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Totally chiral maps and hypermaps of small genus

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ABSTRACT

An orientably regular hypermap is totally chiral if it and its mirror image have no non-trivial common quotients. We classify the totally chiral hypermaps of genus up to 1001, and prove that the least genus of any totally chiral hypermap is 211, attained by twelve orientably regular hypermaps with monodromy group A_7 and type $(3,4,4)$ (up to triality). The least genus of any totally chiral map is 631, attained by a chiral pair of orientably regular maps of type $\{11,4\}$, together with their duals; their monodromy group is the Mathieu group M_{11}. This is also the least genus of any totally chiral hypermap with non-simple monodromy group, in this case the perfect triple covering $3\,.\,A_7$ of A_7. The least genus of any totally chiral map with non-simple monodromy group is 1457, attained by 48 maps with monodromy group isomorphic to the central extension $2\,.\,Sz(8)$.

1. Introduction

Maps on compact surfaces have for many years been a common topic of interest among those working on finite and discrete groups, on Riemann surfaces, and in areas as diverse as combinatorics and hyperbolic geometry. Their generalisations to hypermaps (surface embeddings of hypergraphs), introduced by Cori [Cor] in 1975, have recently attracted considerable attention in view of Grothendieck's theory of *dessins d'enfants* [Gro], in which Belyĭ's theorem [Bel] characterises the Riemann surfaces defined (as projective algebraic curves) over the field of algebraic numbers as those obtained in a canonical way from hypermaps. This means that these combinatorial objects play a ma-

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jor role in relating the Galois theory of algebraic number fields to the Teichmüller theory of Riemann surfaces (see [JSin], for instance).

Chirality is the phenomenon which occurs when an oriented object, such as a map, hypermap or Riemann surface, is not isomorphic to its mirror image under any orientation-preserving transformations. This concept is of great importance in science: for instance, a molecule and its mirror image may have very different chemical properties. It is also important in mathematics: for instance, the well-known Poincaré dodecahedral space and Weber–Seifert space both occur as chiral pairs of oriented 3-manifolds [Mon, §§3.7, 3.8, 3.13], [WS].

Within the category of oriented hypermaps, the most symmetric objects are the orientably regular (or rotary) hypermaps, those for which the orientation-preserving automorphism group acts transitively on darts. Such a hypermap $\mathcal{H}$ is said to be *regular* (or *reflexible*) if it has an orientation-reversing automorphism, so that $\mathcal{H}$ is isomorphic to its mirror image $\overline{\mathcal{H}}$; otherwise it is *chiral*. Early classifications of maps and hypermaps showed chirality to be a rather unusual phenomenon for low genus. For instance, there are no chiral hypermaps on the sphere, and on the torus they form three easily described infinite families, whose types are permutations of $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$. For each genus $g > 1$ there are (up to isomorphism) only finitely many orientably regular hypermaps, and when g is small relatively few of these are chiral. For instance, there are no chiral maps of genus g for $2 \leqslant g \leqslant 6$, the first examples as g increases being the chiral pair of Edmonds maps of genus 7 and type $\{7,7\}$, with monodromy group (and hence automorphism group) isomorphic to the affine group $AGL_1(8)$, together with a chiral pair of maps of type $\{6,9\}$ and their duals, with a monodromy group of order 54. Among the proper hypermaps (those of type (p,q,r) with $p,q,r > 2$) there are no chiral pairs of genus 2, and (up to triality) just one of genus 3, having type $(3,3,7)$ and a monodromy group of order 21; similarly, there is just one pair each for $g = 4, 6$ and 7 and there are none for $g = 5$. A computer classification by Conder [Con07] up to genus 101 has shown that chirality becomes more common as g increases, though regularity still predominates in this range. He has shown that there are, up to isomorphism and duality, 3378 regular orientable maps of genus g for $2 \leqslant g \leqslant 101$, but only 594 chiral pairs. Similarly for proper hypermaps in this range there are (up to isomorphism and triality) 14647 regular orientable hypermaps and 2191 chiral pairs. At present it is still not clear what the asymptotic behaviour is as $g \to \infty$.

In [BJNŠ], Nedela, Škoviera and the present authors introduced numerical and algebraic measures of chirality, called the *chirality index* and the *chirality group*, which indicate how much a hypermap differs from its mirror image. Here we consider the most extreme form of chirality, called *total chirality*, in which these two hypermaps are as unlike as possible, in the sense of having no non-trivial common quotients; this is a combinatorial analogue of two integers being mutually coprime. As noted in [BJNŠ], although there are infinitely many totally chiral hypermaps, those of small genus seem to be very rare. Indeed, the least genus of such a hypermap described there is 481, having type $(7,3,7)$ with monodromy group isomorphic to the alternating group A_7 (see Theorem 13 of [BJNŠ] and the remark following Theorem 14). Our aim here is to classify the totally chiral hypermaps of genus up to 1001 (see Theorem B and the tables in Appendix A), and in particular to describe the totally chiral hypermaps and maps of least genus. These genera turn out to be surprisingly large, namely 211 and 631 respectively (see Theorems A and C, and Examples 1, 2 and 5). The monodromy groups of the hypermaps and maps achieving these lower bounds are respectively isomorphic to A_7 and to the Mathieu group M_{11}. These groups are both simple, and in Theorem D we show that the least genus of any totally chiral hypermap with a composite monodromy group is also 631; in this case the monodromy group is the perfect triple covering group $3 \,.\, A_7$ of A_7, and the hypermaps are unbranched triple coverings of those of genus 211 mentioned above.

Our method is based on the observation that the monodromy group G of a totally chiral hypermap must be a perfect covering group of a non-abelian finite simple group S which is not of type $L_2(q)$. Such groups S are comparatively rare: for instance, there are only ten of order less than 10^5. The restriction $g \leqslant 1001$, together with the Hurwitz bound, implies that $|G| \leqslant 84000$, so that determining the relevant covering groups G is not difficult, using the Schur multiplier and the representation theory of S to deal with central and non-central extensions. Finding the chiral hypermaps associated with G is equivalent to classifying, up to automorphisms, the generating pairs for G which are not simultaneously inverted by any automorphism; in specific cases this can be done by character theory, using the character tables and other information in the *ATLAS of Finite Groups* [CCNPW], but in order to deal with the numerous possibilities which need to be eliminated we have also made extensive use of GAP [GAP]. If G is simple then total chirality is equivalent to chirality, and in other cases the groups G are sufficiently close to being simple that it is not difficult to determine which of the associated chiral hypermaps are totally chiral.

At the end of the paper we describe some constructions of infinite families of totally chiral hypermaps. By the nature of some constructions (and one of them given in [BJNŠ]) it is not properly a surprise that there are infinitely many totally chiral hypermaps. However, what is unexpected is that every totally chiral hypermap (and map) has infinitely many totally chiral hypermaps as unbranched coverings (Corollary 8).

2. Hypermaps

Throughout this paper, all hypermaps are assumed to be finite and oriented, and all automorphisms are assumed to be orientation-preserving. A hypermap $\mathcal{H}$ of type (l,m,n) can be regarded as a transitive permutation representation of the triangle group

$$\Delta = \Delta(l,m,n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle$$

on a set of darts: the hypervertices, hyperedges and hyperfaces correspond to the cycles of X, Y and Z, with incidence given by non-empty intersection; equivalently $\mathcal{H}$ corresponds to a conjugacy class of subgroups K of finite index in Δ, the stabilisers of darts, called *hypermap subgroups*. The monodromy group $\mathrm{Mon}\,\mathcal{H}$ of $\mathcal{H}$ is the group of permutations of the darts induced by Δ, and the automorphism group $\mathrm{Aut}\,\mathcal{H}$ is its centraliser in the symmetric group.

We say that $\mathcal{H}$ is *orientably regular* (or rotary) if $\mathrm{Aut}\,\mathcal{H}$ acts transitively on the darts, or equivalently K is a torsion-free normal subgroup of Δ, in which case $\mathrm{Mon}\,\mathcal{H} \cong \mathrm{Aut}\,\mathcal{H} \cong \Delta/K$. Thus the orientably regular hypermaps with a given monodromy group G correspond to the orbits of $\mathrm{Aut}\,G$ on generating triples for G of type (l,m,n) (that is, triples (x,y,z) of elements of order l,m,n that generate G and satisfy $xyz = 1$), or equivalently on pairs of generators x and y of G such that x, y and xy have orders l,m and n. Such a hypermap is denoted by $(G;x,y)$, and it is isomorphic to $(G';x',y')$ if and only if there is an isomorphism $G \to G'$ taking x to x' and y to y'. The darts of $(G;x,y)$ can then be identified with the elements of G, permuted regularly, and the hypervertices, hyperedges and hyperfaces with the cosets of the cyclic subgroups generated by x, y and z; incidence corresponds to non-empty intersection of cosets, and local orientation is determined by the successive powers of these generators. Maps are hypermaps of type $(l,2,n)$.

For the rest of this paper we will let $\mathcal{H}$ denote an orientably regular hypermap of type (l,m,n) with monodromy group G. If $N_G(l,m,n)$ is the number of the above generating triples in G, then since $\mathrm{Aut}\,G$ acts freely on them the number of such hypermaps is $N_G(l,m,n)/|\mathrm{Aut}\,G|$. They all have genus

$$g = 1 + \frac{|G|}{2}\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right). \tag{2.1}$$

Quotients $\mathcal{H}'$ of $\mathcal{H}$ correspond to subgroups K' of Δ containing K, and the degree d of the covering $\mathcal{H} \to \mathcal{H}'$ is equal to the index $|K' : K|$ of K in K'. If $\mathcal{H}$ and $\mathcal{H}'$ have Euler characteristics χ and χ', and genera g and g', then

$$\chi = d\chi' - b \leqslant d\chi',$$

where b is the total order of branching, so

$$g = 1 + d(g' - 1) + \frac{b}{2} \geqslant 1 + d(g' - 1), \tag{2.2}$$

with equality in either case if and only if the covering is unbranched.

Machì's operations [Mac] rename the hypervertices, hyperedges and hyperfaces of a hypermap, thus permuting l, m and n, without changing the surface or the monodromy group; the hypermaps obtained from $\mathcal{H}$ in this way are its *associates*. For instance, replacing (x, y, z) with (x, z, y^z) gives a hypermap $\mathcal{H}^{(12)}$ of type (l, n, m) in which the hyperedges and hyperfaces of $\mathcal{H}$ have been transposed. The relationship between $\mathcal{H}$ and its associates is sometimes known as *triality*.

The mirror image $\overline{\mathcal{H}}$ of $\mathcal{H}$ corresponds to the image $\overline{K}$ of K under the automorphism of Δ inverting X and Y. We say that $\mathcal{H}$ is *regular* if $\overline{\mathcal{H}} \cong \mathcal{H}$, that is, $\overline{K} = K$, or equivalently G has an automorphism inverting x and y; otherwise $\mathcal{H}$ is *chiral*. The reflection operation $\mathcal{H} \mapsto \overline{\mathcal{H}}$ commutes with Machì's operations, and together they generate a group Γ of twelve hypermap operations, isomorphic to $S_3 \times C_2$. Under the action of Γ, each orientably regular hypermap lies in an orbit of length dividing 12, all sharing the same genus and monodromy group. In classifying hypermaps with specific properties, we will generally pick one representative from each orbit of Γ, usually choosing the type so that $l \leqslant m \leqslant n$. In particular, if l, m and n are all distinct, then regular and chiral hypermaps of type (l, m, n) lie in orbits of length 6 and 12 respectively.

The *chirality index* $|K\overline{K} : K|$ of $\mathcal{H}$, introduced in [BJNŠ], is a measure of the chirality of $\mathcal{H}$, equal to 1 if and only if $\mathcal{H}$ is regular. The most extreme form of chirality occurs when the *co-chirality index* $|\Delta : K\overline{K}|$ of $\mathcal{H}$ is 1, or equivalently $K\overline{K} = \Delta$, so that $\mathcal{H}$ and $\overline{\mathcal{H}}$ have no non-trivial common quotients; in this case we say that $\mathcal{H}$ is *totally chiral*. Note that $K\overline{K}$ is a normal subgroup of Δ containing K, so if G is simple then $\mathcal{H}$ must be either regular or totally chiral (see [BJNŠ, Lemma 11]).

We finish this section with two examples, which we will later show to be the totally chiral hypermaps of least genus.

Example 1. The permutations $x = (1, 2, 3)(4, 5, 6)$ and $y = (1, 2)(3, 5, 6, 7)$ clearly generate a transitive subgroup of the alternating group A_7. Since $x^2y = (1, 5, 4, 7, 3)$, and no transitive proper subgroup of A_7 has order divisible by 5, it follows that x and y generate A_7. We will denote the resulting orientably regular hypermap $(A_7; x, y)$ by $\mathcal{A}_7{}^{[1]}$, since it is our first example with monodromy group A_7. Since $z := (xy)^{-1} = (2, 3, 7, 5)(4, 6)$ it has type $(3, 4, 4)$; since $|A_7| = 7!/2 = 2520$, Eq. (2.1) implies that it has genus 211. Now A_7 has automorphism group S_7, acting on A_7 by conjugation, and it is easily seen that no element of S_7 conjugates both x and y to their inverses, so $\mathcal{A}_7{}^{[1]}$ is chiral. Since A_7 is simple, it follows that $\mathcal{A}_7{}^{[1]}$ is totally chiral, as are all members of its Γ-orbit. In fact $\mathcal{A}_7{}^{[1]} \cong \mathcal{A}_7{}^{[1](12)}$ since $g = (1, 6, 3, 5, 2, 4)$ conjugates the pair (x, y) to (x, z), so this orbit contains six totally chiral hypermaps of genus 211, forming a chiral pair of each of the types $(3, 4, 4)$, $(4, 3, 4)$ and $(4, 4, 3)$.

Example 2. Similar arguments show that the permutations $x = (1, 2, 3)(4, 5, 6)$ and $y = (1, 5)(2, 4, 6, 7)$ determine a totally chiral hypermap $\mathcal{A}_7{}^{[2]}$ of type $(3, 4, 4)$ and genus 211 with monodromy group A_7. Again we have $\mathcal{A}_7{}^{[2]} \cong \mathcal{A}_7{}^{[2](12)}$, so this Γ-orbit contains six hypermaps of genus 211. These are distinct from those in Example 1 since there is no element of S_7 which conjugates the generators x and y to those in Example 1 or their inverses.

In Lemma 5 and Corollary 6 we will give a more general method for proving chirality and non-isomorphism of orientably regular hypermaps.

3. Counting triples and hypermaps

We will show that the hypermaps $\mathcal{A}_7{}^{[1]}$, $\mathcal{A}_7{}^{[2]}$, $\overline{\mathcal{A}_7{}^{[1]}}$ and $\overline{\mathcal{A}_7{}^{[2]}}$ are the only orientably regular hypermaps of type $(3, 4, 4)$ with monodromy group A_7. For this, and in many other places in this paper, we need to count generating triples of a given type in a finite group. If the group is not too large, one can easily do this by using a program such as GAP. It is also possible to do this by hand, and thus to check the computer calculations, by using character tables. Specifically, if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are conjugacy classes in a finite group G, then the number of solutions of $xyz = 1$ in G, with $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, is given by the formula

$$\frac{|\mathcal{X}|\,.\,|\mathcal{Y}|\,.\,|\mathcal{Z}|}{|G|}\sum_{\chi}\frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}, \tag{3.1}$$

where χ ranges over the irreducible complex characters of G (see [Ser, §7.2] for this and other similar results). Such a triple generates G if and only if it is not contained in any maximal subgroup of G, so knowledge of the maximal subgroups allows one to count the generating triples obtained from the classes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. By summing over all triples of classes of elements of orders l, m and n we then obtain the total number $N_G(l,m,n)$ of generating triples of type (l,m,n). As an example, we have the following result. (In the proof, and throughout this paper, we use *ATLAS* notation [CCNPW] for finite groups, together with their conjugacy classes and their characters; in particular, $L_n(q)$ denotes the projective special linear group $PSL_n(q) = SL_n(q)/\{\pm I\}$ over the field of q elements.)

Proposition 1. *The group $G = A_7$ has* 20160 *generating triples of type* $(3,4,4)$.

Proof. The elements of order 3 in A_7 form two conjugacy classes, namely the 70 elements of class 3A (in *ATLAS* notation) with cycle-structure 31^4 in the natural representation of A_7, and the 280 elements of class 3B with cycle-structure 3^21. There is a single class 4A of 630 elements of order 4, with cycle-structure 421. It is easy to see that if x and y are in classes 3A and 4A, and the subgroup they generate is transitive, then xy cannot have order 4; thus there are no generating triples of type $(3,4,4)$ with x in 3A, so we may assume that x is in 3B. Applying formula (3.1) with x in 3B and y and z in 4A, we find from the character table of A_7 in [CCNPW] that the resulting number of triples is

$$\frac{280\times 630^2}{2520}\left(1+\frac{(-1)\times 1^2}{35}\right)=42840=17|G|,$$

with only the characters χ_1 and χ_9, of degrees 1 and 35, making non-zero contributions to the sum. It is easy to see that if $\langle x, y\rangle$ is intransitive it must have two orbits, of lengths 1 and 6 or 3 and 4. In the first case, x and y lie in exactly one of the seven conjugates of A_6 in A_7; now the character table of A_6 shows that it has 1080 triples of type $(3,4,4)$ with x having cycle-structure 3^2, so we obtain $7560 = 3|G|$ such triples in A_7. In the second case x and y lie in exactly one of the $\binom{7}{3} = 35$ conjugates of the even subgroup of $S_3 \times S_4$ in A_7; by inspection, this group contains $6 \times 24 = 144$ such triples, so we obtain $5040 = 2|G|$ triples of this type. The only proper subgroups of G of order divisible by $3, 4$ and 7 are the two conjugacy classes of 15 subgroups isomorphic to $L_2(7)$, so if x and y generate a transitive proper subgroup it must be one of these. The character table of $L_2(7)$ shows that it contains 336 such triples, so the number of them in A_7 is $30 \times 336 = 10080 = 4|G|$. This leaves $(17-3-2-4)|G| = 8|G| = 20160$ triples generating G. □

Corollary 2. *The hypermaps $\mathcal{A}_7{}^{[1]}$, $\overline{\mathcal{A}_7{}^{[1]}}$, $\mathcal{A}_7{}^{[2]}$ and $\overline{\mathcal{A}_7{}^{[2]}}$ described in Examples* 1 *and* 2 *are the only orientably regular hypermaps of type* $(3,4,4)$ *with monodromy group* A_7.

Proof. Since A_7 has automorphism group S_7 of order 7!, it follows from Proposition 1 and the remarks in Section 2 that there are $20160/7! = 4$ such hypermaps, so they must be the four described in Examples 1 and 2. □

Clearly the corresponding result also applies to the hypermaps of type $(4,3,4)$ and $(4,4,3)$ described in Examples 1 and 2.

4. Numerical bounds

In order to prove our main theorems, we need an argument which restricts the triples that we have to consider. There are no chiral hypermaps on the sphere, and those on the torus are not totally chiral since their monodromy groups are not perfect (see Lemma 4(b)). Thus a totally chiral hypermap must have genus $g > 1$, so its type (l, m, n) must be a 'hyperbolic' triple, that is,

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1.$$

We can then write (2.1) as

$$g = 1 + \frac{|G|}{\lambda}, \tag{4.1}$$

where

$$\lambda := 2\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right)^{-1}$$

depends only on (l, m, n). If $g \leqslant 211$ (the genus in Examples 1 and 2) then (4.1) gives

$$\lambda \geqslant \frac{|G|}{210}, \tag{4.2}$$

and this, together with the following lemma, can be used to show that various triples cannot correspond to totally chiral hypermaps of smaller genus.

Lemma 3.

(a) *The only hyperbolic types* (l, m, n) *with* $\lambda \geqslant 936/35 = 26.742\ldots$ *are the following* (*up to permutations of* l, m *and* n):

$$(2, 3, n) \textit{ for } n = 7, 8, 9 \textit{ and } 10, \textit{ and } (2, 4, 5).$$

(b) *The only hyperbolic types* (l, m, n) *with* $936/35 = 26.742\ldots > \lambda \geqslant 20.16$ *are the following* (*up to permutations of* l, m *and* n):

$$(2, 3, n) \textit{ for } n = 11, 12, 13 \textit{ and } 14, \textit{ and } (2, 4, 6) \textit{ and } (3, 3, 4).$$

(c) *The only hyperbolic types* (l, m, n) *with* $\lambda \geqslant 12$ *and* $l, m, n \leqslant 7$ *are the following* (*up to permutations of* l, m *and* n):

$$(2, 3, 7), (2, 4, 5), (2, 4, 6), (2, 4, 7), (2, 5, 5), (2, 5, 6), (2, 5, 7), (2, 6, 6), (3, 3, 4),$$

$$(3, 3, 5), (3, 3, 6) \textit{ and } (3, 4, 4).$$

The proof of this is a straightforward but tedious exercise in elementary arithmetic, and is therefore omitted. The rather strange bounds on λ in parts (a) and (b) are required in the proofs of Theorems A and B. The relevant types are listed in Table 1: on the left are those in part (a) and (after a gap) part (b), and on the right those in part (c), in each case in decreasing order of λ. The first column gives the type, the second column gives the number of types formed by applying triality, and the third column gives the corresponding value of λ.

A more extensive table, including the values of λ for all the arithmetic triangle groups, can be found in [BJ]; this omits $(2, 3, 13)$ and $(2, 5, 7)$ since the associated triangle groups are not arithmetic groups.

Table 1
Hypermap types and values of λ appearing in Lemma 3.

Type	No.	λ (a)
(2, 3, 7)	6	84
(2, 3, 8)	6	48
(2, 4, 5)	6	40
(2, 3, 9)	6	36
(2, 3, 10)	6	30
		(b)
(2, 3, 11)	6	$132/5 = 26.4\ldots$
(2, 3, 12)	6	24
(2, 4, 6)	6	24
(3, 3, 4)	3	24
(2, 3, 13)	6	$156/7 = 22.2\ldots$
(2, 3, 14)	6	21

Type	No.	λ (c)
(2, 3, 7)	6	84
(2, 4, 5)	6	40
(2, 4, 6)	6	24
(3, 3, 4)	3	24
(2, 5, 5)	3	20
(2, 4, 7)	6	$56/3 = 18.6\ldots$
(2, 5, 6)	6	15
(3, 3, 5)	3	15
(2, 5, 7)	6	$140/11 = 12.7\ldots$
(2, 6, 6)	3	12
(3, 3, 6)	3	12
(3, 4, 4)	3	12

5. The totally chiral hypermaps of least genus

Before we prove our first main result we establish the following consequences of totally chirality.

Lemma 4. *Let $\mathcal{H}$ be a totally chiral hypermap.*

(a) *Any non-trivial orientably regular quotient of $\mathcal{H}$ is also totally chiral.*
(b) *The monodromy group G of $\mathcal{H}$ is a perfect group.*
(c) *G does not have any simple group $L_2(q)$ as a quotient group.*

Proof. (a) If $\mathcal{H}$ is totally chiral, then so is any orientably regular quotient of $\mathcal{H}$, since its hypermap subgroup of Δ contains that corresponding to $\mathcal{H}$.

(b) An abelian group has an automorphism inverting all its elements, so it cannot be the monodromy group of a totally chiral hypermap. It therefore follows from (a) that G has no non-trivial abelian quotients, so G is perfect.

(c) As shown in [BJNŠ, Proposition 8, Corollary 10], any pair of generators of $L_2(q)$ is inverted by an automorphism of that group, so (c) also follows immediately from (a). □

The converse of Lemma 4(a) is false, since an orientably regular covering of a totally chiral hypermap need not be totally chiral (see Example 8 in Section 11). We will prove a partial converse in Proposition 6.

It follows from Lemma 4 that G is a covering of a non-abelian simple group $S \ncong L_2(q)$. For small orders, there are very few such simple groups, for instance only four of order less than 20160. This forms the basis of the proof of our first main result:

Theorem A. *The least genus of any totally chiral hypermap is* 211, *attained only by the twelve hypermaps formed from $\mathcal{A}_7^{[1]}$ and $\mathcal{A}_7^{[2]}$ by taking associates and mirror images. They all have monodromy group A_7 and type* $(3, 4, 4)$, $(4, 3, 4)$ *or* $(4, 4, 3)$.

Proof. Let G be the monodromy group of a totally chiral hypermap of least genus g. By Examples 1 and 2 we have $g \leqslant 211$, so $|G| \leqslant 84(g-1) \leqslant 17640$ by the Hurwitz bound. By Lemma 4(b) G is perfect, and moreover G must be simple, for otherwise Lemma 4(a) implies that a proper epimorphic image of G would be the monodromy group of a totally chiral hypermap $\mathcal{H}'$, and by (2.2) this would have genus $g' < g$. We therefore consider the non-abelian simple groups of order at most 17640 as possible monodromy groups of totally chiral hypermaps. According to the *ATLAS* the simple groups within this range are:

(a) $L_2(q)$ for various odd $q \leqslant 31$ and even $q \leqslant 16$,
(b) A_7 of order $2520 = 2^3 . 3^2 . 5 . 7$,
(c) $L_3(3)$ of order $5616 = 2^4 . 3^3 . 13$,
(d) $U_3(3)$ of order $6048 = 2^5 . 3^3 . 7$,
(e) M_{11} of order $7920 = 2^4 . 3^2 . 5 . 11$.

We can immediately eliminate case (a) by Lemma 4(c). We will eliminate cases (c), (d) and (e) by using the inequality (4.2) and Lemma 3(a).

First let $G = L_3(3)$, so $|G|/210 = 5616/210 = 936/35 = 26.742\ldots$. By Lemma 3(a) the only types satisfying $\lambda \geqslant |G|/210$ are $(2,3,n)$ for $n = 7$, 8, 9 and 10, and $(2,4,5)$. However $(2,3,7)$, $(2,3,9)$, $(2,3,10)$ and $(2,4,5)$ are impossible since G has no elements of order 5, 7 or 9, and a calculation similar to the proof of Proposition 1, using GAP or the character table of $L_3(3)$, shows that G has no generating triples of type $(2,3,8)$.

If G is the unitary group $U_3(3)$ then $|G|/210 = 28.2$, so by Lemma 3(a) and Table 1 the same five types must be considered. In this case $(2,3,10)$ and $(2,4,5)$ are impossible since G has no elements of order 5, and we find that there are no generating triples of the other three types.

If G is the Mathieu group M_{11} then $|G|/210 = 37.714\ldots$, so the only types to consider are $(2,3,7)$, $(2,3,8)$ and $(2,4,5)$. The first is impossible since G has no elements of order 7, and there are no generating triples of the other two types.

Thus $G = A_7$, so $|G|/210 = 12$. Since A_7 has no elements of order greater than 7 it follows that the only types we need to consider are the twelve listed in Lemma 3(c). The methods used earlier show that there are no generating triples of types $(2,3,7)$, $(2,4,5)$, $(2,4,6)$, $(2,5,5)$, $(2,5,6)$, $(2,6,6)$ or $(3,3,4)$ in A_7. There are 10080 each of types $(2,4,7)$ and $(2,5,7)$, giving two hypermaps in each case, of genus $g = 136$ and 199; by considering generating pairs one sees that these are both regular. There are 5040 each of types $(3,3,5)$ and $(3,3,6)$, giving one hypermap in each case, of genus 169 and 211; by their uniqueness these are necessarily regular. Finally there are 20160 generating triples of type $(3,4,4)$, giving the four hypermaps of genus 211 described in Examples 1 and 2. Applying triality gives the other eight hypermaps of genus 211 described there. □

Remark. One could alternatively regard the smallest totally chiral hypermaps as those having the smallest monodromy group. It is clear from the proof of Theorem A that A_7 is the smallest monodromy group which can occur, so in this sense the smallest totally chiral hypermaps are those with monodromy group A_7. These include all those of genus 211 appearing in Theorem A, but also other totally chiral hypermaps of higher genus such as those of genus 481 mentioned in Section 1; Table 2 lists all those of genus $g \leqslant 1001$.

6. Testing hypermaps for isomorphism and reflexibility

The method of proof of Theorem A can be extended to classify all the totally chiral hypermaps of comparatively small genus. We shall do this for genus $g \leqslant 1001$, though in principle it is possible to go further.

Before proceeding with the classification, we describe a simple method for determining isomorphism or regularity of hypermaps, which we will use repeatedly. Given an orientably regular hypermap $\mathcal{H}$ with monodromy group G, any transitive permutation representation $\rho : G \to S_k$, with a point stabiliser $H \leqslant G$, gives rise to a quotient hypermap $\mathcal{H}^* = \mathcal{H}/H$ with k darts. This hypermap is orientable, but not orientably regular unless H is normal in G; it has monodromy group $G^* = \rho(G) \cong G/\ker(\rho)$. We call $\mathcal{H}^*$ a *faithful quotient* of $\mathcal{H}$ if ρ is faithful, or equivalently H has trivial core $\bigcap_{g\in G} H^g = \ker(\rho)$ in G, so that $G^* \cong G$. In these circumstances, $\mathcal{H}$ is uniquely determined by $\mathcal{H}^*$ as its minimal orientably regular covering, in the sense that any orientably regular covering of $\mathcal{H}^*$ is also a covering of $\mathcal{H}$.

The objective is to find a faithful quotient $\mathcal{H}^*$ of $\mathcal{H}$ which adequately reflects certain properties of $\mathcal{H}$, so that it can be used as a simpler substitute for $\mathcal{H}$ in specific calculations (see Lemma 5 and Corollary 6, for instance). For convenience it is desirable (though not always possible) that $\mathcal{H}^*$ should be *spherical*, that is, have genus 0, and that the degree $k = |G : H|$ should be small. For simple groups G, the *ATLAS* gives useful information about maximal subgroups and hence transitive representations of comparatively small degrees, all of them necessarily faithful.

If k is sufficiently small, it can be useful to represent $\mathcal{H}^*$ visually by its *Walsh map* [Wal]. This construction (which can be applied to any hypermap) produces a bipartite map on the same surface, with black and white vertices, edges and faces corresponding to the hypervertices, hyperedges, darts and hyperfaces of $\mathcal{H}^*$, and with the rotation of edges around each black or white vertex determined by the cyclic order of darts in the corresponding hypervertex or hyperedge. Isomorphisms of hypermaps correspond to colour-preserving isomorphisms of their Walsh maps, and it often easy to see from a diagram whether or not these exist.

Lemma 5. *Let $\mathcal{H}$ and $\mathcal{H}'$ be orientably regular hypermaps with monodromy groups isomorphic to G, and let H be a subgroup of G with trivial core.*

(a) *If $\mathcal{H}/H \cong \mathcal{H}'/H$ then $\mathcal{H} \cong \mathcal{H}'$.*
(b) *If $\mathcal{H} \cong \mathcal{H}'$ and every subgroup of G equivalent under* Aut G *to H is conjugate in G to H, then $\mathcal{H}/H \cong \mathcal{H}'/H$.*

Proof. (a) The hypermap subgroups K and K' of Δ corresponding to $\mathcal{H}$ and $\mathcal{H}'$ are the cores in Δ of those corresponding to $\mathcal{H}/H$ and $\mathcal{H}'/H$. If these quotients are isomorphic then their hypermap subgroups are conjugate and hence have the same core, so $K = K'$ and hence $\mathcal{H} \cong \mathcal{H}'$.

(b) If $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic then they correspond to the same (normal) hypermap subgroup K of Δ, with $\Delta/K \cong G$. Let L and L' be the subgroups of Δ, containing K, which correspond to H under the isomorphisms of $\mathrm{Mon}\,\mathcal{H}$ and $\mathrm{Mon}\,\mathcal{H}'$ with Δ/K. Then L/K and L'/K are equivalent under automorphisms of Δ/K, so by the hypothesis they are conjugate in Δ/K and hence L and L' are conjugate in Δ; but these are the hypermap subgroups corresponding to $\mathcal{H}/H$ and $\mathcal{H}'/H$, so these quotients are isomorphic. □

Corollary 6. *Let $\mathcal{H}$ be an orientably regular hypermap with monodromy group G, and let H be a subgroup of G with trivial core.*

(a) *If $\mathcal{H}/H \cong \overline{\mathcal{H}/H}$ then $\mathcal{H}$ is regular.*
(b) *If $\mathcal{H}$ is regular and every subgroup of G equivalent under* Aut G *to H is conjugate to H, then $\mathcal{H}/H \cong \overline{\mathcal{H}/H}$.*

Proof. This follows immediately from Lemma 5, with $\mathcal{H}' = \overline{\mathcal{H}}$. □

When testing hypermaps for isomorphism or regularity, Lemma 5 and Corollary 6 can be useful in replacing large hypermaps with much smaller ones, as shown by the following example.

Example 3. The hypermap $\mathcal{H} = \mathcal{A}_7{}^{[1]}$ in Example 1 has monodromy group $G = A_7$; the subgroup $H = A_6$, a point stabiliser in the natural representation of A_7, has trivial core, and its conjugates form the unique conjugacy class of subgroups of index 7 in A_7, so it satisfies the hypotheses of Lemma 5 and Corollary 6. The hypermap $\mathcal{A}_7{}^{[1]}$ has $|A_7| = 2520$ darts, whereas $\mathcal{A}_7{}^{[1]}/A_6$ has only 7; it is easy to use the permutations x and y to draw the Walsh map for $\mathcal{A}_7{}^{[1]}/A_6$; Fig. 1(a) shows that this map is not isomorphic to its mirror image, so $\mathcal{A}_7{}^{[1]}$ is chiral by Corollary 6(b). Similarly Fig. 1(b) shows that the hypermap $\mathcal{A}_7{}^{[2]}$ in Example 2 is chiral. Since these two maps cannot be transformed into each other by reflection and/or transposition of colours, it follows from Lemma 5(b) that the hypermaps in Examples 1 and 2 form different orbits of Γ.

Parts (b) of Lemma 5 and Corollary 6 can fail if there is a second conjugacy class of subgroups of G which are equivalent to H under outer automorphisms of G. The following example is instructive.

Example 4. The group $G = L_3(3)$ has 33696 generating triples of type $(3, 3, 4)$. Since $\mathrm{Aut}\,G = PGL_3(3)$ has order 11232, these correspond to three orientably regular hypermaps of genus 235 with monodromy group $L_3(3)$. Two of these form a chiral pair $\mathcal{H}_1 = \mathcal{L}_3(3)^{[1]}$ and $\mathcal{H}_2 = \overline{\mathcal{H}_1} \cong \mathcal{H}_1{}^{(01)}$, which are

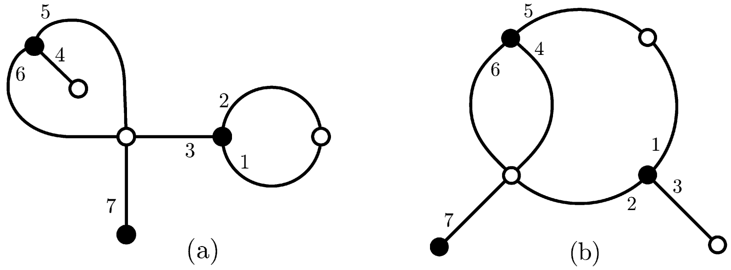

Fig. 1. The quotient hypermaps $\mathcal{A}_7^{[1]}/A_6$ and $\mathcal{A}_7^{[2]}/A_6$.

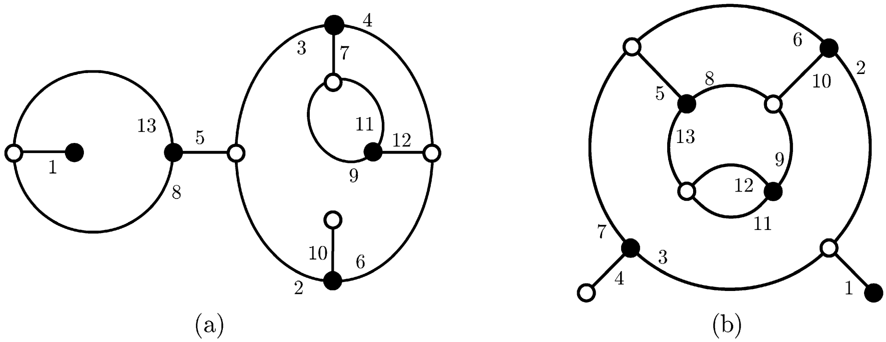

Fig. 2. The quotient hypermaps $\mathcal{L}_3(3)^{[1]}/H$ and $\mathcal{H}_3/H$.

totally chiral since $L_3(3)$ is simple, whereas the third is regular and self-dual. We can represent $L_3(3)$ faithfully and transitively on the 13 points of the projective plane over the field F_3, with point stabiliser $H \cong AGL_2(3)$. The faithful quotient $\mathcal{H}_1/H$, shown in Fig. 2(a), corresponds to the generating pair $x = (2, 6, 10)(3, 7, 4)(5, 13, 8)(9, 12, 11)$, $y = (1, 13, 8)(2, 3, 5)(4, 12, 6)(7, 9, 11)$, and $\overline{\mathcal{H}_1}/H$, the mirror image of Fig. 2(a), corresponds to their inverses. These two quotient hypermaps form a chiral pair, but we cannot use Corollary 6(b) to deduce the same for their coverings $\mathcal{H}_1$ and $\overline{\mathcal{H}_1}$ since the outer automorphism of $L_3(3)$ (induced by the point-line duality of the projective plane) transposes the point stabilisers with a second conjugacy class of subgroups, the stabilisers of lines. Instead we can argue that $\mathcal{H}_1$ is chiral because $w(x, y) = (x^{-1}y)^2(yxy)^2(xy^{-1})^3x(xy)^2[x, y]$ is the identity permutation while $w(x^{-1}, y^{-1}) = (1, 13, 2, 6, 9, 12, 11, 10, 8, 4, 3, 7, 5)$ has order 13, where $[x, y] = x^{-1}y^{-1}xy$ is the commutator of x and y, so $\mathcal{H}_1$ and $\overline{\mathcal{H}_1}$ correspond to distinct normal subgroups of Δ. By applying triality we obtain a Γ-orbit of six totally chiral hypermaps of genus 235 with monodromy group $L_3(3)$, forming three chiral pairs of types $(3, 3, 4)$, $(4, 3, 3)$ and $(3, 4, 3)$.

In the case of the third hypermap $\mathcal{H}_3$, we have the same x as for $\mathcal{H}_1 = \mathcal{L}_3(3)^{[1]}$ but now $y = (1, 2, 3)(5, 6, 7)(8, 9, 10)(11, 12, 13)$. The quotient hypermap $\mathcal{H}_3/H$, shown in Fig. 2(b), is again chiral, since a reflection transposes the vertex-colours, but in this case the hypermap itself is regular. As in the case of $\mathcal{L}_3(3)^{[1]}$, Corollary 6(b) does not apply to H. In both cases, if we want a subgroup which satisfies the condition in Corollary 6(b), we could take H to be the stabiliser of an incident point-line pair: this has index 52 in G, so it gives quotient hypermaps with four times as many darts.

7. Classifying totally chiral hypermaps of low genus

Before using the techniques developed in earlier sections to prove our main theorem, we first need to explain some central extensions which appear in its statement and proof; see [CCNPW, §4.1] or [Hup, §V.23] for further details. A group G is a *central extension* of a group S by a group N if G has a central subgroup N such that $G/N \cong S$; this extension is *proper* if N is contained in the commutator subgroup G' of G. The groups N appearing in proper central extensions of S are all quotients of a group $\mathrm{Mult}(S)$, the *Schur multiplier* of S. If S is perfect then there is a unique proper central extension $G = \hat{S}$ of S by $\mathrm{Mult}(S)$; this is the *universal covering group* of S, and any proper central extension of S is a quotient of $\hat{S}$ by some subgroup of $\mathrm{Mult}(S)$.

Each alternating group $S = A_n$ $(n \geqslant 5)$ has a unique proper central extension G by $N = C_2$, denoted by $2 . A_n$. One can construct this extension by embedding A_n, as the group of even permutation matrices, in the Lie group $SO(n)$; this has simply connected covering group $\mathrm{Spin}(n)$, a central extension of $SO(n)$ by its fundamental group $\pi_1 SO(n) \cong C_2$ [Che, §XII], and $2 . A_n$ is the inverse image of A_n in $\mathrm{Spin}(n)$. Schur [Sch] showed that if $n \neq 6, 7$ then $\mathrm{Mult}(A_n) \cong C_2$, so that $2 . A_n$ is the universal covering group $\hat{A}_n$ of A_n. However, if $n = 6$ or 7 then $\mathrm{Mult}(A_n) \cong C_6$, so there are proper central extensions $d . A_n$ of A_n by C_d for $d = 2, 3$ and 6. The triple covering group $3 . A_7$ can be constructed from the well-known Hoffman–Singleton graph [HS], [Cam, §3.6], which has the simple unitary group $U_3(5)$ as a subgroup of index 2 in its automorphism group. The stabiliser in $U_3(5)$ of one of the 50 vertices is isomorphic to A_7, and its inverse image under the natural epimorphism $SU_3(5) \to U_3(5)$, taking matrices to projective transformations, is the required group $3 . A_7$, with centre $N \cong C_3$ consisting of the scalar matrices αI where $\alpha^3 = 1$ in the Galois field F_{25}.

Theorem B. *The totally chiral hypermaps of genus $g \leqslant 1001$ are those listed in Tables 2, 3, 4, 5, 6, 7 and 8 (see Appendix A), together with those formed from them by taking mirror images and associates. In each case the monodromy group is a simple group A_7, $L_3(3)$, $U_3(3)$, M_{11} or $Sz(8)$, or the proper central extension $d . A_7$ of A_7 by C_d for $d = 2$ or 3.*

Proof. In order to extend our method of proof of Theorem A up to genus 1001, we need to consider monodromy groups G of order at most $84 \times 1000 = 84000$, and we need to modify inequality (4.2) to give $\lambda \geqslant |G|/1000$.

By Lemma 4(b) the monodromy group G of a totally chiral hypermap $\mathcal{H}$ is perfect, so if N is any maximal normal subgroup of G then G/N is a non-abelian simple group S; by Lemma 4(a) S is the monodromy group of a totally chiral hypermap $\mathcal{H}' = \mathcal{H}/N$, which has genus $g' \leqslant 1001$ by (2.2), and by Lemma 4(c) we have $S \ncong L_2(q)$. In addition to the simple groups A_7, $L_3(3)$, $U_3(3)$ and M_{11} considered in the proof of Theorem A, the only other non-abelian simple groups $S \ncong L_2(q)$ with $|S| \leqslant 84000$ are:

$A_8 \cong L_4(2)$ and $L_3(4)$, both of order $20160 = 2^6 . 3^2 . 5 . 7$,
the symplectic group $S_4(3) \cong U_4(2)$ of order $25920 = 2^6 . 3^4 . 5$,
the Suzuki group $Sz(8)$ of order $29120 = 2^6 . 5 . 7 . 13$, and
the unitary group $U_3(4)$ of order $62400 = 2^6 . 3 . 5^2 . 13$.

We will first show that of these five extra groups, only $S = Sz(8)$ can occur. It is easiest to deal with the largest groups first, since they correspond to the fewest triples.

If $S = U_3(4)$ then $\lambda \geqslant |S|/1000 = 62.4$, and Lemma 3 and Table 1 show that the only type satisfying this condition is $(2, 3, 7)$. Since $U_3(4)$ has no elements of order 7, it follows that neither this group S nor any of its coverings G can appear as the monodromy group of a totally chiral hypermap of genus $g \leqslant 1001$.

Similarly, if $S = S_4(3)$, with elements of orders $1, 2, 3, 4, 5, 6, 9$ and 12, it follows from Lemma 3 and Table 1 that the only possible types are $(2, 3, 9)$ and $(2, 4, 5)$. If $S = L_3(4)$, with elements of orders $1, 2, 3, 4, 5$ and 7, only $(2, 3, 7)$, $(2, 4, 5)$ and $(3, 3, 4)$ are possible. In the case of $S = A_8$, which has elements of orders $1, 2, 3, 4, 5, 6, 7$ and 15, we obtain just these three types, together with $(2, 4, 6)$. However, GAP shows that none of these three groups S has a generating triple of any of these listed types, so they can be eliminated.

This leaves the simple groups $S = A_7$, $L_3(3)$, $U_3(3)$, M_{11} and $Sz(8)$, together with their covering groups G, as possible monodromy groups of totally chiral hypermaps of genus at most 1001. Now the quotient hypermap $\mathcal{H}'$ is totally chiral, so by Theorem A it has genus $g' \geqslant 211$; it then follows from (2.2) that $\mathcal{H}$ has genus $g \geqslant 210|N| + 1$. Since $g \leqslant 1001$ this implies that $|N| \leqslant 4$. Now groups of order at most 4 have solvable automorphism groups, whereas G is perfect, so N is in the centre of G, and is isomorphic to a quotient of the Schur multiplier $\mathrm{Mult}(S)$ of S. By [CCNPW] the groups $S = L_3(3)$, $U_3(3)$ and M_{11} all have trivial Schur multipliers, so if S is one of these then $G = S$. Since $\mathrm{Mult}(A_7) \cong C_6$ and $\mathrm{Mult}(Sz(8)) \cong C_2 \times C_2$, the only other possibilities are that $S = A_7$ and $N \cong C_1, C_2$ or C_3, or that $S = Sz(8)$ and $N \cong C_1, C_2$ or $C_2 \times C_2$.

If $G = Sz(8)$ then $\lambda \geqslant 29.12$. Since Suzuki groups have no elements of order 3, Lemma 3 and Table 1 show that the only possible type is $(2, 4, 5)$. As shown in [JS], there are four totally chiral hypermaps of this type with monodromy group $Sz(8)$, forming two chiral pairs of genus 729 (see Table 6). Any covering of $Sz(8)$ with $|N| \geqslant 2$ would give genus $g \geqslant 1457$ by (2.2), and this is outside our range.

If $S = M_{11}$ then $\lambda \geqslant 7.92$. We have not listed all the triples satisfying this inequality, but since M_{11} has elements of orders $1, 2, 3, 4, 5, 6, 8$ and 11, the only ones we need consider are $(2, 3, 11)$, $(2, 4, 6)$, $(2, 4, 8)$, $(2, 4, 11)$, $(2, 5, 5)$, $(2, 5, 6)$, $(2, 5, 8)$, $(2, 5, 11)$, $(2, 6, 6)$, $(2, 6, 8)$, $(2, 6, 11)$, $(2, 8, 8)$, $(3, 3, 4)$, $(3, 3, 5)$, $(3, 3, 6)$, $(3, 3, 8)$, $(3, 3, 11)$, $(3, 4, 4)$, $(3, 4, 5)$, $(3, 4, 6)$ and $(4, 4, 4)$. For the first three types, together with $(2, 5, 5)$, $(2, 5, 6)$, $(2, 6, 6)$, $(3, 3, 4)$, $(3, 3, 5)$, $(3, 3, 6)$, $(3, 3, 11)$, $(3, 4, 4)$ and $(4, 4, 4)$, GAP shows that there are no generating triples. For the remaining types the corresponding hypermaps, as found by GAP, are listed in Table 5.

Similarly the group $G = U_3(3)$ has elements of orders $1, 2, 3, 4, 6, 7, 8$ and 12, with $\lambda \geqslant 6.048$, so we obtain only the types $(2, 4, 6)$, $(2, 4, 7)$, $(2, 4, 8)$, $(2, 4, 12)$, $(2, 6, 6)$, $(2, 6, 7)$, $(2, 6, 8)$, $(2, 6, 12)$, $(2, 7, 7)$, $(2, 7, 8)$, $(2, 7, 12)$, $(2, 8, 8)$, $(2, 8, 12)$, $(3, 3, 4)$, $(3, 3, 6)$, $(3, 3, 7)$, $(3, 3, 8)$, $(3, 3, 12)$, $(3, 4, 4)$, $(3, 4, 6)$, $(3, 4, 7)$, $(3, 4, 8)$ and $(4, 4, 4)$. The first six of these, together with $(2, 6, 12)$, $(2, 8, 8)$, $(3, 3, 4)$, $(3, 4, 4)$ and $(4, 4, 4)$, correspond to no generating triples. The types $(2, 6, 7)$, $(2, 6, 8)$, $(2, 7, 7)$ and $(2, 8, 12)$ are each realised by $12096 = |\mathrm{Aut}\, G|$ generating triples, giving one regular hypermap in each case, while the types $(2, 7, 8)$ and $(2, 7, 12)$ are realised by 24192 generating triples, in each case giving two regular hypermaps. The hypermaps corresponding to the remaining types are listed in Table 4.

The elements of $G = L_3(3)$ have order $1, 2, 3, 4, 6, 8$ or 13. In this case $\lambda \geqslant 5.616$, so the only possible types are $(2, 3, 13)$, $(2, 4, 6)$, $(2, 4, 8)$, $(2, 4, 13)$, $(2, 6, 6)$, $(2, 6, 8)$, $(2, 6, 13)$, $(2, 8, 8)$, $(2, 8, 13)$, $(2, 13, 13)$, $(3, 3, 4)$, $(3, 3, 6)$, $(3, 3, 8)$, $(3, 3, 13)$, $(3, 4, 4)$, $(3, 4, 6)$, $(3, 4, 8)$, $(3, 4, 13)$, $(3, 6, 6)$, $(4, 4, 4)$ and $(4, 4, 6)$. The types $(2, 4, 6)$, $(2, 4, 8)$, $(2, 6, 6)$ and $(2, 6, 8)$ correspond to no generating triples, while $(2, 3, 13)$, $(2, 4, 13)$ and $(2, 6, 13)$ are each realised by $22464 = 2|\mathrm{Aut}\, G|$ generating triples, giving rise to two regular hypermaps in each case. Type $(2, 8, 8)$ is realised by 11232 triples, giving one regular hypermap, type $(2, 8, 13)$ is realised by 44928 triples, giving four regular hypermaps, while type $(2, 13, 13)$ is realised by 89856 triples, giving eight regular hypermaps. The hypermaps corresponding to the remaining types are listed in Table 3.

If $G = A_7$, with elements of orders $1, 2, 3, 4, 5, 6$ and 7 and with $\lambda \geqslant 2.52$, the possible types are $(2, 6, 6)$, $(2, 6, 7)$, $(2, 7, 7)$, $(3, 3, 6)$, $(3, 3, 7)$, $(3, 4, 4)$, $(3, 4, 5)$, $(3, 4, 6)$, $(3, 4, 7)$, $(3, 5, 5)$, $(3, 5, 6)$, $(3, 5, 7)$, $(3, 6, 6)$, $(3, 6, 7)$, $(3, 7, 7)$, $(4, 4, 4)$, $(4, 4, 5)$, $(4, 4, 6)$, $(4, 4, 7)$, $(4, 5, 5)$, $(4, 5, 6)$, $(4, 5, 7)$, $(4, 6, 6)$, $(4, 6, 7)$, $(4, 7, 7)$, $(5, 5, 5)$, $(5, 5, 6)$, $(5, 5, 7)$, $(5, 6, 6)$, $(5, 6, 7)$, $(5, 7, 7)$, $(6, 6, 6)$, $(6, 6, 7)$, $(6, 7, 7)$ and $(7, 7, 7)$. There are no generating triples of type $(2, 6, 6)$. For type $(2, 6, 7)$ there are $10080 = 2|\mathrm{Aut}\, G|$ generating triples, giving rise to two regular hypermaps, while for type $(2, 7, 7)$ there are 15120 triples giving three regular maps, for type $(3, 3, 6)$ there are 5040 triples giving one regular hypermap, and for type $(3, 3, 7)$ there are 10080 triples giving two regular hypermaps. The hypermaps corresponding to the remaining types are listed in Table 2.

There is a unique perfect double covering $G = 2 \,.\, A_7$ of A_7, with elements of orders 1, 2, 3, 4, 5, 6, 7, 8, 10, 12 and 14. Using $\lambda \geqslant 5.04$, we find totally chiral hypermaps of genus $g \leqslant 1001$ for the following types: there is one totally chiral pair and one regular hypermap of type $(3, 5, 5)$ and genus 673; one totally chiral pair and two regular hypermaps of type $(3, 5, 7)$ and genus 817; two totally chiral pairs and one regular hypermap of type $(3, 5, 8)$ and genus 862; two totally chiral pairs and three regular hypermaps of type $(3, 7, 7)$ and genus 961; finally two totally chiral pairs and one regular hypermap of type $(3, 5, 14)$ and genus 997. (There are also a few non-totally chiral hypermaps, with chirality index 2, but only for genera $g > 1001$: the least such genus is 1216, corresponding to two chiral pairs of type $(4, 7, 8)$.) The totally chiral hypermaps are listed in Table 7.

Similarly, there is a unique perfect triple covering $G = 3 \,.\, A_7$, with elements of orders 1, 2, 3, 4, 5, 6, 7, 12, 15 and 21. In this case we find chiral hypermaps of genus $g \leqslant 1001$ for the following types: there is one non-totally chiral pair and one regular hypermap of type $(3,3,5)$ and genus 505; four non-totally chiral pairs and one regular hypermap of type $(3,3,6)$ and genus 631; two totally chiral pairs of type $(3,4,4)$ and genus 631; two totally chiral pairs and one regular hypermap of type $(3,4,5)$ and genus 820; finally three totally chiral pairs, one non-totally chiral pair and one regular hypermap of type $(3,4,6)$ and genus 946. Here the non-totally chiral hypermaps all have chirality index 3. The chiral hypermaps are listed in Table 8. □

In dealing with $2\,.\,A_7$, by starting with the presentation

$$A_7 = \langle a, b \mid a^{-3}(ab^2)^4 = b^5 = (b^2a^{-1}ba)^2 = 1\rangle$$

in [CRKMW] we found a convenient presentation

$$2\,.\,A_7 = \langle x, y \mid x^{14} = y^{10} = (xy^{-1})^3 = [y^5, x] = 1,\ (xy)^4 = (y^3x^{-1}yxy^{-1})^2 = y^5\rangle.$$

This group has a unique involution, namely the central involution y^5 $(= x^7)$, so a subgroup H has trivial core if and only if it has odd order. The largest such subgroups have order 21 and index 240, and they form a single conjugacy class represented by $H = \langle xy^5, yx^{-1}y^{-1}xyxy^2x^{-1}\rangle$, so this subgroup can be used in applications of Lemma 5 and Corollary 6. The fact that $\mathrm{Aut}(2\,.\,A_7) \cong S_7$ allows us to count the hypermaps of various types with monodromy group $2\,.\,A_7$.

8. The totally chiral maps of least genus

Maps are simply hypermaps of type $(l,2,n)$ (that is, of map type $\{n,l\}$ in the notation of [CM, Chapter 8]), so they are duals of hypermaps of type $(2,l,n)$. The classification in Theorem B and its associated tables can therefore be used to determine the totally chiral maps of genus $g \leqslant 1001$, and in particular it tells us those of least genus, described in more detail in the following example.

Example 5. Table 5 shows that M_{11} is the monodromy group of an orientably regular hypermap $\mathcal{M}_{11}{}^{[1]}$ of type $(2,4,11)$, and hence of an orientably regular map of type $\{11,4\}$. One can use character theory to show that there are exactly two such maps. The group M_{11} has a single conjugacy class of 990 elements of order 4, a single conjugacy class of 165 elements of order 2, and two conjugacy classes each of 720 elements of order 11. By formula (3.1) it follows from the character table in [CCNPW] that the number of triples of type $(4,2,11)$ in this group is

$$2 \times \frac{990 \times 165 \times 720}{7920}\left(1 + \frac{2 \times 2 \times -1}{10} + \frac{1 \times -3 \times 1}{45}\right) = 2|M_{11}|,$$

with only the characters χ_1, χ_2 and χ_9 of degrees $1, 10$ and 45 contributing to the sum. No maximal subgroup of M_{11} contains elements of orders 4 and 11, so each such triple generates M_{11}. Since M_{11} has no outer automorphisms it follows that there are two orientably regular maps of type $\{11,4\}$ with monodromy group M_{11}. By (2.1) these maps have genus 631. They form a chiral pair $\mathcal{M}$ and $\overline{\mathcal{M}}$ since the elements of order 11 in M_{11} are not inverted by any automorphism. Alternatively, one can use the natural representation of degree 11 of M_{11} and note that the faithful quotient $\mathcal{M}_{11}{}^{[1]}/M_{10}$ of $\mathcal{M}_{11}{}^{[1]}$ by the point stabiliser $H = M_{10}$ is chiral, as shown in Fig. 3 with $x = (2,10)(4,11)(5,7)(8,9)$ and $y = (1,4,3,8)(2,5,6,9)$; now Corollary 6(b) applies to H since all automorphisms of M_{11} are inner, so we have a chiral pair. Since M_{11} is simple these two maps are totally chiral, as are their associates, which include a chiral pair of maps of type $\{4,11\}$. Table 5 shows all the totally chiral hypermaps of genus at most 1001 with monodromy group M_{11}, including $\mathcal{M}_{11}{}^{[1]}$ of type $(2,4,11)$, a dual of one of the chiral pair. The table also shows that there are totally chiral maps of genus 694, 826, 829, 961 and 991 with monodromy group M_{11}.

To summarise, we have proved:

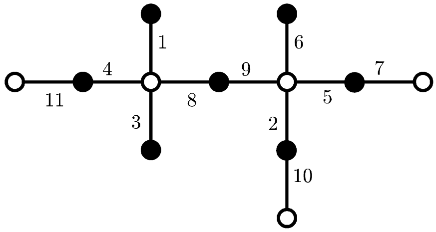

Fig. 3. The faithful quotient $\mathcal{M}_{11}{}^{[1]}/M_{10}$.

Theorem C. *The least genus of any totally chiral map is* 631, *attained by four maps with monodromy group isomorphic to the Mathieu group* M_{11}. *They consist of a chiral pair of type* $\{11,4\}$ *and their duals of type* $\{4,11\}$.

It may seem strange that M_{11} appears in Theorem C, and not a smaller perfect group such as $d\,.\,A_7$ ($d=1,2,3$), $L_3(3)$ or $U_3(3)$. The reason is simple: there are no chiral maps with these monodromy groups. For instance, there are very few orientably regular maps with monodromy group A_7: there are just two regular maps of type $\{6,7\}$ and their duals, of genus 241, and one self-dual regular map of type $\{7,7\}$ and genus 271. In the case of $2\,.\,A_7$ there are no orientably regular maps at all, only hypermaps, since this group has only one involution, lying in the centre. All the orientably regular hypermaps with monodromy group $L_3(3)$ are regular, the least genus being 253 attained by two regular maps of type $\{3,13\}$ and their duals. The same applies to $U_3(3)$, the least genus in this case being 577 attained by a regular map of type $\{6,7\}$ and its dual, and also to $3\,.\,A_7$, where the least genus is 721, attained by two regular maps of type $\{6,7\}$ and their duals.

9. The totally chiral composite hypermaps (and maps) of least genus

Let us define an orientably regular hypermap $\mathcal{H}$ to be *simple* if its monodromy group G is a simple group, and *composite* otherwise. By Theorem B, the totally chiral composite hypermaps of genus $g \leqslant 1001$ are those in Tables 7 and 8 corresponding to the central extensions $G = 2\,.\,A_7$ and $3\,.\,A_7$ of A_7, and by inspection the least genus of such a hypermap is 631. More specifically, we have:

Theorem D. *The least genus of any totally chiral composite hypermap is* 631, *attained by twelve hypermaps formed from* $3\,.\,\mathcal{A}_7{}^{[1]}$ *and* $3\,.\,\mathcal{A}_7{}^{[2]}$ *by taking mirror images and associates. They all have monodromy group* $3\,.\,A_7$ *and type* $(3,4,4)$, $(4,3,4)$ *or* $(4,4,3)$.

Remarks. 1. The quotients of these hypermaps by the centre $N \cong C_3$ of $3\,.\,A_7$ are the twelve hypermaps of genus 211 in the Γ-orbits of $\mathcal{A}_7{}^{[1]}$ and $\mathcal{A}_7{}^{[2]}$, appearing in Examples 1 and 2 and Theorem A.

2. Fig. 4(a) shows the faithful quotient of $3\,.\,\mathcal{A}_7{}^{[1]}$ by a subgroup $H \cong S_5$ of index 63 in $3\,.\,A_7$. This Walsh map is on a torus, and it is easy to see a translation which is a fixed-point-free automorphism of order 3 of the map, induced by the centre $N \cong C_3$ of the automorphism group $3\,.\,A_7$. The quotient of this map by N (Fig. 4(b)) is the faithful quotient, again on a torus, of the orientably regular hypermap $\overline{\mathcal{A}_7{}^{[1]}} = 3\,.\,\mathcal{A}_7{}^{[1]}/N$, with monodromy group A_7, by H; the torus in Fig. 4(a) is tessellated by three copies of a square fundamental region for N.

3. Although 631 is the least genus of any totally chiral hypermap with monodromy group $3\,.\,A_7$, there are chiral hypermaps of smaller genus with this monodromy group. The least such genus is 505, attained by a chiral pair $\mathcal{H}$, $\overline{\mathcal{H}}$ and a regular hypermap, all of type $(3,3,5)$, together with their associates. The quotient hypermap $\mathcal{H}/N \cong \overline{\mathcal{H}}/N$ of genus 169 does not contradict the minimality of the genus 211 in Table 2, since it is regular; thus $\mathcal{H}$ has chirality index 3 and is not totally chiral. Among the orientably regular hypermaps with monodromy group $3\,.\,A_7$, the genus 631 is also attained by four chiral pairs and a regular hypermap, all of type $(3,3,6)$, together with their associates. Again, the chiral hypermaps have chirality index 3, so they are not totally chiral.

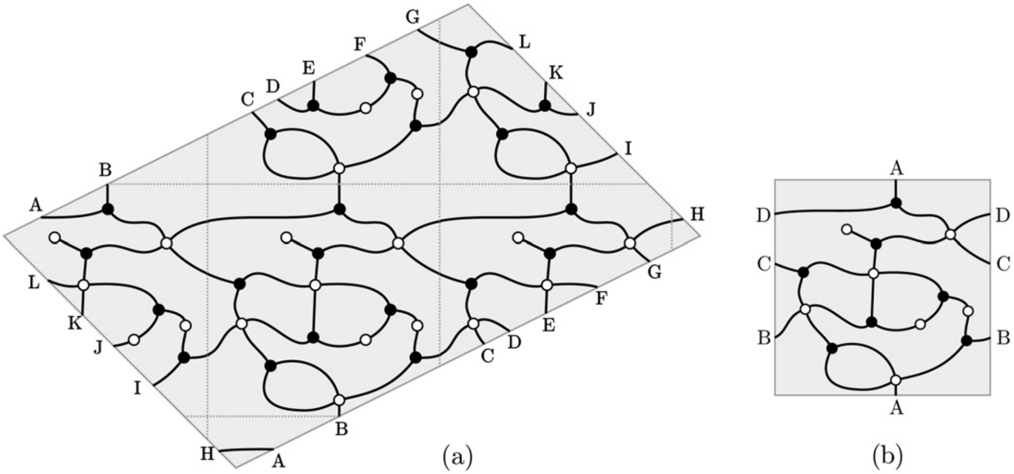

Fig. 4. (a) The faithful quotient of the totally chiral composite hypermap $3\,.\,\mathcal{A}_7{}^{[1]}$ by H. (b) The faithful quotient $\overline{\mathcal{A}_7{}^{[1]}}/H$. Both are on tori.

4. It is strange that this genus 631 is also the least genus of any totally chiral map (see Section 8), although there is no apparent direct connection between the two sets of hypermaps attaining these lower bounds, between their monodromy groups M_{11} and $3\,.\,A_7$, between the corresponding triangle groups $\Delta(2,4,11)$ and $\Delta(3,4,4)$, or between the Riemann surfaces and algebraic curves associated with these hypermaps in Grothendieck's theory of *dessins d'enfants*.

5. It may also seem strange that it is $3\,.\,A_7$, and not the smaller group $2\,.\,A_7$, which is the monodromy group of the totally chiral composite hypermaps of least genus. The reason is that $2\,.\,A_7$ has no generating triples with large values of λ, corresponding to hypermaps of low genus: there are none of type $(2,m,n)$ for any m and n (since the only involution is in the centre), none of type $(3,3,n)$ for any $n<10$, and none of type $(3,4,n)$ or $(4,4,n)$ for any n. The least genera of orientably regular and chiral hypermaps with monodromy group $2\,.\,A_7$ are respectively 589, attained by a regular hypermap of type $(3,3,10)$, and 673, attained by a totally chiral pair and a regular hypermap, all of type $(3,5,5)$.

Although it takes us outside our self-imposed range $g \leqslant 1001$, we conclude this section by computing the least genus of any totally chiral composite map.

Theorem E. *The least genus of any totally chiral composite map is* 1457, *attained by* 48 *maps with monodromy group* $2\,.\,Sz(8)$. *They consist of* 12 *chiral pairs of type* $\{4,5\}$, *together with their duals of type* $\{5,4\}$.

Proof. The *ATLAS* gives a presentation

$$\langle x, y \mid x^2 = y^4 = (xy)^5 = \left(xy^2\right)^7 = \left(xyxy^{-1}xy^2\right)^7 = 1\rangle$$

for the double covering $G = 2\,.\,S$ of the Suzuki group $S = Sz(8)$, showing that G is the monodromy group of an orientably regular hypermap $\mathcal{H}$ of type $(2,4,5)$ and genus 1457. This is a double covering of an orientably regular hypermap $\overline{\mathcal{H}}$ of that type and of genus 729. Applying a triality operation gives a corresponding pair of maps $\mathcal{M}$ and $\overline{\mathcal{M}}$ of type $\{4,5\}$ and of these genera, with monodromy groups G and S. The chirality group of $\mathcal{M}$ is a normal subgroup of G, so it is either trivial, the central subgroup of order 2, or G; the first two cases can be eliminated since they would imply that $\mathcal{M}$ is chiral, whereas no element of order 4 in S is inverted by any automorphism [Suz]. Thus $\mathcal{M}$ is totally chiral (and $\overline{\mathcal{H}}$ is one of the four totally chiral hypermaps of type $(2,4,5)$ and genus 729 appearing in the proof of Theorem B and in Table 8).

Now suppose that $\mathcal{M}'$ is a totally chiral composite map of genus $g' \leqslant 1457$. Being composite, it must be a d-sheeted covering, for some $d \geqslant 2$, of a simple hypermap $\mathcal{M}^*$, which is totally chiral by Lemma 4(a). By Theorem D, $\mathcal{M}^*$ has genus $g^* \geqslant 631$, so $g' = d(g^*-1)+1 \geqslant 630d+1$, giving $d=2$ and hence $g^* \leqslant 729$. Now Theorem B and the associated Tables 2–6 for the simple groups A_7, $L_3(3)$, $U_3(3)$, M_{11} and $Sz(8)$ show that the only possibilities are $g^* = 631$, 694 or 729, corresponding to hypermaps of type $(2,4,11)$, $(2,5,8)$ or $(2,4,5)$, with monodromy groups M_{11}, M_{11} or $Sz(8)$ respectively. Since $d=2$ the d-sheeted covering is central, whereas M_{11} has trivial Schur multiplier, so the first two cases cannot arise here. It follows that $g^* = 729$, $g = 1457$ and the monodromy group of $\mathcal{M}$ is the unique proper double covering G of S.

A calculation with GAP shows that there are $698880 = 24|S|$ generating triples of type $(2,4,5)$ in G. Since $\operatorname{Aut} G \cong G/Z(G) \cong S$ it follows that we obtain 24 totally chiral maps $\mathcal{M}$ of type $\{4,5\}$, forming 12 chiral pairs, together with their 24 duals. □

Remark. The Schur multiplier of the group $S = Sz(8)$ is isomorphic to $C_2 \times C_2$, so the universal covering group $\hat{S} = 2^2 . S$ has three central subgroups of order 2, the quotient by each giving a double covering $2 . S$ of S. The outer automorphism group C_3 of S, induced by the Galois group of the field F_8, lifts to a group of outer automorphisms of $\hat{S}$, permuting these three central subgroups transitively, so the corresponding covering groups $2 . S$ are isomorphic to each other, and each can be taken to be G.

Each epimorphism from $\Delta = \Delta(2,4,5)$ to S factors through $\hat{S}$, so each normal subgroup N of Δ with quotient S contains a normal subgroup M of Δ with $\Delta/M \cong \hat{S}$ and $N/M \cong C_2 \times C_2$. It follows that there are three normal subgroups K of Δ with $M < K < N$, corresponding to three totally chiral maps $\mathcal{M}$ of type $\{4,5\}$, genus 1457 and monodromy group $\Delta/K \cong G$, which are double coverings of the map $\overline{\mathcal{M}}$ of genus 729 corresponding to N. However, these are not the only double coverings of $\overline{\mathcal{M}}$ with these properties: the group Δ has a derived group Δ' of index 2, so each of these groups K satisfies $N/(K \cap \Delta') \cong C_2 \times C_2$, giving three subgroups of index 2 in N containing $K \cap \Delta'$. All three are normal in Δ, since their images are central in $\Delta/(K \cap \Delta') \cong G \times C_2$. One of them is $N \cap \Delta'$, with $\Delta/(N \cap \Delta') \cong S \times C_2$, one is K, with $\Delta/K \cong G$, and the third (let us call it K^*) also has quotient group $\Delta/K^* \cong G$. The six groups K and K^* correspond to six double coverings $\mathcal{M}$ of $\overline{\mathcal{M}}$ with monodromy group G, so the four totally chiral maps $\overline{\mathcal{M}}$ of type $\{4,5\}$ and genus 729 enumerated earlier lift to 24 totally chiral maps $\mathcal{M}$ of type $\{4,5\}$ and genus 1457. The same applies to their duals, thus accounting for the 48 maps in Theorem E.

10. Infinite families

So far we have concentrated on individual examples or finite families of totally chiral hypermaps, but it is possible to give uniform constructions of infinite families.

Example 6. It is shown in [JS] that each of the Suzuki groups $Sz(q)$ ($q = 2^e$, odd $e \geqslant 3$) is the monodromy group of a family of orientably regular maps of type $\{4,5\}$ and genus

$$1 + \frac{|Sz(q)|}{40} = 1 + \frac{q^2(q^2+1)(q-1)}{40}.$$

These maps and their duals are chiral, since the elements of order 4 in $Sz(q)$ are not inverted by any automorphism of the group; since $Sz(q)$ is simple they are totally chiral. There is a similar construction of totally chiral maps of type $\{3,7\}$ based on the Ree groups $Re(3^e)$, which are simple for all odd $e \geqslant 3$ [Jon].

Example 7. Theorem 13 of [BJNŠ] describes a totally chiral hypermap with monodromy group A_n for each $n \geqslant 7$. If n is odd, one can take the generators $x = (1\,2\,\ldots\,n)$ and $y = (1\,2\,4)$, giving a hypermap of type $(n, 3, n)$ and genus $1 + \frac{1}{6}(n-1)!(n-3)$. If n is even, one can take $x = (1\,2\,\ldots\,n-1)$ and $y = (1\,n)(2\,3)$, in which case the type is $(n-1, 2, n-1)$ and the genus is $1 + \frac{1}{4}(n-2)!n(n-5)$. (Values $n < 7$ are excluded by Lemma 4, since A_n is solvable for $n \leqslant 4$, while $A_5 \cong L_2(4)$ and $A_6 \cong L_2(9)$.)

One can also produce infinite families of totally chiral hypermaps as coverings of a single hypermap.

Proposition 7. *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be orientably regular hypermaps corresponding to normal subgroups K and L of Δ, where $K \geqslant L$ and K/L is a solvable group of order coprime to $|\Delta : K|$. If $\mathcal{H}$ is totally chiral then so is $\tilde{\mathcal{H}}$.*

Proof. Suppose that $\mathcal{H}$ is totally chiral and $\tilde{\mathcal{H}}$ is not, so $K\overline{K} = \Delta$ but $M := L\overline{L} < \Delta$. Recall that $\overline{K}$ is the image of K under the automorphism of Δ inverting the generators X and Y.

We first show that $KM = \Delta$, or equivalently $K\overline{L} = \Delta$. Since $\Delta = K\overline{K}$ we have $\Delta/K\overline{L} \cong \overline{K}/(\overline{K} \cap K\overline{L})$, which is an epimorphic image of $\overline{K}/\overline{L} \cong K/L$ and is therefore solvable. On the other hand, $\Delta/K\overline{L}$ is also an epimorphic image of the perfect group Δ/K, so it is trivial. Thus $\Delta = K\overline{L} = KM$.

It follows that $\Delta/M = KM/M \cong K/K \cap M$ with $K \cap M \geqslant L$, so Δ/M is a quotient of K/L and is therefore solvable, of order coprime to $|G|$. It follows that Δ has a proper normal subgroup $N \geqslant M$ such that Δ/N is an elementary abelian p-group P for some prime p not dividing $|\Delta : K|$. We have $KN = \Delta$, so $Q := K/K \cap N \cong \Delta/N \cong P$. Since K is isomorphic to the fundamental group $\pi_1\mathcal{H}$ of $\mathcal{H}$ it follows that Q is a quotient of the first homology group $H_1(\mathcal{H}; F_p) \cong K/K'K^p$ over the field F_p, where K' is the commutator subgroup of K and K^p is the subgroup generated by its pth powers. Now $H_1(\mathcal{H}; F_p)$ is an F_pG-module where $G = \Delta/K$, with the action of G as $\mathrm{Aut}\,\mathcal{H}$ on $H_1(\mathcal{H}; F_p)$ identified with its induced action by conjugation on $K/K'K^p$. Since p is coprime to $|G|$, Maschke's theorem implies that $H_1(\mathcal{H}; F_p)$ is a direct sum of irreducible submodules. We have $\Delta/K \cap N = KN/K \cap N = (K/K \cap N) \times (N/K \cap N) = Q \times (N/K \cap N)$, so G, acting as $N/K \cap N$, acts trivially on Q. Thus the principal F_pG-module has non-zero multiplicity as a direct summand of $H_1(\mathcal{H}; F_p)$. However, it follows from a theorem of Sah [Sah, Theorem 3.5] that if a finite group G has an orientation-preserving action on a compact orientable surface S, and F is a field of characteristic not dividing $|G|$, then the dimension of the subspace of $H_1(S; F)$ fixed by G is twice the genus of S/G. When S is the underlying surface of an orientably regular hypermap with automorphism group G this genus is 0, so the principal module has zero multiplicity. This contradiction proves the result. □

(Sah's theorem is stated and proved in [Sah] for finite groups acting conformally on compact Riemann surfaces. In the above proof, S inherits a Riemann surface structure, preserved by G, from that of the simply connected Riemann surface — in this case the hyperbolic plane — on which the triangle group Δ acts conformally.)

Corollary 8. *Each totally chiral hypermap has infinitely many totally chiral covering hypermaps of the same type.* (*In particular, any totally chiral map has infinitely many totally chiral covering maps of the same type.*)

Proof. Let $\mathcal{H}$ be a totally chiral hypermap of genus g, corresponding to a subgroup K of Δ as in Proposition 7. Then K is a surface group of genus $g \geqslant 1$ (in fact $g \geqslant 211$ by Theorem A), so there are infinitely many characteristic subgroups L of K such that K/L is a solvable group of finite order coprime to $|\Delta : K|$: for instance we could take $L = K'K^n$, so that $K/L \cong C_n^{2g}$, for any integer n coprime to $|\Delta : K|$. Each such L is normal in Δ, so the corresponding hypermap $\tilde{\mathcal{H}}$ is orientably regular, and of the same type as $\mathcal{H}$; by Proposition 7 it is totally chiral. □

The existence of totally chiral hypermaps with monodromy groups $2\,.\,A_7$ or $3\,.\,A_7$ (see Tables 7 and 8) shows that the conclusion of Proposition 7 can sometimes be valid even if $|K : L|$ and $|\Delta : K|$ are not coprime. However, in this case there are many counterexamples, such as the following:

Example 8. Table 2 shows that there is a totally chiral hypermap $\mathcal{H}$ of type $(4,4,4)$ and genus 316 with monodromy group $G = A_7$. Let K be the corresponding hypermap subgroup of $\Delta = \Delta(4,4,4)$, and let $L = K \cap \Delta'$, so $K/L \cong \Delta/\Delta' \cong C_4 \times C_4$. Then $L\bar{L} = \Delta'$, so the hypermap $\tilde{\mathcal{H}}$ corresponding to L is not totally chiral. The point is that Sah's theorem does not apply as in the proof of Proposition 7 when $|G|$ is divisible by p $(= 2$ here).

Appendix A

Here we present the tables which accompany Theorem B, listing all the totally chiral hypermaps of genus $g \leqslant 1001$. There is one table for each of the possible monodromy groups $G = A_7$, $L_3(3)$, $U_3(3)$, M_{11}, $Sz(8)$, $2 \,.\, A_7$ and $3 \,.\, A_7$. In each table we describe one representative of each Γ-orbit, where $\Gamma \cong S_3 \times C_2$ is the group of hypermap operations generated by triality and reflection. We have chosen each representative to have type (l, m, n) with $l \leqslant m \leqslant n$, so the Γ-orbits containing maps are those with $l = 2$. In each case we give the genus of the hypermaps and the number of them in the orbit.

Table 2
The chiral hypermaps with monodromy group A_7 and $g \leqslant 1001$.

Type	g	Trip.	Refl.	Chir.	$\mathcal{H}$
(3, 4, 4)	211	20160	0	4	$\mathcal{A}_7^{[1]} = (A_7; (1,2,3)(4,5,6), (1,2)(3,5,6,7))$
					$\mathcal{A}_7^{[2]} = (A_7; (1,2,3)(4,5,6), (1,5)(2,4,6,7))$
(3, 4, 5)	274	25200	1	4	$\mathcal{A}_7^{[3]} = (A_7; (1,2,3)(4,5,6), (1,3,5,7)(4,6))$
					$\mathcal{A}_7^{[4]} = (A_7; (1,2,3)(4,5,6), (1,4,6,5)(2,7))$
(3, 4, 6)	316	15120	1	2	$\mathcal{A}_7^{[5]} = (A_7; (1,2,3)(4,5,6), (1,7,3,6)(2,4))$
(4, 4, 4)	316	60480	0	12	$\mathcal{A}_7^{[6]} = (A_7; (1,2,3,4)(5,6), (1,3)(2,5,6,7))$
					$\mathcal{A}_7^{[7]} = (A_7; (1,2,3,4)(5,6), (1,5)(2,4,7,3))$
					$\mathcal{A}_7^{[8]} = (A_7; (1,2,3,4)(5,6), (1,4)(2,3,5,7))$
					$\mathcal{A}_7^{[9]} = (A_7; (1,2,3,4)(5,6), (1,6,4,2)(5,7))$
					$\mathcal{A}_7^{[10]} = (A_7; (1,2,3,4)(5,6), (1,6,7,5)(3,4))$
					$\mathcal{A}_7^{[11]} = (A_7; (1,2,3,4)(5,6), (1,7)(3,4,6,5))$
(3, 5, 5)	337	15120	1	2	$\mathcal{A}_7^{[12]} = (A_7; (1,2,3)(4,5,6), (1,7,6,5,4))$
(3, 4, 7)	346	40320	0	8	$\mathcal{A}_7^{[13]} = (A_7; (1,2,3), (2,7,6,4)(3,5))$
					$\mathcal{A}_7^{[14]} = (A_7; (1,2,3)(4,5,6), (1,4,5,6)(3,7))$
					$\mathcal{A}_7^{[15]} = (A_7; (1,2,3)(4,5,6), (1,7,6,3)(2,4))$
					$\mathcal{A}_7^{[16]} = (A_7; (1,2,3)(4,5,6), (1,5)(2,7,3,4))$
(4, 4, 5)	379	55440	3	8	$\mathcal{A}_7^{[17]} = (A_7; (1,2,3,4)(5,6), (1,4,5,6)(3,7))$
					$\mathcal{A}_7^{[18]} = (A_7; (1,2,3,4)(5,6), (1,4)(3,6,5,7))$
					$\mathcal{A}_7^{[19]} = (A_7; (1,2,3,4)(5,6), (1,4)(2,5,6,7))$
					$\mathcal{A}_7^{[20]} = (A_7; (1,2,3,4)(5,6), (1,4,7,5)(2,3))$
(3, 5, 7)	409	45360	3	6	$\mathcal{A}_7^{[21]} = (A_7; (1,2,3)(4,5,6), (1,6,4,7,5))$
					$\mathcal{A}_7^{[22]} = (A_7; (1,2,3)(4,5,6), (1,5,3,7,4))$
					$\mathcal{A}_7^{[23]} = (A_7; (1,2,3)(4,5,6), (3,7,6,4,5))$
(4, 4, 6)	421	30240	2	4	$\mathcal{A}_7^{[24]} = (A_7; (1,2,3,4)(5,6), (2,4,7,6)(3,5))$
					$\mathcal{A}_7^{[25]} = (A_7; (1,2,3,4)(5,6), (1,7,4,6)(2,5))$
(4, 5, 5)	442	20160	0	4	$\mathcal{A}_7^{[26]} = (A_7; (1,2,3,4)(5,6), (1,7,6,5,2))$
					$\mathcal{A}_7^{[27]} = (A_7; (1,2,3,4)(5,6), (2,7,6,4,3))$
(3, 6, 7)	451	20160	2	2	$\mathcal{A}_7^{[28]} = (A_7; (1,2,3)(4,5,6), (1,4,5)(2,6)(3,7))$

Table 2 (*continued*)

Type	g	Trip.	Refl.	Chir.	$\mathcal{H}$
(4, 4, 7)	451	110880	6	16	$\mathcal{A}_7{}^{[29]} = (A_7; (1,2,3,4)(5,6), (2,7,6,4)(3,5))$
					$\mathcal{A}_7{}^{[30]} = (A_7; (1,2,3,4)(5,6), (1,6,2,5)(4,7))$
					$\mathcal{A}_7{}^{[31]} = (A_7; (1,2,3,4)(5,6), (1,7,6,4)(2,5))$
					$\mathcal{A}_7{}^{[32]} = (A_7; (1,2,3,4)(5,6), (1,7,6,3)(2,4))$
					$\mathcal{A}_7{}^{[33]} = (A_7; (1,2,3,4)(5,6), (1,3,4,5)(2,7))$
					$\mathcal{A}_7{}^{[34]} = (A_7; (1,2,3,4)(5,6), (1,2,4,7)(3,6))$
					$\mathcal{A}_7{}^{[35]} = (A_7; (1,2,3,4)(5,6), (1,2,6,3)(5,7))$
					$\mathcal{A}_7{}^{[36]} = (A_7; (1,2,3,4)(5,6), (2,5)(3,7,4,6))$
(3, 7, 7)	481	65520	7	6	$\mathcal{A}_7{}^{[37]} = (A_7; (1,2,3), (1,5,2,7,4,6,3))$
					$\mathcal{A}_7{}^{[38]} = (A_7; (1,2,3)(4,5,6), (1,6,2,5,3,7,4))$
					$\mathcal{A}_7{}^{[39]} = (A_7; (1,2,3)(4,5,6), (1,2,5,6,3,4,7))$
(4, 5, 6)	484	20160	2	2	$\mathcal{A}_7{}^{[40]} = (A_7; (1,2,3,4)(5,6), (1,3,7,2,6))$
(4, 5, 7)	514	90720	0	18	$\mathcal{A}_7{}^{[41]} = (A_7; (1,2,3,4)(5,6), (1,6,4,7,5))$
					$\mathcal{A}_7{}^{[42]} = (A_7; (1,2,3,4)(5,6), (1,5,7,4,6))$
					$\mathcal{A}_7{}^{[43]} = (A_7; (1,2,3,4)(5,6), (1,2,4,6,7))$
					$\mathcal{A}_7{}^{[44]} = (A_7; (1,2,3,4)(5,6), (1,3,4,5,7))$
					$\mathcal{A}_7{}^{[45]} = (A_7; (1,2,3,4)(5,6), (1,5,3,7,4))$
					$\mathcal{A}_7{}^{[46]} = (A_7; (1,2,3,4)(5,6), (1,7,5,3,6))$
					$\mathcal{A}_7{}^{[47]} = (A_7; (1,2,3,4)(5,6), (1,5,2,4,7))$
					$\mathcal{A}_7{}^{[48]} = (A_7; (1,2,3,4)(5,6), (1,2,6,3,7))$
					$\mathcal{A}_7{}^{[49]} = (A_7; (1,2,3,4)(5,6), (1,2,3,5,7))$
(5, 5, 6)	547	20160	2	2	$\mathcal{A}_7{}^{[50]} = (A_7; (1,2,3,4,5), (1,7,5,3,6))$
(4, 6, 7)	556	40320	4	4	$\mathcal{A}_7{}^{[51]} = (A_7; (1,2,3,4)(5,6), (1,5)(2,3,6)(4,7))$
					$\mathcal{A}_7{}^{[52]} = (A_7; (1,2,3,4)(5,6), (1,7,6)(2,4)(3,5))$
(5, 5, 7)	577	60480	4	8	$\mathcal{A}_7{}^{[53]} = (A_7; (1,2,3,4,5), (1,6,4,7,5))$
					$\mathcal{A}_7{}^{[54]} = (A_7; (1,2,3,4,5), (1,2,4,6,7))$
					$\mathcal{A}_7{}^{[55]} = (A_7; (1,2,3,4,5), (1,2,5,6,7))$
					$\mathcal{A}_7{}^{[56]} = (A_7; (1,2,3,4,5), (1,3,6,5,7))$
(4, 7, 7)	586	100800	0	20	$\mathcal{A}_7{}^{[57]} = (A_7; (1,2,3,4)(5,6), (1,6,4,2,7,5,3))$
					$\mathcal{A}_7{}^{[58]} = (A_7; (1,2,3,4)(5,6), (1,5,2,6,3,7,4))$
					$\mathcal{A}_7{}^{[59]} = (A_7; (1,2,3,4)(5,6), (1,3,5,7,2,4,6))$
					$\mathcal{A}_7{}^{[60]} = (A_7; (1,2,3,4)(5,6), (1,5,2,7,4,6,3))$
					$\mathcal{A}_7{}^{[61]} = (A_7; (1,2,3,4)(5,6), (1,6,2,3,4,7,5))$
					$\mathcal{A}_7{}^{[62]} = (A_7; (1,2,3,4)(5,6), (1,3,4,5,7,2,6))$
					$\mathcal{A}_7{}^{[63]} = (A_7; (1,2,3,4)(5,6), (1,6,7,2,5,3,4))$
					$\mathcal{A}_7{}^{[64]} = (A_7; (1,2,3,4)(5,6), (1,6,3,4,2,7,5))$
					$\mathcal{A}_7{}^{[65]} = (A_7; (1,2,3,4)(5,6), (1,5,3,7,4,2,6))$
					$\mathcal{A}_7{}^{[66]} = (A_7; (1,2,3,4)(5,6), (1,5,3,6,2,7,4))$
(5, 6, 7)	619	35280	3	4	$\mathcal{A}_7{}^{[67]} = (A_7; (1,2,3,4,5), (1,4)(2,5,6)(3,7))$
					$\mathcal{A}_7{}^{[68]} = (A_7; (1,2,3,4,5), (1,4)(2,6)(3,5,7))$
(5, 7, 7)	649	120960	12	12	$\mathcal{A}_7{}^{[69]} = (A_7; (1,2,3,4,5), (1,4,5,6,2,7,3))$
					$\mathcal{A}_7{}^{[70]} = (A_7; (1,2,3,4,5), (1,3,4,7,2,5,6))$
					$\mathcal{A}_7{}^{[71]} = (A_7; (1,2,3,4,5), (1,4,6,2,7,3,5))$
					$\mathcal{A}_7{}^{[72]} = (A_7; (1,2,3,4,5), (1,4,6,7,2,3,5))$
					$\mathcal{A}_7{}^{[73]} = (A_7; (1,2,3,4,5), (1,2,5,6,3,4,7))$
					$\mathcal{A}_7{}^{[74]} = (A_7; (1,2,3,4,5), (1,6,7,2,5,3,4))$
(6, 7, 7)	691	35280	5	2	$\mathcal{A}_7{}^{[75]} = (A_7; (1,2,3)(4,5)(6,7), (1,5,2,7,4,6,3))$
(7, 7, 7)	721	115920	15	8	$\mathcal{A}_7{}^{[76]} = (A_7; (1,2,3,4,5,6,7), (1,4,5,6,7,2,3))$
					$\mathcal{A}_7{}^{[77]} = (A_7; (1,2,3,4,5,6,7), (1,5,7,4,6,2,3))$
					$\mathcal{A}_7{}^{[78]} = (A_7; (1,2,3,4,5,6,7), (1,6,7,4,2,5,3))$
					$\mathcal{A}_7{}^{[79]} = (A_7; (1,2,3,4,5,6,7), (1,6,7,2,5,3,4))$

Table 3
The chiral hypermaps with monodromy group $L_3(3)$ and $g \leqslant 1001$.

Type	g	Trip.	Refl.	Chir.	$\mathcal{H}$	(x, y)
(3, 3, 4)	235	33696	1	2	$\mathcal{L}_3(3)^{[1]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,13,8)(2,3,5)(4,12,6)(7,9,11)$
(3, 3, 6)	469	44928	2	2	$\mathcal{L}_3(3)^{[2]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,9,5)(2,4,6)(7,10,12)(8,13,11)$
(3, 4, 4)	469	44928	0	4	$\mathcal{L}_3(3)^{[3]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,8)(2,9,7,5)(4,11)(6,13,12,10)$
					$\mathcal{L}_3(3)^{[4]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,2,5,12)(3,4)(6,11,8,7)(10,13)$
(3, 3, 8)	586	112320	4	6	$\mathcal{L}_3(3)^{[5]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,10,7)(2,9,12)(3,5,8)(6,11,13)$
					$\mathcal{L}_3(3)^{[6]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,8,11)(2,6,13)(3,12,7)(5,10,9)$
					$\mathcal{L}_3(3)^{[7]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,3,8)(2,11,5)(4,9,7)(6,12,10)$
(3, 4, 6)	703	44928	2	2	$\mathcal{L}_3(3)^{[8]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,8)(2,10,5,3)(4,11)(6,12,9,7)$
(4, 4, 4)	703	33696	1	2	$\mathcal{L}_3(3)^{[9]}$	$x = (2,11)(3,12,9,7)(4,10,8,6)(5,13)$ $y = (1,8,12,6)(2,4,5,3)(7,9)(10,11)$
(3, 3, 13)	721	112320	8	2	$\mathcal{L}_3(3)^{[10]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,5,4)(3,7,8)(6,11,9)(10,13,12)$
(3, 4, 8)	820	112320	4	6	$\mathcal{L}_3(3)^{[11]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,12,4,3)(2,10,5,7)(6,8)(9,13)$
					$\mathcal{L}_3(3)^{[12]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,9)(2,11,12,8)(3,5)(6,10,7,13)$
					$\mathcal{L}_3(3)^{[13]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,2,4,3)(5,6,12,13)(7,11)(9,10)$
(3, 6, 6)	937	67392	2	4	$\mathcal{L}_3(3)^{[14]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,2,8,9,11,12)(3,13,10)(6,7)$
					$\mathcal{L}_3(3)^{[15]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,12,13)(2,4,3,5,8,6)(9,10)$
(4, 4, 6)	937	67392	4	2	$\mathcal{L}_3(3)^{[16]}$	$x = (2,11)(3,12,9,7)(4,10,8,6)(5,13)$ $y = (1,12,2,5)(3,7,4,11)(6,8)(9,13)$
(3, 4, 13)	955	157248	8	6	$\mathcal{L}_3(3)^{[17]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,12)(2,6,3,9)(4,10,11,5)(7,13)$
					$\mathcal{L}_3(3)^{[18]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,12,2,5)(3,7,4,11)(6,8)(9,13)$
					$\mathcal{L}_3(3)^{[19]}$	$x = (2,6,10)(3,7,4)(5,13,8)(9,12,11)$ $y = (1,4,5,2)(3,12)(6,11)(7,13,10,8)$

Table 4
The chiral hypermaps with monodromy group $U_3(3)$ and $g \leqslant 1001$.

Type	g	Trip.	Refl.	Chir.	$\mathcal{H}$	(x, y)
(3, 3, 6)	505	48384	0	4	$\mathcal{U}_3(3)^{[1]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,16,23)(2,25,20)(3,18,22)(4,11,24)(5,15,9)(6,27,8)(7,13,10)(14,19,21)(17,28,26)$
					$\mathcal{U}_3(3)^{[2]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,21,13)(2,20,28)(3,15,10)(4,7,5)(8,17,26)(9,18,22)(11,19,23)(12,25,16)(14,27,24)$

Table 4 (*continued*)

Type	g	Trip.	Refl.	Chir.	$\mathcal{H}$	(x, y)
(3, 3, 7)	577	120960	2	8	$\mathcal{U}_3(3)^{[3]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,4,2)(3,8,20)(5,10,11)(6,14,25)(7,26,22)(9,23,28)(12,24,16)(13,17,19)(15,21,18)$
					$\mathcal{U}_3(3)^{[4]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,22,11)(2,15,19)(3,13,7)(4,8,23)(5,20,18)(6,27,12)(9,10,17)(14,24,16)(21,26,28)$
					$\mathcal{U}_3(3)^{[5]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,8,4)(2,20,16)(3,15,24)(5,17,11)(6,18,9)(7,19,13)(10,14,27)(12,25,28)(21,23,26)$
					$\mathcal{U}_3(3)^{[6]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,21,20)(2,14,28)(3,19,22)(4,13,5)(6,11,9)(7,27,24)(8,10,15)(12,23,26)(16,17,18)$
(3, 3, 8)	631	120960	4	6	$\mathcal{U}_3(3)^{[7]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,14,16)(2,6,17)(3,20,13)(4,11,12)(5,23,7)(8,24,26)(9,15,25)(10,21,18)(19,27,28)$
					$\mathcal{U}_3(3)^{[8]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,21,11)(3,9,25)(4,23,18)(5,14,20)(6,12,8)(7,10,22)(13,26,15)(16,24,27)(17,19,28)$
					$\mathcal{U}_3(3)^{[9]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,5,13)(2,7,12)(3,17,28)(4,25,20)(6,14,15)(8,26,18)(9,23,19)(10,22,24)(11,27,16)$
(3, 3, 12)	757	72576	4	2	$\mathcal{U}_3(3)^{[10]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,20,24)(2,5,13)(3,19,25)(4,28,14)(6,16,10)(7,21,27)(8,23,11)(9,12,17)(15,18,26)$
(3, 4, 7)	829	84672	5	2	$\mathcal{U}_3(3)^{[11]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,7,26,11)(2,14)(3,12)(4,18,13,22)(5,21,8,19)(6,17,9,23)(10,15,24,27)(16,20,28,25)$
(3, 4, 8)	883	72576	2	4	$\mathcal{U}_3(3)^{[12]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,11,26,7)(2,14)(3,12)(4,22,13,18)(5,19,8,21)(6,23,9,17)(10,27,24,15)(16,25,28,20)$
					$\mathcal{U}_3(3)^{[13]}$	$x = (2,17,25)(3,5,7)(4,21,14)(6,15,26)(8,18,13)(9,20,28)(10,12,19)(11,24,16)(22,23,27)$ $y = (1,13,27,14)(2,24)(3,25,19,22)(4,21)(5,20,28,7)(6,11,26,18)(8,10,17,12)(9,16,15,23)$

Table 5
The chiral hypermaps with monodromy group M_{11} and $g \leqslant 1001$.

Type	g	Trip.	Refl.	Chir.	$\mathcal{H}$	(x, y)
(2, 4, 11)	631	15840	0	2	$\mathcal{M}_{11}{}^{[1]}$	$x = (2, 10)(4, 11)(5, 7)(8, 9)$ $y = (1, 4, 3, 8)(2, 5, 6, 9)$
(2, 5, 8)	694	47520	0	6	$\mathcal{M}_{11}{}^{[2]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 3, 11, 5, 2)(4, 8, 9, 6, 7)$
					$\mathcal{M}_{11}{}^{[3]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 3, 6, 9, 11)(2, 5, 4, 8, 7)$
					$\mathcal{M}_{11}{}^{[4]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 5, 4, 7, 10)(2, 9, 6, 11, 8)$
(2, 6, 8)	826	47520	0	6	$\mathcal{M}_{11}{}^{[5]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 5, 9)(2, 6, 7, 10, 4, 8)(3, 11)$
					$\mathcal{M}_{11}{}^{[6]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 6, 5, 10, 3, 4)(2, 8, 7)(9, 11)$
					$\mathcal{M}_{11}{}^{[7]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 7, 8, 5, 2, 3)(4, 9)(6, 10, 11)$
(3, 3, 8)	826	31680	0	4	$\mathcal{M}_{11}{}^{[8]}$	$x = (3, 10, 6)(4, 8, 9)(5, 7, 11)$ $y = (1, 9, 4)(2, 6, 10)(3, 8, 5)$
					$\mathcal{M}_{11}{}^{[9]}$	$x = (3, 10, 6)(4, 8, 9)(5, 7, 11)$ $y = (1, 2, 8)(3, 10, 9)(4, 11, 7)$
(2, 5, 11)	829	15840	0	2	$\mathcal{M}_{11}{}^{[10]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 3, 11, 7, 10)(2, 8, 5, 9, 6)$
(3, 4, 5)	859	39600	1	4	$\mathcal{M}_{11}{}^{[11]}$	$x = (3, 10, 6)(4, 8, 9)(5, 7, 11)$ $y = (1, 11, 7, 6)(2, 4, 3, 10)$
					$\mathcal{M}_{11}{}^{[12]}$	$x = (3, 10, 6)(4, 8, 9)(5, 7, 11)$ $y = (1, 9, 11, 2)(5, 10, 8, 7)$
(2, 6, 11)	961	47520	0	6	$\mathcal{M}_{11}{}^{[13]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 11, 10, 2, 9, 7)(3, 4)(5, 8, 6)$
					$\mathcal{M}_{11}{}^{[14]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 3, 8, 7, 5, 2)(4, 11)(6, 9, 10)$
					$\mathcal{M}_{11}{}^{[15]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 8, 3)(2, 11)(4, 6, 5, 7, 9, 10)$
(2, 8, 8)	991	31680	0	4	$\mathcal{M}_{11}{}^{[16]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 4, 11, 10, 5, 6, 2, 3)(8, 9)$
					$\mathcal{M}_{11}{}^{[17]}$	$x = (3, 7)(4, 8)(5, 10)(6, 11)$ $y = (1, 4, 8, 9, 10, 2, 5, 6)(7, 11)$
(3, 4, 6)	991	39600	1	4	$\mathcal{M}_{11}{}^{[18]}$	$x = (3, 10, 6)(4, 8, 9)(5, 7, 11)$ $y = (1, 4, 3, 8)(2, 5, 6, 9)$
					$\mathcal{M}_{11}{}^{[19]}$	$x = (3, 10, 6)(4, 8, 9)(5, 7, 11)$ $y = (1, 3, 10, 11)(2, 8, 5, 4)$

The central extension $2 \,.\, A_7$ has a faithful permutation representation of degree 240 on the cosets of $H = \langle xy^5, yx^{-1}y^{-1}xyxy^2x^{-1}\rangle$ given by

$$\begin{aligned} x = {} & (1, 2)(3, 11, 39, 9, 38, 118, 30, 7, 29, 21, 5, 20, 49, 12) \\ & (4, 16, 66, 195, 98, 25, 6, 8, 34, 127, 225, 143, 42, 10) \\ & (13, 53, 171, 51, 79, 130, 96, 31, 121, 200, 120, 134, 72, 54) \end{aligned}$$

Table 6
The chiral hypermaps with monodromy group $Sz(8)$ and $g \leqslant 1001$.

Type	g	Trip.	Refl.	Chir.	$\mathcal{H}$	(x, y)
(2, 4, 5)	729	349440	0	4	$\mathcal{S}z(8)^{[1]}$	$x =$ (1, 2)(3, 4)(5, 7)(6, 9)(8, 12)(10, 13)(11, 15)(14, 19)(16, 21)(17, 23) (18, 25)(20, 28)(22, 31)(24, 33)(26, 35)(27, 32)(29, 37)(30, 39) (34, 43)(36, 46)(38, 48)(41, 51)(42, 44)(45, 55)(47, 50)(49, 58) (52, 60)(53, 61)(54, 59)(56, 62)(57, 63)(64, 65) $y =$ (1, 3, 5, 8)(4, 6, 10, 14)(7, 11, 16, 22)(9, 12, 17, 24)(13, 18, 26, 36) (15, 20, 29, 38)(19, 27, 31, 28)(21, 30, 40, 50)(23, 32, 41, 52) (25, 34, 44, 54)(33, 42, 53, 43)(35, 45, 56, 63)(37, 47, 51, 46) (39, 49, 59, 60)(48, 57, 55, 58)(61, 64, 62, 65)
					$\mathcal{S}z(8)^{[2]}$	$x =$ (2, 45)(3, 27)(4, 49)(5, 24)(6, 10)(7, 38)(8, 25)(9, 51)(11, 39) (12, 63)(13, 60)(14, 55)(15, 44)(16, 48)(17, 19)(18, 53)(20, 26) (21, 61)(22, 57)(23, 35)(28, 40)(29, 52)(30, 50)(31, 34)(32, 62) (33, 46)(36, 54)(37, 47)(41, 58)(42, 56)(43, 65)(59, 64) $y =$ (1, 12, 10, 27)(2, 37, 33, 3)(4, 21, 43, 59)(5, 60, 14, 35)(6, 52, 62, 41) (7, 46, 32, 64)(8, 18, 15, 29)(9, 24, 48, 25)(11, 30, 53, 40) (13, 26, 47, 19)(16, 61, 57, 42)(17, 55, 39, 50)(20, 34, 36, 58) (23, 45, 54, 65)(28, 56, 49, 44)(31, 51, 63, 38)

(14, 56, 131, 157, 108, 27, 107, 32, 88, 73, 153, 45, 44, 57)

(15, 61, 41, 140, 93, 202, 124, 33, 52, 23, 90, 142, 186, 62)

(17, 70, 167, 84, 219, 109, 129, 35, 75, 170, 137, 174, 150, 71)

(18, 26, 103, 149, 101, 230, 132, 36, 43, 146, 106, 67, 169, 74)

(19, 77, 159, 47, 158, 114, 55, 37, 133, 172, 116, 227, 97, 78)

(22, 86, 65, 69, 144, 216, 91, 40, 138, 60, 128, 100, 190, 87)

(24, 94, 155, 240, 135, 237, 113, 28, 112, 236, 213, 81, 207, 95)

(46, 85, 119, 203, 154, 175, 83, 115, 68, 50, 168, 204, 197, 99)

(48, 163, 235, 145, 224, 126, 238, 117, 233, 231, 102, 192, 64, 164)

(58, 177, 212, 176, 208, 226, 220, 122, 193, 206, 182, 160, 229, 178)

(59, 180, 147, 221, 211, 80, 210, 123, 201, 104, 189, 205, 76, 181)

(63, 185, 141, 188, 139, 173, 156, 125, 194, 92, 198, 89, 165, 161)

(82, 151, 218, 187, 166, 152, 191, 136, 110, 217, 228, 162, 111, 215)

(105, 209, 239, 214, 222, 183, 179, 148, 232, 184, 223, 196, 199, 234),

$y =$ (1, 3, 13, 35, 8, 2, 7, 31, 17, 4)(5, 22, 88, 87, 41, 9, 40, 56, 91, 23)

(6, 26, 104, 150, 44, 10, 43, 147, 109, 27)(11, 45, 154, 236, 115, 29, 108, 204, 155, 46)

(12, 34, 101, 230, 119, 30, 16, 67, 169, 50)(14, 58, 179, 176, 123, 32, 122, 234, 182, 59)

(15, 63, 189, 149, 126, 33, 125, 221, 106, 64)(18, 75, 171, 141, 132, 36, 70, 200, 92, 74)

$(19, 79, 167, 84, 21, 37, 134, 170, 137, 39)(20, 81, 148, 172, 52, 38, 135, 105, 159, 61)$

$(24, 96, 68, 198, 114, 28, 54, 85, 188, 97)(25, 99, 117, 173, 144, 42, 83, 48, 165, 100)$

$(47, 160, 55, 90, 156, 116, 208, 78, 140, 161)(49, 162, 180, 239, 233, 118, 166, 201, 184, 163)$

$(51, 95, 226, 220, 86, 120, 113, 229, 178, 138)(53, 72, 183, 128, 93, 121, 130, 199, 69, 142)$

$(57, 111, 224, 127, 197, 107, 152, 192, 66, 175)(60, 177, 157, 219, 194, 65, 193, 153, 174, 185)$

$(62, 187, 203, 71, 202, 124, 228, 168, 129, 186)(73, 181, 158, 110, 215, 131, 210, 227, 151, 191)$

$(76, 206, 237, 133, 98, 80, 212, 207, 77, 143)(82, 139, 214, 211, 190, 136, 89, 223, 205, 216)$

$(94, 225, 209, 146, 222, 112, 195, 232, 103, 196)(102, 218, 164, 213, 235, 145, 217, 238, 240, 231).$

Table 7 is based on these generators.

Table 7
The totally chiral hypermaps with automorphism group $2 . A_7$ and $g \leqslant 1001$.

Type	g	#Trip.	Refl.	Chir.	$\mathcal{H}$	(a, b)	$\mathcal{H}/C_2$
(3, 5, 5)	673	15120	1	2	$2\mathcal{A}_7^{[1]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = y^{-1}x^2yx^3$	$\mathcal{A}_7^{[12]}$
(3, 5, 7)	817	20160	2	2	$2\mathcal{A}_7^{[2]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = xyx^3y^{-1}x$	$\overline{\mathcal{A}_7^{[23]}}$
(3, 5, 8)	862	25200	1	4	$2\mathcal{A}_7^{[3]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = y^{-2}$	$\mathcal{A}_7^{[4](12)}$
					$2\mathcal{A}_7^{[4]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = x^{-2}y^{-1}x^{-3}$	$\mathcal{A}_7^{[3](12)}$
(3, 7, 7)	961	35280	3	4	$2\mathcal{A}_7^{[5]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = yxy$	$\overline{\mathcal{A}_7^{[39]}}$
					$2\mathcal{A}_7^{[6]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = yx^{-2}y^{-1}$	$\overline{\mathcal{A}_7^{[38]}}$
(3, 5, 14)	997	25200	1	4	$2\mathcal{A}_7^{[7]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = x^2yx^3$	$\overline{\mathcal{A}_7^{[21]}}$
					$2\mathcal{A}_7^{[8]}$ (totally chiral)	$a = yxy^{-1}x^3yx^{-2}$ $b = x^2y^{-1}x^{-7}$	$\overline{\mathcal{A}_7^{[22]}}$

For $3 . A_7$ the following permutation representation of degree 63 satisfy Lemma 5 and Corollary 6, and Table 8 that follows is based on these generators.

$$x = (1, 30, 36)(2, 29, 46)(3, 27, 47)(4, 31, 44)(5, 34, 53)(7, 18, 19)(8, 32, 43)(9, 23, 48)(10, 22, 49)$$
$$(11, 24, 42)(12, 25, 50)(14, 28, 17)(15, 33, 51)(16, 26, 52)(20, 35, 21)(37, 62, 40)(38, 59, 58)$$
$$(39, 55, 60)(45, 61, 63),$$
$$y = (1, 42, 21, 15, 45)(2, 34, 3, 23, 55)(4, 11, 40, 33, 18)(5, 36, 48, 63, 46)(6, 43, 56, 12, 38)$$
$$(7, 44, 10, 37, 28)(8, 13, 24, 51, 54)(9, 19, 29, 53, 31)(14, 41, 32, 57, 49)(16, 17, 39, 47, 22)$$
$$(25, 26, 59, 60, 27)(30, 50, 35, 58, 61).$$

Table 8
The totally chiral hypermaps with automorphism group $3 \,.\, A_7$ and $g \leqslant 1001$.

Type	g	#Trip.	Refl.	Chir.	$\mathcal{H}$	(a, b)
(3, 4, 4)	631	20160	0	4	$3\,.\,\mathcal{A}_7^{[1]}$ $(\mathcal{H}/C_3 = \overline{\mathcal{A}_7^{[1]}})$	$a = yx^{-1}y^{-2}xyxyx$ $b = y^{-1}x^{-1}y^2xyx^{-1}y^{-1}$ (totally chiral)
					$3\,.\,\mathcal{A}_7^{[2]}$ $(\mathcal{H}/C_3 = \mathcal{A}_7^{[2]})$	$a = yx^{-1}y^{-2}xyxyx$ $b = y^{-1}xy^2xyxy^2x^{-1}yx$ (totally chiral)
(3, 4, 5)	820	25200	1	4	$3\,.\,\mathcal{A}_7^{[3]}$ $(\mathcal{H}/C_3 = \mathcal{A}_7^{[4]})$	$a = yx^{-1}y^{-2}xyxyx$ $b = yxy^{-1}xy^{-1}x^{-1}yxy^{-2}x^{-1}y^2xy^{-1}$ (totally chiral)
					$3\,.\,\mathcal{A}_7^{[4]}$ $(\mathcal{H}/C_3 = \mathcal{A}_7^{[3]})$	$a = yx^{-1}y^{-2}xyxyx$ $b = y^{-1}xy^{-1}x^{-1}y^{-2}xy^{-2}$ (totally chiral)
(3, 4, 6)	946	45360	1	8	$3\,.\,\mathcal{A}_7^{[5]}$ $(\mathcal{H}/C_3 = \overline{\mathcal{A}_7^{[5]}})$	$a = yx^{-1}y^{-2}xyxyx$ $b = xy^{-1}x^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y$ (totally chiral)
					$3\,.\,\mathcal{A}_7^{[6]}$ $(\mathcal{H}/C_3 = \mathcal{A}_7^{[5]})$	$a = yx^{-1}y^{-2}xyxyx$ $b = y^{-1}x^{-1}y^2x^{-1}yx^{-1}y^2xyx^{-1}y^{-1}xy^{-1}x^{-1}$ (totally chiral)
					$3\,.\,\mathcal{A}_7^{[7]}$ $(\mathcal{H}/C_3 = \mathcal{A}_7^{[5]})$	$a = yx^{-1}y^{-2}xyxyx$ $b = y^{-1}xy^2x^2y^{-3}x^{-1}y^{-1}x^{-1}$ (totally chiral)

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