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Stability and Accuracy of Finite Element Methods for Flow Acoustics: II Two Dimensional Propagation

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

FLUID DYNAMICS AND ACOUSTICS GROUP

Stability and Accuracy of Finite Element Methods for Flow Acoustics: II. Two Dimensional Propagation

by

G Gabard and R J Astley

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Authorized for issue by
Professor R J Astley, Group Chairman

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ABSTRACT

The dispersion properties of finite element models for aeroacoustic propagation based on the convected scalar Helmholtz equation and on the Galbrun equation are examined. The current study focusses on the effect of the mean flow on the dispersion and amplitude errors present in the the discrete numerical solutions. A general two-dimensional dispersion analysis is presented for the discrete problem on a regular unbounded mesh, and results are presented for the particular case of one dimensional acoustic propagation in which the wave direction is aligned with the mean flow. The magnitude and sign of the mean flow is shown to have a significant effect on the accuracy of the numerical schemes. Quadratic Helmholtz elements in particular are shown to be much less effective for downstream—as opposed to upstream—propagation, even when the effect of wave shortening or elongation due to the mean flow is taken into account. These trends are also observed in solutions obtained for simple test problems on finite meshes. A similar analysis of two-dimensional propagation is presented in an accompanying report.

I. INTRODUCTION

This report is the second part of a study intended to analyze the accuracy of two finite element methods for aeroacoustic propagation. The first numerical method considered here is based on the full potential theory which describes the propagation of acoustic waves on a potential mean flow by mean of the acoustic velocity potential. The second numerical method is based on Galbrun equation which relies on the Lagrangian perturbation of the displacement to describe acoustic wave on a general mean flow. Both these methods are investigated for two-dimensional time-harmonic problems.

The accuracy of these methods are assessed by comparing the dispersion properties of the numerical models with those of the physical model. To apply the dispersion analysis, we must restrict to infinite, periodic meshes and to uniform mean flows which are written

$$v_0 = Mc_0\cos(\alpha)\, e_x + Mc_0\sin(\alpha)\, e_y \quad ,$$

where c_0 is the speed of sound, M the Mach number and α the mean flow direction with respect to the mesh axes (see figure 1). With a uniform mean flow the full potential theory reduces to the convected Helmholtz equation. The solution $u(x)$ of the problem is sought as a plane wave

$$u(x_{n,m}^{(p)}) = u_0^{(p)}\exp\left(ik\cdot x_{n,m}^{(p)}\right) \quad ,$$

with

$$k = k\cos(\theta) + k\sin(\theta) \quad ,$$

where $x_{n,m}^{(p)}$ denotes the nodes of the mesh, $u_0^{(p)}$ is the wave amplitudes and k is the real wavenumber with direction θ and magnitude k (see figure 1). The vector $U = (u_0^{(1)}, u_0^{(2)}, \ldots)$ and k are solutions of an eigenvalue problem devised from the difference equations of the numerical model (see Section 2 in the first report). If a solution $\left(\tilde{k}, \tilde{U}\right)$ corresponds to a physical mode, we can define the dispersion error E_d and the amplitude error E_a

$$E_d = \frac{\left|\tilde{k}h - kh\right|}{kh} \quad , \quad E_a = \frac{\left\|\tilde{U} - U\right\|}{U} \quad ,$$

where h is the node spacing. The dispersion error E_d describes the deviation of the numerical wavenumber form its exact value while E_a represents the error on the amplitude of the wave. A solution $\left(\tilde{k}, \tilde{U}\right)$ which does not correspond to any physical modes is a spurious numerical modes. If such modes exist, the numerical model is likely to be unstable.

A detailed account of the dispersion analysis and the finite element methods as well as an overview of related works are given in the first part of the report.

These errors are analyzed with respect to the parameters θ, M, α and the number of points per wavelength. For one-dimensional problems, the latter is simply $2\pi/(kh)$. For two-dimensional problems, the definition of the number of points per wavelength is more ambiguous since the typical node spacing in the direction of the wave depends on the angle θ. However, if one takes this effect into account, the interpretation of the results becomes much more involved. So, following other contributors (such as Harari[2]o r Deraemaeker *et al.*[1]), we define the number of points per wavelength as $2\pi/(kh)$ irrespective of the wave direction.

The first part of the report was limited to one-dimensional propagation, that is waves propagating along the mesh axes ($\theta = \alpha = 0$). We discussed the effects of the mean flow speed and the number of points per wavelength on the numerical accuracy. One of the main findings of the first part is that the quadratic Helmholtz elements exhibit very important increases of the dispersion and amplitude errors for certain numbers of points per wavelength in the downstream case. This is produced by an 'aliasing' error which is also discussed in the present report (see section IV). For the Galbrun elements, it was found that the stability is Mach number dependent

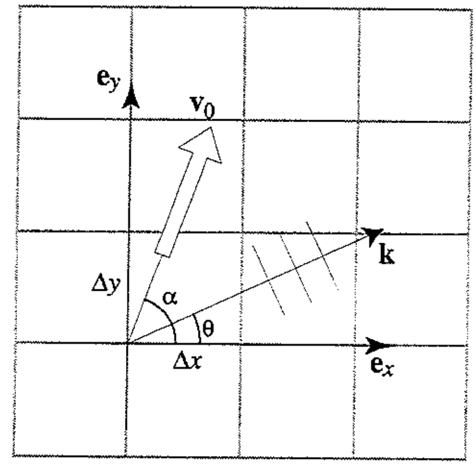

Figure 1 Description of the mesh axes, the mean flow direction α and the wave direction θ.

and that the quadratic element Q9-4c does not improve the accuracy compared to the linear element T4-3c.

This part of the report is concerned with two-dimensional acoustic propagation. We focus on the influence of the two additional parameters arising only in the two-dimensional case, namely the mean flow direction θ and the wave propagation direction α. It is well known that the use of meshes in any numerical methods introduces preferential directions in space, and hence produces numerical anisotropy. Furthermore, the physical problem of wave propagating on amean flow is anisotropic since the dispersion properties of the waves depend on the angle between the mean flow and the wave directions. However, in the numerical model, this physical anisotropy is also combined with the additional anisotropy introduced by the numerical methods. The numerical anisotropy with respect to θ and α is the main concern of this report.

The rest of this report is divided in 4 parts. The effects of the wave direction and the mean flow direction are discussed respectively in section II and III. The occurence of the aliasing error is presented in section IV. Section V summarizes the results of this report.

II. EFFECT OF THE WAVE DIRECTION

First, consider the anisotropy of the finite element models with respect to the wave direction θ. To illustrate more clearly this property, the mean flow direction is fixed ($\alpha = 0$) and we present results for four Mach numbers $M = 0, 0.25, 0.5$ and 0.75. In all cases, we have 10 points per wavelength, that is $2\pi/(kh) = 10$. The dispersion and amplitude errors are given in figures 6 to 9.

First, consider the no flow case (figures 6 and 7). The dispersion errors with the Helmholtz elements (Q4, Q8 and Q9) are maximum for $\theta = 0, \pi/2, \pi, 3\pi/2$ and minimum for $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. Thus, these elements are less accurate when the wave propagates along the mesh directions and more accurate when the wave propagates along the diagonals of the elements. This property has already been noticed by Harari[2] and Deraemaeker[1]. An important consequence is that, if the wave propagation directions can be anticipated, the mesh should be aligned so that the diagonals of the elements coincide with the wave direction in order to improve the accuracy. The dispersion error with the quadratic elements is (almost exactly) one order of magnitude smaller than with linear element. The Q8 and Q9 elements give very similar dispersion errors, the Q9 element being very slightly more accurate along the diagonals. However, the two quadratic elements are very different with respect to the amplitude error. The Q9 element is much more accurate and more isotropic than the Q8 element. So, the accuracy and efficiency can be improved by using the 9 node element rather than the 8 node element. It is especially true if one uses condensation to remove the central degree of freedom of the Q9 element. This finding is of particular importance since the Q8 element is the most widespread element in commercial codes for acoustics.

The T4-3c Galbrun element gives quite different results depending on the mesh pattern. Logically, the mesh A is more anisotropic with a maximum of error along the 45 direction of that mesh. With the mesh B, the dispersion and amplitude errors are minimum for $\theta = 0, \pi/2, \pi, 3\pi/2$ and maximum $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ (this is the contrary of the Helmholtz elements). Globally, the results lie between the linear and quadratic Helmholtz elements. The Q9-4c element gives a dispersion error similar to the Q4 element with a more pronounced anisotropy. The amplitude error is larger than the T4-3c element. This obviously demonstrates that the Q9-4c does not improve the efficiency of the Galbrun finite element model compared to the T4-3c element.

The trends observed in the no flow case are also present with mean flow but combined with other effects, see figures 6 to 9.

For the Helmholtz elements, an interesting propery is that the dispersion error with mean flow can be obtained by multiplying the dispersion error in the no flow case by a factor $(1 - M\cos(\alpha - \theta))$ similar to the Doppler factor. So, when the Mach number is increased, the dispersion is reduced in the downstream direction and increased in the upstream direction (this effect is also shown in the next section). It is important to note that this effect is not a consequence of the wavelength modification by the Doppler effect since all the results are given with a prescribed number of points per wavelength.

The influence of the mean flow on the amplitude error is less obvious: E_a is increased both in the upstream and downstream directions. However, the aliasing error is also well observed with $M = 0.75$: important increases are present in the downstream direction for the dispersion and amplitude errors (see figures 8 to 9). This is very clear with E_a with a peak of three order of magnitude in the range $[-\pi/4; \pi/4]$. The occurence of the aliasing error is discussed with more details in section IV.

With the T4-3c element, the effect of the mean flow on the anisotropy does not follow a simple trend. Especially with $M = 0.25$, where the dispersion error is very low (even better than the Q8 and Q9 elements), this is due to the fact that E_d is almost reduced to zero by the mean flow effect (see figures 14 and 18 in the first part of this report). The amplitude error tends to increase in the direction of the flow ($\theta = 0, \pi$) and to decrease in the direction normal to the flow ($\theta = \pm\pi/2$). For the mesh A, with $M = 0.5$ and 0.75,th e anisotropy due to the mesh alignement is also clearly present. It is worth noting that the results obtained with the mesh B are always comparable to or better than the Q4 element.

With mean flow, the Q9-4c element gives results ranging from the Q9 element to the Q4 depending on θ. However, the amplitude error is slightly changed by the mean flow speed, and, in almost all cases, remains higher than the other elements.

III. EFFECT OF THE MEAN FLOW DIRECTION

We now turn to the effect of the mean flow direction with respect to the mesh axis on the dispersion properties. We consider the three cases $M = 0.25, 0.5$ and 0.75. In all cases, we have 10 points per wavelength and the wave propagates along the mesh axis ($\theta = 0$). The dispersion and amplitude errors plotted against α are given in figures 10 and 11.

For the Helmholtz elements, the dispersion error is found to be exactly proportional to $(1 - M\cos(\alpha - \theta))$, and, obviously, this effect is not due to the anisotropy produced by the mesh. This factor $(1 - M\cos(\alpha - \theta))$ similar to a Doppler factor has already been reported in the previous section. The amplitude errors with the Q8 and Q9 elements are maximum for $\theta = 0$ and are significantly reduced for $\theta = \pm\pi/2$. In fact, these effects can be considered as unrelated to the anisotropy of the mesh and, hence, are only due to the effect of the mean flow direction on the wave propagation. It can be seen more clearly when E_d and E_a are plotted as function of θ and $\alpha - \theta$ (see figures 2 and 3). It is clear, especially for the dispersion error, that the two errors are the combinations of the anisotropy with respect to θ and a Doppler effect depending on $\alpha - \theta$. An important consequence is that the anisotropy of these elements with respect to the mean flow direction is small compared to the anisotropy with respect to the wave direction θ. And thus, for the Helmholtz elements, aligning the mesh on the mean flow direction will not improve the accuracy. The aliasing error is also clearly visible with $M = 0.75$ both for the dispersion and amplitude errors.

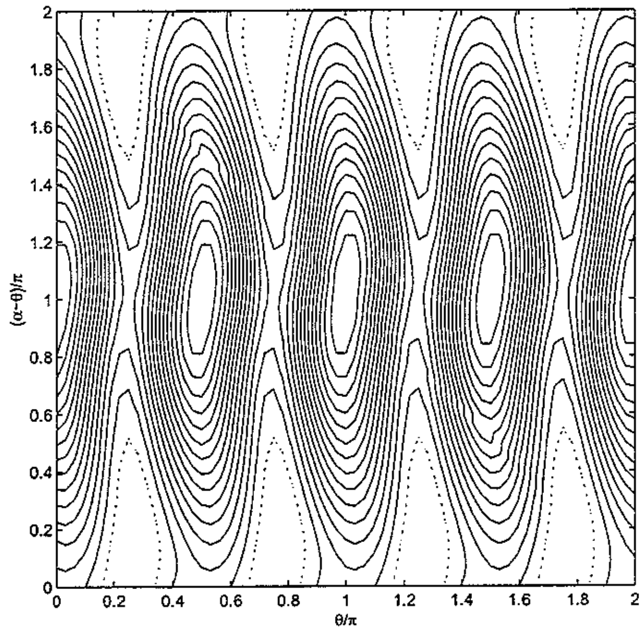

Figure 2 Dispersion error E_d (in %) as a function of θ and $\alpha - \theta$ with for the Q9 element for $M = 0.5$ and 10 points per wavelength: 15 levels between 0.04% and 0.22%, dotted line: the lowest level.

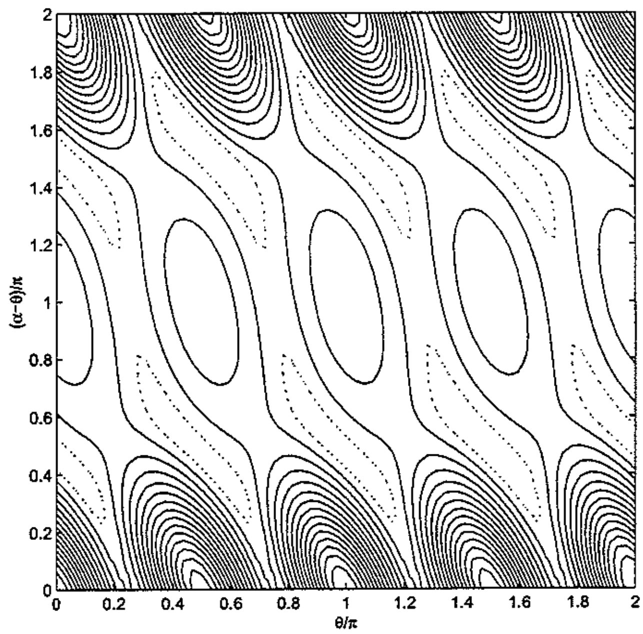

Figure 3 Amplitude error E_a (in %) as a function of θ and $\alpha - \theta$ with for the Q9 element for $M = 0.5$ and 10 points per wavelength: 15 levels between 0.08% and 1.2%, dotted line: the lowest level.

With the T4-3c and the Q9-4c Galbrun elements, the influence of the mean flow direction on E_d and E_d is more complex, and there is clearly an important anisotropy of the numerical model with respect to α. It may be worth noting that both the dispersion and amplitude errors are significantly reduced for $\theta = \pm\pi/2$ when the Mach number is increased. Finally, the Q9-4c element gives worse results than the T4-3c element, especially at low Mach numbers.

IV. THE ALIASING ERROR

It has been shown in the first part of this report that, when the wave propagates along the mesh axis ($\theta = 0$), quadratic elements for the Helmholtz equation can be inaccurate for certain numbers

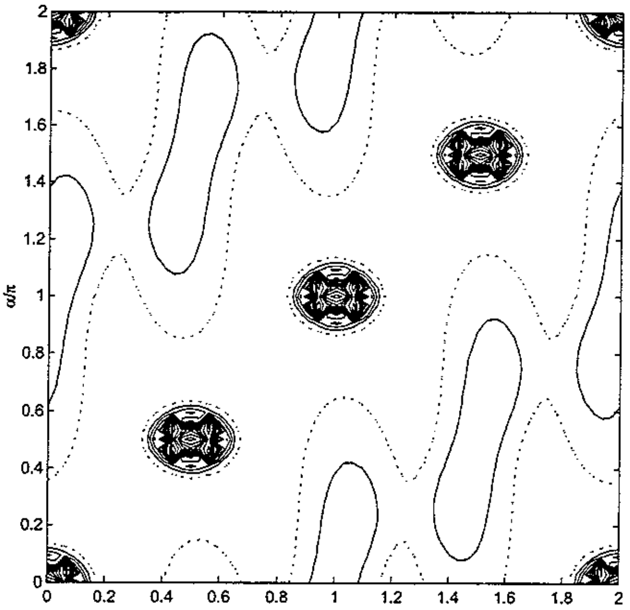

Figure 4 Dispersion error E_d (in %) as a function of θ and α with the Q9 element for $M = 0.75$ and 10 points per wavelength: 15 levels between 0.1% and 1.5%, dotted line: the lowest level.

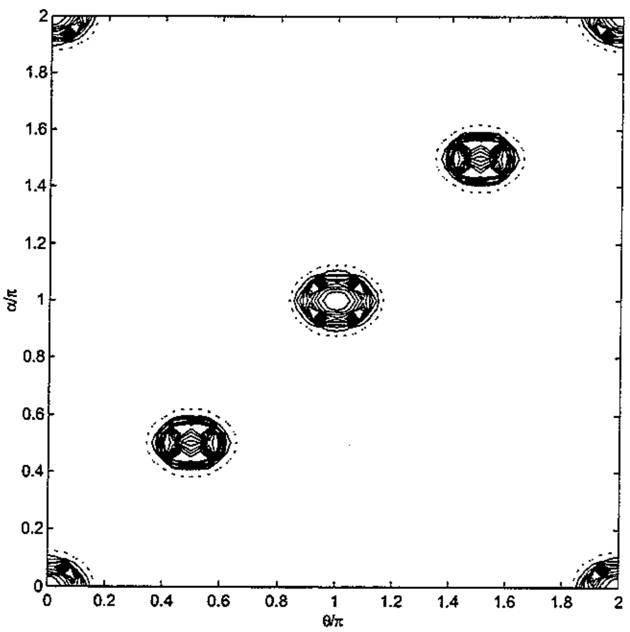

Figure 5 Amplitude error E_a (in %) as a function of θ and α with the Q9 element for $M = 0.75$ and 10 points per wavelength: 15 levels between 12% and 186%, dotted line: the lowest level.

of points per wavelength. This is due to the fact that one physical mode is described by two distinct (but equivalent) modes and this results in an 'aliasing' error. Furthermore, the aliasing error has also been noticed in the two previous sections where α or θ are varied while this other parameter is set to zero. So, it is important to determine whether the aliasing error is specific to one-dimensional problems or if it can also be observed in two- or three-dimensional problems. Here, we discuss the conditions which produce the aliasing error in two-dimensions.

To that end, we consider the Q9 element with $M = 0.75$, and the dispersion and amplitude errors are plotted against α and θ for 10 points per wavelength (see figures 4 and 5).

On both these figures, the aliasing error is clearly identified by peaks along the axis $\alpha =.\theta$ In fact, the aliasing error occurs when *both* the mean flow direction θ and the wave direction α are close to the mesh axes (0, $\pi/2$, π and $3\pi/2$). More precisely, α and θ have to be in the range $\pm\pi/10$ from the mesh axis directions which represents a fairly large angle. We have also checked that these 'spots' of errors reduce and disappear when the number of points per wavelength is increased.

V. CONCLUSION

We can conclude that the Helmholtz elements are more accurate when the wave propagates along the elements diagonal. If it is possible to estimate the direction of waves prior to a simulation, the mesh should be designed so that the diagonal of the elements coincide with these directions.

With Helmholtz elements, the dispersion and amplitude errors are very slightly influenced by the direction of the mean flow with respect to the mesh. So, only a limited accuracy improvement is to be expected by aligning the mesh with the mean flow direction.

Furthermore, in a two-dimensional problem, the aliasing error is likely to appear when the mean flow, the wave direction and the mesh are almost aligned. These requirements can partly explain why this problem has not been clearly identified in previous works. To avoid the aliasing error,o ne should align the finite elements diagonals with the wave directions or, if it is not possible, with the mean flow. The former solution is preferable because it will also improve the accuracy, but, in general, it is not possible to define a simple wave direction. The latter solution is systematicaly applicable since the mean flow is always known prior to the acoustic simulation.

Finally, the 9 node element should be prefered to the 8 node element in order to reduce amplitude error compared.

For the T4-3c Galbrun element, the results obtained with the alternating mesh are better than the mesh A. Thus, one should reduce the anisotropy of the meshes in order to improve the efficiency of these elements. Finally, the Q9-4c element is clearly outperformed by the other elements (including the T4-3c element).

By combining the conclusions drawn in the first and second parts of this study, it is possible to get a global account of what can be expected from the finite element methods based on the full potential theory and the Galbrun equation.

Furthermore, the results of this study suggest that there is a need to devise other finite element methods to cope with the problems outlined by the dispersion analysis: the aliasing error for the Helmholtz elements and the stability problems for the Galbrun elements.

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[2] I. Harari. Reducing spurious dispersion, anisotropy and reflection in finite element analysis of time-harmonic acoustics. *Computer Methods in Applied Mechanics and Engineering*, 140:39–58, 1997.

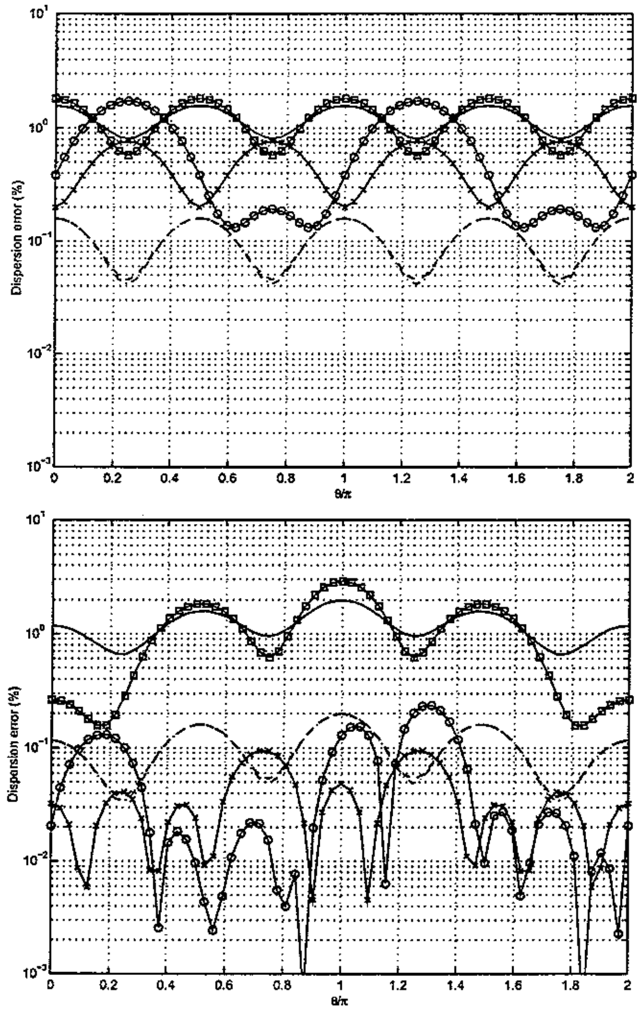

Figure 6 Dispersion error E_d (in %) as a function of θ with $M = 0$ (top) and $M = 0.25$ (bottom). Helmholtz elements: Q4 (——), Q8 (— —), Q9 (— ·) and Galbrun elements: T4-3c with mesh A (–⊖–), T4-3c with mesh B (–×–), Q9-4c (–⊟–).

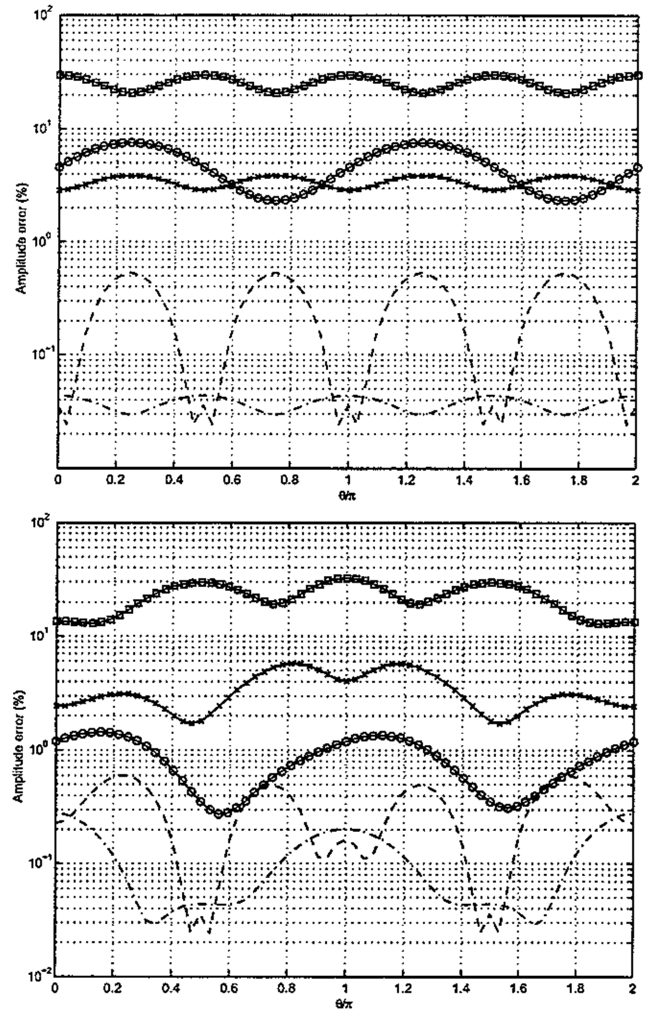

Figure 7 Amplitude error E_a (in %) as a function of θ with $M = 0$ (top) and $M = 0.25$ (bottom). Helmholtz elements: Q8 (— —), Q9 (— ·) and Galbrun elements: T4-3c with mesh A (–⊖–), T4-3c with mesh B (–×–), Q9-4c (–⊟–).

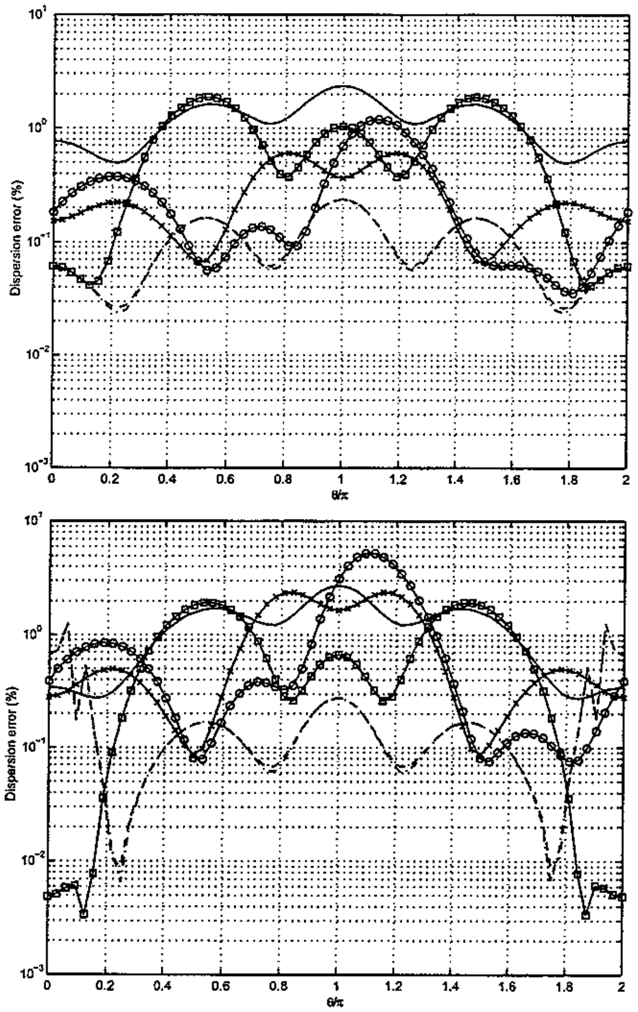

Figure 8 Dispersion error E_d (in %) as a function of θ with $M = 0.5$ (top) and $M = 0.75$ (bottom). Helmholtz elements: Q4 (——), Q8 (– –), Q9 (– ·) and Galbrun elements: T4-3c with mesh A (–⊖–), T4-3c with mesh B (–×–), Q9-4c (–⊟–).

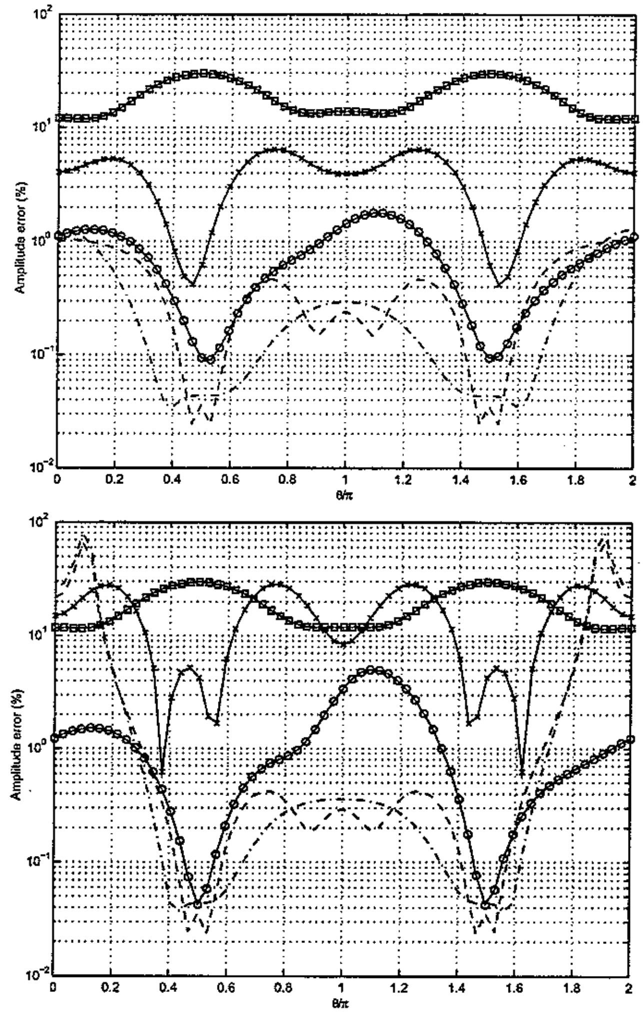

Figure 9 Amplitude error E_a (in %) as a function of θ with $M = 0.5$ (top) and $M = 0.75$ (bottom). Helmholtz elements: Q8 (– –), Q9 (– ·) and Galbrun elements: T4-3c with mesh A (–⊖–), T4-3c with mesh B (–×–), Q9-4c (–⊟–).

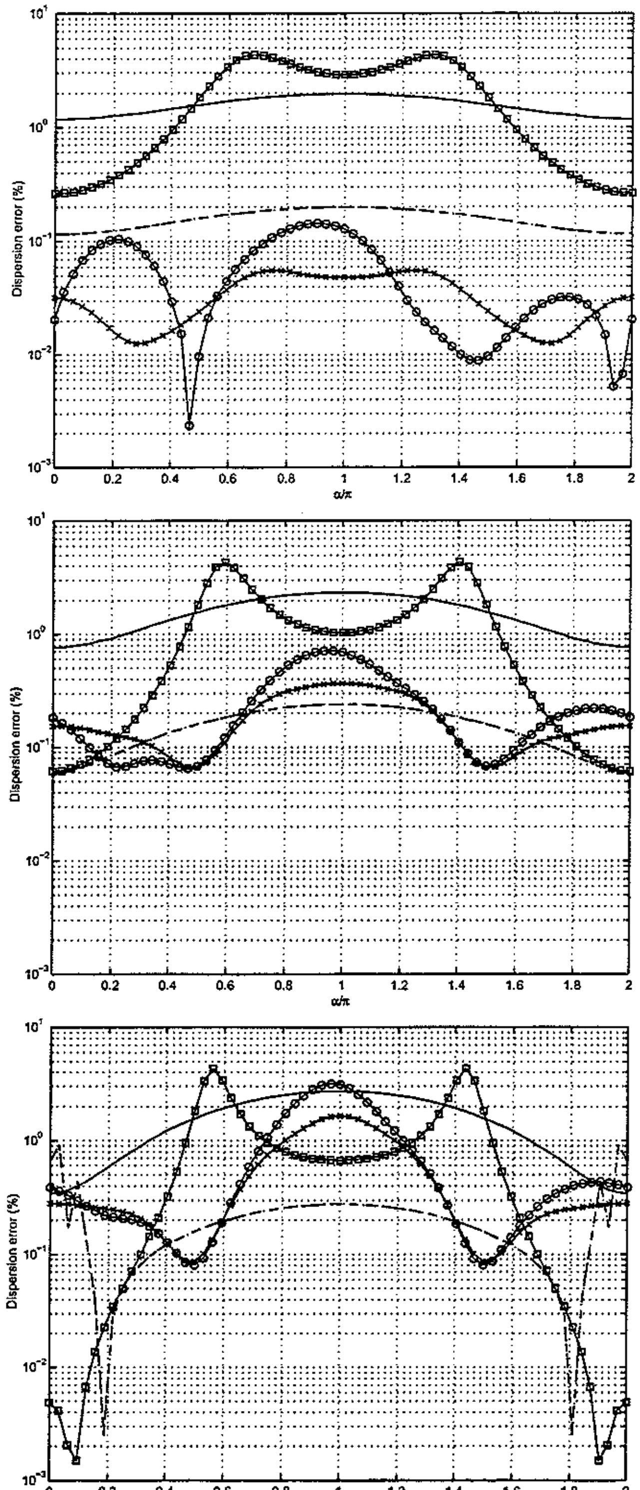

Figure 10 Dispersion error E_d (in %) as a function of α with $M = 0.25$ (top), $M = 0.5$ (center) and $M = 0.75$ (bottom). Helmholtz elements: Q4 (——), Q8 (— —), Q9 (— ·) and Galbrun elements: T4-3c with mesh A (–⊖–), T4-3c with mesh B (–×–), Q9-4c (–⊟–).

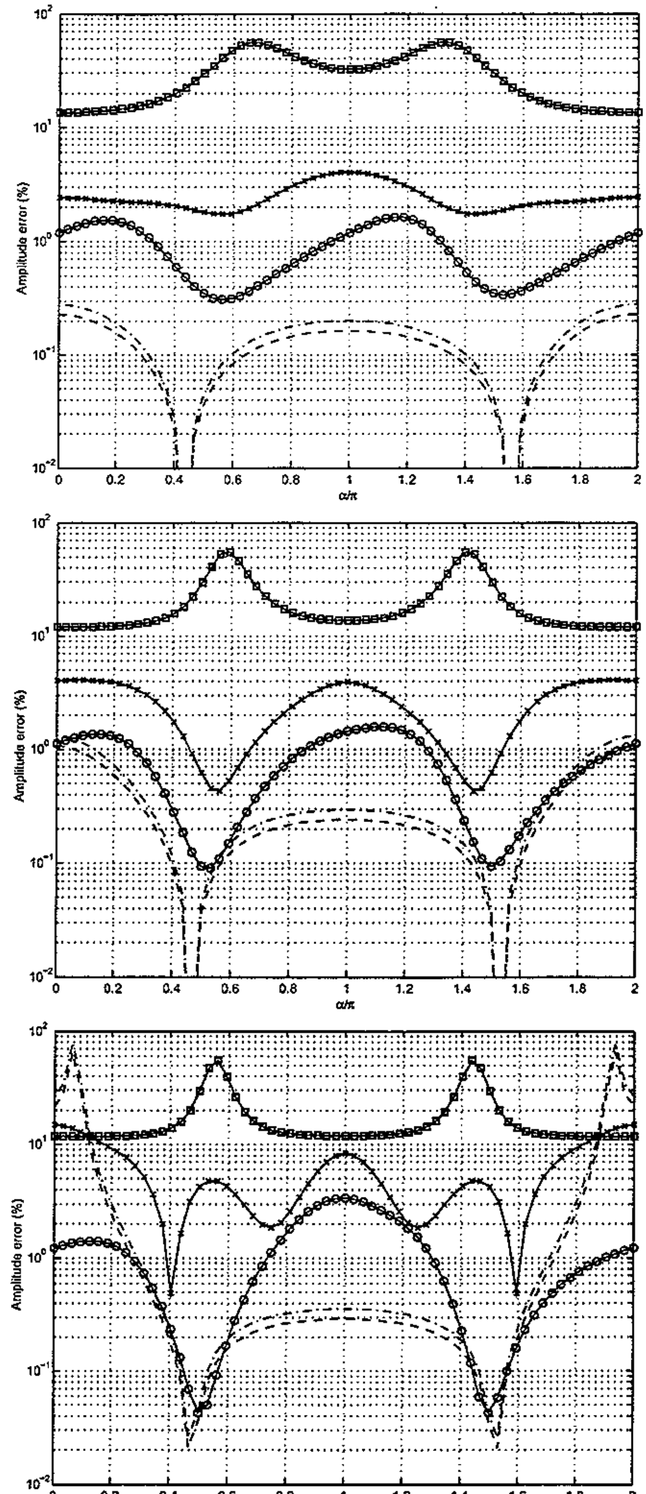

Figure 11 Amplitude error E_a (in %) as a function of α with $M = 0.25$ (top), $M = 0.5$ (center) and $M = 0.75$ (bottom). Helmholtz elements: Q8 (— —), Q9 (— ·) and Galbrun elements: T4-3c with mesh A (–⊖–), T4-3c with mesh B (–×–), Q9-4c (–⊟–).