent to refractive index variations in the scattering medium, since both of them result in a modification of the optical path inside the composite cavity. The obtained phase variation was related to the power distribution \( dW \) inside the grating, which
is effectively approximated by an exponential decay in the case of a simple uniform structure and for propagation near the Bragg condition. The numerical simulations and the theoretical derivation presented show a good agreement for wavelengths close to the Bragg resonance, while less satisfactory results are obtained at the band-gap edges. No discussion of the corresponding properties for out-of-band propagation or of the transmitted light is presented. The general theory developed in Section 3.4.1 provides a simple explanation of these preliminary results on the basis of a theory of much wider applicability. Close correspondence between the local reflection time delay and the power evolution inside the grating is found inside the band-gap. \( R \simeq 1 \), and Eqs. (3.15) and (3.28) show that \( N_R \sim N_{TOT} \propto dW \) under these conditions. Conversely, the existence of two different contributions to the total stored energy becomes apparent in transmission or away from the Bragg wavelength, and a correct identification of the contribution of each final state (reflected or transmitted) to the total power is necessary in order to quantify the effect of the phase perturbation.

It must be finally noted that all the different approaches to phase errors in gratings reviewed in this Section are limited to the analysis of the perturbation of the reflection coefficient. The possibility to apply the time delay derivation to the analysis of perturbations in transmission is a further advantage of this approach. The practical applicability is rather limited, since Bragg gratings are typically designed for operation in reflection. But the obtained results are still useful from a theoretical point of view, and they can be applied to the experimental verification of superluminal effects in periodic structures. They can also be of importance in WDM or sensor applications where concatenated or superimposed gratings are used, since inevitably certain wavelengths will transverse the full length of gratings corresponding to adjacent channels.

4.3 Application: sensitivity of typical gratings

The theoretical approach developed in Section 3.4.1 is now applied to a number of typical grating structures. First, the approximate analysis and the correct results from transfer matrix computations are compared for a limited number of significant wavelengths. Later, the practical distributions of the sensitivity to phase errors of these periodic structures are shown. They provide both a useful tool for the design and manufacture of complex gratings, and also information about the way these gratings work, boosting the understanding of the obtained designs (especially when inverse scattered gratings are considered).

In every example, a standard phase error \( d\phi = 20 \) mrad is considered. This phase shift corresponds to a \( \Delta z \simeq 3.4 \) nm displacement in the longitudinal modulation of the refractive index in a grating with 1550 nm central wavelength. The choice of this value is somewhat arbitrary. On one hand, \( d\phi \) is small enough to satisfy the assumption of “weak measurement” introduced in Section 3.4.2, and to better understand the limitations the developed theory is applicable within. Indeed, large deviations from the correct results
are expected for large $d\phi$, while the corresponding limit for small $d\phi$ is only qualitatively set by theory, and directly addressing the problem is necessary. On the other hand, the chosen value is larger than typical phase errors in high quality gratings. Indeed, assuming a coherence length of the writing system of $L_{coh} \simeq 10$ cm, as discussed in Section 2.4.4, the standard deviation of the phase of the modulation pattern over a single period $\Lambda$ is $3$ mrad only (see [122] for details). The probability of introducing a $20$ mrad error in the grating is negligible. But a large number of phase errors are typically present in a real grating, and their cumulative effect has to be taken into account. This problem is not addressed in this Section, since the emphasis is on the understanding of the sensitivity of the grating to a single perturbation. It is therefore preferable to use a larger phase error in order to show the resulting effects more clearly. A low quality grating with coherence length of $1$ cm only is considered [122], and the standard deviation of the phase over one period is increased to $10$ mrad. The choice of a phase error of magnitude equal to twice the standard deviation of the process is therefore natural, given that its probability is roughly $0.3\%$ according to error theory.

4.3.1 Uniform gratings: accuracy of the model

Eqs. (3.38) and (3.39) have been applied to the analysis of a uniform grating perturbed with a phase shift $d\phi = 20$ mrad along the grating length, as previously described. The length of the considered structure is $L_{gr} = 10$ mm, the effective refractive index is $n_{eff} = 1.45$, and the modulation of the refractive index is $\delta n = 1.48 \times 10^{-4}$. The Bragg wavelength is set to $\lambda_{Bragg} = 1550$ nm. The strength of the grating is $R_{MAX} \simeq 0.99$, and the bandwidth between the first zeros is $\Delta \lambda \simeq 0.23$ nm. The corresponding results are shown in Figs. 4.2 and 4.3 in reflection and in transmission, respectively. As in Section 3.3.1, only a few wavelengths have been considered here as significant examples. Using the same notation as in Fig. 3.5, $\lambda_1$ is the Bragg wavelength (upper row), $\lambda_3$ is close to the first transmission resonance ($T_{gr} \simeq 1$), where the time delay is longer (central row), and $\lambda_4$ is on the first sidelobe peak (lower row). The normalised amplitude perturbations $\Delta R_{gr} / R_{gr}$ and $\Delta T_{gr} / T_{gr}$, consistently with Eq. (4.3), are represented in the left column, while the phase perturbation $\Delta \theta$ is shown in the right column. The results obtained using the correct transfer matrix approach (blue lines) and the approximated time delay approach (red lines) are compared.

If the characteristics at the Bragg wavelength $\lambda_1$ are considered, it is apparent that a perfect agreement in the reconstruction of the resulting phase change is obtained both in reflection and transmission (see (b)). As already discussed, an almost exponential decay of the effect is found in reflection, closely following the power distribution inside the scattering region. Light is mainly reflected in the first part of the grating and cannot sample the entire structure. Conversely, a symmetric shape is obtained in transmission. A detailed analysis was given in Section 3.3.1 and will not be considered here. It should be noted, anyway, that the experimental detection of a smaller phase shift induced by a perturbation in the centre of the grating with respect to a perturbation on the
Chapter 4 Applications of time delay distributions

Figure 4.2: Amplitude (left column) and phase (right column) variations induced in reflection by a $\delta \Phi = 20$ mrad phase error introduced in a uniform grating in the position $z$. $L_{gr} = 10$ mm, $\delta n = 1.48 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. $\lambda_1$ (upper row): Bragg wavelength; $\lambda_3$ (central row): close to first transmission resonance; $\lambda_4$ (lower row): first sidelobe peak. See Fig. 3.5 for details. (Blue lines): correct computations using the transfer matrix method; (red lines): local time delay approach.

Edges would give direct evidence of shorter interaction times of light in the periodic structure compared to free space propagation. Possibly, a direct proof of superluminal local time-of-flight can be obtained if a sufficiently strong grating is considered. Such a striking agreement seems not to be actually confirmed by the corresponding amplitude variations, shown in (a). The time delay approach largely underestimates the extent of the perturbation, but the longitudinal profiles can be shown to be similar and the absolute value of the error is anyway negligible. Indeed, the plot is magnified on a $10^5$ scale in reflection and on a $10^3$ scale in transmission in order to make the difference visible. Therefore, this result perfectly agrees with the theoretical analysis of Steinberg. Nevertheless, the difference in the magnitude of the amplitude change is expected to play an important role when a large number of phase errors are considered and their
Figure 4.3: Amplitude (left column) and phase (right column) variations induced in transmission by a $d\phi = 20$ mrad phase error introduced in a uniform grating in the position $z$. $L_{gr} = 10$ mm, $\delta n = 1.48 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. $\lambda_1$ (upper row): Bragg wavelength; $\lambda_3$ (central row): close to first transmission resonance; $\lambda_4$ (lower row): first sidelobe peak. See Fig. 3.5 for details. (Blue lines): correct computations using the transfer matrix method; (red lines): local time delay approach.

More interesting results are obtained close to the transmission resonance ($\lambda = \lambda_3$). The corresponding distributions are shown in (d) and (c) for the phase and amplitude variation, respectively. As already commented, an oscillatory shape is obtained in reflection, with a sign variation which is directly related to the un-intuitive idea of negative time-of-flight inside the last portion of the grating. Very large interaction times are also obtained in specific sections of the structure ($z \sim 3$ mm in this case), which shows that sensitivity to an external perturbation is very much position-dependent as soon as the considered wavelength is outside the band-gap region. It is also clear that a limited agreement is found in this case over a large section of the grating. The comparison with the result for the relative reflectivity variation $\frac{\Delta R_{gr}}{R_{gr}}$ in Fig. 4.2 (c) shows that a very
large modification of $R_{gr}$ is obtained in this case despite the small phase shift $d\phi$ introduced ($\frac{\Delta R_{gr}}{R_{gr}} \simeq 20\%$). The reflectivity change (and therefore the corresponding error in the estimation of the phase change) is particularly large in the centre of the grating, due to the fact that the local round-trip reflection coefficient $\rho$ is maximum in this case and any modification in the round-trip phase is therefore magnified (using the same physical understanding described for chirped gratings in Section 3.3.2). Conversely, a much better fit of the correct phase shift is obtained in transmission (see Fig. 4.3 (d)), despite the symmetric change $\Delta T_{gr} = -\Delta R_{gr}$ in the transmissivity. But the corresponding relative change is 2 orders of magnitude lower, since $T_{gr}(\lambda_3) \gg R_{gr}(\lambda_3)$. The actual perturbation of the system is negligible and a good agreement in the computation of the phase variation is expected.

The analysis of the sensitivity on the first sidelobe peak ($\lambda = \lambda_4$, (e) and (f) in Figs. 4.2 and 4.3) can be interpreted in a similar way. In reflection, the relative amplitude perturbation (e) is one order of magnitude smaller ($\frac{\Delta R_{gr}}{R_{gr}} \simeq 10^{-2}$) with respect to $\lambda_3$. The corresponding reconstruction of the phase perturbation profile is largely satisfactory all over the grating length. Conversely, a much worse agreement is found in transmission despite the even smaller relative amplitude variation ($\frac{\Delta T_{gr}}{T_{gr}} \simeq 5 \times 10^{-3}$). This contradiction is only fictitious, and related to the relative positions of the peaks and the nodes of the oscillatory behaviour of $\Delta \theta(z)$. Indeed, the analysis of Eqs. (D.6) and (D.3) shows that the local time delay approach gives a perfect approximation close to points where $\rho(\bar{z}) = 0$, i.e., close to transmission resonances for the left or right reflector $r_1(\bar{z})$ or $r_2(\bar{z})$ of the corresponding effective cavity. These particular conditions actually correspond to the nodes of the distributions ($N_{R}(\bar{z}) = 2$ in reflection and $N_{T}(\bar{z}) = 1$ in transmission). In Fig. 4.2, the maximum reflection amplitude variation is obtained close to $z = \bar{z}$, and the corresponding error in $\Delta \theta_R$ is greatly reduced by the aforementioned effect. Far from $z = \bar{z}$, a reduced $\frac{\Delta R_{gr}}{R_{gr}}$ is found and a small error is obtained also in this case. In Fig. 4.3, the relative positions in transmission are dual, so that the effect of the amplitude perturbation is maximised and a larger error is found. It must be noted that the same applies to the previous discussion about $\lambda_3$ in reflection (see Fig. 4.2 (c) and (d)).

The previous discussion shows that the local time delay approach is accurate to a large degree in the description of the effects of phase errors in a uniform grating. Very small amplitude sensitivity to small phase errors is obtained close to the Bragg wavelength of the grating, and also the phase of reflected or transmitted light is rather insensitive in large sections of the grating. A similarly small effect is obtained near the sidelobe peaks, while a much higher sensitivity of both amplitude and phase is typical of transmission resonances, i.e., of the reflection zeros.

Particular attention has to be paid to wavelengths that are corresponding to the transmission resonances of the grating. In these cases, the effects described with reference to Fig. 4.2 (c) and (d) are further magnified by the very low reflectivity of the unperturbed grating, theoretically zero if an ideal structure is considered. An “infinite"
sensitivity to any phase error is expected since, however small the phase perturbation
could be, it is already enough to completely change the phase balance which leads to the
resonance. According to the general theoretical framework described in Section 3.4.2,
it is not possible to introduce any weak measurement that is not able to perturb the
system dramatically, so that the same concept of local time delay has a very critical
physical meaning here. Formally, Eq. (3.14) can be computed even in this limiting case,
but the obtained distribution is characterised by sharp peaks and discontinuities, with
very large positive and negative values. Appendix B shows that these local times still
integrate to the correct limit value of the reflection phase time, even though it must be
noted that (rigorously) no time delay can be associated with a reflected component of
amplitude $R_{gr} = 0$. But no practical applicability of the corresponding distributions
can be found, and completely wrong results are obtained if the approximate expression
given by Eq. (3.36) is used to estimate the effect of phase errors.

The alternative formulations reviewed in Section 4.2.1 prove much more effective in
this case. They are directly derived under the Born approximation and, indeed, Eq. (4.5)
is linear, showing that a single scattering event related to the perturbation is taken into
account. Therefore, they are also expected to be fairly accurate for out-of-band and
generally for very low reflectivity values. Fig. 4.4 shows the comparison between the
linearised approach by Uno and Adachi [171] (green lines), the time delay approach (red
lines), and the correct computation (blue lines) for three different wavelengths. (a) and
(b) are referred to in-band propagation. As commented in Fig. 4.2, the time delay ap-
proach is effective, while the linearised method is inadequate in describing the amplitude
variation due to the multiple scattering in the strong reflection conditions. Conversely,
(c) and (d) shows the corresponding results exactly on a transmission resonance, as is
clear from the very low reflectivity $R_{gr} = -66$ dB. The time delay method is completely
wrong here in the estimation of both the amplitude and phase variations, with a huge
perturbation forecasted and close to the actual numerical precision of the simulation
(note that undersampling of the $\Delta\theta_R$ occurs in (d)). The perturbation approach, in
contrast, is shown to be highly accurate (error $< 1\%$) in amplitude analysis even in
these extreme conditions, while a more significant error is still obtained in phase (error
$\sim 0.5$ rad). Finally, for propagation out-of-band at higher reflectivities, i.e., close to the
sidelobe peaks, the two methods are shown in (e) and (f) to be fairly equivalent and
sufficiently accurate.

Therefore, the effect of a phase error in a uniform grating is better described by a
mixed approach in which the exponential expression (3.36) is used for wavelengths inside
the band-gap of the grating, and the linearised expression (4.13) is used outside the band-
gap. This result also shows that a better estimation of the perturbation obtained with a
distribution of phase errors $\sum_z d\phi(z)$ is expected for out-of-band wavelengths and using
Eq. (4.13), since the linear approximation is proven to be very accurate in these cases.
4.3.2 Apodised gratings

The sensitivity to phase errors of different types of Bragg gratings is now analysed using the previously described mixed approach. A sample phase shift $d\theta = 20$ mrad is supposed to be introduced inside the grating in all the considered cases, and the corresponding phase and amplitude variations are shown for either in-band or out-of-band propagating wavelengths.

An apodised grating is the first example considered. A raised cosine apodisation profile is used, similarly to Section IV in Feced and Zervas [122]. The grating is $L_{gr} = 10$ mm long, $\lambda_{Bragg} = 1550$ nm, and has a maximum refractive index modulation $\delta n = 2.98 \times 10^{-4}$. The peak reflectivity is $R_{MAX} = 0.99$. Fig. 4.5 shows the variation $\Delta R_{gr}$.
Figure 4.5: Contour plot of the amplitude variation $\Delta R_{gr}$ induced in reflection by a $d\phi = 20$ mrad phase error. A raised cosine apodisation is considered and shown on the right as a function of the position $z$. $L_{gr} = 10$ mm, $\delta n = 2.98 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. The unperturbed grating’s reflectivity $R_{gr}$ is shown on the top plot. The contour levels are separated by $\Delta = 0.01$ dB, and the corresponding colormap is shown on the top-right corner.

(in dB) obtained using Eq. (4.4) when the grating is perturbed with the phase shift $d\phi$. The apodisation profile $\delta n(z)$ is shown on the right, while the unperturbed reflectivity $R_{gr}$ plot on the top shows that in-band propagation is considered in this case. The corresponding amplitude variation $\Delta R$ is shown in the main window using a contour plot. The distance between adjacent contour levels is $\Delta = 0.01$ dB, and the colormap used is represented on the top-right corner. It is apparent that a very limited sensitivity to the perturbation is obtained all over the main reflection band, with a variation smaller than 0.01 dB for $\delta \lambda = \lambda - \lambda_{Bragg} \in [-0.1, 0.1]$ nm. An increased sensitivity is found at the edges of the reflection band, but the maximum variation is still below 1 dB even for wavelengths highly detuned from the Bragg condition and with reflectivity as low as $R_{gr} \sim -15$ dB. More interestingly, the (minor) effect of the perturbation is to introduce a slight shift in the position of the main reflection band, as shown by the complementary colours at the band’s edges. The reflectivity is further reduced on the blue side ($A_1$, blue contour lines) if the phase error is introduced at the centre of the grating, while it is correspondingly increased on the red side ($A_2$, red contour lines). A red shift of the reflection band results. This antisymmetric wavelength dependence is expected, since the phase shift can be interpreted as a free space propagation region (as commented in Fig. 4.1). Therefore, symmetric wavelengths with respect to $\lambda_{Bragg}$...
experience anti-symmetric phase shifts, and complementary effects are obtained in the small perturbation limit.

The sensitivity to the position of the phase error shows that the effect is symmetric with respect to the grating centre, and a shift introduced at either \( z = 2 \text{ mm} \) (\( B_1 \)) or \( z = 8 \text{ mm} \) (\( B_2 \)) has the same effect on the grating’s characteristics. This occurs for both highly reflected wavelengths inside the reflection band and highly transmitted wavelengths at the corresponding edges. The obtained result is expected in the second case, since a small depletion of the forward propagating wave \( U(z) \) occurs. Conversely, it is quite counter-intuitive inside the band-gap, since reflected light penetrates the structure up to the peak of the apodisation profile, and only a small part reaches the position \( B_2 \) with respect to the one reaching the position \( B_1 \). This sensitivity is anyway consistent with the analytical formulas in Appendix A, which show a symmetric \( \text{Im} \{ \mathcal{N}_R \} \) when symmetric structures are considered (as clear from Eq. (A.9b) in the case \( T_1 \) and \( T_2 \) are swapped).

The corresponding out-of-band sensitivity is analysed in Fig. 4.6 using Eq. (4.5). It is convenient to show the perturbed reflectivity \( \tilde{R}_{gr} \) rather than the reflectivity variation \( \Delta \tilde{R}_{gr} \). The shift in the reflection band previously discussed is clear from the deformation of the orange contour levels (which means at a reflectivity level \( R_{gr} \approx -20 \text{ dB} \)). The impact on the in-band characteristics is confirmed to be negligible, with minor changes in the -30 dB bandwidth in case the grating is used for channel filtering applications. The out-of-band rejection of the grating (|\( \delta \lambda \)| > 0.3 nm) is shown to be more affected by the presence of phase errors. Large areas in yellow (\( \tilde{R}_{gr} \in [-40, -35] \text{ dB} \)) are obtained at different wavelengths and for errors in different positions \( z \), while the unperturbed reflectivity is always lower than -40 dB in these wavelength ranges. An increased out-of-band reflectivity is typically obtained, especially if the error is localised close to the apodisation peak \( z = 5 \text{ mm} \). Moreover, the reflectivity notches corresponding to the transmission resonances of the unperturbed grating are partially filled, and \( \tilde{R}_{gr} > -55 \text{ dB} \) is typically found. Indeed, the perfect resonance conditions are easily disturbed by the presence of a phase shift, and perfect tunnelling of the structure is no longer possible. In particular, this effect is apparent on the higher order zeros of the reflectivity (|\( \delta \lambda \)| \( \sim 0.424 \text{ nm} \), |\( \delta \lambda \)| \( \sim 0.504 \text{ nm} \) in Fig. 4.6). At the same time, it is even possible to create resonances (\( \tilde{R}_{gr} < -80 \text{ dB} \)) at wavelengths that are normally partially reflected in an unperturbed structure, as shown by the points \( A_1 \) and \( A_2 \).

Another feature to be noted is the antisymmetric shape of the perturbation with respect to the Bragg wavelength, which is more apparent in this case than inside the band-gap. If a wavelength \( \lambda_1 = \lambda_{\text{Bragg}} + \delta \lambda \) experiences an increased reflectivity, i.e., a sidelobe peak is obtained, the corresponding wavelength \( \lambda_2 = \lambda_{\text{Bragg}} - \delta \lambda \) presents a dip at very low \( \tilde{R}_{gr} \) (see the points \( B_1 \) and \( A_1 \)). As an example, if the effect of the perturbation \( \delta r_{gr} = \tilde{r}_{gr} - r_{gr} \) is comparable in amplitude with the reflection coefficient \( r_{gr} \) of the perfect structure, constructive interference at \( \lambda_1 \) results in a 3 dB increase of the perturbed reflectivity, while destructive interference at \( \lambda_2 \) would produce a reflection
notch. The difference between corresponding wavelengths can therefore be very large (up to 40 dB between $B_1$ and $A_1$), even though $10 - 20$ dB are more typical for this amplitude $d\phi$ of the perturbation.

Moreover, even the periodicity of the sidelobes is largely affected by the phase error, if the actual tails of the apodisation profile are disregarded and $z \in (2,8)$ mm respectively is considered. The periodicity is halved if $z \sim 5$ mm ($A_1$, dashed line), since the grating is actually split into two sections of halved length. This effect is commonly known in DFB structures, where a $\pi$ phase shift is introduced in the centre of the grating (or close to it), but it is shown to be present here even for much smaller deviations from the ideal coherence conditions inside the grating. Conversely, a different periodic behaviour is found at $z = 7$ mm ($C_1$, dashed line). The first two sidelobes are still present and with a position similar to the unperturbed structure, while the third sidelobe is completely missing and a corresponding minimum is found (even if at a relatively high reflectivity level, $\tilde{R}_{gr} \simeq -55$ dB). Indeed, these complex features can be interpreted again considering an equivalent interference between the normal grating reflection coefficient $r_{gr}$ and the perturbation $\delta r_{gr}$. Interference is constructive on the second sidelobe (small reflectivity increase), and is destructive on the first and third sidelobes. The final effect is small on the first one since $|r_{gr}| \gg |\delta r_{gr}|$ (a clear sidelobe is still present), and large on the third since $|r_{gr}| \simeq |\delta r_{gr}|$ and almost complete interference occurs. This interpretation is confirmed by the reflectivity profile $\tilde{R}_{gr}$ at the corresponding $-\delta \lambda$. It is more similar to

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure4_6.png}
\caption{Mesh of the perturbed reflectivity $\tilde{R}_{gr}$ of a raised cosine apodised grating (shown on the right). $L_{gr} = 10$ mm, $\delta n = 2.98 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. The unperturbed grating’s reflectivity $R_{gr}$ is shown on the top plot. A $d\phi = 20$ mrad phase error is considered. The corresponding colormap is shown on the top-right corner.}
\end{figure}
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Figure 4.7: Contour plot of the phase variation $\Delta \theta_R$ induced in reflection by a $d\phi = 20$ mrad phase error. A raised cosine apodisation is considered and shown on the right. $L_{gr} = 10$ mm, $\delta n = 2.98 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. The unperturbed grating’s reflectivity $R_{gr}$ is shown on the top plot. Propagation inside the band-gap of the grating is considered. The contour levels are separated by $\Delta = 2$ mrad, and the corresponding colormap is shown on the top-right corner.

The phase perturbation $\Delta \theta_R$ for the previously described grating is represented in Fig. 4.7, where Eq. (4.4) has been used. As plotted in the top graph, only propagation

the one described with reference to the point $A_1$, due to the dual interference conditions and the different relative magnitudes of the unperturbed and perturbed contributions.

The previous analysis shows that a $d\phi = 20$ mrad phase error is not sufficient to significantly alter the properties of a typical apodised grating. The out-of-band characteristics are largely distorted, but still at very low reflectivity levels which are not important for standard applications of gratings. The complex deformations found do not allow a simple identification of the position of the phase error from the analysis of the perturbed reflectivity, even if common trends useful to increase the intuitive understanding can be found. Indeed, a halved periodicity in the sidelobes has been shown to be highly correlated with errors in the centre of the grating, while the presence of patterns such as “two sidelobes up and one down” is typical of phase shifted gratings in which the phase error is located at $z \sim 2/3L_{gr}$. It is clear that larger phase shifts cause a more substantial increase in the out-of-band reflectivity level. $d\phi = 40$ mrad (corresponding to a $|\Delta z| \sim 0.013\Lambda \sim 6.8$ nm in the periodic pattern of the grating) can be shown to be already sufficient to produce sidelobe peaks in excess of -30 dB, limiting the use of such a grating for telecommunications applications as band filters.

The phase perturbation $\Delta \theta_R$ for the previously described grating is represented in Fig. 4.7, where Eq. (4.4) has been used. As plotted in the top graph, only propagation
inside the band-gap is considered here. The phase variation at the entrance of the grating is $\Delta \theta_R = 40$ mrad due to the double pass experienced by the reflected light. As already seen in Section 4.2.1, the perturbation closely follows the spatial distribution of energy inside the scattering structure in this case. Propagating deeper inside the grating, the perturbation is further reduced and eventually no sensitivity is found after the peak of the apodisation profile, due to the very limited field that penetrates the whole structure and is finally reflected for $R_{gr} = 0.99$. The idea of introducing an effective reflection point is shown to be quite arbitrary when apodised gratings are considered, since light is progressively scattered back from the grating due to the smooth increase in the apodisation profile. Only the very central band shows contour levels parallel to the $x$-axis, and therefore an almost uniform distribution of energy and time delay. A large deformation is found closer to the band-gap edges. It corresponds to an increased time delay and to the corresponding dispersive characteristic of these kinds of gratings [88], which makes them suitable for band filtering applications only with relatively narrow-band signals to avoid temporal deformations of the pulses.

Phase sensitivity is much larger outside the reflection band, and an almost antisymmetric behaviour is found with respect to the peak of the apodisation profile. Increased phase perturbation is obtained if the error is before the centre of the structure, while decreased phase perturbation is found after it, in accordance with the counter-intuitive idea of negative time delays discussed in Section 3.3.1. The out-of-band profile of the phase perturbation is not shown here due to the limited significance of the phase information at very low reflectivity levels. It is anyway worth pointing out that the corresponding values of $\Delta \theta_R$ can be as large as $\pm \pi$. This is expected from the fact that the peaks and troughs of the sidelobes change their position when an error is introduced, as previously remarked, so that the sign changes are expected and required to keep consistency even if a small phase perturbation is introduced.

4.3.3 Square dispersionless gratings

The second example of grating sensitivity analysis considered is an inverse scattered design. A square dispersionless filter is examined, similar to the one shown in Fig. 3.4. The filter bandwidth is $\Delta \lambda = 0.2$ nm and it is centred at the Bragg wavelength $\lambda_{Bragg} = 1550$ nm. The maximum reflectivity is set to $R_{MAX} = 0.99$, and the grating’s length is $L_{gr} = 80$ mm. The filling factor of the filter is $\eta_B = 0.8$. On the other hand, the time delay ripple of this grating is not negligible, and can be as large as $\pm 3$ ps at the band-gap edges (see Fig. 4.9, top plot).

Comparing the amplitude and time delay characteristics of the above-mentioned grating with the one shown in Fig. 3.4 (c), it is apparent that a very large penalty is obtained from the point of view of the dispersive properties of the inverse scattered grating, while a smaller degradation is found in the reflectivity profile of the grating. The analysis of the corresponding apodisation profiles (Fig. 3.4 (b) and Fig. 4.8, right plot) shows that the design presented here is much shorter than the previous one ($L_{gr} = 80$ mm
The number of oscillations after the main apodisation peak is practically the same in both cases, while the length reduction is obtained by having only three oscillations rather than six before the peak. The portion of the grating that is disregarded is at very low levels of refractive index change $\delta n$ (typically, $\delta n < 2 \times 10^{-6}$), and it is not expected, intuitively, to have a large influence on the properties of the grating. Indeed, these values are also not easily reproduced in the fabrication of fibre Bragg gratings due to the intrinsic noise floor of the process, related to fluctuations both in the writing process and in the intrinsic properties of the fibre. The possibility of neglecting them in the design stage would be a clear advantage for the reproducibility and the effectiveness of the manufacturing process.

The results of the local time delay analysis are useful in order to understand the described features. Fig. 4.8 shows the phase variation obtained in the reflection coefficient $r_{gr}$ when a $d\phi = 20$ mrad phase shift is introduced in the structure. Only propagation in the band-gap is considered in this case, as shown by the very flat reflectivity $R_{gr}$ (corresponding to oscillations within 0.5 dB in transmissivity), and Eq. (4.4) is used. Light is found to be practically confined within the first part of the grating over the considered bandwidth, with a sharp transition to zero corresponding to the rising edge of the main apodisation peak at $z \sim 25$ mm. The flat time delay characteristic of a dispersionless grating does not automatically imply this shape for the local time delay.
distribution. It would also be possible to have a very different penetration of the field at different wavelengths inside the band-gap, with a correspondingly smooth transition spread out over a much longer length-scale and possibly over multiple oscillations of the apodisation profile. This hypothetical solution would be closer to the one previously described for a raised cosine apodisation (Fig. 4.7). Indeed, the inverse scattered grating is very similar to a standard apodised design, if a single lobe in the apodisation profile is considered. The effect of the lobes added after the peak of the apodisation profile would be to match the dispersion produced by the main reflection.

On the contrary, Fig. 4.8 demonstrates that the main apodisation peak is sufficiently strong to reflect all the light at all the wavelengths. The second part of the grating has no influence whatsoever on the dispersive properties of the structure. Therefore, the dispersion equalisation in a dispersionless grating is performed by the part of the apodisation profile before the effective reflection point. This region has been largely shortened in this design with respect to the ones in Fig. 3.4, which directly justifies the degraded time delay ripple $\tau_R$. Any modification in this part is directly transferred into a perturbation of the phase profile of the reflected light, with spectral oscillations that are different in amplitude, periodicity, and position depending on the distance from the effective reflection point.

This effect is more apparent in Fig. 4.9, where the corresponding distribution of the
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time delay variation, computed using the definition of the phase time delay given by Eq. (2.40), is visualised. It is readily found from Eq. (3.38) that

\[ \Delta \tau_R(z) = \frac{\partial \Delta \theta_R(z)}{\partial \omega} = d\phi \frac{\partial \text{Re}\{N_R(z)\}}{\partial \omega}. \]  

(4.20)

A single phase error \( d\phi = 20 \text{ mrad} \) is sufficient to produce a time delay change as big as \( \pm 0.7 \text{ ps} \) for the wavelengths at the edges of the reflection band of the grating. This occurs even if the error in the periodic pattern of the grating is only \( \Delta z \sim 4 \text{ nm} \) long, which corresponds to a double pass time of \( 0.04 \text{ fs} \) in the perturbed region. An antisymmetric distribution with respect to the Bragg wavelength is found, with an insignificant sensitivity only for propagation exactly at \( \lambda_{\text{Bragg}} \). The time delay ripple \( \tau_R \) due to the shortening of the grating, on the contrary, is shown in Fig. 4.9, top plot, to be symmetric. This mismatch prevents the simple solution to completely correct the residual time delay ripple by introducing a suitable distribution of phase errors in the first 20 mm of the superstructured grating. Possibly, only either the red (\( \lambda > \lambda_{\text{Bragg}} \)) or the blue (\( \lambda < \lambda_{\text{Bragg}} \)) bands of the spectrum can be flattened with this approach, but with a corresponding worsening of the dispersive properties on the complementary band.

Positive and negative lobes in the apodisation profile do not correspond to different signs in the obtained time delay sensitivity. Conversely, the final effect is more directly related to the wavelength mismatch with respect to the Bragg condition, and to the distance between the position of the perturbation and the effective reflection point. Fig. 4.9 shows that a small effect results when the phase perturbation is on a side-peak of the apodisation profile \( \delta n \), while a larger time delay sensitivity (blue and yellow areas) is obtained close to the zeros. They correspond to the regions in which \( \pi \) phase shifts have to be introduced in the grating in order to realise the (effective) negative index variation shown in the grating profile [30]. An immediate practical importance is found, since these positions are more likely to be sources of phase errors during the fabrication process of a complex fibre Bragg grating. If the continuous grating writing system is used to fabricate these superstructures [109], dithering and changing the local periodicity of the multiple exposures is necessary to shift the modulation pattern, so that an accurate positioning of the zeros is not easy, and correspondingly \( d\phi \) errors are introduced. It must be noted that, anyway, this is a characteristic feature of sinc-sampled superstructured gratings [30], and of gratings presenting sign changes in their \( \delta n \) profile in general (not shown here). The high sensitivity is therefore not related to the use of apodisation designs obtained using inverse scattering techniques.

The sensitivity of the reflectivity \( R_{gr} \) of the grating to a phase perturbation is considered in Fig. 4.10. Again, in-band propagation is analysed for the moment according to Eq. (4.4), and the variation of the reflectivity \( \Delta R_{gr} \) is shown in dB. The mesh is actually scaled to dB×10^3, proving that a completely negligible sensitivity is obtained in this case. Deformation of the characteristics of the reflection band is not an issue in practical grating manufacture, but Fig. 4.10 is still interesting since it gives an appreciation of the opposite dependence on the position of the error inside the grating with
Figure 4.10: Mesh of the reflectivity change $\Delta R_{gr}$ of an inverse scattered, square dispersionless grating (shown on the right) when a $d\phi = 20$ mrad phase error is introduced. $L_{gr} = 80$ mm, $\delta n = 1.88 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. The unperturbed grating’s reflectivity $R_{gr}$ is shown on the top plot. Only in-band propagation is considered. The corresponding colormap is shown on the top-right corner.

Figure 4.10: Mesh of the reflectivity change $\Delta R_{gr}$ of an inverse scattered, square dispersionless grating (shown on the right) when a $d\phi = 20$ mrad phase error is introduced. $L_{gr} = 80$ mm, $\delta n = 1.88 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. The unperturbed grating’s reflectivity $R_{gr}$ is shown on the top plot. Only in-band propagation is considered. The corresponding colormap is shown on the top-right corner.

It is important to notice that the region of the grating following the reflection point is more significant in this case, showing a higher sensitivity despite the fact that almost no field can penetrate the grating up to the corresponding position. This means that the $\delta n(z)$ lobes after the main peak are mainly responsible for the amplitude characteristics of the superstructured grating. The periodicity of the $\Delta R_{gr}$ variation for increasing values of $z$, and the fact that the larger sensitivity is progressively shifted towards the
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Figure 4.11: Mesh of the perturbed reflectivity $\tilde{R}_{gr}$ of an inverse scattered, square dispersionless grating (shown on the right) when a $d\phi = 20$ mrad phase error is introduced. $L_{gr} = 80$ mm, $\delta n = 1.88 \times 10^{-4}$, and $R_{MAX} \sim 0.99$. The unperturbed grating’s reflectivity $R_{gr}$ is shown on the top plot. Only out-of-band propagation is considered on both sides of the reflection band. The corresponding colormap is shown on the top-right corner.

edges of the reflection band, suggest that apodisation ripples further inside the grating are responsible for squared shape of the reflectivity typical of these inverse scattered designs. This understanding is also consistent with the picture of a Gires-Tournois interferometer [124, pag. 150], where elements placed further away from the high reflectivity mirror have shorter free spectral range, and therefore can account for faster variations in the amplitude spectrum of the overall structure.

Finally, the out-of-band characteristics of the inverse scattered, square dispersionless gratings are analysed in Fig. 4.11 according to Eq. (4.5). Both the red (left) and blue (right) sides are shown, while the reflected wavelengths have been disregarded. The unperturbed grating’s reflectivity is also plotted, and it shows the extent of the transition bandwidth (defined as the region where a reflectivity $-30$ dB < $R_{gr}$ < $-1$ dB is found) and the high level of suppression of the sidelobes ($R_{gr} < -35$ dB). As in Fig. 4.6, orange areas are related to an out-of-band reflectivity close to $R_{gr} = -30$ dB, typically required for filtering applications of fibre Bragg gratings. It is apparent that the deformation occurring with phase errors placed outside the main apodisation peak is very limited and practically negligible for this error amplitude $d\phi$. The corresponding profile of the variation $\Delta R_{gr}$ (not shown) shows that the usual antisymmetric shape in $\lambda$ is obtained, and that the ratio between the sensitivity before and after the apodisation peak is very
similar to the in-band case previously discussed. In contrast to propagation inside the band-gap, the structure is more sensitive to phase errors located inside the main lobe of $\delta n(z)$ in this case, with variation of up +8 dB corresponding to the sidelobe peaks, and much larger variation close to the transmission resonances, as expected from the discussion in Section 4.3.2. Typically, the interference between the original grating’s contribution and the perturbation contribution causes the transmission resonances to be cancelled, but a few cases of increased tunnelling occur even for this complex grating. They are represented by blue dots in Fig. 4.11, corresponding to $\tilde{R}_{gr} \sim -80$ dB for very specific wavelengths and positions; a few examples are highlighted by points $A_i$. The complex features shown for $z \in (20, 35)$ mm respectively can be interpreted using the same simplified model discussed in the previous Section.

It is important to notice that the perturbation of the main reflection band and of the transition region ($0.11 \text{ nm} < |\delta \lambda| < 0.125 \text{ nm}$) is much less apparent in this case than in the grating apodised with a raised cosine profile (see Fig. 4.6) for the same amplitude of the perturbation $d\phi$. This means that a square (dispersionless) grating is not more sensitive to phase errors than a standard apodised design, if the device is used as a spectral filter. The improved selectivity result in a design that, besides being more complex, does not have tighter error-control manufacturing requirements.

### 4.3.4 Chirped gratings

Chirped gratings for dispersion compensation applications are the final example that is going to be considered. This class of devices has been shown to be mature for commercial applications of fibre Bragg grating technology, so that an analysis of the corresponding sensitivity to perturbations introduced in the fabrication process is very important. The discussion in Section 3.3.2 already shows the main characteristics of the distribution of the local time delay, and therefore the expected phase sensitivity to phase errors. But the focus was on the physical properties of linearly chirped gratings rather than on the typical issues that affect dispersion compensating gratings. These aspects are covered here in more detail.

The sensitivity of the reflectivity $\tilde{R}_{gr}$ both inside and outside the band-gap will not be considered here since the reflectivity profile is not the major problem of these devices. In the first case, the in-band sensitivity $\Delta R_{gr}$ is negligible when a very flat reflectivity is obtained, as shown by both Fig. 4.5 and 4.10. Similar results are also expected for flat-top dispersion compensators, so that the corresponding analysis is omitted. In the second case, combined dispersion compensation and wavelength filtering is indeed an advantage in terms of system simplicity, and allows the reduction of the insertion losses of the overall compensating module. Apodised chirped gratings typically show a large out-of-band attenuation (see Fig. 3.8 as an example), however the large roll-off at the band edges is not a problem when long gratings are used. Indeed, the present lack of proper packaging when $L_{gr} > 200$ mm means that appropriate guard-bands have to be used anyway, relaxing the stability requirements of the grating and, therefore,
the need of high out-of-band suppression close to the reflection band. Moreover, the possibility of compensating the full gain bandwidth of an EDFA with a single grating has been experimentally proven only recently [119], with the ability to write $L_{gr} > 10$ m gratings by translating the fibre on a fibre drum and correspondingly intensity modulating the writing UV beam. Different techniques have been proposed for multi-channel compensation using more conventional grating lengths ($L_{gr} < 1$ m), such as a single, 1 m long chirped grating [37], concatenated long gratings able to cover up to a 18 nm bandwidth [174], or Moiré superstructured gratings [31]. But band filtering is still necessary in all these cases, and it is often implemented using alternative (cheaper, but coarser) optical technologies. A different approach is given by dispersion compensation on a single channel basis, using non-linear chirp and apodisation profiles obtained using inverse scattering techniques [49]. Even in this case, spectral interleavers are typically used to provide relaxed requirements on the wavelength stability of the components inside the dispersion compensating module and to reduce costs. Additional filtering is provided by these interleavers, and the out-of-band reflectivity of the chirped grating is not as important as in the previously described examples.

Conversely, the analysis of the phase and time delay sensitivity is more important for these devices, given their main application in dispersion compensation. Fig. 4.12 shows
the phase sensitivity $\Delta \theta_R$ of a typical linearly chirped grating when a $d\phi = 20$ mrad perturbation is considered. It has been computed using Eq. (3.38). The analysed grating is similar to the one described in Feded and Zervas [122], Section V. Its length is $L_{gr} = 150$ mm, a linear chirp is present and the chirp rate is $CR = -0.065$ nm/cm, corresponding to compensation of a total dispersion $D_{TOT} = -1500$ ps/nm. A raised cosine apodisation profile that extends over 20 mm at both ends is used to suppress the ripple in the time delay characteristic. The apodisation (blue) and chirp (red) profiles are shown on the right in Fig. 4.12. The maximum refractive index modulation is $\delta n = 6.4 \times 10^{-5}$, the reflectivity of the grating is $R_{MAX} = 0.9$, and the available grating bandwidth is 0.6 nm. The time delay response of the structure is shown in the top plot, showing a negligible time delay ripple. It must be noted that the drawbacks of the apodisation process are represented by a reduced bandwidth efficiency of the grating and by a low frequency deformation of the time delay response, partially apparent in the plot for $\delta \lambda \sim +0.3$ nm. The corresponding group delay ripple $GDR$ defined in Eq. (2.41) is as large as 13 ps. This deviation may cause degradation in the performance of the device due to the residual dispersion after compensation, especially at high bit rate, $R = 40$ Gb/s, where the time slot is only 25 ps long.

The mesh of the sensitivity pattern does not show unexpected features. Light is highly reflected in the position in which the Bragg condition $\lambda_{Bragg}(z) = 2n_{eff} \Lambda(z)$ is fulfilled, and a negligible effect is found if the perturbation is deeper inside the grating (blue region in Fig. 4.12). Conversely, the region before the band-gap is sensitive to the perturbation, even though no resonance peaks with very large or small variations are obtained. Typically, a single double pass is experienced by reflected light, with larger deviations only for limited detuning from the local Bragg wavelength $\lambda_{Bragg}(z)$ (see Section 3.3.2). The effect of the apodised region is apparent, since no residual oscillations are found close to the entrance of the grating. Even the deformation in the linear characteristic of the time delay can be traced. It is related to the overlap of the apodised section with the local band-gap, easily recognised by the transition from the red area to the blue one which corresponds to evanescent propagation-attenuation. A different efficiency of the reflection process results in this position with respect to the rest of the grating, and a different effective reflection position is therefore found.

A linear dependence of the contour levels with wavelength and position is clearly shown here. It confirms the intuitive understanding discussed in Section 3.3.2 and Fig. 3.11, according to which a simple spatial shift in the sensitivity profile is found by changing the considered wavelength. Well defined physical meaning for this dependence can be found if a (deterministic) uniform distribution $d\Phi(z)$ of phase errors is considered, and using Eq. (3.40). The linear dependence gives an almost linear phase perturbation $\Delta \theta_R(\lambda)$ by integrating the sensitivity profile over $z$. All the phase ripples are largely averaged, and only the double pass contribution before the band-gap is important. But a linear phase variation corresponds to a simple shift of the time delay profile according to Eq. (2.40). This is consistent with the physical understanding of this perturbation,
since the previous phase error distribution can be thought of as a uniform change in the background refractive index $n_{\text{eff}}$. A local detuning of the central Bragg wavelength is found, and the previous result is confirmed.

More significant results are obtained if more complex error distributions $d\Phi(z)$ are taken into account. Large bandwidth dispersion compensators are typically much longer than uniform and apodised gratings used for band filtering and signal conditioning, and longer than typical inverse scattering superstructures, such as the one described in the previous Section. Keeping phase coherence inside the whole grating is therefore a much harder task, and phase deviations from the ideal profile are typically obtained. In particular, it is interesting to consider the effect of periodic phase perturbations $d\Phi(z, \Lambda)$, where $\Lambda$ is the periodicity of the phase errors. A periodic (or, generally, quasi-periodic) change in the local periodicity of the grating with respect to the nominal value is likely to happen during the manufacturing process of a fibre Bragg grating. The fluctuations of the fibre parameters are often periodic due to the oscillations produced by instabilities in the fibre drawing process. At the same time, fluctuations in the writing set-up due to acoustic vibrations in the optical bench or to other sources (such as turbulent flow in the laser cooling pipelines, see [109]) are likely to be characterised by certain resonant frequencies, which are directly imprinted in the grating pattern.

The effect of these periodic perturbations on the group delay ripple of chirped gratings is considered in detail here. Indeed, the main limitation to the practical application of grating-based dispersion compensators is the still unsatisfactory characteristic of the grating’s time delay, and, in particular, to the impact of the group delay ripple. Apart from noise and manufacturing imperfection, $GDR$ is also a typical feature of uniform, linearly chirped grating, as seen in Section 3.3.2. The use of suitable apodisation profiles at the very ends of the grating, such as the one in Fig. 4.12, is proven effective to reduce the corresponding spectral oscillations at high frequency. Unfortunately, theoretical [175] and experimental [176] evidence has recently shown that the high frequency ripple is not the major cause of system performance degradation. Indeed, high correlation has been shown between the eye-closure in a typical optical system and phase (rather than group delay) ripple, so that high frequency components of $GDR$ are partially suppressed by integration over the bandwidth of the signal. Conversely, low frequency $GDR$ ripple introduces higher system penalties, especially if the corresponding periodicity matches the bandwidth of the optical signal ($80$ GHz $\rightarrow \Lambda = 0.64$ nm for $R = 10$ Gb/s transmission, $320$ GHz $\rightarrow \Lambda = 2.56$ nm for $R = 40$ Gb/s transmission). A deep understanding of the sources of $GDR$ is therefore necessary. An approach based on the WBK approximation of the coupled-wave equations (2.15) was originally proposed in Sumetsky et al. [177] to study the effect of periodic perturbation $d\Phi(z, \Lambda)$ on the $GDR$. It has the advantage of providing closed formulas, but it is difficult to visualise the corresponding effect and to understand where the perturbation is effective. The time delay analysis proposed here has complementary characteristics, since it does not provide simple expressions, but an immediate, visual understanding of the phenomenon, and it also allows a local analysis.
inside the grating length.

Fig. 4.13 shows the time delay sensitivity of the linearly chirped grating previously analysed. An oscillatory shape is found, whose periodicity and amplitude depend on the detuning with respect to the local Bragg wavelength \( \lambda_{\text{Bragg}}(z) \) (see Section 3.3.2). Larger sensitivity is found in the region of the grating preceding the band-gap, as obvious from the previous discussion about Fig. 4.12. The effect of the introduction of a periodic phase noise with periodicity \( \tilde{\Lambda} \) is intuitively derived. Using Eq. (3.40), the total time delay variation \( \Delta \tau_{R,TOT} \) is given by

\[
\Delta \tau_{R,TOT}(\lambda) = \int_0^{L_{gr}} d\phi \cos \left( \frac{2\pi z}{\tilde{\Lambda}} + \tilde{\vartheta} \right) \Delta \tau_R(\lambda,z)dz, \tag{4.21}
\]

where \( d\phi \) is the peak-to-peak amplitude of the phase ripple, \( \tilde{\Lambda} \) and \( \tilde{\vartheta} \) are the period and the phase of the perturbation, respectively, and \( \Delta \tau_R \) is the sensitivity of the unperturbed structure. Periodic spatial noise produces periodic spectral ripple since the sensitivity pattern is shifted with respect to the perturbation pattern by changing the wavelength, and therefore the previous superposition integral gives positive or negative results depending on the reciprocal phases. An almost linear mapping of the periodicity of the perturbation ripple into the spectral periodicity of \( GDR \) results from the linear shape of the sensitivity contour lines. It is also clear that increasing the chirp rate \( CR \)
Chapter 4 Applications of time delay distributions

of the grating results in less steep contour lines in $\Delta \tau_R(\lambda, z)$, and increased periodicity in the induced ripple.

If very high spatial frequencies are considered, the principle of stationary phase can be invoked in solving Eq. (4.21). Significant time delay perturbation is obtained only when the sensitivity distribution presents oscillations with periodicity similar to the perturbation’s, while all the other contributions are largely averaged and can be disregarded. Only at the edges of the reflection band very high frequency components are obtained (see points $A_1$ and $A_2$ in Fig. 4.13), corresponding to large detuning from the local band-gap, but their amplitude is negligible on the red side ($\delta \lambda > 0$ nm, $A_2$) due to the limited penetration of light if strong gratings are considered. The creation of an effective cut-off wavelength $\lambda_c$ is therefore apparent for each $\hat{\Lambda}$ larger than the extension of the main peak of the sensitivity distribution $\tilde{\Lambda}_0$, shown in red and marked with arrows in Fig. 4.13, and roughly corresponding to the penetration length inside the band-gap. This occurs for $\hat{\Lambda} \lesssim \tilde{\Lambda}_0 = 15$ mm in the considered example. The linear dependence between $\lambda_c$ and $\hat{\Lambda}$ found in Sumetsky et al. [177] is immediately obvious in this context. The previous comments also show that a limited dependence of this effective cut-off on the strength of the grating is expected. Indeed, stronger gratings present a reduced penetration length in the band-gap, so that the typical periodicity of the sensitivity ripple is shorter for a given wavelength, and $\lambda_c$ is increased. At the same time, reduced averaging occurs due to the faster detuning of the local periodicity of the sensitivity distribution. A smoother transition between the two regions (above and below cut-off) is therefore also expected. This feature is not explicitly discussed in the theory developed in Sumetsky et al. [177], even though it can be derived also from that point of view by taking into account that the spatial extension of the band-gap region and, therefore, the position of the secondary Bragg reflection point $z_{st}$ defined there, depends on the strength of the grating [26].

It is also clear from Fig. 4.13 that a certain cut-off $\hat{\Lambda}_c = \hat{\Lambda}_c(\lambda)$ in the phase perturbation periodicity is found for each wavelength $\lambda$. Phase perturbations with periods $\hat{\Lambda}$ ranging from this cut-off to $\tilde{\Lambda}_0 \simeq 15$ mm present similar features, since they are phase-matched to the sensitivity pattern in a certain region of the grating (different for different wavelengths). This region is shorter close to the band-gap, i.e., for long periods (see $\tilde{\Lambda}_0$ in Fig. 4.13), and longer for larger detunings, i.e., for shorter periods (see arrow $B$ in Fig. 4.13). Similar amplitudes of the ripple are expected, since the longer phase-matched regions also correspond to smaller effects of the perturbation. This qualitative idea agrees with the results in [177, Fig. 3.a], and quantitative data can be obtained if sample periodic distributions are considered in Eq. (4.21). The presented analysis shows that a different effect is obtained if the periodic perturbation does not extend over the whole length of the grating. Indeed, a highly suppressed ripple is found if the two distributions do not overlap in a phase-matched region, while practically the full amplitude of the ripple result if they overlap and the perturbation extends to the whole phase-matched section. In this case, large group delay ripple modifications are localised
around a limited range of wavelengths.

If, finally, a very low spatial periodicity is considered \( \tilde{\Lambda} \gg \tilde{\Lambda}_0 \), oscillations in the GDR response are still found, and with amplitude similar to perturbations at \( \tilde{\Lambda}_0 \). Most of the interaction comes from the overlap with the large sensitivity peak before the bandgap region, while higher frequency ripple is averaged and gives a minor contribution to the final effect.

4.4 Design optimisation of gratings using phase perturbations

In the previous Section, evidence has been given to prove that the local time delay approach is effective in increasing the understanding of the main features of typical Bragg grating devices. But the analysis of the local sensitivity to perturbations is not only important to characterise the effect of manufacturing imperfections. Besides being a tool for analysis, it can also be used as a tool for correction of these non-ideal features, or (even more interestingly) for synthesis of gratings with strict spectral characteristics.

The first example of this approach is directly given by the original work about perturbation analysis in scattering structures by Uno and Adachi [171]. Indeed, the effect of the perturbation on a known structure was used to infer how to deform a uniform apodisation profile in order to obtain pre-defined spectral characteristics. Nowadays, this problem has been elegantly solved by using rigorous and computationally efficient layer-peeling algorithms (see Chapter 5). In particular, grating length constraints, bandwidth limitation in the reconstruction process, and finite resolution of the numerical implementation limit the capabilities of the layer-peeling algorithm to perfectly synthesise the target shape of the filter. Residual ripples in the amplitude and/or group delay characteristics are often found, which might not be ideal for certain applications. But, often, a certain spectral feature is much more important than the others for a particular application, so that it would be greatly advantageous to improve the reconstruction of that feature and sacrifice the accuracy in the other parameters. Typical inverse scattering implementations do not allow such an extra flexibility. The original idea in Uno’s work is still valid in this case, and the use of sensitivity distribution \( N_R(\lambda, z) \) provides a theoretically powerful means for designing phase perturbation profiles tailored for correction of undesired spectral characteristics derived from either manufacturing errors or design limitations.

Group delay ripple correction is the most important example of an application of this technique. On one hand, Section 4.3 shows that amplitude sensitivity to phase perturbations is typically negligible if the in-band characteristics are considered. Very large perturbations would be required to significantly modify the reflectivity profile, and the assumptions used to derive the local time delay analysis might not be fulfilled. Wrong correction profiles would be designed in this case. On the other hand, group delay ripple has been shown to be the major cause of performance limitation for many differ-
ent grating-based devices, such as dispersionless filters, see Section 4.3.3, or dispersion compensators, see Section 4.3.4. A limited deformation in the amplitude spectrum can be tolerated in these applications, if improved dispersion characteristics are obtained. A trade-off between the two effects might be necessary especially if a large number of gratings are cascaded [57].

Particular focus is given here to dispersion compensation devices. The previous discussion about the impact of GDR and phase ripple in a typical communication system shows that simple designs of dispersion compensators based on linearly chirped, apodised profiles are not the best solution. High frequency ripple is indeed suppressed by apodisation, but low frequency GDR is actually magnified, as is clear from the GDR profile shown in Fig. 4.13, top plot. Inverse scattered designs [42, 49] or special post-processing of the gratings [172] has been shown to be necessary in order to provide acceptable performance. In particular, the latter solution is highly attractive, since it can also be used to correct manufacturing errors. The fact that such a post-processing is typically performed by using unmodulated UV light, i.e., changing the background index in the grating, makes the local time delay approach naturally suitable for the corresponding analysis. In [172], conversely, the WKB approximation of linearly chirped gratings developed in [177] is applied to the definition of the DC index profile that has to be used to correct for a localised, Gaussian peak in the group-delay ripple. Again, this method is useful since it gives closed formulas for the correction scheme, and gives evidence of the dependence of the correction shape on the main parameters of the grating (chirp rate $CR$, grating strength $\delta n$, FWHM amplitude of the ripple peak). But its applicability is limited to Gaussian shapes, and multiple steps are necessary before the GDR can be reduced below acceptable levels.

**4.4.1 Exact approach**

The local time delay approach is expected to provide a much more direct solution. Assuming that the deviation from the linear time delay characteristic of the considered grating is $\Delta \tau_{R,TOT}(\lambda)$, the global phase perturbation profile $d\Phi(z)$ necessary to flatten the GDR response can be theoretically obtained by inverting Eq. (4.21), when a generic profile $d\Phi(z)$ is used rather then a single spatial frequency $\tilde{\Lambda}$,

$$\Delta \tau_{target}(\lambda) = -\Delta \tau_{R,TOT}(\lambda) = \int_0^{Lgr} d\Phi(z) \Delta \tau_R(\lambda, z) dz.$$  

(4.22)

Eq. (4.22) shows that the reconstruction is nothing but a deconvolution problem. In order to invert the previous expression, we are looking for a bi-dimensional function $M = M(\lambda, z)$ that fulfills

$$d\Phi(z) = \int_{\lambda} \Delta \tau_{target}(\lambda) M(\lambda, z) d\lambda.$$  

(4.23)
where the integration is performed over all the significant wavelengths in the spectrum, and the equality holds for $z \in [0, L_{gr}]$. By substituting (4.23) into (4.22), it is found that

$$
\Delta \tau_{\text{target}}(\lambda) = \int_{0}^{L_{gr}} d\Phi(z) \Delta \tau_{R}(\lambda, z) dz
$$

and therefore

$$
\int_{0}^{L_{gr}} M(\lambda, z) \Delta \tau_{R}(\lambda, z) dz = \delta(\lambda),
$$

where the general properties of the Dirac $\delta$ have been used, and the previous relation must hold for every wavelength. Eq. (4.25) is the starting point for the application of the time delay analysis to gratings’ design. The nature of $M(\lambda, z)$ is apparent if the corresponding discrete representation, suitable for numerical computation, is considered:

$$
M \Delta \tau_{R} = I \Rightarrow M = \Delta \tau_{R}^{-1},
$$

where $\Delta \tau_{R}^{-1}$ has to be intended as the pseudo-inverse of the matrix since, in general, $\Delta \tau_{R}$ is not square due to the different number of samples in space (number of sections) and in wavelength. Ultimately, the inversion procedure can be implemented numerically using minimum least square (MLS) approximations, implemented in the MATLAB® package using the function $\text{pinv}$.

The linearly chirped, raised cosine apodised grating described in Section 4.3.4 is considered as a practical example. The technology for mass production of this device is well established and much simpler compared to a set-up such as the continuous grating writing system, necessary to manufacture complex, inverse scattered designs. The availability of a simple post-processing technique allows this partially unsatisfactory device to be turned into a perfect dispersion compensator by correcting for the residual $GDR$. But useless results are obtained if the (theoretically) correct procedure just described is applied to the correction of the $GDR$ of a real grating. Fig. 4.14 shows the resulting profile of the average refractive index variation $\Delta n_{\text{eff}}$ to be applied to the grating (a). The amplitude of the values obtained has no physical meaning, and the corresponding oscillations have a periodicity of a few hundreds microns. Despite the meaningless result, the group delay ripple expected after the correction and shown in (b) has a negligible amplitude, roughly six orders of magnitude lower than the starting one. It must be noted that this $GDR$ profile has been obtained using Eq. (4.23) to compute the time delay change, and not by simulating the perturbed grating. Indeed, it is obvious that any approximation used in the analysis is not fulfilled with such a huge perturbation. Nevertheless, this result is shown here to prove that the procedure used for the computation of Eq. (4.25) is numerically correct, since an almost perfect reconstruction of the
target GDR shape is obtained.

Physical reasons limit the applicability of this (supposedly) general result. As discussed in Section 4.3.4, a cut-off wavelength \( \lambda_c \) exists in linearly chirped gratings for any given periodicity \( \tilde{\Lambda} \) of a phase perturbation, and in parallel a cut-off period \( \tilde{\Lambda}_c \) (and a corresponding cut-off period in the GDR ripple) exists for any given wavelength \( \lambda \) (see Sumetsky et al. [177, Fig. 5.b]). Any GDR frequency component above such a cut-off is highly attenuated, so that a very large \( d\Phi(z, \tilde{\Lambda}) \rightarrow \Delta n_{\text{eff}} \) perturbation is necessary to perform the corresponding correction. Further proof of this interpretation is given by the structure of the singular values of the matrix \( \Delta \tau \) to be inverted. A huge difference is found, with values spanning over 11 orders of magnitude. From the point of view of numerical analysis, the corresponding matrix is said to be ill-conditioned, and poor, “noise-like” results are obtained when matrix inversion is performed. This is indeed clear in Fig. 4.14 (a). Given the previous physical understanding, large singular values are expected to be related to low frequency components, while small singular values are related to the highly amplified, high frequency components.

A largely improved result is obtained if the smaller singular values are disregarded in the computation of the pseudo-inverse matrix \( M \), as shown in Fig. 4.15. The same parameters as in the previous example are considered, but only the largest 138 singular values out of \( \sim 400 \) are used in the computation of the inverse matrix by introducing a suitable threshold value. A clearly defined profile of the perturbation is found, with reasonable values of the corresponding DC refractive index change. High frequency ripple components are present, but their amplitude is limited and their characteristic period is roughly 1 mm. The expected residual GDR is shown in (b) and is not negligible, since the 1 ps peak-to-peak oscillations are still present in the long-wavelength region (where the large peak to be corrected is in the unperturbed design). But the typical frequency of the final GDR is higher, so that the corresponding peak-to-peak phase ripple is reduced from 0.4 rad to 0.02 rad within the reflection band of the dispersion compensating grating, and the eye-opening penalty is limited to less than 0.04 dB according to the results in [175].
The choice of the threshold is pseudo-empirically optimised by considering different values and monitoring the Fourier spectrum of the corresponding correction refractive index profiles. Decreasing the number of singular values used in the matrix inversion first reduces the high frequency spatial components to physically reasonable-manufacturable values. The quality of the obtained GDR compensation is degraded by this operation, but not to a significant extent since only high frequency GDR components are largely affected. After the optimum value, no further suppression of the high frequency ripple is obtained, while the GDR degradation becomes significant. Monitoring this change allows the optimum threshold value to be established. The result in Fig. 4.15 represents the best trade-off between simplicity of the correction profile and final performance.

The suggested design method is quite effective and general, since it can be applied to every parameter and to every type of grating once the corresponding sensitivity distributions are known. But the identification and use of the singular values related to the low frequency components is not trivial, and the previously described procedure for the corresponding selection has to be checked in different cases and has to be automated. Further work is needed to understand how the conditioning properties of the sensitivity matrix affect the reconstruction, how they are affected by the spectral and spatial sampling performed on the grating, and if more accurate reconstructions are ultimately possible using this correct method.

### 4.4.2 Approximated iterative approach

A different way of estimating the perturbation profile $d\Phi(z)$ which produces the desired GDR change $\Delta \tau_{\text{target}}$ can be inferred from the analysis of Fig. 2 in [172]. It suggests that a kind of duality between the shape of the phase perturbation and the shape of the group delay variation exists in the case of linearly chirped Bragg gratings. The distribution of $\Delta \tau_R(\lambda, z)$ shown in Fig. 4.13 gives an intuitive understanding of this unexpected similarity. Indeed, an almost linear relationship exists between $z$ and $\lambda$, with only minor differences found at the edges of the reflectivity band (close to $\delta \lambda \simeq 0.3$ nm). If a
certain spatial perturbation $d\Phi(z)$ results in a spectral variation $\tau_{\text{target}}(\lambda)$, it is also true that a spectral variation $d\Phi(\lambda)$ will be the result of a spatial perturbation $\tau_{\text{target}}(z)$ because of linearity. But, comparing Eqs. (4.22) and (4.23), this means that $\Delta\tau_{\text{R}}(\lambda, z) \propto M(\lambda, z)$ and that a first approximation of the function $d\Phi(z)$ can be obtained by simply computing

$$d\Phi(z) \simeq k \int_\lambda \Delta\tau_{\text{target}}(\lambda) \Delta\tau_{\text{R}}(\lambda, z) d\lambda.$$  \hspace{1cm} (4.27)

$k$ is a pseudo-empirical constant that accounts for the described proportionality and makes Eq. (4.27) dimensionally correct. If a true linear relationship holds, $k$ is simply the inverse of the proportionality factor. Unfortunately, this is not the case due to the finite bandwidth $\delta\lambda$ over which such a linear dependence occurs (i.e., only over the actual grating band-gap), but the actual value of $k$ can be chosen in order to minimise the error introduced by the approximation. Indeed, from Eq. (4.25)

$$\delta(\lambda) = \int_0^{L_{\text{gr}}} M(\lambda, z) \Delta\tau_{\text{R}}(\lambda, z) \simeq k \int_0^{L_{\text{gr}}} (\Delta\tau_{\text{R}}(\lambda, z))^2 dz \simeq k\chi(\lambda),$$  \hspace{1cm} (4.28)

and therefore $k$ is obtained by minimising the functional distance (i.e., in a least mean square sense) between $\delta(\lambda)$ and $k\chi(\lambda)$ (sinc-like function due to the finite bandwidth).

The empirical way the previous expression has been derived shows that this approach is by no means general, and a small error is obtained by using Eq. (4.27) rather than (4.23). Further improvement in this approximated design of the perturbation is obtained by using an iterative approach. The residual GDR after each iteration is used as a new target function to be compensated in the next cycle, and all the corresponding phase error distributions $d\Phi_i(z)$ are finally summed up given the linearity assumed in Eq. (4.27). Possibly, the sensitivity distribution $\Delta\tau_{\text{R}}$ can be recomputed for the perturbed grating after each cycle, and used in the next iteration in order to decrease the error introduced by the use of the local time delay approach to perturbation analysis. This option has been shown to slightly improve the final result for a given number of cycles, but a limited number of iterations are possible in this case due to the much longer computation time. Therefore, this option will not be further considered.

Fig. 4.16 shows the result for the same linearly chirped grating previously analysed after 20 iterations. The perturbation design (a) fairly resembles the result in Fig. 4.15, with a large chirped region introduced at the beginning of the grating to compensate for the large deviation from linear dispersion compensation on the red side of the reflection band, and smaller modifications further inside the grating mainly to compensate for the extra high frequency ripple introduced by the peak of the compensation profile at \(z \simeq 20\ \text{mm}\). The amplitude of this ripple is slightly larger in this case, and indeed a remarkably better cancellation of the residual GDR is obtained for wavelengths reflected deeper inside the grating (\(\delta\lambda < 0.15\ \text{nm}\)). Improved performance is anyway obtained all over the grating bandwidth, and the peak-to-peak amplitude of the corresponding
phase ripple is found to be 0.004 rad, with a further factor of 20 reduction. Negligible eye-opening penalty results.

The potentialities of this approximated approach seems to be very attractive for a fast and low-complication implementation of the correction scheme. But a few problems are still present even in this case. Firstly, the determination of the scaling constant $k$ used in (4.27) is not trivial. The profile of the correction contribution to the $GDR$ is indeed different from the target profile due to the intrinsic approximations used, so that a best fit has to be found. Simple methods based either on equalising the peak values of these functions or finding the better value in a “least mean square” sense can be used, but no ideal results are obtained in either case. In particular, an incorrect estimation of $k$ due to the particular shape of the $GDR$ functions can even lead to magnified ripple with increasing number of iterations and to instabilities in the algorithm, and empirically optimised damping factors have to be introduced to prevent divergence or oscillations in this case. No automated procedure to produce an optimised result is possible even using this approach. Large dependence of the final result is also found by changing the wavelength range over which the compensation procedure is carried on. Secondly, not all the possible $GDR$ distributions can be effectively corrected in this way, since only $d\Phi(z) = \Delta \tau_R(\lambda, z)$ functions are used in the iterative design. Given the fact that these functions are not an orthogonal basis, it is obvious that not every generic target perturbation $d\Phi(z)$ can be correspondingly synthesised.

### 4.4.3 Comparison and performances

The actual potentialities and drawbacks of the correction scheme proposed are summarised in Fig. 4.17. The DC refractive index profiles derived previously and shown in Figs. 4.15 and 4.16 are added to the unperturbed grating, and the direct computation of the grating spectral characteristics is performed using the transfer matrix method. Indeed, the $GDR$ reduction discussed previously was only inferred from the perturbation analysis, which is only valid in a small error regime as discussed in Section 3.4.2. Its applicability in this case is not obvious due to the large number of phase errors introduced.
in the ideal structure. At the same time, the corresponding amplitude perturbation introduced in the grating by the imaginary part of local time delay has to be derived, in order to understand if the induced deformation is acceptable for typical dispersion compensation devices. The results are shown in Fig. 4.17 (a) and (b), respectively. As far as \( GDR \) is concerned, the low frequency ripple is confirmed to be highly suppressed over the reflection bandwidth of the device, and a very flat \( GDR \) is obtained. Linear dispersion compensation at the nominal dispersion value of 1500 ps/nm is performed. On the red wavelength edge, the huge deformation of the dispersion response is mostly corrected, even though a residual 2 ps ripple is retained. The high frequency \( GDR \) is almost perfectly suppressed by the iterative algorithm (the typical peak-to-peak variation is \( \sim 0.05 \) ps), while the ideal correction scheme is unable to reduce it below 0.4 ps. Nevertheless, the corresponding phase deviation, which is the important parameter in terms of system performance, can be shown to be very similar and to present a maximum peak-to-peak ripple of \( \sim 0.01 \) rad only. This value is satisfactory for most applications.

The corresponding deformation of the reflectivity spectrum is shown to be more important in this case. In-band flatness is lost, but only a small deviation is found on the red side, where the largest perturbation is introduced. The transmissivity is reduced by 0.4 dB if a \( \pm 0.3 \) nm band is considered, but the loss is increased to 0.7 dB if the full \( \pm 0.35 \) nm bandwidth is taken into account. Such a variation can be tolerated as long as a limited number of similar devices are concatenated.

A further example of the potentialities of this technique is represented by its application to the fine tuning of inverse scattered gratings. As already discussed, inverse scattered designs are not flexible, in the sense that the algorithm simply tries to reproduce the chosen target spectrum. A suitable modification of the target is therefore necessary to optimise their performance, but such a modification is typically not obvious and empirical tuning is needed. The perturbation approach is a valid alternative in this case. A single channel, second-order dispersion compensator grating is considered as an example in Fig. 4.18. The grating is similar to the experimental demonstration of
Figure 4.18: Mesh of the time delay variation $\Delta \tau_R$ of an inverse scattered, single channel dispersion compensating grating when a $d\phi = 20$ mrad phase error is introduced. The profiles of the grating are shown on the right: (blue line) apodisation; (red line) chirp. $L_{gr} = 100$ mm, $\delta n_{MAX} = 8.5 \times 10^{-5}$, and $R_{MAX} \sim 0.9$. The group delay ripple $GDR$ is shown on the top plot. Only in-band propagation is considered. The corresponding colormap is shown on the top-right corner.

This device by Durkin et al. [49]. The grating length is $L_{gr} = 100$ mm, the in-band reflectivity is $R_{gr} = 0.9$, the filling factor is $\eta_B = 0.75$, and the grating is compatible with a 50 GHz WDM channel spacing (corresponding to a flat bandwidth of $\sim 0.3$ nm). The total dispersion to be compensated is 1360 ps/nm, corresponding to propagation through 80 km of standard SMF fibre with $D_\lambda = 17$ ps/nm/km. The inverse scattered design is shown on the right inset in Fig. 4.18, where both amplitude (blue) and chirp (red) profiles are plotted. The residual deviation from linear dispersion characteristic is shown on the top. The $GDR$ is very small compared to the classical linearly chirped design considered previously, but it can still be as large as 2 ps inside the available signal bandwidth. Phase perturbation is a possible solution to further decrease it and minimise the corresponding phase ripple (which is shown in Fig. 4.19 (b), blue line). The $\Delta \tau_R(\lambda, z)$ sensitivity of this grating to a phase error is also shown in Fig. 4.18 for a $d\phi = 20$ mrad. $GDR$ variation as large as $\pm 2.5$ ps is obtained with this phase error.

Comparing this sensitivity mesh with the linearly chirped grating’s in Fig. 4.13, it is apparent that the two distributions are very similar for $\delta \lambda < 0.05$ nm, despite the completely different shape of the grating profiles. This means that the obtained sensitivity distribution with parallel contour lines is typical of a linear dispersion compensator and is directly correlated to the spectral characteristics of the overall device. A larger deformation is obtained on the red wavelength side, i.e., for wavelengths reflected at the front
end of the structure. It is therefore expected that worse estimation of the phase profile $d\Phi(z)$ to correct the residual GDR is obtained in this case using the iterative approximated method based on Eq. (4.27), while the correct method based on Eq. (4.23) should give better results. Fig. 4.19 shows the corresponding designs. The DC refractive index profiles obtained with the two methods are plotted in (a) in red and green, respectively. 20 iterations are used in the approximated approach, and a suitable damping factor is found to optimise the final performance. A finer wavelength resolution ($d\lambda = 0.5$ pm rather than 1 pm) is required in the exact method to improve the conditioning properties of the matrix $\Delta \tau_R$ and have a better computation of its pseudo-inverse. Tweaking of the number of singular values to be used is necessary even here to prevent large oscillations related to high frequency components. The residual phase ripple (shown in (b) using the same colour coding) is comparable in both designs, with a maximum peak-to-peak variation reduced from 15 mrad to $2 - 2.5$ mrad all over the reflection band. A corresponding significant improvement of performance is found with respect to the original design. It is apparent that the exact method (green) allows such an improvement to be obtained with a much simpler profile, with reduced peak $\Delta n_{\text{eff}}$ variation and reduced ripple at high spatial frequencies. The corresponding amplitude distortions are shown in (c) and (d) for the reflectivity and the transmissivity, respectively. Both the in-band and the out-of-band properties of the grating are preserved given the small perturbation introduced, with negligible variations in the transmissivity and minor modifications to

![Figure 4.19](image-url)
the reflectivity, at unimportant levels anyway ($R_{gr}$ is changed only below -50 dB).

It is therefore apparent that the mixed used of inverse scattering and phase correction techniques is effective and allows the design of complex filters to be easily optimised. Both the described algorithms are useful when dispersion compensating devices are considered, and the exact method can be possibly applied in a general sense to different grating-based devices and to amplitude optimisation, besides phase fine-tuning considered here.

4.5 Conclusions

Local time delay has been shown to be a powerful tool to interpret how sensitive to perturbations a certain scattering structure is. The general theory developed in Chapter 3 has been applied to the analysis of the effect of a phase shift in the periodic pattern of a grating. Different classes of typical gratings have been directly addressed, taking particularly into account applications such as band filtering and dispersion compensation, and good agreement with the physical understanding obtained previously is found.

Different approaches proposed in the past have also been extensively reviewed, and an optimised, mixed formulation has been defined. Perturbation techniques have been considered in detail, and excellent agreement has been found. In particular, these results are equivalent to a linearised form of the corresponding result of the local time delay approach, and an improved fit of the corrected data obtained with the transfer matrix approach has been obtained for out-of-band propagation. Therefore, it is convenient to use the exponential form derived in Section 3.4.1 for in-band wavelengths, and a linearised form for out-of-band wavelengths. The Born approximation is automatically taken into account in the last case because of the linearity of the obtained expression. The effect of multiple scattering from the perturbation is neglected, and a reduced error is introduced in the perturbed coefficients.

Three main areas of interest have been found for the obtained sensitivity analysis. Robustness analysis when the errors introduced by the manufacturing process are known, characterisation of such errors from experimental data if they are not known, or conscious use of errors for fine tuning of experimental gratings or theoretical designs are possible applications. In all cases, the understanding of the local properties is fundamental in order to tailor the global properties of the structure and suit the tight requirements of practical devices.

The results of the sensitivity analysis confirm that a large dependence is found on both the wavelength which is considered, and the position of the error. The accuracy in the manufacturing process has to be higher in certain regions of the grating or in others depending on which spectral characteristics are more important. As significant examples, it has been shown that the amplitude sensitivity inside the band-gap is typically negligible for all the considered designs. Much larger modifications are obtained on the sidelobes of the gratings, especially when filtering applications are considered. An aver-
age increase in the out-of-band reflectivity is found, with also complex deformations of
the sidelobes structure. Simple physical interpretation of the effect is obtained, since the
Born approximation applies in this case. Intuitive understanding of the characteristics
of the phase errors introduced is also possible from the analysis of such a deformation.

Interestingly, complex inverse scattered designs (square dispersionless filters) are
proven not to be more sensitive to errors than simpler apodised structures. But the
regions close to large phase shifts in the inverse scattered designs present the largest
sensitivity and, typically, they are also the most likely sources of phase errors in the
manufacturing process. Careful manufacturing is therefore necessary. At the same time,
the proposed analysis gives a deeper insight in the way these complex structures work.
The apodisation ripples before the main lobe are responsible for the flat in-band time
delay, while ripples after the main lobe are mainly responsible for the squareness of the
obtained design, although they do not affect the dispersive properties of the structure.
As a different example, the effect of periodic phase perturbations has been analysed
in chirped gratings. The occurrence of a cut-off wavelength for a given perturbation
periodicity, and of a cut-off periodicity for a given wavelength (already identified in
previous works) is easily visualised with the sensitivity approach, so that an immediate
understanding of corresponding main features is possible.

The potentialities of this approach as a design tool have also been considered. Orig-
inally proposed as a solution for the design of complex gratings, the perturbation ap-
proach is useful to tune the result of the inverse scattering algorithm when a perfect
reconstruction is not possible. This typically occurs when length limitations are consid-
ered. But if a spectral parameter is more important than its conjugated momentum, i.e.,
phase (time delay) rather than amplitude, a suitable phase perturbation profile can be
found to correct it. It is simply obtained by deconvolving the local time delay distribu-
tion from the target spectral deformation. Successful application to chirped gratings has
been demonstrated, considerably improving the performances of both apodised, linearly
chirped structures and inverse scattered structures with respect to phase ripple. Unfor-
unately, the proposed approach presents important limitations, form both a physical
and a numerical point of view. In the first case, not all the spectral modifications can be
synthesised by using a phase-only perturbation profile, due to symmetry properties that
have to be fulfilled when variations in the phase pattern are considered. A clear exam-
ple of this is represented by square dispersionless gratings, in which only antisymmetric
profiles (with respect to the Bragg wavelength) can be obtained. In the second case,
the process of matrix inversion necessary in the computation can introduce very large
oscillations, which prevent the identification of the proper profile. This phenomenon
has a well defined physical meaning in chirped gratings, and it can be tamed by a suit-
able selection of the singular values used in the computation of the inverse matrix. But
further work is necessary in order to apply it to a generic grating and a generic target
perturbation.
Chapter 5

Inverse Scattering in lossy waveguides

5.1 Introduction

A large number of different manufacturing techniques for production of fibre Bragg gratings with complex coupling coefficient profiles are nowadays available [34, 36, 59, 60, 61, 62], as described in Section 2.4. An almost complete control of apodisation, chirp, and discrete phase shifts inside a grating is possible, and gives substantial flexibility to the grating design process. Very complex refractive index profiles can be designed and realised, and new and elaborate grating applications are always being found (see Section 1.1). A tool for the design of such complex structures is necessary to take full advantage of the potentialities offered by fibre Bragg gratings.

As described in Section 2.1.2, the discrete layer-peeling algorithm is the most used and powerful technique available and is well developed. New design approaches with promising features have just been proposed and demonstrated [70, 71, 72] after this Chapter was completed, and they will not be considered here. Several demonstrations of the potentialities of layer-peeling designs have been reported: grating filters with high filling factor, high out-of-band sidelobe suppression, and dispersion-less characteristics [47, 48]; dispersion compensators with 2nd order [49], 3rd order [51], or combined compensation over a broad bandwidth; generic amplitude and phase profiles necessary to provide accurate pulse reshaping at high reflectivity levels (in order to minimise insertion losses) [53, 54, 55]. Further examples of inverse scattering designs and realisations of specifically tailored gratings were also given (multichannel gratings [52], square-pulse generation and multiplication [56]), showing how the availability of a powerful design tool can drive research towards new applications. Only inverse scattering designs allow fibre gratings to be effectively used in Dense WDM applications with 50 GHz or less signal spacing, multiple functions to be combined within the same device, or enable customised applications in telecommunications or lasers technology.

Layer-peeling approaches have also been successfully applied to the reconstruction
of the grating profile, starting from experimentally measured data [169, 170, 178], and to the correction of systematic errors in the manufacturing process [179], even though fundamental limitations due to noise-related instabilities have been found when very strong gratings are considered [92]. Only a negligible part of the light launched in the grating samples the far end of the structure when the reflectivity is very high, so that its contribution is actually comparable to the numerical noise in the reconstruction process or to the intrinsic measurement noise. No correct identification of the far end of the grating is possible when the strength of the section preceding the reconstructed layer is higher than $R_{gr} \approx 0.97 - 0.98$ [169, 170]. Measuring the grating from both ends and combining the reconstructed parts is an effective way to extend this technique to stronger gratings (up to $R_{gr} \sim 0.9991$) or to reduce the noise in the obtained profile [170]. Reducing the strength of the grating by chirping it using an external thermal gradient has also been recently proposed for application in strong DFB grating characterisation [180], even though the result is not particularly satisfactory. Convergence problems are also present in the design of ultra-strong gratings with minimum transmissivity lower than -40 dB, solely due to numerical error.

In all the described applications, the grating writing process is now so precise that manufacturing errors, despite still being present (as discussed in the previous Chapters), are not the only limitation on the quality of the final spectral response. Even in the characterisation process, experimental uncertainty is no longer the only reason for an unsatisfactory reconstruction. The restrictions of the model considered in the inverse scattering algorithm contribute in a significant way to the non-ideal features obtained in the realisation and reconstruction of complex devices. In particular, the inverse scattering algorithm assumes that only two counter-propagating modes are coupled by the (unknown) quasi-periodic modulation of the medium, and that propagation inside the perturbed region is lossless. In the first case, coupling to cladding modes and radiation modes of the grating [18, 19] is disregarded since special fibres are typically used in order to minimise the power exchange in the grating towards spurious modes. However, the realisation of wide-band gratings for dispersion compensation still suffers from this problem, and only practical “recipes” for the pre-emphasis of the grating strength were found in order to compensate for cladding mode losses [181].

In the second case, background losses in the waveguide where the grating is written are assumed to be negligible, given the very low intrinsic losses ($\sim 0.2$ dB/km) characteristic of standard fibres. However, this is not always the case in practical gratings. UV-induced losses introduced in the writing process are typically present [182, 183], and they are particularly important when photosensitisation techniques such as hydrogen loading are used. It also applies to the analysis of gratings written in active fibres, such as a grating written in a wavelength region corresponding to an absorption line of the material for DFB laser applications, or in waveguides (not fibres) having higher background losses, especially in UV-written planar lightwave circuits [184] or surface relief gratings [185, 186]. Either constant background loss or loss proportional to the grating
strength are possibly present and need to be taken into account. A new inverse scattering algorithm for the reconstruction of the loss/gain profile in fibre Bragg gratings has been proposed very recently [69]. This method assumes that the loss/gain profile of the grating is independent from the refractive index profile and has also to be identified from the reflection data. It is powerful and general, but it requires the experimental characterisation of the grating from both sides. In many cases, this is an over-complication, since the loss characteristics of the grating are already known with a sufficient accuracy and are constant along the grating length.

Therefore, the standard layer-peeling technique has been extended to the design of gratings when known losses inside the structure are taken into account. Both cladding mode loss and propagation loss have been considered separately. The problem of cladding mode loss compensation is discussed in Section 5.2. A new transfer matrix approach is developed in Section 5.2.2 for the direct computation of the grating spectra when cladding mode coupling is considered. The availability of faster simulation tools allows the inverse scattering techniques to be applied even to this class of gratings. An iterative algorithm is presented in Section 5.2.4 for cladding mode loss equalisation and grating design. This approach is inherently approximate, but no exact algorithm can be derived due to the nature of the boundary conditions. The design of gratings in lossy media is considered in Section 5.3. The feasibility limit of these devices and the required deformation of the target spectrum in order to pre-compensate for propagation losses are described. Physical understanding of the underlying mechanisms is given by using the results of the previous Chapters about local time delay and perturbed gratings.

5.2 Inverse scattering with cladding mode loss compensation

5.2.1 The problem of cladding mode loss

Usually, light propagation inside a Bragg grating is modelled by considering a structure with a single mode. This fundamental mode is mainly confined in the core, and only the coupling of this forward-propagating mode with the corresponding backward-propagating mode is taken into account because of the synchronous approximation, as seen in Section 2.2.1. These assumptions lead to the well known coupled-wave equations (2.10), which have been extensively studied and for which efficient computational techniques have been developed (a review is presented in Section 2.2.2).

When the grating spectrum is considered over a very large bandwidth away from the fundamental Bragg resonance, or when long, chirped gratings are modelled, the single mode assumption does not hold anymore and propagation of cladding modes inside the grating has to be taken into account [17, 18, 88]. Cladding modes are higher order modes of the guiding fibre that exist when the fibre cladding is surrounded by a medium with a refractive index lower than that of the glass, such as air. They are mainly guided by the total internal reflection at the glass-air boundary, and cannot propagate over
long distances when the fibre is coated with a material of index equal to or higher than that of the glass. But the fibre has usually to be stripped in the grating writing process in order to avoid the high absorption of typical coating materials at the near-UV laser wavelengths (248 nm, 244 nm, 193 nm) used, and coupling with cladding modes cannot be neglected. Even when cladding-mode suppression techniques are employed, the suppression is not complete and the effect of cladding modes has to be taken into account.

The typical cladding mode features are related to the resonant coupling between the forward-propagating core mode and backward-propagating cladding modes when a micrometre long grating period $\Lambda$ is considered. In transmission, these resonances appear as additional losses on the short wavelength side of the main grating band. Indeed, the effective refractive index of the cladding mode $n_{\text{eff,cm}}$ is lower than that of the core mode $n_{\text{eff,co}}$ (the cladding mode mainly extends in the cladding region) and, considering the synchronous approximation, the corresponding Bragg condition for optimum coupling is given by

$$\lambda_{\text{Bragg,cm}} = (n_{\text{eff,co}} + n_{\text{eff,cm}})\Lambda < \lambda_{\text{Bragg,co}}. \quad (5.1)$$

In unchirped gratings, the detuning between the core and cladding resonances is usually so high that independent synchronous approximations can be considered in different wavelength ranges. The effect on the overall transmissivity spectrum is obtained by simply multiplying the individual frequency responses. No corresponding peaks are found in the reflectivity spectrum, since the power coupled into backward-propagating cladding modes is extinguished by scattering losses, bending losses, and ultimately by leakage losses when the cladding mode reaches the coated part of the fibre. Negligible reflectivity distortion is related to cladding mode coupling in this case.

The presence of cladding mode losses in chirped gratings affects the grating performance in a much more complicated way, when the grating bandwidth is larger than the separation $\delta\lambda_{\text{cm}}$ of the core-cladding Bragg resonances. In transmission, cladding mode losses interfere with the core mode within the main reflection band and introduce additional transmission losses on the short-wavelength side. Because of reciprocity (2.34c), this effect is independent of the grating input side. The corresponding effect on the reflection spectrum, however, depends on the grating input side. Entering the chirped grating from the “blue” end (longer local period at the end, $\Lambda(0) < \Lambda(L_{gr})$), the reflection spectrum is unaffected since effective cladding mode coupling occurs only after the core mode resonance (additional losses are limited to transmitted light). Entering the chirped grating from the “red” end (longer local period at the beginning, $\Lambda(0) > \Lambda(L_{gr})$), cladding mode coupling results in a pronounced reflection spectrum slope on the short-wavelength side. For short wavelengths, the core mode is first phase-matched with the cladding mode, attenuated by the corresponding additional losses, and only then reflected, while long wavelengths experience no additional losses. The resulting spectral imbalance can be as large as a few dB [181], and it seriously limits the applicability of long, chirped gratings for multichannel dispersion compensation.
Different solutions were proposed in the past. The most obvious one is to re-coat the fibre where the grating is written, but this process is time-consuming and the mechanical strength of the fibre is reduced. Another possibility is to avoid fibre stripping by either using different UV writing wavelengths [187] or using special UV transparent coatings [188]. In the first case, expensive and non-standard laser equipment is needed, and a completely different characterisation of the photosensitivity properties of the material has to be obtained. In the second, special coatings might introduce problems when mechanical or aging properties of the gratings are considered, besides being more expensive. Special fibre designs can be used in order either to suppress coupling to cladding modes (such as photosensitive cladding fibres) [189, 190] or to shift the cladding mode resonances away from the core mode (high NA fibres) [191], so that they do not interfere with the grating spectrum. In both cases, the design and production of these fibres presents an increased complexity and cost. Splicing is often not trivial, and large losses are introduced because of the modal mismatch with standard fibres. Moreover, thermal stability and aging properties of the gratings have to be specifically studied, since they strongly depend on the fibre composition. Finally, modified grating writing techniques were also proposed, for instance using tilted gratings that are rotationally symmetric [192]. Increased complexity and reduced reliability in the manufacture have to be tackled in this case. It is therefore appealing to avoid all these problems by directly designing gratings with cladding mode loss compensation already accounted for, so that standard SMF and standard grating writing techniques can be used.

5.2.2 Modelling propagation with cladding modes included

The first step in order to obtain an efficient inverse scattering design of gratings with cladding mode loss compensation is a fast and reliable tool for the simulation of gratings with cladding modes included. To account for the complex effect of cladding modes on the performance of chirped fibre Bragg gratings, the basic coupled-mode theory introduced in Section 2.2.1 is extended to consider the interaction and power exchange between core-cladding and cladding-cladding fibre modes due to the grating chirp. The extended theory starts from Eq. (2.5) and simply includes the possibility of synchronous coupling with backward-propagating and quasi-phase-matched cladding modes besides the usual core mode coupling. It was formally developed by Erdogan [18] and specifically applied to chirped grating simulation and equalisation by Finazzi and Zervas [100]. Two approximations are usually introduced in order to simplify the final set of coupled differential equations. First, no forward-propagating cladding modes are considered inside the structure. Second, the cladding mode cross-coupling is considered to take place only through the forward-propagating core mode. These assumptions are justified by the fact that the coupling constant between core-cladding and cladding-cladding modes is much weaker than the corresponding core-core parameter (i.e. the usual $\kappa$ in Section 2.2.1), as is shown by the theoretical derivation under the hypothesis of perturbation confined in the fibre core (only the fibre core is assumed to be a photosensitive region) in Erdogan
Starting from Eq. (2.5) and taking into account the forward-propagating core mode \( A_0 \), the backward-propagating core mode \( B_0 \), and \( C \) backward-propagating cladding modes \( B_1, \ldots, B_C \), the following set of extended coupled-wave equations (hereon referred to as ECWEs) is derived by using the synchronous approximation:

\[
\frac{d}{dz} \begin{bmatrix} A_0(z) \\ B(z) \end{bmatrix} = \begin{bmatrix} j\sigma_0(z) & q^T(z)D(e^{-j(\beta_0+\beta_c-2\pi/\Lambda)z}) \\ q^*(z)D(e^{j(\beta_0+\beta_c-2\pi/\Lambda)z}) & D(-j\sigma_c(z)) \end{bmatrix} \begin{bmatrix} A_0(z) \\ B(z) \end{bmatrix}, \tag{5.2}
\]

In order to simplify the form of the ECWEs, the following notations are used in Eq. (5.2) and throughout this section:

\[
B = \begin{bmatrix} B_0 \\ \vdots \\ B_N \end{bmatrix} \text{ column vector of elements } B_c, \text{ with } c = 0, \ldots, C, \tag{5.3}
\]

\[
D(x_c) = \begin{bmatrix} x_1 \\ \vdots \\ x_C \end{bmatrix} \text{ diagonal matrix with elements } x_c. \tag{5.4}
\]

\( \sigma_c(z) \) and \( q_c(z) = j\kappa_c(z)e^{j\phi(z)} \) are the self-coupling and the complex cross-coupling coefficients of the grating, respectively, for the generic mode \( c \), with \( c = 0, \ldots, C \). The derivation of the actual expressions for \( \sigma_c \) and \( \kappa_c \) for a generic fibre index profile are not considered in this work. Analytical formulas for a step-index fibre were derived by Erdogan [18]. The oscillatory term \( e^{-j(\beta_0+\beta_c-2\pi/\Lambda)z} \) that multiplies the coupling coefficient \( q_c(z) \) takes into account the dephasing between the coupled modes \( A_0(z), B_c(z) \), and the grating pattern \( \Lambda \), and it is directly related to momentum conservation in the grating. As already seen in Section 2.2.1, it is convenient to define the effective forward and backward field-envelopes \( U_0, V_0, \ldots, V_C \):

\[
\begin{bmatrix} U_0(\delta_0, z) \\ V(\delta_c, z) \end{bmatrix} = \begin{bmatrix} A_0(z)e^{j\delta_0z} \\ B(z)e^{-j\delta_cz} \end{bmatrix} \Rightarrow \begin{bmatrix} E_{+0}(z) \\ E_{-0}(z) \end{bmatrix} = \begin{bmatrix} U_0(\delta_0, z)e^{j\pi z/\Lambda} \\ V(\delta_c, z)e^{-j\pi z/\Lambda} \end{bmatrix}, \tag{5.5}
\]

where \( \delta_c = \beta_c - \pi/\Lambda \) is the frequency detuning from the Bragg condition for the mode \( c \), so that Eq. (5.2) can be written as

\[
\frac{d}{dz} \begin{bmatrix} U(\delta_0, z) \\ V(\delta_c, z) \end{bmatrix} = \begin{bmatrix} j\hat{\sigma}_0(\delta_0, z) & q^T(z) \\ q^*(z)D(-j\hat{\sigma}_c(z)) \end{bmatrix} \begin{bmatrix} U(\delta_0, z) \\ V(\delta_c, z) \end{bmatrix} = A(\delta_c, z) \cdot \begin{bmatrix} U(\delta_0, z) \\ V(\delta_c, z) \end{bmatrix}, \tag{5.6}
\]

with \( \hat{\sigma}_c = \sigma_c + \beta_c - \pi/\Lambda \) effective detuning. The matrix \( A \) is the coefficient matrix of the differential equation system (5.6) and contains all the characteristics of the simulated grating.

As for the simpler 2x2 system described in Eq. (2.15), the ECWEs (5.2) cannot be solved analytically for a generic choice of the self- and cross-coupling longitudinal profiles. The direct integration is possible using the Runge-Kutta algorithm with boundary
conditions $U(\delta_0,0) = 1$ and $V(\delta_c, L_{gr}) = 0$. This method was used till now for the grating spectra computation when cladding modes are included [18, 100], but it has a low efficiency and is computationally intensive. No real application was ever developed, and empirical recipes were used to solve the equalisation problems already described in practical, chirped, long gratings [181, 193].

Moreover, even restricting the attention to a uniform grating, no analytical expressions such as Eqs. (2.22) and (2.23) or an explicit transfer matrix $T$ are available. Therefore, no fast grating computation via matrix multiplication was proposed in the nonuniform case until now. However, if Eq. (5.6) is considered in the uniform case, the coefficient matrix $A$ of the differential equation system has no direct dependence on the differentiation variable $z$. The ECWEs reduce to a linear differential equation system with constant coefficients, and the general form of the solution is easily obtained from the theory of differential equations

$$\frac{da(z)}{dz} = A \cdot a(z) \quad \Rightarrow \quad a(z) = e^{zA} \cdot a(0), \quad (5.7)$$

assuming that $a(0)$ is known because of the boundary conditions (general solution of a Cauchy problem). The transfer matrix $T$ of the system is directly derived from Eq. (5.7) and is given by

$$a(\Delta) = e^{\Delta A} \cdot a(0) \quad \Rightarrow \quad T = e^{-\Delta A}. \quad (5.8)$$

Therefore, the standard transfer matrix algorithm can be applied to the simulation of nonuniform gratings with cladding modes by splitting the grating into almost uniform sections of length $\Delta$, deriving the corresponding coefficient matrix $A$, and calculating the exponential matrix $e^{-\Delta A}$.

Unfortunately, even though the general form of the coefficient matrix $A$ is quite simple (doubly bordered diagonal matrix) due to the approximations used in the definition of the extended coupled wave equations, no analytical form exists for the exponential matrix. Only a numerical implementation is possible, and it requires the computation of the eigenvalues and eigenvectors of the matrix $A$. Given the diagonal eigenvalues matrix $\tilde{A} = \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_C \end{bmatrix}$ and the matrix $X$ formed by columns of eigenvectors $x_1, \ldots, x_C$, the transfer matrix is

$$T = e^{-\Delta A} = X e^{-\Delta \tilde{A}} X^{-1} = X \begin{bmatrix} e^{-\gamma_1 \Delta} & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & e^{-\gamma_C \Delta} \end{bmatrix} X^{-1}. \quad (5.9)$$

Equation (5.9) is not as efficient as Eq. (2.26) in the computation, but can still provide a significant reduction of the simulation time with respect to standard Runge-Kutta-based routines, as will be shown later.

A further simplification of the problem is obtained if the discrete-reflector approach is applied to the ECWEs. As explained in Feced et al. [42] and in Section 2.2.2, every section of the grating is considered as a single, point reflector, and the grating is ap-
proximated by a stack of such reflectors separated by free space propagation regions. The propagation matrix $T_{\Delta}$ is obtained by letting $q \to 0$, and it is easily calculated by recognizing that the coefficient matrix is diagonal:

$$ -\Delta A \xrightarrow{q \to 0} \begin{bmatrix} -j\sigma_0\Delta & 0 \\ 0 & D(j\sigma_c\Delta) \end{bmatrix} \Rightarrow T_{\Delta} = \begin{bmatrix} e^{-j\sigma_0\Delta} & 0 \\ 0 & D(e^{j\sigma_c\Delta}) \end{bmatrix}. \quad (5.10) $$

The transfer matrix of the point-scatterer $T^S$ is derived from $T$ assuming a complex coupling coefficient $q \to \infty$ and maintaining the product $q\Delta$ finite. If Eq. (5.8) is manipulated using this approximation:

$$ -\Delta A \xrightarrow{q \to \infty} \begin{bmatrix} -j\sigma_0\Delta & -qT\Delta \\ -q^*\Delta & D(j\sigma_c\Delta) \end{bmatrix} \Rightarrow T^S = \begin{bmatrix} 0 & -qT\Delta \\ -q^*\Delta & 0 \end{bmatrix}. \quad (5.11) $$

The particular form of the simplified matrix (5.11) allows the computation of the exponential matrix (5.8) using the Taylor series expansion

$$ e^{-\Delta A} = I + (-\Delta A) + \frac{(-\Delta A)^2}{2!} + \frac{(-\Delta A)^3}{3!} + \cdots = \sum_{m=0}^{+\infty} \frac{(-\Delta A)^m}{m!}. \quad (5.12) $$

Indeed, independent recursion formulae are derived for even and odd elements in the series,

$$ (-\Delta A)^{2m} = (\gamma\Delta)^{2m} \begin{bmatrix} 1 & 0 \\ 0 & q^*q^T/\gamma^2 \end{bmatrix}, \quad \text{even, } m > 0, \quad (5.13) $$

$$ (-\Delta A)^{2m+1} = (\gamma\Delta)^{2m} \begin{bmatrix} 0 & -q^T\Delta \\ -q^*\Delta & 0 \end{bmatrix}, \quad \text{odd, } m \geq 0, \quad (5.14) $$

where the coefficient $\gamma^2 = q^Tq^* = \sum_{c=0}^{C} |q_c|^2$ has been introduced. Using Eqs. (5.12), (5.13), and (5.14), and remembering the Taylor expansions for sinh($x$) and cosh($x$), the result for the point-scattering matrix is

$$ T^S = I + \sum_{m=0}^{+\infty} \frac{1}{\gamma\Delta (2m+1)!} \left( \begin{bmatrix} 0 & -q^T\Delta \\ -q^*\Delta & 0 \end{bmatrix} + \left( \sum_{m=0}^{+\infty} \frac{(\gamma\Delta)^{2m}}{(2m)!} - 1 \right) \begin{bmatrix} 1 & 0 \\ 0 & q^*q^T/\gamma^2 \end{bmatrix} \right) $$

$$ = I - \sinh(\gamma\Delta) \begin{bmatrix} 0 & q^T/\gamma \\ q^*/\gamma & 0 \end{bmatrix} + [\cosh(\gamma\Delta) - 1] \begin{bmatrix} 1 & 0 \\ 0 & q^*q^T/\gamma^2 \end{bmatrix} \quad (5.15) $$

and therefore

$$ T^S = \begin{bmatrix} \cosh(\gamma\Delta) & -(q^T/\gamma)\sinh(\gamma\Delta) \\ -(q^*/\gamma)\sinh(\gamma\Delta) & I + (q^*q^T/\gamma^2)[\cosh(\gamma\Delta) - 1] \end{bmatrix}. \quad (5.16) $$

The availability of the analytical expressions of $T^S$ and $T_{\Delta}$ allows a time-efficient direct computation of the grating with cladding modes. As already discussed in Section 2.2.2,
the use of discrete reflectors introduces a spectral deformation in the grating due to the overlooking of multiple scattering inside each subgrating, but the computation errors can be neglected if a sufficiently small sampling period $\Delta$ is used (see Fig. 5.1 for an example). The choice of a correct section length $\Delta$ is of major importance in the discrete-reflector method especially for ensuring that no aliasing is introduced in the simulated bandwidth by the quantisation process. From Eq. (2.48), the fundamental period $\Delta \lambda$ induced by quantisation is given by

$$\Delta = \frac{\pi}{\Delta \delta} \simeq \frac{\pi \lambda^2}{2 \pi n_{\text{eff}} \Delta \lambda} \Rightarrow \Delta \lambda \simeq \frac{\lambda^2}{2 n_{\text{eff}} \Delta} = \frac{\lambda^2 N}{2 n_{\text{eff}} L_{gr}},$$

where $L_{gr}$ is the grating length and $N$ is the number of sections. If the spectrum of both the core and the cladding modes is not completely included in the bandwidth $\Delta \lambda$ (i.e., if the amplitude of any mode at the bandwidth edge cannot be neglected), aliasing occurs and the correct shape is not obtained. In particular, if the last cladding mode is only partially included in the blue side of the spectrum, the corresponding replica created by quantisation is shifted by $+\Delta \lambda$ and will overlap the core mode reflection bandwidth on the red side. This effect prevents a correct evaluation of the grating characteristics for the red wavelengths that are not perturbed by cladding modes.

### 5.2.3 Simulation of gratings with cladding modes

Different kinds of chirped gratings were considered in order to compare the different simulation techniques in terms of speed and accuracy. First, a chirped uniform grating with nominal Bragg wavelength $\lambda_{\text{Bragg}} = 1550$ nm, length $L_{gr} = 50$ mm, refractive index variation $\delta n = 2.7 \times 10^{-4}$, and chirp rate $CR = -0.6$ nm/cm is considered in Fig. 5.1. These parameters correspond to a 3 nm bandwidth, an average minimum transmissivity $T_{\text{min}} \simeq -18$ dB, and total dispersion $D_{\text{TOT}} \simeq -1610$ ps/nm. Three cladding modes are included in the simulation, located 2 nm below the fundamental Bragg wavelength and separated by 0.5 nm. The cladding mode to core mode coupling constant ratio $q_c/q_0$ is 0.5 for all the modes, with $c = 1, 2, 3$. This overestimated value (compared to typical values for a single mode fibre, see Erdogan [18]) is used in order to highlight the interplay between the different energy transfer mechanisms involved in the multimodal propagation in the grating. The grating is simulated in the bandwidth 1545 to 1552 nm using a wavelength resolution of 5 pm, which results in 1401 different wavelengths to be evaluated.

The direct integration of the ECWEs is performed using a variable-step, 4th order Runge-Kutta algorithm (implemented in MATLAB® software). For the transfer matrix approach using the exact transfer matrix (see Eq. (5.9)), the grating is split in 200 sections and the exponential matrix is computed using the standard “expm” MATLAB® function. For the point-scattering approximation, three different runs are considered, with 400, 600, and 800 sections, respectively. The minimum number of sections according to Eq. (5.17) is given by $N_{\text{min}} \simeq 393$ in order to avoid aliasing. The reflection spectrum
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Figure 5.1: (a) Reflectivity spectra of a chirped unapodised grating with $\lambda_{\text{Bragg}} = 1550$ nm, $L_{gr} = 50$ mm, $\delta n = 2.7 \times 10^{-4}$, and $CR = -0.6$ nm/cm. Three cladding modes are included in the simulation, separated from the core mode by 2 nm (green), 2.5 nm (red), and 3 nm (cyan), and with coupling constant $q_c/q_0 = 0.5$. The different simulation techniques are indistinguishable. (b) Magnified plot of the core mode reflectivity in a restricted wavelength range. The Runge-Kutta method and the exact transfer matrix method are still indistinguishable (blue). The discrete-reflector method presents the expected minor differences, with improved agreement when an increased number of sections is considered.

<table>
<thead>
<tr>
<th>Runge-Kutta</th>
<th>Exact Transfer Matrix</th>
<th>Discrete-Reflector Matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable-step</td>
<td>200 sections</td>
<td>800 sections</td>
</tr>
<tr>
<td>36 h</td>
<td>10 min 12 s</td>
<td>4 min 13 s</td>
</tr>
</tbody>
</table>

Table 5.1: Number of sections used and corresponding computation time for a uniform, linearly chirped grating with three cladding modes, described in the text. The different simulation techniques are considered.

obtained for the core mode reflectivity (blue) and for the cladding modes (green, red, cyan, respectively) is shown in Fig. 5.1 (a). No differences can be seen on this scale between the various simulations. As expected, the cladding mode spectra overlap the main reflection band and affect the grating reflectivity response. If a magnified scale is considered in the core mode reflection band (see Fig. 5.1 (b)), the exact transfer matrix approach still has a perfect agreement with the direct integration of the differential equations, while the approximated discrete-reflector approach shows minor differences limited to 0.05 dB when 400 sections are used in the computation. The disagreement is reduced to less than 0.02 dB when the number of sections is increased to 800.

All the previous simulations were performed using MATLAB® code on a Pentium III® 550 MHz machine running Windows 98SE®, and the resulting computation time is shown in Table 5.1. The numerical integration of the ECWEs is impractical even for a $L_{gr} = 50$ mm-long chirped grating, while the transfer matrix approach proposed here is almost two orders of magnitude faster and reduces the computation time to reasonable levels. A further factor of 2.5 is gained with the use of the discrete-reflector method, even though 4 times more sections are simulated to prevent aliasing and guarantee a good approximation. As a further means of comparison, the simulation of the same grating without any cladding modes and with 200 sections takes 6.5 s when the analytic
expression for the transfer matrix is used, and 8 min 15 s when the exponential matrix (5.8) is computed numerically (~75 times slower).

Another example is given in Fig. 5.2. The same grating parameters as before are considered, but a 15% sine apodisation is introduced at each grating end in order to reduce the reflectivity and time delay ripple. The Runge-Kutta integration is always performed using a variable-step algorithm, while 500 sections and 800 sections are used for the exact matrix and the discrete-reflector approaches, respectively. The number of sections required in the exact transfer matrix method is increased from 200 to 500 in order to properly approximate the apodisation profile at the grating ends. A variable sampling period $\Delta_i$ would allow a reduction of the total number of sections to be computed (higher sampling rate where the grating is apodised), but no adaptive-step algorithm has been implemented. The simulation time necessary for the sine-apodised grating computation is shown in Table 5.2.

The discrete-reflector approach is as efficient as in the previous case, since the overall number of sections $N$ and wavelengths $M$ involved in the matrix multiplication does not change. The exact transfer matrix method has the same complexity $O(MN)$, while the direct integration is more efficient in this case, but is still one order of magnitude slower than the matrix multiplication algorithm. This improved efficiency is possibly related to
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the smoother field distributions inside the apodised grating with respect to the uniform case (as predicted in [21, 27] and confirmed by the time delay simulations presented in Section 3.3.2, see Figs. 3.10 and 3.13). The presence of fewer field oscillations enables the use of longer integration steps in the Runge-Kutta algorithm, that therefore converges more quickly.

The transfer matrix algorithms present an asymptotical complexity $O(M N C^2)$, where $(C + 1)$ is the number of coupled differential equations, with $C$ the total number of considered modes. $M N$ gives the number of matrix multiplications to be performed, while $C^2$ is the size of the considered matrices. This assumption has been numerically confirmed by simulating the previous gratings in a small frequency range and with an increasing number of cladding modes. However, for large values of $C$ the coefficient matrix $A$ is a sparse matrix in a doubly bordered diagonal form. An optimised eigenvalues-eigenvectors decomposition can possibly be found and an improved computation efficiency obtained for the exact transfer matrix method. On the contrary, increasing the product $M N$ gives worse performances in a practical implementation compared to the theoretical predictions, due to the storage and numerical manipulation of longer arrays of coefficients.

However, the simulation of 1 m-long dispersion compensating devices with bandwidth up to 8 nm and including cladding mode losses still requires a very long computation time. Given the required dispersion $D_{\text{TOT}}$ over a bandwidth $\Delta \lambda$, the grating length $L_{gr} \propto D_{\text{TOT}} \Delta \lambda$ is fixed. Increasing the grating bandwidth, the number of wavelengths $M$ increases linearly for a fixed wavelength accuracy $d\lambda$, while the number of sections $N$ increases quadratically due to both the linear dependence $L_{gr} \propto \Delta \lambda$ and the $1/\Delta \lambda$ dependance of the section length $\Delta$ required according to Eq. (2.48). Moreover, larger bandwidths mean higher numbers of cladding modes $C$ to be accounted for, with a square root dependance $C \propto \sqrt{\Delta \lambda}$ due to the uneven spacing of the modes [17, 18]. Therefore, the overall complexity as a function of the grating bandwidth is $O(M N C^2) = O(\Delta \lambda \Delta \lambda^2 \Delta \lambda) = O(\Delta \lambda^4)$, i.e., very demanding when the bandwidth of a typical chirped grating is considered.

As an example, the simulation of the 0.75 m-long dispersion compensating grating measured by Durkin et al. [181] is considered. The grating is designed to compensate for 50 km of standard fibre and has a dispersion $D_{\text{TOT}} = -850$ ps/nm. A 5% sine-apodisation is assumed, and the obtained bandwidth is 8.5 nm. The Bragg wavelength separation from the core mode is estimated using the approximate analysis proposed by Mizrahi and Sipe [17] for a step-index fibre. Sixteen different cladding modes are included in the simulation, and the relative coupling constants $q_c/q_0$ are estimated from the theoretical analysis developed by Erdogan [18]. The odd-mode coupling coefficients are in the $(0.05 - 0.2) q_0$ range, while the even-mode coupling coefficients are almost negligible. The simulation is performed in the [1545.5 - 1554.5] nm wavelength range with a $d\lambda = 10$ pm resolution, and $N = 18000$ sections are required in order to avoid aliasing in the simulated bandwidth. From the previous example and from the $O(M N C^2)$ complex-
A distinctive feature of Fig. 5.3, that is also observed in Figs. 5.1 and 5.2, is the discrete, step-like loss in the core mode reflectivity spectra caused by each cladding mode contribution. This is in contrast with the cladding mode losses experimentally measured, that are shown to be almost linearly increasing (in logarithmic scale) for increasing separation from the core mode Bragg wavelength. A possible explanation for the obtained disagreement is given by a nonuniform UV-induced refractive index change in the transverse profile of the fibre. In this case, the cladding mode coupling constants are different from the ideal situation modelled by Erdogan [18], with reduced coupling to the odd-modes and increased coupling to the even-modes. A more regular spectral distribution and strength of the cladding modes results, and better fitting of the experimental data would be obtained. Other possible reasons are radiation mode coupling or polarisation dependence of the characteristics of the grating.

5.2.4 Equalisation of cladding mode loss

The simulation techniques previously described is now applied to the equalisation of cladding mode losses through an inverse scattering approach [194]. In order to obtain a reasonably good equalisation of the grating reflection spectrum when light is launched from the "red" side of the grating (the only case of interest considered hereon, as already described in Section 5.2.3), the reflectivity of the grating is increased in the region where cladding mode losses are present, while it is possibly reduced in the unaffected part of the spectrum. Only an empirical method for evaluating the necessary grating deformation has been proposed by Durkin et al. [181] and numerically verified by Finazzi and Zervas [100]. But only an inverse-scattering-based algorithm is suitable for loss compensation in complex structures, when nonuniform reflectivity or dispersion characteristics

![Figure 5.3: (a) Reflectivity spectra of a chirped sine-apodised grating with $\lambda_{Bragg} = 1550$ nm, $L_{gr} = 750$ mm, $\delta n = 1 \times 10^{-4}$, and $CR = -0.1138$ nm/cm. A 5% sine apodisation is used to reduce the reflectivity and time delay ripples. Sixteen cladding modes are included in the simulation. (b) Magnified plot of the core mode reflectivity, showing the expected 2.5 dB cladding mode losses.](image)
are considered. In this Section, only the problem of flat reflectivity, linear dispersion compensating gratings is addressed, but the same approach can be applied to the design of a generic grating.

The reflection spectrum inside a chirped grating can be approximately described as the cascade of a lossy element (which accounts for cladding mode losses) and of a grating without any cladding modes included, as shown in Fig. 5.4. Coupling to the cladding modes is effective in the first part of the grating, and power is coupled on the backward-propagating core mode only afterwards. Less power is reflected in the short-wavelength range because less power actually “enters” the grating in that wavelength range. In order to have a flat reflectivity through the whole grating bandwidth, the grating reflectivity cannot exceed the minimum transmissivity $\min(T_{loss})$ of the lossy element in the reflection bandwidth, if passive media are considered. If the required reflectivity is higher than the transmissivity $\min(T_{loss})$, no equalisation can be obtained without the introduction of additional insertion loss $IL$.

The previous discussion describes how to design a pre-compensated grating with flat in-band response. The target reflectivity spectrum $R_{\text{target}}(\delta)$ has to be normalised to the maximum number of photons that can be reflected, which is given in this case by the cladding mode transmissivity $T_{loss}(\delta)$ and not by a white spectrum with $T_{in} = 1$. Therefore, the transformation $T_{loss} \rightarrow 1$ has to be considered and gives

$$R_{\text{comp}}(\delta) = \frac{R_{\text{target}}(\delta)}{T_{loss}(\delta)}. \quad (5.18)$$
Since $R_{\text{comp}}(\delta)$ can be greater than 1, the new target spectrum has to be rescaled to a physically realisable maximum reflectivity $R_{\text{MAX}} < 1$, so that

$$R_{\text{comp}}(\delta) = R_{\text{comp}}(\delta) \frac{R_{\text{MAX}}}{\max[R_{\text{comp}}(\delta)]},$$

(5.19)

The corresponding additional insertion loss $IL$ with respect to the original target spectrum $R_{\text{target}}(\delta)$ is given by

$$IL = R_{\text{MAX}} \max[R_{\text{comp}}(\delta)] = R_{\text{MAX}} \min \left[ \frac{T_{\text{loss}}(\delta)}{R_{\text{target}}(\delta)} \right].$$

(5.20)

Therefore, the knowledge of the cladding mode losses of a given grating allows the optimum pre-compensated shape to be determined, which is then used in the grating synthesis through the usual inverse scattering algorithm.

Unfortunately, no direct relation between the ideal grating reflection profile and the associated cladding mode losses can be established, and an independent estimation of $T_{\text{loss}}$ is required. This problem is simply solved by running the standard inverse scattering algorithm on the unperturbed grating and with no cladding modes, and then by computing the related cladding mode spectra using either the exact or the discrete-reflector transfer matrix approach. The point-scattering approach described in Section 5.2.2 is used in the following, since it is remarkably faster than the other methods and is still reasonably accurate. Once the profile $q(z)$ of the grating is known, the direct computation is performed by recursively using the transfer matrix relation

$$\begin{bmatrix} U_0(0) \\ V(0) \end{bmatrix} = T_S T_\Delta \begin{bmatrix} U_0(\Delta) \\ V(\Delta) \end{bmatrix} = \begin{bmatrix} \cosh \gamma \Delta & -\frac{q^T \sqrt{\gamma^2}}{\gamma} \sinh \gamma \Delta \\ -\frac{q^T \sqrt{\gamma^2}}{\gamma} \sinh \gamma \Delta & I + \frac{q^T q}{\gamma^2} (\cosh \gamma \Delta - 1) \end{bmatrix} \begin{bmatrix} e^{-j\beta_0 \Delta} & 0 \\ 0 & D(e^{j\beta_0 \Delta}) \end{bmatrix} \begin{bmatrix} U_0(\Delta) \\ V(\Delta) \end{bmatrix},$$

(5.21)

where $I$ is the identity matrix and $\gamma^2 = q^T q = \sum_{c=0}^{N} |q_c|^2$. $T_S$ is the discrete reflector matrix, and $T_\Delta$ is the free space propagation matrix. The boundary conditions of the problem are given by $U_0(0) = 1$ and $V(L_{gr}) = 0$, since no power is assumed to be scattered back from the region $z > L_{gr}$ following the grating.

The intuitive approach proposed is inherently approximate and an iterative algorithm is required in order to perform an accurate spectral equalisation. In the second cycle, the standard inverse scattering algorithm is always used to reconstruct the grating profile, but it is applied to the new compensated spectrum $r_{\text{comp}}(\delta)$ obtained using Eqs. (5.18) and (5.19). However, deforming the grating target spectrum deforms the corresponding loss at the same time. For instance, reducing the grating reflectivity reduces the effective coupling to cladding modes, and loss over-compensation results in the final reflectivity spectrum. Agreement between the pre-compensated grating profile and the estimated loss is reached by iterating the described procedure. The cladding mode losses of the last
Figure 5.6: Iterative identification of a chirped grating with cladding mode loss compensation. The target grating parameters are: $R_{gr} = 0.9$, $\Delta \lambda = 3.2$ nm, $D_{TOT} = -1700$ ps/nm, $L_{gr} = 800$ mm, $\eta_B = 0.97$. (a) pre-compensated spectrum $R_{comp}$ fed to the inverse scattering algorithm; (b) corresponding grating spectrum with cladding modes included (discrete-reflector matrix approach). (blue line), starting spectra; (green line), spectra after 1 iteration; (red line), spectra after 2 iterations.

Simulated (and pre-compensated) grating are used to design the target grating for the next step. The iterations are carried on until a user-defined goal equalisation is achieved in the bandwidth of interest.

Fig. 5.6 shows how the iterative method works. The grating characteristics are described in Section 5.2.5, but they are unimportant for the present discussion. The evolution of the compensated target $r_{comp}$ is represented in Fig. 5.6 (a), while the resulting spectrum after inverse scattering and direct scattering computation is plotted in (b). In the first cycle (blue line), a chirped grating with reflectivity $R_{gr} = 0.9$ and in-band flatness within 0.01 dB is considered as a target, but the actual reflectivity is highly distorted on the “blue” side and is as low as $R_{gr} = 0.47$. A new target grating is obtained (green line) by considering the corresponding cladding mode losses, with increased reflectivity where cladding mode losses are present ($R_{MAX} = 0.98$) and reduced strength on the “red” side ($R_{gr} \simeq 0.6$) in order to keep the grating physically realisable in passive media. The resulting grating spectrum, shown in (b), green line, is over-compensated as previously discussed, even though the spectral imbalance within the grating bandwidth has been reduced. When the new losses are taken into account and a new iteration is performed (red lines), a stronger grating is synthesised ($R_{gr} \simeq 0.79$), but the obtained spectrum is now under-compensated since the estimated coupling to cladding modes is lower than it should be (it refers to a $R_{gr} = 0.6$ grating while the actual reflectivity is $R_{gr} \simeq 0.79$). The iteration of the described procedure allows the obtained spectral imbalance to be progressively reduced until the required in-band flatness is reached, but it is also clear that a large number of cycles are needed before convergence due to over-shooting. 9 iterations are necessary in the presented example before the in-band reflectivity is flattened with the target 0.01 dB. This is particularly important due to the intensive computation still required by the direct scattering part of the proposed algorithm.

In order to reduce the over-shoot and improve the algorithm convergence, a damping
factor $\Gamma_i$ is considered in the computation of Eq. (5.18) at every iteration $i$. As already discussed, over-oscillations are related to the change in the reflectivity of the target grating and ultimately, to the insertion loss $IL$ that is introduced. When the insertion loss $IL$ is changed at the end of each cycle, it is already known that the cladding mode losses will also be changing correspondingly. Deriving a suitable estimation of this change allows a faster convergence to be obtained. First, the variation in the coupling coefficient $\kappa$ of the grating due to the new value of $IL$ is estimated by using the simple relation

$$R_{gr} \simeq \tanh^2 (\kappa L_{eff}),$$

where a uniform grating approximation of the chirped grating is used [6] and $L_{eff}$ is the corresponding effective length given by Eq. (3.32). As discussed in Section 3.3.2, this approximation is coarse and physically questionable, but it is acceptable for the present analysis. If the cladding mode coupling is weak, the Born approximation holds and a simple proportionality is obtained between the estimated coupling coefficient $\kappa$ and the reflection coefficient $r_{cm,c} = \frac{V_c(0)}{U_0(0)}$ of each cladding mode, so that the resulting total cladding mode loss $T_{loss}$ is simply given by

$$T_{loss} \simeq 1 - \sum_c |r_{cm,c}|^2,$$

and the new, estimated insertion losses $\tilde{IL}$ are computed using Eq. (5.20). The described procedure is iterated until a convergence is found and an optimised value for $IL$ is obtained (and therefore for the target spectrum $r_{comp}$ to be used in the next cycle of the inverse scattering algorithm). Ideally, a single cycle would be required to guess the correct target spectrum if this estimation method was perfect. However, this heuristic procedure assumes that no change in the longitudinal profile $q(z)$ occurs when the insertion losses are changed. This is incorrect, which means that multiple iterations are still required before a perfect compensation is obtained. Nevertheless, a significant decrease in the number of cycles is found (they are reduced from 9 to 4 in the example previously shown), with a corresponding improvement in the computation time required by the inverse scattering algorithm.

The described approach is graphically summarised by the flow chart diagram in Fig. 5.7. The use of an iterative solution is required by the particular inverse scattering problem itself and an exact solution is obtained only asymptotically. This fundamental difference with respect to the usual inverse scattering problem described in Section 2.3 is related to the different boundary conditions. Indeed, the knowledge of $r(\delta,0)$ gives a complete characterisation of the grating when no cladding modes are considered, since both $U(\delta,0)$ and $V(\delta,0)$ are known and the reconstruction of the coupling constant profile can start from $z = 0$. On the contrary, the boundary conditions for the problem described here are given by the fundamental mode reflection coefficient $V_0(\delta,0)/U_0(\delta,0) = r_{comp}(\delta)$ at the beginning of the grating (in $z = 0$), and by the cladding mode reflection coefficients $r_{cm,c}(\delta,L_{gr}) = 0, c = 1, \ldots, C$, at the end.
of the grating (in $z = L_{gr}$). Therefore, no $q(z)$ identification can be performed since $V_c(\delta,0)/U_0(\delta,0)$ are unknown and, therefore, the contributions of the cladding modes to propagation cannot be directly accounted for.

A different approach to inverse scattering design is given by the extension of the basic recursion (2.46) based on 2x2 transfer matrices to the discrete-reflector matrices (5.10) and (5.16) with cladding modes included. Obviously, the ratio $q/q_0$ is assumed to be known from the fibre geometry or from a previous cladding mode experimental characterisation. The core mode coupling coefficient $q_0$ is the only parameter to be determined layer by layer. Nevertheless, this modified algorithm still requires the reflection coefficients of all the propagating modes in $z = 0$ to be known, which have to be estimated using the same procedure proposed earlier in this Section. In this case, the iteration does not modify the target spectrum, but only the cladding mode spectra estimation. This approach presents major drawbacks. First, it is inherently slower, since the computationally intensive multiplication of $(C+1)\times(C+1)$ matrices has to be performed in both the inverse scattering and the direct simulation parts of the algorithm. Second, the convergence of the modified inverse scattering method is found to be highly sensitive to the initial conditions. The approximate estimation of the cladding mode reflection spectra is often not sufficiently accurate and, therefore, the algorithm fails to converge. For these reasons, this alternative procedure is not discussed in further detail here.

5.2.5 A practical example

The procedure described in the previous Section is now applied to the design of a practical second-order dispersion compensating grating. A standard SMF fibre is considered in order to demonstrate that special fibres are no longer necessary for rejection of
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135

\( \delta \lambda \) (nm) \( \kappa_c / \kappa_0 \)

(a)
core mode
odd modes
even modes

1548 1549 1550 1551 1552

\( \lambda \) (nm) \( R_{gr} \) (dB)

2.9 dB

(b)

Figure 5.8: (a) Relative position \( \delta \lambda \) and coupling strength \( \kappa_c / \kappa_0 \) of all the modes of a SMF fiber: \( n_{cl} = 1.45, NA \approx 0.12, \lambda_{cut-off} = 1250 \) nm, and core diameter \( d \approx 7.95 \) \( \mu \)m. Only modes marked in colour are computed in the following. (b) Corresponding core and cladding mode reflectivities for the grating described in the text: \( \lambda_{Bragg} = 1550 \) nm, \( D_{TOT} = -1700 \) ps/nm, \( \Delta \lambda = 3.2 \) nm, \( \eta_B = 0.97 \), and \( R_{gr} = 0.9 \).

Cladding mode losses. The typical parameters for this step index fibre are \( n_{cl} = 1.45, n_{co} - n_{cl} = 0.005, NA \approx 0.12, \lambda_{cut-off} = 1250 \) nm, and core diameter \( d \approx 7.95 \) \( \mu \)m. The wavelength positions of the cladding modes and the corresponding coupling strengths are computed as in [18] and are shown in Fig. 5.8 (a). Coupling to even modes is negligible for wavelength detuning up to 5 nm, while a much stronger scattering occurs to odd modes of the fibre, with \( \kappa_c < 0.3 \kappa_0 \). The separation between the core mode and cladding modes is only \( \delta \lambda_{cm} = 1.1 \) nm, which explains why SMF fibres are not used for the manufacture of wide-band chirped gratings. The grating to be synthesised is designed for compensation of the second-order dispersion (group delay slope) of 100 km of standard fibre, resulting in an overall dispersion \( D_{TOT} = -1700 \) ps/nm. The grating’s central wavelength is set to \( \lambda_{Bragg} = 1550 \) nm, and its bandwidth is \( \Delta \lambda = 3.2 \) nm in order to accommodate 8 channels on a 50 GHz grid or 16 channels on a 25 GHz grid. An out-of-band extinction of 30 dB within 5 GHz from the operational bandwidth of the grating is also targeted, resulting in a filling factor \( \eta_B = 0.97 \). The nominal grating strength is \( R_{gr} = 0.9 \), but the maximum strength of the pre-compensated spectrum is set to \( R_{MAX} = 0.98 \).

According to the above parameters, only the 7 cladding modes marked in colour in Fig. 5.8 (a) are significant and are included in the computation of the equalised grating. Indeed, the other modes are either too weakly coupled to produce any relevant distortion (the weaker mode considered, with \( \kappa_c = 0.03 \kappa_0 \), results in a maximum reflectivity of -26 dB for the corresponding cladding mode contribution) or spectrally too far away to affect the reflection band of the chirped grating. Fig. 5.8 (b) shows the result of the direct computation of the \( L_{gr} = 800 \) mm-long grating designed with standard inverse scattering techniques and using the exact transfer matrix method given by Eq. (5.8). Cladding modes are included and their spectrum is also shown using the same colour coding. An unacceptable 2.9 dB spectral imbalance in the final reflectivity is obtained (see black line), which means that cladding mode equalisation is necessary if SMF fibre
The results of the iterative inverse scattering procedure are shown in Fig. 5.9. In (a), the synthesised spectrum at the end of each cycle is plotted with different colours. The grating in-band reflectivity is already flattened within 0.03 dB after only 3 iterations, and almost perfect equalisation is obtained after the forth cycle. The point-scattering method in Eq. (5.21) is used for a faster computation; however, the simulation of the grating with the required resolution (number of sections $N \approx 6600$ and number of wavelengths $M \approx 5200$) takes about 4 hours using MATLAB® code on a Pentium IV® 2 GHz machine running Windows XP®. Despite being not yet ideal, the proposed algorithm runs on a desktop machine within a reasonably limited time, and it is now apparent why a minimisation of the number of iterations is particularly important. The corresponding profile of the equalised grating is represented in Fig. 5.9 (b). The chirp profile $\Delta \Lambda$ (cyan line) does not show a large deformation with respect to standard designs, while the apodisation profile (black line) is highly nonlinear, with the expected increased strength towards the end of the structure. High sidelobe suppression and grating squareness are obtained by means of the oscillatory apodisation profile in the grating and in the grating tail [42]. Only the use of an “exact” inverse scattering approach allows the computation of such a complex grating profile. The grating strength in the unperturbed region is $q_0 \approx 90$ m$^{-1}$ (corresponding to $\delta n \approx 4.4 \times 10^{-5}$), while the peak strength at the very end of the grating is $q_0 \approx 175$ m$^{-1}$. As a means of comparison, the standard inverse scattering design gives a coupling coefficient $q_0 \approx 125$ m$^{-1}$.

Finally, the grating spectrum is computed using the exact matrix formulation. The resulting reflectivity and transmissivity are shown in Fig. 5.10 (a) (black and magenta lines, respectively), while a magnified view of the in-band reflectivity and the group delay ripple $GDR$ are plotted in (b) (black and green lines, respectively). The grating has a sidelobe suppression larger than 35 dB and the required squareness is fulfilled, and a minimum transmissivity $T_{min} \approx -18.5$ dB is necessary in order to equalise the whole spectrum at a -5.1 dB level, corresponding to a flat reflectivity $R_{gr} \approx 0.69$. The final insertion loss is as large as $IL \approx 1.2$ dB. This is the trade-off to avoid any
spectral deformation related to cladding mode losses. For any stronger grating, there are simply not enough extra-photons to be reflected back on the core mode, since they are already scattered into the cladding modes [100]. This is a physical limitation of the considered system and it is not related to the implementation of the inverse scattering algorithm discussed here. Indeed, starting from any target grating reflectivity higher than $R_{gr} = 0.69$ would result in exactly the same final synthesised structure. $R_{gr} = 0.69$ is simply the upper bound for the available reflectivity, given the target spectrum (for both amplitude and time delay characteristics) to be obtained. Fig. 5.10 (b) proves that a very good equalisation is obtained, with a minor ripple and slope still limited within the target 0.01 dB figure-of-merit. Even the $GDR$ plot shows the good quality of the designed grating. The ripple is limited within $\pm 2$ ps over the whole bandwidth, with a very small fast ripple superimposed to a residual third-order dispersion. The glitches corresponding to the various cladding modes are the most apparent feature. They are related to the changes in the local reflectivity to compensate for the increased cladding mode losses and the use of the point-scattering approximation rather than the exact computation in the design procedure. Indeed, a phase mismatch exists between the two approaches for larger detunings, so that a non-perfect estimation of the position of the cladding modes is obtained. Possible solutions are either to decrease the section length $\Delta$ or to directly use the exact method for the direct scattering during the iterative inverse scattering. In both cases, improved spectral characteristics correspond to a longer computation time, which might not be acceptable.

5.3 Inverse scattering with propagation loss

5.3.1 Propagation losses and extension of the IS algorithm

In the previous Section, the inverse scattering algorithm was successfully applied to cladding mode loss compensation using an iterative procedure. No actual modification of the method developed by Skaar et al. [43] was performed. In this Section, a straightforward extension is introduced, allowing the same algorithm to be used in the design of
complex gratings when propagation losses are present and the propagation constant is
given by $\beta = \beta_0 + j\alpha_0$. Propagation losses are not a main concern when standard fibres
are considered, but they can become significant due to UV-induced material changes,
or in active fibres, or when gratings are written in different waveguide structures with
intrinsically higher background losses. A spectral deformation of the obtained grating
with respect to the design target is found in this case, since losses modify the interfer-
ence conditions between different paths inside the complex scattering structure on which
the standard inverse scattering algorithm relies. Including the effect of losses directly
in the design step allows, again, a perfect compensation to be obtained, improving the
characteristics of the resulting devices. In all the considered cases, the loss coefficient
$\alpha_0$ is assumed to be known before the actual synthesis procedure.

If fibre Bragg gratings are considered, the presence of a loss coefficient $\alpha_0 \neq 0$
can be related both to UV-induced losses and to material absorption. UV-induced
propagation losses were experimentally measured in a hydrogen loaded fibre illuminated
with $\lambda_{UV} = 244$ nm FRED Argon laser radiation \cite{182}. $\alpha_{dB} = 20$ dB/m power losses
(where $\alpha_{dB} = -10 \log(e^{-2\alpha_0 L})$, with $L = 1$ m) were obtained with a UV-fluence of
1.22 kJ/cm$^2$, which corresponds to an estimated refractive index variation $\delta n \simeq 4 \times 10^{-4}$.
They are commonly attributed to the long-wavelength tail of the UV-activated $OH$
absorption band centred at 1390 nm and are shown to be proportional (to a large extent)
to the induced refractive index variation \cite{195}. An independent estimation based on the
spectral slope in a chirped grating written in a hydrogen loaded fibre \cite{118} gives a value
of $\alpha_{dB} \simeq 4$ dB/m with a refractive index change $\delta n \simeq 1.25 \times 10^{-4}$. The obtained
difference is easily explained by the different (and not specified) fibre photosensitivity,
loading conditions, and grating writing conditions. A further measurement is given by
LaBrake et al. \cite{183}, which reports a detailed analysis of the absorption loss changes of
both a germano-silicate and a boron-codoped fibre during the various steps of the $H_2$
photosensitisation and UV-exposure processes. Lower UV-fluence was used in this study
(0.436 kJ/cm$^2$), and 4 dB/m attenuation in the 1550 nm wavelength range was obtained
for the germano-silicate fibre after UV-exposure and before thermal annealing. This
data is in reasonable agreement with the previously reported values. At the same time,
boron-codoped fibres are shown to give $\alpha_{dB} = 2$ dB/m losses, and very different changes
are obtained after 10 minutes annealing at 300° C. The attenuation of the first fibre is
indeed reduced to roughly 2 dB/m, while the second one is increased above 3 dB/m.
The actual induced loss is therefore also related to fibre composition and to the various
processing the fibre passes through during the grating manufacture process. Giving a
sensible number for the UV-induced losses is not possible in a general sense, but it is
definitely confirmed that they should be accounted for in grating design if outstanding
grating performances have to be achieved. As an upper limit, $\alpha_{dB} \sim [10 - 20]$ dB/m is
considered when hydrogen loaded fibre is used.

Another possible application of a lossy inverse scattering algorithm is grating design
or grating reconstruction (starting from experimentally measured reflectivity spectra)
close to a material absorption line, as for DFB lasers in Er-Yb doped fibres. For a typical Er-doping concentration of 500 ppm, the absorption at $\lambda = 1550$ nm is again estimated in the $\alpha_{DB} \simeq 10$ dB/m range. Finally, the availability of an extended algorithm can find application in the design of complex grating structures where intrinsic guiding losses are much higher than in fibre gratings. Gratings are currently written in UV-induced waveguides [184], and the recent demonstration of the possibility to use the same “continuous grating writing technique” used for fibre gratings opens up the possibility to manufacture highly complex designs also in planar structures [196]. In this case, the background losses of the waveguide are significant and typically in the 20 dB/m range. Surface relief gratings are a possible different solution applicable in planar optics [186]. In this case, both background losses of the waveguide and scattering to radiation modes due to the grating are present, resulting in very large losses (4 dB over a grating length $L_{gr} = 16$ mm are reported in the previously cited work). It is clear that such losses must be taken into account in the design of nonuniform gratings.

Since the proportionality between UV-induced losses and refractive index change is experimentally demonstrated, both a propagation loss $\alpha_0 = \alpha_0(z)$ proportional to the local grating strength $\delta n(z)$ and a constant loss $\alpha_0$ are considered in the following. The first condition is met when the grating apodisation profile is simply obtained by changing the total UV-fluence during the grating writing, while the second one applies to different types of losses (absorption, scattering,...) or to UV-induced losses when the so-called “complete apodisation technique” is used. In this case, the total UV-fluence is maintained constant over the whole grating length in order to prevent detrimental chirping effects due to background index variation, as shown by Eq. (2.17). Apodisation, chirping, and phase shifts are introduced in an independent way, for instance by multiple exposure dephasing in the “continuous grating writing” technique.

As seen in Chapter 2 and Section 2.3, the inverse scattering algorithm is based on the discrete-reflector approximation of the coupled wave equations (2.15). If lossy propagation is considered, the same set of differential equations is obtained from the Helmholtz equation (2.2) using the same approximations. The only difference is given by the complex nature of the propagation constant $\beta$ already discussed, and therefore the trivial update of the inverse scattering algorithm consists in modifying the recursion (2.46) in

$$r(\delta, z_i + \Delta) = \frac{-\varrho_i + r(\delta, z_i)}{1 - \varrho_i^* r(\delta, z_i)} e^{-j2\beta_0,i \Delta e^{2\alpha_0,i \Delta}} = \frac{-\varrho_i + r(\delta, z_i)}{1 - \varrho_i^* r(\delta, z_i)} e^{-j2\beta_0,i \Delta G}, \quad (5.24)$$

i.e., in multiplying the original target spectrum $r(\delta, z_i + \Delta)$ to be synthesised by the gain factor $G = e^{2\alpha_0,i \Delta}$ related to field propagation in the lossy medium. In general, $\alpha_0,i$ can be position dependent, and the local value can be estimated from the local grating strength $\varrho_i$ if the “proportional-loss” case is considered. As intuitively expected, amplification of the required grating reflectivity occurs as the reference plane moves away from the beginning of the structure. Enhanced grating reflectivity is required to
compensate for the excess losses induced by propagation, as clearly visualised by Fig. 5.5 with reference to cladding mode losses. Because of the gain factor $G$, the reflectivity $R(\delta, z_i)$ that has still to be synthesised can become higher than 1 in a certain position $z_i$. Then, the algorithm fails to converge since $R > 1$ is not physically achievable in a passive medium. As shown in Fig. 5.5, the obvious solution is to introduce an additional insertion loss $IL$. The reflectivity of the initial target grating is reduced to a physically realisable value and no spectral deformations are introduced. The actual value of $IL$ can be optimised by using an iterative procedure once the maximum reflectivity $R_{MAX}$ of the effective, lossless grating is fixed.

A re-analysis of the inverse scattering requirements discussed in Section 2.3 with respect to the grating design length $L_{gr}$ is necessary after this consideration. As already discussed, overestimation of $L_{gr}$ is usually a useful procedure in order to reduce temporal aliasing, and gives grating leading and (eventually) trailing tails at very low refractive index levels. But if the same windowed-apodised impulse response is fed into the lossy algorithm and a constant loss $\alpha_0$ is considered, very high insertion losses are needed in order to compensate for the initial, low reflectivity section $L_{in}$ of the grating. Indeed, no actual grating is present and the recursion relation (5.24) simply becomes $r(\delta, z_i + \Delta) = r(\delta, z_i)G$, so that $IL \simeq e^{2\alpha_0L_{in}}$ is required. The longer the starting section $L_{in}$ of the grating, the higher the insertion losses of the synthesised device are. Therefore, the choice of the better windowing and apodising technique described in Section 2.3 must take into account a further parameter (insertion losses) when a constant propagation loss is included. This effect is particularly important for square-dispersionless gratings, such as the one shown in Fig. 3.4 (b). They present a sinc-like shape with a refractive index ripple at very low $\delta n$ values at the beginning of the grating. This section was shown in Section 4.3.3 to mainly affect the in-band GDR of the grating (see Fig. 4.9), but produces high insertion losses if $\alpha_0 \neq 0$ is considered. A trade-off between device loss, grating reflectivity shape (especially when square filters with high filling factor $\eta_B$ are considered) and in-band GDR has to be found, and the final result is very much case-dependent. Therefore, this problem is not addressed in detail here.

### 5.3.2 Practical examples

As a possible example of inverse scattering design when lossy propagation is included, a square, linear dispersion compensating grating is considered. The inverse scattering algorithm parameters are not optimised in order to obtain “perfect” in-band flatness and minimum time delay ripple. The main purpose of this Section is to show how and why the local characteristics of the grating are changed by including loss compensation. The inverse scattered grating is $L_{gr} = 100$ mm-long, with in-band reflectivity $R_{gr} = 0.9$, nominal bandwidth $\Delta \lambda = 0.3$ nm, and filling factor $\eta_B = 0.75$. It is designed to compensate the dispersion of 80 km of standard optical fibre, which gives a nominal dispersion $D_{TOT} = -1360$ ps/nm.

The grating refractive index and chirp profiles resulting from the normal inverse
scattering algorithm are shown in Fig. 5.11 (a). The grating strength and the local chirp grow smoothly in the first section in order to guarantee the requested sidelobe suppression (ideally infinite) and spectral steepness (related to the filling factor). With respect to the flat $\delta n$ and linear chirp of standard dispersion compensating gratings, the central part of the inverse scattered grating presents oscillations in both strength and chirp. These additional features enable the compensation for the in-band time delay ripple and for the reflectivity ripple related to Gibbs oscillations created by the square shape of the filter. Finally, a smooth transition characterises the last section of the grating. The grating profile is slightly clipped at $z = 100$ mm, showing that the chosen length $L_{gr}$ is just sufficient to accommodate the whole grating. The corresponding in-band reflectivity is shown in Fig. 5.11 (b). The solid line corresponds to the lossless condition, and the expected “optimum” filter is found. The dotted line corresponds to a constant background loss $\alpha_{\text{dB}} = 10$ dB/m (hereon referred to as CL), while the dash-dotted line corresponds to a proportional loss $\alpha_{\text{dB}} = 17$ dB/m for an induced refractive index change $\delta n = 10^{-4}$ (PL).

The different characteristics of the two kinds of losses are easily described. In the first section ($L_{in} \simeq 30$ mm), almost no back-reflection occurs and a 0.6 dB additional loss is introduced in the whole bandwidth by the 10 dB constant loss CL ($\alpha_{\text{dB}} \cdot 2L_{in} \simeq 0.6$ dB). Conversely, very low losses are introduced in the PL case, as shown by the long-wavelength side of the spectrum. Shifting towards the short-wavelength side, i.e., moving towards wavelengths reflected deeper inside the grating, losses are almost linearly increasing in the CL case. This linear slope is expected since the total loss has to be proportional to the effective path length in the lossy region, and therefore to the (linear) reflection time delay $\tau_R(\lambda)$. A less well defined behaviour characterises the PL case. The steeper loss slope is related to a slightly higher average loss (13 dB/m at the grating centre), but no direct understanding of the spectral features can be gained at a first sight. Better physical insight is given by taking into account the local time delay distributions introduced in Chapter 3, as discussed later.

If losses are directly taken into account in the inverse scattering algorithm, the grating profiles resulting for the constant (green) and proportional (magenta) case are represented in Figs. 5.11 (c) and 5.11 (e), respectively. In both cases, the grating strength is increased in the last part of the grating, as better seen by direct comparison in Fig. 5.12 (a). In order to minimise the insertion losses, the last section of the grating has a much higher local reflectivity with respect to the original grating, as shown by the grating transmissivity in Fig. 5.12 (b). The maximum grating reflectivity is limited to $R_{\text{MAX}} \simeq 0.994$ (corresponding to $T_{\text{MIN}} \simeq -22$ dB). Nevertheless, the simulation length $L_{gr} = 100$ mm is not sufficient for such a stronger grating and huge clipping of the impulse response occurs. A further $L = 20$ mm-long section is added at the end of the grating to compensate for the multiple reflections and reduce the reflectivity ripple and time delay ripple obtained. Conversely, both the lossy cases present a reduced strength at the beginning of the grating. This reduction corresponds to the insertion losses $IL$.
Chapter 5 Inverse Scattering in lossy waveguides

Figure 5.11: Comparison between inverse scattering results under different loss conditions: (blue) lossless medium; (green) $\alpha_{dB} = 10$ dB/m constant loss; (magenta) proportional loss with $\alpha_{dB} = 17$ dB/m $\leftrightarrow$ $\delta_n = 10^{-4}$. The grating design parameters are: $L_{gr} = 100$ mm; $R_{gr} = 0.9$; $\Delta\lambda = 0.3$ nm; $\eta_B = 0.75$; $D_{TOT} = -1360$ ps/nm. (left column) refractive index and chirp profiles $\delta n(z)$ and $\Delta\Lambda(z)$. (right column) corresponding reflectivity spectra. The dotted and the dash-dotted lines correspond to the lossless-designed grating when constant and proportional losses, respectively, are included. The insertion losses $IL$ related to lossy designs are shown.

necessary to obtain a physically realisable equalised grating.

If the chirp profile $\Delta\Lambda(z)$ is considered, Figs. 5.11 (c) and 5.11 (e) show that smaller oscillations in the local grating period are necessary to flatten the time delay ripple in the lossy cases. In order to understand this feature, the time delay analysis of chirped gratings discussed in Section 3.3.2 proves helpful. The local time delay distribution presents a ripple that is related to optimum constructive or destructive interference between different paths in the effective Fabry-Pérot cavity. But cavity losses disturb the interfering conditions, and therefore both the local time delay ripple and the associated spectral ripple are reduced. Therefore, a smaller compensation is requested in the design
The equalised resulting gratings are shown in Figs. 5.11 (d) and 5.11 (f) for the constant-loss (green) and proportional-loss (magenta) cases, respectively. Flat in-band spectra are still possible, but at lower reflectivity levels as stressed by the corresponding insertion losses $IL$. When CL are considered, a reduced grating reflectivity as big as 0.98 dB is necessary, while it is limited in the PL case to “only” 0.61 dB. In both cases, the final flat level is 0.44 dB higher than the minimum in-band reflectivity $\min(R(\lambda))$ obtained from the lossless design and including losses. This characteristic is related to the already discussed increase in the local grating reflectivity from $R = 0.9$ to the design $R_{MAX} = 0.994 \simeq 1$ in order to compensate for the short-wavelength side losses. It must be noted that the $IL$ values just described have limited significance, since they are related to a starting reflectivity level that is not physically realisable in the lossy medium analysed. Even if a $R = 0.99$ grating had initially to be designed, the same final result would have been produced by inverse scattering, with higher $IL$ values. If a non-realisable structure is designed and insertion losses have to be introduced, the output of the inverse scattering algorithm always represents the best possible realisation of the initial impulse response under the design constraints, i.e., the given lossy propagation conditions. Therefore, the reflectivity level $\hat{R}$ obtained is the physically important quantity, since it sets the limit of physical realisability for the design grating shape in the lossy medium as $R = 1$ in a lossless medium.

The physical explanation of the final inverse scattered grating shape is now considered. First, the constant-loss case is analysed using the same approach that was used for cladding mode losses. The synthesised refractive index profile is the result of the normal inverse scattering computation of an effective pre-compensated target grating, as shown by Fig. 5.5. The final lossy grating can be directly designed without using any modified algorithm or iteration, since a proper loss estimation for the final structure can already be inferred from the original target spectrum.
Indeed, the theory discussed in detail in Section 3.4 shows that a small perturbation induced in a certain position \(z\) causes a deformation of the grating spectrum which is proportional to the time spent locally by light in the perturbed region. In this case, the considered perturbation is given by a constant loss coefficient \(\alpha_0(z) = \alpha_0\), and the spectral deformation is nothing but the corresponding loss \(\alpha_{\text{tot}} = \Delta R\). From Eq. (3.41):

\[
\frac{r_{\text{pert}}(z)}{r} = e^{-\alpha_0 dz N_R(z)} \Rightarrow \Delta R(z) = -2\alpha_0 dz \text{Re}\{N_R(z)\}, \quad (5.25)
\]

\[
\Delta \phi(z) = -\alpha_0 dz \text{Im}\{N_R(z)\}. \quad (5.26)
\]

If the lossy region extends to the whole grating length \(L_{gr}\) and the small perturbation approximation is still satisfied, the corresponding reflectivity and phase variations in the grating (i.e., loss and time delay perturbations) are given by

\[
\Delta R = -2\alpha_0 \frac{c}{n_{\text{eff}}} \text{Re}\{\tau_R\} \quad \rightarrow \quad \alpha_{\text{tot}} = -2\alpha_0 L_{\text{eff}}, \quad (5.27)
\]

\[
\Delta \phi = -\alpha_0 \frac{c}{n_{\text{eff}}} \text{Im}\{\tau_R\} \quad \rightarrow \quad \Delta \tau = -\alpha_0 \frac{c}{n_{\text{eff}}} \frac{d}{d\omega} \text{Im}\{\tau_R\}, \quad (5.28)
\]

where the dependence on the integrated time delay of the grating is emphasised and an effective propagation length \(L_{\text{eff}}\) introduced. It has to be stressed that both the real and the imaginary parts of the time delay \(\tau_R\) of the target grating spectrum \(r = \sqrt{R}e^{j\omega\tau_R}\) are known. Indeed [197]:

\[
r = |r| e^{j\theta_R} = e^{j(\theta_R - j \ln |r|)} = e^{j\hat{\theta}}, \quad (5.29)
\]

\[
\tau_R = \frac{d\hat{\theta}}{d\omega} = \begin{cases} 
\text{Re}\{\tau_R\} = \frac{d\phi_R}{d\omega}, \\
\text{Im}\{\tau_R\} = -\frac{d \ln |r|}{d\omega} = -\frac{1}{|r|} \frac{d |r|}{d\omega} = -|r| \frac{dR}{d\omega}.
\end{cases} \quad (5.30)
\]

If flat filter design is considered (as for the grating in Fig. 5.11), the in-band contribution of \(\text{Im}\{\tau_R\}\) is negligible since \(dR/d\omega \approx 0\). No significant change in the time delay is caused by the introduction of losses, apart from the ripple reduction already discussed, and incidentally confirmed by the analysis of Eqs. (5.28) and (5.30). As a result, Eq. (5.27) gives a perfect estimation of the total loss even for the pre-compensated grating, since both gratings are characterised by the same time delay \(\tau_R\). Even if non-flat spectra are considered, the previous relations allow a reasonable estimate of the total losses of the final filter to be made, starting from the designed time delay. Therefore, perfect pre-compensation is obtained by using the algorithm described in Section 5.2.4, the lossless target grating \(r\), and Eq. (5.27), as shown in Fig. 5.13. It has been numerically verified that the method presented gives the same final result as the modified inverse scattering algorithm.

If proportional losses are considered, magnitude and spectral shape of the induced losses (as in Fig. 5.11 (b), dash-dotted line) can be modelled and understood using the previous framework. In this case, Eq. (5.25) still holds, but a position dependent loss coefficient \(\alpha_0(z)\) has to be taken into account. Given the profile of the grating...
coupling constant, the local reflection time delay $\tau_R(z)$ is evaluated using Eq. (3.14), and integration over the grating length $L_{gr}$ has to be performed in order to find the correct value of the total loss $\alpha_{tot}$. Numerical simulations show perfect agreement between the described approach and the direct grating computation with proportional loss for the grating previously considered.

But, unfortunately, no direct algorithm can be derived from this analysis as is possible for the constant-loss case. Indeed, any pre-compensation results in a deformation of the grating profile $\delta n(z)$ and, therefore, in different local losses $\alpha_0(z)$ and different local time delay $\tau_R(z)$. No inference about the total loss of such a modified grating can be made from the starting target spectrum. An iterative algorithm based on lossless inverse scattering and lossy direct grating computation is the only possible solution if this approach has to be pursued. The strict analogy of this case with the cladding mode compensation described in Section 5.2.4 is apparent, since in both problems the actual loss is proportional to the local scattering coefficient ($\alpha_0(z) \propto \delta n(z) \propto q(z)$ here, $q_c \propto q_0(z)$ for cladding mode coupling). The main difference is the assumption that no further interactions occur between lost power and core-propagating modes in this case, giving a loss spectrum that is independent of propagation. On the contrary, cladding modes are always coupled to core modes and affect their propagation, resulting in a position dependent shape of the induced loss. The modified, lossy inverse scattering algorithm presented in this Section obviously provides far better performances with respect to an iterative solution, since it is faster and inherently not approximated once the target impulse response is defined. It corresponds to the alternative cladding mode compensation algorithm briefly described at the end of Section 5.2.4. But the boundary conditions are perfectly defined in this case, and an efficient implementation is possible and effective.
5.4 Conclusions

Layer-peeling techniques used for the design of complex inverse scattered gratings have been extended by taking into account lossy propagation inside the waveguide structure. Different sources of losses have been considered in the analysis, namely, uniform losses due to either background losses of the guiding structure (intrinsic or related to the UV writing process) or to propagation in an active material (absorption losses), or variable losses in which the effect is proportional to the strength of the grating coupling coefficient, such as cladding mode losses or, again, UV-induced losses.

A modified, iterative inverse scattering algorithm is presented and proven to be effective in the compensation of cladding mode coupling and equalisation of the grating spectrum of linearly chirped gratings designed for dispersion compensation. The losses introduced by cladding modes are estimated and the grating reflectivity spectrum is correspondingly deformed in order to pre-compensate for these losses. The standard layer-peeling method is used for the synthesis of the modified target spectrum, while an iterative procedure is used for the estimation of cladding mode losses. Indeed, the effect of cladding modes is known only when the grating profile is known, since the boundary conditions simply impose that no power is backward-propagating on cladding modes after the grating. The proposed solution is to first assume that no cladding modes are present, compute the ideal grating profile, obtain the corresponding losses when cladding modes are included, pre-deform the target spectrum, and repeat the whole procedure. But changing the target spectrum also changes the cladding mode losses, so that imperfect compensation is obtained and the described loop has to be repeated several times before a perfect equalisation occurs. A suitable approach for damping the over-oscillations of the described algorithm is also developed, which allows the number of cycles before convergence to be largely reduced, improving the computational efficiency of the grating synthesis.

Using this method, the quantitative design of complex inverse scattering gratings is possible even if cladding modes are present. In particular, the design of a 3.2 nm dispersion compensating grating has been considered. True linear dispersion compensation, ultra-high filling factor and high sidelobe suppression to minimise crosstalk, and excellent in-band flatness are obtained with cladding modes included in the analysis. The cladding mode structure of a standard SMF fibre is considered, and despite the small separation of the cladding mode spectrum from the core mode ($\delta\lambda = 1.1$ nm) and the large coupling to even cladding modes (as large as 30% of the core mode coupling), an excellent result is obtained. This is the first time special fibres or special writing techniques are proven not to be necessary for the manufacture of large-band dispersion compensating gratings, even though the manufacturing complexity of the grating is increased and the use of a complex set-up which allows complete control over apodisation and chirp is necessary.

A faster method for the computation of the direct scattering of gratings with cladding modes has also been proposed. The analysis of a simplified model of propagation in the
fibre (only backward-propagating cladding modes are considered and multiple scattering between cladding modes is neglected) allows the identification of a simple transfer matrix method for the computation of propagation in the structure. The method is a direct extension of the well known formalism used for standard gratings, it is exact in the limit of small section lengths, but no analytical expression for the transfer matrix is available. It can only be expressed in the form of an exponential matrix. Numerical computation of eigenvalues and eigenvectors is still necessary, which limits the efficiency of the algorithm. Nevertheless, a large improvement is obtained (larger than one order of magnitude) with respect to standard integration using numerical routines (such as the Runge-Kutta algorithm). An approximated, point-scattering formalism has also been developed, which allows close formulas to be obtained for the transfer matrices. Further improved computational efficiency results since the problem is reduced to a simple matrix multiplication. The complexity of this algorithm is found to be $O(MNC^2)$, where $M$, $N$, and $C$ are the numbers of wavelengths, sections, and cladding modes to be computed, respectively. However, the analytical expressions obtained are complicated and shorter section lengths are typically needed to minimise the discretisation error introduced by the approximation and to prevent aliasing in the computed spectrum. As a means of comparison, the design of the previously cited grating (4 iterations of the algorithm) took about 1 day of computation time, while longer time would be required by a single computation of the direct scattering problem if standard numerical approaches were used.

If constant or proportional propagation losses are considered, the extension of the layer-peeling algorithm is straightforward, since a complex propagation constant has only to be used when propagating the fields inside the grating. Losses are assumed to be known, and attenuations in the order of 10-20 dB/m are shown to be sensible numbers for this effect. Perfect compensation of losses is obtained, even though more complex design trade-offs have to be found with respect to the lossless case. Indeed, higher reflectivity corresponding to spectral regions with larger losses is necessary to minimise the insertion loss of the device, but also implied are longer structures to compensate for multiple scattering. Longer grating lengths allow a better grating spectrum to be synthesised, with sharper edges if required and reduced in-band oscillations, but they also introduce higher losses in the structure, which means that a reduced reflectivity may be required. An optimum balance between the different effects is necessary, but no general recipe is available and a case-by-case tuning has to be performed, increasing the overall complexity of the design procedure.

In both the described cases, the target spectrum of the grating is not only deformed, but it may have to be rescaled to lower reflectivity levels by introducing extra insertion losses in order to keep the maximum reflectivity level below $R_{\text{gr}} = 1$, i.e., to keep the grating physically realisable in passive media. This effect is not a limitation of the design procedures introduced here, but is a physical limitation due to the presence of losses in the considered system. It simply means that the starting target spectrum is
not realisable in that particular (lossy) guiding structure, as a grating with $R_{gr} > 1$ is not possible in passive media. The proposed algorithms allow the identification of the grating with the required spectral shape and time delay characteristic which presents the minimum insertion losses and is still feasible given the waveguide losses.
Chapter 6

Inverse scattering applications: CDMA systems

6.1 Introduction

As was pointed out in the Introduction, the new manufacturing techniques for fibre Bragg grating production allow these devices to be used for new and different applications from originally imagined. A typical example is given by Optical Code-Division Multiple Access (OCDMA) systems [198, 199]. This is a well known format for the data transmission in mobile networks, and recently it has gained significant interest also in the field of optical communications. Different users are able to transmit on the same band and in the same time slot by coding the information with special orthogonal sequences, which typically requires an accurate control of amplitude and/or phase of the encoded waveform. The complete control on the spectral content of each wavelength component reflected by a FBG, and the capacity to synthesise almost arbitrary waveforms starting from simple laser pulses, is therefore of the greatest importance for CDMA technology.

Grating technology has been successfully demonstrated in a number of different CDMA systems, that use either amplitude [200, 201, 202, 203] or phase [204, 205, 206] to encode the signal in either the frequency or the temporal domain. A simple approach is to use simple (uniform or apodised) gratings as discrete blocks to build up the actual code. However, the potentialities to scale the coding capabilities to either longer or more complicated codes, which provide superior orthogonality performance and enable a larger number of users to share the same frequency-time slot, are limited due to manufacturing and handling issues. The use of complex grating superstructures (SSFBGs) fabricated using the continuous grating writing technique is the most promising approach [205], since almost arbitrary complexity can be achieved together with physical integration and shorter length requirements. This means that shorter waveforms are obtained using codes of the same complexity, i.e., higher data rates can be transmitted and higher throughput obtained. Fourier-transform designs have typically been used for CDMA applications to date, since they do not require advanced inverse scattering techniques and
they minimise the manufacturing complexity of the device [206]. But this approach does not take full advantage of the present potentialities of SSFBGs, since uniform gratings with discrete phase shifts embedded are sufficient for phase encoding in the time domain (which is the most successful application). Much more complicated superstructures have already been demonstrated [47, 49, 51], with complex apodisation and chirp profiles in addition to precise coherent control of the periodic pattern of the grating.

New designs are therefore required, either implementing more advanced coding, improving the performance limitations intrinsic to Born approximation approaches, or combining multiple functionalities within the same grating. The two latter problems are specifically tackled in this Chapter by use of the layer-peeling algorithm as a design tool. Insertion losses are inevitable when using low reflectivity gratings, such as the Fourier-transformed ones, and are a serious problem in practical system operation due to the higher level of amplification (i.e., added noise) required. Increasing the amplitude of the refractive index modulation is not possible, since multiple reflections inside the stronger structure deform the encoded waveform and prevent a successful CDMA operation. Only inverse scattered profiles that automatically compensate for these multiple scatterings can solve the problem effectively. On the other hand, the gratings used for encoding/decoding can also be interpreted as general signal processors, so that more than a single system function can be included in the same structure. Dispersion compensation is considered in this work, since it involves chirping of the grating while CDMA coding typically involves apodisation and phase shifts only. The highest possible SSFBG complexity is achieved, with a reduced system complexity (less components are needed) and possible improved performance (due to the reduced total insertion losses). In both cases, attention is also paid to design superstructures that are still compatible with the present manufacturing capabilities and practical issues in system operation. Length requirements and limitations in the apodisation, chirp, and discrete phase shift profiles are taken into account, so that the best trade-off between complexity, reliability, and performance can be identified.

The Chapter is structured as follows. A brief summary of the main characteristics of optical CDMA technology is given in Section 6.2, together with a more detailed review of coherent phase coding techniques, and grating-based implementations. The advantages of the SSFBG approach are highlighted. Section 6.3 deals with the design of encoding/decoding gratings with higher reflectivities, and the different designs are compared in terms of the orthogonality properties of the resulting waveforms. Dispersion compensation embedded with CDMA coding is considered in Section 6.4, showing that more stringent limitations have to be faced in this case. Finally, Section 6.5 describes the performance of these gratings and their sensitivity to variations in both the grating parameters (due to imperfect fabrication) and the operational parameters (due to random fluctuations in the system).
6.2 CDMA transmission systems

The development of optical transmission systems with higher capacity and functionalities has become increasingly important over recent years due to the explosive growth of the Internet and the increasing demands of high-speed optical communications. Research has mainly been focussed on the use of wavelength division multiplexing (WDM), optical time-division multiplexing (TDM), or hybrid approaches in order to boost the system capacity. Now that terabit-per-second systems have been demonstrated in the laboratory [207], interest is beginning to grow in investigating processing and routing of signals directly in the optical domain. In this way, many of the bottlenecks currently imposed by optoelectronic conversion and electronic processing of data can be removed.

An alternative multiplexing scheme such as optical code-division multiple access (OCDMA) is very attractive from this point of view. OCDMA is a spread-spectrum technique widely used in mobile communications and applied to optical communications only in the last ten years. It allows a large number of separate users to share the same broad transmission bandwidth and time slot through the allocation of specifically designed address codes. Such codewords are orthogonal (or quasi-orthogonal) and therefore minimally interfering at the detector. Retrieval of the original information is possible in a synchronous or asynchronous way, and a limited degradation of system performance occurs as the number of transmitting users increases. OCDMA technology is still at a relatively immature stage of development, but its inherent capacity for higher connectivity, flexible bandwidth usage, higher granularity and scalability are key issues within future optical networks. Improved crosstalk performance, relaxed frequency stability and timing requirements on optical sources, asynchronous access, and improved system security are further advantages of this multiplexing format.

6.2.1 CDMA general overview

Different techniques are used to code and spectrally spread a data signal. In frequency-encoded (FE) CDMA, the available optical bandwidth $\Delta f$ is sliced into $L$ different frequency bands, where $L$ is the length of the considered codeword. An appropriate coding signature is then applied to either the amplitude [208] or the relative phase [209] of each band. The energy of the starting pulse is spread over an encoded waveform of duration $\tau_{\text{enc}} \simeq L/\Delta f$ and has statistical properties very similar to white noise [209]. In direct-sequence (DS) CDMA, data is directly coded in the time domain [198]. Each data bit is converted into a pseudo-random sequence of shorter pulses, commonly referred to as chips. The duration of each chip is $T_{\text{chip}} \simeq 1/\Delta f$ and the encoded waveform is $\tau_{\text{enc}} \simeq L/\Delta f$ long. Again, either the amplitude [199] or the phase [210] of each chip are used to carry the necessary code information. Both the aforementioned techniques are inherently based on uni-dimensional coding, since either the frequency or the time content of the signal are coded. Given a code family, longer codes provide better correlation properties between different codewords and finally better system performance.
On the other hand, slower data rates or larger optical bandwidth (i.e., more demanding optical processing requirements) have to be used, since the code length $L$ determines the spreading factor and, ultimately, relates the transmission data rate to the total optical bandwidth. In frequency-hopping (FH) CDMA, the previous approaches are merged and two-dimensional coding is considered using wavelength-hopping/time-spreading schemes [211, 212]. The carrier frequency of the data signal is rapidly and pseudo-randomly switched between different frequencies within the hopping bandwidth according to a well-defined time-frequency sequence. The increased number of coding coordinates provides more flexibility in the design of code families and relaxes the requirements on code length and processing speed. At the same time, statistical analyses have been reported showing that these codes theoretically outperform the previous approaches in terms of number of users at a given bit error rate (BER) [200, 213].

Simple receivers based on matched filtering were demonstrated in all the described approaches. The correlation receiver became popular in OCDMA systems due to its theoretical single-user optimality and due to the potential for all-optical processing. The optical processing allows the electronic processing speed and the detector electrical bandwidth to limit only the user bit rate, without limiting the spread spectrum bandwidth. The incoming data stream is first correlated in the optical domain with a locally generated copy of the desired signature waveform. If the codewords are matched, each spectral-temporal component is properly re-phased and the spread signal is temporally recompressed in a single, high-power pulse of duration $\tau \approx 1/\Delta f$; otherwise, a noise-like sequence results. The optical signal at the output of the optical correlator is detected with an intensity photodetector and then the electrical photo-count signal is compared to a threshold value to recover the transmitted information [214]. The correlation detector has a limited performance with respect to optimal receivers in multiple user configurations. The final OCDMA system performance is therefore poor, with a limited spectral efficiency and a substantial degradation with increasing number of interfering users. Typically, BER values well above the error free performance $BER = 10^{-9}$ are obtained for a number of active users far from the overall number of available signature codes [213, 215, 216], showing that code capabilities are not fully exploited. The user codes are known in a multiple access system, but this information is ignored with a matched filtering receiver. This is a major limitation, since spread spectrum systems suffer from an extra noise factor known as multiple access interference (MAI) besides the usual thermal and shot noise contributions in the photodetector. MAI is due to the imperfect orthogonality of the used signature sequences, it is power independent, and it is typically the limiting factor of CDMA performance, as shown by the noise floor in the BER-received power graphs in [217, 218, 219].

The MAI-limiting strategies developed in mobile communications involve the use of optimal, maximum-likelihood receivers, requiring multiple correlators, multistage architectures, fully coherent signal acquisition, and large signal processing capabilities. Most of these requirements cannot be met in optical CDMA. Firstly, multi-Gbit/s data rates
typical of optical communication links prevent intensive on-the-fly post-processing of the received data streams. This is not a physical limitation of CDMA systems, since a custom realisation of specialised digital processors can fulfil such a demanding processing power. As an example, powerful forward error correction (FEC) codes (Reed-Solomon codes, BCH codes, TCM codes) are applied in mobile communications to the digital data stream in a concatenate coding configuration. Impressive results are obtained, since error free performance is reached starting from a BER as high as $2 \times 10^{-4}$ using a RS(255,239,8) code [220, Section 4.3]. This solution is now available for up-to 10 Gb/s SONET-SDH based optical systems. Nevertheless, the development of these products is expensive and is attractive only if mass production is required, which is not the case for OCDMA systems at the present stage. Secondly, the intensity-based nature of optical photodiodes washes out all phase information after detection, limiting the possibility of complex data manipulation. A completely optical realisation of the optimal receiver is necessary. Back in 1994, an optical implementation of multiuser detection was theoretically proposed for amplitude-coded, direct-sequence CDMA [221], but no optical multistage detector has ever been realised and tested. Losses, stability requirements, limitations in optical integration, ultimately costs of current optical technologies and passive components push towards easier and more reliable, though sub-optimal, receiver solutions. Alternatively, OCDMA configurations based on coherent optical communications were proposed in order to boost the system performance [222]. The theoretical advantages over incoherent detection schemes are apparent, but the present unavailability of local optical oscillators sufficiently stable in wavelength and phase makes these systems purely hypothetical.

Therefore, theoretical analysis and experimental testing of different coding formats were mainly considered in order to mitigate the effect of MAI with minimum decoder complications. Intensity photodetection issues and available optical technologies were taken into account by designing highly customised code families. These codes must have good properties in terms of signature sequence orthogonality, they must be easily encoded and decoded on optical waveforms by means of cheap and reliable optical components, and they must allow simple but effective all-optical decoding schemes. The reader is referred to the following ORC internal report [220, Chapter 2] for a detailed review of the possible coding schemes and related optical technologies.

### 6.2.2 CDMA with coherent phase coding

A different approach to optical CDMA is to directly process the information in the optical domain using passive but coherent devices [210, 223]. This does not mean that coherent optical transmission is required. The idea is to take advantage of the fact that random phase variations over the fibre optic link occur on a much longer timescale with respect to the typical duration of the encoded waveform. If a coherent phase pattern is imprinted on the transmitted pulse, the same pattern arrives at the receiver and information is retrieved by using a matched filter. All the coding schemes previously developed
for RF applications and based on multipolar coding can be used (Gold codes [224], quadrupolar codes [225], multipolar extensions [226]). These codes are characterised by superior correlation properties, larger code cardinalities [227], and efficient use of the energy of the starting pulse (all the chips carry energy, while incoherent codes use OOK format [198]). Even though optimal receiver algorithms cannot be applied in the optical domain, their performances are expected to largely exceed those of incoherent coding schemes and to be comparable with bi-dimensional codes with larger code dimensions. Intensity detection has still to be performed at the receiver, but matched and unmatched codewords are already discriminated in the optical domain, and the code does not rely on the averaging properties of the photodetector. In this case, MAI noise is not simply additive as in incoherent schemes [214], since different contributions can actually add or subtract to each other depending on the interfering conditions. A completely different noise analysis is therefore necessary, based on the original work by Salehi et al. [209]. It has been developed in Ghiringhelli and Zervas [220]. The different noise features have often been disregarded in the literature, and attention has been paid mainly to practical demonstrations of coherent schemes under very limited MAI noise conditions. On one hand, the practical limits of these approaches have been defined and claims of possible outstanding CDMA system operation have been drawn up. On the other, no direct experimental proof of these claims have been reported, and their actual effectiveness is still very much an open question.

Coherent CDMA is possible by coding the starting pulse both in time and in frequency. Tight coherence requirements have to be imposed on the encoding-decoding devices, since any unmatched phase variation results in an effective deviation from the code signature and, therefore, in an unrecoverable degradation of the power of the code. In frequency-encoded CDMA, both free-space [209] and AWG [228] implementations were demonstrated. A strict control on each optical path has to be performed since a fixed phase relation must exist between the different wavelength components before the encoding process. For the same reason, femtosecond laser sources must be considered, since only mode-locked pulses offer the required phase coherence together with a sufficiently large encoding bandwidth. No cheap implementation is therefore possible. The final performance of these systems is anyway remarkable according to the early work of Salehi et al. [209]. Using 80 fs transform-limited pulses, Gold bipolar coding, and a code length \( L = 512 \), the system can support up to 100 users at error rate \( \text{BER} \approx 10^{-10} \) and individual transmission rate \( R = 1 \text{Gb/s} \). But such an appealing theoretical result has not been matched by any practical demonstration. AWG operation with \( L = 255 \) was reported by Tsuda et al. [228], but only with a single transmitting user. Moreover, the experimental data presented show a cross-correlation extinction lower than 10 dB on the photodetected intensity signal, much smaller than the theoretically expected value (which is in the 15 dB range according to the result of Table A.1 in [220]). Manufacturing issues are still a limiting factor for the practical application of this technology.

In direct-sequence CDMA, the reciprocal phase between different chips has to be
fixed by means of a sub-wavelength stabilisation of the optical paths. No coherence requirements are theoretically set on the source in this case. Even if a coherence time $\tau_{\text{coh}} < T_{\text{chip}}$ is used, the encoder-decoder pair re-matches all the components of the starting pulse on an instant-by-instant basis, therefore in conditions where relative coherence is always fulfilled. This is an advantage when long chip durations are considered, since cheap sources can be used. However, good performances are obtained only using short chip durations according to the results in Ghiringhelli and Zervas [220], so that coherent mode-locked sources have to be used anyway. Coherent delay line implementations based on ladder networks have been considered, with earlier demonstrations based on discrete fibre components [210, 223] and more recent experiments based on integrated optics [229, 230, 231]. Fibre experiments are easier to set up in a laboratory environment in order to prove the feasibility of the CDMA concept, but practical use is impossible due to the high sensitivity to any environmental variation. PLC are more suited for this application since all the devices are integrated on a single substrate. Tunable taps based on integrated Mach-Zehnder interferometers and active phase control are added directly on the optical chip [229], allowing active stabilisation and reconfigurable operation of the encoders-decoders. Integrated optics also allows very short chip durations to be used, since the actual delay is accurately controlled by designing the optical waveguides on the substrate. Typically, 5 ps delay lines were used in the experimental demonstrations cited earlier (corresponding to a length difference of $\sim 1 \text{ mm}$), but much higher resolutions are possible. As an example, the path difference inside the AWG used in [228] is $\sim 77 \mu\text{m}$, corresponding to a temporal resolution of $\sim 0.4 \text{ ps}$.

The only fibre-based approach is represented by fibre Bragg gratings. The single fibre implementation helps in keeping the device compact and, therefore, easier to package and stabilise against external perturbations. But discrete array implementations [200] are not suitable in this case, since control of the relative phases of the light reflected from each subgrating is necessary. UV trimming techniques can be used to correct the optical path separating adjacent gratings, but characterisation is not straightforward and no practical demonstration was attempted. The segmented-composite gratings described in [204] are a more promising technology, especially for mass production. Three-phase code signatures of length $L = 8$ were used (these codewords were obtained by an exhaustive algorithmic search given the small dimensionality of the problem), but multipolar codes are easily applicable. The drawback of this approach is represented by the complexity of realisation of the customised phase mask. Stitching errors [232], limited overall grating length, and limited scalability to longer $L$ due to the accumulation of phase errors are the main problems to be faced, and they seriously limit the practical use of the written gratings.

### 6.2.3 CDMA using SSFBGs: applications and design challenges

A further FBG-based approach to CDMA is to use the superstructured gratings technology available at the ORC. If a weak grating is considered, the impulse response of
the SSFBG follows directly the superstructured profile of the refractive index change of the grating. Application in direct-sequence CDMA systems is immediate. OOC-based, amplitude-coded gratings [206] are manufactured by simply turning on or off the UV laser light in different chip positions. The various subgratings are tightly packed in space, so that shorter encoded waveforms result, longer codes can be used for a given data rate, and packaging and stabilisation are easier. Moreover, all the chips are written in the fibre with a single process, with a reduced manufacturing complexity and ultimately lower cost. A similar analysis applies to frequency-encoded CDMA based on SSFBG technology. If each wavelength in the code is written in a separate length of fibre, the possibility of changing the pitch of the grating without changing the phase mask allows a single manufacturing process even in this case. Technological problems exist, since the maximum detuning in the grating pitch is limited by self-apodisation effects [109]. This technique is therefore applicable only to coding schemes with a relatively narrow band (typically below 10 nm), and optimisation of the strength of the different bands may be required. SSFBG technology even allows different wavelengths to be written on the same region of fibre (Moiré gratings [31]), further reducing the length of the device, but with a considerable increase in the complexity of the apodisation profile. Despite the previous discussion, no experimental demonstration of these superstructures has been attempted to date. In the former case, the simplicity of the code results in a limited system performance (see, for instance, [233]), and the possibility of easily upgraded operation to bipolar coding schemes soon turned the attention towards more appealing configurations. In the latter, the increased manufacturing complication does not correspond to any expected boost in the system performance.

Coherent DS-CDMA is the most natural environment for the application of SSFBGs, given the flexibility in the choice of the code signatures and the improved performance. Both bipolar and quadrupolar coding schemes have been successfully demonstrated in multi-user operation [205, 234], and further increase in the number of coding phases (or application of bi-dimensional amplitude-phase codes) is theoretically possible, even though not demonstrated yet. Scalability to long codes has already been shown, with \( L = 255 \) quadrupolar implementations that match the longest CDMA coding sequence ever shown to date (see [228]), and are much longer than any other grating-based demonstration. Superstructured grating technology offers several more advantages over other grating-based approaches. The quality of the phase mask is not an issue, since the complex code profile does not depend on it, and the grating length is not limited by the mask size. The size of the translation stage used in the writing process is the ultimate limit, even though the coherence length of the writing set-up is typically more important for complex phase shifted gratings such as CDMA encoders and decoders. The coherence length \( L_{coh} \) is defined as the length over which phase coherence is maintained inside the grating despite noise and random fluctuations in either the imprinting process or the fibre parameters [109, 122]. This technology also has potential for the production of low cost devices, given the flexibility of the manufacturing process, even if the added
functionality is paid for with a more complex fabrication technique which does not pay
off if simple structures are manufactured. The main disadvantages of superstructured
gratings are the high insertion losses of these devices, which they share with any other
grating-based implementation, and the limitations in the spatial resolution of the writing
process, which do not allow the manufacture of encoders with a very short chip duration.
These technological issues are discussed in detail in [220, Section 3.5], where it is shown
that features below 150-200 µm, which correspond to a chip duration $T_{chip} \sim 1.6 - 2$ ps,
are not achievable. This means that a high data rate and a very long code cannot be
achieved simultaneously.

If frequency-based coherent implementations are considered, SSFBG technology can
be theoretically used, but there are a few major drawbacks that actually limit its ap-
lication. First of all, the total bandwidth that can be processed is limited by cladding
mode losses on the short wavelength side of the grating, since all the partial subgratings
(one for each frequency band) are superimposed in the superstructure to keep a correct
phase relation. Secondly, the typical spatial features are on a very short scale. Using the
previous numbers, no more than 10 nm gratings are possible, and with a correspond-
ing apodisation profile at the limit of the present manufacturing capabilities. Similar
considerations apply to bi-dimensional coding performed via SSFBGs.

It is therefore clear why attention has been paid to the realisation of a coherent direct-
sequence CDMA encoders and decoders based on SSFBGs. The increasing complexity
of the experimental demonstrations and the still impressive match between theoretical
predictions and manufactured devices show that the potentialities of the underlying
technology have not yet been fully exploited. Precise phase shifts are embedded into the
grating modulation pattern, but no apodisation is introduced. Much more complicated
structures have already been demonstrated using the same writing system. The main
purpose of this work is to analyse whether more complex designs can be used to improve
the performance of DS-CDMA systems, but still fulfilling the limitations of SSFBG
technology.

Two aspects can be taken into account. On one hand, more complex coding schemes
are possible, either further increasing the number of phases in the code signatures, or
using bi-dimensional amplitude-phase constellations similar to QAM in digital transmis-
sion theory. Very little research has been published in this field, especially since studies
on mobile communications have been directed towards more sophisticated coding ap-
proaches (block codes, convolutional codes, TCM codes, CPM). These are suitable in
the RF domain, but are not compatible with an optical implementation at the present
moment. The possibility of using 8-phase codes rather than the bipolar or quadrupolar
codes has been considered [220, Appendix A]. The larger cardinality of these codes
provides more flexibility in the choice of the optimum subset for a given system config-
uration (starting pulse width, chip duration, level of dispersion in the system), but the
actual optimisation is not within the scope of this work. Conversely, further increas-
ing the number of coding phases is found not to give a practical advantage in terms of
system performance, at least if the average properties of the codes are considered. The increased complexity in generating the code and manufacturing the gratings is not fully justified, and therefore only quadrupolar CDMA implementations will be used.

On the other hand, the application of inverse scattering techniques to CDMA gratings is a more interesting area of investigation. In particular, insertion losses are one of the main disadvantages of this technology. Typically, the maximum reflectivity is kept below $R_{gr} = 0.25$ in order to work in the Born approximation limit, which means a net loss of at least $6 + 6$ dB for the encoder and decoder stages. Actually, the insertion loss of each grating is much higher and typically in the 10-15 dB range, since the average reflectivity (weighted over the spectrum of the starting pulse) is lower than the peak one. If the reflectivity is increased, the multiple scattering deforms the impulse response of the grating and therefore the quality of the encoding process. Intuition does not give a clear understanding of the process, since multiple scattered light from the front chips will interfere in different ways with light reflected deeper into the grating, depending on the phase pattern imprinted by the code. Only the layer-peeling algorithm described in Chapter 5 can take all the paths into account and reconstruct the (complex) apodisation profile from the (simple) impulse response that is targeted. This analysis is presented in Section 6.3, and the resulting designs are compared to the standard approach in terms of both pure performance and manufacturing complication.

Another possible application of inverse scattering techniques is to include multiple functionalities into a single grating. The natural wavelength selectivity of the encoding-decoding gratings already allows them to be used as band filters in a mixed WDM-CDMA configuration. Experimental demonstrations were already obtained for both 4 channels [234] and 16 channels [235] mixed operation. No added power penalty from adjacent wavelength channels was reported, showing the good level of filtering provided. Dispersion compensation is another typical application of FBGs. The challenging task is to combine the encoding-decoding functionalities with dispersion compensation of the transmitted span, typically 100 km of standard SMF fibre. The encoded bandwidth plays a significant role in this case, so that both the pulse and the chip durations have to be taken into account in order to optimise the performance of the resulting system. Moreover, grating manufacturing and packaging issues cannot be disregarded, and the corresponding trade-offs need to be considered in the design. The analysis of multifunction, dispersion compensating encoders and decoders is discussed in Section 6.4. The use of a fixed dispersion compensation scheme in a CDMA system may be questioned, since reconfigurability is one of the key advantages of this approach with respect to TDM and WDM. However, network topologies in which a certain code signature is assigned to a fixed node and flexibility is given by the fact that each node may be transmitting or not in an asynchronous way are still interesting. The length of each link is known in this case, and fixed dispersion compensation can be included in the encoder and/or decoder. Finally, the inverse scattered gratings are compared to standard Fourier transform designs both in terms of performance, complexity, and sensitivity of the system.
to typical parameters in Section 6.5.

A last consideration must be taken into account. The codes used in this analysis are designed to work in a purely discrete (numerical) environment, with an ideal $\delta(\xi)$ starting pulse and squared chip waveforms. Under these assumptions, they provide optimal correlation properties and homogeneous performances inside the code set. Given two randomly chosen codewords, their auto- and cross-correlations are fairly identical. But these conditions do not apply in the real system modelled in the following, and very different results are obtained when different code subsets are selected. Moreover, the results also depend on the actual choice of the different parameters, so that certain subsets are optimal in certain conditions, but they can have very bad performances in different conditions. An example of these features is given in Appendix F, considering the design of a dispersion compensating grating. This peculiarity has to be taken into account in order to fairly compare the results.

6.3 High reflectivity DS-CDMA gratings

6.3.1 Why are complex designs needed?

High insertion loss is one of the main problems of DS-CDMA systems based on SSF-BGs. As described in Section 6.2.3, successful encoding of the code signature onto the transmitted waveform occurs only if the reflectivity of the grating is low, so that light penetrates the whole structure and multiple scattering is negligible. Moreover, the spectrum of the encoding grating presents large oscillations due to the interference between the different chips. If the shape of the incoming pulse is not perfectly matched, large spectral filtering occurs, and the encoding process is not efficient. Increasing the peak reflectivity of the grating results in a net gain at the output, but the corresponding waveform deformation due to multiple scattering is detrimental for the quality of the encoding-decoding process, limiting the orthogonality of the codewords used and the performance of the system.

The encoders and decoders have to be pre-deformed in the high reflectivity limit in order to compensate for this effect and obtain ideal encoded waveforms. Inverse scattering techniques are the natural approach to this problem. They theoretically allow the “perfect” apodisation and chirp profiles of the target gratings to be obtained, but a limited design bandwidth and the length requirements can reduce the capability of the algorithm, as described in Section 2.3. The resulting design is inherently approximated, and suitable trade-offs are needed between the length of the device, the maximum reflectivity to be obtained, the manufacturing complexity, and the orthogonality properties of the resulting codewords. A proper approach to how it is better to approximate these structures has also to be found.

In the following, a grating system already demonstrated [235] is taken as a practical example. The considered system was used to prove the feasibility of WDM-CDMA operation and the possibility to use the SSFBGs as spectral filters as well as temporal
encoders, but for the purpose of this work only single wavelength operation is considered. Quadrupolar, family $A$ code signatures of length $L = 15$ were considered in the experiment and reported in the paper. The corresponding phases for the codes $Q1$ and $Q2$ are shown in Table 6.1. The temporal duration of each chip is 50 ps, so that a $\sim 800$ ps waveform was transmitted on the optical link and $R = 622$ Mb/s operation was theoretically possible without inter-symbol interference at the receiver. The length of the corresponding grating is $L_{gr} = 8.22$ cm, and a low reflectivity ($R_{gr} = 0.25$), Fourier transform design is used. The grating has a uniform apodisation profile and fixed phase shifts along its length, according to the phase pattern shown in Table 6.1. In this regime, the impulse response is expected to be a simple replica of the modulation profile of the grating. As far as the source pulses are considered, 20 ps pulses were generated by external modulation of CW sources (either a DFB laser or a tunable external cavity semiconductor laser). These pulses are relatively long in order to reduce the insertion losses of the CDMA encoders and decoders, but they are still shorter than the chip duration, so that averaging of the phase content of each chip does not occur.

<table>
<thead>
<tr>
<th>$Q1$</th>
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<td>$3\pi/2$</td>
<td>$3\pi/2$</td>
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<td>$0$</td>
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</table>

Table 6.1: Code signatures used in [235] and throughout this Section.
Chapter 6 Inverse scattering applications: CDMA systems

The correlation traces at the output of the decoder are shown in Fig. 6.1 for both a grating reflectivity $R_{gr} = 0.25$ (upper row) and $R_{gr} = 0.9$ (lower row). The autocorrelations of the code $Q1$ are shown in the upper row, and the cross-correlations $Q2 - Q1^*$ in the lower row (the codeword $Q2$ is transmitted and received by the decoder matched to $Q1$). The ideal traces are in blue, while the actual traces encoded by the SSFBGs are in red. A good correspondence is found in the low reflectivity implementation. In (a), the peak power is even higher if the gratings’ impulse responses are used (peak $\{I_{Q1 - Q1^*}\} = -23.75$ dB vs. -24.76 dB, with a 1 dB increase), and a small temporal imbalance occurs with higher correlation values before the peak and lower after the peak. The extinction in the auto-correlation is practically constant, with a limited reduction from 12.98 dB to 12.72 dB. In (b), similar features are obtained for the cross-correlation trace, with a final extinction practically constant (from 8.21 dB to 8.25 dB). Also, no increase in the duration of the decoded waveform is found, proving that multiple scattering is not important at this reflectivity. These features show that a very good approximation of the ideal operation is obtained and the use of approximated impulse responses is not always detrimental. This occurs because the code signatures were optimised for operation with SSFBGs, and because a small code subset of cardinality 2 is used. In general, the final result is largely dependent on the actual codewords used.

Very different results are obtained when the peak strength of the grating is increased to $R_{gr} = 0.9$, but a uniform apodisation profile is still kept (see Fig. 6.1 (c) and (d)). A large difference is found with the ideal correlation traces, with a much higher average level in the decoded waveforms. The gratings are reflecting more energy than expected, but not quite with the correct phase relations given by the code signature. Indeed, the received peak power is 2.7 dB higher than the ideal case (peak $\{I_{Q1 - Q1^*}\} = -8.9$ dB vs. -13.63 dB), but the out-of-peak signal is 7.6 dB and 7.2 dB higher for the auto- and cross-correlations, respectively, with a degradation larger than 4.5 dB. An unacceptable collapse in the performance of the CDMA system is expected \[220\]. Furthermore, a long tail in the cross-correlation signals is produced (d), so that each interfering signal at the detector will affect more than a single bit slot, causing inter-symbol interference (ISI) and further increasing the multiple access interference (MAI) noise. This extra back-reflected energy is readily understood by remembering that a $\text{tanh}^2$ relation exists between the refractive index modulation $\delta n$ and the reflectivity $R_{gr}$ of a uniform grating \[88\], and applying this result to the present case to a first approximation. Saturation occurs at high $R_{gr}$, so that an increase in $\delta n$ much larger than $\sqrt{0.25 \over 0.9}$ is necessary to boost the reflectivity up to the new value. Therefore, the low reflectivity components are raised much more than expected (since they are still in the linear regime), providing extra energy but also deforming the ideal spectrum of the code, with reduced final orthogonality.
6.3.2 Inverse scattered design

The need for using inverse scattering designs is apparent. A very long grating length ($L_{IS} = 500$ mm) is first assumed in order to reduce the effect of temporal aliasing. The grating is shortened after reconstruction is complete by applying a suitable windowing function of the desired length $L_{gr}$. The presence of discrete phase jumps in the impulse response requires a very large bandwidth to be used to prevent the occurrence of Gibbs oscillations that would deform the reconstructed grating. Very high resolution in the temporal-spatial response is necessary. But a different approach can be used in this case. The finite resolution of the writing set-up (discussed in Section 2.4) does not allow very fast variations to be introduced in either the apodisation or the chirp profiles, so that any reproducible feature has to be on a scale $\gtrsim 200 \mu$m. Therefore, the target impulse response is pre-deformed by convolving it with a smoothing function of suitable duration (2 ps hyperbolic secant shape in the present case). The corresponding bandwidth requirements are relaxed, the inverse scattering reconstruction is easier and less noisy, while negligible differences are expected in a final, manufactured grating. After this procedure, a $\Delta f = 2000$ GHz bandwidth with $df = 125$ MHz resolution is used in the IS algorithm, corresponding to a spatial resolution $\Delta = 52 \mu$m and a total impulse response duration of 8000 ps. The maximum reflectivity of the superstructured grating is set to $R_{MAX} = 0.9$.

Fig. 6.2 (a) shows in blue the resulting apodisation profile of the encoding grating for the code signature $Q1$. The generalisation of the Fourier transform design is represented by the first $\sim 90$ mm of the grating. The average refractive index change $\delta n$ is still at low levels ($\delta n < 2.5 \times 10^{-5}$). There is no need to use special photosensitive fibres or hydrogen loading techniques in the manufacturing process. This is related to the length of the device determined by the CDMA temporal spreading pattern. However, its strength is not constant inside each chip, since less light penetrates the structure and higher reflectivity is necessary towards the end of the chip to keep the impulse response constant, as shown in (b), blue line. At the beginning of the next chip, the
local reflectivity is decreased due to the different interfering conditions related to the localised phase jumps in the structure. A simple pattern cannot be found to relate the variations in the grating strength to the actual phase shifts, and intuition does not help in the design. This pseudo-random behaviour in the grating is due to the pseudo-random characteristics of the code itself. Conversely, the chirp profile of the grating (not reported here, but shown in Fig. 6.4 (b) for $L_{gr} < 90$ mm) has the same features as the Fourier transform design, with quasi-discrete phase changes between different chips (actually, they are spread over the length of the smoothing function used in the design).

After this region, the ideal inverse scattered grating shows a very long tail with reduced, but not negligible strength. The chirp profile is also more complex, with a continuous chirp introduced. The reason is clear if a shortened version of the IS design is taken into account. The red lines in Fig. 6.2 refer to the same grating, but with a length reduced to $L_{gr} = 89$ mm in order to compare it with the low reflectivity design and to avoid the need for more complex packaging. A hyperbolic tangent windowing function is applied to the last 5 mm of the grating in order to minimise the effect of clipping. The corresponding impulse response in (b), red line, is still very good in the first $\sim 800$ ps, since it is completely determined by the initial $\sim 80$ mm of the grating because of causality. But multiple scattering still occurs after this region, causing more light to be reflected at successive times. If no compensation is introduced by increasing the physical length of the grating, this scattering gives rise to the impulse response tail. If the grating length is increased, further light is reflected from deeper into the grating and can destructively interfere with the previous contribution, cancelling the tail out. Perfect compensation is not possible with a finite length superstructure, but the spurious tail can be suppressed down to the level required for its contribution to the final performance to be negligible.

A trade-off between grating length and quality of the encoding process has to be found. The imperfect cancelling of the tail of the impulse response causes an increased ISI after the decoder, but it also results in a reduced peak reflectivity of the grating. Indeed, the spectral features of the encoded sequence are fairly matched throughout the main grating lobe (where most of the reflected energy comes from) and also the sidelobes, but the reflectivity on the peaks is not as high as from nominal design. This means that the tail of the grating also helps in reflecting part of the light back. Indeed, the shortened grating (red line) has a maximum reflectivity reduced to $R_{MAX} = 0.657$, and no improvement is found with the same length requirements even starting from higher reflectivity gratings (with $R_{MAX}$ up to 0.99). This means that it is not possible to get to higher reflectivities without deforming the temporal shape of the encoded pulse for $\xi < 800$ ps, i.e., without changing the actual code signature.

The design of the decoder has also to be considered to understand the effect of each approach, since CDMA operation always requires the use of an encoder/decoder pair. As described in Section 6.2.1, simple matched filtering techniques are used at the detector. In the low reflectivity limit, the decoder is the very same encoder, but used from the
Figure 6.3: Decoded waveforms (in intensity) using inverse scattered designs. (a) matched signal $Q_1 - Q_1^*$; (b) interfering signal $Q_2 - Q_1^*$. The ideal gratings’ output is shown in blue; the shortened gratings’ output is in red.

opposite side [206]. Ideal operation occurs if $h_{dec}(\xi) = h_{enc}^*(-\xi)$, where $h_{enc}$ and $h_{dec}$ are the impulse responses of the encoding and decoding gratings, respectively. In the high reflectivity designs this is not true, due to the different interfering conditions related to the sequence of chips. New decoder designs are obtained using the previously described approach, and similar considerations can be drawn. For this reason, the gratings are not shown here, while the decoded waveforms are considered directly in Fig. 6.3 (a) and (b) for matched and interfering signals $Q_1 - Q_1^*$ and $Q_2 - Q_1^*$, respectively. The blue lines still refer to the ideal inverse scattered gratings, and the red ones to the shortened versions ($L_{gr} \sim 90$ mm). A limited difference between the two approaches is found. The degradation in the correlation properties of the code signature is negligible, as expected from the very good reconstruction of the first 800 ps of the encoded waveform. The cross-correlation extinction is the same as in the ideal case (8.34 dB), and the peak auto-correlation of the system based on shortened gratings is just 0.05 dB lower despite the reduced peak reflectivity (peak$\{I_{Q_1-Q_1^*}\} = -13.66$ dB). Indeed, a small bandwidth range has very high reflectivity, while most of the spectrum is at lower reflectivity levels and is already very well reconstructed by the 90 mm long grating. Therefore, very little energy is lost in the shortening process. The increased duration of the correlation traces is the most noticeable difference. The energy after the correlation peak is spread out over a much longer time duration, with an exponential decay in the intensity signal (linear decay in the dB representation). This means that a communication system based on these SSFBGs is not ISI-free, even if the interfering signal on the next bit is found in the -52 dB/-56 dB range for the auto- and cross-correlation signals, respectively. This noise level is approximately 40 dB lower than the signal level. This contribution is usually neglected in a standard WDM-TDM system, since ASE noise from the optical amplifiers is dominant in the optical system. But the high degree of coherence of these CDMA systems results in a much more detrimental beat noise in the encoded-decoded waveforms [220]. It manifests itself mainly as amplitude noise, and a 0.24 dB intensity ripple is found on the signal peak due to ISI even with the reported suppression. This level can be tolerated in the present case, since any other interfering signal introduces
a much larger MAI (extinction in the cross-correlation is just above 8 dB, giving an intensity ripple of $\sim 2$ dB). But, in general, ISI has to be accounted for, and longer gratings have possibly to be used to suppress the ISI-generated MAI.

High reflectivity, inverse scattered designs are clearly a valuable tool for improving the performance of the $L = 15$ chips DS-CDMA system considered so far. The superstructured gratings are more complicated, since both chirp and apodisation profiles must be accurately controlled. But the spatial extent of each chip is far above the resolution limit of the writing system (actually, $L_{\text{chip}} > 4$ mm), and the phase shifts to be introduced are almost discrete. Amplitude and phase features are practically disentangled and the manufacturing process is relatively easy. No penalty in the quality of the encoding process is found, and actually the comparison between Fig. 6.1 (a) and Fig. 6.3 (a) shows that the high reflectivity IS design is even better than the low reflectivity standard design in reproducing the ideal features of the code. A 11.09 dB gain in the received power is obtained on the auto-correlation peak, very close the expected 12 dB despite an effective reflectivity of the IS gratings limited to $R_{\text{MAX}} \sim 0.7$. Moreover, the grating length has not to be increased with respect to the Fourier transform design.

### 6.3.3 Improved iterative design

The comparison of Fig. 6.1 (c) and Fig. 6.3 (a) shows that a higher peak power is obtained without design complications if a lower extinction is tolerated. Unfortunately, the problem is to define a proper target spectrum to be given to the algorithm. Lower insertion losses are targeted, but limited orthogonality degradation is needed and the grating still has to be physically feasible. Since only the peak reflectivities are not properly reconstructed in the previous example and a minor effect results, the reflectivity of the grating can be further increased and then kept physically realisable ($R_{\text{MAX}} < 1$) by suitably deforming the frequency bands with higher $R_{gr}$ only. Unfortunately, this simple approach tackles the problem only in frequency, so that the resulting impulse response is typically no more causal. Either it has to be shifted and windowed, but a longer gratings results, or it has to be directly clipped, but the final spectral deformation is typically unacceptable.

A more elaborate approach is to apply an iterative algorithm. Given the results of the first step (the standard IS just presented), it is possible to use them as a starting point for a second step, where either (1) the reconstructed grating profile $\delta n_{IS}$ is multiplied by a constant in order to obtain a stronger grating, or (2) the obtained spectrum $R_{IS}$ is first rescaled to $R_{\text{MAX}} = 0.9$ and then a new iteration of the IS algorithm is performed. The corresponding designs are shown in Fig. 6.4, where both apodisation (a) and chirp (b) profiles are presented. The red line refers to the previously described design, while the green and black lines are the results for cases (1) and (2), respectively. The corresponding maximum reflectivity is boosted from $R_{\text{MAX}} = 0.66$ (standard IS) to 0.79 (case 1) and 0.80 (case 2), with a limited increase in the maximum refractive index modulation required ($\delta n < 3.3 \times 10^{-5}$). Minor modifications are obtained in the chirp of the encoding...
Figure 6.4: Different iterative designs of the encoder grating Q1. (a) apodisation profile; (b) chirp profile. (red) standard inverse scattering design $\delta n_{IS}$; (green) $\delta n = \delta n_{IS} \times 1.25$; (black) new $\delta n_{IS,2}$ starting from $R_{\text{target}} = R_{IS} \times k$ to get $R_{MAX} = 0.9$.

Figure 6.5: Decoded waveforms (in intensity) using the iterative designs. (a) matched signal $Q1 - Q1^*$; (b) interfering signal $Q2 - Q1^*$. (red) standard inverse scattering design; (green) $\delta n = \delta n_{IS} \times 1.25$; (black) $R_{\text{target}} = R_{IS} \times k$ to get $R_{MAX} = 0.9$.

grating, while the apodisation function is quite different. The approach requiring two iterations of the IS algorithm (black) results in sharper features inside each chip, since the higher local reflectivity requires more light to be reflected deeper inside the chip to keep the impulse response constant.

The corresponding decoded waveforms are represented in Fig. 6.5 using the same colour coding. Both methods produce a further significant decrease in the insertion losses, with the auto-correlation peak improving by 2.7 dB (1) and 2.4 dB (2), respectively (see Table 6.2). More noticeably, the large gain does not correspond to a worse orthogonality between the encoded waveforms, meaning that the whole spectrum is almost uniformly boosted and the deformation is minimised. Solution (1) is suboptimal, since both the out-of-peak maximum value of the auto-correlation and the peak of the cross-correlation are increased by more than the corresponding increase in the signal peak (3.25 dB and 2.83 dB versus 2.7 dB). Conversely, solution (2) allows losses to be reduced, and also the performance of the code to be slightly improved (2.28 dB and 1.73 dB versus 2.3 dB). The quality of the encoding/decoding process is also proven by the limited degradation of the waveform tail. These results cannot be generalised since the final effect is very much dependent on the actual parameters used (codewords,
Chapter 6 Inverse scattering applications: CDMA systems

Table 6.2: Peak values, out-of-peak auto-correlation values, and cross-correlation values for the iterative designs of high reflectivity gratings shown in Fig. 6.5.

<table>
<thead>
<tr>
<th></th>
<th>peak({I_{Q_1-Q_1^*}})</th>
<th>max({I_{Q_1-Q_1^*}})</th>
<th>max({I_{Q_2-Q_1^*}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard IS</td>
<td>13.66 dB</td>
<td>-26.72 dB</td>
<td>-22 dB</td>
</tr>
<tr>
<td>Solution (1)</td>
<td>-10.97 dB</td>
<td>-23.37 dB</td>
<td>-19.17 dB</td>
</tr>
<tr>
<td>Solution (2)</td>
<td>-11.27 dB</td>
<td>-24.34 dB</td>
<td>-20.27 dB</td>
</tr>
</tbody>
</table>

Figure 6.6: Inverse scattered high reflectivity design for a \(L = 63\), \(T_{\text{chip}} = 6.4\) ps CDMA encoding grating. (a) apodisation profile; (b) chirp profile.

chip and pulse durations), as described for a particular case in Appendix F, but they are a clear example that further performance improvements are possible by slightly deforming the target spectrum. This also confirms that the code signatures commonly used are not optimised for operation in an optical CDMA system (better performance is obtained with a spectrum deformed from the ideal target), and that a custom search of optimal codewords for each system implementation may lead to improved designs. This is typically applicable to the design of systems with small cardinality.

The approaches derived so far have been applied to a fairly simple case, since only 16 chips are coded by the grating and the duration of each chip is very long (\(T_{\text{chip}} = 50\) ps, corresponding to \(\sim 5\) mm long chips). Using the same technique with more complex (and powerful) coding schemes does not present any specific problem, and the same features are obtained in every case. The practical possibility to manufacture and package the obtained designs is the only issue. Packaging can be a serious problem if the length of the grating exceeds 20 cm, but it is good practice to keep the length of the device as short as possible and below 10 cm. Even in this case, the actual duration of the encoded waveform may constitute the limiting factor, if no ISI is accepted. But it has just been shown that inverse scattered designs can be reduced to roughly the same length as standard Fourier transform approaches without significant performance loss. Therefore, length requirements are specific to the particular system scheme chosen (code length, chip duration, ultimately data rate) and not of the improved designs presented here. Conversely, chip duration requirements are different with respect to standard approaches. As already discussed, the continuous grating writing set-up has a finite spatial resolution, which sets the practical limit for the separation between phase shifts to
an estimated $\min\{T_{\text{chip}}\} \approx 1.6 - 2$ ps. But this value has to be relaxed if high reflectivity IS designs are considered. The apodisation profile is not uniform and sufficient resolution has to be left for a proper definition of the transitions shown in Figs. 6.2 and 6.4. It is therefore reasonable to assume a chip duration limit $T_{\text{chip}} = 6.4$ ps for these IS gratings. This value is compatible with previous experimental demonstrations using bipolar codes [205] with code length $L = 63$ and data rate $R = 1.25$ Gb/s. An example of the corresponding design is shown in Fig. 6.6. The increased complexity due to the shorter length scale ($L_{\text{chip}} = 660$ $\mu$m) is apparent, but the low refractive index change required, the lack of a complex chirp profile, and the small amplitude oscillations make this design still possible to be manufactured. Conversely, longer codes cannot be considered unless the transmission rate is reduced or ISI is accepted.

6.4 Dispersion compensating DS-CDMA gratings

The discussion developed so far assumes that the optical channel is transparent, in the sense that propagation does not introduce any modification/deformation in the transmitted waveforms. But, apart from bandwidth filtering due to the in-line optical components and non-linear effects, which cannot be accounted for if the actual layout of the deployed system is unknown, attenuation and wavelength dispersion are intrinsic properties that cannot be neglected. Attenuation can be compensated only by active devices, while dispersion compensation has been successfully demonstrated using chirped SSFBGs, on both a multiple channel basis [31, 37] and a single channel basis [49].

It is therefore natural to include dispersion compensation in the design of the encoding and/or decoding superstructured gratings. Multi-function operation is obtained, with coding, bandwidth filtering, and dispersion compensation performed by the same device. System design simplification and reduced insertion losses are the main advantages of this approach. The resulting IS designs are not trivial and intuitive, and they require an accurate control of both apodisation and chirp of the grating. They fully exploit the capabilities of the writing set-up, while the designs analysed in the previous Section do not, since they are almost chirp free.

6.4.1 A simple example

As a first example, the $L = 16$ system considered previously is upgraded to support dispersion compensation. According to the discussion at the end of Section 6.2.3, this functionality is included only in the decoder grating, while the encoder is kept unchanged. The target maximum reflectivity for the gratings is again set to $R_{\text{MAX}} = 0.9$, and compensation of the dispersion accumulated along $L_{\text{link}} = 100$ km of standard fibre with $D_{\lambda} = 17$ ps/nm/km is assumed. The grating length requirements are slightly relaxed in this case, and $L_{\text{gr}} = 120$ mm designs are considered. This length is still well within the SSFBGs manufacturing capabilities, it is compatible with standard grating
packaging, and it allows dispersion compensation over a maximum time window

$$\Delta \xi_{tot} = \frac{2n_{eff} L_{gr}}{c},$$

(6.1)

where the approximated formula, valid in the low reflectivity case, is used. With $L_{gr} = 120$ mm, Eq. (6.1) gives $\Delta \xi_{tot} \approx 1160$ ps. This means that the bandwidth, that is effectively compensated for dispersion, is limited and given by

$$\Delta \lambda_{disp} = \frac{2n_{eff} L_{gr}}{c D_{\lambda} L_{link}} \Rightarrow \Delta f_{disp} = \frac{2n_{eff} L_{gr}}{2\pi \beta_2 L_{link}} \geq \frac{LT_{chip}}{2\pi \beta_2 L_{link}},$$

(6.2)

where the last inequality holds since the grating has to be longer than in the dispersion-less case, and where $\beta_2$ is the second-order dispersion of the fibre used for transmission (see Eq. (2.42)). $\Delta \lambda_{disp} \approx 0.68$ nm for the considered optical link (corresponding to $\Delta f_{disp} \approx 85$ GHz), while the bandwidth of the CDMA system (considered between the first zeros of the spectral envelope) is $\Delta f \approx 40$ GHz, as shown in Fig. 6.7 (a), green line. The bandwidth $\Delta f_{disp}$ is sufficient to compensate for the dispersion of the main lobe and the first sidelobes of the spectrum, so that a large part of the dispersed energy is recompressed in a large auto-correlation peak. Nevertheless, the spectral envelope has a sinc-like shape for rectangular chips. Higher order sidelobes are attenuated only as $1/\Delta f$, with the third sidelobe still having a peak reflectivity of -23 dB. The energy in these sidelobes is still dispersed after the decoder and spread out to a longer timescale (see next Section for details), so that the actual duration of the dispersed encoded waveform is longer than $\Delta \xi_{tot}$.

The target impulse response $h_{dec}(\xi)$ of the decoder has to be suitably shortened, and a minimum impact on CDMA operation is required. A first possibility is to cut $h_{dec}(\xi)$ before inverse scattering the grating, but this typically causes a large spectral deformation in the synthesised grating, as already discussed in the previous Section. Conversely, it is convenient to first inverse scatter the target impulse response assuming a very long grating is considered, and then to shorten it to the required length by using an apodisation function at both edges. The position of the actual grating inside the original IS profile can be optimised in order to minimise spectral deformations and, especially, maximise the resulting orthogonality between the obtained waveforms. In this case, the length of the reconstructed grating is first set to $L_{IS} = 500$ mm, and then it is shortened to the target length $L_{gr} = 120$ mm using $tanh$ apodisation functions. The result of the inverse scattering design is shown in Fig. 6.7 (a) and (b), green lines, for the apodisation and the chirp profiles, respectively.

The maximum reflectivity of the grating is $R_{MAX} = 0.748$, and the maximum refractive index change is not increased with respect to the previous designs in Fig. 6.4 (a). On the contrary, the spatial features are completely different. An intuitive design would require a linearly chirped grating (dispersion compensation) with a discrete pattern of phase shifts superimposed (CDMA coding). But no clear slope in the chirp profile is present in Fig. 6.4 (b), and phase shifts are distributed over a long length scale (1 mm
rather than the resolution limit of the set-up of \( \sim \) 200 \( \mu \)m discussed in the previous Section). No direct identification of each chip transition is possible any longer. The amplitude profile is also much more complex, with periodicity of the main oscillations still in the range of the phase shifts’ separation (\( \sim \) 5 mm), but with a large amplitude noise superimposed and characterised by a \( \sim \) 2 mm ripple. A detailed analysis for the reasons of this increased complication is described in the next Section. However, the resulting grating is still within the manufacturing capabilities, even though the combined amplitude and phase variations and the reduced spatial separation of the grating features makes it much more complicated than the previously reported superstructures. It is therefore convenient to find different designs that still allow good performance, but require easier-to-manufacture profiles.

The most obvious solution is to smooth the oscillations in Fig. 6.7 (b) and (c), green lines, by convolving with a smoothing function either (directly) the obtained profile or the target impulse response. The red lines in Fig. 6.7 show the result when a 20 ps FWHM hyperbolic secant function is used. The effect on the spectrum of the decoding grating is shown in (a). Sidelobes are suppressed below -30 dB, but a deformation inside the main lobe also occurs and reduced orthogonality can result from this. Increased insertion losses are also expected, since a larger portion of the incoming signal is filtered out by the decoder. The inverse scattering of this modified design is straightforward, since the modified target impulse response is shorter (see next Section for details); it is shown in Fig. 6.7 (b) and (c), red lines. Both amplitude and chirp ripples at high spatial frequencies are highly suppressed, and a simpler design is found. The oscillations in the apodisation function have a period of 8 mm or longer, and the chirp profile is smoothed out, with discrete phase jumps substituted by a continuous phase variation along the grating length. The maximum strength of the grating is not affected by the different design (\( R_{\text{MAX}} = 0.76 \)).

A quantitative analysis of the trade-off between manufacturing complexity and performance in an encoding/decoding system is given by the auto- and cross-correlation functions. The dispersion compensated designs in Fig. 6.7 are used as decoders, while the high reflectivity designs in Fig. 6.4 are considered for the encoder gratings. Fig. 6.8
shows the resulting decoded waveforms in case of both matched (a) and unmatched (b) operation. The blue line shows the output of a decoder without any dispersion compensation after propagation through the $L_{\text{link}} = 100$ km optical link. The auto-correlation peak is still recognisable, which means that the dispersion is still tolerated in this system due to its relatively small bandwidth. Indeed, no dispersion compensation was required in the experimental demonstration by Teh et al. [235], where propagation through 50 km of standard fibre was considered. However, the peak features are blurred and a 4 dB loss in the peak power is found. Moreover, both out-of-peak auto-correlation and cross-correlation extinctions are worsened by 1.7 dB and 2.95 dB, respectively. Dispersion compensation is not necessary, but highly preferable in order to preserve the system performances. The green lines show the result using the standard IS design with dispersion compensation included. The FWHM duration of the auto-correlation pulse is $T_{\text{FWHM}} \sim 43$ ps, practically equal to the dispersionless case for which $T_{\text{FWHM}} \sim 42$ ps, and the peak power is increased due to the stronger gratings obtained. The orthogonality of the considered codewords is preserved, showing an almost ideal operation despite the large clipping of the impulse response. Longer decoded waveforms are the main drawback, and the extinction on the next bit, i.e., for $\Delta \xi = 1600$ ps assuming a data rate $R = 622$ Mb/s, is reduced from 40 dB to 33 dB. The red lines show the corresponding correlations for the modified design. The convolution with the smoothing function results in a further increased duration of the main peak ($T_{\text{FWHM}} \sim 44$ ps), in a correspondingly decreased peak power (-0.76 dB), since energy is spread out over a slightly longer pulse, and in further improved correlation properties (a not negligible 1 dB gain is obtained). No improvement with respect to the waveform tail is found.

The second approach has to be preferred in this case, since it provides easier gratings to be manufactured and better performances. Indeed, a 20 ps smoothing function is still short with respect to the chip duration $T_{\text{chip}} = 50$ ps, and no averaging of the phase content of multiple chips takes place. Therefore, the orthogonality of the code is preserved, and the waveforms are completely (or mostly) rephased over the limited...
length of the grating due to its lower dispersion. The reduced peak power is not a real problem, since the decoded waveform has to be received by a photodetector with limited bandwidth. This problem is discussed in detail in Ghiringhelli and Zervas [220, Section 3.4], but it is obvious that the lower peak and larger $T_{FWHM}$ are partially compensating each other, so that similar results are obtained upon integration over the detector time.

### 6.4.2 How does dispersion compensation work?

In order to understand the chirp and amplitude ripples on a short spatial scale of the inverse scattered grating in Fig. 6.7, it is convenient to limit the analysis to the low reflectivity regime. As seen in Section 6.3.2, the main effect of an increased reflectivity is a non-uniform deformation of the strength of the grating, which is not important for the present discussion and is disregarded. Fourier transform arguments can therefore be applied. The target spectrum of the decoder is given by

$$H_{dec}(f) = H^*_{enc}(f)e^{j\beta_2(2\pi f)^2L_{\text{link}}},$$

(6.3)

where $H_{enc}$ is related to the CDMA code signature and the quadratic phase term refers to the contribution of dispersive propagation through a fibre with length $L_{\text{link}}$ and second-order dispersion $\beta_2$. The impulse response of the decoder is

$$h_{dec}(\xi) = h^*_{enc}(-\xi) \ast \frac{e^{-j\xi^2/2\beta_2L_{\text{link}}}}{\sqrt{-j2\pi\beta_2L_{\text{link}}}} = h^*_{enc}(-\xi) \ast h_{\text{disp}}(\xi),$$

(6.4)

where $h_{\text{disp}}(\xi)$ is a Gaussian phase function related to dispersion compensation. In the Born approximation, Eq. (6.4) also gives the shape of the resulting apodisation and chirp profiles, and they can be interpreted as the convolution of the original grating with $h_{\text{disp}}(\xi)$. The spatial extent of each chip is increased in order to balance the corresponding pulse spreading due to dispersive propagation. But the contributions of adjacent chips now overlap, and interference occurs making the resulting grating response more complex. It is convenient to analyse the contribution of each chip separately.

If the convolution integral in Eq. (6.3) is used for a single chip with constant amplitude and phase (rectangular pulse of duration $T_{\text{chip}}$),

$$h_{\text{chip}}(\xi) = rect\left(\frac{\xi}{T_{\text{chip}}}\right) \ast \frac{e^{-j\xi^2/2\beta_2L_{\text{link}}}}{\sqrt{-j2\pi\beta_2L_{\text{link}}}},$$

(6.5)

the function $h_{\text{chip}}$ shown in Fig. 6.9 (a) results, using the data of the example in Section 6.4.1, where $p = \xi/T_{\text{chip}}$ is the distance measured in number of chips. A sinc envelope shape is obtained (blue line) with zeros corresponding to

$$\Delta \phi_{\text{disp}}(T_{\text{chip}}) = \phi_{\text{disp}}(\xi_m + T_{\text{chip}}) - \phi_{\text{disp}}(\xi_m) = \frac{2\xi_m T_{\text{chip}} + T_{\text{chip}}^2}{2\beta_2 L_{\text{link}}} = 2m\pi,$$

(6.6)
Figure 6.9: (a) Amplitude (blue) and phase (green) of the dispersed-chip function $h_{\text{disp}}(\xi)$ for the example discussed in Section 6.4 (green line). Bottom axes: time $\xi$ (ps); top axes: number of chips $p$. A and B show the temporal extent of the various chips if the centre or the start of the grating are considered, respectively. (b) Periodicity $\Delta \xi$ of the phase of $h_{\text{disp}}$, expressed in number of chips $\Delta p = \Delta \xi/T_{\text{chip}}$.

where $\phi_{\text{disp}}$ is the phase of the dispersion compensation function $h_{\text{disp}}$. Eq. (6.6) is based on the fact that the amplitude is averaged to zero when a $2m\pi$ phase change is obtained over the integration time $T_{\text{chip}}$, irrespectively of the value of the absolute phase. This is rigorously true if the amplitude profile is constant and if the phase can be linearised within the chip duration. Minor corrections occur in the general case. Therefore, the zeros are

$$\hat{\xi}_m = \frac{1}{2} \left[ \frac{2m\pi \cdot 2\beta_2 L_{\text{link}}}{T_{\text{chip}}} - T_{\text{chip}} \right] \simeq \frac{2m\pi \beta_2 L_{\text{link}}}{T_{\text{chip}}} , \quad (6.7)$$

where the last expression is valid only if $4\pi \beta_2 L_{\text{link}} \gg T_{\text{chip}}^2$. If a chip pair separated by a number of chips $p \simeq \hat{p}_m = \frac{\hat{\xi}_m}{T_{\text{chip}}}$ is considered, interference is still present, but it is expected to be limited due to the large averaging. Unfortunately, the response of each dispersion compensated chip decreases with a $1/\xi$ dependence only, so that large $m$ values are theoretically necessary in Eq. (6.6) before this contribution becomes negligible. A large deviation from the intuitive shape is expected even for chips separated by $p \simeq (m + 0.5)\hat{p}_m$, $m \geq 1$, which correspond to the sidelobe peaks in the $\text{sinc}$ function. Finally, the largest amount of interference (lowest averaging) results from chips very close to each other ($p \ll \hat{p}_1$).

$\hat{\xi}_1 \gg 247$ ps is found in the case shown in Fig. 6.7, corresponding to a number of chips $\hat{p}_1 \approx 5$. All the chips inside the code signature are superimposed and interfere with each other. This situation is shown by the arrows in Fig. 6.9 (a), which show the extent of the grating when the contributions to a chip at the centre of the grating (A) or on the grating edge (B) are considered. In A, interference mainly occurs with the nearest chips and (with reduced efficiency) with the chips at the very end of the structure, while it is negligible between chips separated by $\sim 300$ ps. In B, it is shown that chips near the edges affect the whole grating structure, with peaks for $\xi = 400$ ps and $\xi = 700$ ps and zeros for 300 ps and 550 ps.
The relative phases of the different components have to be characterised in order to understand the interference patterns between different chips. On one hand, different chips carry different coding information, but this value is constant along the dispersion compensated waveform and its effect on interference ripple is negligible. On the other, the phase of the function $h_{\text{chip}}$ is rapidly changing, roughly following the average phase of $h_{\text{disp}}$ over the integration path, as shown in Fig. 6.9 (a), green dashed line. The periodicity of the phase variation is not constant due to the quadratic dependence of the dispersive phase with respect to $\xi$ shown in Eq. (6.4). It is ultimately related to the chirp introduced by dispersion compensation, which is equivalent to a stretched phase along $h_{\text{chip}}$. This phase is changing slowly if the chips in the considered pair are very close, but a large variation inside the duration of a single chip is already obtained for $\xi = 150$ ps (in the example considered here). Interference is constructive at the beginning of the chip and destructive at the end of it (for instance), causing a large variation in the apodisation profile on a scale now smaller than the chip length. This is better visualised by considering the distance $\Delta \xi$ between points with approximately equal phases, and therefore by the distance over which a full oscillation occurs, as in Fig. 6.9 (b). $\Delta \xi$ is given by

$$\Delta \phi_{\text{chip}}(\xi) = \phi_{\text{chip}}(\xi + \Delta \xi) - \phi_{\text{chip}}(\xi) \simeq 2\pi \implies \Delta \xi(\xi) = -\xi + \sqrt{\xi^2 + 4\beta_2 L_{\text{link}}}, \quad (6.8)$$

where the expression in (6.6) has been inverted with respect to $\Delta \xi$ and $\phi_{\text{chip}} \simeq \phi_{\text{disp}}$ has been assumed. This position is fairly correct for $\xi \neq \tilde{\xi}_m$, while almost discrete $\pi$ phase shifts occur for $\xi \simeq \tilde{\xi}_m$ due to the sign change in the $\text{sinc}$ envelope function. $\Delta p = \frac{\Delta \xi}{t_{\text{chip}}}$ applies. For $p < 5$, less than a full oscillation is found in the chip length, and only for $p > 10$ do more than 2 oscillations result. In the first case, similar interfering conditions are retained along the chip length, and small oscillations occur even if the amplitude of $h_{\text{chip}}$ is large. The spatial features in the apodisation profile are still of the order of the chip length. This is typical of the chips at the centre of the grating (A), and indeed the centre of the inverse scattered design in Fig. 6.7 (b), green line, shows the less oscillating structure. In the second case, which is characteristic for chips on the side of the grating (B), $p$ can be up to 10-15. The amplitude of the oscillations is limited, but the spatial features are of the order of 2 mm, which corresponds to $\Delta p \simeq 0.40$. Indeed, this occurs for $p = 13, 14$, i.e., on a peak in the interfering amplitude function ($\xi \sim 700$ ps).

The characteristics of the obtained design are therefore simply explained by considering a pair of interfering chips, even if all the chips are superimposed in this case. An intuitive explanation is obtained by visualising the interference as a sum of pseudo-random phasors (the interfering components), each of them with a different amplitude related to the $\text{sinc}$ envelope function and rotating at a different speed with time (due to the quadratic dependence of the phase). Most of these phasors are rotating at very low speed in the previous case (chip pairs close to each other). Only a few have much higher speed since most of the contributions in between have negligible amplitude due to the zeros in the envelope (see Fig. 6.9). These last components are expected to largely
determine the fast ripple in the sum vector.

If the amount of dispersion is increased or the chip duration is decreased (longer $L_{\text{link}}$ or larger bandwidth to be compensated, respectively), $\xi_m$ increases and more chips are expected to effectively interfere since less averaging is present over the chip duration. In this case, the number of significant chips is not related to the $1/\xi$ dependence of $h_{\text{disp}}$, but to the actual grating length $L_{\text{gr}}$ and therefore by the time window $\Delta\xi_{\text{tot}}$ over which dispersion compensation occurs. Clipping $h_{\text{disp}}$ results in a partial compensation of the dispersed waveform, and therefore the compensation process is less effective. The use of shorter chip durations is critical under grating length restrictions, and a large deviation from the theoretical performances of the code is expected in a real dispersion compensated system with small $T_{\text{chip}}$.

Considering the periodicity of the apodisation ripple in the design grating, both the amplitude and the phase oscillations in Eqs (6.6) and (6.8) are on a shorter length scale if the dispersion is lower ($L_{\text{link}} < 100$ km in the example in Fig. 6.9). Lower dispersion produces superstructured gratings that are more complex to manufacture, since the interfering conditions are already rapidly changing within the same chip for very low values of $\rho$. Therefore, features on a 2 mm scale are expected over all the grating length if, for instance, $L_{\text{link}}$ is set to 20 km ($\rho_1 \simeq 1$), with possibly faster features on the grating’s edges. Conversely, both $\xi_m$ and $\Delta\xi(\xi)$ are increased if the amount of dispersion to compensate is increased ($L_{\text{link}} > 100$ km), with a faster growth in $\xi_m$. More chips are interfering even though only the first lobe of the amplitude function in Fig. 6.7 is considered, and the phase variation within a single chip is reduced for a given $\xi$ (lower chirp indeed corresponds to higher dispersion). Therefore, the interference patterns are less oscillating and simpler gratings are obtained, with a clearer disentanglement between the effect of the code phase shifts and the linear chirp related to dispersion compensation. In particular, this occurs when the phase change is smaller than the discrete phase shift, and the phase jumps are easily recognised within the phase pattern.

In general, a typical optical link has $L_{\text{link}} = 100$ km, so that the previous discussion has a limited applicability. Conversely, the analysis of the effect of changing the chip duration $T_{\text{chip}}$ is more interesting. The chip duration is directly determined by the code length $L$ and by the transmission data rate $R$ in a dispersion free system. The temporal duration of the decoded waveform is given by $\tau_{\text{dec}} \simeq 2LT_{\text{chip}}$, and

$$\tau_{\text{dec}} < T_{\text{bit}} \Rightarrow 2LT_{\text{chip}} < \frac{1}{R}$$

(6.9)

applies when no ISI is present at the receiver and the bandwidth of the system is minimised. The presence of dispersion compensation makes the choice more flexible. The grating length has to be relaxed with respect to dispersionless operation in order to recompress the dispersed waveform. But no perfect compensation is possible using a grating length $L_{\text{gr}}$ compatible with manufacturing and packaging requirements, according to the results in Eqs. (6.1) and (6.7). A trade-off must be found by tuning $T_{\text{chip}}$, taking into account that the phase periodicity $\Delta\xi(\xi)$ is not affected by $T_{\text{chip}}$ for a given
separation $\xi < LT_{chip}$. It is only determined by the dispersion $\beta_2 L_{link}$, according to Eq. (6.8). A longer $T_{chip}$ gives shorter $h_{chip}$ functions and, therefore, better dispersion compensation. But changes in $\phi_{chip}$ are faster if the same L is used since chips are placed further apart, and an increased ripple due to interference results. A shorter $T_{chip}$ is expected to give worse performance, but smaller phase shifts and easier grating features are forecasted.

Optimisation of $T_{chip}$ has to be performed by targeting a practically manufacturable SSFBG. Complex apodisation and chirp features have not been demonstrated on a scale close to the theoretical resolution of the system ($\sim 200 - 300 \mu m$), as for purely phase shifted structures [234] or purely apodised structures [236]. In the following, a $\sim 2$ mm scale for a full oscillation is typically assumed (corresponding to $\sim 20$ ps). This requirement is not very tight, but it allows the structure to be both chirped and apodised at the same time without pushing the manufacturing capabilities to the ultimate limit. $T_{chip} > 20$ ps is also consistent with the bandwidth that can be compensated using $L_{gr} = 120$ mm long grating. It corresponds to a $\Delta f = 100$ GHz bandwidth between the first zeros, not much above the limit set by Eq. (6.2). A $T_{chip} = 20$ ps limit is very strict, and reduces the possibility to use multi-function SSFBGs to a few cases of limited performance and interest (short code lengths or low transmission rates). Moreover, a non-trivial relation exists between chip duration and oscillations in the designed apodisation profile, and the simple analysis used previously is not expected to be applicable to shorter values of $T_{chip}$. All the summing phasors have similar amplitudes due to the larger $\tilde{\xi}_1$ (especially for very low $T_{chip}$). They are also characterised by a small rotating speed, since the total temporal distance $LT_{chip}$ is small with respect to $\tilde{\xi}_1$ in Eq. (6.7). Variations in the sum vector are not easily associated with any particular component, and the presence of the pseudo-random phase shifts related to the code makes things even more cumbersome.

Therefore, the spectral content of the resulting apodisation function is computed.
by performing a Fourier transform on $|h_{\text{dec}}(\xi)|$ defined in Eq. (6.4). The periodicity $\Delta \xi_{-20 \text{ dB}}$, that gives a 20 dB attenuation in the spectrum with respect to the peak DC component, is shown in Fig. 6.10 (a) for different values of $T_{\text{chip}}$ (from 4 ps to 60 ps), code lengths $L$ (from $L = 15$, yellow lines, to $L = 127$, magenta lines), and actual codewords, given the random nature of the codes. $\Delta \xi_{-20 \text{ dB}}$ cannot be considered as a real cut-off frequency, but it gives an indicative estimation of the minimum periodicity that is required to manufacture the grating. In (a), the ripple periodicity at 20 dB attenuation is plotted with respect to the overall duration $LT_{\text{chip}}$ of the dispersionless waveform. The points do not sit on a well defined trendline due to the random nature of the problem, but a clearly defined behaviour is found. As a “rule of thumb”, $\Delta \xi_{-20 \text{ dB}} \propto (L \cdot T_{\text{chip}})^{-1}$, which empirically shows that the maximum distance between two interfering points is the only important parameter even in this case, irrespectively of the number of chips and of the bandwidth of the grating. The explanation is clear in (b), where the phase difference $\Delta \phi_{\text{MAX}}$ is defined as

$$\Delta \phi_{\text{MAX}} = \phi_{\text{chip}}(L \cdot T_{\text{chip}} + \Delta \xi_{-20 \text{ dB}}) - \phi_{\text{chip}}(L \cdot T_{\text{chip}}) = \frac{2LT_{\text{chip}} \Delta \xi_{-20 \text{ dB}} + \Delta \xi_{-20 \text{ dB}}^2}{2 \beta_2 L_{\text{link}}},$$

and therefore it is the maximum phase variation experienced over a single ripple period, when the two interfering contributions are placed further apart. Typically, $\Delta \phi_{\text{MAX}} \simeq 2\pi$, especially if longer code lengths are considered ($L = 63$, black lines, and $L = 127$, magenta lines). This corresponds to a single oscillation in the ripple. Larger deviations from this simple formula are obtained for shorter codes, with $\Delta \phi_{\text{MAX}} \simeq \frac{3\pi}{2}$ when longer chips are used ($LT_{\text{chip}} \gtrsim 800$ ps), and $\Delta \phi_{\text{MAX}} \simeq \frac{3\pi}{2}$ when shorter chips are used ($LT_{\text{chip}} \lesssim 400$ ps). The first case is actually equivalent to the analysis in Fig. 6.9, where it was shown that $\Delta \phi_{\text{chip}} \simeq 2\pi$. This apparent disagreement is possibly due to the approximated definition of the cut-off frequency at -20 dB.

From Fig. 6.10 (a), $LT_{\text{chip}} < 700$ ps is required in order to have a physically manufacturable grating, and actually $LT_{\text{chip}} < 350$ ps is convenient in order to make the superstructure easier (it corresponds to $\Delta \xi \sim 30$ ps). This fact sets an important limitation on the possible choices of code lengths. Indeed, $L = 255$ (not considered so far) is demonstrated not to be a feasible solution, since the corresponding $T_{\text{chip}} \simeq 1.5$ ps is below the resolution of the writing set-up, and the corresponding dispersionless encoding gratings cannot be manufactured. $L = 127$ also results in very demanding encoding structures, even though $T_{\text{chip}} \simeq 3.1$ ps SSFBGs have already been experimentally demonstrated [234]. The corresponding duration of the dispersed waveform (between the first zeros) is given by $2\xi_1 \simeq 8000$ ps, which is 7 times longer than the actual compensation length provided by the $L_{\text{gr}} = 120$ mm grating considered here. It is therefore convenient to consider a shorter code length $L = 63$ as a limiting case. The reduced dispersion results in a better recompression of the energy by the decoder, with a smaller performance degradation with respect to the ideal case. A longer code naturally provides better cor-
relation properties, so that worse dispersion compensation does not necessarily imply worse final performance. A significative example of this balance is given in Section 6.5, where the performances of the $L = 63$ and $L = 31$ codes are compared for a given manufacturing complexity of the decoder grating, i.e., for a given $LT_{chip}$. $T_{chip} = 6.4$ ps or longer makes the manufacturing of the encoder gratings relatively straightforward, reducing the overall complexity of the system and ultimately its cost. Finally, this choice is compatible with previous experimental demonstrations, so that a reduced number of newly manufactured gratings is necessary to prove experimentally the feasibility of the proposed designs. Despite the fact that no experimental demonstration has been performed to date, this aspect has also to be considered in the definition of a target design.

6.4.3 $L_{gr}$-limited designs

According to the previous discussion, the design of dispersion compensating CDMA gratings has been applied to a system using $L = 63$ chips long codewords. A $T_{chip} = 6.4$ ps chip duration and a $T_{pulse} = 1.8$ ps starting pulse duration are assumed in the design and performance evaluation in order to keep the manufacturing easy and to make the gratings compatible with previous experiments. Other important parameters are the total grating length, which is set to $L_{gr} = 120$ mm in order to be compatible with available packaging modules, the maximum target reflectivity, which is set to $R_{MAX} = 0.9$, and the total dispersion of the system. 100 km of Corning SMF-28 fibre is the optical span to be compensated. The nominal dispersion of this fibre is $D_\lambda = 15.68$ ps/nm/km, and an overall dispersion of 1568 ps/nm is obtained. Using Eq. (6.2), the compensated bandwidth is $\Delta \lambda_{disp} \simeq 0.74$ nm, while the bandwidth of the main lobe of the spectrum is $\Delta \lambda \simeq 2.52$ nm. A large part of the energy of the dispersed waveform cannot be rephased by the dispersion compensating decoder grating because of this length requirement, and a large penalty in the quality of the encoding-decoding process is already forecast. The effective refractive index of the grating is $n_{eff} = 1.45$, and the central wavelength is $\lambda_{Bragg} = 1555$ nm. The proposed CDMA system operates in perfectly dispersion compensated conditions at $R = 1.25$ Gb/s without any ISI at the receiver.

If the results obtained in the design of the $L = 16$ CDMA grating in Section 6.4.1 are considered, the preferred solution is to convolve the target impulse response of the grating with a suitable smoothing function before performing the inverse scattering. But the requirements of the present problem are more strict and, ultimately, different. Fig. 6.10 shows that the obtained superstructures are expected to be easier to manufacture, since the temporal duration of the encoded waveform is reduced from $LT_{chip} \simeq 800$ ps to 400 ps. It corresponds to a $\sim 50\%$ increase in the fast ripple $\Delta \xi_{-20 \, \text{dB}}$ of the apodisation function. At the same time, reducing the spectral content of the target grating to -30 dB at the edges of the compensated bandwidth requires the use of a smoothing function with FWHM duration $T_{smooth} = 15$ ps, which is 2.3 times larger than the chip duration itself. A massive averaging of the code properties is obtained, together with
Fig. 6.11 shows the corresponding result, using the same inverse scattering parameters as in Section 6.4.1. In (a), the blue line refers to the ideal spectrum to be synthesised, while the red line is the spectrum obtained using the described approach. The spectral filtering obtained is apparent, and a large part of the starting energy of the pulse is lost even if the peak reflectivity is raised to $R_{MAX} = 0.86$. Indeed, the peak power in the decoded waveform is -37.46 dB, where this value is normalised to the peak power of the starting pulse, and a $R_{gr} = 0.25$ encoder design based on the Born approximation is assumed. The corresponding value is -33.15 dB for a low reflectivity system based on $R_{gr} = 0.25$ encoding and decoding gratings and with independent dispersion compensation, i.e., 4.31 dB higher despite the smaller peak reflectivity of the decoder. The corresponding FWHM durations of the auto-correlation peak pulse are $T_{FWHM} = 17.2$ ps and $T_{FWHM} \sim 6$ ps. The increased duration is due to the convolution with the smoothing pulse, as easily confirmed by simply checking the FWHM duration of a pulse resulting from the convolution of a $T_{smooth}$ and a $T_{chip}$ pulses. At the same time, the value of auto-correlation extinction drops from a theoretical 12 dB to an unsatisfactory 9.09 dB. The auto-correlation extinction only refers to the visibility of the peak in case matched operation occurs, and is strictly important only for synchronisation issues in a CDMA system. But it also gives indications about how the orthogonality of the code signatures is preserved during the encoding/decoding process and during transmission. A large degradation means that bad system performances are obtained when multiple access interference is present [220].

In Fig. 6.11 (b) and (c), the corresponding apodisation and chirp designs are shown. The green lines are the result of the Born approximation (the profile is proportional to the theoretical impulse response), while the red line is the actual inverse scattered grating obtained from the truncated impulse response. The obtained superstructure is limited well within the length requirements, and a clear disentanglement between the
Figure 6.12: Design of multifunction gratings with integrated dispersion compensation and CDMA decoding. $L = 63$, $T_{chip} = 6.4$ ps, $D_{TOT} = -1568$ ps/nm. The grating length is limited to $L_{gr} = 120$ mm by clipping the ideal profile. (a) Spectrum of the target grating (blue) and of the designed superstructure (red). (b) Apodisation profile $\delta n(z)$. (c) Chirp profile $\Delta \Lambda(z)$ (nm).

The effect of chirp and the code-related phase shifts is found. The IS profile is different from the low reflectivity approximation only in the very last section, showing an increased ripple that helps to cancel multiple scattering from the main peaks. This feature is even more apparent in (c), since the different chirp slope clearly shows that contributions from spectral components reflected earlier in the grating are dominating over the low spectral components reflected only at the end. A large part of the apodisation profile is at very low levels of refractive index modulation, which is not always an advantage given the limited dynamic range of the writing set-up and, especially, its intrinsic noise floor (see Section 2.4 and [109]). A large relative error is expected, also because tight focussing of the writing beam (and therefore reduced averaging) is necessary in order to reproduce the ripple with high spatial frequency.

The first approach described in Section 6.4.1 gives more satisfactory results in this case. It is convenient to simply compensate all the dispersion introduced by propagation and, afterwards, to clip the obtained design to $L_{gr}$. This procedure seems very coarse, but any effect related to imperfect smoothing of the desired response, or numerical aliasing in the design process, is found to be negligible with respect to the very large dispersion to be compensated (compared to the available length of the grating). The final performance is simply determined by the quality of dispersion compensation. Fig. 6.12 shows the corresponding grating obtained with this method. The spectral content of the grating is always limited (see (a), red line) given the necessity to spread the different wavelengths in time, but an almost perfect reconstruction of the spectral characteristics of the code is now possible over all the compensated bandwidth $\Delta \lambda_{disp}$. Reduced insertion losses are obtained even though the actual peak reflectivity is $R_{MAX} \approx 0.80$ (the peak power is now up to -30.96 dB), and the auto-correlation extinction is actually increased to 12.90 dB. On the other hand, the FWHM duration of the peak pulse is still longer than in the ideal case and equal to $T_{FWHM} \sim 11.1$ ps. A possible explanation is given by the partial dispersion compensation experienced by the components at the edges of the grating bandwidth ($\delta \lambda \approx \pm 0.45$ nm in Fig. 6.12 (a)).
The corresponding design is shown in Fig. 6.12 (b) and (c). The blue line refers to the ideal reconstructed grating, while the red line represents the clipped part. The choice of the starting point of the grating depends on both the actual apodisation function and on the spectral characteristics. On one hand, it is convenient to avoid amplitude discontinuities by clipping the ideal grating close to a zero. A short smoothing function (typically, with a hyperbolic tangent shape) is used to further reduce Gibbs oscillations in the spectrum, even though the corresponding effect is minor. On the other hand, the starting point determines which portion of the spectrum of the code is actually dispersion compensated. Changing it from $z = 0$ in Fig. 6.12 moves the spectrum to the blue side (short wavelengths) if $z > 0$ and to the red side if $z < 0$, as is clear from the chirp profile in (c). Different peak powers and different extinctions are obtained, with variations up to 1 dB by fine tuning the position $z$. They largely depend on the pseudo-random pattern of the code signature used, so that a codeword-by-codeword optimisation is necessary. In the example presented, the central part of the spectrum is used ($\delta \lambda \simeq 0$ is the central wavelength), but it is possible that shifted spectra are convenient in different cases.

Unfortunately, the search for the optimal solution in terms of best codeword orthogonality and largest peak powers typically requires very time consuming algorithms to be performed, given the huge number of parameters involved. Even assuming that the grating’s design parameters are fixed, an optimised subset of the desired cardinality $|C|$ (depending on the actual system implementation) has to be identified, and this already presents a binomial number of possible solutions $\binom{L}{|C|}$. Fast algorithms are available which enable the reduction in the number of configurations to be checked (for example, see [213]). But the available combinations rise to $\binom{L}{|C|} n_{\text{clip}}$ if each codeword has to be scanned along $z$ to determine the best clipping position, where $n_{\text{clip}}$ is the number of starting positions considered. This is not a tractable problem already for $L = 63$ and $|C| = 4$, and even suitably modified fast search algorithms are still very demanding. Therefore, a suboptimal scheme was used to identify a proper subset, where the clipping position was individually optimised for each codeword by maximising the out-of-peak extinction, and the best subset was chosen from these codewords using the criteria described in Appendix F. A detailed description of the corresponding performances is given in the next Section.

6.5 Practical application of dispersion compensating CDMA gratings

6.5.1 Performance comparison

An optimised subset of cardinality $|C| = 4$ of dispersion compensating CDMA gratings based on the $L = 63$ quadrupolar code is obtained in Appendix F. The chip duration is $T_{\text{chip}} = 6.4$ ps and the total dispersion to be compensated is $D_{\text{TOT}} = -1568$ ps/nm. Here, the corresponding performance is analysed and compared with standard designs.
The encoding gratings are based on a simple Fourier transform design, resulting in a phase-shifted only superstructure with distance between adjacent chips $\Delta L_{\text{chip}} = 660 \, \mu\text{m}$. Either (nominal) $R_{gr} = 0.25$ (2) or $R_{gr} = 0.9$ (3) gratings are possible, with corresponding $\delta n = 2.25 \times 10^{-5}$ or $\delta n = 4.27 \times 10^{-5}$. Alternatively, the high reflectivity inverse scattering approach is possible (4), as seen in Section 6.3.2. An example of a 63 chip, high reflectivity encoder is shown in Fig. 6.6. The resulting superstructure has a complex apodisation profile, but amplitude variations are not large and the chirp profile is rather simple since the phase shifts are still discrete. The grating complexity is therefore comparable (or lower) than already published examples [236].

Table 6.3 summarises the most relevant parameters of the various systems. The peak power values peak$\{I_{Q_i-Q_i^*}\}$ are again normalised to the peak value of the transmitted pulse, and the actual spectral response of the grating is used in every case. (1) shows the result for a system comparable with configurations already demonstrated. Fourier transform gratings are considered, and independent compensation of the dispersion of the optical link is assumed. A large dependence on the actual codeword used is found, with almost 2 dB imbalance in the peak received power in matched conditions $(Q_i-Q_i^*)$, and even larger inhomogeneity in the correlation properties of the subset. As a means of comparison, the pure code provide a minimum cross-correlation of $\sim 13$ dB, which is practically maintained considering the practical realisation using SSFBGs for this subset. In (2), the system implementation is similar to the one discussed in Section 6.4.3, with dispersion compensating decoders and low reflectivity encoders. The high reflectivity design guarantees an increased received power, even though the actual advantage is limited due to the imperfect dispersion compensation possible in these conditions. This is more apparent in the cross-correlation properties, since the corresponding loss is high and close to 2 dB, and results in a large penalty for operation in asynchronous CDMA conditions [220]. Conversely, a much better uniformity within the subset is found as expected, since the subset is optimised for operation in dispersion limited conditions.
The maximum power imbalance is limited to $< 0.4$ dB, even though this uniformity is penalised with a slightly lower peak power, as shown by Table F.2 in Appendix F. Further degradation is expected if stronger encoding gratings are considered (3), but a limited 0.4 dB decrease in the orthogonality of the considered codewords is obtained. This variation is not negligible, but it can be tolerated given the 5 dB advantage in the received power correspondingly obtained. This is significant especially if the optical and electrical noises in the system are the dominant cause of performance limitation and a “few users” CDMA configuration is used (as typically occurs in laboratory demonstrations). The use of high reflectivity IS gratings is proven in (4) to largely solve this problem, restoring the cross-correlation properties to the dispersion limited value and further boosting the peak power at the receiver. The increased manufacturing complexity is justified in this case. Finally, the duration of the auto-correlation peak $T_{FWHM}$ is not significantly changed under the different system configurations proposed, showing that it is mainly related to partial dispersion compensation as previously discussed. At the same time, the extinction of the correlation traces on the next bit (separated by 800 ps if a data rate of $R = 1.25$ Gb/s is considered) is still kept below 36 dB in all the configurations, and the degradation with respect to a dispersion free case, as described in Section 6.3.2, is not significant. This confirms that the energy missing on the correlation peak is spread out by dispersion all over the temporal duration of the decoded waveform, providing an almost uniform degradation of the correlation properties.

The comparison of the correlation data in Table 6.3 with the theoretical values (see [220, Table A.1], valid for pure code, discrete operation) suggests that the $\sim 2$ dB loss is connected to an actual shortening of the effective length of the code. The minimum cross-correlation is $\sim 11$ dB if a pure code with $L = 31$ is considered, in agreement with the previous data. However, the analysis in Section 6.4.2 is not adequate to explain this result. A different point of view can be used, still based on the Born approximation. Perfect dispersion compensation is assumed to occur over all the signal’s bandwidth, and the limited grating length is assumed to actually window it to $\Delta\lambda_{disp}$, as from Eq. (6.2). This means that the ideal decoded waveform is convolved with a smoothing function whose actual shape depends on the kind of spectral windowing ($sinc$ shape for a perfect rectangular window, more smoothed and not oscillating if a certain roll off is present, as shown in Fig. 6.11). A possible estimation of the width of the main lobe of this function is given by $1/\Delta f_{disp}$, which corresponds to $\sim 10.8$ ps in the present case. Therefore, pairs of chips are blurred together by convolution, decreasing the orthogonality of the code signatures. If an approximated linear relationship is assumed, the effective length of the code due to partial dispersion compensation is given by

$$L_{eff} = \frac{T_{chip}}{1/\Delta f_{disp}} = 63.6 \frac{.4}{10.8} \approx 37.3,$$

which roughly agrees with the obtained results despite the gross approach.

On this basis, the design of an equivalent subset of cardinality $|C| = 4$ was considered
using $L = 31$ long codewords and $T_{\text{chip}} = 12.8$ ps. Similar spatial features are expected for the dispersion compensating decoder grating due to the similar overall length of the encoded waveform. Conversely, easier fabrication of the encoder grating is obtained due to the longer chip duration and to the reduced number of phase shifts. A new, optimised subset is derived using the same approach applied in Appendix F. The total dispersion $D_{\text{TOT}}$ and the grating length $L_{\text{gr}}$ are kept unchanged, and low reflectivity encoding gratings are considered. The peak power of the auto-correlation trace is lowered to $-33 \pm 0.2$ dB, with a 2 dB loss with respect to the $L = 63$ case, but the duration of the peak pulse is correspondingly increased from $\sim 11.5$ ps to 14.3 ps. This means that the amount of energy recompressed in the peak is roughly equal, and similar results are obtained at the output of a real (finite bandwidth) photodetector. The cross-correlation properties are also practically unchanged, being bounded between an optimum 12.45 dB and a limiting 11.03 dB. The reduced bandwidth of the grating allows an almost perfect compensation of dispersion over the main lobe of the encoded signal, and preserves the code properties with high fidelity.

Therefore, the need for using longer codes seems to be not justified in these working conditions, but the longer code length $L = 63$ can still be necessary if a higher number of orthogonal codewords are needed. The results in Table F.2 show that larger subsets are penalised by lower cross-correlations, and larger penalties are expected when the code subset gets closer to the maximum cardinality of the code $L + 1$ due to the lack of uniformity of the continuous, grating-based realisation.

### 6.5.2 Sensitivity to operational parameters

Sensitivity of the obtained design to different parameters in system operation is another interesting aspect to be checked. In particular, sensitivity to a mismatch in the propagation length to be compensated $L_{\text{link}}$, to the duration of the starting input pulse $T_{\text{pulse}}$, and to a mismatch in the central wavelengths of the encoding and decoding gratings are analysed.

Fig. 6.13 shows the variation of the parameters already outlined in Table 6.3 with the length $L_{\text{link}}$ to be compensated. The value of $L_{\text{link}}$ is changed from 96 km to 104 km, giving a maximum mismatch $D_{\lambda} \Delta L_{\text{link}} \sim 62$ ps/nm with respect to the nominal dispersion compensation in the decoding gratings ($L_{\text{link}} = 100$ km). The black dots refer to the standard implementation with external dispersion compensation and low reflectivity gratings (1), while the blue crosses, green circles, and red triangles refer to the systems (2), (3), and (4), respectively, in Table 6.3. The variation in the duration of the decoded pulse in matched operation is shown in (a). In the perfectly compensated system, the pulse gets very large with a small residual dispersion, with an oscillatory behaviour possibly due to interference between different chips. If a dispersion compensating decoder is used, the pulse broadens only slightly with up to a 2 km mismatch. Indeed, the typical oscillations in the decoder impulse response are already above 20 ps to simplify grating manufacture (as seen at the end of Section 6.4.2), so that $D_{\lambda} \Delta L_{\text{link}} \Delta \lambda_{\text{disp}} = 23$ ps is
still comparable and has limited effect. Conversely, very distorted pulses are obtained with a 4 km mismatch, and their width is massively increased for certain code signatures (up to $\sim 45$ ps, consistently with $D_\lambda \Delta L_{link} \Delta \lambda_{disp} = 46$ ps). Residual dispersion becomes the dominant effect. The corresponding values of cross- and auto-correlations, shown in (c) and (d), respectively, are largely compromised. The received signal is still correctly identified, but extinction is reduced to 4-5 dB only, and no operation is possible in multiple user configurations. This feature is characteristic of the code itself, since the same trend is found in all the four configurations considered. In (b), the peak power of the decoded waveform is shown. Again, differences among the different systems are very limited, with the exception of the standard case (1) that typically presents the worst degradation. An almost linear dependence with $\Delta L_{link}$ is found, with a decreased performance uniformity ($\sim 0.6$ dB/km for the best case and $\sim 1.15$ dB/km for the worst case).

The random nature of the used codes does not allow recognition of well defined trends, and a case-by-case optimisation is required. An almost perfect matching between the design specifications and the actual parameters is necessary to provide acceptable performances. A 1 km mismatch already gives a variation of 1 dB in the correlation properties. This limited tolerance is even more problematic due to the impossibility to
tune the actual amount of dispersion compensation. Any inverse scattering design is specifically tailored for a certain operation, and the complex interplay between all the different dispersed chips changes completely by changing the total amount of dispersion (as apparent from Section 6.4.2). Conversely, the standard design is more sensitive to changes in the total dispersion of the link, but optimal performance can be recovered with use of tunable dispersion compensating devices [237]. Therefore, the system with multi-function SSFBG and a reduced number of components is penalised by a lack of agility of the resulting configuration.

The initial pulse duration $T_{pulse}$ is another parameter whose value can be changed during system operation. A 1.8 ps hyperbolic secant source was considered in the designs in Section 6.4.3 to keep compatibility with the actual mode-locked laser used in previous experimental demonstrations. A longer pulse duration results in a reduced bandwidth and in reduced insertion losses for the gratings, since spectral filtering is reduced. However, performance degradation is expected. The effect is similar to the one discussed in Fig. 6.11. Indeed, all the impulse responses of the cascaded components are convolved in a linear system, and the starting pulse itself can be compared to the smoothing function applied to the decoder. This intuitive understanding is better analysed in Fig. 6.14, where the same configurations previously considered are compared for different values of $T_{pulse}$. In (a), the duration $T_{FWHM}$ is practically constant for the
dispersion compensated decoders for $T_{\text{pulse}} \leq 4$ ps, and a linear dependance is found for longer pulses. Negligible differences between the different configurations are found except (1), which shows an always increasing linear trend. In the first cases, the pulse duration is mainly limited by the imperfect dispersion compensation for short $T_{\text{pulse}}$ (as discussed in Section 6.4.3), and therefore is not a function of the original pulse. The bandwidth of the encoding/decoding system is reduced by increasing $T_{\text{pulse}}$, and the dispersion-related distortion becomes negligible with respect to the actual convolution of the original pulse with the chip duration. This is confirmed by the fact that the inhomogeneity of this parameter within the code subset is not worsened by increasing $T_{\text{pulse}}$.

In Fig. 6.14 (b), the peak power in the auto-correlation signal is shown to grow less than linearly (i.e., less than logarithmic in dB) with $T_{\text{pulse}}$, and saturation occurs. A worse recompression of the available energy occurs for longer $T_{\text{pulse}}$ because of the increased degree of averaging of the code properties. This effect is partially balanced at short $T_{\text{pulse}}$ by the better dispersion compensation. In analogy with Eq. (6.11), an effective length of the code $L_{\text{eff}}$ can be introduced, but the available bandwidth is determined in this case by the bandwidth of the starting pulse and not by the actual dispersion compensated bandwidth. The spreading of the values gets larger with $T_{\text{pulse}}$, meaning that a reduced homogeneity in the properties of the chosen subset is obtained. This is also confirmed by (c), where the cross-correlation properties are shown. For the standard configuration (black dots), the expected progressive degradation is found, but in the other cases the minimum value gets worse with increasing averaging, and, surprisingly, the maximum value is actually increased. Pairs of code signatures are more orthogonal when a longer pulse is used. The subset is therefore no longer optimised, and it is also clear that cross-correlation is limited by the residual dispersion in some cases (maximum values) and by the level of code averaging in other cases (minimum values). Again, no general trend can be found due to the pseudo-random characteristics of the codes. Finally, (d) shows the extinction in the auto-correlation trace, which confirms the comments made with reference to (b). An initial improvement is obtained, followed by a limited worsening. No increased spreading of the values is found. Conversely, the standard system shows a continuous degradation, since no improved dispersion compensation occurs to balance the effect of code averaging.

The previous discussion shows that $T_{\text{pulse}} = 1.8$ ps is not optimal for the system parameters that were chosen, i.e., given the chip duration and the amount of dispersion to compensate for. $T_{\text{pulse}} \simeq 4$ ps provides a further 5 dB improvement in the received power with negligible changes in the correlation properties of the subset. Further increase in $T_{\text{pulse}}$ has to be considered with reference to the actual optical signal-to-noise ratio on the channel, since it helps in decreasing the insertion losses of the CDMA components, but this is also paid for by a partially reduced CDMA performance.

The actual wavelength of the pulsed source and its possible chirp are further factors that can affect system operation. However, their final effect is rather unimportant. In the
first case, a shift in the spectrum of the starting pulse results in a slightly different balance between the different spectral components in the decoded signal. Small variations in the correlation properties are expected. If short $T_{\text{pulse}}$ are considered, the spectrum is largely flat over the whole CDMA bandwidth, and the small sensitivity to $T_{\text{pulse}}$, shown in Fig. 6.14, provide a further (indirect) confirmation. As far as the chirp is considered, the pulsed source or the external modulation of a CW source can possibly result in a residual, typically linear, chirp in the pulse. This means that, for a given pulse duration, the actual spectrum of the source is larger than expected (see Agrawal [238]). Therefore, increased insertion losses are obtained, with up to 5.3 dB loss for $C = 2$ (where the chirp parameter $C$ is defined in [238, page 67]). This loss is almost code independent, consistent with the results in Fig. 6.14 (b) for very short pulses. Indeed, chirped pulses can be thought of as shorter pulses given their bandwidth and disregarding the actual temporal chirp. Actually, this assumption is largely correct, since the chirp itself can be assimilated to a further propagation in the optical link corresponding to an effective distance

$$L_{\text{link,eff}} \simeq \left( \frac{T_{\text{FWHM}}}{1.763} \right)^2 \frac{1}{\beta_2C} = \frac{L_D}{C} = 24 \, \text{m}, \quad (6.12)$$

where $L_D$ is the dispersion length, $C = 2$, and the same parameters used throughout this Section are considered. The results of Fig. 6.13 confirm that this further compensation mismatch is absolutely negligible.

Sensitivity to a wavelength mismatch between the encoder and the decoder gratings is the final parameter to be checked. Perfect wavelength tuning is possible during system calibration, but wavelength drifts due either to environmental changes or to simple aging of the gratings [239] are likely to happen in the system. Fig. 6.15 shows the trends of the same parameters already analysed for $L_{\text{link}}$ and $T_{\text{pulse}}$. The wavelength detuning is varied from $\delta \lambda = -12 \, \text{pm}$ to $\delta \lambda = +12 \, \text{pm}$. The larger drift is already sufficient to completely destroy the orthogonality of the considered codewords, at least for certain codeword pairs. Large dependence of the correlation properties on the actual pair is found in (c) and (d), with up to 5 dB imbalance in the worst cases. As usual, this is related to the random nature of the code signatures used and to the imperfect dispersion compensation. However, wavelength misalignment can actually help in certain fortunate cases. An acceptable drift is given by $\delta \lambda = \pm 4 \, \text{pm}$, since the performances are degraded by (only) $\sim 1 \, \text{dB}$. At the same time, also the decoded pulse duration $T_{\text{FWHM}}$ (a) and the corresponding peak power (b) are decreased, showing that a noise-like waveform is obtained after decoding due to the collapse of the properties of the code itself. This is typical of DS-CDMA operation, while the use of complex grating designs does not produce any degradation with respect to simpler system implementations. The reason is apparent if the spectrum of the encoded waveform is considered, as in Fig. 6.12. The typical period of a spectral oscillation is $\delta \lambda \simeq 20 \, \text{pm}$, so that any shift larger than 10 pm is sufficient to completely scramble the matched filtering operation. With analogy to
the discussion in Section 6.4.2, this periodicity ultimately depends on the total temporal length of the encoded waveform $L_{T_{\text{chip}}}$, since it gives rise to the fastest ripple components related to the interference between the different chips. A $2\pi$ phase shift is required to produce a full oscillation, and

$$\Delta \phi = \Delta \omega \Delta \xi = \left( \frac{2\pi c}{\lambda^2} \delta \lambda \right) (LT_{\text{chip}}) = 2\pi \quad \Rightarrow \quad \delta \lambda = \frac{\lambda^2}{cLT_{\text{chip}}} \approx 20 \text{ pm} \quad (6.13)$$

in the case under study. This requirement is particularly important when long gratings are considered (i.e., longer waveforms). Both the acceptable wavelength shift is tighter, according to (6.13), and the stabilisation of the device is more complicated. The previous discussion also shows that CDMA systems are even more demanding than WDM systems from a stabilisation point of view. Stability requirements are simply shifted from the sources to the encoding and decoding devices.

### 6.6 Conclusions

Inverse scattering techniques have been applied in this Chapter to the design of superstructured gratings for application in direct-sequence CDMA systems. The code is directly imprinted in the time domain by changing either the amplitude or the phase
of the optical waveform reflected by the grating. The intrinsic coherence of fibre Bragg
gratings is directly transferred onto coherent waveforms, so that powerful bipolar code
signatures are effectively used to improve the performance of the system with respect to
standard optical implementations based on simple intensity coding (OOCs).

Scalability of the optical codes used, flexibility of the manufacturing set-up, and
higher quality of the encoded and decoded waveforms with respect to other reported op-
tical technologies are the key advantages of SSFBGs for this application. The simplicity
of the encoding and decoding gratings is a further advantage of the SSFBG approach.
A simple linear relationship connects the grating spatial profile to its impulse response
in the low reflectivity Born regime, so that the coding structures are characterised by
uniform apodisation and discrete phase shifts. Low local reflectivity in the grating is
necessary to allow light to penetrate the whole structure, to get a complete mapping of
the code in the reflected waveform, and to avoid distortions related to multiple scatter-
ing. However, this causes large insertion losses for these devices. This problem is further
amplified by the large spectral filtering of the starting pulse fed into the grating, since
short pulses (shorter than the chip duration of the code) are needed to prevent blurring
of the phase and/or amplitude characteristics.

Inverse scattering designs are the obvious solution of this problem. Multiple scat-
tering is automatically taken into account in the layer-peeling algorithm, and an almost
perfect reconstruction of the desired impulse response, i.e., of the encoded waveform,
is possible even at high reflectivity levels. The resulting superstructure presents a non-
uniform apodisation profile within each chip of the code. But (1) the chirp structure of
the gratings is practically unchanged, with almost discrete phase shifts between the dif-
f erent chips, and (2) the inverse scattered gratings can be shortened to the same length
of its low reflectivity counterpart with negligible effect on the resulting CDMA opera-
tion. A limited increase in the manufacturing complexity is obtained. The use of similar
writing lengths does not increase the coherence length requirements for the set-up, and
the quality of the gratings and their intrinsic coherence are not expected to be affected.
The preserved length of the grating is explained by causality. Perfect reconstruction of
the ideal waveform is obtained in the considered part of the grating. The tail, instead,
typically compensates for multiple scattering due to the high local reflectivity of the
designed structure, which produces a tail in the impulse response. The corresponding
deformation in the encoding/decoding process and the resulting increased duration of
the decoded waveform (possibly producing inter-symbol interference between adjacent
transmitted bits) have been shown not to seriously affect the quality of the system. It
has also been found that the tail resulting from inverse scattering provides extra scat-
tering in order to reach a high reflectivity. The maximum reflectivity of the inverse
scattered design is limited to the $R_{gr} = 0.65 - 0.8$ range without this tail, even if the
target reflectivity is much higher. Nevertheless, the loss with respect to ideal opera-
tion is negligible, since all the remaining part of the spectrum at lower $R_{gr}$ is actually
boosted with respect to low reflectivity design specifications. Up to $11.09$ dB increase in
the received peak power is obtained for a system with $L = 15$ code, $T_{\text{chip}} = 50$ ps chip duration, and $T_{\text{pulse}} = 20$ ps pulse duration. The insertion losses of the encoding/decoding set-up are considerably limited, while no degradation in the correlation properties of the codewords is found. The received power is further boosted if an iterative design is used. The gain is raised to 13.48 dB by slightly deforming the spectrum of the encoded waveform with respect to its ideal shape. An improvement of the correlation properties is also found (the cross-correlation extinction passes from 8.25 dB to 9 dB).

This clearly shows that the code used is not optimal for operation using SSFBGs, and a search of optimised codewords based on the actual characteristics of the system is possible. Better codewords rather than Gold codes can be found, especially if subsets of small cardinality are targeted. This is similar to standard approaches in mobile communications, where the transfer function of the channel (i.e., of both the physical transmission media and the transmitting/receiving components) is taken into account in the design of the proper code to be used. It is a possible area of investigation if the potentialities of high reflectivity SSFBGs have to be fully exploited for DS-CDMA.

Dispersion compensation integrated with CDMA decoding has also been shown using inverse scattering techniques. The main advantage of this approach is the reduction of the number of components that have to be integrated within the CDMA receiver, i.e., reduced costs and module size. The intuitive idea that the resulting design is the simple sum of a chirped grating and of a standard decoder is not correct. A complex interplay between dispersed wavelengths and phase shifts is found. Recognition of the individual chips is practically impossible, as is a simple identification of an effective reflection point for the different wavelengths. This effect is shown to be particularly important when the total dispersion over the grating bandwidth is small. At the same time, the complexity in the apodisation and chirp profile is maximum in these conditions, with fast amplitude oscillations on a length scale shorter than the actual chip duration.

These designs are mainly limited by grating length. Long superstructures are needed to recompress the dispersed wavelength components, especially when short chip durations are considered, i.e., larger CDMA bandwidths. Unfortunately, very short chip durations require very long grating lengths ($T_{\text{chip}} = 6.4$ ps gives $L_{gr} \approx 400$ mm if 100 km of SMF fibre have to be compensated, for example). The corresponding grating is difficult to manufacture, especially because of the high coherence requirements of CDMA superstructures. Longer linear dispersion compensators have been experimentally realised (up to 1 m), but in no case complex apodisation and chirp were required over such a length. Even if manufacturing problems could be tackled, packaging is eventually a major issue. Length is therefore limited and only compensation of dispersion over a limited bandwidth is obtained. Different approaches have been analysed. Smoothing the impulse response of the decoder, i.e., limiting the grating bandwidth, is effective when almost complete compensation is possible (longer chip durations). Better performance and easier to manufacture grating profiles are obtained. Conversely, it is better to simply design very long superstructures and clip them to the desired length afterwards when
large approximations are necessary. Perfect compensation is obtained within the chosen bandwidth, while no compensation is provided to out-of-band wavelength components. The choice of the actual bandwidth, i.e., of the clipping position, has to be optimised case-by-case for the particular subset of code signatures used.

The increased reflectivity obtained with these inverse scattered designs does not improve the final received power, but still helps to compensate for the dispersed energy that is not recompressed by the decoder. At the same time, the resulting correlation extinction is decreased and performances are worsened. A net loss of 2-2.5 dB is obtained with respect to standard designs if \( L_{gr} = 120 \) mm gratings instead of \( L_{gr} \sim 400 \) mm are used. Apart from this expected degradation, inverse scattered designs do not show any increased sensitivity to changes of the operational parameters. Actually, they were shown to be more robust in a few conditions.

The reduced flexibility of the resulting system is a further disadvantage of inverse scattered designs for CDMA application. Low reflectivity approaches have recently been proven compatible with dynamically reconfigurable devices \([240]\), in which the encoder/decoder codeword is actively changed by means of simple electrical switches. Tunable dispersion compensators have also been demonstrated, so that the length \( L_{link} \) between the encoder/decoder modules can be changed without the need to change the matched grating in the receiver. The use of inverse scattered devices does not allow any post-fabrication tuning or modification of the obtained superstructure, since a direct correspondence between a certain portion of the grating and certain features in the impulse response no longer applies. Any change in a single chip, or in the total dispersion to be compensated, affects the whole portion of the grating after the tuning point, so that an unacceptable distortion of the rest of the response is obtained.

A final comment has to be made about the range of possible system implementations these designs are available for. Because of the limited resolution of the writing set-up, a chip duration \( T_{chip} \gtrsim 2 \) ps is typically required for a standard low reflectivity SSFBG. But the increased complexity of inverse scattered designs requires the previous value to be relaxed. When high reflectivity gratings are considered, only the apodisation profile is more complex, while the phase shifts are still quasi-discrete. Therefore, \( T_{chip} \gtrsim 6 \) ps is a reasonable limit, since the resulting features are comparable to already demonstrated complex superstructures \([236]\). Dispersion compensating gratings provide, on the contrary, simultaneous variations in both apodisation and chirp. The continuous grating writing set-up enables the realisation of this kind of grating \([49]\), but with a much shorter length and with smaller ripples. The upgrade to the very complex designs necessary in this case is not trivial. The requirements need to be greatly relaxed, and \(~ 20 - 30 \) ps ripple periods in the impulse response of the grating are considered as a reasonable limit. This is obtained using configurations in which the overall length of the encoded waveform is \( LT_{chip} < 700 \) ps (350 ps if the more conservative position is considered), if dispersion over 100 km is considered. Chip duration is not a limit in this case, even though the corresponding requirements for the encoding grating have to be considered.
again. As far as performance degradation is concerned, different systems with similar total duration $LT_{\text{chip}}$ have similar final characteristics. A reduced code length results in theoretically worse orthogonality between codewords, but the increased chip duration allows better dispersion compensation for a given grating length. The two effects are found to balance each other. The amount of dispersion and the available grating length are the only parameters which finally determine the performance of dispersion compensated systems. Shorter codes are characterised by simpler encoding gratings, while longer codes may provide better results if subsets of large cardinality have to be used.

The previous discussion about $L_g$ and $T_{\text{chip}}$ sets also the limits for the code lengths $L$ that can be used with inverse scattered designs. If high reflectivity gratings are considered, $L \leq 127$ is necessary to keep the grating length within the 100 mm limit with 6.4 ps chips. $L = 255$ already requires $L_g > 150$ mm gratings, which are at the limit of the typical coherence length of the writing system [122] and are difficult to package and stabilise. This limit has to be shifted to $L \leq 63$ if dispersion compensation is included, especially if longer gratings are not considered as a possible choice.

It is therefore clear that the presented designs are innovative, they allow the exploitation of the full potentialities of the SSFBG’s technology, and they can contribute to improve the performance of DS-CDMA systems. But it is also apparent that they are not the ultimate solutions to the limitations of CDMA systems, and their actual domain of applicability is still rather limited given the technological issues of SSFBG fabrication and stabilisation.
Chapter 7

Conclusions and future work

7.1 Summary of thesis

The availability of effective manufacturing techniques for the production of complex fibre Bragg gratings drives research towards a better understanding of the way these structures work and towards the refinement of their design in order to take into account second-order effects that were previously disregarded. A large number of models are available to describe propagation in periodic structures, and the main issues are largely understood. Nevertheless, intuition often fails when complex gratings have to be analysed, so that it is difficult to discriminate which parts of a superstructured grating accomplishes which function. Further analysis of the fundamental properties of Bragg gratings is therefore needed, together with the identification of intuitive parameters which can constitute a guideline for the comprehension of these gratings.

7.1.1 Local properties in gratings and their applications

This thesis provides a new tool for the analysis of gratings in terms of a local point of view. Instead of looking at the overall features of the structure, its local properties are taken into account and, in particular, the time spent locally by light inside each section of the grating is derived. The grating is divided in two parts, preceding and following the position whose local features are under investigation. A multiple reflection approach is used to calculate all the possible classical paths for both transmitted and reflected light. This effective Fabry-Pérot analysis allows a field decomposition in terms of components whose traversal time inside the layer is well defined, and the average time spent in the layer by transmitted or reflected light can be evaluated. A generally complex-valued time is obtained, but clear physical meaning has been given to both its real and imaginary parts. The real part is related to the actual traversal time, while the imaginary part gives the extent of the response of this “time-delay measurement” on the system, given by the corresponding amplitude variation. Finally, the dwell time in the structure is derived by appropriately weighting and summing up the two contributions. The dwell time is analytically shown to be always real, positive, and directly related to
the power distribution inside the grating in small index-contrast gratings.

The correctness of the described approach has been analytically proven for uniform gratings. Perfect numerical agreement with the standard definition of phase times is also found for arbitrarily complex designs. A possible measurement technique to obtain an experimental confirmation of these results has also been discussed. Scanning a small phase shift along the grating produces a variation of the output optical phase and power proportional to the real and imaginary parts of the local time delay, respectively. The experimental demonstration of the proposed technique has been reported with respect to the imaginary part of the time delay.

This measurement technique shows that the derived local properties can be interpreted as a form of sensitivity of the grating to an external perturbation. This perturbation can either be undesired, as for errors introduced in the periodic pattern by manufacturing imperfections, or controlled, as when a pre-determined modification of the nominal profile of the grating is introduced. This is important, from an engineering point of view, to correct previous manufacturing imperfections (fine tuning of manufactured gratings to improve their specifications), or to compensate for limitations of the design. The same understanding of sensitivity can also be used as a tool for comparing different grating designs in terms of robustness to phase errors, and therefore for choosing the one that is expected to be more convenient. The (apparently) odd theoretical formalism based on complex times finds therefore a number of useful practical applications.

As an example, the design of a linearly chirped grating for dispersion compensation is considered. Complex inverse scattering solutions are available and have been experimentally demonstrated, but a continuous grating writing set-up is required for their fabrication. An optimised linear time delay response is obtained here by using a standard phase mask scanning technique, and post-processing the grating by introducing the phase perturbation profile derived from the analysis of the local sensitivity of the grating. The final result is highly improved with respect to the practical figure-of-merit of dispersion compensating devices (i.e., the low-frequency group delay ripple), even though not as satisfactory as a real reverse-engineered design. Phase and amplitude cannot both be controlled by using a phase-only perturbation. However, all the complexity is transferred into the realisation of the amplitude masks used in the apodisation and post-processing. Simpler, faster, and more reliable fabrication are obtained, which is ideal for mass production of devices.

### 7.1.2 Inverse scattering design extensions and applications

The described local analysis also finds valuable application in the understanding and the improvement of advanced grating designs. The standard layer-peeling algorithm has been extended in order to enable the design of complex grating structures in lossy media. The main sources of propagation losses in a periodic structure have been considered: constant losses related to UV-writing in hydrogenated fibres or to background
losses of the waveguide (significant when non-fibre-based devices are taken into account); nonuniform losses occurring when a variable UV-fluence is used in grating inscription; absorption losses in active media (for applications in active devices such as DFB lasers); and cladding mode losses, important in chirped structures such as dispersion compensators. All these features that limit the quality of complex gratings are now properly accounted for in the design step. The practical solution is to pre-deform the spectral response that is targeted in order to compensate for the extra-loss. However, not every spectral response obtainable in a lossless medium is physically realisable when propagation losses are included. In this case, the spectral imbalance related to losses is traded-off with a uniform loss over the full bandwidth of the device (i.e., additional insertion losses). The spectral characteristic of the designed grating is undistorted with respect to the target, and is simply reproduced at lower reflectivity levels.

Cladding mode losses are perfectly compensated by the use of an iterative procedure, due to the nature of the boundary conditions. This approach is approximate, but excellent results are obtained with a limited number of iterations. The standard layer-peeling algorithm is used for the synthesis stage, while a new transfer matrix method is implemented for the direct simulation of the grating when cladding mode coupling is considered. Both the exact matrix expression and an approximated one based on the assumption of discrete-scattering are proposed. The latter method is the fastest, and it is considerably faster than direct numerical integration of the extended coupled-wave equation. It is accurate if a sufficiently short length is chosen for each grating in the piecewise-uniform approximation. This technique has been applied to the design of a practical device for wide-band dispersion compensation. For the first time, the proposed design is compatible with the realisation of such a device using a standard SMF fibre. Conventional grating profiles cannot be used due to the very short separation of the cladding mode resonances in these fibres (only 1.1 nm), which causes large wavelength dependent losses on the short-wavelength side. The new algorithm allows the realisation of a perfectly flat dispersion compensator, even though the maximum reflectivity of this device is limited to $R_{gr} = 0.69$.

For other kinds of losses considered, a fast and direct lossy implementation of the inverse scattering algorithm has been derived. Optimum loss compensation is obtained, but a case-dependent tuning of the simulation parameters (grating length, windowing-apodising of the target impulse response) is often necessary. A trade-off between reflectivity, time delay characteristics, and insertion losses has to be found, but no optimised recipe has been obtained to date. The effect of losses in the algorithm is clearly understood by using the local properties of gratings, since loss can be interpreted as a local perturbation of the ideal, lossless grating. A correction profile for the amplitude distortion can be estimated starting from the local time delay distributions, and a modified lossy design is obtained in this way. The two approaches produce equivalent results when constant losses are considered, giving a further check to the correctness (and fruitfulness) of the grating analysis on a local scale. Conversely, this technique cannot be
used when losses proportional to the local strength of the grating are present. Any modification of the grating profile corresponds to a modification in the losses that cannot be accounted for in this case. The lossy layer-peeling algorithm has to be used and gives excellent results, while the local properties of the final, synthesised structure simply help the understanding of the obtained design.

Finally, the inverse scattering algorithm has been applied to the design of optimised superstructured gratings for optical CDMA applications. The target of the analysis is to show that complex grating designs are useful in many areas and can constitute a large step forward with respect to other optical technologies available. The complexity of the codes used in CDMA systems, together with the need for highly coherent waveforms when phase coding is considered, make superstructured gratings naturally suited for these applications. Inverse scattering is shown to enable a full utilisation of the potentialities inherent to the fibre grating approach. Complex apodisation profiles, with nonuniform strength inside each chip of the code, allows the reflectivity of these coding gratings to be increased, with peaks up to $R_{gr} \sim 0.80$. The insertion losses of the encoding/decoding pair is correspondingly reduced by more than 12 dB, significantly improving the system performance since less amplification is necessary at the receiver. An iterative optimisation of the target spectral response allows better gratings to be synthesised even in this case. Complex chirp is added in the superstructured grating if combined encoding and dispersion compensation is targeted. The system architecture is simplified, and further reduced insertion losses are obtained by using such multifunction devices. Unfortunately, these designs are at the limit of the manufacturing capabilities, so that chip durations as short as in standard low reflectivity gratings cannot be obtained, and dispersion compensation greatly suffers from limitations in the length of the grating (related to the necessity for accurate wavelength stabilisation of the coding grating). Nevertheless, the designed gratings have satisfactory characteristics and are fully compatible with the continuous grating writing technology. The comparison with standard implementations shows that similar properties and sensitivity to typical perturbations are obtained, with much lower insertion losses, but also lower orthogonality between codewords due to the imperfect dispersion compensation.

### 7.2 Future work

The work developed in this thesis is far from covering the various topics considered in an exhaustive way. A large amount of possible future developments can be foreseen starting from the presented analysis. This concerns both the techniques proposed for the first time, that can be applied to a number of different applications not addressed here, and some (more or less) obvious extensions that can be considered in order to enlarge their field of applicability.
7.2.1 Improvements to local time delay analysis

The local time delay approach can be compared with other recent studies from a theoretical point of view. Winful [144] describes gratings as resonant structures, and shows that phase times should be considered as times of interaction with the cavity (energy storage and release) rather than as times-of-flight, since evanescent waves do not formally propagate. The proposed deviation is correct for symmetric structures, but extension to general gratings is not trivial. Possible relations with the local time delay distributions described here can be found, and different interpretations of these results are likely to be obtained in this case. Ware et al. [150] show that propagation in a linear dielectric is very effectively described (again) by energy exchange between the incoming radiation and the medium, and that the temporal response of the medium is related to an instantaneous spectrum of the incoming pulse (only the portion of the pulse seen by the medium is responsible for energy storage and release because of causality). Extension of this framework to scattering structures is appealing and can lead to a better identification of the effect of local time delays, if an analogue of the instantaneous spectrum response can be defined. Tracking of a pulse on a temporally and spatially resolved base is the ultimate possibility of such an approach.

Further extensions to the described model can also be considered. In particular, dispersive features related to the effective guidance properties of generic waveguides (in fibre or planar configurations) have not been explicitly taken into account. Little contribution is expected for small index-contrast structures, since the relative bandwidth of these gratings is typically negligible with respect to the central wavelength, and dispersion can be disregarded. But a derivation with dispersive properties fully accounted for is necessary when multilayers (thin-films or stacks of semiconductors) are considered, given their large contrast, short length, and bandwidth spanning hundreds of nanometres. Direct analysis of grating structures in lossy or gain media constitutes a further possible development of this approach. Lossy propagation does not require any major modification to the presented model, since it is simply accounted for by using complex propagation constants in the effective Fabry-Pérot cavity. Conversely, a proper definition of an equivalent “total time delay” in this case and its relationship with energy distribution and losses inside a lossy grating is less intuitive. The extension of the local time delay analysis to active materials with gain is even more interesting, since it would allow a direct approach to the design of active devices in “hot” cavity conditions, with particular focus on DFB lasers threshold characterisation. However, the presence of gain cannot be included in a straightforward way. As an example, the geometrical series involved in the multiple path computation can fail to converge. Physically, this condition is expected to correspond to above-threshold DFB operation, since it means that, on average, photons do not leave the grating’s cavity. But a precise connection between number of photons, stimulated (and eventually spontaneous) emission, and time delay has not been identified yet, and further theoretical analysis is needed.

From an experimental point of view, the measurement of the real part of the local
time delay in fibre Bragg gratings is of primary interest. It would allow a further direct proof of the validity of the proposed theory, but especially it would give evidence to unfamiliar concepts such as superluminal propagation of light or negative time delay. Less noisy results can be obtained for the measurement of the imaginary part by considering uniform gratings of higher quality, more localised perturbations (by means of beam focussing), and eventually improving the stability of the experimental set-up. The possibility of in-house fabrication of longer and/or stronger high-quality gratings may allow higher superluminal velocities to be measured than reported [163] with limited added complication.

### 7.2.2 Improvements to sensitivity analysis and further applications

A possible further application of the sensitivity analysis of gratings is the reconstruction of measured gratings. The availability of a finite measured bandwidth limits the use of inverse scattering techniques in this case, and unsatisfactory results are obtained for very strong gratings due to amplification of errors in the layer-peeling process. But the real grating can be interpreted as an ideal grating plus a noise contribution to be identified. Knowledge of the in-band components (which are highly affected by noise in the reconstruction process) is not required, even though it would provide additional bounds to the search. Conversely, the out-of-band components are not highly attenuated by the grating and sample the whole structure, and they are theoretically sufficient to infer the distribution of errors by minimising a suitable error function. Such a procedure can be particularly time-consuming if a high spatial resolution is required, and the final result is not guaranteed to be correct, in the sense that it is only the best fit in the available bandwidth and under strict approximations. But it can still provide valuable information for cross-checking results from different approaches.

Different techniques can also be used for the determination of an optimised perturbation profile to be applied to grating post-processing or fine tuning of a certain design, as in the case of linear dispersion compensators. The direct inversion of the sensitivity matrix is not possible due to the divergent contribution of the high frequency components. If a rigorous approach is limited, a random approach based on the implementation of genetic algorithms can prove to be more suitable. Starting from a reasonable profile as initial guess, such as the exact design derived using only very low frequency components (and therefore possibly not ideal), random modifications of the phase error distribution are used in each step. An optimal solution is found after a large number of iterations by always selecting the subset of profiles that are “healthier”. In this case, the “health” parameter is represented by a suitable performance monitor. This approach is attractive because a perfect reconstruction of the target is not necessary, and it is not even necessary to find an approximation of it that has good performance and an easy-to-synthesise shape, since the algorithm is going to take care of it. The drawbacks of this approach are represented by the computation time, usually much longer than typical deterministic methods, and by possible problems of convergence if the algorithm parameters (number
of representatives in each generation, extent of the mutations, selection criteria) are not properly chosen.

### 7.2.3 Improvements to inverse scattering and further applications

The extensions proposed to the standard inverse scattering algorithm, together with the new approaches to mixed integral solution/layer-peeling techniques [70] and non-rigid implementations [71, 72], show that there is still the possibility for further theoretical development of these inverse scattering techniques applied to propagation in periodic structures. The standard method can be applied to the design of complex gratings with specifically tailored properties (typically, pulse reshaping for signal generation or telecommunication applications, or in all-optical signal processing, for instance combined with nonlinear techniques), and such a potentiality has been exploited only in a few cases so far. Moreover, new interesting physical properties can be studied by taking advantage of the possibility to design very special structures (see [54]).

The layer-peeling techniques proposed in this thesis can be further improved, and new applications besides the few considered examples can be found. The design of practical dispersion compensating square-filters or specially tailored profiles with cladding mode loss compensation is feasible using these tools. Further work is needed in order to optimise the convergence of the iterative algorithm and decrease the computational time for very long, broad bandwidth gratings. The time delay characteristic of the gratings obtained has also to be addressed. A time delay compensation is likely to be necessary in order to minimise the small but possibly annoying deviations from the ideal time delay profile caused by cladding mode losses and reflectivity compensation when using the point-scattering approximation. A possible further improvement is represented by a faster implementation of the exact transfer matrix formalism. Indeed, the form of the exponential matrix is rather simple (doubly bordered diagonal matrix) and sparse when a large number of cladding modes are included, so that an optimised algorithm for the computation of eigenvalues and eigenvectors can possibly be found.

A more interesting application of the lossy algorithm is grating reconstruction from measured data. The possibility of accurately characterising the grating spatial features starting from the easily measured reflectivity and time delay data is important from a practical point of view, since it allows valuable feedback to be obtained for the optimisation of grating writing process. In this case, inverse scattering techniques can be extended to the reconstruction of gratings written in both highly lossy guiding structures (as surface relief gratings in planar optics) and absorbing materials, such as DFB lasers in cold cavity conditions. Recently, a new algorithm that performs both the reconstruction of the grating strength and of the local losses has been published [69]. It is definitely a more general solution, but it requires accurate and consistent measurements of the grating under test from both sides. The algorithm used in this thesis requires only a single measurement, although this occurs only since “a priori” assumptions about the loss characteristics are used.
7.2.4 Inverse scattering in different areas of optics

At the same time, new design problems can be tackled. Design of complex long-period gratings has been performed using the same layer-peeling approach [65, 66, 67, 94]. Very little work has been developed in this area to date and it is mainly applied to gain flattening devices, but refining the synthesis procedure can further improve the design and minimise the residual amplitude ripple related to the finite length of the grating. The inclusion in the layer-peeling procedure of the effect of non-sinusoidal modulations of the refractive index is also interesting (binary gratings’ characteristics were studied [241, 242], but more complex base-cells can be used given the present grating writing techniques). In particular, the extension of the integral layer-peeling approach developed by Rosenthal and Horovitz [70] seems suitable for this improved design technique. Moreover, recent non-rigid implementations can be applied to co-propagating structures, enabling a better optimisation of the gratings, especially when length or amplitude limitations have to be considered. The lossy designs considered in this thesis are also likely to be useful, given the higher propagation losses of cladding modes with respect to the core mode. Inclusion of propagation losses has recently been shown to allow new interesting designs to be realised, with new characteristic features [243]. The implementation of a lossy inverse scattering algorithm may allow a direct synthesis and more striking effects to be obtained and exploited.

Different fields of optics can also benefit from the application of co-propagating layer-peeling techniques, such as second harmonic generation in periodically poled materials [66]. The parallelism between core and cladding modes and pump and second harmonic signals is apparent, but no direct analysis of the nonlinear equations and correspondent application of the layer-peeling method has been developed. Complex periodic modulations of the nonlinearity can be obtained both using domain engineered \( \text{LiNbO}_3 \) (by suitably varying the duty cycle of the amplitude mask pattern used in the poling process) or using periodically poled glasses, especially with new UV-erasure techniques [244, 245]. Square-band wavelength converters are an example of another possible application of the inverse scattering method.

Multi-dimensional layer-peeling approaches are another area of research. A first example is the design of gratings in which the local properties in certain grating sections have to be defined (such as sensitivity to external perturbations) together with the grating spectrum. These structures are interesting for application in tunable/reconfigurable devices with improved performances, both using short-period gratings [237, 246] and long-period gratings. Thermo-optical tuning is possible in the first case, while simple tuning of the propagation constant and propagation losses of cladding modes is obtained in the second case by using different index matching materials surrounding the cladding. Another example is given by the combined design of mixed short- and long-period structures with improved (and possibly novel) characteristics, which have recently been proposed [247]. This approach can also be applied to the design of quasi-phase-matched nonlinearities embedded into Bragg gratings, for which enhanced nonlinear interactions
[143] and dramatically improved figure-of-merit [248] have been already proven. Finally, extension to bi-dimensional structures is also appealing. Generalisation of Bragg scattering design to two dimensions can be applied to both bulk gratings and dynamically reconfigurable MEMS devices, when a particular spatial impulse response is required. More interestingly, automated fibre design is possibly tackled by these techniques. Directly obtaining the traversal refractive index profile, once the position of the guided modes, and/or the shape of the modal profile, and/or their dispersive characteristics have been defined, is highly appealing for a huge number of applications, from standard telecommunication devices to lasers, to power delivery. In axially symmetric fibres only the radial coordinate has to be designed, while complex bi-dimensional arrangements, such as in holey fibres, undoubtedly require more sophisticated (and possibly very different) inverse scattering implementations. Non-rigid implementations are expected to be necessary for holey fibre design, since only two different refractive index materials (glass and air) are typically available. Design of photosensitive fibres is also appealing, given the possibility to shape the strength and position of both core and cladding modes. A final area of interest is represented by the simultaneous design of the guiding profile of the fibre and the coupling profile of a periodic structure imprinted in it. Possible applications to the design of novel fibre devices are foreseen, taking the recent proposal of few-mode fibre based, grating assisted devices as an example [249].

It is clear that the implementation of the possible extensions of the inverse scattering algorithm just described is not obvious and, in many cases, it may even not be feasible since some problems might be inappropriate. Nevertheless, they all represent interesting areas for future investigation.
Appendix A

Derivation of time delay expressions

A monochromatic excitation $|\psi_{IN}\rangle \rightarrow E_{IN} = e^{i(\beta z - \omega_0 t)}$ is considered in the following in order to derive the expressions for the local time delays. Using the Eq. (3.1) formalism and considering the time delay as a generic complex number, Eq. (3.11b) allows the computation of the time delay in transmission:

$$
\langle \psi_T | \psi_T \rangle = \frac{|t_1 + t_2|^2}{1 - |\rho|^2} = T_{gr}, \quad (A.1a)
$$

$$
\langle \psi_T | \hat{\tau}_T(z) | \psi_T \rangle = d\tau_0 \frac{|t_1 + t_2|^2}{1 - |\rho|^2} \left[ \frac{1}{1 - \rho} - \frac{1}{1 + \rho} \right], \quad (A.1b)
$$

where $\rho = r_1 - r_2$ is the round-trip reflection coefficient, $\tau_{T,m}(z) = (2m - 1)d\tau_0$ is the time delay for each field component $|\psi_{T,m}(s)\rangle$, and the phase shift $\phi$ associated with the propagation through the considered layer $s$ has been disregarded since its corresponding length is $dz \to 0$. $d\tau_0$ is the time spent in the layer in a single pass, and $(2m - 1)$ is the number of passes before the photon is transmitted, as shown in Fig. 3.3 (a). Combining Eqs. (A.1a) and (A.1b) according to Eq. (3.1), the expression for the transmission time delay $\tau_T(z)$ in layer $s$ is

$$
\tau_T(z) = \frac{\langle \psi_T | \hat{\tau}_T(z) | \psi_T \rangle}{\langle \psi_T | \psi_T \rangle} = d\tau_0 \frac{1 + \rho}{1 - \rho}, \quad (A.2)
$$

from which the number of passes $N_T(z)$ through the position $z$ inside the grating is readily found

$$
N_T(z) = \frac{1 + \rho}{1 - \rho}, \quad (A.3)
$$
In general, $N_T(z)$ is a complex number whose real and imaginary parts can be expressed as

$$\text{Re}\{N_T(z)\} = T_{gr} \frac{1 - R_1R_2}{T_1T_2}, \quad (A.4a)$$

$$\text{Im}\{N_T(z)\} = 2\text{Im}\{\rho\} \frac{T_{gr}}{T_1T_2}, \quad (A.4b)$$

where $T_1, R_1$ and $T_2, R_2$ are the transmissivity and reflectivity of the left and right reflectors, respectively. Equation (A.4a) shows that the real part of the transmission time delay is always positive.

The reflection time delay $\tau_R(z)$ can be computed with an analogous derivation starting from Eq. (3.12b):

$$\langle \psi_R | \psi_R \rangle = |r_{gr}|^2 = R_{gr} = (1 - T_{gr}), \quad (A.5a)$$

$$\langle \psi_R | \tau_R(z) | \psi_R \rangle = r_{gr}^* \left[ r_{1+} \cdot 0 + \sum_{m=1}^{\infty} 2m d\tau_0 \ t_{1+} t_{1-} [r_{1-}r_2]^{m-1} \right]$$

$$= r_{gr}^* 2d\tau_0 \frac{t_{1+} t_{1-} - r_2}{|1 - \rho|^2}. \quad (A.5b)$$

$|\psi_{R,0}(z)\rangle$ is reflected before reaching layer $s$ and therefore $\tau_{R,0}(z) = 0$, while all the other components pass $2m$ times through $s$, as shown in Fig. 3.3 (b). The desired expression is

$$\tau_R(z) = \frac{\langle \psi_R | \tau_R(s) | \psi_R \rangle}{\langle \psi_R | \psi_R \rangle} = 2d\tau_0 \frac{r_{gr}^* \frac{(t_{1+} t_{1-} - r_2)}{|1 - \rho|^2}}{|r_{gr}|^2}. \quad (A.6)$$

With algebraic manipulation and using the reciprocity relation $r_{1+}^* = -r_{1-}^*$, valid in the lossless case only, see Eq. (2.36), Eq. (A.6) can be simplified as follows:

$$\tau_R(z) = 2d\tau_0 \frac{T_1}{1 - \rho \rho - R_1}. \quad (A.7)$$

The number of passes in reflection $N_R(z)$ is given by

$$N_R(z) = 2 \frac{\rho}{1 - \rho \rho - R_1}, \quad (A.8)$$

and the corresponding real and imaginary parts are found to be

$$\text{Re}\{N_R(z)\} = \frac{T_{gr}}{T_1T_2} \left[ 1 - R_1R_2 + \frac{R_2 - R_1}{R_{gr}} \right], \quad (A.9a)$$

$$\text{Im}\{N_R(z)\} = -2\text{Im}\{\rho\} \frac{T_{gr} T_{gr}}{T_1T_2 R_{gr}}, \quad (A.9b)$$

where the relations $|\rho|^2 = R_1R_2$ and $R_{gr} = \frac{|R_1 - \rho|^2}{|1 - \rho|^2} R_1$, easily derived from the definition of the reflection coefficient $r_{gr}$ (3.12b), have been used.

It is interesting to point out that Eq. (2.40), which represents the general relationship connecting the reflection and transmission time delays computed with the phase
Appendix A Derivation of time delay expressions

derivatives in the lossless case, is still valid even if it is applied on a layer-by-layer basis. Indeed, using Eqs. (A.4a) and (A.9a) and taking into account that the simple substitutions $R_2 \to R_1$ and $R_1 \to R_2$ apply when computing $\text{Re}\{N_{R-}(z)\}$ from $\text{Re}\{N_{R+}(z)\}$, it is readily found that

$$\text{Re}\{N_{R+}(z)\} + \text{Re}\{N_{R-}(z)\} = 2 \frac{T_{gr}}{T_1 T_2} [1 - R_1 R_2] = 2 \text{Re}\{N_T(z)\}.$$  \hspace{1cm} (A.10)

The total time $\tau_{tot}(z)$ spent in section $s$ by a photon is given by the average value of $\tau_T(z)$ and $\tau_R(z)$, weighted by the final state probability $T_{gr}$ or $R_{gr}$. Using Eqs. (A.4) and (A.9), its concise expression results in

$$\tau_{tot}(z) = \frac{\tau_T(z) T_{gr} + \tau_R(s) R_{gr}}{d\tau_0 T_{gr}} [1 - R_1 R_2 + R_2 - R_1] = d\tau_0 T_{gr} \frac{1 + R_2}{1 - R_2},$$ \hspace{1cm} (A.11)

and the number of passes $N_{tot}(z)$ can be formally defined even in this case as

$$N_{tot}(z) = T_{gr} \frac{1 + R_2}{1 - R_2}.$$  \hspace{1cm} (A.12)
Appendix B

**Re\{τ_{f}\} = τ_{\text{phase}} in uniform gratings: analytic proof**

The equivalence between the multiple path approach and the classical phase time delay \(τ_{\text{phase}} = \frac{∂θ}{∂ω}\) introduced with Eq. (2.40) can be analytically proved only in the special case of a uniform grating, where closed expressions are available for the reflection (2.22) and transmission (2.23) coefficients.

Taking into account the approach described in Chapter 3, the time delay accumulated over the entire grating is given by Eq. (3.17). If the transmission time delay \(\text{Re}\{τ_{T}\}\) is considered, it is convenient to express Eq. (A.4a) in terms of the left and right reflector transmissivities \(T_{1}(z)\) and \(T_{2}(z)\):

\[
\text{Re}\{τ_{T}\} = \int_{0}^{L_{\text{gr}}} \frac{n_{\text{eff}}}{c} T_{\text{gr}} \frac{T_{1}(z) + T_{2}(z) - T_{1}(z)T_{2}(z)}{T_{1}(z)T_{2}(z)} \, dz.
\] (B.1)

The generic transmissivity \(T\) for a uniform grating of length \(Δ\) is directly obtained from Eq. (2.23)

\[
T = |t|^{2} = \frac{(γ/κ)^{2}}{\cosh^{2}(γΔ) - (δ/κ)^{2}},
\] (B.2)

where \(Δ = L_{\text{gr}}, z, (L_{\text{gr}} - z)\) for \(T_{\text{gr}}, T_{1}(z), T_{2}(z)\), respectively, \(δ\) is the effective detuning from the Bragg wavelength, \(κ\) is the grating coupling constant, and \(γ = \sqrt{κ^{2} - δ^{2}}\), as defined in Section 2.2.2. Therefore, it is possible to write with algebraic manipulations

\[
\text{Re}\{τ_{T}\} = \frac{n_{\text{eff}}}{c} \frac{(γ/κ)^{2}}{\cosh^{2}(γL_{\text{gr}}) - (δ/κ)^{2}} \int_{0}^{L_{\text{gr}}} \frac{\cosh^{2}(γz) + \cosh^{2}[γ(L_{\text{gr}} - 2z)] - 1 - (δκ)^{2}}{(γ/κ)^{2}} \, dz
\]
which results in
\[
\text{Re}\{\tau_T\} = \frac{n_{\text{eff}} L_{gr}}{c} \frac{[\sinh(2\gamma L_{gr})]/2\gamma L_{gr} - (\hat{\sigma}/\kappa)^2}{\cosh^2(\gamma L_{gr}) - (\hat{\sigma}/\kappa)^2}.
\] (B.3)

If the reflection time delay \(\text{Re}\{\tau_R\}\) is considered, it is apparent from Eq.(A.9a) that
\[
\text{Re}\{\tau_R\} = \text{Re}\{\tau_T\} + \frac{n_{\text{eff}}}{c} T_{gr} \int_0^{L_{gr}} \frac{R_2(z) - R_1(z)}{T_1(z) T_2(z)} \, dz = \text{Re}\{\tau_T\},
\] (B.4)
since the integrated function is odd with respect to \(z = L_{gr}/2\) in a symmetric structure, as easily shown considering that \(R_1(z) = R_2(L_{gr} - z)\), and integrates to 0 over the grating length \(L_{gr}\). Therefore, it is straightforward from Eq. (A.11) to show that even \(\tau_{\text{tot}} = \text{Re}\{\tau_T\}\).

The derivation of the phase time delay is algebraically more involved. Starting from the analytic expression for the coefficients \(t_{gr}\) and \(r_{gr}\) of a uniform grating \[88\], it is easy to recognise that the transmission and reflection phases \(\theta_T\) and \(\theta_R\) are equal apart from a constant factor. Therefore, the corresponding transmission and reflection time delays \(\tau_{\text{phase}} = \frac{\partial \theta}{\partial \omega}\) are equal and equal to the total time delay, as is clear from Eq. (3.15). The corresponding expression is
\[
\tau_{\text{phase}} = \frac{\partial}{\partial \omega} \left\{ \arctan \left( \frac{\hat{\sigma}}{\gamma} \tanh(\gamma L_{gr}) \right) \right\}.
\] (B.5)

\(\frac{\partial \hat{\sigma}}{\partial z} = \frac{n_{\text{eff}}}{c}\) applies under the assumption that the dispersion of the effective refractive index \(n_{\text{eff}}\) can be disregarded compared to the dispersive contribution of the grating, and therefore for gratings with bandwidth much smaller than the central wavelength (approximation valid in a small-index contrast limit, see also Section 3.2.3). This is also consistent with the previous derivation of Eq. (B.3), where the average refractive index \(n_{\text{eff}}\) has been used and the longitudinal modulation \(n_{\text{eff}}(z)\) disregarded (valid, again, only in a small-index contrast limit). Performing the differentiation in the previous expression, the total time delay is found to be
\[
\tau_{\text{phase}} = \frac{\partial \hat{\sigma}}{\partial \omega} \frac{\gamma^2 \cosh^2(\gamma L_{gr})}{\kappa^2 \cosh^2(\gamma L_{gr}) - \hat{\sigma}^2} \left[ \frac{\kappa^2}{\gamma^3} \tanh(\gamma L_{gr}) - \frac{\hat{\sigma}^2 L_{gr}}{\gamma^2 \cosh^2(\gamma L_{gr})} \right] = \frac{n_{\text{eff}}}{c} \frac{1}{\kappa^2 \cosh^2(\gamma L_{gr}) - \hat{\sigma}^2} \left[ \frac{\kappa^2}{\gamma} \sinh(\gamma L_{gr}) \cosh(\gamma L_{gr}) - \hat{\sigma}^2 L_{gr} \right],
\]
and finally [250]
\[
\text{Re}\{\tau_R\} = \frac{n_{\text{eff}} L_{gr}}{c} \frac{[\sinh(2\gamma L_{gr})]/2\gamma L_{gr} - (\hat{\sigma}/\kappa)^2}{\cosh^2(\gamma L_{gr}) - (\hat{\sigma}/\kappa)^2}
\] (B.6)
The equivalence between \(\tau_{\text{tot}}\), \(\text{Re}\{\tau_T\}\), \(\text{Re}\{\tau_R\}\), and \(\tau_{\text{phase}}\) at every wavelength is analytically proved by Eqs. (B.3) and (B.6) under the described approximation.
Appendix C

Local dwell time derivation and physical interpretation

The dwell time $\tau_D$ associated with a generic cavity can be interpreted as the time necessary to empty the cavity itself in steady state conditions. The same physical meaning can be extended to the local dwell time $\tau_D(z) = \tau_{\text{tot}}(z)$ introduced by Eq. (3.28) by considering a generic layer $s$ of length $dz$ and the associated Fabry-Pérot cavity. The stored energy is $dW(z)dz$ and the flux leaving the layer is given by $S_{\text{OUT}}(z) = S_+(z) + S_-(z)$, where both the counter-propagating fields $E_+(z)$ and $E_-(z)$ given by Eq. (3.20) are considered. But $S_{\text{OUT}}(z)$ takes into account even photons that will be scattered during the propagation and will re-enter layer $s$. This means that these photons do not actually contribute to energy removal from the analysed layer and therefore must not be accounted for in the dwell time computation. The fields associated with photons that will not be further scattered and re-enter the layer $s$ are therefore

\begin{align*}
\tilde{E}_+(z) &= E_+(z) t_2 = E_T, \\
\tilde{E}_-(z) &= E_-(z) t_1 - E_\text{IN} r_{1+} = E_R,
\end{align*}

and correspond to the transmitted and reflected fields as easily verified from Eqs. (3.11b), (3.12b), and (3.20). It is worth noting that interference in the backward direction between photons leaving section $s$ and photons scattered back before reaching $s$ has to be considered. Using Eq. (C.1) and considering the correct Poynting vector fluxes, the outgoing associated flux $\tilde{S}_+(z) + \tilde{S}_-(z) = S_T + S_R$ is constant, independent of the layer position, and equal to $S_{\text{IN}}$ because of energy conservation. Therefore, applying the general definition of dwell time to each layer, the local dwell time $\tau_D(z)$ is

\[ \tau_D(z) = \frac{dW(z)dz}{\tilde{S}_+ + \tilde{S}_-} = \frac{W(z)}{S_{\text{IN}}}, \]

and the formal relation introduced in Eqs. (3.28) is proven.
Appendix D

Derivation of $\tilde{r}_{gr}$ and $\tilde{t}_{gr}$ in a phase shifted grating

Let us start considering how the transmission coefficient $t$ of a grating is affected by introducing an infinitesimal phase shift $d\phi$ in a defined position $z$. Using the formalism of Section 3.2.2 and Eq. (3.11b):

$$
\tilde{t}_{gr} = \frac{t_1 t_2 e^{j\phi}}{1 - r_1 r_2 e^{2j\phi}} = \frac{t_1 t_2}{1 - \rho e^{2j\phi}},
$$

$$
t_{gr} = \frac{t_1 t_2}{1 - r_1 r_2} = \frac{t_1 t_2}{1 - \rho},
$$

where $d\phi = 0$ in the unperturbed grating and $\rho = r_1 - r_2$. Using the following Taylor expansions:

$$
e^x \simeq 1 + x,
$$

$$
\frac{1}{1 + x} \simeq 1 - x,
$$

valid in the $x \to 0$ limit, it is possible to write

$$
\frac{\tilde{t}_{gr}}{t_{gr}} = e^{j\phi} \frac{1 - \rho}{1 - \rho e^{2j\phi}} \simeq e^{j\phi} \frac{1 - \rho}{1 - \rho - j2d\phi \rho}
$$

$$
= e^{j\phi} \frac{1}{1 - j2d\phi \frac{\rho}{1 - \rho}} \simeq e^{j\phi} \left[ 1 + j2d\phi \frac{\rho}{1 - \rho} \right]
$$

$$
\simeq [1 + jd\phi] \left[ 1 + j2d\phi \frac{\rho}{1 - \rho} \right] \simeq 1 + jd\phi + j2d\phi \frac{\rho}{1 - \rho}
$$

$$
= 1 + jd\phi \frac{1 + \rho}{1 - \rho} \simeq e^{j\phi \frac{1 + \rho}{1 - \rho}},
$$

where $\rho \approx r_1 - r_2 e^{j2d\phi}$ for an infinitesimal perturbation. $\frac{1 + \rho}{1 - \rho} = N_T$ can therefore be identified as the average number of passes that light experiences in the perturbed region, as is clear considering the grating sides, where $N_T = 1$. From Eq. (3.7), the transmission time delay (A.2) is obtained in a completely independent way.

Considering the reflection coefficient, from Eq. (3.12b) and using the same approxi-
Appendix D Derivation of \( \tilde{r}_{gr} \) and \( \tilde{t}_{gr} \) in a phase shifted grating

As before:

\[
\tilde{r}_{gr} = \frac{r_{1+} + r_2 T_{1+} e^{j 2 \phi}}{1 - \rho e^{j 2 \phi}}, \quad (D.4)
\]

\[
\tilde{t}_{gr} = \frac{r_{1+} + r_2 T_{1+}}{1 - \rho}. \quad (D.5)
\]

Using Eq. (2.36), it can be proved that \( t_{1+} t_2 - r_1 - r_2 = \frac{t_{1+}}{r_{1+}} = -\frac{t_{1-}}{r_{1+}} \) and therefore

\[
\frac{\tilde{r}_{gr}}{r_{gr}} = \frac{r_{1+} - r_2 e^{j 2 d \phi} \frac{r_1}{r_{1+}}}{r_{1+} - r_2 \frac{T_{1+}}{T_{1+}} \frac{1 - \rho}{1 - \rho e^{j 2 d \phi}}}
\]

\[
= \frac{R_1 - \rho e^{j 2 d \phi}}{R_1 - \rho} \frac{1 - \rho}{1 - \rho e^{j 2 d \phi}}
\]

\[
\simeq \left[ 1 - j 2 d \phi \frac{\rho}{R_1 - \rho} \right] \left[ 1 + j 2 d \phi \frac{\rho}{1 - \rho} \right]
\]

\[
\simeq \left[ 1 - j 2 d \phi \frac{\rho}{R_1 - \rho} + j 2 d \phi \frac{\rho}{1 - \rho} \right]
\]

\[
= 1 + j 2 d \phi \frac{T_1}{R_1 - \rho} \frac{\rho}{1 - \rho} \simeq e^{j 2 d \phi \frac{T_1}{R_1 - \rho}} \frac{\rho}{1 - \rho}, \quad (D.6)
\]

where \( |r_{1+}|^2 = R_1 \). Following the same approach used in transmission, it is possible to define an equivalent number of passes \( N_R \) and an average reflection time delay, which is found to agree with Eq. (A.6).
Appendix E

Derivation of the perturbation approach in gratings

The perturbation analysis developed by Uno and Adachi [171] starts directly from the Helmholtz equation (2.2). In the following, the same approach is developed using the formalism described in Chapter 2, in order to obtain an expression which is consistent with the notation of this work.

Assuming that the field distributions \( E(z) \) and \( \tilde{E}(z) \) are the solution of the Helmholtz equation for two different refractive index profiles \( n \) and \( \tilde{n} \) (in the form of Eq. (2.3)), it is possible to write

\[
\frac{d}{dz} \left[ \tilde{E}(z) \frac{dE(z)}{dz} - E(z) \frac{d\tilde{E}(z)}{dz} \right] \cdot e_0(x, y) = \left[ \tilde{E}(z) \frac{d^2E(z)}{dz^2} - E(z) \frac{d^2\tilde{E}(z)}{dz^2} \right] \cdot e_0(x, y)
\]

\[
= k_0^2 \left[ \tilde{n}^2(x, y, z) - n^2(x, y, z) \right] \tilde{E}(z)E(z)e_0(x, y),
\]

(E.1)

where \( e_0(x, y) \) is the transverse profile of the fundamental mode and Eq. (2.2) has been used for both \( n \) and \( \tilde{n} \), separately. Integrating over the transverse plane and taking into account the definitions of \( \sigma \) and \( \kappa \) in Eqs. (2.11) and (2.12), the previous expression becomes

\[
\frac{d}{dz} \left[ \tilde{E} \frac{dE}{dz} - E \frac{d\tilde{E}}{dz} \right] = 2\beta \left\{ (\tilde{\sigma} - \sigma) + 2 \left[ \tilde{\kappa} \cos \left( \frac{2\pi z}{\Lambda} + \tilde{\phi}_P \right) \right. \right.
\]

\[
- \left. \kappa \cos \left( \frac{2\pi z}{\Lambda} + \phi_P \right) \right\} \tilde{E}E,
\]

(E.2)

where \( \Lambda \) is the nominal periodicity of the grating and \( \beta = \frac{2\pi}{\Lambda} n_{\text{eff}} \) is the propagation constant in the unperturbed grating. Eq. (E.2) is the general extension of Uno’s formula for the case of generic profiles of the self-coupling coefficient \( \sigma \), of the coupling coefficient \( \kappa \), and of the pattern phase \( \phi_P \). It can be integrated over the grating length \( z \in [0, L_{gr}] \) in order to derive the expression for the reflection coefficient, and taking into account
that the boundary conditions are

\[ E(0) = 1 + r, \quad E(L_{gr}) = t, \]

\[ \frac{dE}{dz} \bigg|_{z=0} = j\beta(1 - r), \quad \frac{dE}{dz} \bigg|_{z=L_{gr}} = j\beta t, \]

it is easily found that

\[ \tilde{r} = r + j \int_0^{L_{gr}} \left\{ (\tilde{\sigma} - \sigma) + 2 \left[ \tilde{\kappa} \cos \left( \frac{2\pi z}{\Lambda} + \tilde{\phi}_P \right) - \kappa \cos \left( \frac{2\pi z}{\Lambda} + \phi_P \right) \right] \right\} \tilde{E}Edz \]

\[ = r + j \int_0^{L_{gr}} \left[ (\sigma - \tilde{\sigma}) + 2 \text{Im} \left\{ (\tilde{q} - q)e^{j\frac{2\pi z}{\Lambda}} \right\} \right] \tilde{E}Edz, \tag{E.3} \]

where \( q = j\kappa e^{j\phi_P} \) is the corresponding complex coupling coefficient defined in Eq. (2.16).

The previous expression is absolutely correct, in the sense that no approximations have been introduced so far, and it is valid for a generic choice of the profiles \( \tilde{q} \) and \( q \) of the gratings.

From a perturbation point of view, it is obvious that \( q \) is interpreted as the profile of the unperturbed structure, while \( \tilde{q} = q + \delta q \) takes into account the effect of a perturbation \( \delta q \) of the original structure. Only the electric field distribution \( E(z) \) of the unperturbed grating is known in this case. An estimation of the perturbed reflection coefficient \( \tilde{r} \) is obtained by assuming that the variation in the field distribution due to the (small) error \( \delta q \) is negligible, and \( \tilde{E}(z) = E(z) \) is set to a first approximation. A closed formula depending only on the characteristics of the original grating and of the perturbation is found,

\[ \tilde{r} = r + j \int_0^{L_{gr}} 2 \text{Im} \left\{ \delta q e^{j\frac{2\pi z}{\Lambda}} \right\} E^2dz, \tag{E.4} \]

and an alternative formulation is obtained considering the forward- and backward-propagating envelopes \( U \) and \( V \) defined in Eqs. (2.13) and (2.14)

\[ \tilde{r} = r + j \int_0^{L_{gr}} \left[ \delta q^*e^{-j\frac{2\pi z}{\Lambda}} - \delta q e^{j\frac{2\pi z}{\Lambda}} \right] \left[ U^2(z)e^{j\frac{2\pi z}{\Lambda}} + V^2(z)e^{-j\frac{2\pi z}{\Lambda}} + 2U(z)V(z) \right] dz \]

\[ = r - \int_0^{L_{gr}} \left[ \delta q^*U^2 - \delta qV^2 + 2U(z)V(z) \left( \delta q^*e^{-j\frac{2\pi z}{\Lambda}} - \delta q e^{j\frac{2\pi z}{\Lambda}} \right) \right] dz \]

\[ \simeq r - \int_0^{L_{gr}} \left[ \delta q^*U^2 - \delta qV^2 \right] dz, \tag{E.5} \]

where the synchronous approximation has been used to drop the last, fast oscillating terms in the previous expression.

Eq. (4.9) is valid for a generic distribution of amplitude and phase errors inside the grating. Conversely, the time delay approach is particularly suitable for the interpretation of single phase errors. If it is assumed that the perturbation \( \delta q \) is simply related to a shift \( d\phi_P \) in the periodic modulation of the refractive index and localised in a certain
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position $z = \tilde{z} \in [0, L_{gr}]$, the corresponding expression is given by

$$\delta q(z) = \tilde{q} - q = j\kappa e^{j\phi} \left( e^{j\phi_p \Theta(z-\tilde{z})} - 1 \right) \simeq q \left[ j\phi_p \Theta(z-\tilde{z}) \right], \quad (E.6)$$

where $q$ is the complex coupling coefficient of the original grating, $\tilde{\kappa} = \kappa$ since the strength of the grating is not changed, and the phase $\tilde{\phi} = \phi + d\phi_p \Theta(z-\tilde{z})$ is shifted by an amount $d\phi_p$ after the phase error position $\tilde{z}$. Indeed, $\Theta(x)$ is the Heaviside step function. Using the formalism introduced in Section 3.2.2 and with reference to Fig. 4.1, the corresponding phase change $d\phi$ in the fields propagating inside the grating is given by

$$d\phi = \beta(-|\Delta z|) = -\beta \frac{d\phi_p}{2\pi} \Lambda = -\frac{d\phi_p}{2\pi} \frac{2\pi n_{eff}}{\lambda} \frac{\lambda}{2n_{eff}} = -\frac{d\phi_p}{2}, \quad (E.7)$$

and it is directly related to the shift in the modulation pattern. The Bragg relation $\lambda = 2n_{eff}\Lambda$ was used in the derivation of Eq. (E.7).

It is now possible to derive the expression for the perturbed reflection coefficient using this perturbation approach. Using (E.5), (E.6), and (E.7), it is possible to write

$$\tilde{r} = r - j2d\phi \int_{\tilde{z}}^{L_{gr}} [q^*U^2 + qV^2] \, dz$$

$$= r - j2d\phi \int_{\tilde{z}}^{L_{gr}} \left[ \left( \frac{dV}{dz} + j\hat{\sigma}V \right) U + \left( \frac{dU}{dz} - j\hat{\sigma}U \right) V \right] \, dz$$

$$= r - j2d\phi \int_{\tilde{z}}^{L_{gr}} \frac{d[UV]}{dz} \, dz = r + j2d\phi U(\tilde{z})V(\tilde{z}) \, dz, \quad (E.8)$$

where $U \frac{V}{dz} + V \frac{U}{dz} = \frac{d[UV]}{dz}$ and $V(L_{gr}) = 0$ because of the boundary conditions that are used. Finally, the forward- and backward-propagating envelopes in the perturbation position are derived from their definition and from Eq. (3.20)

$$U(\tilde{z}) = \frac{t_{1+}}{1-\rho} e^{-j\frac{\Delta z}{\Lambda}}, \quad (E.9)$$

$$V(\tilde{z}) = \frac{t_{1+r_2}}{1-\rho} e^{j\frac{\Delta z}{\Lambda}}, \quad (E.10)$$

where the left and right transmission and reflection coefficients are referred to the layer $\tilde{z}$. The practical expression to be used is

$$\tilde{r} = r + j2d\phi \frac{t_{1+r_2}}{(1-\rho)^2} = r + j\phi_p rN_R. \quad (E.11)$$

The definition of the local number of passes $N_R$ in Eq. (3.14) has been used to simplify the final result and to show the remarkable agreement found with Eq. (3.36) derived using the local time delay approach. Indeed, assuming a small perturbation and using
a first order approximation

\[ \tilde{r} = re^{j\phi_N} \simeq r (1 + j\phi_N), \tag{E.12} \]

which is also a partial result obtained in the derivation of Eq. (D.6). It is therefore expected that the linearised expression can practically outperform the one in exponential form in certain conditions.
Appendix F

Code-related issues: a design example

Even if the attention is restricted to quadrupolar codes only, there are still some code-related issues worth analysing. As an example, the design of multifunction superstructured Bragg gratings can be considered (CDMA encoding/decoding + high reflectivity + dispersion compensation). The detailed analysis is presented elsewhere (Section 6.4.3), and only the practical issue of an appropriate signature code choice is discussed here. It is assumed in the design that propagation through 100 km of SMF-28 fibre ($D_{\lambda} = 15.68$ ps/nm/km) has to be compensated, that family $A$ code signatures with optimised phases and code length $L = 63$ are considered, that the mode-locked laser gives transform-limited, hyperbolic secant with $T_{FWHM} = 1.8$ ps, pulses and that the chip duration is $T_{chip} = 6.4$ ps. The grating length is limited to $L_{gr} = 120$ mm. Systematic computation of all the auto- and cross-correlation waveforms is performed. Autocorrelation peaks are normalised to the corresponding uncoded system values and give a measure of the insertion loss of the encoder-decoder pair. The average peak power after the intensity photodetector is as low as -31.1 dB, but, more noticeably, values are spread over a [-28, -37] dB range. This is obviously not acceptable for practical system implementations due to the huge power imbalance. An optimised subset of small cardinality has to be computed. Optimisation is performed by retaining the subset with higher minimum extinction value, i.e., better code performance, and under the further constraint of balanced autocorrelation peak power. If a subset of cardinality as small as 4 is considered and power imbalance is kept within 0.5 dB, the best signature sequences are labelled “1”, “11”, “47”, and “51”. They are obtained using the characteristic polynomial $h(x) = x^6 + 2x^3 + 3x + 1$ with generator $\alpha = 1$ according to the formalism in Sun and Leib [251]. The corresponding correlation values are shown in Table F.1.

It must be noted that the peak power of these sequences is below the average value reported previously. This trade-off has to be accepted in order to keep the orthogonality between all the codewords in the subset to the highest level possible. Nevertheless, poor consistency is found in the extinction values, with oscillations of more than 2 dB.
even within an optimised subset. If the subset cardinality is increased, the performance degradation is shown in Table F.2 considering the same parameters as before. For $|C| = 12$ the maximum peak power imbalance had to be increased to 0.6 dB since no subset exists with a 0.5 dB constraint. The collapse of the orthogonality properties is not catastrophic, and it is surprising that the actual averaged received power increases with subset size. This means that most of the signature sequences have consistent properties at sub-optimal cross-correlation values, and only the very small subset previously described is not fully penalised by the system implementation.

Table F.2 shows that the cross-correlation penalty with respect to pure-code values due to the practical implementation starts from 4 dB and increases with increased subset size. It has to be mentioned that with simpler, low-reflectivity, Fourier-transform designed implementations, cross-correlation extinctions of practical waveforms are in the -15 dB range, i.e., very close to the ideal value (or even better in a few cases). Then, this 4 dB performance degradation is mainly related to the high complexity of the multifunction operation and the physical length limitation of the grating, which does not allow complete dispersion compensation. Conversely, the further degradation with increasing subset size code properties can be ascribed to the use of continuous waveforms, since the code properties have been designed to be homogeneous for discrete, digital operation only. In a practical implementation phase transitions are not abrupt and depend

<table>
<thead>
<tr>
<th>Subset cardinality</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Peak (dB)</td>
<td>-31.58</td>
<td>-31.52</td>
<td>-31.37</td>
<td>-29.85</td>
<td>-29.85</td>
</tr>
<tr>
<td>Extinction (dB)</td>
<td>11.01</td>
<td>10.24</td>
<td>9.29</td>
<td>8.81</td>
<td>8.50</td>
</tr>
<tr>
<td>Extinction (code, dB)</td>
<td>15.34</td>
<td>15.02</td>
<td>14.40</td>
<td>14.37</td>
<td>14.37</td>
</tr>
</tbody>
</table>

Table F.2: Average peak power and degradation of the cross-correlation extinction with increasing subset cardinality. Corresponding extinctions for the pure code are also shown. Subsets are independently optimised. Values refer to power signals.
very much on the pulse shape and actual phase profile of the encoder/decoder structure. Custom design of the code signatures is necessary in order to tailor the characteristics of a given CDMA system, but it has never been attempted due to the lack of clear guidelines and is beyond the scope of this work. Alternatively, chip profile optimisation can be performed, following analogous work for mobile communications’ applications [252]. This is theoretically achievable in superstructured grating based CDMA systems by properly designing the chirp transition between adjacent chips in the grating, and it is consistent with the possibility of taking full advantage of the capabilities of the grating continuous writing set-up. This is a possibility of further development, but it will not be addressed in this work.

The need for such a custom analysis is further stressed by comparing the code performance results in slightly different system implementations. For instance, the temporal duration of each chip can be varied from $T_{\text{chip}} = 6.4 \text{ ps}$ to $9 \text{ ps}$ and $12.8 \text{ ps}$. The aim is to reduce the optical bandwidth of the encoded signal and therefore to have a better dispersion compensation under the constraint of a fixed grating length. If the code subset derived for the 6.4 ps case is used, the minimum extinction is raised to 11.03 dB and 12.04 dB, respectively. But if newly optimised subsets are computed, these values are further improved to 12.23 dB and 13.21 dB, with a significant 1 dB gain.

This simple consideration has major impact on the design of a superstructured grating based, DS-CDMA system, when multiple functionalities such as high reflectivity and dispersion compensation have to be included within the same structure. Indeed, a really high number of design parameters are available, most importantly, the code length, the maximum grating length, the chip duration, the shape and the squareness of the phase transitions in the grating, and the duration of the incoming laser pulse. For each combination, a different subset of the original code is expected to be optimal in terms of cross-correlation properties. But even if the code subset is identified, performance cannot be the only driving parameter in the choice. Grating manufacturing issues have to be considered. The grating’s amplitude and chirp variations have to be practically manufacturable, especially if very fast oscillations in the parameters are required. Noise in the manufacturing process should be accounted for, and the sensitivity of the possible different structures to such noise and to small variations in the system parameters computed.
List of Publications

Journal papers


Conference presentations


Invited conference presentations

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