CHAPTER 14

LINKING GEOMETRY AND ALGEBRA IN THE SCHOOL MATHEMATICS CURRICULUM

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INTRODUCTION

This paper could have been entitled *Linking Algebra and Geometry in the School Mathematics Curriculum*. After all, algebra does come before geometry in the dictionary. Yet there are a number of reasons why it might be advantageous to begin this paper with these two components of mathematics in reverse alphabetic order: it switches attention to geometry (rather than bolstering the tendency for algebra to dominate the school mathematics curriculum); it implies that geometry can provide insight into other aspects of mathematics; and it is indicative of how the development of digital technologies has seen a resurgence in interest in geometry and in techniques for visualizing mathematics (Jones, 2000, 2002). For these reasons, and more, the focus of this paper is linking geometry and algebra - and how, through such linking, the mathematics curriculum (and hence the teaching and learning experience) might be strengthened.

Jones, K. (2010), Linking geometry and algebra in the school mathematics curriculum. In Z. Usiskin, K. Andersen & N. Zotto (Eds) *Future Curricular Trends in School Algebra and Geometry*. Charlotte, NC: Infoage. pp203-215. ISBN: 9781607524724

THE RELATIONSHIP BETWEEN GEOMETRY AND ALGEBRA

That algebra can tend to dominate the school mathematics curriculum is apparent in many ways, one example being the work of the U.S. National Mathematics Advisory Panel which was directed to focus on “the preparation of students for entry into, and success in, algebra” (U.S. National Mathematics Advisory Panel, 2008, p. 8). Yet it is worth reflecting on the words of people like Coxeter, Bell, and Atiyah (an ABC of renowned mathematicians, taken in reverse alphabetic order). It was Coxeter, the famous geometer, who replied with the following advice when asked what would most improve upper secondary or college level mathematics teaching: “I think that, by being careful, we could probably do the same amount of calculus and linear algebra in less time, and have some time left over for nice geometry” (Coxeter quoted in Logothetti and Coxeter, 1980).

The mathematician Eric Bell noted that “With a literature much vaster than those of algebra and arithmetic combined, and at least as extensive as that of analysis, geometry is a richer treasure house of more interesting and half-forgotten things, which a hurried generation has no leisure to enjoy, than any other division of mathematics” (Bell quoted in Coxeter and Greitzer 1967, p. 1).

At his Fields Lecture at the World Mathematical Year 2000 Symposium (Toronto, Canada, June 7-9, 2000), the celebrated mathematician Michael Atiyah argued that “…spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool” (Atiyah, 2001).

In the history of mathematics there has, it seems, been a somewhat (and sometimes) uneasy relationship between geometry and algebra (Atiyah, 2001; Charbonneau, 1996; Giaquinto, 2007; Kvasz, 2005). According to Atiyah (2001), fundamental to what can seem like a dichotomy is that “algebra is concerned with manipulation in time, and geometry is concerned with space. These are two orthogonal aspects of the world, and they represent two different points of view in mathematics. Thus the argument or dialogue between mathematicians in the past about the relative importance of geometry and algebra represents something very, very fundamental”. Yet while algebra provides powerful techniques for mathematics, Atiyah sees a danger that “when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning”.

These are some of the reasons for focusing on linking geometry and algebra, for recognising the important role that geometrical thinking has in mathematics, and for strengthening the teaching and learning of mathematics through finding ways of building on students' spatial intuition and spatial perception. That it remains vital to do these things might be surmised from considering the case of the school mathematics curriculum in England.

THE SCHOOL MATHEMATICS CURRICULUM: THE CASE OF ENGLAND

The introduction, in 1988, of a statutory national curriculum in England cemented existing UK practice that "while it is convenient to break mathematics down …[into areas such as number, algebra, geometry, statistics].., it is important to remember that they do not stand in isolation from each other (UK DES, 1988, p. 3). In this way, mathematics in UK schools is generally presented as an integrated subject, although students may well experience a curriculum diet of mathematics taught as a series of separate topics (of algebra, geometry, and so on) of varying lengths (of perhaps four to six weeks each). Parallel to the introduction of a statutory national curriculum, a system of national testing for students aged 7, 11, and 14 was instigated, augmenting existing national testing at 16 and 18.

In the period from 1995-2000, this form of statutory curriculum and national testing became more and more entrenched, leading to an increase in forms of school accountability through the publication of, for example, "league tables" of schools (based on their national test results). At this time, also, international comparisons such as TIMSS began to have an increasing impact; so much so that the UK Government launched its *National Numeracy Strategy* in 1998 (Department for Education and Employment, 1998a; 1998b; 1999). This numeracy strategy sought to address perceived weaknesses in the teaching of mathematics, particularly at the elementary school level, and focused primarily on skills of calculation and computation. Geometry received hardly a mention (Jones & Mooney, 2003). At the same time, there were emerging concerns about mathematics teaching at the secondary school level, particularly regarding perceived inadequacies in the preparation for proof at University level (London Mathematical Society, 1995; Engineering Council, 1999).

During this period, the International Commission on Mathematical Instruction (ICMI) study on the teaching and learning of geometry was taking place (Mammana and Villani, 1998), with, amongst many other issues, a consideration of the relationship between deductive and intuitive approaches to solving geometrical problems (Jones, 1998) and the nature and role of proof in the context of dynamic geometry software (Hoyles and Jones, 1998).

In the period 2000-2005, the statutory curriculum for England was revised. The revision of the mathematics curriculum included more explicit stipulations regarding proof, and some further encouragement for links within mathematics and across subjects. National testing continued in much the same form, with school accountability in the form of league tables of schools published in the national media becoming even more ingrained. The national *numeracy* strategy was extended into secondary schools as the national *mathematics* strategy (Department for Education and Employment, 2001).

During this period, the UK Royal Society and Joint Mathematical Council instigated a working group on the teaching and learning of geometry from age 11 to 19 (Royal Society, 2001). The report of this working group stressed the far-reaching importance of geometry within and beyond the school mathematics curriculum and was widely welcomed. Amongst the themes of the report were an emphasis on conjecturing and proving, on the importance of spatial thinking and visualizing, plus the benefits of linking geometry with other areas of mathematics, and on the powerful role of digital technology. A number of the report's recommendations have already been enacted within the UK school system, with some being illuminated through a UK Government initiative on algebra and geometry (Qualifications and Curriculum Authority, 2004). This initiative sponsored six modest curriculum development projects, with the overall report stressing that "making connections between different mathematical concepts is important for developing understanding [of mathematics]" (*ibid*, p. 25). The report, in summarising the six individual projects, offered two suggestions of ways of linking geometry and algebra, one being to exploit the capacity of dynamic geometry software to provide novel ways of visualizing algebraic relationships, the second being to use different approaches to tackle the same problem. Such ways of linking geometry and algebra are illustrated below.

Since 2005, the UK has revised its statutory curriculum again. This time, while established school "subjects" remain, there is less emphasis on specifying the curriculum in terms of subjects (Qualifications and Curriculum Authority, 2005). Despite this, the system of national testing and the entrenched school accountability remains (with the continuing use of league tables) even though an increasing volume of evidence suggests such a system narrows the curriculum (to the testing regime) and thence stifles innovation in curriculum and limits teachers' professional autonomy (for an overview of the state of mathematics teaching in the UK, see Ofsted, 2008).

Around this time, ICMI study 17 on technology examined, amongst many other things, the design of digital technologies for different geometries (Jones, Mackrell, & Stevenson, 2009), and the European Union funded projects on the teaching of three dimensional geometry (Christou, Jones, Mousoulides, & Pittalis, 2006) and the teaching of calculus with dynamic geometry software (Zachariades, Jones, Giannakoulias, Biza, Diacoumopoulos, & Souyoul, 2007).

All this suggests that while international comparisons of mathematical achievement can lead to a government being committed to implementing strict regimes of statutory curricula and student testing, it can happen that reports from outside bodies and from research can have an influence such that, over time, some account starts to be taken of under-represented aspects of the mathematics curriculum.

MAKING CONNECTIONS BETWEEN DIFFERENT MATHEMATICAL CONCEPTS

As the QCA report on algebra and geometry (Qualifications and Curriculum Authority, 2004) indicates, one way of linking geometry and algebra is to exploit the capacity of dynamic geometry software to provide novel ways of visualizing algebraic relationships. As an illustration, teachers in a Hampshire school (in England) worked on a project in which their students used dynamic geometry software to plot quadratic functions that match the flight of a basketball, providing their students with hands-on experiences of how the various algebraic coefficients affect the shape of the graph – as illustrated in Figure 1.

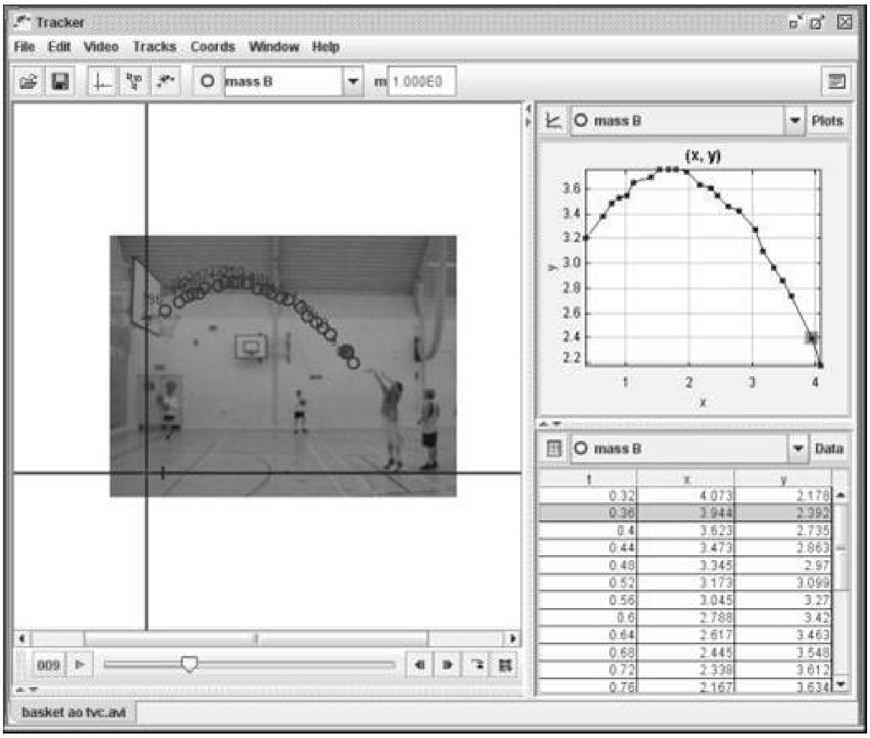

Figure 1. Plotting the trajectory of a basketball

Another way of linking geometry and algebra is to use different approaches to tackle the same problem. To illustrate this, consider the oft-repeated claim that one of the oldest problems in number theory is to find Pythagorean triples, triples of whole numbers (a, b, c) which fulfill the Pythagorean relation $a^2 + b^2 = c^2$. Yet if a stance is maintained that this is *solely* a problem in number theory, then one outcome is likely to be the omission of the link between Pythagorean triples and the integer size of the radius of the incircle of a right triangle. While not wishing to give too much away to anyone unfamiliar with the construction illustrated in Figure 2, using dynamic geometry to construct the figure and dragging the vertices of the triangle to integer values of the sides of the triangle might suggest a connection to integer values of the radius of the incircle. With, in this way, a conjecture of a theorem being generated, then a small amount of algebra might suffice to prove such a theorem.

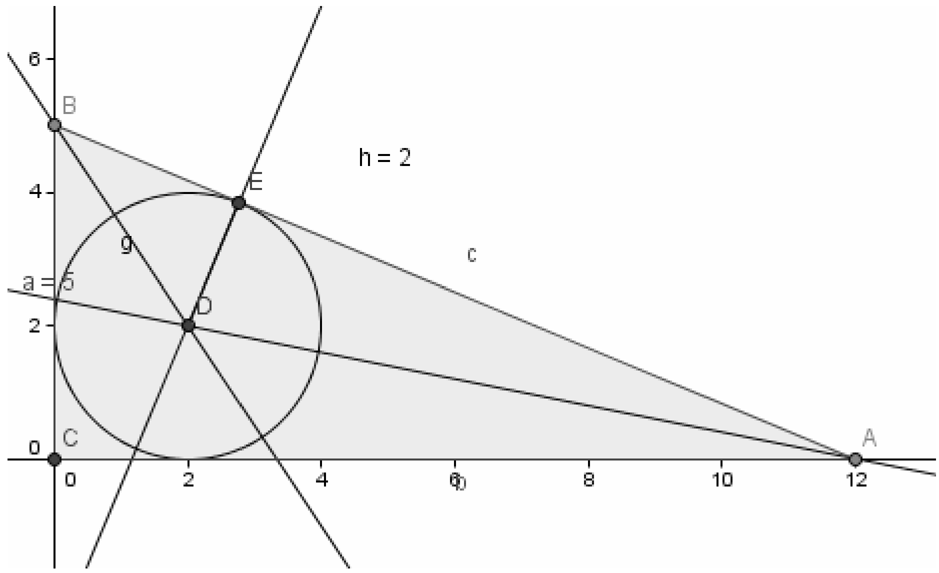

Figure 2. Pythagorean triples and integer values of the radius of the incircle

It is notable that, in making connections between different mathematical concepts, both of the approaches illustrated in this section of the paper utilize digital technologies.

THE POWER OF GEOMETRY TO BRING CONTEMPORARY MATHEMATICS TO LIFE

A familiar occurrence for many mathematics teachers around the world is students being heard to ask about the usefulness of whatever part of mathematics they are studying. No doubt teachers continually devise inventive attempts to address such questions, yet one thing that might help is to consider how the power of geometry can bring contemporary mathematics to life; examples include double bubbles, black holes, and flags.

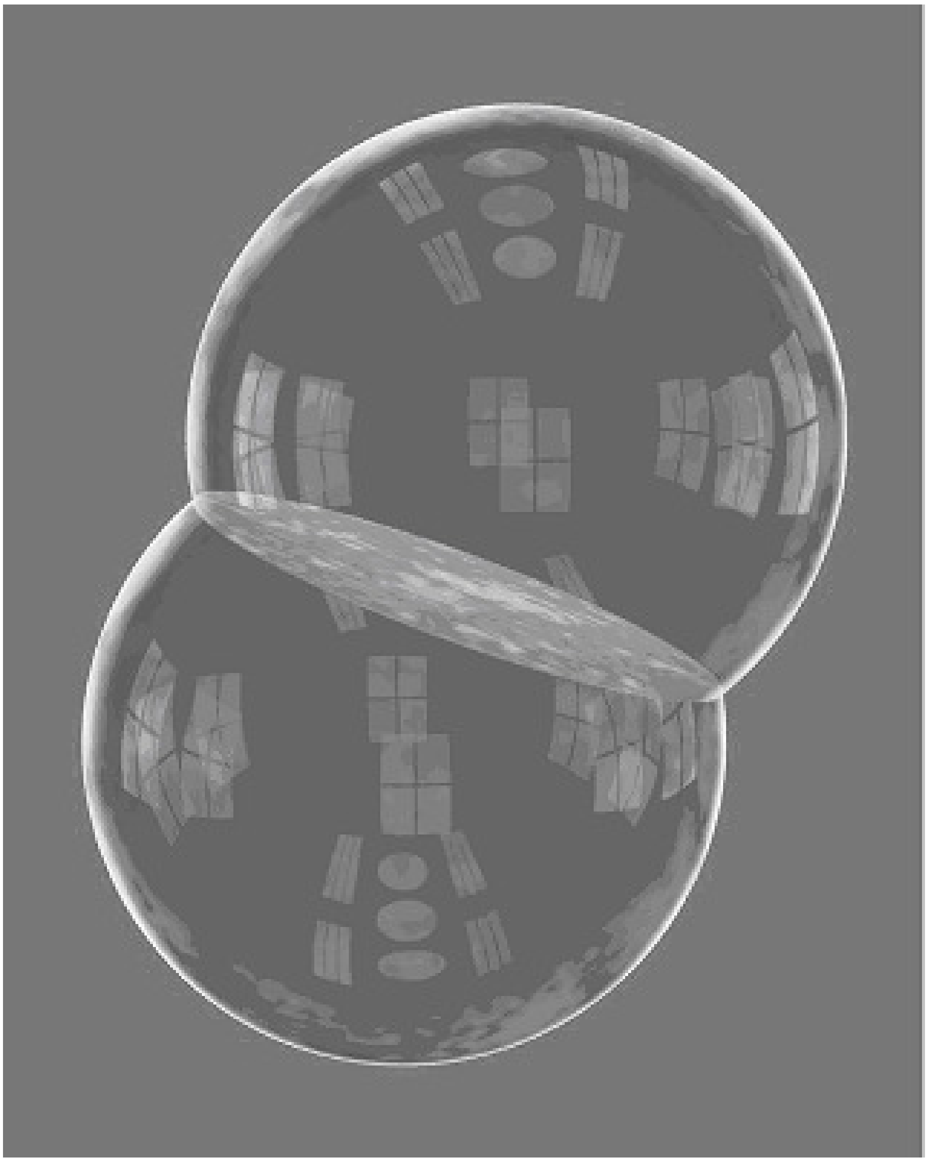

Figure 3. A standard double bubble of equal volumes

A double bubble is a pair of bubbles which intersect and are separated by a membrane bounded by the intersection, as illustrated in Figure 3. It had been conjectured that two partial spheres of the same radius that share a boundary of a flat disk separating two volumes of air use a total surface area that is less than any other arrangement. This equal-volume case was proved in 1995. When the bubbles

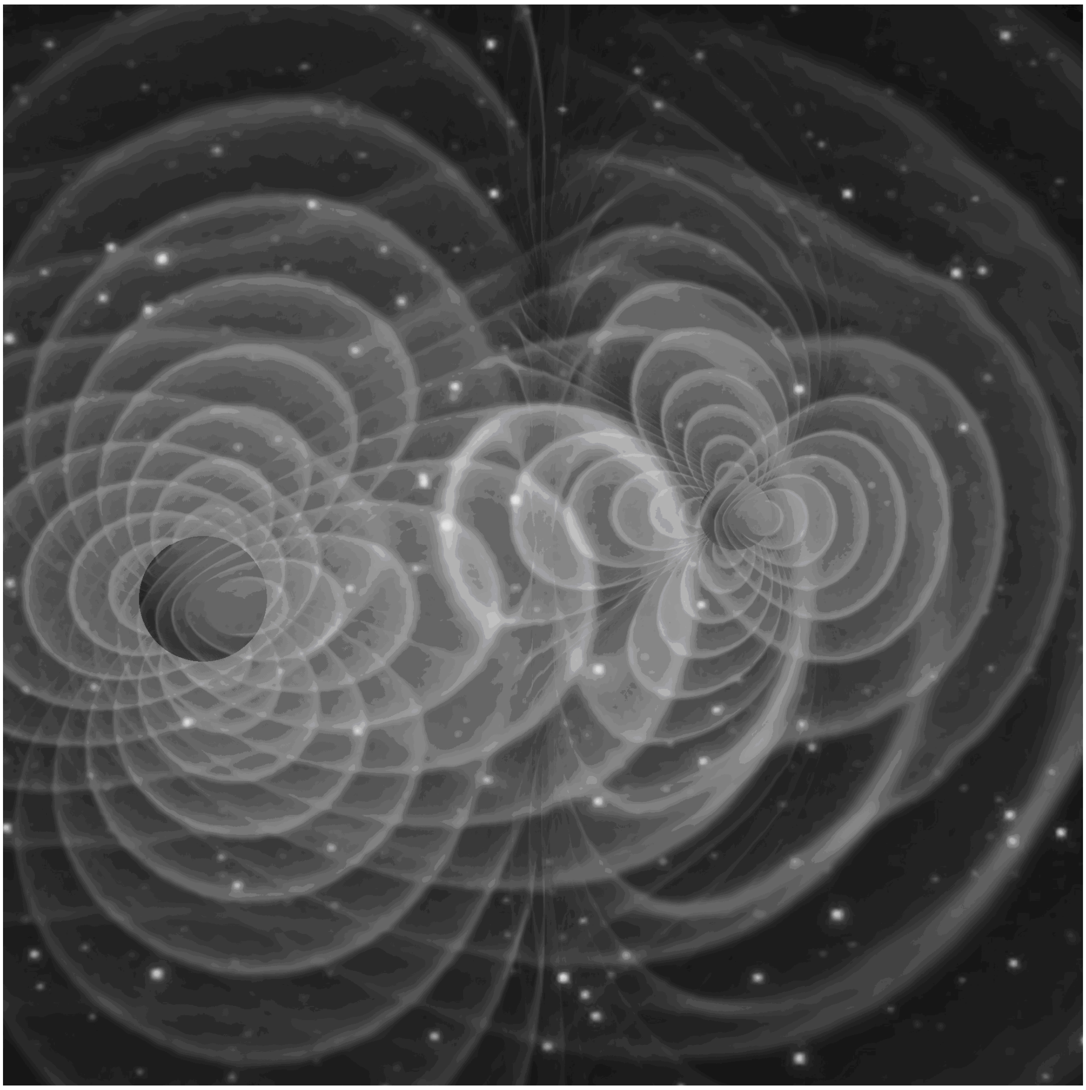

Figure 4. Merging black holes

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are unequal in size, it has been shown that the separating boundary which minimizes the total surface area is itself a portion of a sphere. Corresponding conjectures about triple bubbles remain open. For more information on such bubble problems, see Brubaker et al (2008).

In 2006, NASA scientists reached a breakthrough in computer modeling that allowed them to simulate what gravitational waves from merging black holes look like (NASA, 2006). The three-dimensional simulations, illustrated by Figure 4, are the largest astrophysical calculations ever performed on a NASA supercomputer.

The design of flags is sometimes mentioned in the school mathematics classroom, perhaps during a topic on symmetry. Yet modeling the movement of a flag mathematically is of interest to mathematicians interested in dynamic systems (such mathematics involves the analytic, asymptotic and numerical solution of non-linear partial singular integro-differential equations with Cauchy Kernels).

Figure 5: a fluttering US flag

In these ways, the power of geometry can be used to bring contemporary mathematics to life. Mentioning these things in the mathematics classroom might mean that learners of mathematics look differently at bubbles, astronomic entities, and flags.

LOOKING TO THE FUTURE

In England, a new curriculum for schools began to be implemented in September 2008. This new curriculum is intended to "give schools greater flexibility to tailor learning to their learners' needs" and as such there is "less prescribed subject content" (QCA, 2007, p. 4). While students are still taught "essential subject knowledge", the new curriculum "balances subject knowledge with the key concepts and processes that underlie the discipline of each subject"

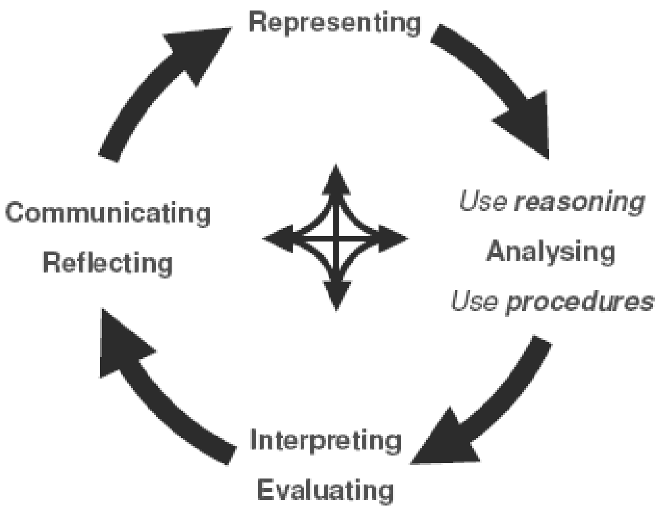

Figure 6: key concepts and processes in the new curriculum for England

(QCA, 2009). In terms of mathematics, these "key concepts and processes" are set out in Figure 6.

This increasing focus on concepts and processes provides new opportunities to ensure that the full potentialities of geometric and algebraic approaches are used for the true benefit of student learning. Already what can appear as the opposing tendencies of geometry and algebra are being blurred in mathematics. For example, Artin, one of the leading algebraists of the 20th century, gave rise to the contemporary use of the term geometric algebra through his book of that title (Artin, 1957). Current applications of geometric algebra include computer vision, biomechanics and robotics, and spaceflight dynamics. Then there is algebraic geometry, the study of geometries that come from algebra. This occupies a central place in contemporary mathematics and has multiple conceptual connections with such diverse fields as complex analysis, topology, and number theory.

The term *concinnity* is most often used for the harmonious or purposeful reinforcement of the various parts of a work of art (with generally the higher the art form, the higher the degree of concinnity). Yet concinnity comes from the Latin *concintas*, meaning skillfully put together, and can apply to any object or situation (even though it is most commonly used in the discussion of music where an example of concinnity might be when the various parts of a piece of music - melody, harmony, rhythm, on so on - reinforce each other).

In the future, we might look for greater concinnity in the mathematics curriculum, especially in terms of the harmonious/purposeful reinforcement of mathematical thinking through the linking of geometry and algebra. Such an approach might be supported by Giaquinto's (2007) view that, from an epistemological perspective, "the algebraic-geometric contrast, so far from being a dichotomy, represents something more like a spectrum".

CONCLUDING COMMENTS

In conclusion, it is worth further reflecting on the words of Coxeter, Bell, and Atiyah. A greater concinnity in the mathematics curriculum through the linking of geometry and algebra might enable us, as Coxeter advised, to "do the same amount of calculus and linear algebra in less time, and have some time left over for nice geometry" (quoted in Logothetti and Coxeter, 1980). Given that, as Bell argues, "geometry is a richer treasure house ... than any other division of mathematics" (quoted in Coxeter and Greitzer 1967, p. 1), this points to how geometry can be such a rich source of ideas for teaching mathematical thinking. What is more, as Atiyah indicates, "geometry is actually such a powerful part of mathematics - not only for things that are obviously geometrical, but even for things that are not" (Atiyah, 2001).

As Atiyah (1982) put it "The educational implications of this are clear. We should aim to cultivate and develop both modes of thought. It is a mistake to overemphasize one at the expense of the other and I suspect that geometry has been suffering in recent years. The exact balance is naturally a subject for detailed debate and must depend on the level and the ability of the students involved. The main point that I have tried to get across is that geometry is not so much a branch of mathematics but a way of thinking that permeates all branches".

Usiskin (2004) put it this way "the soul of mathematics may lie in geometry, but algebra is its heart" - and, of course, one needs both a heart and a soul. For after all, as is commonly recognized, without geometry, life is pointless.

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