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A computational approach to 1-dimensional representations of finite W-algebras associated to simple Lie algebras of exceptional type

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ABSTRACT

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

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A COMPUTATIONAL APPROACH TO 1-DIMENSIONAL REPRESENTATIONS OF FINITE W-ALGEBRAS ASSOCIATED TO SIMPLE LIE ALGEBRAS OF EXCEPTIONAL TYPE

by Glenn Ubly

Let $\mathfrak{g}$ be a simple complex Lie algebra and let e be a nilpotent element of $\mathfrak{g}$. It was conjectured by Premet in [P07i] that the finite W-algebra $U(\mathfrak{g}, e)$ admits a 1-dimensional representation, and further work [L10, P08] has reduced this conjecture to the case where $\mathfrak{g}$ is of exceptional type and e lies in a rigid nilpotent orbit in $\mathfrak{g}$. Using the PBW-theorem for $U(\mathfrak{g}, e)$ we give an algorithm for determining a presentation for $U(\mathfrak{g}, e)$ which allows us to determine the 1-dimensional representations for $U(\mathfrak{g}, e)$. Implementing this algorithm in GAP4 we verify the conjecture in the case that $\mathfrak{g}$ is of type G_2, F_4 or E_6. Using a result of Premet in [P08], we can use these results to deduce that reduced enveloping algebras of those types admit representations of minimal dimension, and using the explicit presentations we can determine for which characteristics this will hold. Further, we show that we can determine the 1-dimensional representations of $U(\mathfrak{g}, e)$ from a smaller set of relations than is required for a presentation. From calculating these sets of relations, we show that in the case that $\mathfrak{g}$ is of type E_7 and e lies in any rigid nilpotent orbit, or in the case that $\mathfrak{g}$ is of type E_8 and e lies in one of 14 (out of 17) rigid nilpotent orbits, that $U(\mathfrak{g}, e)$ admits a 1-dimensional representation.

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Author's declaration

I, Glenn Ubly, declare that the thesis entitled *A computational approach to* 1-*dimensional representations of finite* W*-algebras associated to simple Lie algebras of exceptional type* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: *On* 1*-dimensional representations of finite* W*-algebras associated to simple Lie algebras of exceptional type*, S. M. Goodwin, G. Röhrle and G. Ubly, arXiv:0905.3714 v2 (2009). To appear in the London Mathematical Society Journal of Computation and Mathematics.

Signed...

Date...

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1

Introduction

The study of finite W-algebras begins in [DT93], and the definition used here was first used by Premet in [P02]. For a complex semisimple Lie algebra $\mathfrak{g}$ with nilpotent element e, we associate the finite W-algebra $U(\mathfrak{g}, e)$, an infinite dimensional associative algebra. There has been a great deal of recent interest in the representation theory of finite W-algebras. It was proved in [P07ii] that $U(\mathfrak{g}, e)$ always admits finite-dimensional representations. In an earlier paper it was conjectured that $U(\mathfrak{g}, e)$ always admits a representation of dimension 1 [P07i, Conjecture 3.1 (1)]. This was verified for $\mathfrak{g}$ of classical type by Losev in [L10]. Also, in [P08], Premet reduces the conjecture to the case that e lies in a nilpotent orbit of $\mathfrak{g}$ which does not arise by Lusztig–Spaltenstein induction, i.e. a *rigid* nilpotent orbit of $\mathfrak{g}$. This thesis is a contribution to the completion of a proof that $U(\mathfrak{g}, e)$ admits a 1-dimensional representation.

The representation theory of finite W-algebras is closely connected to the representation theory of *reduced enveloping algebras*, which are certain quotients of the universal enveloping algebras of modular Lie algebras. Let $\Bbbk$ be the algebraically closed field of characteristic $p \gg 0$, let $\mathfrak{g}_{\mathbb{Z}}$ be the Chevalley $\mathbb{Z}$-form of $\mathfrak{g}$ and let $\mathfrak{g}_{\Bbbk} = \mathfrak{g}_{\mathbb{Z}} \otimes \Bbbk$. The Kac–Weisfeiler conjecture, proved by Premet in [P95], states that for $\xi \in \mathfrak{g}_{\Bbbk}^*$, any module of the reduced enveloping algebra $U_\xi(\mathfrak{g}_{\Bbbk})$ has dimension divisible by p^{d_ξ} where d_ξ is half the dimension of the coadjoint orbit of ξ. In [P08, Theorem 1.4], Premet proves that if $U(\mathfrak{g}, e)$

admits a 1-dimensional representation then there exists a $U_\xi(\mathfrak{g}_\Bbbk)$-module with dimension p^{d_ξ}. This makes use of a modular analogue $U(\mathfrak{g}_\Bbbk, e_\Bbbk)$ of $U(\mathfrak{g}, e)$.

Chapter 2 contains the necessary background on Lie algebras and nilpotent orbits. In Chapter 3 we give the definition and construction of $U(\mathfrak{g}, e)$ and we state a PBW theorem which shows how we can obtain a presentation for $U(\mathfrak{g}, e)$. In Chapter 4 we give an algorithm for constructing a presentation for $U(\mathfrak{g}, e)$. This takes a nilpotent element $e \in \mathfrak{g}$ and gives an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$, and from this we can determine a set of generators and relations for $U(\mathfrak{g}, e)$ in terms of a carefully chosen ordered basis for $\mathfrak{g}$. In Chapter 5 we show how the presentation obtained by the methods of Chapter 4 can be used to determine all 1-dimensional representations for $U(\mathfrak{g}, e)$. We also show that in many cases, we can determine the 1-dimensional representations from a much smaller set of relations than is necessary for a presentation. We also consider the implications for the representation theory of reduced enveloping algebras. In Chapter 6 we give the results of the application of the algorithms of Chapter 4 to $U(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a simple Lie algebra of type G_2, F_4, E_6 or E_7, and e lies in a rigid nilpotent orbit in $\mathfrak{g}$. We see that in these cases $U(\mathfrak{g}, e)$ admits either one or two 1-dimensional representations. For $\mathfrak{g}$ of type G_2, F_4 or E_6 and e in a rigid nilpotent orbit in $\mathfrak{g}$ we have explicit presentations for $U(\mathfrak{g}, e)$, so we can see for which p we are able to define $U(\mathfrak{g}_\Bbbk, e_\Bbbk)$ and hence determine when $U_\xi(\mathfrak{g}_\Bbbk)$ (where ξ corresponds to $e_\Bbbk$ under an identification of $\mathfrak{g}_\Bbbk$ with $\mathfrak{g}_\Bbbk^*$) has a module of dimension p^{d_ξ}. We also give this for 5 of the 7 non-zero rigid nilpotent orbits in $\mathfrak{g}$ of type E_7. For the remaining 2 non-zero rigid nilpotent orbits for type E_7 and for 14 of the 17 non-zero rigid nilpotent orbits for type E_8 we are able to determine the 1-dimensional representations but without further calculation we are not able to determine the condition on p which allows us to draw conclusions about the associated reduced enveloping algebras. In Appendices A to F we give an implementation of the algorithms of Chapter 4 in GAP4.

The code given in the appendices is available online at `http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Lehrstuhl-VI/ubly-thesis.html`.

2

Preliminaries

In this chapter we give the necessary definitions and results on Lie algebras. The material here is standard and can be found in [H72], [S87], [CM93], [J62] and [C93]. Let k denote a field.

2.1 Lie algebras – elementary definitions and results

Definition 2.1.1. A *Lie algebra* over k is a vector space $\mathfrak{g}$ over k with a bilinear operation $[\cdot\,,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying for all $x, y, z \in \mathfrak{g}$:

1. $[x, x] = 0$,

2. $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$.

For example, given any associative algebra A over k, we can take the bracket to denote the commutator, i.e. $[x, y] = xy - yx$ for $x, y \in A$. This can easily be seen to satisfy the above axioms, thus defining a Lie algebra over k.

For the rest of this section, $\mathfrak{g}$ denotes a finite-dimensional Lie algebra over k. For $x \in \mathfrak{g}$ write $[x, \mathfrak{g}]$ for the set $\{[x, y] \mid y \in \mathfrak{g}\}$, and for a subset $A \subseteq \mathfrak{g}$ write $[A, \mathfrak{g}]$ for $\{[x, y] \mid x \in A \text{ and } y \in \mathfrak{g}\}$.

Definition 2.1.2. We call a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ a *subalgebra* of $\mathfrak{g}$ if for all $x, y \in \mathfrak{h}$ we have $[x, y] \in \mathfrak{h}$.

Definition 2.1.3. Call a subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ an *ideal* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{a}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$.

Definition 2.1.4. Call $\mathfrak{g}$ *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Definition 2.1.5. The *centre* of $\mathfrak{g}$ is the set $\{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$.

It is clear that the centre of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$.

Definition 2.1.6. For a subset (or element) A of $\mathfrak{g}$, the *centralizer* of A in $\mathfrak{g}$ is the set $\mathfrak{g}^A := \{x \in \mathfrak{g} \mid [x, A] = 0\}$.

Definition 2.1.7. The *normalizer* of a subspace A of $\mathfrak{g}$ is the set $N(A) = \{x \in \mathfrak{g} \mid [x, A] \subseteq A\}$.

It is clear that the normalizer of a subspace of $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$.

Definition 2.1.8. Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie algebras. A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$ is a *homomorphism* of Lie algebras. If the map ϕ is also a bijection, then ϕ is an *isomorphism* of Lie algebras. A homomorphism $\mathfrak{g} \to \mathfrak{g}$ is an *endomorphism* of $\mathfrak{g}$.

We denote by $\mathfrak{gl}_n(k)$ the *general linear* Lie algebra; that is $n \times n$ matrices with entries in k with the commutator $[x, y] = xy - yx$. More generally, write $\mathfrak{gl}(V)$ for the Lie algebra of endomorphisms of V, a vector space over k.

Definition 2.1.9. We call a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}_n(k)$ or $\mathfrak{g} \to \mathfrak{gl}(V)$ a *representation* of $\mathfrak{g}$.

Equivalently, given a representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, the action of $\mathfrak{g}$ on V by $x.v = \phi(x)(v)$ allows us to view V as a $\mathfrak{g}$-module. Note that the ideals of $\mathfrak{g}$ are precisely the kernels of homomorphisms from $\mathfrak{g}$.

Definition 2.1.10. We call $\mathfrak{g}$ *simple* if $\mathfrak{g}$ is non-abelian and contains no ideals other than $\mathfrak{g}$ and 0.

For any element x in $\mathfrak{g}$, we define the map $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ by $\operatorname{ad} x(y) = [x, y]$ for all $y \in \mathfrak{g}$. The map $\operatorname{ad} x$ is an endomorphism of $\mathfrak{g}$. Call this the *adjoint action* of x on $\mathfrak{g}$. This gives a map $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ which sends $x \in \mathfrak{g}$ to $\operatorname{ad} x \in \mathfrak{gl}(\mathfrak{g})$; call this the *adjoint representation* of $\mathfrak{g}$.

Definition 2.1.11. We call a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ a *derivation* of $\mathfrak{g}$ if it satisfies the following for all $a, b \in \mathfrak{g}$:

$$\delta([a, b]) = [\delta(a), b] + [a, \delta(b)].$$

We can easily see that for $x \in \mathfrak{g}$, $\operatorname{ad} x$ is a derivation on $\mathfrak{g}$. The space of derivations of $\mathfrak{g}$ forms a Lie algebra with the commutator operation.

Definition 2.1.12. We call an element $x \in \mathfrak{g}$ *nilpotent* if $\operatorname{ad} x$ is a nilpotent endomorphism of $\mathfrak{g}$. That is, if there is $n > 0$ such that $(\operatorname{ad} x)^n(\mathfrak{g}) = [x, [x, \cdots [x, \mathfrak{g}]]] = 0$ (where there are n sets of brackets).

Definition 2.1.13. We define the *derived series* of $\mathfrak{g}$ by putting $\mathfrak{g}^{(0)} = \mathfrak{g}$ and for $i \geq 1, \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$. If there is some $n \geq 0$ with $\mathfrak{g}^{(n)} = 0$ then we say $\mathfrak{g}$ is *solvable*.

Definition 2.1.14. For a Lie algebra $\mathfrak{g}$, we call the unique maximal solvable ideal the *radical* of $\mathfrak{g}$.

Definition 2.1.15. We call a Lie algebra $\mathfrak{g}$ *semisimple* if its radical is 0.

Definition 2.1.16. We call a Lie algebra $\mathfrak{g}$ *reductive* if the radical of $\mathfrak{g}$ is equal to the centre of $\mathfrak{g}$.

We immediately have that a simple Lie algebra is semisimple and that a semisimple Lie algebra is reductive.

Definition 2.1.17. For a Lie algebra $\mathfrak{g}$, we define the *lower central series* by putting $\mathfrak{g}^0 = \mathfrak{g}$ and for $i \geq 1$, $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$. If there is some $n \geq 0$ with $\mathfrak{g}^n = 0$ then we say $\mathfrak{g}$ is *nilpotent*.

Theorem 2.1.18 (Engel's theorem)**.** *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if each element of $\mathfrak{g}$ is nilpotent.*

For the remainder of this section, we assume that k has characteristic 0 and is algebraically closed.

Theorem 2.1.19 (Ado's theorem)**.** *Each finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(V)$, for some finite-dimensional vector space V.*

Definition 2.1.20. We call an element $x \in \mathfrak{g}$ *semisimple* if the endomorphism $\operatorname{ad} x$ is diagonalizable.

We can write $x \in \mathfrak{g}$ uniquely as a sum $x_s + x_n$ where x_s is semisimple, x_n is nilpotent and x_s and x_n commute. This is the *Jordan–Chevalley decomposition.*

Let $\operatorname{Tr}(A)$ denote the trace of an endomorphism A of $\mathfrak{g}$.

Definition 2.1.21. We define the *Killing form* on $\mathfrak{g}$ to be the map $\kappa : \mathfrak{g}\times\mathfrak{g} \to k$ given by

$$\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$$

for $x, y \in \mathfrak{g}$.

We note that as $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$, the Killing form is symmetric and we deduce that the Killing form is associative; that is for $x, y, z \in \mathfrak{g}$, we have $\kappa([x, y], z) = \kappa(x, [y, z])$. The Killing form gives the following criterion for $\mathfrak{g}$ to be semisimple.

Theorem 2.1.22. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is non-degenerate.*

Theorem 2.1.23. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is a direct sum of a finite set of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ of $\mathfrak{g}$ such that each $\mathfrak{a}_i$ is a simple Lie algebra.*

To a Lie algebra over k (where k can be any field) we can associate an infinite-dimensional associative algebra over k, containing k, called the *universal enveloping algebra* of $\mathfrak{g}$, denoted $U(\mathfrak{g})$. There is a natural equivalence between the category of $\mathfrak{g}$-modules and the category of $U(\mathfrak{g})$-modules. We give the definition of $U(\mathfrak{g})$ in terms of a universal property, and then a more explicit construction which is more useful for the remainder of this thesis.

Definition 2.1.24. For a Lie algebra $\mathfrak{g}$, the *universal enveloping algebra* $U = U(\mathfrak{g})$ of $\mathfrak{g}$ is an associative algebra with 1 with a map $i : \mathfrak{g} \to U$ satisfying

$$i[x, y] = i(x)i(y) - i(y)i(x) \tag{2.1.25}$$

such that given any U' and $i' : \mathfrak{g} \to U'$ satisfying (2.1.25) there is a unique homomorphism $\phi : U \to U'$ with $\phi \circ i = i'$.

Given a Lie algebra $\mathfrak{g}$ over k, we can construct the universal enveloping algebra of $\mathfrak{g}$ as follows. Let $T(\mathfrak{g})$ denote the *tensor algebra* of $\mathfrak{g}$, that is we put $T^n(\mathfrak{g}) = \mathfrak{g}^{\otimes n} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (with n copies of $\mathfrak{g}$) with $T^0(\mathfrak{g}) = k$. Then the tensor algebra is $T(\mathfrak{g}) = \oplus_{i \geq 0} T^i(\mathfrak{g})$ with the natural associative multiplication of elements of $T(\mathfrak{g})$: for $u = u_1 \otimes \cdots \otimes u_m \in T^m(\mathfrak{g})$ and $v = v_1 \otimes \cdots \otimes v_n \in T^n(\mathfrak{g})$ we have $u \otimes v = u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n \in T^{n+m}(\mathfrak{g})$. Let I be the 2-sided ideal of $T(\mathfrak{g})$ generated by the elements $x \otimes y - y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$. Then we can take $U(\mathfrak{g})$ to be the quotient $T(\mathfrak{g})/I$. We write an element $x_1 \otimes \cdots \otimes x_m + I$ of this quotient as $x_1 \cdots x_m$. Given an ordered basis of $\mathfrak{g}$ (which we do not require to be finite-dimensional here) we have a basis for $U(\mathfrak{g})$ as follows.

Theorem 2.1.26 (Poincaré–Birkhoff–Witt)**.** *Let $x_1, x_2, \ldots$ be an ordered basis of $\mathfrak{g}$. Then the elements $x_{i_1} \ldots x_{i_m}$ where $i_1 \leq \cdots \leq i_m$, together with $1 \in k$, form a basis of $U(\mathfrak{g})$.*

We call such a basis a *PBW-basis* of $U(\mathfrak{g})$. Note that we can identify $\mathfrak{g}$ with $T^1(\mathfrak{g})$, and we do not distinguish in our notation between elements of $\mathfrak{g}$ and the corresponding elements in $U(\mathfrak{g})$.

2.2 Root systems and the classification of semisimple Lie algebras

We can classify semisimple Lie algebras in terms of their associated root systems. In this section we define root systems in an abstract sense and then show how root systems are used to classify semisimple Lie algebras.

Definition 2.2.1. For a Euclidean space E with standard inner product $(\cdot\,,\cdot)$, a *root system* is a subset Φ of E satisfying the following:

- Φ is a finite set, spanning E and $0 \notin \Phi$;
- for each $\alpha \in \Phi$ the only multiples of α in Φ are $\pm\alpha$;
- for each $\alpha \in \Phi$ there is a reflection s_α such that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\Phi) = \Phi$;
- if $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle := 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Definition 2.2.2. For a root system Φ in a Euclidean space E there is a finite subgroup W of isometries of E generated by the reflections s_α for $\alpha \in \Phi$; call this the *Weyl group* of Φ.

Definition 2.2.3. Let Φ, Φ' be root systems in Euclidean spaces E, E' respectively. An isomorphism of vector spaces $\phi : E \to E'$ sending Φ to Φ' is an *isomorphism of root systems* if for any roots $\alpha, \beta \in \Phi$ we have $\langle \alpha, \beta \rangle = \langle \phi(\alpha), \phi(\beta) \rangle$.

Definition 2.2.4. The *rank* of a root system $\Phi \subset E$ is the dimension of the Euclidean space E.

Given a root system Φ, we can choose a subset $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ (where l is the rank of Φ) such that Δ spans E and each root $\beta \in \Phi$ can be expressed as a finite sum $\beta = \sum_{i=1}^{l} c_i \alpha_i$ where either all coefficients c_i are non-negative integers (call such roots *positive*) or all coefficients c_i are non-positive integers (call such roots *negative*). We call the roots $\alpha_1, \ldots, \alpha_l$

simple roots. Denote by Φ^+ the set of positive roots, and by $\Phi^- = -\Phi^+$ the set of negative roots. We have $\Phi = \Phi^+ \cup \Phi^-$.

Definition 2.2.5. A root system Φ is called *irreducible* if it cannot be expressed as a union of 2 non-empty subsets: $\Phi = \Phi_1 \cup \Phi_2$, where each root in Φ_1 is orthogonal to each root in Φ_2.

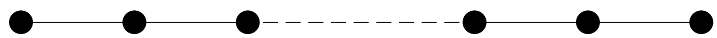
For a root system Φ of rank l with a set of simple roots $\{\alpha_1, \ldots, \alpha_l\}$, we define the *Cartan matrix* associated to Φ to be the $l \times l$ matrix with each integer $\langle \alpha_i, \alpha_j \rangle$ in position (i, j). Note that any Cartan matrix is non-singular, and is independent of the choice of simple roots, other than the order. The Cartan matrix determines the root system up to isomorphism.

For a root system Φ we define the associated *Dynkin diagram* as follows. Choose a set of simple roots $\alpha_1, \ldots, \alpha_l$ in Φ. To each α_i we have a vertex (which we also label α_i) and we join vertices α_i and α_j with $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges for $i \neq j$. In the case that the lengths of the roots α_i and α_j are not equal we add an arrow to the edges joining vertices α_i and α_j in the direction towards the vertex corresponding to the shorter root.

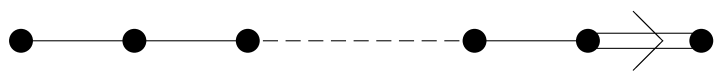
The Dynkin diagram associated to a root system Φ is connected if and only if Φ is irreducible. We can uniquely decompose a root system Φ into a union of pairwise orthogonal irreducible root systems. The classification of irreducible root systems is therefore equivalent to the classification of connected Dynkin diagrams. From geometrical consideration of the root systems in the Euclidean space E, we can deduce the following.

Theorem 2.2.6. *If Φ is an irreducible root system then its associated Dynkin diagram is one of the following:*

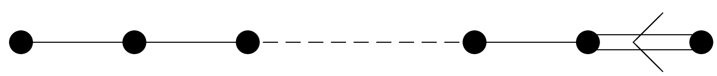
Type A_n $(n \geq 1)$:



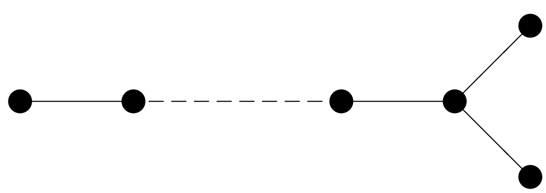
Type B_n $(n \geq 2)$:



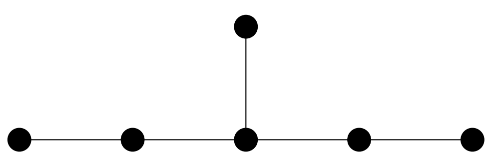
Type C_n $(n \geq 3)$:



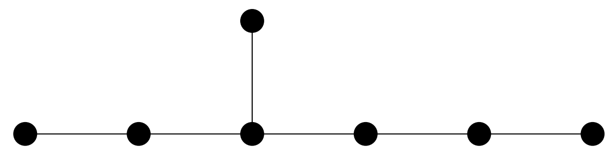
Type D_n $(n \geq 4)$:



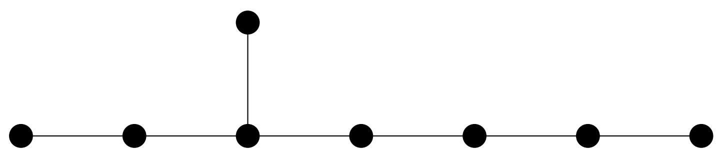
Type E_6:



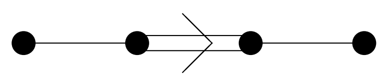
Type E_7:



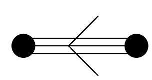
Type E_8:



Type F_4:



Type G_2:



We now describe how to associate a root system to a semisimple Lie algebra $\mathfrak{g}$.

Definition 2.2.7. A *Cartan subalgebra* (CSA) of $\mathfrak{g}$ is a nilpotent subalgebra equal to its normalizer in $\mathfrak{g}$.

We have that Cartan subalgebras always exist for semisimple $\mathfrak{g}$. Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then the dimension of $\mathfrak{h}$ is the rank of $\mathfrak{g}$. We can decompose $\mathfrak{g}$ as a direct sum of $\mathfrak{h}$ and the 1-dimensional $\mathfrak{h}$-modules $\{\mathfrak{g}_\alpha \mid \alpha \in \Phi\}$, where for $x \in \mathfrak{g}_\alpha$ for some α, we have that the adjoint action of $h \in \mathfrak{h}$ on x is scalar multiplication. We have

$$[h, x] = \alpha(h)x$$

for $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$. Thus α lies in $\mathfrak{h}^*$, the dual space of the Cartan subalgebra. The set Φ of such α forms a root system in the Euclidean space $\mathfrak{h}^*$ of dimension rank $\mathfrak{g}$. This root system is independent (up to isomorphism) of the choice of Cartan subalgebra, so we may refer to the root system of a semisimple Lie algebra without specifying a Cartan subalgebra. We have the following classification of semisimple Lie algebras over k in terms of the associated root systems and Dynkin diagrams.

Theorem 2.2.8.

- *A semisimple Lie algebra is simple if and only if its root system is irreducible.*
- *Two semisimple Lie algebras are isomorphic if and only if they have the same Dynkin diagrams.*
- *Every root system is the root system of some semisimple Lie algebra.*

2.3 Nilpotent orbits in semisimple $\mathfrak{g}$

The material in this section is contained mostly in [CM93]. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and let G be the *adjoint group* of $\mathfrak{g}$; that is the connected component of the group of automorphisms of $\mathfrak{g}$. We have that $\mathfrak{g}$ is the Lie algebra of G. In this section we consider the G-orbits (referred to simply as *orbits*) in $\mathfrak{g}$. References to *conjugacy* of subalgebras of $\mathfrak{g}$ refer to the action of G. The action of G preserves the semisimplicity or nilpotency of $x \in \mathfrak{g}$. We consider the semisimple and nilpotent orbits of G in turn.

Theorem 2.3.1. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with associated Weyl group W. Then the set of semisimple orbits in $\mathfrak{g}$ is parameterized by $\mathfrak{h}/W$.*

In particular, there are infinitely many semisimple orbits in $\mathfrak{g}$. We now show that there are finitely many nilpotent orbits in $\mathfrak{g}$. The following theorem tells us that any non-zero nilpotent element in $\mathfrak{g}$ lies in a subalgebra isomorphic to $\mathfrak{sl}_2$.

Theorem 2.3.2 (Jacobson–Morozov). *[CM93, Theorem 3.3.1] Let $e \in \mathfrak{g}$ be nilpotent. Then there are elements $h, f \in \mathfrak{g}$ such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$.*

If $e, h, f \in \mathfrak{g}$ span a subalgebra isomorphic to $\mathfrak{sl}_2$ then we say (e, h, f) is an $\mathfrak{sl}_2$-*triple*. The adjoint action of G on $\mathfrak{g}$ can be naturally extended to an action on $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. There is a bijection between the non-zero nilpotent orbits in $\mathfrak{g}$ and the G-orbits of $\mathfrak{sl}_2$-triples. Given an $\mathfrak{sl}_2$-triple (e, h, f), we can decompose $\mathfrak{g}$ into a direct sum of $\operatorname{ad} h$ eigenspaces:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j),$$

where

$$\mathfrak{g}(j) = \{x \in \mathfrak{g} \mid [h, x] = jx\}.$$

We call this the *Dynkin grading* of $\mathfrak{g}$ associated to the $\mathfrak{sl}_2$-triple (e, h, f).

Definition 2.3.3. The *height* of a nilpotent orbit in $\mathfrak{g}$ corresponding to an $\mathfrak{sl}_2$-triple (e, h, f) is the maximum N such that in the associated Dynkin grading $\mathfrak{g}(N) \neq 0$.

We can choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ which contains h. This determines a root system Φ for $\mathfrak{g}$. Choose a set $\Delta \subset \Phi$ of simple roots, and label the vertex of the Dynkin diagram corresponding to each root $\alpha \in \Delta$ with the value $\alpha(h)$. Each $\alpha(h)$ lies in $\{0, 1, 2\}$. This gives us the *weighted Dynkin diagram* for the $\mathfrak{sl}_2$-triple (e, h, f), or equivalently (by the bijection above) for the nilpotent element e. For any $\mathfrak{g}$, the element 0 forms a nilpotent orbit. The weighted Dynkin diagram for the zero orbit is obtained by labelling each vertex with 0.

We have the following theorem.

Theorem 2.3.4. *The weighted Dynkin diagram of a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is a complete invariant. That is, two orbits $\mathcal{O}, \mathcal{O}'$ are equal if and only if their associated weighted Dynkin diagrams are equal.*

Due to the range of values the labels can take, we immediately get an upper bound of $3^{\operatorname{rank}\mathfrak{g}}$ on the number of weighted Dynkin diagrams, and hence on the number of nilpotent orbits in $\mathfrak{g}$. This bound is not achieved by any semisimple $\mathfrak{g}$.

We next show how the nilpotent orbits are classified in simple $\mathfrak{g}$. There is a partial order on the nilpotent orbits of $\mathfrak{g}$ given by the rule $\mathcal{O} \geq \mathcal{O}'$ if $\overline{\mathcal{O}} \supseteq \overline{\mathcal{O}'}$, where $\overline{\mathcal{O}}$ denotes the Zariski closure of $\mathcal{O}$. There are 4 canonical nilpotent orbits in a simple Lie algebra $\mathfrak{g}$ (which may coincide for small $\mathfrak{g}$) which are determined by their position in the partial order. The zero orbit is the least element in this partial order. There is a unique nilpotent orbit $\mathcal{O} \neq 0$ with $\mathcal{O} \leq \mathcal{O}'$ for all orbits $\mathcal{O}' \neq 0$ in $\mathfrak{g}$. This is called the *minimal orbit* in $\mathfrak{g}$, and is denoted $\mathcal{O}_{\text{min}}$. There is a unique maximal nilpotent orbit, called the *regular orbit*, denoted $\mathcal{O}_{\text{reg}}$, and this has dimension $\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g})$. There is a unique nilpotent orbit $\mathcal{O} \neq \mathcal{O}_{\text{reg}}$ with $\mathcal{O} \geq \mathcal{O}'$ for all nilpotent orbits $\mathcal{O}' \neq \mathcal{O}_{\text{reg}}$. This is called the *subregular orbit* in $\mathfrak{g}$, and is denoted $\mathcal{O}_{\text{subreg}}$. The subregular orbit has dimension $\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g}) - 2$.

We begin with the classical types. In each of the 4 families of simple Lie algebras, the nilpotent orbits are parameterized by partitions of a positive integer which depends on the rank of $\mathfrak{g}$. We denote a partition of N by a tuple $[d_1, \ldots, d_m]$ of non-negative integers in non-increasing order where $\sum_{i=1}^m d_i = N$. We may assume any d_q for $q > m$ is zero. We call each d_i in a partition a *part*, and if a part d_i is repeated a_i times we may write $d_i^{a_i}$ in the partition. The set of partitions of N is denoted by $\mathcal{P}(N)$. The *dominance ordering* on partitions of N is a partial order defined by the following: we have $[a_1, \ldots, a_m] \geq [b_1, \ldots, b_m]$ if $\sum_{i=1}^q a_i \geq \sum_{i=1}^q b_i$ for all q. In each of the following cases, where we denote the orbits corresponding to the partitions $\mathbf{a} = [a_1, \ldots, a_m]$, $\mathbf{b} = [b_1, \ldots, b_m]$ by $\mathcal{O}_{\mathbf{a}}$ and $\mathcal{O}_{\mathbf{b}}$ respectively, we have $\mathcal{O}_{\mathbf{a}} \geq \mathcal{O}_{\mathbf{b}}$ if and only if $\mathbf{a} \geq \mathbf{b}$.

Type A_n

For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, the nilpotent orbits of $\mathfrak{g}$ are in bijective correspondence with $\mathcal{P}(n+1)$.

Type B_n

For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the nilpotent orbits of $\mathfrak{g}$ are in bijective correspondence with the partitions in $\mathcal{P}(2n+1)$ in which even parts occur with even multiplicity.

Type C_n

For $\mathfrak{g} = \mathfrak{sp}_{2n}$, the nilpotent orbits of $\mathfrak{g}$ are in bijective correspondence with the partitions in $\mathcal{P}(2n)$ in which odd parts occur with even multiplicity.

Type D_n

For $\mathfrak{g} = \mathfrak{so}_{2n}$, the nilpotent orbits of $\mathfrak{g}$ are in bijective correspondence with the set of partitions in $\mathcal{P}(2n)$ in which even parts occur with even multiplicity except that each partition with only even parts corresponds to two distinct nilpotent orbits.

Exceptional type

For a simple Lie algebra $\mathfrak{g}$ of type G_2, F_4, E_6, E_7 or E_8 we do not have a neat parameterization of the nilpotent orbits as in the case where $\mathfrak{g}$ is of classical type. We make use of Bala–Carter theory to classify the nilpotent orbits in $\mathfrak{g}$ of exceptional type. We require some further terminology.

Definition 2.3.5. A *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$.

For example, if we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, with associated root system Φ and a choice of simple roots $\Delta \subset \Phi$ determining a set of positive roots Φ^+, then

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

is a Borel subalgebra of $\mathfrak{g}$. Any two Borel subalgebras of $\mathfrak{g}$ are conjugate, so in particular any Borel subalgebra is conjugate to $\mathfrak{b}$.

For a subset X of Φ we denote by $\langle X \rangle$ the root system generated by X.

Definition 2.3.6. A *parabolic subalgebra* of $\mathfrak{g}$ is a subalgebra which contains a Borel subalgebra.

A subset $\Psi \subseteq \Delta$ gives an example of a parabolic subalgebra containing the above Borel:

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Phi^+ \\ -\alpha \in \langle \Psi \rangle}} \mathfrak{g}_\alpha.$$

Such subsets of Δ parameterize the parabolic subalgebras of $\mathfrak{g}$ which contain $\mathfrak{b}$. Thus for a simple Lie algebra $\mathfrak{g}$ there are $2^{\operatorname{rank} \mathfrak{g}}$ conjugacy classes of parabolic subalgebras of $\mathfrak{g}$.

Definition 2.3.7. The *Levi decomposition* of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the *nilradical* (i.e. the unique maximal nilpotent ideal) of $\mathfrak{p}$ and $\mathfrak{l}$ is the corresponding *Levi subalgebra* of $\mathfrak{g}$.

Note that $[\mathfrak{l}, \mathfrak{l}]$ is a semisimple Lie algebra. For the parabolic subalgebra $\mathfrak{p}$ above, the Levi decomposition is $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ where

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Psi \rangle} \mathfrak{g}_\alpha$$

and

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+ \setminus (\langle \Psi \rangle \cap \Phi^+)} \mathfrak{g}_\alpha.$$

Definition 2.3.8. A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ is called *distinguished* if $\dim \mathfrak{l} = \dim(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])$.

We can now state the Bala–Carter Theorem [C93, Theorem 5.9.5], which gives us a classification of the nilpotent orbits in a simple Lie algebra $\mathfrak{g}$.

Theorem 2.3.9. *There is a natural bijection between the nilpotent orbits in* $\mathfrak{g}$ *and conjugacy classes of pairs* $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ *where* $\mathfrak{l}$ *is a Levi subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{p}_{\mathfrak{l}}$ *is a distinguished parabolic subalgebra of* $[\mathfrak{l}, \mathfrak{l}]$*.*

Given a nilpotent element $e \in \mathfrak{g}$, by Theorem 2.3.2 we have an $\mathfrak{sl}_2$-triple (e, h, f) and the resulting Dynkin grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$. We take $\mathfrak{l}$ to be the minimal Levi subalgebra of $\mathfrak{g}$ which contains e, and then the parabolic subalgebra

$$\mathfrak{p}_{\mathfrak{l}} = \bigoplus_{j\geq 0} \mathfrak{g}(j) \cap [\mathfrak{l}, \mathfrak{l}],$$

is distinguished in $[\mathfrak{l}, \mathfrak{l}]$. Then the conjugacy class of $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ corresponds to the nilpotent orbit containing e.

We label the nilpotent orbit corresponding to a pair $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ as $X_N(a_i)$, where X_N is the type of the Dynkin diagram associated to the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$, and i is the number of simple roots of $\mathfrak{g}$ which are roots of the Levi factor of the parabolic subalgebra $\mathfrak{p}_{\mathfrak{l}}$. We omit the part (a_i) in the case that $i = 0$. We must also distinguish the cases where we have isomorphic but non-conjugate Levi subalgebras $\mathfrak{l}$. In the case that $\mathfrak{g}$ has different root lengths, then we distinguish the case with the shorter root length by $\tilde{X}_N$. If the Levi subalgebras $\mathfrak{l}$ of $\mathfrak{g}$ are isomorphic but non-conjugate and cannot be distinguished by root lengths then if $i = 0$ we distinguish the labels with either one or two primes (for example in type E_7), and if $i \neq 0$ then we write b_i instead of a_i for one label (for example in type E_8). This labelling is sufficient to distinguish all nilpotent orbits in $\mathfrak{g}$ where $\mathfrak{g}$ is a simple Lie algebra of exceptional type. These are listed in [CM93, Chapter 8.4]. The numbers of nilpotent orbits in $\mathfrak{g}$ of type G_2, F_4, E_6, E_7 and E_8 are 5, 16, 21, 45 and 70 respectively.

We now give a brief description of *Lusztig–Spaltenstein induction* on nilpotent orbits [LS79]. Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{p}$ ba a parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$. The Levi factor $\mathfrak{l}$ is reductive, so $[\mathfrak{l}, \mathfrak{l}]$ is semisimple and the nilpotent elements (and orbits) of $\mathfrak{l}$ and $[\mathfrak{l}, \mathfrak{l}]$ coincide. Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in $\mathfrak{l}$ under the action of the connected Levi subgroup L of G where $\mathrm{Lie}(L) = \mathfrak{l}$. Then there is a unique nilpotent orbit $\mathcal{O}_\mathfrak{g}$ in $\mathfrak{g}$ whose intersection with $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ is open and dense in $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$. We say $\mathcal{O}_\mathfrak{g}$ in *induced* from $\mathcal{O}_\mathfrak{l}$, and write $\mathcal{O}_\mathfrak{g} = \mathrm{Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$.

Definition 2.3.10. A nilpotent orbit $\mathcal{O}_\mathfrak{g}$ is called *rigid* if there is no proper parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that we can obtain $\mathcal{O}_\mathfrak{g}$ by induction from a nilpotent orbit $\mathcal{O}_\mathfrak{l}$ in $\mathfrak{l}$ where $\mathfrak{l}$ is a Levi factor of $\mathfrak{p}$.

Rigid orbits turn out to be of particular significance for the representation theory of finite W-algebras so for $\mathfrak{g}$ of classical type we give a criterion in terms of the associated partition for a nilpotent orbit to be rigid, and for $\mathfrak{g}$ of exceptional type we give explicit lists of the rigid nilpotent orbits, along with their dimensions.

Type A_n

The only rigid nilpotent orbit in $\mathfrak{sl}_{n+1}$ is the zero orbit.

Type B_n

The rigid nilpotent orbits in $\mathfrak{so}_{2n+1}$ are those whose partition $[d_1, \dots, d_m] \in \mathcal{P}(2n+1)$ satisfies $d_{i+1} \leq d_i \leq d_{i+1} + 1$ for $i = 1, \dots, m-1$, and no odd part has multiplicity 2.

Type C_n

The rigid nilpotent orbits in $\mathfrak{sp}_{2n}$ are those whose partition $[d_1, \dots, d_m] \in \mathcal{P}(2n)$ satisfies $d_{i+1} \leq d_i \leq d_{i+1} + 1$ for $i = 1, \dots, m-1$, and no even part has multiplicity 2.

Type D_n

The rigid nilpotent orbits in $\mathfrak{so}_{2n}$ are those whose partition $[d_1, \dots, d_m] \in \mathcal{P}(2n)$ satisfies $d_{i+1} \leq d_i \leq d_{i+1} + 1$ for $i = 1, \dots, m-1$, and no odd part has multiplicity 2.

Table 2.1: Rigid orbits for type G_2.

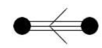
Bala–Carter label	Dynkin diagram 	Dimension
A_1	0 1	6
$\tilde{A}_1$	1 0	8

Table 2.2: Rigid orbits for type F_4.

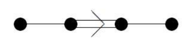
Bala–Carter label	Dynkin diagram 	Dimension
A_1	1 0 0 0	16
$\tilde{A}_1$	0 0 0 1	22
$A_1 + \tilde{A}_1$	0 1 0 0	28
$A_2 + \tilde{A}_1$	0 0 1 0	34
$\tilde{A}_2 + A_1$	0 1 0 1	36

Table 2.3: Rigid orbits for type E_6.

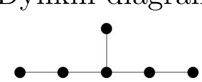
Bala–Carter label	Dynkin diagram 	Dimension
A_1	1 0 0 0 0 0	22
$3A_1$	0 0 0 1 0 0	40
$2A_2 + A_1$	0	54

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Table 2.3 – continued from previous page

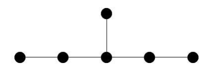
Bala–Carter label	Dynkin diagram 	Dimension
	1 0 1 0 1	

Table 2.4: Rigid orbits for type E_7.

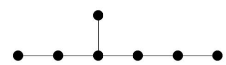
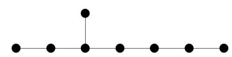
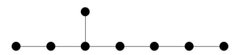
Bala–Carter label	Dynkin diagram 	Dimension
A_1	0 1 0 0 0 0 0	34
$2A_1$	0 0 0 0 0 1 0	52
$(3A_1)'$	0 0 1 0 0 0 0	64
$4A_1$	1 0 0 0 0 0 1	70
$A_2 + 2A_1$	0 0 0 1 0 0 0	82
$2A_2 + A_1$	0 0 1 0 0 1 0	90
$(A_3 + A_1)'$	0 1 0 1 0 0 0	92

Table 2.5: Rigid orbits for type E_8.

Bala–Carter label	Dynkin diagram 	Dimension
A_1	0 0 0 0 0 0 0 1	58
$2A_1$	0 1 0 0 0 0 0 0	92
$3A_1$	0 0 0 0 0 0 1 0	112
$4A_1$	1 0 0 0 0 0 0 0	128
$A_2 + A_1$	0 1 0 0 0 0 0 1	136
$A_2 + 2A_1$	0 0 0 0 0 1 0 0	146
$A_2 + 3A_1$	0 0 1 0 0 0 0 0	154
$2A_2 + A_1$	0 1 0 0 0 0 1 0	162
$A_3 + A_1$	0 0 0 0 0 1 0 1	164
$2A_2 + 2A_1$	0 0 0 0 1 0 0 0	168
$A_3 + 2A_1$	0 0 1 0 0 0 0 1	172
$D_4(a_1) + A_1$	1 0 0 0 0 0 1 0	176
$A_3 + A_2 + A_1$	0 0 0 1 0 0 0 0	182

Continued on next page

Table 2.5 – continued from previous page

Bala–Carter label	Dynkin diagram 	Dimension
$2A_3$	0 1 0 0 1 0 0 0	188
$A_4 + A_3$	0 0 0 1 0 0 1 0	200
$D_5(a_1) + A_2$	0 0 1 0 0 1 0 1	202
$A_5 + A_1$	0 1 0 1 0 0 0 1	202

3

Finite W-algebras

Here we define the finite W-algebra associated to a nilpotent element e of the Lie algebra $\mathfrak{g}$ of G, a simple simply-connected algebraic group over $\mathbb{C}$. We use the definition given by Premet in [P02]. The definition used by de Boer and Tjin [DT93] was shown in [D^{3}HK] to be equivalent to that used here, and in [L10], Losev uses a further definition in terms of Fedosov quantization [L10, Subsection 2.2], which is shown to be equivalent [L10, Corollary 3.3].

3.1 Some definitions

Let e be a non-zero nilpotent element of $\mathfrak{g}$. By Theorem 2.3.2 we can choose h and f such that (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. We let $(\cdot\,,\cdot)$ denote the form $\frac{1}{\kappa(e,f)}\kappa(\cdot\,,\cdot)$. This form is non-degenerate, symmetric and invariant – properties inherited from κ. Define $\chi \in \mathfrak{g}^*$ by $\chi(x) = (e, x)$ for $x \in \mathfrak{g}$.

The $\mathfrak{sl}_2$-triple gives a decomposition of $\mathfrak{g}$ into $\operatorname{ad} h$ weight spaces:

$$\mathfrak{g}(j) = \{x \in \mathfrak{g} \mid [h, x] = jx\}$$

and the Dynkin grading on $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$. Note that as $\mathfrak{g}$ is finite-dimensional, $\mathfrak{g}(\pm j) = 0$ for all large enough j.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ containing h. Let $\Phi \subset \mathfrak{t}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and let $\Pi \subset \Phi$ be a set of simple roots. Write $\mathfrak{g}^e$ for the centralizer of e in $\mathfrak{g}$, and let $\mathfrak{t}^e = \mathfrak{g}^e \cap \mathfrak{t}$.

We define a bilinear non-degenerate alternating form on $\mathfrak{g}(-1)$ by

$$\langle x, y \rangle = (e, [x, y]).$$

These properties follow easily from the properties of the Killing form. It follows that $\mathfrak{g}(-1)$ has even dimension, and that we can choose a basis $z_1, \ldots, z_s, z'_1, \ldots, z'_s$ such that

$$\langle z'_i, z'_j \rangle = 0 = \langle z_i, z_j \rangle \quad \text{and} \quad \langle z'_i, z_j \rangle = \delta_{ij}$$

for all $1 \leq i, j \leq s$, i.e. a *Witt basis*. Write $\mathfrak{g}(-1)^0$ for the subspace spanned by the z'_i. Then $\mathfrak{g}(-1)^0$ is a *Lagrangian* subspace of $\mathfrak{g}(-1)$, that is a maximal isotropic subspace with respect to the form $\langle \cdot\,, \cdot \rangle$. One way to choose the Witt basis is to make a choice of positive roots in the *restricted root system* Φ^e [BG07], which is defined to be the set of roots in Φ restricted to $\mathfrak{t}^e$ (excluding those whose restriction to $\mathfrak{t}^e$ is zero), and take the corresponding root vectors in $\mathfrak{g}(-1)$ to be $z_1, \ldots, z_s$ and the corresponding negative root vectors (possibly scalar multiples) to be $z'_1, \ldots, z'_s$. For our purposes here all that is required is that $\mathfrak{g}(-1)^0$ is Lagrangian. We define

$$\mathfrak{m} := \mathfrak{g}(-1)^0 \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i),$$

a nilpotent subalgebra of $\mathfrak{g}$. We consider the restriction of χ to $\mathfrak{m}$. We have $\mathfrak{m} \subseteq \oplus_{j \leq -1} \mathfrak{g}(j)$, so for $x, y \in \mathfrak{m}$, if we have either x or y in $\oplus_{j \leq -2} \mathfrak{g}(j)$ then $[x, y] \in \oplus_{j \leq -3} \mathfrak{g}(j)$ and hence $\chi([x, y]) = 0$. We therefore have

$$\begin{aligned} (e, [\mathfrak{m}, \mathfrak{m}]) &= (e, [\mathfrak{g}(-1)^0, \mathfrak{g}(-1)^0]) \\ &= \langle \mathfrak{g}(-1)^0, \mathfrak{g}(-1)^0 \rangle \\ &= 0 \end{aligned}$$

by the construction of $\mathfrak{g}(-1)^0$. We can therefore extend the action of χ to $U(\mathfrak{m})$, where for an element $x = x_{i_1} \cdots x_{i_k} \in U(\mathfrak{m})$ we have $\chi(x) = \chi(x_{i_1}) \cdots \chi(x_{i_k})$. We denote the corresponding 1-dimensional $U(\mathfrak{m})$-module by $\mathbb{C}_\chi$. Let $1_\chi \in \mathbb{C}_\chi$.

We define the induced module $Q_\chi := \mathrm{Ind}_{U(\mathfrak{m})}^{U(\mathfrak{g})} \mathbb{C}_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$.

Definition 3.1.1. We now define the *finite W-algebra* associated to $\mathfrak{g}$ and e to be

$$U(\mathfrak{g}, e) := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{\mathrm{op}},$$

the opposite algebra of $U(\mathfrak{g})$-module endomorphisms of Q_χ.

Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by all $x - \chi(x)$ for $x \in \mathfrak{m}$. We can identify Q_χ with the space of cosets $U(\mathfrak{g})/I_\chi$. There is an associative action of $\mathfrak{g}$ on $U(\mathfrak{g})/I_\chi$ given by $x(y + I_\chi) = xy + I_\chi$ for $x \in \mathfrak{g}$ and $y + I_\chi \in U(\mathfrak{g})/I_\chi$. The left ideal I_χ is stable under the action of $\mathrm{ad}\, x$ for $x \in \mathfrak{m}$ so we can define the adjoint action of $\mathfrak{m}$ on $U(\mathfrak{g})/I_\chi$ by $\mathrm{ad}\, x(y + I_\chi) = [x, y] + I_\chi$.

From Frobenius reciprocity we can identify $U(\mathfrak{g}, e)$ with the space of $U(\mathfrak{m})$-module homomorphisms $\mathbb{C}_\chi \to Q_\chi$. It follows that elements $\phi \in U(\mathfrak{g}, e)$ are determined by the value $\phi(1_\chi) \in U(\mathfrak{g})/I_\chi$, and also that we can identify $U(\mathfrak{g}, e)$ with the elements of Q_χ for which the associative action of $\mathfrak{m}$ is scalar multiplication by $\chi(x)$ for $x \in \mathfrak{m}$. For $x \in \mathfrak{m}$ and $y + I_\chi \in U(\mathfrak{g})/I_\chi$ we have

$$\begin{array}{rcl} x(y + I_\chi) & = & \chi(x)y + I_\chi \\ & \Updownarrow & \\ xy - yx + I_\chi & = & \chi(x)y - yx + I_\chi \\ & \Updownarrow & \\ [x, y] + I_\chi & = & y\underbrace{(\chi(x) - x)}_{\in I_\chi} + I_\chi = I_\chi. \end{array}$$

We are therefore identifying $U(\mathfrak{g}, e)$ with the subspace

$$\{y + I_\chi \mid [x, y] \in I_\chi \text{ for all } x \in \mathfrak{m}\} \tag{3.1.2}$$

of $U(\mathfrak{g})/I_\chi$.

Composition of endomorphisms determined by $\phi_y(1_\chi) = y+I_\chi$ and $\phi_{y'}(1_\chi) = y' + I_\chi$ in $U(\mathfrak{g}, e)$ is given by

$$(\phi_{y'}\phi_y)(u + I_\chi) = uyy' + I_\chi.$$

Expressed in terms of the above identification with the space of cosets (3.1.2), this is

$$(y + I_\chi) \cdot (y' + I_\chi) = yy' + I_\chi,$$

which is the natural operation of multiplication of cosets. This justifies why we take the opposite algebra of endomorphisms in Definition 3.1.1. From now on we view $U(\mathfrak{g}, e)$ as the subspace of $U(\mathfrak{g})/I_\chi$ invariant under the adjoint action of $\mathfrak{m}$.

It is straightforward to see that $U(\mathfrak{g}, e)$ is closed under this multiplication of cosets. For $y_1 + I_\chi$ and $y_2 + I_\chi$ in $U(\mathfrak{g}, e)$ and $x \in \mathfrak{m}$ we have

$$[x, y_1y_2] = (xy_1 - y_1x)y_2 + y_1(xy_2 - y_2x),$$

so to show that $[x, y_1y_2] \in I_\chi$ for all $x \in \mathfrak{m}$ it is sufficient to show that $I_\chi y_2 \subseteq I_\chi$. We have

$$\begin{aligned}(x - \chi(x))y &= yx - \chi(x)y + [x, y] \\ &= y(x - \chi(x)) + [x, y] \in I_\chi.\end{aligned}$$

It is shown in [GG02] that we do not need to take a Lagrangian subspace of $\mathfrak{g}(-1)$ to be $\mathfrak{m}\cap\mathfrak{g}(-1)$. We can take any isotropic subspace $\mathfrak{a} \subset \mathfrak{g}(-1)$, and let $\mathfrak{a}^\perp$ denote the subspace $\{x \in \mathfrak{g}(-1) \mid \langle x, a\rangle \text{ for all } a \in \mathfrak{a}\}$. Then we define I'_χ to be the left ideal of $U(\mathfrak{g})$ generated by all $x-\chi(x)$ for $x \in \mathfrak{a}\oplus\bigoplus_{j\leq -2} \mathfrak{g}(j)$, and we define $Q'_\chi := U(\mathfrak{g})/I'_\chi$. Then the space of elements of Q'_χ invariant under the adjoint action of $\mathfrak{a}^\perp \oplus \bigoplus_{j\leq -2} \mathfrak{g}(j)$ is isomorphic to $U(\mathfrak{g}, e)$. In particular, we can take $\mathfrak{a} = 0$, so $\mathfrak{a}^\perp = \mathfrak{g}(-1)$. This makes the procedure of finding a suitably ordered basis of $\mathfrak{g}$ significantly easier, but computationally, finding a presentation becomes harder, as we are looking for elements of a larger space Q'_χ invariant under the action of a larger subalgebra $\mathfrak{a}^\perp \oplus \bigoplus_{j\le -2} \mathfrak{g}(j)$.

We observe that by the conjugacy of $\mathfrak{sl}_2$-triples [CM93, Chapter 3], the isomorphism class of $U(\mathfrak{g}, e)$ depends only on the nilpotent orbit of e and not on the choices of f and h.

There is a natural embedding of $U(\mathfrak{g}, e)$ in $U(\mathfrak{g})$. By (3.1.2) we see that the image of the centre of the universal enveloping algebra, denoted $\mathcal{Z}(\mathfrak{g})$, is a subalgebra of $U(\mathfrak{g}, e)$. From [K78] we have equality in the case that e lies in the regular orbit in $\mathfrak{g}$. At the opposite end of the partial order on the nilpotent orbits, if $e = 0$ then we can see that $U(\mathfrak{g}, e) \cong U(\mathfrak{g})$. In the case that e lies in the minimal orbit in $\mathfrak{g}$, there is a presentation of $U(\mathfrak{g}, e)$ given in [P07i, Theorem 6.1].

3.2 PBW-theorem

In order to give a PBW theorem for $U(\mathfrak{g}, e)$, we need a filtration of $U(\mathfrak{g}, e)$. In order to define this filtration, we need to choose an ordered homogeneous basis of $\mathfrak{g}$ satisfying certain properties.

Let $\mathfrak{p}$ be the parabolic subalgebra $\bigoplus_{j\ge 0} \mathfrak{g}(j)$ of $\mathfrak{g}$ determined by our $\mathfrak{sl}_2$-triple. Then from [J04, Section 5.8] we know that $\mathfrak{g}^e \subseteq \mathfrak{p}$. We choose a homogeneous basis $x_1, \dots, x_m$ of $\mathfrak{p}$ such that $x_1, \dots, x_d$ is a basis of $\mathfrak{g}^e$. As above, we can choose a basis $z_1, \dots, z_s, z'_1, \dots, z'_s$ of $\mathfrak{g}(-1)$ where $z'_1, \dots, z'_s$ span a Lagrangian subspace of $\mathfrak{g}(-1)$ with respect to the form $\langle \cdot, \cdot \rangle$. Let $x_{m+1}, \dots, x_{m+s}$ be $z_1, \dots, z_s$ and let $x_{m+s+1}, \dots, x_{m+2s}$ be $z'_1, \dots, z'_s$. In what follows the choice for the remaining terms of the basis is made for reasons of computational convenience. Note that f lies in $\ker \chi|_{\mathfrak{g}(-2)}$. We can choose a basis of $\mathfrak{t}^e$-weight vectors of $\ker \chi|_{\mathfrak{g}(-2)}$, including f. We complete our basis $x_1, \dots, x_n$ including these elements.

We can make all of the above choices for the basis elements to be weight vectors for $\mathfrak{t}^e$ and eigenvectors for $\operatorname{ad} h$. For each $i = 1, \dots, n$, let $n_i \in \mathbb{Z}$ be such that $x_i \in \mathfrak{g}(n_i)$ and let $\beta_i \in \Phi^e$ be such that $x_i \in \mathfrak{g}_{\beta_i}$.

The tables in Chapter 6 show bases according to these specifications for each non-zero nilpotent orbit in $\mathfrak{g}$ of type G_2, and for each non-zero rigid nilpotent orbit for $\mathfrak{g}$ of type F_4, E_6 and E_7.

We have a basis of Q_χ given by the set of cosets $x^{\mathbf{a}}+I_\chi = x_1^{a_1}\cdots x_{m+s}^{a_{m+s}}+I_\chi$, where $\mathbf{a}=(a_1,\dots,a_{m+s})\in\mathbb{Z}_{\geq 0}^{m+s}$. For $\mathbf{a}\in\mathbb{Z}_{\geq 0}^{m+s}$, we define

$$|\mathbf{a}| = \sum_{i=1}^{m+s} a_i \quad \text{and} \quad |\mathbf{a}|_e = \sum_{i=1}^{m+s} a_i(n_i+2).$$

We say that $x^{\mathbf{a}}+I_\chi\in Q_\chi$ has *Kazhdan degree* $|\mathbf{a}|_e$. This restricts to $U(\mathfrak{g},e)$, and we write $F_iU(\mathfrak{g},e)$ for the span of all elements $x^{\mathbf{a}}+I_\chi\in U(\mathfrak{g},e)$ where $|\mathbf{a}|_e\leq i$. Note that $F_iU(\mathfrak{g},e)=0$ for all $i<0$.

We now give the PBW-theorem for $U(\mathfrak{g},e)$. This combines [P02, Theorem 4.6] and [P07i, Lemma 2.2], written in terms of our interpretation of $U(\mathfrak{g},e)$ as a subalgebra of $U(\mathfrak{g})/I_\chi$.

Theorem 3.2.1. *Let $x_1,\dots,x_n$ be a basis of $\mathfrak{g}$ as described above. Then we have the following:*

1. *There is a set of generators for $U(\mathfrak{g},e)$ given by*

$$\Theta_i = \left(x_i + \sum_{|\mathbf{a}|_e\leq n_i+2} \lambda^i_{\mathbf{a}} x^{\mathbf{a}}\right) + I_\chi,$$

   for $i=1,\dots,d$ where the coefficients $\lambda^i_{\mathbf{a}}\in\mathbb{Q}$ are zero when $a_{d+1}=\cdots=a_{m+s}=0$, or if $|\mathbf{a}|_e=n_i+2$ and $|\mathbf{a}|=1$. The coefficients $\lambda^i_{\mathbf{a}}$ are uniquely determined by the choice of ordered basis $x_1,\dots,x_n$ of $\mathfrak{g}$ and the above vanishing conditions.

2. *The Θ_i are weight vectors for $\mathfrak{t}^e$ with weight β_i.*

3. *The monomials $\Theta^{\mathbf{a}}=\Theta_1^{a_1}\cdots\Theta_d^{a_d}$ with $\mathbf{a}\in\mathbb{Z}_{\geq 0}^d$ form a basis of $U(\mathfrak{g},e)$.*

4. *We have $[\Theta_i,\Theta_j]\in F_{n_i+n_j+2}U(\mathfrak{g},e)$ for $i,j=1,\dots,d$. Moreover, if*

$[x_i, x_j] = \sum_{k=1}^{d} \mu_{ij}^k x_k$ in $\mathfrak{g}^e$, then

$$[\Theta_i, \Theta_j] = \sum_{k=1}^{d} \mu_{ij}^k \Theta_k + q_{ij}(\Theta_1, \dots, \Theta_d) \mod F_{n_i+n_j} U(\mathfrak{g}, e),$$

where q_{ij} is a polynomial with coefficients in $\mathbb{Q}$, and zero constant and linear terms.

The uniqueness claimed in part (1) follows from the proof of [P02, Theorem 4.6], though is not given in the statement of that result. The conditions on the coefficients $\lambda_{\mathbf{a}}^i$ can be given more clearly by stating that the expression for Θ_k contains no term of Kazhdan degree greater than $n_k + 2$, no term is in $U(\mathfrak{g}^e)$ other than the leading term x_k, and it contains no term x_j (for $j \neq k$) of Kazhdan degree equal to $n_k + 2$.

Theorem 3.2.1 is fundamental to the methods described in Chapter 4 to calculate presentations for $U(\mathfrak{g}, e)$ and to deduce the existence of 1-dimensional representations as in Chapter 5. From [P07ii, Lemma 4.1] we have that these commutator relations are sufficient for a presentation of $U(\mathfrak{g}, e)$. From the anti-symmetry of the commutator, a full presentation of $U(\mathfrak{g}, e)$ is given by the $\binom{d}{2}$ relations $[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \dots, \Theta_d)$ where $d = \dim(\mathfrak{g}^e)$, $(i > j)$ and F_{ij} is a polynomial in d indeterminates. We reduce the amount of relations needed for a presentation in Section 5.3.

4

An algorithm for finding a presentation of $U(\mathfrak{g}, e)$

For a nilpotent element e in a simple complex Lie algebra $\mathfrak{g}$, the finite W-algebra $U(\mathfrak{g}, e)$ depends, up to isomorphism, on the adjoint orbit of e. For $\mathfrak{g}$ of exceptional type, the nilpotent orbits are listed, along with the associated weighted Dynkin diagrams, in [CM93, Ch. 8]. This chapter describes an algorithm which takes as its input a weighted Dynkin diagram associated to a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$, and returns a presentation for the finite W-algebra $U(\mathfrak{g}, e)$ where e lies in $\mathcal{O}$. The GAP4 implementation of this is detailed in Appendices A to F.

4.1 Finding an $\mathfrak{sl}_2$-triple

Let $\mathfrak{g}$, $\mathfrak{t}$, Φ and Π be as in Section 3.1. Let l denote the rank of $\mathfrak{g}$, and $\Pi = \{\alpha_1, \dots, \alpha_l\}$. We can construct a Chevalley basis of $\mathfrak{g}$, $\{e_\alpha \mid \alpha \in \Phi\} \cup \{h_\alpha = [e_\alpha, e_{-\alpha}] \mid \alpha \in \Pi\}$. So we have a basis of $\mathfrak{t}$ given by $h_i := h_{\alpha_i}$ for $i = 1, \dots, l$.

Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. Write $D = (D_1, \dots, D_l)$ for the weights on the nodes of the Dynkin diagram (with the order of nodes corresponding to the order of simple roots in Π) for $\mathfrak{g}$ associated to $\mathcal{O}$. From D we have a decomposition $\Phi = \bigcup_{j\in\mathbb{Z}} \Phi(j)$, where

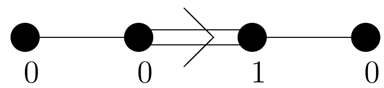
$$\Phi(j) = \{\sum_{i=1}^{l} a_i\alpha_i \in \Phi \mid \sum_{i=1}^{l} a_i D_i = j\}.$$

There is a unique element $h \in \mathfrak{t}$ such that h is part of an $\mathfrak{sl}_2$-triple corresponding to the nilpotent orbit $\mathcal{O}$. To find $h = \sum_{j=1}^{l} \lambda_j h_j$ we need values of λ_j satisfying $[\sum_{j=1}^{l} \lambda_j h_j, \alpha_i] = D_i$ for $i = 1, \dots, l$. This is equivalent to finding a column vector λ such that $C\lambda = D^T$, where C is the Cartan matrix corresponding to $\Pi \subset \Phi$. Uniqueness follows as C is non-singular.

The element h determines the Dynkin grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(j)$. Note that the Dynkin grading coincides with the grading given by letting $\mathfrak{g}(j)$ be the span of the Chevalley basis elements $\{e_\alpha \mid \alpha \in \Phi(j)\}$ for $j \neq 0$ and let $\mathfrak{g}(0)$ be the span of $\mathfrak{t}$ and $\{e_\alpha \mid \alpha \in \Phi(0)\}$. We now require elements e and f in $\mathfrak{g}$ such that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. We can take e to be a sum of e_α for $\alpha \in \Phi(2)$. We find this as follows.

Write $\Gamma_j = \{e_\alpha \mid \alpha \in \Phi(j)\}$. For a subset $A \subseteq \Gamma_2$ let $A' = \{e_{-\alpha} \mid e_\alpha \in A\} \subseteq \Gamma_{-2}$. We require $A \subseteq \Gamma_2$ such that h lies in the span of $\{\sum_{e_\alpha \in A}[e_\alpha, e_{-\beta}] \mid e_\beta \in A\}$. For such A we have coefficients a_α for $e_\alpha \in A$ satisfying $h = \sum_{e_\beta\in A} a_\beta \big(\sum_{e_\alpha\in A}[e_\alpha, e_{-\beta}]\big)$. Then we can take $f = \sum_{e_\beta\in A} a_\beta e_{-\beta}$. This gives our $\mathfrak{sl}_2$-triple (e, h, f). We can always make our choice of $A \subseteq \Gamma_2$ so that the a_β are positive integers.

Example 4.1.1. As an example, we take $\mathfrak{g}$ to be the simple Lie algebra of type F_4, and we take the nilpotent orbit with Bala–Carter label $\tilde{A}_1 + A_2$. We use the Chevalley basis $b_1, \dots, b_{52}$ for $\mathfrak{g}$ given by GAP4, where we have simple root vectors $b_1, \dots, b_4$, positive root vectors $b_1, \dots, b_{24}$ and negative root vectors $b_{25}, \dots, b_{48}$. We have a basis for the Cartan subalgebra $\mathfrak{t}$ given by $h_i = b_{48+i} = [b_i, b_{i+24}]$ for $i = 1, \dots, 4$. This orbit has weighted Dynkin diagram:



With the order of simple roots used by GAP4 (note that this differs from the order more commonly used, in Bourbaki [B07, Ch.4-6, Plate VII] and elsewhere) we have $D = (0, 0, 1, 0)$, and the Cartan matrix

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -2 & 2 \end{pmatrix}.$$

We require $\lambda_1, \dots, \lambda_4$ satisfying

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

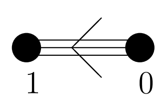
So we have $h = \sum_{i=1}^{4} \lambda_i b_{i+48} = 3b_{49} + 4b_{50} + 6b_{51} + 8b_{52}$. This gives the following decomposition: $\mathfrak{g} = \bigoplus_{-4 \leq j \leq 4} \mathfrak{g}(j)$, where $\mathfrak{g}(2)$ is spanned by $\Gamma_2 = \{b_{10}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16}, b_{17}, b_{18}, b_{20}\}$. By checking each subset of Γ_2 directly, we can check that for $A \subseteq \Gamma_2$ to have h in the span of $\{\sum_{e_\alpha \in A} [e_\alpha, e_{-\beta}] \mid e_\beta \in A\}$ to be one of $\{b_{14}, b_{15}, b_{16}\}$, $\{b_{10}, b_{17}, b_{18}\}$, $\{b_{13}, b_{15}, b_{17}\}$, $\{b_{10}, b_{14}, b_{20}\}$, $\{b_{12}, b_{13}, b_{20}\}$ or $\{b_{12}, b_{16}, b_{18}\}$. Each of these gives an $\mathfrak{sl}_2$-triple with all coefficients in $\mathbb{Z}$ (though this will not always be the case), so we may choose to take $e = b_{14} + b_{15} + b_{16}$. We have $h = [b_{14}, b_{38}] + 2[b_{15}, b_{39}] + 2[b_{16}, b_{40}]$, so we get $f = b_{38} + 2b_{39} + 2b_{40}$, and we easily verify that (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.

4.2 Finding generators for $U(\mathfrak{g}, e)$

Given an $\mathfrak{sl}_2$-triple (e,h,f) in our Lie algebra $\mathfrak{g}$, we have $\mathfrak{t}^e = \mathfrak{g}^e \cap \mathfrak{t}$, the non-degenerate form $(\cdot\,,\cdot) = \frac{1}{\kappa(e,f)}\kappa(\cdot,\cdot)$, the Dynkin grading for $\mathfrak{g}$, the map $\chi(\cdot) = (e,\cdot)$ and the alternating form $\langle u,v\rangle = (e,[u,v])$ on $\mathfrak{g}(-1)$. We can determine an ordered basis of $\mathfrak{g}$ according to the conditions of Section 3.2 such that each element in the basis is a weight vector for $\mathfrak{t}^e$ and for h. This defines the subalgebra $\mathfrak{m}$ and the left ideal I_χ of $U(\mathfrak{g})$.

From Theorem 3.2.1, we determine the generators of $U(\mathfrak{g},e)$ by finding for each $i = 1,\dots,d$ the coefficients $\lambda^i_{\mathbf{a}}$ for each $\mathbf{a}$ such that $|\mathbf{a}|_e \le n_i + 2$ and $\lambda^i_{\mathbf{a}}$ can be non-zero by the conditions of Theorem 3.2.1(2). Let $A_i \subset \mathbb{Z}^{m+s}_{\ge 0}$ be the set of all such $\mathbf{a}$ for Θ_i. We require that $[x,\Theta_i] = 0 + I_\chi$ for all $x \in \mathfrak{m}$. Let B be a subset of $\{x_{m+s+1},\dots,x_n\}$ such that B generates $\mathfrak{m}$. We take B to be minimal in order to reduce the amount of computation required. For each generator $y \in B$ we calculate $[y, x_i + \sum_{\mathbf{a}\in A_i} \lambda^i_{\mathbf{a}} x^{\mathbf{a}} + I_\chi] = [y, x_i + I_\chi] + \sum_{\mathbf{a}\in A_i} \lambda^i_{\mathbf{a}} [y, x^{\mathbf{a}} + I_\chi]$ in $U(\mathfrak{g})/I_\chi$. This can be rewritten as $\sum_{0\le|\mathbf{b}|_e\le n_i+2} q_{\mathbf{b}}(\{\lambda^i_{\mathbf{a}} \mid \mathbf{a}\in A_i\}) x^{\mathbf{b}} + I_\chi$, where $\mathbf{b} \in \mathbb{Z}^{m+s}_{\ge 0}$, and $q_{\mathbf{b}}$ is a linear polynomial in the coefficients $\lambda^i_{\mathbf{a}}$ where $\mathbf{a} \in A_i$. The condition that $[y,\Theta_i] \in I_\chi$ is met by setting each polynomial $q_{\mathbf{b}} = 0$ and solving. From Theorem 3.2.1(1) there is a unique solution to this system of equations, which determines the generator Θ_i.

Example 4.2.1. Let $\mathfrak{g}$ be the simple Lie algebra of type G_2, with Chevalley basis (given by GAP4) comprising $b_1,\dots,b_6$ positive root vectors with simple short root vector b_1 and simple long root vector b_2; negative root vectors $b_7,\dots,b_{12}$; and a Cartan subalgebra $\mathfrak{t}$ spanned by $b_{13} = [b_1,b_7]$ and $b_{14} = [b_2,b_8]$. Let $\mathcal{O}$ denote the nilpotent orbit of $\mathfrak{g}$ given by the weighted Dynkin diagram



By Section 4.1 we have $h = 2b_{13} + 3b_{14}$, and as $\mathfrak{g}(2)$ is the span of b_4, we must take $e = b_4$ and $f = b_{10}$. Then $\mathfrak{g}^e$ has dimension 6, and is spanned by the elements $b_6, b_5, b_4, b_2, b_8, b_{14}$, with $\mathfrak{t}^e$ spanned by the single element b_{14}. We complete a basis of $\mathfrak{p} = \bigoplus_{j\geq 0}\mathfrak{g}(j)$ with the elements b_1, b_3, b_{13}. We divide the Killing form by $\kappa(e, f) = 24$ for the form $(\cdot\,,\cdot)$. The subspace $\mathfrak{g}(-1)$ is spanned by b_9 and b_7, so we may append b_9 to the basis, then take $\mathfrak{g}(-1)^0$ to be $\mathbb{C}b_7$, and complete the basis with $b_{10} = f$, b_{11}, b_{12}. This ordered basis is given, with values for n_i and β_i in Table 6.3. We now relabel the basis elements as $x_1, \ldots, x_{14}$ in this order. We have that $\mathfrak{m}$ is generated by the set $\{x_{11}, x_{12}, x_{14}\}$. We find the A_i by listing the monomials $x^{\mathbf{a}}$ for which the Kazhdan degree is at most the maximum of the $n_i + 2$ (in this case the upper bound is 5) and selecting the monomials $x^{\mathbf{a}}$ where the conditions of Theorem 3.2.1(1) are satisfied, and label these (in some order) as $A_i^2, \ldots, A_i^{|A_i|+1}$, and put $A_i^1 = 1$. Write $\lambda^i_{\mathbf{a}} = \lambda^i_j$ where $A_i^j = x^{\mathbf{a}}$. We have

$$A_1 \;=\; \{x_8,\ x_4x_7,\ x_4x_{10},\ x_6x_8,\ x_8x_9,\ x_4x_6x_{10},\ x_4x_9x_{10}\}.$$

We calculate $[x_{11}, x_1 + \sum_{\mathbf{a}\in A_1}\lambda^i_{\mathbf{a}}x^{\mathbf{a}}] = [x_{11}, x_1 + \sum_{2\leq j\leq |A_1|+1}\lambda^1_j A_1^j]$ and project into $U(\mathfrak{g})/I_\chi$ to give the condition

$$\begin{array}{l}(-3\lambda^1_5 + 2\lambda^1_7)x_4x_6 + (-\lambda^1_3 - 3\lambda^1_6 + 2\lambda^1_8)x_4x_9\\ +(-3\lambda^1_2 + 2\lambda^1_4 - 3\lambda^1_5 - 2\lambda^1_7 + 4\lambda^1_8)x_4 \in I_\chi,\end{array}$$

giving 3 linear polynomials in the coefficients λ^1_j. We obtain the rest from calculating $[x_{12}, x_1 + \sum_{2\leq j\leq|A_i|+1}\lambda^i_jA_i^j]$ and $[x_{14}, x_1 + \sum_{2\leq j\leq|A_i|+1}\lambda^i_jA_i^j]$ to get the conditions

$$(-2\lambda^1_3 + \lambda^1_8)x_4x_{10} + (-1 + \lambda^1_6)x_8 \in I_\chi$$

and

$$(-2 + \lambda^1_5)x_6 + (-1 + \lambda^1_6)x_9 + (\lambda^1_2 + \lambda^1_3 + \lambda^1_5 + \lambda^1_6) \in I_\chi.$$

Putting each of this set of 8 polynomials equal to zero yields the unique solution $(\lambda_2^1,\ldots,\lambda_8^1) = (-4,1,-4,2,1,3,2)$, giving

$$\begin{aligned}\Theta_1 &= (x_1 - 4x_8 + x_4x_7 - 4x_4x_{10} + 2x_6x_8 + x_8x_9 + 3x_4x_6x_{10}\\ &\quad +2x_4x_9x_{10}) + I_\chi.\end{aligned}$$

Similarly, to calculate Θ_2 we have

$$\begin{aligned}A_2 &= \{x_7,\ x_{10},\ x_3x_{10},\ x_5x_8,\ x_6x_7,\ x_6x_{10},\ x_7x_9,\ x_9x_{10},\ x_4x_5x_{10},\\ &\quad x_6^2x_{10},\ x_6x_9x_{10},\ x_8x_{10}^2,\ x_9^2x_{10},\ x_4x_{10}^3\},\end{aligned}$$

and for $k = 11, 12, 14$ we calculate $[x_k, x_2 + \sum_{2\leq j\leq |A_2|+1} \lambda_j^2 A_2^j]$ and project into $U(\mathfrak{g})/I_\chi$. For $k = 11$ this gives

$$\begin{aligned}&(\lambda_{10}^2 + 3\lambda_{13}^2 + 6\lambda_{15}^2)x_4x_{10}^2 + (-3\lambda_5^2 + 2\lambda_{10}^2)x_4x_5 + (2\lambda_{11}^2)x_6^2\\ &+(-\lambda_6^2 + 2\lambda_{12}^2)x_6x_9 + (-2\lambda_4^2 + \lambda_5^2 + 4\lambda_{13}^2)x_8x_{10} + (-\lambda_8^2 + 2\lambda_{14}^2)x_9^2\\ &+(-1 + 2\lambda_4^2)x_3 + (2\lambda_7^2 - 4\lambda_{11}^2 + 4\lambda_{12}^2)x_6\\ &+(-\lambda_2^2 - \lambda_5^2 + \lambda_6^2 + 2\lambda_9^2 - 2\lambda_{12}^2 + 8\lambda_{14}^2)x_9\\ &+(2\lambda_3^2 - 2\lambda_7^2 + 4\lambda_9^2 + 2\lambda_{11}^2 - 4\lambda_{12}^2 + 8\lambda_{14}^2) \in I_\chi;\end{aligned}$$

for $k = 12$ we have

$$\begin{aligned}&(-3\lambda_4^2 - 2\lambda_6^2 + \lambda_{12}^2)x_6x_{10} + (-2\lambda_4^2 - 2\lambda_8^2 + 2\lambda_{14}^2)x_9x_{10}\\ &+(\lambda_1^2 + \lambda_8^2)x_7 + (-2\lambda_2^2 + 2\lambda_8^2 + \lambda_9^2 + 8\lambda_{13}^2 + \lambda_{14}^2)x_{10} \in I_\chi;\end{aligned}$$

and for $k = 14$ we have

$$(-\lambda_4^2 + \lambda_{13}^2)x_{10}^2 + (\lambda_1^2 + \lambda_5^2)x_5 \in I_\chi.$$

This set of 16 polynomials has a unique solution

$$(\lambda_2^2,\ldots,\lambda_{15}^2) = (2,-3,\tfrac{1}{2},-1,-1,1,-1,\tfrac{5}{2},-\tfrac{3}{2},0,-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}),$$

so we have

$$\begin{aligned}\Theta_2 &= (x_2 + 2x_7 - 3x_{10} + \tfrac{1}{2}x_3x_{10} - x_5x_8 - x_6x_7 + x_6x_{10} - x_7x_9 + \tfrac{5}{2}x_9x_{10} \\ &\quad -\tfrac{3}{2}x_4x_5x_{10} - \tfrac{1}{2}x_6x_9x_{10} + \tfrac{1}{2}x_8x_{10}^2 - \tfrac{1}{2}x_9^2x_{10} + \tfrac{1}{2}x_4x_{10}^3) + I_\chi.\end{aligned}$$

To determine Θ_3, we have

$$A_3 = \{x_9,\ x_6x_9,\ x_8x_{10},\ x_9^2,\ x_4x_{10}^2\}.$$

The condition that $[x_k, x_3 + \sum_{2\le j\le |A_3|+1} \lambda_j^3 A_3^j] \in I_\chi$ for $k = 11, 12, 14$ gives us the following: for $k = 11$ we get

$$(-3\lambda_4^3 + 4\lambda_6^3)x_4x_{10} + (-2 + 2\lambda_4^3)x_8 \in I_\chi;$$

for $k = 12$ we get

$$(-3 + \lambda_3^3)x_6 + (-2 + 2\lambda_5^3)x_4x_9 + (\lambda_2^3 + 4\lambda_4^3 + \lambda_5^3)x_9 \in I_\chi;$$

and for $k = 14$ we get

$$(-1 + \lambda_4^3)x_{10} \in I_\chi.$$

From these 6 polynomials we have the unique solution

$$(\lambda_2^3, \ldots, \lambda_6^3) = (-5, 3, 1, 1, \tfrac{3}{4}),$$

so we have

$$\Theta_3 = (x_3 - 5x_9 + 3x_6x_9 + x_8x_{10} + x_9^2 + \tfrac{3}{4}x_4x_{10}^2) + I_\chi.$$

In determining Θ_4, we find that A_4 is empty, and we verify that $[x_k, x_4] \in I_\chi$ for $k = 11, 12, 14$. We therefore have $\Theta_4 = x_4 + I_\chi$.

We have $A_5 = \{x_{10}^2\}$, and $[x_{11}, x_5 + \lambda_2^5 x_{10}^2 + I_\chi] = (1 + 4\lambda_2^5) + I_\chi$, so we have $\lambda_2^5 = -\frac{1}{4}$ and $\Theta_5 = x_5 - \frac{1}{4}x_{10}^2 + I_\chi$. It is straightforward to check that $[x_k, \Theta_5] \in I_\chi$ for $k = 12, 14$.

For the final generator, we see that A_6 is empty, and $[x_k, x_6] \in I_\chi$ for $k = 11, 12, 14$, so we have $\Theta_6 = x_6 + I_\chi$, and our set of generators is complete.

4.3 Finding relations for $U(\mathfrak{g}, e)$

Given a set of generators $\Theta_1, \ldots, \Theta_d$ of $U(\mathfrak{g}, e)$ (where $d = \dim(\mathfrak{g}^e)$), we find polynomials F_{ij} for $1 \leq i \leq j \leq d$ for which $[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \ldots, \Theta_d)$ by the following method. We write $(\boldsymbol{\Theta})$ for $(\Theta_1, \ldots, \Theta_d)$, and for $i = 1, \ldots, d$ write $\Theta_i = u_i + I_\chi$ where $u_i \in U(\mathfrak{g})$.

For generators Θ_i and Θ_j, we calculate $[u_i, u_j]$ as an element of $U(\mathfrak{g})$ and project into $U(\mathfrak{g})/I_\chi$. Recalling that $[\Theta_i, \Theta_j] \in F_{n_i+n_j+2}U(\mathfrak{g}, e)$, we get an expression of the form

$$[\Theta_i, \Theta_j] = \sum_{|\mathbf{a}|_e \leq n_i+n_j+2} \mu_{\mathbf{a}}^{i,j} x^{\mathbf{a}} + I_\chi,$$

with coefficients $\mu_{\mathbf{a}}^{i,j} \in \mathbb{Q}$. If all $\mu_{\mathbf{a}}^{i,j} = 0$ then the generators Θ_i and Θ_j commute and $F_{ij} = 0$. Otherwise, we write the desired polynomial as a sum of polynomials F_{ij}^k homogeneous with respect to the Kazhdan degree:

$$F_{ij}(\boldsymbol{\Theta}) = \sum_{k=0}^{n_i+n_j+2} F_{ij}^k(\boldsymbol{\Theta}).$$

We find the F_{ij}^k in decreasing order of k, from $k = n_i + n_j + 2$ to $k = 0$. Let $A_{n_i+n_j+2}$ be the set of all $\mathbf{a}$ such that $\mu_{\mathbf{a}}^{i,j} \neq 0$, $|\mathbf{a}|_e = n_i+n_j+2$ and $a_q = 0$ for all $q > d$. If $A_{n_i+n_j+2}$ is empty, then there is no $\mathbf{a}$ such that $|\mathbf{a}|_e = n_i + n_j + 2$ and $\mu_{\mathbf{a}}^{i,j} \neq 0$. Otherwise there is some $\mathbf{a}$ with $|\mathbf{a}|_e = n_i + n_j + 2$, $\mu_{\mathbf{a}}^{i,j} \neq 0$ and $a_q \neq 0$ for some $q > d$, and so the term $\mu_{\mathbf{a}}^{i,j} x^{\mathbf{a}}$ must occur in some term $\mu_{\mathbf{b}}^{i,j} \Theta^{\mathbf{b}}$ in $F^{n_i+n_j+2}(\boldsymbol{\Theta})$ (expressed as a coset in $U(\mathfrak{g})/I_\chi$) with $\mathbf{b} \in \mathbb{Z}_{\geq 0}^d$ and $|\mathbf{b}|_e = n_i + n_j + 2$, which appears in $F_{ij}^{n_i+n_j+2}(\boldsymbol{\Theta})$. So we have in $F_{ij}^{n_i+n_j+2}(\boldsymbol{\Theta})$ a term $\mu_{\mathbf{b}}^{i,j} \Theta^{\mathbf{b}} \neq 0$ with $|\mathbf{b}|_e = n_i + n_j + 2$, and $b_q = 0$ for all $q > d$, and we have that $A_{n_i+n_j+2}$ is non-empty. We put $F_{ij}^{n_i+n_j+2}(\boldsymbol{\Theta}) = \sum_{\mathbf{a} \in A_{n_i+n_j+2}} \mu_{\mathbf{a}}^{i,j} \Theta^{\mathbf{a}}$.

We have

$$[\Theta_i, \Theta_j] - F_{ij}^{n_i+n_j+2}(\mathbf{\Theta}) = \sum_{|\mathbf{a}|_e \leq n_i+n_j+1} \mu_{\mathbf{a}}^{i,j,n_i+n_j+1} x^{\mathbf{a}} + I_\chi,$$

for some coefficients $\mu_{\mathbf{a}}^{i,j,n_i+n_j+1}$ (which are not the same as the coefficients $\mu_{\mathbf{a}}^{i,j}$). From the construction of the set $A_{n_i+n_j+1}$ we can reduce the range of the sum on the right. We now let $A_{n_i+n_j+1}$ be the set of $\mathbf{a}$ such that $\mu_{\mathbf{a}}^{i,j,n_i+n_j+1} \neq 0$, $|\mathbf{a}|_e = n_i + n_j + 1$ and $a_q = 0$ for all $q > d$, and write $F_{ij}^{n_i+n_j+1}(\mathbf{\Theta}) = \sum_{\mathbf{a} \in A_{n_i+n_j+1}} \mu_{\mathbf{a}}^{i,j} \Theta^{\mathbf{a}}$. Given $F_{ij}^{n_i+n_j+2}, F_{ij}^{n_i+n_j+1}, \ldots, F_{ij}^{k}$, we have

$$[\Theta_i, \Theta_j] - \sum_{q=k}^{n_i+n_j+2} F_{ij}^{q}(\mathbf{\Theta}) = \sum_{|\mathbf{a}|_e \leq k-1} \mu_{\mathbf{a}}^{i,j,k-1} x^{\mathbf{a}} + I_\chi,$$

from which we define A_{k-1} to be the set of $\mathbf{a}$ for which $\mu_{\mathbf{a}}^{i,j,k-1} \neq 0$, $|\mathbf{a}|_e = k-1$ and $a_q = 0$ for all $q > d$, and we let $F_{ij}^{k-1}(\mathbf{\Theta}) = \sum_{\mathbf{a} \in A_{k-1}} \mu_{\mathbf{a}}^{i,j,k-1} \Theta^{\mathbf{a}}$. Inductively, this gives the polynomial F_{ij}.

It follows from Theorem 3.2.1 parts (2) and (4) that the sets A_i above are either empty for all even i or for all odd i, but this does not affect the above process.

Example 4.3.1. Let $\mathfrak{g}$, e and $U(\mathfrak{g}, e)$ be as in Example 4.2.1, with the set of PBW generators $\Theta_1, \ldots, \Theta_6$. We find $F_{1,2}$ as follows. Note that $n_1+n_2+2 = 8$. We have

$$\begin{aligned}
[\Theta_1, \Theta_2] = {} & (-18 - \tfrac{3}{2}x_1x_{10} - 6x_3 + 5x_3x_4x_5 - 2x_3x_4x_{10}^2 + \tfrac{11}{2}x_3x_6 - 3x_3x_6x_9 \\
& -x_3x_6^2 - x_3x_8x_{10} + 4x_3x_9 - x_3x_9^2 - \tfrac{1}{2}x_3^2 + 33x_4x_5 - \tfrac{69}{2}x_4x_5x_6 \\
& +15x_4x_5x_6x_9 + 9x_4x_5x_6^2 + 5x_4x_5x_8x_{10} - 25x_4x_5x_9 + 5x_4x_5x_9^2 \\
& -6x_4x_6x_9x_{10}^2 + 21x_4x_6x_{10}^2 - 3x_4x_6^2x_{10}^2 - \tfrac{13}{2}x_4x_7x_{10} - 2x_4x_8x_{10}^3 \\
& +10x_4x_9x_{10}^2 - 2x_4x_9^2x_{10}^2 - 9x_4x_{10}^2 + 6x_4^2x_5x_{10}^2 - \tfrac{9}{2}x_4^2x_5^2 - \tfrac{3}{2}x_4^2x_{10}^4 \\
& +42x_6 - 3x_6x_8x_9x_{10} + \tfrac{23}{2}x_6x_8x_{10} - 53x_6x_9 + 22x_6x_9^2 - 3x_6x_9^3 \\
& -30x_6^2 - x_6^2x_8x_{10} + 26x_6^2x_9 - \tfrac{11}{2}x_6^2x_9^2 + 6x_6^3 - 3x_6^3x_9 - x_7x_8
\end{aligned}$$

$$+\tfrac{15}{2}x_8x_9x_{10} - x_8x_9^2x_{10} - 15x_8x_{10} - \tfrac{1}{2}x_8^2x_{10}^2 + 30x_9 - \tfrac{37}{2}x_9^2 + 5x_9^3$$
$$-\tfrac{1}{2}x_9^4) + I_\chi.$$

From the terms in the above expression we get

$$\{x^{\mathbf{a}} \mid \mathbf{a} \in A_8\} \;=\; \{x_3^2,\ x_3x_6^2,\ x_3x_4x_5,\ x_4x_5x_6^2,\ x_4^2x_5^2\},$$

and we have

$$\begin{aligned} F_{1,2}^8(\boldsymbol{\Theta}) &= \sum_{\mathbf{a}\in A_8} \mu_{\mathbf{a}}^{1,2,8}\Theta^{\mathbf{a}} \\ &= -\tfrac{1}{2}\Theta_3^2 - \Theta_3\Theta_6^2 + 5\Theta_3\Theta_4\Theta_5 + 9\Theta_4\Theta_5\Theta_6^2 - \tfrac{9}{2}\Theta_4^2\Theta_5^2. \end{aligned}$$

We subtract this polynomial from the commutator, to get:

$$\begin{aligned} [\Theta_1, \Theta_2] - F_{1,2}^8(\boldsymbol{\Theta}) &= (-18 - 6x_3 + 7x_3x_6 + \tfrac{93}{4}x_4x_5 - \tfrac{69}{2}x_4x_5x_6 \\ &\quad + \tfrac{111}{8}x_4x_6x_{10}^2 + \tfrac{279}{16}x_4x_{10}^2 + 42x_6 + 7x_6x_8x_{10} - 53x_6x_9 \\ &\quad + 7x_6x_9^2 - 30x_6^2 + 21x_6^2x_9 + 6x_6^3 - 6x_8x_{10} + 30x_9 \\ &\quad - 6x_9^2) + I_\chi. \end{aligned}$$

We can check that this expression has Kazhdan degree 6, and so A_7 is empty and $F_{1,2}^7 = 0$. We now find $F_{1,2}^6$. We have

$$\{x^{\mathbf{a}} \mid \mathbf{a} \in A_6\} \;=\; \{x_3x_6,\ x_6^3,\ x_4x_5x_6\},$$

and so

$$\begin{aligned} F_{1,2}^6(\boldsymbol{\Theta}) &= \sum_{\mathbf{a}\in A_6} \mu_{\mathbf{a}}^{1,2,6}\Theta^{\mathbf{a}} \\ &= 7\Theta_3\Theta_6 + 6\Theta_6^3 - \tfrac{69}{2}\Theta_4\Theta_5\Theta_6. \end{aligned}$$

Subtracting these polynomials from the commutator we get:

$$[\Theta_1, \Theta_2] - \sum_{k=6}^{8} F^k_{1,2}(\mathbf{\Theta}) = (18 - 6x_3 + \tfrac{93}{4}x_4x_5 - \tfrac{165}{16}x_4x_{10}^2 + 42x_6 - 18x_6x_9 - 30x_6^2 - 6x_8x_{10} + 30x_9 - 6x_9^2) + I_\chi.$$

As before, we have that A_5 is empty so $F^5_{1,2} = 0$. We now find $F^4_{1,2}$. We have

$$\{x^{\mathbf{a}} \mid \mathbf{a} \in A_4\} = \{x_3,\ x_6^2,\ x_4x_5\},$$

and so

$$F^4_{1,2}(\mathbf{\Theta}) = \sum_{\mathbf{a} \in A_4} \mu^{1,2,4}_{\mathbf{a}} \Theta^{\mathbf{a}} = -6\Theta_3 - 30\Theta_6^2 + \tfrac{93}{4}\Theta_4\Theta_5.$$

Subtracting further:

$$[\Theta_1, \Theta_2] - \sum_{k=4}^{8} F^k_{1,2}(\mathbf{\Theta}) = (-18 + 42x_6) + I_\chi.$$

From this we easily see that $F^3_{1,2} = 0$, $F^2_{1,2}(\mathbf{\Theta}) = 42\Theta_6$, $F^1_{1,2} = 0$ and $F^0_{1,2}(\mathbf{\Theta}) = -18$; we can verify that

$$[\Theta_1, \Theta_2] - \sum_{k=0}^{8} F^k_{1,2}(\mathbf{\Theta}) \in I_\chi,$$

so we have

$$F_{1,2}(\mathbf{\Theta}) = -\tfrac{1}{2}\Theta_3^2 - \Theta_3\Theta_6^2 + 5\Theta_3\Theta_4\Theta_5 + 9\Theta_4\Theta_5\Theta_6^2 - \tfrac{9}{2}\Theta_4^2\Theta_5^2 + 7\Theta_3\Theta_6 + 6\Theta_6^3 - \tfrac{69}{2}\Theta_4\Theta_5\Theta_6 - 6\Theta_3 - 30\Theta_6^2 + \tfrac{93}{4}\Theta_4\Theta_5 + 42\Theta_6 - 18.$$

We now find $F_{1,3}$. We have

$$\begin{aligned}[\Theta_1, \Theta_3] &= (-3x_1 + 6x_1x_6 - 3x_2x_4 - \tfrac{3}{2}x_3x_4x_{10} + 3x_4x_5x_8 + 9x_4x_6x_7 \\ &\quad + \tfrac{27}{2}x_4x_6x_9x_{10} - 27x_4x_6x_{10} + 18x_4x_6^2x_{10} - 6x_4x_7 + 3x_4x_7x_9 \\ &\quad - \tfrac{3}{2}x_4x_8x_{10}^2 - \tfrac{15}{2}x_4x_9x_{10} + \tfrac{3}{2}x_4x_9^2x_{10} + 9x_4x_{10} + \tfrac{9}{2}x_4^2x_5x_{10} \\ &\quad - \tfrac{3}{2}x_4^2x_{10}^3 - 48x_6x_8 + 6x_6x_8x_9 + 12x_6^2x_8 + 48x_8 - 12x_8x_9) + I_\chi.\end{aligned}$$

We have $n_1 + n_3 + 2 = 7$, and

$$F_{1,3}^7(\mathbf{\Theta}) = -3\Theta_2\Theta_4 + 6\Theta_1\Theta_6.$$

We subtract this polynomial from the commutator, to get:

$$\begin{aligned}[\Theta_1, \Theta_3] - F_{1,3}^7(\mathbf{\Theta}) &= (-3x_1 - 9x_4x_6x_{10} - 3x_4x_7 - 6x_4x_9x_{10} + 12x_4x_{10} \\ &\quad -6x_6x_8 + 12x_8 - 3x_8x_9) + I_\chi.\end{aligned}$$

We see that $F_{1,3}^6 = 0$ and $F_{1,3}^5(\mathbf{\Theta}) = -3\Theta_1$, and

$$[\Theta_1, \Theta_3] - F_{1,3}^7(\mathbf{\Theta}) - F_{1,3}^5(\mathbf{\Theta}) \in I_\chi.$$

Thus $F_{1,3}(\mathbf{\Theta}) = -3\Theta_2\Theta_4 + 6\Theta_1\Theta_6 - 3\Theta_1$. We now find $F_{2,3}$. We have

$$\begin{aligned}[\Theta_2, \Theta_3] &= (-3x_1x_5 + \tfrac{3}{4}x_1x_{10}^2 + 12x_2 - 6x_2x_6 - 3x_3x_6x_{10} + 3x_3x_{10} \\ &\quad -3x_4x_5x_7 - 6x_4x_5x_9x_{10} + 3x_4x_5x_{10} - \tfrac{3}{4}x_4x_6x_{10}^3 + \tfrac{3}{4}x_4x_7x_{10}^2 \\ &\quad + \tfrac{3}{2}x_4x_9x_{10}^3 - 6x_4x_{10}^3 - 3x_5x_8 - 3x_5x_8x_9 - 12x_6x_7 + 6x_6x_7x_9 \\ &\quad - \tfrac{3}{2}x_6x_8x_{10}^2 - 18x_6x_9x_{10} + 3x_6x_9^2x_{10} + 24x_6x_{10} + 6x_6^2x_7 \\ &\quad + 3x_6^2x_9x_{10} - 6x_6^2x_{10} + 9x_7 - 3x_7x_9 + \tfrac{3}{4}x_8x_9x_{10}^2 + 15x_9x_{10} \\ &\quad - 3x_9^2x_{10} - 18x_{10}) + I_\chi,\end{aligned}$$

and so $F^7_{2,3}(\boldsymbol{\Theta}) = -3\Theta_1\Theta_5 - 6\Theta_2\Theta_6$. Subtracting, we get

$$\begin{aligned}[\Theta_2, \Theta_3] - F^7_{2,3}(\boldsymbol{\Theta}) &= (12x_2 + 6x_3x_{10} - 18x_4x_5x_{10} + 6x_4x_{10}^3 - 12x_5x_8 \\ &\quad -12x_6x_7 - 6x_6x_9x_{10} + 12x_6x_{10} + 24x_7 - 12x_7x_9 \\ &\quad +6x_8x_{10}^2 + 30x_9x_{10} - 6x_9^2x_{10} - 36x_{10}) + I_\chi \\ &= 12\Theta_2,\end{aligned}$$

so $F_{2,3}(\boldsymbol{\Theta}) = -3\Theta_1\Theta_5 - 6\Theta_2\Theta_6 + 12\Theta_2$. We have $[\Theta_1, \Theta_4] = 0$, so $F_{1,4}(\boldsymbol{\Theta}) = 0$. Next,

$$\begin{aligned}[\Theta_2, \Theta_4] &= (x_1 + 3x_4x_6x_{10} + x_4x_7 + 2x_4x_9x_{10} - 4x_4x_{10} + 2x_6x_8 - 4x_8 \\ &\quad +x_8x_9) + I_\chi \\ &= \Theta_1,\end{aligned}$$

so $F_{2,4}(\boldsymbol{\Theta}) = \Theta_1$. Next, we have

$$[\Theta_3, \Theta_4] = \tfrac{15}{2}x_4 - 9x_4x_6 + I_\chi,$$

so $F^4_{3,4}(\boldsymbol{\Theta}) = -9\Theta_4\Theta_6$, and

$$\begin{aligned}[\Theta_3, \Theta_4] - F^4_{3,4}(\boldsymbol{\Theta}) &= \tfrac{15}{2}x_4 + I_\chi \\ &= \tfrac{15}{4}\Theta_4,\end{aligned}$$

so $F_{3,4}(\boldsymbol{\Theta}) = -9\Theta_4\Theta_6 + \frac{15}{2}\Theta_4$. We also have

$$\begin{aligned}[\Theta_1, \Theta_5] &= (x_2 + \tfrac{1}{2}x_3x_{10} - \tfrac{3}{2}x_4x_5x_{10} + \tfrac{1}{2}x_4x_{10}^3 - x_5x_8 - x_6x_7 - \tfrac{1}{2}x_6x_9x_{10} \\ &\quad +x_6x_{10} + 2x_7 - x_7x_9 + \tfrac{1}{2}x_8x_{10}^2 + \tfrac{5}{2}x_9x_{10} - \tfrac{1}{2}x_9^2x_{10} - 3x_{10}) + I_\chi \\ &= \Theta_2,\end{aligned}$$

and we get $F_{1,5}(\boldsymbol{\Theta}) = \Theta_2$. We have $[\Theta_2, \Theta_5] = 0$, so $F_{2,5}(\boldsymbol{\Theta}) = 0$. Next, for

$F_{3,5}$ we have

$$[\Theta_3, \Theta_5] = (-\tfrac{51}{2}x_5 + 9x_5x_6 - \tfrac{9}{4}x_6x_{10}^2 + \tfrac{15}{8}x_{10}^2) + I_\chi,$$

so $F_{3,5}^4(\mathbf{\Theta}) = 9\Theta_5\Theta_6$. Subtracting, we get

$$\begin{aligned}[\Theta_3, \Theta_5] - F_{3,5}^4(\mathbf{\Theta}) &= (-\tfrac{51}{2}x_5 + \tfrac{51}{8}x_{10}^2) + I_\chi \\ &= -\tfrac{51}{2}\Theta_5,\end{aligned}$$

which gives $F_{3,5}(\mathbf{\Theta}) = 9\Theta_5\Theta_6 - \frac{51}{2}\Theta_5$. Next, we have

$$[\Theta_4, \Theta_5] = (x_6 + \tfrac{1}{2}) + I_\chi,$$

so $F_{4,5}(\mathbf{\Theta}) = \Theta_6 + \frac{1}{2}$. We have

$$\begin{aligned}[\Theta_1, \Theta_6] &= (-x_1 - 3x_4x_6x_{10} - x_4x_7 - 2x_4x_9x_{10} + 4x_4x_{10} - 2x_6x_8 + 4x_8 \\ &\quad -x_8x_9) + I_\chi \\ &= -\Theta_1,\end{aligned}$$

and we have $F_{1,6}(\mathbf{\Theta}) = -\Theta_1$. Next,

$$\begin{aligned}[\Theta_2, \Theta_6] &= (x_2 + \tfrac{1}{2}x_3x_{10} - \tfrac{3}{2}x_4x_5x_{10} + \tfrac{1}{2}x_4x_{10}^3 - x_5x_8 - x_6x_7 - \tfrac{1}{2}x_6x_9x_{10} \\ &\quad +x_6x_{10} + 2x_7 - x_7x_9 + \tfrac{1}{2}x_8x_{10}^2 + \tfrac{5}{2}x_9x_{10} - \tfrac{1}{2}x_9^2x_{10} - 3x_{10}) + I_\chi \\ &= \Theta_2,\end{aligned}$$

and $F_{2,6}(\mathbf{\Theta}) = \Theta_2$. We have $[\Theta_3, \Theta_6] = 0$, so $F_{3,6}(\mathbf{\Theta}) = 0$. Next, $[\Theta_4, \Theta_6] = -2x_4 + I_\chi = -2\Theta_4$, so $F_{4,6}(\mathbf{\Theta}) = -2\Theta_4$. Finally, we have $[\Theta_5, \Theta_6] = (2x_5 - \frac{1}{2}x_{10}^2) + I_\chi$, so $F_{5,6}(\mathbf{\Theta}) = 2\Theta_5$, completing the presentation for $U(\mathfrak{g}, e)$.

5

One-dimensional representations of $U(\mathfrak{g}, e)$

In [P07ii, Corollary 1.1] Premet proved the existence of finite-dimensional representations for $U(\mathfrak{g}, e)$, but the existence of 1-dimensional representations (i.e. algebra homomorphisms $U(\mathfrak{g}, e) \to \mathbb{C}$) remains open. This is equivalent to Conjecture 3.1(1) in [P07i], which conjectures the existence of an ideal of codimension 1 in $U(\mathfrak{g}, e)$. It was proved by Losev [L10, Theorem 1.2.3(1)] that in the case that when $\mathfrak{g}$ is classical $U(\mathfrak{g}, e)$ always admits a 1-dimensional representation.

Further progress was made towards a proof of the conjecture by Premet in [P08, Theorem 1.1]. This states that the following condition is sufficient for $U(\mathfrak{g}, e)$ to admit a 1-dimensional representation. Let $\mathcal{O} \subset \mathfrak{g}$ be the nilpotent orbit with $e \in \mathcal{O}$. If there is a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with a nilpotent orbit $\mathcal{O}_0 \subset \mathfrak{l}$ such that $\mathcal{O}$ is induced from $\mathcal{O}_0$ and that for $e_0 \in \mathcal{O}_0$, the finite W-algebra $U([\mathfrak{l}, \mathfrak{l}], e_0)$ admits a 1-dimensional representation, then $U(\mathfrak{g}, e)$ also admits a 1-dimensional representation.

These two results reduce the conjecture to the finite number of cases where $\mathfrak{g}$ is of exceptional type and e lies in a rigid orbit of $\mathfrak{g}$. These orbits are given in [S82] and listed in Section 2.3.

5.1 Using a presentation

Given a presentation of $U(\mathfrak{g}, e)$ as in Chapter 4, with generators $\Theta_1, \ldots, \Theta_d$ for $d = \dim(\mathfrak{g}^e)$ and relations $F_{ij}(\mathbf{\Theta}) = [\Theta_i, \Theta_j]$ for $1 \leq i < j \leq d$, it is straightforward to determine the 1-dimensional representations. A representation $\rho : U(\mathfrak{g}, e) \to \mathbb{C}$ is determined by the values $\rho(\Theta_i)$ taken at each generator Θ_i. Given $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$, the map $\rho : U(\mathfrak{g}, e) \to \mathbb{C}$ is defined by setting $\rho(\Theta_i) = \alpha_i$ and extending to sums, scalar multiplication and products $\Theta_{i_1} \cdots \Theta_{i_m}$ for $1 \leq i_1 \leq \cdots \leq i_m \leq d$. This defines a 1-dimensional representation of $U(\mathfrak{g}, e)$ if and only if $F_{ij}(\alpha_1, \ldots, \alpha_d) = 0$ for all $1 \leq i < j \leq d$. So the question of the existence of a 1-dimensional representation of $U(\mathfrak{g}, e)$ is answered by solving a set of rational polynomial equations.

Example 5.1.1. We return to the case where $\mathfrak{g}$ is the simple Lie algebra of type G_2, and $e \in \mathfrak{g}$ is a short root vector, as in Examples 4.2.1, 4.3.1 . We recall the 12 non-zero values of F_{ij}:

$$
\begin{aligned}
F_{1,2}(\mathbf{\Theta}) &= 5\Theta_3\Theta_4\Theta_5 - \tfrac{1}{2}\Theta_3^2 - \Theta_3\Theta_6^2 + 9\Theta_4\Theta_5\Theta_6^2 - \tfrac{9}{2}\Theta_4^2\Theta_5^2 + 7\Theta_3\Theta_6 \\
&\quad - \tfrac{69}{2}\Theta_4\Theta_5\Theta_6 + 6\Theta_6^3 - 6\Theta_3 + \tfrac{93}{4}\Theta_4\Theta_5 - 30\Theta_6^2 + 42\Theta_6 - 18 \\
F_{1,3}(\mathbf{\Theta}) &= 6\Theta_1\Theta_6 - 3\Theta_2\Theta_4 - 3\Theta_1 \\
F_{1,5}(\mathbf{\Theta}) &= \Theta_2 \\
F_{1,6}(\mathbf{\Theta}) &= -\Theta_1 \\
F_{2,3}(\mathbf{\Theta}) &= -3\Theta_1\Theta_5 - 6\Theta_2\Theta_6 + 12\Theta_2 \\
F_{2,4}(\mathbf{\Theta}) &= \Theta_1 \\
F_{2,6}(\mathbf{\Theta}) &= \Theta_2 \\
F_{3,4}(\mathbf{\Theta}) &= -9\Theta_4\Theta_6 + \tfrac{15}{2}\Theta_4 \\
F_{3,5}(\mathbf{\Theta}) &= 9\Theta_5\Theta_6 - \tfrac{51}{2}\Theta_5 \\
F_{4,5}(\mathbf{\Theta}) &= \tfrac{1}{2} + \Theta_6 \\
F_{4,6}(\mathbf{\Theta}) &= -2\Theta_4 \\
F_{5,6}(\mathbf{\Theta}) &= 2\Theta_5.
\end{aligned}
$$

We immediately see that for a 1-dimensional representation ρ of $U(\mathfrak{g}, e)$ given by $\rho(\Theta_i) = \alpha_i$ we must have $\alpha_i = 0$ for $i = 1, 2, 4, 5$ and $\alpha_6 = -\frac{1}{2}$. Substituting into the above, the only remaining non-zero polynomial is $F_{1,2}$, and we have

$$-\tfrac{1}{2}\alpha_3^2 - \tfrac{39}{4}\alpha_3 - \tfrac{189}{4} \quad = \quad 0.$$

Solving, we get $\alpha_3 = -9$ or $\alpha_3 = -\frac{21}{2}$. We therefore have precisely two distinct 1-dimensional representations of $U(\mathfrak{g}, e)$, given by $(\Theta_1, \ldots, \Theta_6) \mapsto (0, 0, -9, 0, 0, -\frac{1}{2})$ and $(\Theta_1, \ldots, \Theta_6) \mapsto (0, 0, -\frac{21}{2}, 0, 0, -\frac{1}{2})$ respectively.

5.2 Reduced enveloping algebras

We now consider a connection between finite W-algebras and the representation theory of modular Lie algebras.

Definition 5.2.1. [J62, Section V.7] A *restricted Lie algebra* of characteristic $p > 0$ is a Lie algebra $\mathfrak{g}_k$ over a field k of characteristic p with a map $a \mapsto a^{[p]}$ such that for $a, b \in \mathfrak{g}_k$ and $\alpha \in k$:

1. $(\alpha a)^{[p]} = \alpha^p a^{[p]}$,

2. $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$, where $is_i(a, b)$ is the coefficient of λ^{i-1} in $a(\mathrm{ad}(\lambda a + b))^{p-1}$, and

3. $[a, b^{[p]}] = a(\mathrm{ad}\, b)^p$.

Denote the universal enveloping algebra of a restricted Lie algebra $\mathfrak{g}_k$ by $U(\mathfrak{g}_k)$. For $\xi \in \mathfrak{g}_k^*$, let J_ξ denote the 2-sided ideal of $U(\mathfrak{g}_k)$ generated by all elements $x^p - x^{[p]} - \xi(x)^p$ for $x \in \mathfrak{g}_k$. We define the *reduced enveloping algebra* $U_\xi(\mathfrak{g}_k)$ to be the quotient $U(\mathfrak{g}_k)/J_\xi$.

Let G and $\mathfrak{g}$ be as in Chapter 3, with a nilpotent element $e \in \mathfrak{g}$, and let $\mathfrak{g}_\mathbb{Z}$ denote the Chevalley $\mathbb{Z}$-form of $\mathfrak{g}$. Let $\Bbbk$ be the algebraic closure of the finite field $\mathbb{F}_p$ for $p \gg 0$ (below we consider more precisely what restriction we wish to place on p). Let $\mathfrak{g}_\Bbbk = \mathfrak{g}_\mathbb{Z} \otimes \Bbbk$, let $G_\Bbbk$ be the simple simply-connected algebraic group with $\mathrm{Lie}(G_{\Bbbk}) = \mathfrak{g}_{\Bbbk}$, and let $e_{\Bbbk} = e \otimes 1 \in \mathfrak{g}_{\Bbbk}$. We use the same notation $(\cdot\,,\cdot)$ for the bilinear form on $\mathfrak{g}$ as for its analogue on $\mathfrak{g}_{\Bbbk}^*$, and let $\chi \in \mathfrak{g}_{\Bbbk}^*$ denote the (rescaled) map from $\mathfrak{g}_{\Bbbk}^*$ to $\Bbbk$ corresponding to $e_{\Bbbk}$. Write $d_\chi = \frac{1}{2}\dim(G_{\Bbbk} \cdot \chi)$ for half of the dimension of the orbit of χ under the coadjoint action of $G_{\Bbbk}$ on $\mathfrak{g}_{\Bbbk}^*$.

In this context, the Kac–Weisfeiler conjecture [KW71], proved by Premet [P95] states that any irreducible representation of the reduced enveloping algebra $U_\chi(\mathfrak{g}_{\Bbbk})$ associated to $\chi \in \mathfrak{g}_{\Bbbk}^*$ has dimension divisible by p^{d_χ}. Given this, it is natural to ask whether $U_\chi(\mathfrak{g}_{\Bbbk})$ has representations with dimension *equal* to p^{d_χ}. A sufficient condition for this in terms of the finite W-algebra $U(\mathfrak{g}, e)$ is given by Premet in [P08, Theorem 1.4], which we state here.

Theorem 5.2.2. *If the finite W-algebra $U(\mathfrak{g}, e)$ admits a 1-dimensional representation then for an algebraically closed field $\Bbbk$ of sufficiently large characteristic p, the reduced enveloping algebra $U_\chi(\mathfrak{g}_{\Bbbk})$ has a simple module of dimension p^{d_χ}.*

Combined with the results of Premet [P08] and Losev [L10], this means that the existence of a 1-dimensional representation for each finite W-algebra $U(\mathfrak{g}, e)$ where $\mathfrak{g}$ is of exceptional type and e lies in a rigid nilpotent orbit of $\mathfrak{g}$ gives the existence of a representation of dimension p^{d_χ} for the associated reduced enveloping algebra $U_\chi(\mathfrak{g}_{\Bbbk})$ provided the characteristic of $\Bbbk$ is sufficiently large.

Premet's proof of Theorem 5.2.2 uses a modular analogue of $U(\mathfrak{g}, e)$, defined over some $\Bbbk = \overline{\mathbb{F}}_p$ for $p \gg 0$. For this we need to repeat the construction of $U(\mathfrak{g}, e)$ over $\mathbb{Z}[D^{-1}]$, where D is a sufficiently large integer, to get $U(\mathfrak{g}_{\mathbb{Z}[D^{-1}]}, e)$ where we have chosen our $\mathfrak{sl}_2$-triple from $\mathfrak{g}_{\mathbb{Z}}$. We first require that the bad primes for $\mathfrak{g}$ are invertible in $\mathbb{Z}[D^{-1}]$, so our first candidate for D is the product of those bad primes. In order that the rescaled Killing form $(\cdot\,,\cdot)$ is defined over $\mathbb{Z}[D^{-1}]$ we may need to increase D to ensure that $\kappa(e, f)$ is invertible in $\mathbb{Z}[D^{-1}]$. For example, if $\mathfrak{g}$ is the simple Lie algebra of type E_7, and e lies in the orbit with Bala–Carter label $(A_3 + A_1)'$, then $\kappa(e, f) = 396$, which has 11 as a factor, which is a good prime for $\mathfrak{g}$, and we must increase

D accordingly. When we choose the basis, we require that the new structure constants are invertible in $\mathbb{Z}[D^{-1}]$, so we may need to increase D further at this point. We also require that the coefficients $\lambda_{\mathbf{a}}$ in the expressions for the generators $\Theta_1, \ldots, \Theta_d$ of $U(\mathfrak{g}, e)$ lie in $\mathbb{Z}[D^{-1}]$, so we may need to increase D again. Similarly, we require that the commutator relations $[\Theta_i, \Theta_j] = F_{ij}(\mathbf{\Theta})$ have all coefficients in $\mathbb{Z}[D^{-1}]$, so it is possible that D may need to be increased here also.

To establish a lower bound on the characteristic p of $\Bbbk$ for Theorem 5.2.2 we require that D does not divide p, but we must also consider each of the finite W-algebras $U([\mathfrak{l}, \mathfrak{l}], e_0)$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $e_0 \in [\mathfrak{l}, \mathfrak{l}]$ is rigid nilpotent. The Levi subalgebras we need to consider are classical, and the nilpotent orbits can be listed using a partition classification found in [CM93, Chapter 5.1] and $\mathfrak{sl}_2$-triples and orbit representatives can be found using the methods of [CM93, Chapter 5.2]. We require that there is a presentation of $U([\mathfrak{l}, \mathfrak{l}], e_0)$ for which p does not divide any denominator, and that p does not divide the (integer) value $\kappa_{\mathfrak{l}}(e_0, f_0)$ where $\kappa_{\mathfrak{l}}$ is the Killing form on $\mathfrak{l}$ and $(e_0, [e_0, f_0], f_0)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{l}$.

Example 5.2.3. We return to the case where $\mathfrak{g}$ is the simple Lie algebra of type G_2, and $e \in \mathfrak{g}$ is a short root vector in the orbit with Bala–Carter label $\tilde{A}_1$. The bad primes for $\mathfrak{g}$ are 2 and 3, and $\kappa(e, f) = 24$. We have the generators $\Theta_1, \ldots, \Theta_6$ from Example 4.2.1 and the relations F_{ij} from Example 4.3.1 and we see that the denominators which occur are 2 and 4. We can therefore define $U(\mathfrak{g}_{\mathbb{Z}[6^{-1}]}, e)$, and we can take any prime $p > 3$ for Theorem 5.2.2.

We conclude that for $\Bbbk = \overline{\mathbb{F}}_p$ where $p > 3$, the reduced enveloping algebra associated to a short root vector $e_{\Bbbk}$ in $\mathfrak{g}_{\Bbbk}$ has a simple module of dimension p^4, where $4 = \frac{8}{2}$ is half the dimension of the coadjoint orbit of χ in $\mathfrak{g}_{\Bbbk}^*$.

5.3 Removing relations

The algorithms of Chapter 4 give a presentation of $U(\mathfrak{g}, e)$ involving $\binom{d}{2}$ relations, which allows us to determine the 1-dimensional representations. In this section, we show that a presentation of $U(\mathfrak{g}, e)$ can be given by fewer relations, and that the task of determining the 1-dimensional representations requires consideration of even fewer relations, thus significantly reducing the amount of calculation required. The results given here appear in [GRU09, Section 3].

Lemma 5.3.1. *Suppose we have a basis for $\mathfrak{g}$ as in Section 3.2 where $\mathfrak{g}^e$ is generated by $x_1, \dots, x_b$ for some $b \leq d$. Then $U(\mathfrak{g}, e)$ is generated by $\Theta_1, \dots, \Theta_b$.*

Proof. We may assume that the basis is chosen so that $n_{b+1} \leq \cdots \leq n_d$ and for $k = b-1, \dots, d$ that x_k lies in the span of the elements $x_1, \dots, x_{k-1}$ and $[x_i, x_j]$ for $1 \leq i, j \leq k-1$. Let W denote the subalgebra of $U(\mathfrak{g}, e)$ generated by $\Theta_1, \dots, \Theta_b$. It is sufficient to show that Θ_k lies in W for each $k = b+1, \dots, d$.

Suppose we have shown $\Theta_{b+1}, \dots, \Theta_{k-1} \in W$. Then from our condition on the n_i we have that $F_{n_k+1}U(\mathfrak{g}, e) \subseteq W$. We can express the basis element $x_k \in \mathfrak{g}^e$ in terms of $x_1, \dots, x_{k-1}$:

$$x_k = \sum_{i,j<k} \nu_{ij}^k [x_i, x_j] + \sum_{i<k} \rho_i^k x_i,$$

for coefficients $\nu_{ij}^k, \rho_i^k \in \mathbb{Q}$. In the case that we have non-zero values for some ρ_i^k, we can change the basis so that x_k is replaced by $x_k - \sum_{i<k} \rho_i^k x_i$, and we get $x_k = \sum_{i,j<k} \nu_{ij}^k [x_i, x_j]$. With this change, the new x_k is still a weight vector for $\operatorname{ad} h$ and the action of $\mathfrak{t}^e$.

From Theorem 3.2.1(4) we can write

$$\sum_{i,j<k} \nu_{ij}^k [\Theta_i, \Theta_j] = \Theta_k + G_k(\Theta_1, \dots, \Theta_{k-1}) + H_k(\Theta_1, \dots, \Theta_{k-1})$$

where $G_k(\Theta_1, \ldots, \Theta_{k-1}) = \sum_{i,j<k} q_{ij}(\Theta_1, \ldots, \Theta_{k-1})$ is a polynomial with coefficients in $\mathbb{Q}$ which lies in $F_{n_k+2}U(\mathfrak{g}, e)$ with zero constant and linear terms, and H_k is a polynomial with coefficients in $\mathbb{Q}$ such that $H_k(\Theta_1, \ldots, \Theta_{k-1}) \in F_{n_k}U(\mathfrak{g}, e)$. We therefore have $H_k(\Theta_1, \ldots, \Theta_{k-1}) \in W$. We can write

$$G_k(\Theta_1, \ldots, \Theta_{k-1}) = \sum_{\substack{|\mathbf{a}| \geq 2 \\ 2 \leq |\mathbf{a}|_e < n_k+1}} \mu_{\mathbf{a}}^k \Theta_1^{a_1} \cdots \Theta_{k-1}^{a_{k-1}}$$

so we have that $G_k(\Theta_1, \ldots, \Theta_{k-1})$ is a sum of products of elements of $F_{n_k+1}U(\mathfrak{g}, e)$ along with $\Theta_1, \ldots, \Theta_b$, and so $G_k(\Theta_1, \ldots, \Theta_{k-1}) \in W$. We also know that each commutator $[\Theta_i, \Theta_j]$ lies in W, so $\Theta_k \in W$ as required. □

Lemma 5.3.1 shows that not all generators $\Theta_1, \ldots, \Theta_d$ are necessary in order to give a presentation of $U(\mathfrak{g}, e)$, but if we just have $\Theta_1, \ldots, \Theta_b$ for $b < d$ then we no longer have a set of PBW generators. Incidentally, we get from the proof of Lemma 5.3.1 an algorithm for finding the generators $\Theta_{b+1}, \ldots, \Theta_d$ which does not require us to determine the large list of monomials which may occur in the expression for Θ_k given in Theorem 3.2.1(1).

Theorem 5.3.2. *Suppose we have a generating set $x_1, \ldots, x_b$ of $\mathfrak{g}^e$ as in Lemma 5.3.1. Then $U(\mathfrak{g}, e)$ is generated by $\Theta_1, \ldots, \Theta_d$ subject only to the relations*

$$[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \ldots, \Theta_d) = F_{ij}(\mathbf{\Theta}),$$

for $i = 1, \ldots, b$ and $j = 1, \ldots, d$, where F_{ij} is a polynomial with coefficients in $\mathbb{Q}$, and $F_{ij}(\mathbf{\Theta}) \in F_{n_i+n_j+2}U(\mathfrak{g}, e)$.

Proof. The case that $b = d$ is immediate from [P07ii, Lemma 4.1] and Theorem 3.2.1(4). For $b < d$ it is sufficient to show that the polynomials $F_{kl}(\mathbf{\Theta})$ for $k, l > b$ are determined by what we shall refer to as *known polynomials* i.e. the polynomials $F_{ij}(\mathbf{\Theta})$ where $1 \leq i \leq b$ and $1 \leq j \leq d$.

We need to show that for $k, l > b$, we can calculate $[\Theta_k, \Theta_l] = F_{kl}(\mathbf{\Theta})$ in terms of the known polynomials F_{ij}. As in the proof of Lemma 5.3.1 we may assume that $n_{b+1} \leq \cdots \leq n_d$ and for $m = b-1, \ldots, d$ that x_m lies in the span of the elements $[x_i, x_j]$ for $1 \leq i, j \leq m-1$. From the anti-symmetry of the relations, we may assume $k < l$. We use induction on the order of the F_{ij} given first by $n_k + n_l$, then by k, and then by l. Assume that we have each $F_{k'l'}$ if either: $n_{k'} + n_{l'} < n_k + n_l$; or if $n_{k'} + n_{l'} = n_k + n_l$ and $k' < k$; or if $n_{k'} + n_{l'} = n_k + n_l$, $k' = k$ and $l' < l$. With polynomials $G_k(\Theta_1, \ldots, \Theta_{k-1})$ and $H_k(\Theta_1, \ldots, \Theta_{k-1})$ and rational coefficients ν_{ij}^k as in the proof of Lemma 5.3.1, we can write

$$\Theta_k = \sum_{i,j<k} \nu_{ij}^k [\Theta_i, \Theta_j] - G_k(\boldsymbol{\Theta}) - H_k(\boldsymbol{\Theta}).$$

For $k, l > b$ we have

$$[\Theta_k, \Theta_l] = \sum_{i,j<k} \nu_{ij}^k [[\Theta_i, \Theta_j], \Theta_l] - [G_k(\boldsymbol{\Theta}), \Theta_l] - [H_k(\boldsymbol{\Theta}), \Theta_l].$$

We show that terms on the right hand side can be expressed as polynomials in $\Theta_1, \ldots, \Theta_d$ using the known polynomials. We know $G_k(\boldsymbol{\Theta}) \in F_{n_k+2} U(\mathfrak{g}, e)$ with no constant or linear terms, and that $\Theta_l \in F_{n_l+2} U(\mathfrak{g}, e)$, so we can apply the Leibniz rule to $[G_k(\boldsymbol{\Theta}), \Theta_l]$, expressing it in terms of polynomials $F_{ij}(\boldsymbol{\Theta})$ polynomials $F_{ij}(\boldsymbol{\Theta})$ given by the inductive hypothesis. We can therefore express $[G_k(\boldsymbol{\Theta}), \Theta_l]$ in terms of the known polynomials. Similarly, $H_k(\boldsymbol{\Theta}) \in F_{n_k} U(\mathfrak{g}, e)$ so we can apply the Leibniz rule and the inductive hypothesis to express $[H_k(\boldsymbol{\Theta}), \Theta_l]$ in terms of the known polynomials as well.

Finally, we consider a term of the form $[[\Theta_i, \Theta_j], \Theta_l]$. From the Jacobi identity we have

$$\begin{aligned} [[\Theta_i, \Theta_j], \Theta_l] &= [\Theta_i, [\Theta_j, \Theta_l]] + [[\Theta_i, \Theta_l], \Theta_j] \\ &= [\Theta_i, F_{jl}(\boldsymbol{\Theta})] + [F_{il}(\boldsymbol{\Theta}), \Theta_j]. \end{aligned}$$

We know the polynomials $F_{il}(\boldsymbol{\Theta})$ and $F_{jl}(\boldsymbol{\Theta})$ from the inductive hypothesis, and by applying the Leibniz rule and the inductive hypothesis again we express $[[\Theta_i, \Theta_j], \Theta_l]$, and hence $[\Theta_k, \Theta_l] = F_{kl}(\boldsymbol{\Theta})$ in terms of the known polynomials. □

In the following example, and also in the others for which the explicit calculations are given below, the order of the basis is not in the form of Lemma 5.3.1 beginning with the generating set for $\mathfrak{g}^e$, but instead is in decreasing order of the Dynkin degree, but this does not impact upon Lemma 5.3.1 and Theorem 5.3.2, except to simplify the notation used.

Example 5.3.3. Returning to the example of $U(\mathfrak{g}, e)$ where $\mathfrak{g}$ is of type G_2 and e is a short root vector, with basis, generators and relations as in Example 4.3.1. We have that $\mathfrak{g}^e$ has dimension $d = 6$, and is generated by the elements x_2, x_3, x_4, x_5 where $x_1 = [x_2, x_4]$ and $x_6 = [x_4, x_5]$. Note that for most cases, the minimal number of generators is significantly less than the dimension of $\mathfrak{g}^e$. To illustrate the above, we find $F_{1,6}(\boldsymbol{\Theta})$ from the other relations. We have

$$\begin{aligned} [\Theta_1, \Theta_6] &= [[\Theta_2, \Theta_4], \Theta_6] \\ &= [\Theta_2, F_{4,6}(\boldsymbol{\Theta})] + [F_{2,6}(\boldsymbol{\Theta}), \Theta_4] \\ &= [\Theta_2, -2\Theta_4] + [\Theta_2, \Theta_4] \\ &= -\Theta_1, \end{aligned}$$

which agrees with our original calculation in Example 4.3.1.

Combining this with Section 5.1, we see that the 1-dimensional representations of $U(\mathfrak{g}, e)$ are determined by the relations given by the polynomials F_{ij} for $1 \leq i \leq b$ and $1 \leq j \leq d$ where $\mathfrak{g}^e$ is generated by $x_1, \ldots, x_b$. From [P07i, Lemma 2.4], we have an embedding of $\mathfrak{t}^e$ in $U(\mathfrak{g}, e)$; we identify $\mathfrak{t}^e$ with its image under this embedding, and we have an adjoint action of $\mathfrak{t}^e$ on $U(\mathfrak{g}, e)$ for which Θ_i in $U(\mathfrak{g}, e)$ has $\mathfrak{t}^e$-weight β_i. The following results show that in order to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, it is sufficient to calculate a much smaller subset of the relations F_{ij}.

Lemma 5.3.4. *Let $\rho : U(\mathfrak{g}, e) \to \mathbb{C}$ be a 1-dimensional representation of $U(\mathfrak{g}, e)$. Then $\rho(\Theta_i) = 0$ for all i such that $\beta_i \neq 0$.*

Proof. Let i be such that $\beta_i \neq 0$. Then we can choose $t \in \mathfrak{t}^e$ such that $\beta_i(t) \neq 0$. So we have

$$\beta_i(t)\rho(\Theta_i) = \rho(\beta_i(t)\Theta_i) = \rho([t, \Theta_i]) = [\rho(t), \rho(\Theta_i)] = 0,$$

and $\rho(\Theta_i) = 0$ as required. □

The following theorem shows that all 1-dimensional representations of $U(\mathfrak{g}, e)$ can be determined by calculating only the commutator relations $[\Theta_i, \Theta_j]$ when $\beta_i + \beta_j = 0$. For $i = 1, \ldots, d$ write

$$\delta_i = \begin{cases} 1 & \text{for} \quad \beta_i = 0 \\ 0 & \text{for} \quad \beta_i \neq 0 \end{cases}$$

and write $\overline{F}_{ij}(T_k \mid \beta_k = 0) = F_{ij}(\delta_1 T_1, \ldots, \delta_d T_d)$ for the evaluation of the polynomial F_{ij} at $T_i = 0$ for $\beta_i \neq 0$.

Theorem 5.3.5. *The* 1*-dimensional representations of* $U(\mathfrak{g}, e)$ *are in bijective correspondence with solutions to the set of polynomial equations* $\overline{F}_{ij}(T_k \mid \beta_k \neq 0) = 0$ *for* $1 \leq i \leq b$ *and* $1 \leq j \leq d$ *and* $\beta_i + \beta_j = 0$*. The solution given by* $T_k = \alpha_k$ *for each* k *with* $\beta_k = 0$ *corresponds to the* 1*-dimensional representation* ρ *determined by setting each* $\rho(\Theta_k) = 0$ *for* k *with* $\beta_k \neq 0$ *and each* $\rho(\Theta_k) = \alpha_k$ *for* k *with* $\beta_k = 0$.

Proof. Given a representation $\rho : U(\mathfrak{g}, e) \to \mathbb{C}$, the values $\rho(\Theta_1), \ldots, \rho(\Theta_d)$ give a solution to all F_{ij}, and by Lemma 5.3.4 this also is a solution to the set of polynomials $\overline{F}_{ij}$ with $1 \leq i \leq b$ and $1 \leq j \leq d$.

Given a solution $(\alpha_k \mid \beta_k = 0)$ to the $\overline{F}_{ij}$, we need to show that there is no F_{ij} with $1 \leq i \leq b$ and $1 \leq j \leq d$ and $\beta_i + \beta_j \neq 0$ such that the evaluation of $F_{ij}(T_1, \ldots, T_d)$ at $T_k = \alpha_k$ for $\beta_k = 0$ and $T_k = \alpha_k = 0$ for $\beta_k \neq 0$ is non-zero. Suppose there are some i, j such that $F_{ij}(\alpha_1, \ldots, \alpha_d) \neq 0$. Write $F_{ij}(\mathbf{\Theta}) = \sum_{|\mathbf{a}|_e \leq n_i + n_j + 2} \lambda_{\mathbf{a}}^{ij} \Theta^{\mathbf{a}}$. Then there is some $\mathbf{a} = (a_1, \ldots, a_d)$

with $\lambda_{\mathbf{a}}^{ij}\Theta^{\mathbf{a}} \neq 0$ and

$$\lambda_{\mathbf{a}}^{ij} \prod_{a_k \neq 0} \alpha_k^{a_k} \neq 0.$$

But then each $\alpha_k^{a_k}$ in the product is non-zero, and so for each k with $a_k \neq 0$, we have $\beta_k = 0$. The polynomial $F_{ij}(\mathbf{\Theta})$ is homogeneous with respect to $\mathfrak{t}^e$-weight, so we must have $\beta_i + \beta_j = 0$, giving a contradiction. □

The above results greatly facilitate the task of finding the 1-dimensional representations for $U(\mathfrak{g}, e)$, however without a full presentation of $U(\mathfrak{g}, e)$ we cannot yet establish a bound on p for Theorem 5.2.2, as there may be further denominators appearing in the relations not calculated.

Example 5.3.6. Again we return to the case where $\mathfrak{g}$ is simple of type G_2 and e is a short root vector, and the generators and relations of $U(\mathfrak{g}, e)$ are as in Example 4.3.1. We have $(\beta_1, \ldots, \beta_6) = (1, -1, 0, 2, -2, 0)$, so for any 1-dimensional representation ρ we must have $\rho(\Theta_k) = 0$ for $k = 1, 2, 4, 5$, and to determine the 1-dimensional representations we need only calculate the relations $F_{1,2}(\mathbf{\Theta})$, $F_{3,6}(\mathbf{\Theta})$ and $F_{4,5}(\mathbf{\Theta})$. We have:

$$\begin{aligned}
\overline{F}_{1,2}(T_3, T_6) &= -\tfrac{1}{2}T_3 - T_3T_6^2 + 7T_3T_6 + 6T_6^3 - 6T_3 - 30T_6^2 + 42T_6 - 18 \\
\overline{F}_{3,6}(T_3, T_6) &= 0 \\
\overline{F}_{4,5}(T_3, T_6) &= \tfrac{1}{2} + T_6,
\end{aligned}$$

which give the same solutions as in Example 5.1.1.

6

Results for rigid nilpotent orbits in exceptional Lie algebras

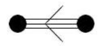
In this chapter we show that for $\mathfrak{g}$ of type G_2, F_4, E_6 or E_7, and e a rigid nilpotent element of $\mathfrak{g}$ the finite W-algebra $U(\mathfrak{g}, e)$ admits either 1 or 2 1-dimensional representations. In the case that $\mathfrak{g}$ is of type G_2, F_4 or E_6 we also give a lower bound on the characteristic p of the field $\Bbbk$ for which the reduced enveloping algebra $U_\chi(\mathfrak{g}_\Bbbk)$ admits a representation of dimension p^{d_χ}. We give for each rigid nilpotent orbit for $\mathfrak{g}$ of type G_2, F_4, E_6 or E_7 a basis according to Section 3.2, an $\mathfrak{sl}_2$-triple for that orbit and minimal generating sets for the subalgebras $\mathfrak{g}^e$ and $\mathfrak{m}$.

6.1 Type G_2

Here we calculate $U(\mathfrak{g}, e)$ for $\mathfrak{g}$ of type G_2 and e lying in each of the 4 non-zero nilpotent orbits (not just the rigid orbits). We summarize in Table 6.1 certain data for these orbits, including the number of 1-dimensional representations for $U(\mathfrak{g}, e)$ in each case and the primes p for which we cannot define $U(\mathfrak{g}_\Bbbk, e_\Bbbk)$ for $\Bbbk$ of characteristic p and hence we cannot apply Theorem 5.2.2

and conclude that $U_\chi(\mathfrak{g}_\Bbbk)$ admits a representation of dimension p^{d_χ}.

Table 6.1: Results for type G_2.

Orbit	Dynkin diagram 	$\kappa(e,f)$	$\dim(\mathfrak{g}^e)$	$\dim(\mathfrak{t}^e)$	# 1-dim reps	Bad primes for $U(\mathfrak{g}_\Bbbk, e_\Bbbk)$
A_1	0 1	8	8	1	1	$2,3$
$\tilde{A}_1$	1 0	24	6	1	2	$2,3$
$G_2(a_1)$	2 0	32	4	0	∞	$2,3$
G_2	2 2	224	2	0	∞	$2,3,7$

Below we give the full details of the presentation of each of the finite W-algebras associated to the 4 non-zero nilpotent orbits in $\mathfrak{g}$. Note that in order to make any conclusions about a lower bound for the characteristic of the field in Theorem 5.2.2, we must consider a presentation of the finite W-algebra associated to a rigid nilpotent element e_0 in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. In this case, the only Levi subalgebra to consider is of type A_1, which contains precisely 1 non-zero nilpotent orbit (which is both minimal and regular), and the associated finite W-algebra has a presentation where the only denominators to occur are powers of 2, which is a bad prime for $\mathfrak{g}$. Thus for rigid nilpotent $e \in \mathfrak{g}$ we can apply Theorem 5.2.2 for $p > 3$.

6.1.1 The orbit A_1

Here we consider the finite W-algebra associated to the minimal nilpotent orbit in $\mathfrak{g}$, with Bala–Carter label A_1, or equivalently, the orbit containing a long root vector. In Table 6.2 we give our choice of basis $x_1, \ldots, x_{14}$ of $\mathfrak{g}$ in terms of the inbuilt Chevalley basis $b_1, \ldots, b_{14}$ in GAP4 for the simple Lie algebra of type G_2. We take our $\mathfrak{sl}_2$-triple to be $(e,h,f) = (b_6, b_{13} +$

$2b_{14}, b_{12}) = (x_1, x_8 + 2x_9, x_{14})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_5,\ x_6,\ x_7\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{12},\ x_{13},\ x_{14}\}$. The subalgebra $\mathfrak{t}^e$ is spanned by x_8. We calculate $\kappa(e, f) = 8$.

Table 6.2: Basis for type G_2, orbit A_1.

				n_i	β_i
		x_1	b_6	2	0
		x_2	b_2	1	-3
		x_3	b_3	1	-1
	$\mathfrak{g}^e$	x_4	b_4	1	1
$\mathfrak{p}$		x_5	b_5	1	3
		x_6	b_1	0	2
		x_7	b_7	0	-2
		x_8	b_{13}	0	0
		x_9	b_{14}	0	0
		x_{10}	b_8	-1	3
		x_{11}	b_9	-1	1
		x_{12}	b_{10}	-1	-1
$\mathfrak{m}$		x_{13}	b_{11}	-1	-3
		x_{14}	b_{12}	-2	0

Following the algorithm of Section 4.2, we calculate the following generators for $U(\mathfrak{g}, e)$:

$$\begin{aligned}
\Theta_1 &= (x_1 + x_2x_{10} + \tfrac{1}{3}x_3x_{11} - \tfrac{1}{9}x_7x_{11}^2 + x_8x_9 - 3x_9 + x_9^2) + I_\chi \\
\Theta_2 &= x_2 + I_\chi \\
\Theta_3 &= (x_3 - \tfrac{2}{3}x_7x_{11}) + I_\chi \\
\Theta_4 &= (x_4 + x_7x_{10} + \tfrac{2}{3}x_8x_{11} + x_9x_{11} - \tfrac{5}{3}x_{11}) + I_\chi
\end{aligned}$$

$$\begin{aligned}
\Theta_5 &= (x_5 - \tfrac{1}{3}x_6x_{11} - x_8x_{10} - x_9x_{10} + 3x_{10} - \tfrac{2}{27}x_{11}^3) + I_\chi \\
\Theta_6 &= (x_6 + \tfrac{1}{3}x_{11}^2) + I_\chi \\
\Theta_7 &= x_7 + I_\chi \\
\Theta_8 &= x_8 + I_\chi.
\end{aligned}$$

And using the algorithm of Section 4.3 we calculate all commutators $[\Theta_i, \Theta_j]$ for $1 \leq i < j \leq 8$:

$$\begin{aligned}
[\Theta_1, \Theta_2] &= -8\Theta_2 + \tfrac{1}{3}\Theta_3\Theta_7 + 2\Theta_2\Theta_8 \\
[\Theta_1, \Theta_3] &= -3\Theta_3 + \tfrac{2}{3}\Theta_4\Theta_7 + \tfrac{2}{3}\Theta_3\Theta_8 + \Theta_2\Theta_6 \\
[\Theta_1, \Theta_4] &= \Theta_5\Theta_7 - \tfrac{2}{3}\Theta_4\Theta_8 + \tfrac{2}{3}\Theta_3\Theta_6 \\
[\Theta_1, \Theta_5] &= \Theta_5 - 2\Theta_5\Theta_8 + \tfrac{1}{3}\Theta_4\Theta_6 \\
[\Theta_1, \Theta_6] &= \tfrac{4}{3}\Theta_6 - \Theta_6\Theta_8 \\
[\Theta_1, \Theta_7] &= -\tfrac{10}{3}\Theta_7 + \Theta_7\Theta_8 \\
[\Theta_1, \Theta_8] &= 0 \\
[\Theta_2, \Theta_3] &= \tfrac{2}{3}\Theta_7^2 \\
[\Theta_2, \Theta_4] &= 2\Theta_7 - \tfrac{2}{3}\Theta_7\Theta_8 \\
[\Theta_2, \Theta_5] &= -\tfrac{2}{9} - \Theta_1 + \tfrac{1}{3}\Theta_6\Theta_7 \\
[\Theta_2, \Theta_6] &= \Theta_3 \\
[\Theta_2, \Theta_7] &= 0 \\
[\Theta_2, \Theta_8] &= 3\Theta_2 \\
[\Theta_3, \Theta_4] &= 2 - 2\Theta_8 + 3\Theta_1 + \tfrac{2}{3}\Theta_8^2 - \tfrac{7}{3}\Theta_6\Theta_7 \\
[\Theta_3, \Theta_5] &= -\tfrac{2}{3}\Theta_6 + \tfrac{2}{3}\Theta_6\Theta_8 \\
[\Theta_3, \Theta_6] &= 2\Theta_4 \\
[\Theta_3, \Theta_7] &= 3\Theta_2 \\
[\Theta_3, \Theta_8] &= \Theta_3 \\
[\Theta_4, \Theta_5] &= \tfrac{2}{3}\Theta_6^2 \\
[\Theta_4, \Theta_6] &= 3\Theta_5
\end{aligned}$$

$$\begin{aligned}
[\Theta_4, \Theta_7] &= 2\Theta_3 \\
[\Theta_4, \Theta_8] &= -\Theta_4 \\
[\Theta_5, \Theta_6] &= 0 \\
[\Theta_5, \Theta_7] &= \Theta_4 \\
[\Theta_5, \Theta_8] &= 3\Theta_5 \\
[\Theta_6, \Theta_7] &= -2 + \Theta_8 \\
[\Theta_6, \Theta_8] &= -2\Theta_6 \\
[\Theta_7, \Theta_8] &= 2\Theta_7.
\end{aligned}$$

Solving the associated set of polynomials we conclude that we have a unique 1-dimensional representation for $U(\mathfrak{g}, e)$, defined by

$$\begin{aligned}
\Theta_1 &\mapsto -\tfrac{2}{9} \\
\Theta_8 &\mapsto 2,
\end{aligned}$$

with $\Theta_i \mapsto 0$ for each other generator. Note that the existence of a 1-dimensional representation in this case was known from [P07i] (as our orbit is minimal), though to verify the details of the presentation given in [P07i, Theorem 6.1] using these methods we would need to make a different choice of basis (but still meeting the conditions of Section 3.2). We can observe that the only denominators occurring in this presentation are powers of 3, which is already excluded as it is a bad prime for $\mathfrak{g}$, so we may define $U(\mathfrak{g}_\Bbbk, e_\Bbbk)$ provided $\Bbbk$ has characteristic $p > 3$.

6.1.2 The orbit $\tilde{A}_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $\tilde{A}_1$, or equivalently, the orbit containing a short root vector. In Table 6.3 we give our choice of basis for $\mathfrak{g}$. The information given here is repeated from the examples in Chapters 4 and 5. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_4, 2b_{13} + 3b_{14}, b_{10}) = (x_3, 3x_6 + 2x_9, x_{12})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_2,\ x_3,\ x_4,\ x_5\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{11},\ x_{12},\ x_{14}\}$. The subalgebra $\mathfrak{t}^e$ is spanned by x_6. We calculate $\kappa(e,f) = 24$.

Table 6.3: Basis for type G_2, orbit $\tilde{A}_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_6	3	1
		x_2	b_5	3	−1
		x_3	b_4	2	0
		x_4	b_2	0	2
		x_5	b_8	0	−2
		x_6	b_{14}	0	0
		x_7	b_1	1	−1
		x_8	b_3	1	1
		x_9	b_{13}	0	0
		x_{10}	b_9	−1	−1
$\mathfrak{m}$		x_{11}	b_7	−1	−1
		x_{12}	b_{10}	−2	0
		x_{13}	b_{11}	−3	1
		x_{14}	b_{12}	−3	−1

We calculate the generators:

$$
\begin{aligned}
\Theta_1 &= (x_1 + 3x_4x_6x_{10} + x_4x_7 + 2x_4x_9x_{10} - 4x_4x_{10} \\
&\quad +2x_6x_8 - 4x_8 + x_8x_9) + I_\chi \\
\Theta_2 &= (x_2 + \tfrac{1}{2}x_3x_{10} - \tfrac{3}{2}x_4x_5x_{10} + \tfrac{1}{2}x_4x_{10}^3 - x_5x_8 - x_6x_7 \\
&\quad -\tfrac{1}{2}x_6x_9x_{10} + x_6x_{10} + 2x_7 - x_7x_9 + \tfrac{1}{2}x_8x_{10}^2 \\
&\quad +\tfrac{5}{2}x_9x_{10} - \tfrac{1}{2}x_9^2x_{10} - 3x_{10}) + I_\chi
\end{aligned}
$$

$$\begin{aligned}
\Theta_3 &= (x_3 + \tfrac{3}{4}x_4x_{10}^2 + 3x_6x_9 + x_8x_{10} - 5x_9 + x_9^2) + I_\chi \\
\Theta_4 &= x_4 + I_\chi \\
\Theta_5 &= (x_5 - \tfrac{1}{4}x_{10}^2) + I_\chi \\
\Theta_6 &= x_6 + I_\chi.
\end{aligned}$$

We calculate all non-zero commutators $[\Theta_i, \Theta_j]$ for $1 \leq i < j \leq 6$:

$$\begin{aligned}
[\Theta_1, \Theta_2] &= 5\Theta_3\Theta_4\Theta_5 - \tfrac{1}{2}\Theta_3^2 - \Theta_3\Theta_6^2 + 9\Theta_4\Theta_5\Theta_6^2 \\
&\quad -\tfrac{9}{2}\Theta_4^2\Theta_5^2 + 7\Theta_3\Theta_6 - \tfrac{69}{2}\Theta_4\Theta_5\Theta_6 + 6\Theta_6^3 \\
&\quad -6\Theta_3 + \tfrac{93}{4}\Theta_4\Theta_5 - 30\Theta_6^2 + 42\Theta_6 - 18 \\
[\Theta_1, \Theta_3] &= 6\Theta_1\Theta_6 - 3\Theta_2\Theta_4 - 3\Theta_1 \\
[\Theta_1, \Theta_5] &= \Theta_2 \\
[\Theta_1, \Theta_6] &= -\Theta_1 \\
[\Theta_2, \Theta_3] &= -3\Theta_1\Theta_5 - 6\Theta_2\Theta_6 + 12\Theta_2 \\
[\Theta_2, \Theta_4] &= \Theta_1 \\
[\Theta_2, \Theta_6] &= \Theta_2 \\
[\Theta_3, \Theta_4] &= -9\Theta_4\Theta_6 + \tfrac{15}{2}\Theta_4 \\
[\Theta_3, \Theta_5] &= 9\Theta_5\Theta_6 - \tfrac{51}{2}\Theta_5 \\
[\Theta_4, \Theta_5] &= \tfrac{1}{2} + \Theta_6 \\
[\Theta_4, \Theta_6] &= -2\Theta_4 \\
[\Theta_5, \Theta_6] &= 2\Theta_5.
\end{aligned}$$

We have two 1-dimensional representations, given by

$$\begin{aligned}
\Theta_3 &\mapsto -9 \\
\Theta_6 &\mapsto -\tfrac{1}{2}
\end{aligned}$$

and

$$\Theta_3 \mapsto -\tfrac{21}{2}$$
$$\Theta_6 \mapsto -\tfrac{1}{2},$$

with all other generators $\Theta_i \mapsto 0$. We can observe that the only denominators occurring in this presentation are powers of 2, which is already excluded as it is a bad prime for $\mathfrak{g}$, so we may define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.1.3 The orbit $G_2(a_1)$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $G_2(a_1)$, that is, the subregular orbit of $\mathfrak{g}$. In Table 6.4 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_2+b_4, 2b_{13}+4b_{14}, b_8+b_{10}) = (x_1, x_9, x_{10})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_2,\ x_3,\ x_4\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{10},\ x_{11},\ x_{12},\ x_{13}\}$. The subalgebra $\mathfrak{t}^e$ is zero. We calculate $\kappa(e, f) = 32$.

Table 6.4: Basis for type G_2, orbit $G_2(a_1)$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_6	4	
		x_2	$b_2 + b_4$	2	
		x_3	$b_3 - 3b_5$	2	
		x_4	b_4	2	
		x_5	b_3	2	
		x_6	b_1	0	
		x_7	b_7	0	
		x_8	b_{13}	0	
		x_9	b_{14}	0	
$\mathfrak{m}$		x_{10}	$b_8 + b_{10}$	-2	
		x_{11}	$3b_8 - b_{10}$	-2	

Continued on next page

Table 6.4 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{12}	b_9	-2	
		x_{13}	b_{11}	-2	
		x_{14}	b_{12}	-4	

We calculate the generators:

$$\begin{aligned}
\Theta_1 &= (x_1 + \tfrac{4}{3}x_2x_6 - \tfrac{2}{3}x_2x_7 - \tfrac{2}{3}x_3x_9 - \tfrac{4}{3}x_4x_6 + \tfrac{4}{3}x_4x_7 - \tfrac{16}{3}x_5 + \tfrac{4}{3}x_5x_8 \\
&\quad + \tfrac{8}{3}x_5x_9 + \tfrac{8}{9}x_6x_7^2 - \tfrac{8}{3}x_6x_8x_9 + \tfrac{8}{3}x_6x_9 - \tfrac{8}{3}x_6x_9^2 - \tfrac{128}{9}x_7 + \tfrac{64}{9}x_7x_8 \\
&\quad - \tfrac{8}{3}x_7x_8x_9 - \tfrac{8}{9}x_7x_8^2 + \tfrac{40}{3}x_7x_9 - \tfrac{8}{3}x_7x_9^2 - \tfrac{8}{27}x_7^3) + I_\chi \\
\Theta_2 &= (x_2 + \tfrac{4}{3}x_6x_7 - \tfrac{20}{3}x_8 + 4x_8x_9 + \tfrac{4}{3}x_8^2 - 12x_9 + 4x_9^2) + I_\chi \\
\Theta_3 &= (x_3 - 4x_6 + 4x_6x_8 + 4x_6x_9 + \tfrac{20}{3}x_7 - \tfrac{8}{3}x_7x_8 - 4x_7x_9) + I_\chi \\
\Theta_4 &= (x_4 + \tfrac{4}{3}x_6x_7 - \tfrac{1}{3}x_7^2 - \tfrac{20}{3}x_8 + 4x_8x_9 + \tfrac{4}{3}x_8^2 - 9x_9 + 3x_9^2) + I_\chi.
\end{aligned}$$

We calculate all non-zero commutators $[\Theta_i, \Theta_j]$ for $1 \leq i < j \leq 6$:

$$\begin{aligned}
[\Theta_1, \Theta_3] &= \tfrac{128}{3}\Theta_4 - 32\Theta_2 + \tfrac{32}{3}\Theta_4^2 + \tfrac{2}{3}\Theta_3^2 - \tfrac{32}{3}\Theta_2\Theta_4 + 2\Theta_2^2 \\
[\Theta_1, \Theta_4] &= 2\Theta_1 - \tfrac{4}{3}\Theta_3\Theta_4 + \tfrac{4}{3}\Theta_2\Theta_3 \\
[\Theta_3, \Theta_4] &= 4\Theta_3 + 3\Theta_1.
\end{aligned}$$

We have infinitely many 1-dimensional representations. The only denominators appearing in this presentation are powers of 3 which is already excluded as it is a bad prime for $\mathfrak{g}$, so we may define $U(\mathfrak{g}_\Bbbk, e_\Bbbk)$ provided $\Bbbk$ has characteristic $p > 3$.

6.1.4 The orbit G_2

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label G_2, that is, the regular orbit of $\mathfrak{g}$. In Table 6.5 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e,h,f) = (b_1+b_2, 6b_{13}+10b_{14}, 6b_7+10b_8) = (x_2, 6x_7+10x_8, x_9)$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_1,\ x_2\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_9,\ x_{10}\}$. The subalgebra $\mathfrak{t}^e$ is zero. We calculate $\kappa(e,f) = 224$.

Table 6.5: Basis for type G_2, orbit G_2.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_6	10	
		x_2	b_1+b_2	2	
		x_3	b_5	8	
		x_4	b_4	6	
		x_5	b_3	4	
		x_6	b_1	2	
		x_7	b_{13}	0	
		x_8	b_{14}	0	
$\mathfrak{m}$		x_9	$6b_7+10b_8$	-2	
		x_{10}	b_7-3b_8	-2	
		x_{11}	$3b_9$	-4	
		x_{12}	b_{10}	-6	
		x_{13}	b_{11}	-8	
		x_{14}	b_{12}	-10	

We calculate the generators:

$$\begin{aligned}\Theta_1 \;=\;& (x_1 - \tfrac{28}{3}x_2x_4 + 196x_2x_5 - \tfrac{784}{9}x_2x_5x_7 - \tfrac{392}{3}x_2x_5x_8 + 10976x_2x_6 \\ & - \tfrac{60368}{9}x_2x_6x_7 + \tfrac{10976}{3}x_2x_6x_7x_8 + \tfrac{21952}{27}x_2x_6x_7^2 - \tfrac{38416}{3}x_2x_6x_8 \\ & + \tfrac{10976}{3}x_2x_6x_8^2 - \tfrac{784}{9}x_2x_6^2 - \tfrac{178250240}{81}x_2x_7 + 2458624x_2x_7x_8 \\ & - \tfrac{7683200}{9}x_2x_7x_8^2 + \tfrac{307328}{3}x_2x_7x_8^3 + \tfrac{2458624}{3}x_2x_7^2 - \tfrac{6146560}{9}x_2x_7^2x_8\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1536640}{9}x_2x_7^2x_8^2-\tfrac{12293120}{81}x_2x_7^3+\tfrac{2458624}{27}x_2x_7^3x_8+\tfrac{1229312}{81}x_2x_7^4\\
&-\tfrac{89125120}{27}x_2x_8+1843968x_2x_8^2-\tfrac{768320}{3}x_2x_8^3+\tfrac{109760}{9}x_2^2x_7\\
&-\tfrac{21952}{3}x_2^2x_7x_8-\tfrac{21952}{9}x_2^2x_7^2+\tfrac{54880}{3}x_2^2x_8-5488x_2^2x_8^2-98x_3+28x_3x_7\\
&+56x_3x_8+\tfrac{26656}{27}x_4+\tfrac{28}{3}x_4x_6-\tfrac{9800}{9}x_4x_7+784x_4x_7x_8+\tfrac{784}{3}x_4x_7^2\\
&-\tfrac{4312}{3}x_4x_8+\tfrac{1568}{3}x_4x_8^2-\tfrac{268912}{9}x_5+\tfrac{1960}{9}x_5x_6-\tfrac{392}{9}x_5x_6x_7-\tfrac{392}{3}x_5x_6x_8\\
&+\tfrac{3150112}{81}x_5x_7-\tfrac{455504}{9}x_5x_7x_8+\tfrac{142688}{9}x_5x_7x_8^2-\tfrac{455504}{27}x_5x_7^2\\
&+10976x_5x_7^2x_8+\tfrac{21952}{9}x_5x_7^3+\tfrac{1575056}{27}x_5x_8-\tfrac{109760}{3}x_5x_8^2+\tfrac{21952}{3}x_5x_8^3\\
&+\tfrac{7}{3}x_5^2-\tfrac{12293120}{9}x_6+\tfrac{183782144}{81}x_6x_7-\tfrac{117399296}{27}x_6x_7x_8+\tfrac{24278912}{9}x_6x_7x_8^2\\
&-\tfrac{1536640}{3}x_6x_7x_8^3-\tfrac{284585728}{243}x_6x_7^2+1536640x_6x_7^2x_8-\tfrac{12293120}{27}x_6x_7^2x_8^2\\
&+\tfrac{21973952}{81}x_6x_7^3-\tfrac{1536640}{9}x_6x_7^3x_8-\tfrac{614656}{27}x_6x_7^4+\tfrac{92813056}{27}x_6x_8\\
&-\tfrac{100803584}{27}x_6x_8^2+\tfrac{4456256}{3}x_6x_8^3-\tfrac{614656}{3}x_6x_8^4+\tfrac{1174432}{81}x_6^2-\tfrac{356720}{27}x_6^2x_7\\
&+\tfrac{76832}{9}x_6^2x_7x_8+\tfrac{71344}{27}x_6^2x_7^2-\tfrac{60368}{3}x_6^2x_8+\tfrac{60368}{9}x_6^2x_8^2+\tfrac{1568}{27}x_6^3\\
&+\tfrac{688414720}{27}x_7+\tfrac{808887296}{9}x_7x_8-\tfrac{7650008576}{81}x_7x_8^2+17210368x_7x_8^3\\
&+\tfrac{8605184}{9}x_7x_8^4+\tfrac{4784482304}{243}x_7^2-\tfrac{3467889152}{81}x_7^2x_8+4302592x_7^2x_8^2\\
&+\tfrac{43025920}{9}x_7^2x_8^3-\tfrac{4302592}{9}x_7^2x_8^4-\tfrac{5077058560}{729}x_7^3-\tfrac{68841472}{27}x_7^3x_8\\
&+\tfrac{456074752}{81}x_7^3x_8^2-\tfrac{8605184}{9}x_7^3x_8^3-\tfrac{189314048}{243}x_7^4+\tfrac{197919232}{81}x_7^4x_8\\
&-\tfrac{55933696}{81}x_7^4x_8^2+\tfrac{86051840}{243}x_7^5-\tfrac{17210368}{81}x_7^5x_8-\tfrac{17210368}{729}x_7^6+\tfrac{344207360}{9}x_8\\
&+\tfrac{2443872256}{27}x_8^2-68841472x_8^3+\tfrac{34420736}{3}x_8^4)+I_\chi
\end{aligned}
$$

$$
\Theta_2 \;=\; (x_2-\tfrac{140}{3}x_7+28x_7x_8+\tfrac{28}{3}x_7^2-84x_8+28x_8^2)+I_\chi.
$$

These generators commute, and as we know, $U(\mathfrak{g},e)$ is isomorphic to $\mathcal{Z}(\mathfrak{g})$, the centre of $U(\mathfrak{g})$, in this case [K78]. We have infinitely many 1-dimensional representations. The only denominators appearing in this presentation are powers of 3 which is already excluded as it is a bad prime for $\mathfrak{g}$. We have that 7 divides $\kappa(e,f)$, so we may define $U(\mathfrak{g}_{\Bbbk},e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p>7$.

6.2 Type F_4

Here we calculate $U(\mathfrak{g}, e)$ for $\mathfrak{g}$ of type F_4 and e lying in each of the 5 non-zero rigid nilpotent orbits. We summarize in Table 6.6 certain data for these orbits, including the number of 1-dimensional representations for $U(\mathfrak{g}, e)$ in each case and the primes p for which we cannot define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$ for $\mathbb{k}$ of characteristic p.

Table 6.6: Results for type F_4.

Orbit	Dynkin diagram •—•⇒•—•	$\kappa(e,f)$	$\dim(\mathfrak{g}^e)$	$\dim(\mathfrak{t}^e)$	# 1-dim reps	Bad primes for $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$
A_1	1 0 0 0	18	36	3	1	$2, 3$
$\tilde{A}_1$	0 0 0 1	36	30	3	1	$2, 3$
$A_1 + \tilde{A}_1$	0 1 0 0	54	24	2	1	$2, 3$
$A_2 + \tilde{A}_1$	0 0 1 0	108	18	1	1	$2, 3$
$\tilde{A}_2 + A_1$	0 1 0 1	162	16	1	2	$2, 3$

From here onwards, the generators are too many and/or have too many terms to be written here, and so are omitted. Similarly, we omit the relations which are not required in order to determine the 1-dimensional representations according to Theorem 5.3.5. This information will, however, be considered in any statements made regarding the lower bound on p for Theorem 5.2.2. The simple summands of the Levi subalgebras of $\mathfrak{g}$ are of types A_1, A_2, B_2, B_3 and C_3. Calculation of presentations of the associated finite W-algebras for rigid e shows that the only denominators which occur are powers of 2 and the values of $\kappa(e, f)$ which occur have prime factors 2, 3 and 5. Thus for rigid nilpotent $e \in \mathfrak{g}$ we can apply Theorem 5.2.2 for $p > 5$.

6.2.1 The orbit A_1

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label A_1, that is, the minimal nilpotent orbit in $\mathfrak{g}$. In Table 6.7 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e,h,f) = (b_{24}, b_{49} + 2b_{50} + 2b_{51} + 3b_{52}, b_{48}) = (x_1, x_{32} + 2x_{33} + 3x_{34} + 2x_{37}, x_{52})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{15},\ x_{16},\ x_{17},\ x_{18},\ x_{19},\ x_{20},\ x_{21}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{45}, \dots, x_{52}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{32},\ x_{33},\ x_{34}\}$. We calculate $\kappa(e,f) = 18$.

Table 6.7: Basis for type F_4, orbit A_1.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{24}	2	$(0,0,0)$
		x_2	b_2	1	$(0,0,-1)$
		x_3	b_6	1	$(0,-2,1)$
		x_4	b_9	1	$(-1,0,0)$
		x_5	b_{11}	1	$(1,-1,0)$
		x_6	b_{13}	1	$(-2,2,-1)$
		x_7	b_{14}	1	$(0,1,-1)$
		x_8	b_{16}	1	$(-2,0,1)$
		x_9	b_{17}	1	$(0,-1,1)$
		x_{10}	b_{18}	1	$(2,0,-1)$
		x_{11}	b_{19}	1	$(-1,1,0)$
		x_{12}	b_{20}	1	$(2,-2,1)$
		x_{13}	b_{21}	1	$(1,0,0)$
		x_{14}	b_{22}	1	$(0,2,-1)$
		x_{15}	b_{23}	1	$(0,0,1)$
		x_{16}	b_1	0	$(2,-1,0)$
		x_{17}	b_{25}	0	$(-2,1,0)$
		x_{18}	b_3	0	$(-1,2,-1)$

Continued on next page

Table 6.7 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{19}	b_{27}	0	$(1,-2,1)$
		x_{20}	b_4	0	$(0,-2,2)$
		x_{21}	b_{28}	0	$(0,2,-2)$
		x_{22}	b_5	0	$(1,1,-1)$
		x_{23}	b_{29}	0	$(-1,-1,1)$
		x_{24}	b_7	0	$(-1,0,1)$
		x_{25}	b_{31}	0	$(1,0,-1)$
		x_{26}	b_8	0	$(1,-1,1)$
		x_{27}	b_{32}	0	$(-1,1,-1)$
		x_{28}	b_{10}	0	$(-2,2,0)$
		x_{29}	b_{34}	0	$(2,-2,0)$
		x_{30}	b_{12}	0	$(0,1,0)$
		x_{31}	b_{36}	0	$(0,-1,0)$
		x_{32}	b_{49}	0	$(0,0,0)$
		x_{33}	b_{51}	0	$(0,0,0)$
		x_{34}	b_{52}	0	$(0,0,0)$
		x_{35}	b_{15}	0	$(2,0,0)$
		x_{36}	b_{39}	0	$(-2,0,0)$
		x_{37}	b_{50}	0	$(0,0,0)$
		x_{38}	b_{26}	-1	$(0,0,1)$
		x_{39}	b_{30}	-1	$(0,2,-1)$
		x_{40}	b_{33}	-1	$(1,0,0)$
		x_{41}	b_{35}	-1	$(-1,1,0)$
		x_{42}	b_{37}	-1	$(2,-2,1)$
		x_{43}	b_{38}	-1	$(0,-1,1)$
		x_{44}	b_{40}	-1	$(2,0,-1)$
$\mathfrak{m}$		x_{45}	b_{47}	-1	$(0,0,-1)$
		x_{46}	b_{46}	-1	$(0,-2,1)$

Continued on next page

Table 6.7 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{47}	b_{45}	-1	$(-1,0,0)$
		x_{48}	b_{43}	-1	$(1,-1,0)$
		x_{49}	b_{44}	-1	$(-2,2,-1)$
		x_{50}	b_{41}	-1	$(0,1,-1)$
		x_{51}	b_{42}	-1	$(-2,0,1)$
		x_{52}	b_{48}	-2	$(0,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, we require the commutators $F_{20,21}$, $F_{16,17}$, $F_{10,19}$ and $F_{2,15}$:

$$\begin{aligned}
[\Theta_{20}, \Theta_{21}] &= -\tfrac{1}{2} + \Theta_{34} \\
[\Theta_{16}, \Theta_{17}] &= -2 + \Theta_{32} \\
[\Theta_{18}, \Theta_{19}] &= \Theta_{33} \\
[\Theta_{2}, \Theta_{15}] &= 6 - 2\Theta_{34} - \Theta_{33} + \Theta_1 - \Theta_{35}\Theta_{36} - \tfrac{1}{2}\Theta_{30}\Theta_{31} - \Theta_{28}\Theta_{29} \\
&\quad -\tfrac{1}{2}\Theta_{26}\Theta_{27} - \tfrac{1}{2}\Theta_{24}\Theta_{25} - \Theta_{20}\Theta_{21}.
\end{aligned}$$

We have one 1-dimensional representation, given by

$$\begin{aligned}
\Theta_1 &\mapsto -5 \\
\Theta_{32} &\mapsto 2 \\
\Theta_{33} &\mapsto 0 \\
\Theta_{34} &\mapsto \tfrac{1}{2},
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_\Bbbk, e_\Bbbk)$ provided $\Bbbk$ has characteristic $p > 3$. As in the case of the minimal orbit in $\mathfrak{g}$ of type G_2 above, the existence of the 1-dimensional representation for $U(\mathfrak{g}, e)$ was known from [P07i], however given the explicit presentation we can also establish the bound on the characteristic of p for the modular case.

6.2.2 The orbit $\tilde{A}_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $\tilde{A}_1$, that is, the orbit containing a short root vector of $\mathfrak{g}$. In Table 6.8 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e,h,f) = (b_{21}, 2b_{49}+2b_{50}+3b_{51}+4b_{52}, b_{45}) = (x_4, 2x_{28}+3x_{29}+4x_{30}+2x_{37}, x_{49})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{14},\ x_{15},\ x_{16},\ x_{17},\ x_{19},\ x_{22},\ x_{23},\ x_{25}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{42}, \dots, x_{49}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{28},\ x_{29},\ x_{30}\}$. We calculate $\kappa(e,f) = 36$.

Table 6.8: Basis for type F_4, orbit $\tilde{A}_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{15}	2	$(-1,0,0)$
		x_2	b_{18}	2	$(-1,0,1)$
		x_3	b_{20}	2	$(0,-2,1)$
		x_4	b_{21}	2	$(0,0,0)$
		x_5	b_{22}	2	$(0,2,-1)$
		x_6	b_{23}	2	$(-1,0,1)$
		x_7	b_{24}	2	$(1,0,0)$
		x_8	b_1	1	$(0,-1,0)$
		x_9	b_5	1	$(0,1,-1)$
		x_{10}	b_8	1	$(-1,-1,1)$
		x_{11}	b_{11}	1	$(1,-1,0)$
		x_{12}	b_{12}	1	$(-1,1,0)$
		x_{13}	b_{14}	1	$(1,1,-1)$
		x_{14}	b_{17}	1	$(0,-1,1)$
		x_{15}	b_{19}	1	$(0,1,0)$
		x_{16}	b_2	0	$(2,0,-1)$

Continued on next page

Table 6.8 – continued from previous page

				n_i	β_i
		x_{17}	b_4	0	$(-1,-2,2)$
		x_{18}	b_6	0	$(1,-2,1)$
		x_{19}	b_{10}	0	$(-1,2,0)$
		x_{20}	b_{13}	0	$(1,2,-1)$
		x_{21}	b_{16}	0	$(0,0,1)$
		x_{22}	b_{26}	0	$(-2,0,1)$
		x_{23}	b_{28}	0	$(1,2,-2)$
	$\mathfrak{g}^e$	x_{24}	b_{30}	0	$(-1,2,-1)$
		x_{25}	b_{34}	0	$(1,-2,0)$
		x_{26}	b_{37}	0	$(-1,-2,1)$
$\mathfrak{p}$		x_{27}	b_{40}	0	$(0,0,-1)$
		x_{28}	b_{50}	0	$(0,0,0)$
		x_{29}	b_{51}	0	$(0,0,0)$
		x_{30}	b_{52}	0	$(0,0,0)$
		x_{31}	b_3	0	$(0,2,-1)$
		x_{32}	b_7	0	$(-1,0,1)$
		x_{33}	b_9	0	$(1,0,0)$
		x_{34}	b_{27}	0	$(0,-2,1)$
		x_{35}	b_{31}	0	$(1,0,-1)$
		x_{36}	b_{33}	0	$(-1,0,0)$
		x_{37}	b_{49}	0	$(0,0,0)$
		x_{38}	b_{25}	-1	$(0,1,0)$
		x_{39}	b_{29}	-1	$(0,-1,0)$
		x_{40}	b_{32}	-1	$(1,1,-1)$
		x_{41}	b_{33}	-1	$(-1,-1,0)$
		x_{42}	b_{43}	-1	$(0,-1,0)$
$\mathfrak{m}$		x_{43}	b_{41}	-1	$(0,1,-1)$
		x_{44}	b_{38}	-1	$(-1,-1,1)$

Continued on next page

Table 6.8 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{45}	b_{36}	-1	$(1,-1,0)$
		x_{46}	b_{39}	-2	$(1,0,0)$
		x_{47}	b_{42}	-2	$(-1,0,1)$
		x_{48}	b_{44}	-2	$(0,2,-1)$
		x_{49}	b_{45}	-2	$(0,0,0)$
		x_{50}	b_{46}	-2	$(0,-2,1)$
		x_{51}	b_{47}	-2	$(1,0,-1)$
		x_{52}	b_{48}	-2	$(-1,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{19,25}$, $F_{16,22}$, $F_{17,23}$, $F_{8,15}$, $F_{9,14}$:

$$\begin{aligned}
[\Theta_{19},\Theta_{25}] &= -1+\Theta_{30}+\Theta_{29}\\
[\Theta_{16},\Theta_{22}] &= \Theta_{28}\\
[\Theta_{17},\Theta_{23}] &= \Theta_{30}\\
[\Theta_{8},\Theta_{15}] &= -18+8\Theta_{30}+6\Theta_{29}+4\Theta_{28}-\Theta_4+4\Theta_{21}\Theta_{27}+4\Theta_{20}\Theta_{26}\\
&\quad +4\Theta_{19}\Theta_{25}\\
[\Theta_{9},\Theta_{14}] &= 24-12\Theta_{30}-14\Theta_{29}-6\Theta_{28}+\Theta_4+4\Theta_{29}\Theta_{30}+2\Theta_{29}^2\\
&\quad +2\Theta_{28}\Theta_{29}-4\Theta_{21}\Theta_{27}-4\Theta_{18}\Theta_{24}-4\Theta_{17}\Theta_{23}.
\end{aligned}$$

We have one 1-dimensional representation given by

$$\begin{aligned}
\Theta_4 &\mapsto -12\\
\Theta_{28} &\mapsto 0\\
\Theta_{29} &\mapsto 1\\
\Theta_{30} &\mapsto 0,
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The coefficients which appear in the presentation of $U(\mathfrak{g}, e)$ are all integers, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.2.3 The orbit $A_1 + \tilde{A}_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $A_1 + \tilde{A}_1$. In Table 6.9 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{17} + b_{22}, 2b_{49} + 3b_{50} + 4b_{51} + 6b_{52}, b_{41} + b_{46}) = (x_4, 2x_{23} + 3x_{24} + 4x_{31} + 6x_{32}, b_{48})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{18},\ x_{19},\ x_{20},\ x_{21},\ x_{22}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{39},\ x_{40},\ x_{41},\ x_{42},\ x_{43},\ x_{44},\ x_{45},\ x_{46},\ x_{48},\ x_{49}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{23},\ x_{24}\}$. We calculate $\kappa(e, f) = 54$.

Table 6.9: Basis for type F_4, orbit $A_1 + \tilde{A}_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{23}	3	$(0, -1)$
		x_2	b_{24}	3	$(0, 1)$
		x_3	b_{16}	2	$(-2, 0)$
		x_4	$b_{17} + b_{22}$	2	$(0, 0)$
		x_5	b_{19}	2	$(-1, 0)$
		x_6	b_{20}	2	$(2, 0)$
		x_7	b_{21}	2	$(1, 0)$
		x_8	b_{22}	2	$(0, 0)$
		x_9	$b_4 + \frac{1}{2}b_{12}$	1	$(0, -1)$
		x_{10}	$b_6 + \frac{1}{2}b_{14}$	1	$(0, 1)$
		x_{11}	b_7	1	$(-1, -1)$
		x_{12}	b_8	1	$(1, -1)$
		x_{13}	b_9	1	$(-1, 1)$
		x_{14}	b_{10}	1	$(-2, -1)$

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Table 6.9 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{15}	b_{11}	1	$(1,1)$
		x_{16}	b_{13}	1	$(-2,1)$
		x_{17}	b_{15}	1	$(2,-1)$
		x_{18}	b_{18}	1	$(2,1)$
		x_{19}	b_2	0	$(0,2)$
		x_{20}	b_3+b_{29}	0	$(-1,0)$
		x_{21}	b_5+b_{27}	0	$(1,0)$
		x_{22}	b_{26}	0	$(0,-2)$
		x_{23}	b_{49}	0	$(0,0)$
		x_{24}	b_{50}	0	$(0,0)$
		x_{25}	b_4	1	$(0,-1)$
		x_{26}	b_6	1	$(0,1)$
		x_{27}	b_1	0	$(2,0)$
		x_{28}	b_3	0	$(-1,0)$
		x_{29}	b_5	0	$(1,0)$
		x_{30}	b_{25}	0	$(-2,0)$
		x_{31}	b_{51}	0	$(0,0)$
		x_{32}	b_{52}	0	$(0,0)$
		x_{33}	b_{28}	-1	$(0,1)$
		x_{34}	b_{31}	-1	$(0,1)$
		x_{35}	b_{32}	-1	$(1,1)$
		x_{36}	b_{34}	-1	$(-1,1)$
		x_{37}	b_{36}	-1	$(2,1)$
		x_{38}	b_{37}	-1	$(2,-1)$
$\mathfrak{m}$		x_{39}	b_{38}	-1	$(0,-1)$
		x_{40}	b_{30}	-1	$(0,-1)$
		x_{41}	b_{35}	-1	$(-1,-1)$
		x_{42}	b_{33}	-1	$(1,-1)$

Continued on next page

Table 6.9 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{43}	b_{42}	-1	$(-2,-1)$
		x_{44}	b_{39}	-1	$(-2,1)$
		x_{45}	b_{40}	-2	$(2,0)$
		x_{46}	b_{43}	-2	$(1,0)$
		x_{47}	b_{44}	-2	$(-2,0)$
		x_{48}	$b_{41}+b_{46}$	-2	$(0,0)$
		x_{49}	$b_{41}-2b_{46}$	-2	$(0,0)$
		x_{50}	b_{45}	-2	$(-1,0)$
		x_{51}	b_{47}	-3	$(0,1)$
		x_{52}	b_{48}	-3	$(0,-1)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{19,22}$, $F_{20,21}$, $F_{5,21}$, $F_{7,20}$, $F_{14,18}$:

$$\begin{aligned}
[\Theta_{19},\Theta_{22}] &= -2+\Theta_{24}\\
[\Theta_{20},\Theta_{21}] &= 2-\Theta_{23}\\
[\Theta_{5},\Theta_{21}] &= 12-\tfrac{9}{2}\Theta_{23}-3\Theta_8+\Theta_4-3\Theta_{20}\Theta_{21}\\
[\Theta_{7},\Theta_{20}] &= 24-6\Theta_{24}-\tfrac{39}{2}\Theta_{23}+3\Theta_8-\Theta_4+3\Theta_{23}\Theta_{24}+3\Theta_{23}^2\\
[\Theta_{14},\Theta_{18}] &= -\Theta_8+3\Theta_{19}\Theta_{22}.
\end{aligned}$$

We have one 1-dimensional representation:

$$\begin{aligned}
\Theta_4 &\mapsto -3\\
\Theta_8 &\mapsto 0\\
\Theta_{23} &\mapsto 2\\
\Theta_{24} &\mapsto 2,
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$ provided $\mathbb{k}$ has characteristic $p > 3$.

6.2.4 The orbit $A_2 + \tilde{A}_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $A_2+\tilde{A}_1$. In Table 6.10 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{14} + b_{15} + b_{16}, 3b_{49} + 4b_{50} + 6b_{51} + 8b_{52}, b_{38} + 2b_{39} + 2b_{40}) = (x_6, 4x_{18} - x_{30} - 4x_{31} + 6x_{32}, x_{47})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{15},\ x_{16},\ x_{17}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{36},\ x_{37},\ x_{38},\ x_{39},\ x_{40},\ x_{41},\ x_{43},\ x_{45},\ x_{46},\ x_{47}\}$. The subalgebra $\mathfrak{t}^e$ is spanned by x_{18}. We calculate $\kappa(e, f) = 108$.

Table 6.10: Basis for type F_4, orbit $A_2 + \tilde{A}_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{22}	4	-2
		x_2	b_{23}	4	0
		x_3	b_{24}	4	2
		x_4	b_{19}	3	-1
		x_5	b_{21}	3	1
		x_6	$b_{14} + b_{15} + b_{16}$	2	0
		x_7	b_{14}	2	0
		x_8	b_{10}	2	-4
		x_9	$b_{12} - 2b_{13}$	2	-2
		x_{10}	$b_{17} - 2b_{18}$	2	2
		x_{11}	b_{20}	2	4
		x_{12}	b_3	1	-3
		x_{13}	$b_5 + b_7$	1	-1
		x_{14}	$b_8 - b_9$	1	1

Continued on next page

Table 6.10 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{15}	b_{11}	1	3
		x_{16}	$b_1 - 2b_2 - b_4$	0	2
		x_{17}	$b_{25} - b_{26} - 2b_{28}$	0	−2
		x_{18}	$b_{49} + 2b_{50} + 2b_{52}$	0	0
		x_{19}	b_{12}	0	−2
		x_{20}	b_{15}	0	0
		x_{21}	b_{17}	0	−2
		x_{22}	b_5	0	−1
		x_{23}	b_8	0	1
		x_{24}	b_1	0	2
		x_{25}	b_2	1	2
		x_{26}	b_6	1	4
		x_{27}	b_{25}	0	−2
		x_{28}	b_{26}	0	−2
		x_{29}	b_{30}	0	−4
		x_{30}	b_{49}	0	0
		x_{31}	b_{50}	0	0
		x_{32}	b_{51}	0	0
		x_{33}	b_{27}	−1	3
		x_{34}	b_{29}	−1	1
		x_{35}	b_{31}	−1	1
$\mathfrak{m}$		x_{36}	b_{35}	−1	−3
		x_{37}	b_{32}	−1	−1
		x_{38}	b_{33}	−1	−1
		x_{39}	b_{34}	−2	4
		x_{40}	b_{36}	−2	2
		x_{41}	b_{37}	−2	2
		x_{42}	b_{41}	−2	−2

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Table 6.10 – continued from previous page

				n_i	β_i
		x_{43}	b_{42}	-2	-2
		x_{44}	b_{44}	-2	-4
		x_{45}	$b_{38} - 2b_{29}$	-2	0
		x_{46}	$b_{39} - b_{40}$	-2	0
$\mathfrak{m}$		x_{47}	$b_{38} + 2b_{39} + 2b_{46}$	-2	0
		x_{48}	b_{43}	-3	1
		x_{49}	b_{45}	-3	-1
		x_{50}	b_{46}	-4	2
		x_{51}	b_{47}	-4	0
		x_{52}	b_{48}	-4	-2

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, we require the commutators $F_{16,17}$, $F_{12,15}$, $F_{10,17}$, $F_{9,16}$, $F_{1,16}$, $F_{3,17}$:

$$
\begin{aligned}
[\Theta_{16}, \Theta_{17}] &= -\tfrac{5}{2} + \Theta_{18} \\
[\Theta_{12}, \Theta_{15}] &= -30 + 6\Theta_{18} - \Theta_7 \\
[\Theta_{10}, \Theta_{17}] &= -84 + 18\Theta_{18} - 6\Theta_7 + 2\Theta_6 + 6\Theta_{16}\Theta_{17} \\
[\Theta_9, \Theta_{16}] &= -\tfrac{93}{2} + 3\Theta_{18} - 6\Theta_7 + 2\Theta_6 + 6\Theta_{16}\Theta_{17} \\
[\Theta_1, \Theta_{16}] &= \tfrac{1557}{4} - \tfrac{333}{4}\Theta_{18} + \tfrac{87}{2}\Theta_7 + 3\Theta_6 + \Theta_2 + \tfrac{9}{2}\Theta_{18}^2 - 72\Theta_{16}\Theta_{17} \\
&\quad +6\Theta_{12}\Theta_{15} + 3\Theta_9\Theta_{16} - 6\Theta_7\Theta_{18} + 9\Theta_{16}\Theta_{17}\Theta_{18} \\
[\Theta_3, \Theta_{17}] &= -\tfrac{3321}{2} + \tfrac{1305}{2}\Theta_{18} - 102\Theta_7 + 51\Theta_6 + \Theta_2 - 63\Theta_{18}^2 + 369\Theta_{16}\Theta_{17} \\
&\quad -3\Theta_{13}\Theta_{14} - 3\Theta_{12}\Theta_{15} + 3\Theta_{10}\Theta_{17} + 3\Theta_9\Theta_{16} + 18\Theta_7\Theta_{18} \\
&\quad -9\Theta_6\Theta_{18} - 54\Theta_{16}\Theta_{17}\Theta_{18}.
\end{aligned}
$$

We have one 1-dimensional representation:

$$\begin{aligned} \Theta_2 &\mapsto \tfrac{1179}{4} \\ \Theta_6 &\mapsto -\tfrac{51}{2} \\ \Theta_7 &\mapsto -15 \\ \Theta_{18} &\mapsto \tfrac{5}{2} \end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$ provided $\mathbb{k}$ has characteristic $p > 3$.

6.2.5 The orbit $\tilde{A}_2 + A_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $\tilde{A}_2 + A_1$. In Table 6.11 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_8 + b_{14} + b_{16}, 4b_{49} + 5b_{50} + 7b_{51} + 10b_{52}, 2b_{32} + 2b_{38} + b_{40}) = (x_8, -7x_{16} + 4x_{28} + 12x_{29} + 10x_{30}, x_{43})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_7,\ x_8,\ x_{13},\ x_{14},\ x_{15}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{35},\ x_{36},\ x_{37},\ x_{38},\ x_{41},\ x_{42},\ x_{43}\}$. The subalgebra $\mathfrak{t}^e$ is spanned by x_{16}. We calculate $\kappa(e, f) = 162$.

Table 6.11: Basis for type F_4, orbit $\tilde{A}_2 + A_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{23}	5	-1
		x_2	b_{24}	5	1
		x_3	b_{20}	4	2
		x_4	b_{21}	4	0
		x_5	b_{22}	4	-2
		x_6	$b_{15} + \frac{1}{2}b_{19}$	3	-1
		x_7	$b_{17} - 2b_{18}$	3	1

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Table 6.11 – continued from previous page

				n_i	β_i
		x_8	$b_8 + b_{14} + b_{16}$	2	0
		x_9	b_{16}	2	0
		x_{10}	$b_4 - b_9$	1	1
		x_{11}	b_6	1	3
	$\mathfrak{g}^e$	x_{12}	$b_7 - b_{13}$	1	-1
		x_{13}	b_{16}	1	-3
		x_{14}	$b_2 - b_{27}$	0	2
		x_{15}	$b_3 - b_{26}$	0	-2
		x_{16}	$b_{50} - b_{51}$	0	0
		x_{17}	b_{15}	3	-1
		x_{18}	b_{17}	3	1
$\mathfrak{p}$		x_{19}	b_8	2	0
		x_{20}	b_{11}	2	2
		x_{21}	b_{12}	2	-2
		x_{22}	b_1	1	1
		x_{23}	b_4	1	1
		x_{24}	b_5	1	-1
		x_{25}	b_{13}	1	-1
		x_{26}	b_2	0	2
		x_{27}	b_3	0	-2
		x_{28}	b_{49}	0	0
		x_{29}	b_{50}	0	0
		x_{30}	b_{52}	0	0
		x_{31}	b_{29}	-1	1
		x_{32}	b_{30}	-1	-3
		x_{33}	b_{31}	-1	1
		x_{34}	b_{37}	-1	1
$\mathfrak{m}$		x_{35}	b_{25}	-1	-1

Continued on next page

Table 6.11 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{36}	b_{28}	-1	3
		x_{37}	b_{33}	-1	-1
		x_{38}	b_{34}	-1	-1
		x_{39}	b_{35}	-2	-2
		x_{40}	b_{36}	-2	2
		x_{41}	$b_{32} - b_{38}$	-2	0
		x_{42}	$b_{38} - 2b_{40}$	-2	0
		x_{43}	$2b_{32} + 2b_{38} + b_{40}$	-2	0
		x_{44}	b_{39}	-3	1
		x_{45}	b_{41}	-3	-1
		x_{46}	b_{42}	-3	-1
		x_{47}	b_{43}	-3	1
		x_{48}	b_{44}	-4	-2
		x_{49}	b_{45}	-4	0
		x_{50}	b_{46}	-4	2
		x_{51}	b_{47}	-5	1
		x_{52}	b_{48}	-5	-1

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, we require the commutators $F_{14,15}$, $F_{11,13}$, $F_{8,16}$, $F_{3,15}$, $F_{7,12}$, $F_{5,14}$, $F_{8,9}$, $F_{4,8}$, $F_{6,7}$, $F_{1,7}$:

$$\begin{aligned}
[\Theta_{14}, \Theta_{15}] &= -\Theta_{16} \\
[\Theta_{11}, \Theta_{13}] &= 38 + 18\Theta_{16} + \Theta_9 + 3\Theta_{14}\Theta_{15} \\
[\Theta_8, \Theta_{16}] &= 0 \\
[\Theta_3, \Theta_{15}] &= \tfrac{441}{2} + \tfrac{27}{4}\Theta_{16} - 36\Theta_9 + 33\Theta_8 + \Theta_4 - \tfrac{27}{4}\Theta_{16}^2 - \tfrac{405}{2}\Theta_{14}\Theta_{15} \\
&\quad + \tfrac{9}{2}\Theta_{11}\Theta_{13} - \tfrac{9}{2}\Theta_9\Theta_{16} + \tfrac{9}{2}\Theta_8\Theta_{16} - \tfrac{81}{2}\Theta_{14}\Theta_{15}\Theta_{16}
\end{aligned}$$

$$
\begin{aligned}
[\Theta_7,\Theta_{12}] &= 3663+2349\Theta_{16}+180\Theta_9-66\Theta_8-2\Theta_4+\tfrac{729}{2}\Theta_{16}^2\\
&\quad+1242\Theta_{14}\Theta_{15}+18\Theta_{11}\Theta_{13}-9\Theta_{10}\Theta_{12}+27\Theta_9\Theta_{16}-12\Theta_8\Theta_{16}\\
&\quad+216\Theta_{14}\Theta_{15}\Theta_{16}\\
[\Theta_5,\Theta_{14}] &= -\tfrac{441}{2}-1728\Theta_{16}+36\Theta_9-33\Theta_8-\Theta_4-702\Theta_{16}^2-\tfrac{1863}{2}\Theta_{14}\Theta_{15}\\
&\quad-\tfrac{9}{2}\Theta_{11}\Theta_{13}+\tfrac{9}{2}\Theta_9\Theta_{16}-\tfrac{9}{2}\Theta_8\Theta_{16}-81\Theta_{16}^3-243\Theta_{14}\Theta_{15}\Theta_{16}\\
[\Theta_8,\Theta_9] &= 0\\
[\Theta_4,\Theta_8] &= 0\\
[\Theta_6,\Theta_7] &= \tfrac{248751}{4}+52245\Theta_{16}+3159\Theta_9-\tfrac{1107}{2}\Theta_8-\tfrac{9}{2}\Theta_4+\tfrac{47871}{4}\Theta_{16}^2\\
&\quad+\tfrac{63099}{4}\Theta_{14}\Theta_{15}+\tfrac{1053}{2}\Theta_{11}\Theta_{13}-\tfrac{1053}{4}\Theta_{10}\Theta_{12}+\tfrac{3159}{2}\Theta_9\Theta_{16}\\
&\quad-\tfrac{27}{2}\Theta_9^2-\tfrac{1485}{4}\Theta_8\Theta_{16}+\tfrac{27}{2}\Theta_8\Theta_9-3\Theta_8^2-\tfrac{9}{2}\Theta_7\Theta_{12}+\tfrac{9}{2}\Theta_5\Theta_{14}\\
&\quad+18\Theta_3\Theta_{15}+\tfrac{3159}{4}\Theta_{16}^3+\tfrac{26487}{8}\Theta_{14}\Theta_{15}\Theta_{16}+81\Theta_{11}\Theta_{13}\Theta_{16}\\
&\quad-\tfrac{243}{4}\Theta_{11}\Theta_{12}\Theta_{15}-\tfrac{81}{2}\Theta_{10}\Theta_{12}\Theta_{16}+162\Theta_9\Theta_{16}^2+\tfrac{243}{2}\Theta_9\Theta_{14}\Theta_{15}\\
&\quad-\tfrac{81}{2}\Theta_8\Theta_{16}^2-\tfrac{297}{4}\Theta_8\Theta_{14}\Theta_{15}\\
[\Theta_1,\Theta_7] &= -3177711-\tfrac{15642639}{4}\Theta_{16}-\tfrac{605313}{4}\Theta_9+12285\Theta_8-648\Theta_4\\
&\quad-\tfrac{10759311}{8}\Theta_{16}^2-\tfrac{6426135}{4}\Theta_{14}\Theta_{15}-15309\Theta_{11}\Theta_{13}-\tfrac{3645}{4}\Theta_{10}\Theta_{12}\\
&\quad-\tfrac{413343}{4}\Theta_9\Theta_{16}+\tfrac{2349}{4}\Theta_9^2+\tfrac{54513}{4}\Theta_8\Theta_{16}-\tfrac{459}{2}\Theta_8\Theta_9+27\Theta_8^2\\
&\quad+\tfrac{297}{2}\Theta_7\Theta_{12}-81\Theta_6\Theta_{10}-18\Theta_6\Theta_7-\tfrac{1701}{2}\Theta_5\Theta_{14}-513\Theta_4\Theta_{16}\\
&\quad+\tfrac{27}{2}\Theta_4\Theta_9-6\Theta_4\Theta_8-648\Theta_3\Theta_{15}+\tfrac{9}{2}\Theta_2\Theta_{12}-\tfrac{1301265}{8}\Theta_{16}^3\\
&\quad-\tfrac{5452191}{8}\Theta_{14}\Theta_{15}\Theta_{16}-972\Theta_{12}^2\Theta_{14}-\tfrac{3159}{2}\Theta_{11}\Theta_{13}\Theta_{16}\\
&\quad+\tfrac{7533}{2}\Theta_{11}\Theta_{12}\Theta_{15}-729\Theta_{10}\Theta_{13}\Theta_{14}-\tfrac{143613}{8}\Theta_9\Theta_{16}^2\\
&\quad-\tfrac{54675}{4}\Theta_9\Theta_{14}\Theta_{15}-\tfrac{243}{2}\Theta_9\Theta_{11}\Theta_{13}+\tfrac{81}{2}\Theta_9^2\Theta_{16}+\tfrac{8991}{4}\Theta_8\Theta_{16}^2\\
&\quad+\tfrac{30699}{4}\Theta_8\Theta_{14}\Theta_{15}+\tfrac{189}{2}\Theta_8\Theta_{11}\Theta_{13}-\tfrac{243}{4}\Theta_7\Theta_{13}\Theta_{14}\\
&\quad+\tfrac{81}{2}\Theta_7\Theta_{10}\Theta_{15}-\tfrac{297}{2}\Theta_5\Theta_{14}\Theta_{16}-81\Theta_4\Theta_{16}^2-162\Theta_3\Theta_{15}\Theta_{16}\\
&\quad-\tfrac{19683}{4}\Theta_{16}^4-\tfrac{566433}{8}\Theta_{14}\Theta_{15}\Theta_{16}^2-\tfrac{243}{2}\Theta_{12}^2\Theta_{14}\Theta_{16}\\
&\quad+\tfrac{729}{2}\Theta_{11}\Theta_{13}\Theta_{16}^2+\tfrac{3645}{4}\Theta_{11}\Theta_{12}\Theta_{15}\Theta_{16}-243\Theta_{10}\Theta_{13}\Theta_{14}\Theta_{16}\\
&\quad-729\Theta_9\Theta_{16}^3-\tfrac{5589}{2}\Theta_9\Theta_{14}\Theta_{15}\Theta_{16}+\tfrac{6075}{4}\Theta_8\Theta_{14}\Theta_{15}\Theta_{16}.
\end{aligned}
$$

We have two 1-dimensional representations:

$$\begin{aligned}\Theta_4 &\mapsto \tfrac{6723}{2}\\ \Theta_8 &\mapsto -150\\ \Theta_9 &\mapsto -38\\ \Theta_{16} &\mapsto 0,\end{aligned}$$

and

$$\begin{aligned}\Theta_4 &\mapsto \tfrac{7119}{2}\\ \Theta_8 &\mapsto -156\\ \Theta_9 &\mapsto -38\\ \Theta_{16} &\mapsto 0,\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$ for each of these. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.3 Type E_6

Here we calculate $U(\mathfrak{g}, e)$ for $\mathfrak{g}$ of type E_6 and e lying in each of the 3 non-zero rigid nilpotent orbits. We summarize in Table 6.12 certain data for these orbits, including the number of 1-dimensional representations for $U(\mathfrak{g}, e)$ in each case and the primes p for which we cannot define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ where $\Bbbk$ has characteristic p.

Table 6.12: Results for type E_6.

Orbit	Dynkin diagram	$\kappa(e,f)$	$\dim(\mathfrak{g}^e)$	$\dim(\mathfrak{t}^e)$	# 1-dim reps	Bad primes for $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$
A_1	1 0 0 0 0 0	24	56	5	1	$2,3$
$3A_1$	0 0 0 1 0 0	72	38	3	1	$2,3$
$2A_2+A_1$	0 1 0 1 0 1	216	24	1	1	$2,3$

The simple summands of the Levi subalgebras of $\mathfrak{g}$ are of types A_1, A_2, A_3, A_4, A_5, D_4 and D_5. Calculation of presentations of the associated finite W-algebras for rigid e shows that the only denominators which occur are powers of 2 and the values of $\kappa(e,f)$ which occur for rigid e have prime factors only 2 and 3. Thus for rigid nilpotent $e \in \mathfrak{g}$ we can apply Theorem 5.2.2 for $p > 3$.

6.3.1 The orbit A_1

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label A_1. In Table 6.13 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e,h,f) = (b_{36}, b_{73} + 2b_{74} + 2b_{75} + 3b_{76} + 2b_{77} + b_{78}, b_{72}) = (x_1, x_{52} + 2x_{53} + 3x_{54} + 2x_{55} + x_{56} + 2x_{57}, x_{78})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{21},\ x_{36},\ x_{37},\ x_{41},\ x_{42},\ x_{45},\ x_{46},\ x_{48},\ x_{49},\ x_{50}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{68}, \dots, x_{78}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{52},\ x_{53},\ x_{54},\ x_{55},\ x_{56}\}$. We calculate $\kappa(e,f) = 24$.

Table 6.13: Basis for type E_6, orbit A_1.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{36}	2	$(0,0,0,0,0)$
		x_2	b_2	1	$(0,0,-1,0,0)$
		x_3	b_8	1	$(0,-1,1,-1,0)$
		x_4	b_{13}	1	$(-1,1,0,-1,0)$
		x_5	b_{14}	1	$(0,-1,0,1,-1)$
		x_6	b_{17}	1	$(1,0,0,-1,0)$
		x_7	b_{19}	1	$(-1,1,-1,1,-1)$
		x_8	b_{20}	1	$(0,-1,0,0,1)$
		x_9	b_{22}	1	$(1,0,-1,1,-1)$
		x_{10}	b_{24}	1	$(-1,0,1,0,-1)$
		x_{11}	b_{25}	1	$(-1,1,-1,0,1)$
		x_{12}	b_{26}	1	$(1,-1,1,0,-1)$
		x_{13}	b_{27}	1	$(1,0,-1,0,1)$
		x_{14}	b_{28}	1	$(-1,0,1,-1,1)$
		x_{15}	b_{29}	1	$(0,1,0,0,-1)$
		x_{16}	b_{30}	1	$(1,-1,1,-1,1)$
		x_{17}	b_{31}	1	$(-1,0,0,1,0)$
		x_{18}	b_{32}	1	$(0,1,0,-1,1)$
		x_{19}	b_{33}	1	$(1,-1,0,1,0)$
		x_{20}	b_{34}	1	$(0,1,-1,1,0)$
		x_{21}	b_{35}	1	$(0,0,1,0,0)$
		x_{22}	b_1	0	$(2,-1,0,0,0)$
		x_{23}	b_3	0	$(-1,2,-1,0,0)$
		x_{24}	b_4	0	$(0,-1,2,-1,0)$
		x_{25}	b_5	0	$(0,0,-1,2,-1)$
		x_{26}	b_6	0	$(0,0,0,-1,-2)$
		x_{27}	b_7	0	$(1,1,-1,0,0)$

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Table 6.13 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{28}	b_9	0	$(-1,1,1,-1,0)$
		x_{29}	b_{10}	0	$(0,-1,1,1,-1)$
		x_{30}	b_{11}	0	$(0,0,-1,1,1)$
		x_{31}	b_{12}	0	$(1,0,1,-1,0)$
		x_{32}	b_{15}	0	$(-1,1,0,1,-1)$
		x_{33}	b_{16}	0	$(0,-1,1,0,1)$
		x_{34}	b_{18}	0	$(1,0,0,1,-1)$
		x_{35}	b_{21}	0	$(-1,1,0,0,1)$
		x_{36}	b_{23}	0	$(1,0,0,0,1)$
		x_{37}	b_{37}	0	$(-2,1,0,0,0)$
		x_{38}	b_{39}	0	$(1,-2,1,0,0)$
		x_{39}	b_{40}	0	$(0,1,-2,1,0)$
		x_{40}	b_{41}	0	$(0,0,1,-2,1)$
		x_{41}	b_{42}	0	$(0,0,0,1,-2)$
		x_{42}	b_{43}	0	$(-1,-1,1,0,0)$
		x_{43}	b_{45}	0	$(1,-1,-1,1,0)$
		x_{44}	b_{46}	0	$(0,1,-1,-1,1)$
		x_{45}	b_{47}	0	$(0,0,1,-1,1)$
		x_{46}	b_{48}	0	$(-1,0,-1,1,0)$
		x_{47}	b_{51}	0	$(1,-1,0,-1,0)$
		x_{48}	b_{52}	0	$(0,1,-1,0,-1)$
		x_{49}	b_{54}	0	$(-1,0,0,-1,1)$
		x_{50}	b_{57}	0	$(1,-1,0,0,-1)$
		x_{51}	b_{59}	0	$(-1,0,0,0,-1)$
		x_{52}	b_{73}	0	$(0,0,0,0,0)$
		x_{53}	b_{75}	0	$(0,0,0,0,0)$
		x_{54}	b_{76}	0	$(0,0,0,0,0)$
		x_{55}	b_{77}	0	$(0,0,0,0,0)$

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Table 6.13 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{56}	b_{78}	0	$(0,0,0,0,0)$
		x_{57}	b_{74}	0	$(0,0,0,0,0)$
		x_{58}	b_{38}	-1	$(0,0,1,0,0)$
		x_{59}	b_{44}	-1	$(0,1,-1,0,0)$
		x_{60}	b_{49}	-1	$(1,-1,0,1,0)$
		x_{61}	b_{50}	-1	$(0,1,0,-1,1)$
		x_{62}	b_{53}	-1	$(-1,0,0,1,0)$
		x_{63}	b_{55}	-1	$(1,-1,1,-1,1)$
		x_{64}	b_{56}	-1	$(0,1,0,0,-1)$
		x_{65}	b_{58}	-1	$(-1,0,1,-1,1)$
		x_{66}	b_{60}	-1	$(-1,0,1,0,1)$
		x_{67}	b_{61}	-1	$(1,-1,1,0,-1)$
$\mathfrak{m}$		x_{68}	b_{62}	-1	$(-1,1,-1,0,1)$
		x_{69}	b_{63}	-1	$(-1,0,1,0,-1)$
		x_{70}	b_{64}	-1	$(1,0,-1,1,-1)$
		x_{71}	b_{65}	-1	$(0,-1,0,0,1)$
		x_{72}	b_{66}	-1	$(-1,1,-1,1,-1)$
		x_{73}	b_{67}	-1	$(1,0,0,-1,0)$
		x_{74}	b_{68}	-1	$(0,-1,0,1,-1)$
		x_{75}	b_{69}	-1	$(-1,1,0,-1,0)$
		x_{76}	b_{70}	-1	$(0,-1,1,-1,0)$
		x_{77}	b_{71}	-1	$(0,0,-1,0,0)$
		x_{78}	b_{72}	-2	$(0,0,0,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{2,21}$, $F_{22,37}$, $F_{26,41}$, $F_{27,42}$, $F_{30,45}$, $F_{31,46}$, $F_{33,48}$,

$F_{34,49}$, $F_{35,50}$, $F_{36,51}$:

$$\begin{aligned}
[\Theta_2,\Theta_{21}] &= 11-\Theta_{55}-2\Theta_{54}-\Theta_{53}+\Theta_1-\Theta_{36}\Theta_{51}-\Theta_{35}\Theta_{50}-\Theta_{34}\Theta_{49}\\
&\quad -\Theta_{33}\Theta_{48}-\Theta_{32}\Theta_{47}-\Theta_{31}\Theta_{46}-\Theta_{29}\Theta_{44}-\Theta_{28}\Theta_{43}-\Theta_{24}\Theta_{39}\\
[\Theta_{22},\Theta_{37}] &= -1+\Theta_{52}\\
[\Theta_{26},\Theta_{41}] &= -1+\Theta_{56}\\
[\Theta_{27},\Theta_{42}] &= -1+\Theta_{53}+\Theta_{52}\\
[\Theta_{30},\Theta_{45}] &= -1+\Theta_{56}+\Theta_{55}\\
[\Theta_{31},\Theta_{46}] &= -2+\Theta_{54}+\Theta_{53}+\Theta_{52}\\
[\Theta_{33},\Theta_{48}] &= -2+\Theta_{56}+\Theta_{55}+\Theta_{54}\\
[\Theta_{34},\Theta_{49}] &= -2+\Theta_{55}+\Theta_{54}+\Theta_{53}+\Theta_{52}\\
[\Theta_{35},\Theta_{50}] &= -2+\Theta_{56}+\Theta_{55}+\Theta_{54}+\Theta_{53}\\
[\Theta_{36},\Theta_{51}] &= -3+\Theta_{56}+\Theta_{55}+\Theta_{54}+\Theta_{53}+\Theta_{52}.
\end{aligned}$$

We have one 1-dimensional representation:

$$\begin{aligned}
\Theta_1 &\mapsto -9,\\
\Theta_{52} &\mapsto 1\\
\Theta_{53} &\mapsto 0\\
\Theta_{54} &\mapsto 1\\
\Theta_{55} &\mapsto 0\\
\Theta_{56} &\mapsto 1,
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The coefficients which appear in the presentation of $U(\mathfrak{g}, e)$ are all integers, so we can define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$ provided $\mathbb{k}$ has characteristic $p > 3$. We again note that the existence of a 1-dimensional representation in this case was known from [P07i], as our orbit is minimal.

6.3.2 The orbit $3A_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $3A_1$. In Table 6.14 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e,h,f) = (b_{29}+b_{30}+b_{31}, 2b_{73}+3b_{74}+4b_{75}+6b_{76}+4b_{77}+2b_{78}, b_{65}+b_{66}+b_{67}) = (x_6, 4x_{36}+3x_{37}+2x_{38}-2x_{47}+2x_{48}+6x_{49}, x_{76})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{27},\ x_{29},\ x_{31},\ x_{32},\ x_{33},\ x_{34}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{59},\ x_{60},\ x_{61},\ x_{62},\ x_{63},\ x_{64},\ x_{65},\ x_{66},\ x_{67},\ x_{68},\ x_{69},\ x_{70},\ x_{74},\ x_{75},\ x_{76}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{36},\ x_{37},\ x_{38}\}$. We calculate $\kappa(e,f) = 72$.

Table 6.14: Basis for type E_6, orbit $3A_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{35}	3	$(0,-1,0)$
		x_2	b_{36}	3	$(0,1,0)$
		x_3	b_{24}	2	$(-1,0,-1)$
		x_4	b_{26}	2	$(1,0,-2)$
		x_5	b_{28}	2	$(-2,0,1)$
		x_6	$b_{29}+b_{30}+b_{31}$	2	$(0,0,0)$
		x_7	b_{30}	2	$(0,0,0)$
		x_8	b_{31}	2	$(0,0,0)$
		x_9	b_{32}	2	$(-1,0,2)$
		x_{10}	b_{33}	2	$(2,0,-1)$
		x_{11}	b_{34}	2	$(1,0,1)$
		x_{12}	b_4	1	$(-1,-1,-1)$
		x_{13}	b_8	1	$(-1,1,-1)$
		x_{14}	b_9	1	$(-2,-1,1)$
		x_{15}	b_{10}	1	$(1,-1,-2)$
		x_{16}	$b_{12}-b_{15}$	1	$(0,-1,0)$
		x_{17}	b_{13}	1	$(-2,1,1)$
Continued on next page					

Table 6.14 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{18}	b_{14}	1	$(1,1,-2)$
		x_{19}	$b_{15}-b_{16}$	1	$(0,-1,0)$
		x_{20}	$b_{17}-b_{19}$	1	$(0,1,0)$
		x_{21}	b_{18}	1	$(2,-1,-1)$
		x_{22}	$b_{19}-b_{20}$	1	$(0,1,0)$
		x_{23}	b_{21}	1	$(-1,-1,2)$
		x_{24}	b_{22}	1	$(2,1,-1)$
		x_{25}	b_{23}	1	$(1,-1,1)$
		x_{26}	b_{25}	1	$(-1,1,2)$
		x_{27}	b_{27}	1	$(1,1,1)$
		x_{28}	b_1+b_5	0	$(2,0,-1)$
		x_{29}	b_2	0	$(0,2,0)$
		x_{30}	b_3+b_6	0	$(-1,0,2)$
		x_{31}	b_7+b_{11}	0	$(1,0,1)$
		x_{32}	$b_{37}+b_{41}$	0	$(-2,0,1)$
		x_{33}	b_{38}	0	$(0,-2,0)$
		x_{34}	$b_{39}+b_{42}$	0	$(1,0,-2)$
		x_{35}	$b_{43}+b_{47}$	0	$(-1,0,-1)$
		x_{36}	$b_{73}+b_{77}$	0	$(0,0,0)$
		x_{37}	b_{74}	0	$(0,0,0)$
		x_{38}	$b_{75}+b_{78}$	0	$(0,0,0)$
		x_{39}	b_{15}	1	$(0,-1,0)$
		x_{40}	b_{19}	1	$(0,1,0)$
		x_{41}	b_1	0	$(2,0,-1)$
		x_{42}	b_3	0	$(-1,0,2)$
		x_{43}	b_7	0	$(1,0,1)$
		x_{44}	b_{37}	0	$(-2,0,1)$
		x_{45}	b_{39}	0	$(1,0,-2)$

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Table 6.14 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{46}	b_{43}	0	$(-1,0,-1)$
		x_{47}	b_{73}	0	$(0,0,0)$
		x_{48}	b_{75}	0	$(0,0,0)$
		x_{49}	b_{76}	0	$(0,0,0)$
		x_{50}	b_{40}	-1	$(1,1,1)$
		x_{51}	b_{44}	-1	$(1,-1,1)$
		x_{52}	b_{45}	-1	$(2,1,-1)$
		x_{53}	b_{46}	-1	$(-1,1,2)$
		x_{54}	b_{48}	-1	$(0,1,0)$
		x_{55}	b_{49}	-1	$(2,-1,-1)$
		x_{56}	b_{50}	-1	$(-1,-1,2)$
		x_{57}	b_{51}	-1	$(0,1,0)$
		x_{58}	b_{52}	-1	$(0,1,0)$
$\mathfrak{m}$		x_{59}	b_{63}	-1	$(-1,-1,-1)$
		x_{60}	b_{59}	-1	$(-1,1,-1)$
		x_{61}	b_{58}	-1	$(-2,-1,1)$
		x_{62}	b_{61}	-1	$(1,-1,-2)$
		x_{63}	b_{53}	-1	$(0,-1,0)$
		x_{64}	b_{54}	-1	$(-2,1,1)$
		x_{65}	b_{57}	-1	$(1,1,-2)$
		x_{66}	b_{55}	-1	$(0,-1,0)$
		x_{67}	b_{56}	-1	$(0,-1,0)$
		x_{68}	b_{60}	-2	$(1,0,1)$
		x_{69}	b_{62}	-2	$(-1,0,2)$
		x_{70}	b_{64}	-2	$(2,0,-1)$
		x_{71}	b_{68}	-2	$(1,0,-2)$
		x_{72}	b_{69}	-2	$(-2,0,1)$
		x_{73}	b_{70}	-2	$(-1,0,-1)$

Continued on next page

Table 6.14 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{74}	$b_{65} - b_{66}$	-2	$(0,0,0)$
		x_{75}	$b_{66} - b_{67}$	-2	$(0,0,0)$
		x_{76}	$b_{65} + b_{66} + b_{67}$	-2	$(0,0,0)$
		x_{77}	b_{71}	-3	$(0,1,0)$
		x_{78}	b_{72}	-3	$(0,-1,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{3,31}$, $F_{9,34}$, $F_{10,32}$, $F_{12,27}$, $F_{28,32}$, $F_{29,33}$, $F_{30,34}$, $F_{31,35}$:

$$\begin{aligned}
[\Theta_3,\Theta_{31}] &= -12 - 3\Theta_{38} + 6\Theta_{37} + 2\Theta_8 + \Theta_7 - \Theta_6 + 3\Theta_{31}\Theta_{35} \\
[\Theta_9,\Theta_{34}] &= -24 + 3\Theta_{38} + 3\Theta_{37} + 3\Theta_{36} + \Theta_8 + 2\Theta_7 - \Theta_6 \\
[\Theta_{10},\Theta_{32}] &= -24 + 3\Theta_{38} + 3\Theta_{37} + 6\Theta_{36} + \Theta_8 - \Theta_7 + 3\Theta_{28}\Theta_{32} \\
[\Theta_{12},\Theta_{27}] &= \tfrac{3}{2} + \Theta_7 - \tfrac{3}{2}\Theta_{31}\Theta_{35} - 3\Theta_{29}\Theta_{33} \\
[\Theta_{28},\Theta_{32}] &= -2 + \Theta_{36} \\
[\Theta_{29},\Theta_{33}] &= -\tfrac{3}{2} + \Theta_{37} \\
[\Theta_{30},\Theta_{34}] &= -2 + \Theta_{38} \\
[\Theta_{31},\Theta_{35}] &= -4 + \Theta_{38} + \Theta_{36}.
\end{aligned}$$

We have one 1-dimensional representation:

$$\begin{aligned}
\Theta_6 &\mapsto -\tfrac{21}{2} \\
\Theta_7 &\mapsto -\tfrac{3}{2} \\
\Theta_8 &\mapsto 0 \\
\Theta_{36} &\mapsto 2 \\
\Theta_{37} &\mapsto \tfrac{3}{2} \\
\Theta_{38} &\mapsto 2,
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.3.3 The orbit $2A_2 + A_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $2A_2 + A_1$. In Table 6.15 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{12} + b_{20} + b_{21} + b_{22} + b_{24}, 4b_{73} + 5b_{74} + 7b_{75} + 10b_{76} + 7b_{77} + 4b_{78}, 2b_{48} + 2b_{56} + 2b_{57} + 2b_{58} + b_{60}) = (x_{11}, -7x_{24} + 4x_{41} + 12x_{42} + 14x_{43} + 10x_{44} + 4x_{45}, x_{64})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $x_{18},\ x_{22},\ x_{23}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{52},\ x_{53},\ x_{54},\ x_{55},\ x_{56},\ x_{57},\ x_{58},\ x_{60},\ x_{61},\ x_{62},\ x_{63},\ x_{64},\ x_{65}\}$. The subalgebra $\mathfrak{t}^e$ is spanned by x_{24}. We calculate $\kappa(e, f) = 216$.

Table 6.15: Basis for type E_6, orbit $2A_2 + A_1$.

				n_i	β_i
		x_1	b_{35}	5	-1
		x_2	b_{36}	5	1
		x_3	b_{30}	4	0
		x_4	b_{32}	4	2
		x_5	b_{33}	4	-2
		x_6	b_{34}	4	0
$\mathfrak{p}$	$\mathfrak{g}^e$	x_7	$b_{23} - b_{26}$	3	-1
		x_8	$b_{26} + b_{31}$	3	-1
		x_9	$b_{27} - b_{28}$	3	1
		x_{10}	$b_{28} + b_{29}$	3	1
		x_{11}	$b_{12} + b_{20} + b_{21} + b_{22} + b_{24}$	2	0
		x_{12}	$b_{12} + b_{20}$	2	0
		x_{13}	$b_{16} - b_{18}$	2	-2

Continued on next page

Table 6.15 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{14}	$b_{17} - b_{25}$	2	2
		x_{15}	$b_{21} + b_{22}$	2	0
		x_{16}	$b_1 + 2b_4 + b_{11} - b_{14}$	1	−1
		x_{17}	$b_4 - b_{14} + b_{15}$	1	−1
		x_{18}	$b_6 + b_7 + 2b_8 - b_9$	1	1
		x_{19}	$b_8 - b_9 + b_{19}$	1	1
		x_{20}	b_{10}	1	−3
		x_{21}	b_{13}	1	3
		x_{22}	$b_2 - b_3 + b_{41}$	0	2
		x_{23}	$b_5 + b_{38} - b_{39}$	0	−2
		x_{24}	$b_{74} + b_{75} - b_{77}$	0	0
		x_{25}	b_{23}	3	−1
		x_{26}	b_{27}	3	1
		x_{27}	b_{12}	2	0
		x_{28}	b_{16}	2	−2
		x_{29}	b_{17}	2	2
		x_{30}	b_{21}	2	0
		x_{31}	b_1	1	−1
		x_{32}	b_4	1	−1
		x_{33}	b_6	1	1
		x_{34}	b_7	1	1
		x_{35}	b_8	1	1
		x_{36}	b_{11}	1	−1
		x_{37}	b_2	0	2
		x_{38}	b_3	0	2
		x_{39}	b_5	0	−2
		x_{40}	b_{38}	0	−2
		x_{41}	b_{73}	0	0

Continued on next page

Table 6.15 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{42}	b_{74}	0	0
		x_{43}	b_{75}	0	0
		x_{44}	b_{76}	0	0
		x_{45}	b_{78}	0	0
		x_{46}	b_{37}	-1	1
		x_{47}	b_{40}	-1	1
		x_{48}	b_{46}	-1	3
		x_{49}	b_{47}	-1	1
		x_{50}	b_{50}	-1	1
		x_{51}	b_{51}	-1	1
$\mathfrak{m}$		x_{52}	b_{42}	-1	-1
		x_{53}	b_{43}	-1	-1
		x_{54}	b_{44}	-1	-1
		x_{55}	b_{45}	-1	-1
		x_{56}	b_{49}	-1	-3
		x_{57}	b_{55}	-1	-1
		x_{58}	b_{52}	-2	2
		x_{59}	b_{53}	-2	-2
		x_{60}	b_{54}	-2	2
		x_{61}	$b_{48} - b_{56}$	-2	0
		x_{62}	$b_{56} - b_{57}$	-2	0
		x_{63}	$b_{57} - b_{58}$	-2	0
		x_{64}	$2b_{48} + 2b_{56} + 2b_{57} + 2b_{58} + b_{60}$	-2	0
		x_{65}	$b_{58} - b_{60}$	-2	0
		x_{66}	b_{61}	-2	-2
		x_{67}	b_{59}	-3	1
		x_{68}	b_{62}	-3	1
		x_{69}	b_{63}	-3	-1

Continued on next page

Table 6.15 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{70}	b_{64}	-3	-1
		x_{71}	b_{65}	-3	-1
		x_{72}	b_{67}	-3	1
		x_{73}	b_{66}	-4	0
		x_{74}	b_{68}	-4	-2
		x_{75}	b_{69}	-4	2
		x_{76}	b_{70}	-4	0
		x_{77}	b_{71}	-5	1
		x_{78}	b_{72}	-5	-1

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, we require the commutators $F_{1,18}$, $F_{4,23}$, $F_{5,22}$, $F_{7,18}$, $F_{8,18}$, $F_{13,22}$, $F_{14,23}$, $F_{16,18}$, $F_{17,18}$, $F_{22,23}$:

$$\begin{aligned}
[\Theta_{22}, \Theta_{23}] &= -4 + \Theta_{24} \\
[\Theta_{16}, \Theta_{18}] &= 132 + 93\Theta_{24} - 2\Theta_{15} + 2\Theta_{12} + \Theta_{11} - 48\Theta_{22}\Theta_{23} \\
[\Theta_{14}, \Theta_{23}] &= -216 - 15\Theta_{24} - \Theta_{15} + \Theta_{12} - 21\Theta_{22}\Theta_{23} \\
[\Theta_{13}, \Theta_{22}] &= 168 - 39\Theta_{24} - \Theta_{15} + \Theta_{12} - 18\Theta_{24}^2 + 15\Theta_{22}\Theta_{23} \\
[\Theta_{17}, \Theta_{18}] &= 42 + 57\Theta_{24} - 4\Theta_{15} - 2\Theta_{12} + 3\Theta_{11} - 30\Theta_{22}\Theta_{23} \\
[\Theta_{7}, \Theta_{18}] &= 1683 + 2988\Theta_{24} - 135\Theta_{15} - 54\Theta_{12} + 54\Theta_{11} + 3\Theta_{3} + 378\Theta_{24}^2 \\
&\quad -3042\Theta_{22}\Theta_{23} - 27\Theta_{20}\Theta_{21} + 6\Theta_{16}\Theta_{18} + 21\Theta_{13}\Theta_{22} \\
&\quad -378\Theta_{22}\Theta_{23}\Theta_{24} \\
[\Theta_{5}, \Theta_{22}] &= 14508 + 693\Theta_{24} - 9\Theta_{15} + 90\Theta_{12} - 3\Theta_{11} - \Theta_{6} - \Theta_{3} \\
&\quad -1269\Theta_{24}^2 + 2034\Theta_{22}\Theta_{23} - 9\Theta_{20}\Theta_{21} - 9\Theta_{16}\Theta_{18} + 9\Theta_{15}\Theta_{24} \\
&\quad +9\Theta_{14}\Theta_{23} + 15\Theta_{13}\Theta_{22} + 9\Theta_{12}\Theta_{24} - 162\Theta_{24}^3 + 405\Theta_{22}\Theta_{23}\Theta_{24}
\end{aligned}$$

$$
\begin{aligned}
[\Theta_4, \Theta_{23}] &= 13500 + 1350\Theta_{24} + 108\Theta_{15} + 9\Theta_{12} - 3\Theta_{11} - \Theta_6 - \Theta_3 \\
&\quad +882\Theta_{22}\Theta_{23} - 9\Theta_{20}\Theta_{21} - 9\Theta_{16}\Theta_{18} + 9\Theta_{15}\Theta_{24} + 3\Theta_{14}\Theta_{23} \\
&\quad -27\Theta_{22}\Theta_{23}\Theta_{24} \\
[\Theta_8, \Theta_{18}] &= 3978 + 2313\Theta_{24} + 180\Theta_{15} + 243\Theta_{12} - 126\Theta_{11} - \Theta_6 - \Theta_3 \\
&\quad +594\Theta_{24}^2 + 738\Theta_{22}\Theta_{23} + 45\Theta_{20}\Theta_{21} + 9\Theta_{17}\Theta_{19} - 12\Theta_{17}\Theta_{18} \\
&\quad +9\Theta_{16}\Theta_{19} - 18\Theta_{14}\Theta_{23} - 15\Theta_{13}\Theta_{22} + 36\Theta_{12}\Theta_{24} \\
&\quad -108\Theta_{22}\Theta_{23}\Theta_{24} \\
[\Theta_1, \Theta_{18}] &= -85158 - 266868\Theta_{24} + 5616\Theta_{15} + 2241\Theta_{12} - 2025\Theta_{11} \\
&\quad +45\Theta_6 - 225\Theta_3 - 55728\Theta_{24}^2 + 185220\Theta_{22}\Theta_{23} - 972\Theta_{20}\Theta_{21} \\
&\quad -216\Theta_{17}\Theta_{19} + 126\Theta_{17}\Theta_{18} + 270\Theta_{16}\Theta_{19} + 810\Theta_{16}\Theta_{18} \\
&\quad +864\Theta_{15}\Theta_{24} - 108\Theta_{14}\Theta_{23} - 747\Theta_{13}\Theta_{22} + 270\Theta_{12}\Theta_{24} \\
&\quad -324\Theta_{11}\Theta_{24} - 9\Theta_{10}\Theta_{17} - 9\Theta_{10}\Theta_{16} - 6\Theta_8\Theta_{18} \\
&\quad -18\Theta_7\Theta_{18} - 6\Theta_5\Theta_{22} - 18\Theta_4\Theta_{23} - 36\Theta_3\Theta_{24} - 2916\Theta_{24}^3 \\
&\quad +45846\Theta_{22}\Theta_{23}\Theta_{24} - 108\Theta_{18}^2\Theta_{23} + 81\Theta_{17}\Theta_{18}\Theta_{24} + 81\Theta_{17}^2\Theta_{22} \\
&\quad +81\Theta_{16}\Theta_{18}\Theta_{24} - 81\Theta_{16}^2\Theta_{22} + 162\Theta_{15}\Theta_{22}\Theta_{23} - 108\Theta_{13}\Theta_{22}\Theta_{24} \\
&\quad +162\Theta_{12}\Theta_{22}\Theta_{23} - 162\Theta_{11}\Theta_{22}\Theta_{23} + 2916\Theta_{22}\Theta_{23}\Theta_{24}^2.
\end{aligned}
$$

We have one 1-dimensional representation:

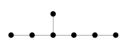
$$
\begin{aligned}
\Theta_3 &\mapsto -44613 \\
\Theta_6 &\mapsto -\tfrac{56130}{3} \\
\Theta_{11} &\mapsto -1056 \\
\Theta_{12} &\mapsto -299 \\
\Theta_{15} &\mapsto -\tfrac{3450}{6} \\
\Theta_{24} &\mapsto 4,
\end{aligned}
$$

with all other generators $\Theta_i \mapsto 0$. The coefficients which appear in the presentation of $U(\mathfrak{g}, e)$ are all integers, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.4 Type E_7

Here we calculate $U(\mathfrak{g}, e)$ for $\mathfrak{g}$ of type E_7 and e lying in each of the 7 non-zero rigid nilpotent orbits. We summarize in Table 6.16 certain data for these orbits, including the number of 1-dimensional representations for $U(\mathfrak{g}, e)$ in each case and the primes p, if known, for which we cannot define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$ where $\mathbb{k}$ has characteristic p.

Table 6.16: Results for type E_7.

Orbit	Dynkin diagram 	$\kappa(e,f)$	$\dim(\mathfrak{g}^e)$	$\dim(\mathfrak{t}^e)$	# 1-dim reps	Bad primes for $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$
A_1	0 1 0 0 0 0 0	36	99	6	1	2, 3
$2A_1$	0 0 0 0 0 1 0	72	81	5	1	2, 3
$(3A_1)'$	0 0 1 0 0 0 0	108	69	3	1	2, 3
$4A_1$	1 0 0 0 0 0 1	144	63	3	1	2, 3
$A_2 + 2A_1$	0 0 0 1 0 0 0	216	51	3	1	2, 3
$2A_2 + A_1$	0 0 1 0 0 1 0	324	43	2	1	
$(A_3 + A_1)'$	0 1 0 1 0 0 0	396	41	3	2	

For the first 5 of these orbits, we have calculated a full presentation, so we can in each of those cases identify for which primes p we cannot define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$, but for the last 2 rigid orbits in the table due to computational limitations we have only sufficient relations to determine the 1-dimensional representations.

The simple summands of the Levi subalgebras of $\mathfrak{g}$ are of types A_1, A_2, A_3, A_4, A_5, D_4, D_5, D_6 and E_6. Calculation of presentations of the associated finite W-algebras for rigid e shows that the only denominators which occur are powers of 2 and 3 and the values of $\kappa(e, f)$ which occur for rigid e have prime factors only 2, 3 and 5. Thus for rigid nilpotent $e \in \mathfrak{g}$ in one of the orbits A_1, $2A_1$, $(3A_1)'$, $4A_1$ or $A_2 + 2A_1$ we can apply Theorem 5.2.2 for $p > 5$. To extend this to all rigid orbits in $\mathfrak{g}$ would require significant further calculation of relations in $U(\mathfrak{g}, e)$ for e in the orbits $2A_2 + A_1$ and $(A_3 + A_1)'$.

6.4.1 The orbit A_1

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label A_1. In Table 6.17 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{63}, 2b_{127} + 2b_{128} + 3b_{129} + 4b_{130} + 3b_{131} + 2b_{132} + b_{133}, b_{126}) = (x_1, 2x_{94} + 3x_{95} + 4x_{96} + 3x_{97} + 2x_{98} + x_{99} + 2x_{100}, x_{133})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{33}, x_{63}, x_{64}, x_{65}, x_{71}, x_{72}, x_{74}, x_{79}, x_{81}, x_{82}, x_{83}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{117}, \ldots, x_{133}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{94}, \ldots, x_{99}\}$. We calculate $\kappa(e, f) = 36$.

Table 6.17: Basis for type E_7, orbit A_1.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{63}	2	$(0, 0, 0, 0, 0, 0)$
		x_2	b_1	1	$(0, -1, 0, 0, 0, 0)$
		x_3	b_8	1	$(0, 1, -1, 0, 0, 0)$
		x_4	b_{14}	1	$(-1, 0, 1, -1, 0, 0)$
		x_5	b_{20}	1	$(1, 0, 0, -1, 0, 0)$
		x_6	b_{21}	1	$(-1, 0, 0, 1, -1, 0)$
		x_7	b_{26}	1	$(1, 0, -1, 1, -1, 0)$

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Table 6.17 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_8	b_{27}	1	$(-1,0,0,0,1,-1)$
		x_9	b_{32}	1	$(0,-1,1,0,-1,0)$
		x_{10}	b_{33}	1	$(1,0,-1,0,1,-1)$
		x_{11}	b_{34}	1	$(-1,0,0,0,0,1)$
		x_{12}	b_{37}	1	$(0,1,0,0,-1,0)$
		x_{13}	b_{38}	1	$(0,-1,1,-1,1,-1)$
		x_{14}	b_{39}	1	$(1,0,-1,0,0,1)$
		x_{15}	b_{42}	1	$(0,1,0,-1,1,-1)$
		x_{16}	b_{43}	1	$(0,-1,0,1,0,-1)$
		x_{17}	b_{44}	1	$(0,-1,1,-1,0,1)$
		x_{18}	b_{46}	1	$(0,1,-1,1,0,-1)$
		x_{19}	b_{47}	1	$(0,1,0,-1,0,1)$
		x_{20}	b_{48}	1	$(0,-1,0,1,-1,1)$
		x_{21}	b_{50}	1	$(-1,0,1,0,0,-1)$
		x_{22}	b_{51}	1	$(0,1,-1,1,-1,1)$
		x_{23}	b_{52}	1	$(0,-1,0,0,1,0)$
		x_{24}	b_{53}	1	$(1,0,0,0,0,-1)$
		x_{25}	b_{54}	1	$(-1,0,1,0,-1,1)$
		x_{26}	b_{55}	1	$(0,1,-1,0,1,0)$
		x_{27}	b_{56}	1	$(1,0,0,0,-1,1)$
		x_{28}	b_{57}	1	$(-1,0,1,-1,1,0)$
		x_{29}	b_{58}	1	$(1,0,0,-1,1,0)$
		x_{30}	b_{59}	1	$(-1,0,0,1,0,0)$
		x_{31}	b_{60}	1	$(1,0,-1,1,0,0)$
		x_{32}	b_{61}	1	$(0,-1,1,0,0,0)$
		x_{33}	b_{62}	1	$(0,1,0,0,0,0)$
		x_{34}	b_2	0	$(2,0,-1,0,0,0)$
		x_{35}	b_3	0	$(0,2,-1,0,0,0)$

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Table 6.17 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{36}	b_4	0	$(-1,-1,2,-1,0,0)$
		x_{37}	b_5	0	$(0,0,-1,2,-1,0)$
		x_{38}	b_6	0	$(0,0,0,-1,2,-1)$
		x_{39}	b_7	0	$(0,0,0,0,-1,2)$
		x_{40}	b_9	0	$(1,-1,1,-1,0,0)$
		x_{41}	b_{10}	0	$(-1,1,1,-1,0,0)$
		x_{42}	b_{11}	0	$(-1,-1,1,1,-1,0)$
		x_{43}	b_{12}	0	$(0,0,-1,1,1,-1)$
		x_{44}	b_{13}	0	$(0,0,0,-1,1,1)$
		x_{45}	b_{15}	0	$(1,1,0,-1,0,0)$
		x_{46}	b_{16}	0	$(1,-1,0,1,-1,0)$
		x_{47}	b_{17}	0	$(-1,1,0,1,-1,0)$
		x_{48}	b_{18}	0	$(-1,-1,1,0,1,-1)$
		x_{49}	b_{19}	0	$(0,0,-1,1,0,1)$
		x_{50}	b_{22}	0	$(1,1,-1,1,-1,0)$
		x_{51}	b_{23}	0	$(1,-1,0,0,1,-1)$
		x_{52}	b_{24}	0	$(-1,1,0,0,1,-1)$
		x_{53}	b_{25}	0	$(-1,-1,1,0,0,1)$
		x_{54}	b_{28}	0	$(0,0,1,0,-1,0)$
		x_{55}	b_{29}	0	$(1,1,-1,0,1,-1)$
		x_{56}	b_{30}	0	$(1,-1,0,0,0,1)$
		x_{57}	b_{31}	0	$(-1,1,0,0,0,1)$
		x_{58}	b_{35}	0	$(0,0,1,-1,1,-1)$
		x_{59}	b_{36}	0	$(1,1,-1,0,0,1)$
		x_{60}	b_{40}	0	$(0,0,0,1,0,-1)$
		x_{61}	b_{41}	0	$(0,0,1,-1,0,1)$
		x_{62}	b_{45}	0	$(0,0,0,1,-1,1)$
		x_{63}	b_{49}	0	$(0,0,0,0,1,0)$

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Table 6.17 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{64}	b_{65}	0	$(-2,0,1,0,0,0)$
		x_{65}	b_{66}	0	$(0,-2,1,0,0,0)$
		x_{66}	b_{67}	0	$(1,1,-2,1,0,0)$
		x_{67}	b_{68}	0	$(0,0,1,-2,1,0)$
		x_{68}	b_{69}	0	$(0,0,0,1,-2,1)$
		x_{69}	b_{70}	0	$(0,0,0,0,1,-2)$
		x_{70}	b_{72}	0	$(-1,1,-1,1,0,0)$
		x_{71}	b_{73}	0	$(1,-1,-1,1,0,0)$
		x_{72}	b_{74}	0	$(1,1,-1,-1,1,0)$
		x_{73}	b_{75}	0	$(0,0,1,-1,-1,1)$
		x_{74}	b_{76}	0	$(0,0,0,1,-1,-1)$
		x_{75}	b_{78}	0	$(-1,-1,0,1,0,0)$
		x_{76}	b_{79}	0	$(-1,1,0,-1,1,0)$
		x_{77}	b_{80}	0	$(1,-1,0,-1,1,0)$
		x_{78}	b_{81}	0	$(1,1,-1,0,-1,1)$
		x_{79}	b_{82}	0	$(0,0,1,-1,0,-1)$
		x_{80}	b_{85}	0	$(-1,-1,1,-1,1,0)$
		x_{81}	b_{86}	0	$(-1,1,0,0,-1,1)$
		x_{82}	b_{87}	0	$(1,-1,0,0,-1,1)$
		x_{83}	b_{88}	0	$(1,1,-1,0,0,-1)$
		x_{84}	b_{91}	0	$(0,0,-1,0,1,0)$
		x_{85}	b_{92}	0	$(-1,-1,1,0,-1,1)$
		x_{86}	b_{93}	0	$(-1,1,0,0,0,-1)$
		x_{87}	b_{94}	0	$(1,-1,0,0,0,-1)$
		x_{88}	b_{98}	0	$(0,0,-1,1,-1,1)$
		x_{89}	b_{99}	0	$(-1,-1,1,0,0,-1)$
		x_{90}	b_{103}	0	$(0,0,0,-1,0,1)$
		x_{91}	b_{104}	0	$(0,0,-1,1,0,-1)$

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Table 6.17 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{92}	b_{108}	0	$(0,0,0,-1,1,-1)$
		x_{93}	b_{112}	0	$(0,0,0,0,-1,0)$
		x_{94}	b_{128}	0	$(0,0,0,0,0,0)$
		x_{95}	b_{129}	0	$(0,0,0,0,0,0)$
		x_{96}	b_{130}	0	$(0,0,0,0,0,0)$
		x_{97}	b_{131}	0	$(0,0,0,0,0,0)$
		x_{98}	b_{132}	0	$(0,0,0,0,0,0)$
		x_{99}	b_{133}	0	$(0,0,0,0,0,0)$
		x_{100}	b_{127}	0	$(0,0,0,0,0,0)$
		x_{101}	b_{64}	-1	$(0,1,0,0,0,0)$
		x_{102}	b_{71}	-1	$(0,-1,1,0,0,0)$
		x_{103}	b_{77}	-1	$(1,0,-1,1,0,0)$
		x_{104}	b_{83}	-1	$(-1,0,0,1,0,0)$
		x_{105}	b_{84}	-1	$(1,0,0,-1,1,0)$
		x_{106}	b_{89}	-1	$(-1,0,1,-1,1,0)$
		x_{107}	b_{90}	-1	$(1,0,0,0,-1,1)$
		x_{108}	b_{95}	-1	$(0,1,-1,0,1,0)$
		x_{109}	b_{96}	-1	$(-1,0,1,0,-1,1)$
		x_{110}	b_{97}	-1	$(1,0,0,0,0,-1)$
		x_{111}	b_{100}	-1	$(0,-1,0,0,1,0)$
		x_{112}	b_{101}	-1	$(0,1,-1,1,-1,1)$
		x_{113}	b_{102}	-1	$(-1,0,1,0,0,-1)$
		x_{114}	b_{105}	-1	$(0,-1,0,1,-1,1)$
		x_{115}	b_{106}	-1	$(0,1,0,-1,0,1)$
		x_{116}	b_{107}	-1	$(0,1,-1,1,0,-1)$
$\mathfrak{m}$		x_{117}	b_{109}	-1	$(0,-1,1,-1,0,1)$
		x_{118}	b_{110}	-1	$(0,-1,0,1,0,-1)$
		x_{119}	b_{111}	-1	$(0,1,0,-1,1,-1)$

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Table 6.17 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{120}	b_{113}	-1	$(1,0,-1,0,0,1)$
		x_{121}	b_{114}	-1	$(0,-1,1,-1,1,-1)$
		x_{122}	b_{115}	-1	$(0,1,0,0,-1,0)$
		x_{123}	b_{116}	-1	$(-1,0,0,0,0,1)$
		x_{124}	b_{117}	-1	$(1,0,-1,0,1,-1)$
		x_{125}	b_{118}	-1	$(0,-1,1,0,-1,0)$
		x_{126}	b_{119}	-1	$(-1,0,0,0,1,-1)$
		x_{127}	b_{120}	-1	$(1,0,-1,1,-1,0)$
		x_{128}	b_{121}	-1	$(-1,0,0,1,-1,0)$
		x_{129}	b_{122}	-1	$(1,0,0,-1,0,0)$
		x_{130}	b_{123}	-1	$(-1,0,1,-1,0,0)$
		x_{131}	b_{124}	-1	$(0,1,-1,0,0,0)$
		x_{132}	b_{125}	-1	$(0,-1,0,0,0,0)$
		x_{133}	b_{126}	-2	$(0,0,0,0,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, we require the commutators $F_{41,71}$, $F_{42,72}$, $F_{44,74}$, $F_{34,64}$, $F_{35,65}$, $F_{51,81}$, $F_{52,82}$, $F_{53,83}$, $F_{49,79}$, $F_{63,93}$, $F_{79,72}$:

$$\begin{aligned}
[\Theta_{41}, \Theta_{71}] &= -1 + \Theta_{96} + \Theta_{95} \\
[\Theta_{42}, \Theta_{72}] &= -1 + \Theta_{97} + \Theta_{96} \\
[\Theta_{44}, \Theta_{74}] &= -1 + \Theta_{99} + \Theta_{98} \\
[\Theta_{34}, \Theta_{64}] &= \Theta_{94} \\
[\Theta_{35}, \Theta_{65}] &= -1 + \Theta_{95} \\
[\Theta_{51}, \Theta_{81}] &= -1 + \Theta_{98} + \Theta_{97} + \Theta_{96} + \Theta_{94} \\
[\Theta_{52}, \Theta_{82}] &= -2 + \Theta_{98} + \Theta_{97} + \Theta_{96} + \Theta_{95} \\
[\Theta_{53}, \Theta_{83}] &= -2 + \Theta_{99} + \Theta_{98} + \Theta_{97} + \Theta_{96}
\end{aligned}$$

$$\begin{aligned}
[\Theta_{49}, \Theta_{79}] &= -2 + \Theta_{99} + \Theta_{98} + \Theta_{97} \\
[\Theta_{63}, \Theta_{93}] &= -4 + \Theta_{99} + 2\Theta_{98} + 2\Theta_{97} + 2\Theta_{96} + \Theta_{95} + \Theta_{94} \\
[\Theta_{2}, \Theta_{33}] &= 31 - \Theta_{98} - 2\Theta_{97} - 4\Theta_{96} - 4\Theta_{95} - 2\Theta_{94} + \Theta_{1} - \Theta_{63}\Theta_{93} \\
&\quad -\Theta_{62}\Theta_{92} - \Theta_{61}\Theta_{91} - \Theta_{60}\Theta_{90} - \Theta_{59}\Theta_{89} - \Theta_{58}\Theta_{88} - \Theta_{57}\Theta_{87} \\
&\quad -\Theta_{55}\Theta_{85} - \Theta_{54}\Theta_{84} - \Theta_{52}\Theta_{82} - \Theta_{50}\Theta_{80} - \Theta_{47}\Theta_{77} - \Theta_{45}\Theta_{75} \\
&\quad -\Theta_{41}\Theta_{71} - \Theta_{35}\Theta_{65}.
\end{aligned}$$

We have one 1-dimensional representation:

$$\begin{aligned}
\Theta_{1} &\mapsto -25 \\
\Theta_{94} &\mapsto 0 \\
\Theta_{95} &\mapsto 1 \\
\Theta_{96} &\mapsto 0 \\
\Theta_{97} &\mapsto 1 \\
\Theta_{98} &\mapsto 0 \\
\Theta_{99} &\mapsto 1,
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The coefficients which appear in the presentation of $U(\mathfrak{g}, e)$ are all integers, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$. We again note that the existence of a 1-dimensional representation in this case was known from [P07i], as our orbit is minimal.

6.4.2 The orbit $2A_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $2A_1$. In Table 6.18 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{57} + b_{60}, 2b_{127} + 3b_{128} + 4b_{129} + 6b_{130} + 5b_{131} + 4b_{132} + 2b_{133}, b_{120} + b_{123}) = (x_4, 2x_{77} + x_{78} + 4x_{79} + 5x_{80} + 2x_{81} + 2x_{90} + 4x_{91}, x_{127})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{42}, x_{43}, x_{44}, x_{45}, x_{46}, x_{47}, x_{48}, x_{61}, x_{62}, x_{63}, x_{64}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{108}, \ldots, x_{130}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{77}, \ldots, x_{81}\}$. We calculate $\kappa(e, f) = 72$.

Table 6.18: Basis for type E_7, orbit $2A_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{49}	2	$(-1,0,0,0,0)$
		x_2	b_{52}	2	$(1,0,-1,0,0)$
		x_3	b_{55}	2	$(0,-1,1,-1,0)$
		x_4	$b_{57}+b_{60}$	2	$(0,0,0,0,0)$
		x_5	b_{58}	2	$(0,1,0,-1,0)$
		x_6	b_{59}	2	$(0,-1,0,1,0)$
		x_7	b_{60}	2	$(0,0,0,0,0)$
		x_8	b_{61}	2	$(0,1,-1,1,0)$
		x_9	b_{62}	2	$(-1,0,1,0,0)$
		x_{10}	b_{63}	2	$(1,0,0,0,0)$
		x_{11}	b_6	1	$(0,0,0,-1,-1)$
		x_{12}	b_{12}	1	$(0,-1,0,0,-1)$
		x_{13}	b_{13}	1	$(0,0,0,-1,1)$
		x_{14}	b_{18}	1	$(0,0,-1,1,-1)$
		x_{15}	b_{19}	1	$(0,-1,0,0,1)$
		x_{16}	b_{23}	1	$(0,1,-1,0,-1)$
		x_{17}	b_{24}	1	$(-1,-1,1,0,-1)$
		x_{18}	b_{25}	1	$(0,0,-1,1,1)$
		x_{19}	b_{27}	1	$(1,-1,0,0,-1)$
		x_{20}	b_{29}	1	$(-1,0,1,-1,-1)$
		x_{21}	b_{30}	1	$(0,1,-1,0,1)$
		x_{22}	b_{31}	1	$(-1,-1,1,0,1)$
		x_{23}	b_{33}	1	$(1,0,0,-1,-1)$
		x_{24}	b_{34}	1	$(1,-1,0,0,1)$
		x_{25}	b_{35}	1	$(-1,1,0,0,-1)$
		x_{26}	b_{36}	1	$(-1,0,1,-1,1)$
		x_{27}	b_{38}	1	$(1,1,-1,0,-1)$

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Table 6.18 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{28}	b_{39}	1	$(1,0,0,-1,1)$
		x_{29}	b_{40}	1	$(-1,0,0,1,-1)$
		x_{30}	b_{41}	1	$(-1,1,0,0,1)$
		x_{31}	b_{42}	1	$(0,0,1,-1,-1)$
		x_{32}	b_{43}	1	$(1,0,-1,1,-1)$
		x_{33}	b_{44}	1	$(1,1,-1,0,1)$
		x_{34}	b_{45}	1	$(-1,0,0,1,1)$
		x_{35}	b_{46}	1	$(0,-1,1,0,-1)$
		x_{36}	b_{47}	1	$(0,0,1,-1,1)$
		x_{37}	b_{48}	1	$(1,0,-1,1,1)$
		x_{38}	b_{50}	1	$(0,0,0,1,-1)$
		x_{39}	b_{51}	1	$(0,-1,1,0,1)$
		x_{40}	b_{53}	1	$(0,1,0,0,-1)$
		x_{41}	b_{54}	1	$(0,0,0,1,1)$
		x_{42}	b_{56}	1	$(0,1,0,0,1)$
		x_{43}	b_1	0	$(2,0,-1,0,0)$
		x_{44}	b_2-b_{68}	0	$(0,1,0,-1,0)$
		x_{45}	b_3	0	$(-1,-1,2,-1,0)$
		x_{46}	b_4+b_{16}	0	$(0,1,-1,1,0)$
		x_{47}	b_5-b_{65}	0	$(0,-1,0,1,0)$
		x_{48}	b_7	0	$(0,0,0,0,2)$
		x_{49}	b_8	0	$(1,-1,1,-1,0)$
		x_{50}	b_9	0	$(0,2,-1,0,0)$
		x_{51}	$b_{10}+b_{22}$	0	$(-1,0,1,0,0)$
		x_{52}	b_{11}	0	$(0,0,-1,2,0)$
		x_{53}	$b_{14}+b_{26}$	0	$(1,0,0,0,0)$
		x_{54}	b_{15}	0	$(-1,1,1,-1,0)$
		x_{55}	b_{17}	0	$(-1,-1,1,1,0)$

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Table 6.18 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{56}	b_{20}	0	$(1,1,0,-1,0)$
		x_{57}	b_{21}	0	$(1,-1,0,1,0)$
		x_{58}	b_{28}	0	$(-1,1,0,1,0)$
		x_{59}	b_{32}	0	$(1,1,-1,1,0)$
		x_{60}	b_{37}	0	$(0,0,1,0,0)$
		x_{61}	b_{64}	0	$(-2,0,1,0,0)$
		x_{62}	b_{66}	0	$(1,1,-2,1,0)$
		x_{63}	$b_{67}+b_{79}$	0	$(0,-1,1,-1,0)$
		x_{64}	b_{70}	0	$(0,0,0,0,-2)$
		x_{65}	b_{71}	0	$(-1,1,-1,1,0)$
		x_{66}	b_{72}	0	$(0,-2,1,0,0)$
		x_{67}	$b_{73}+b_{85}$	0	$(1,0,-1,0,0)$
		x_{68}	b_{74}	0	$(0,0,1,-2,0)$
		x_{69}	$b_{77}+b_{89}$	0	$(-1,0,0,0,0)$
		x_{70}	b_{78}	0	$(1,-1,-1,1,0)$
		x_{71}	b_{80}	0	$(1,1,-1,-1,0)$
		x_{72}	b_{83}	0	$(-1,-1,0,1,0)$
		x_{73}	b_{84}	0	$(-1,1,0,-1,0)$
		x_{74}	b_{91}	0	$(1,-1,0,-1,0)$
		x_{75}	b_{95}	0	$(-1,-1,1,-1,0)$
		x_{76}	b_{100}	0	$(0,0,-1,0,0)$
		x_{77}	b_{127}	0	$(0,0,0,0,0)$
		x_{78}	$b_{128}+b_{130}$	0	$(0,0,0,0,0)$
		x_{79}	b_{129}	0	$(0,0,0,0,0)$
		x_{80}	$b_{130}+b_{131}$	0	$(0,0,0,0,0)$
		x_{81}	b_{133}	0	$(0,0,0,0,0)$
		x_{82}	b_{2}	0	$(0,1,0,-1,0)$
		x_{83}	b_{4}	0	$(0,1,-1,1,0)$

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Table 6.18 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{84}	b_5	0	$(0,-1,0,1,0)$
		x_{85}	b_{10}	0	$(-1,0,1,0,0)$
		x_{86}	b_{14}	0	$(1,0,0,0,0)$
		x_{87}	b_{67}	0	$(0,-1,1,-1,0)$
		x_{88}	b_{73}	0	$(1,0,-1,0,0)$
		x_{89}	b_{77}	0	$(-1,0,0,0,0)$
		x_{90}	b_{128}	0	$(0,0,0,0,0)$
		x_{91}	b_{132}	0	$(0,0,0,0,0)$
		x_{92}	b_{69}	-1	$(0,0,0,1,1)$
		x_{93}	b_{75}	-1	$(0,1,0,0,1)$
		x_{94}	b_{76}	-1	$(0,0,0,1,-1)$
		x_{95}	b_{81}	-1	$(0,0,1,-1,1)$
		x_{96}	b_{82}	-1	$(0,1,0,0,-1)$
		x_{97}	b_{86}	-1	$(0,-1,1,0,1)$
		x_{98}	b_{87}	-1	$(1,1,-1,0,1)$
		x_{99}	b_{88}	-1	$(0,0,1,-1,-1)$
		x_{100}	b_{90}	-1	$(-1,1,0,0,1)$
		x_{101}	b_{92}	-1	$(1,0,-1,1,1)$
		x_{102}	b_{93}	-1	$(0,-1,1,0,-1)$
		x_{103}	b_{94}	-1	$(1,1,-1,0,-1)$
		x_{104}	b_{96}	-1	$(-1,0,0,1,1)$
		x_{105}	b_{97}	-1	$(-1,1,0,0,-1)$
		x_{106}	b_{99}	-1	$(1,0,-1,1,-1)$
		x_{107}	b_{102}	-1	$(-1,0,0,1,-1)$
$\mathfrak{m}$		x_{108}	b_{98}	-1	$(1,-1,0,0,1)$
		x_{109}	b_{101}	-1	$(-1,-1,1,0,1)$
		x_{110}	b_{103}	-1	$(1,0,0,-1,1)$
		x_{111}	b_{104}	-1	$(1,-1,0,0,-1)$

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Table 6.18 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{112}	b_{105}	-1	$(0,0,-1,1,1)$
		x_{113}	b_{106}	-1	$(-1,0,1,-1,1)$
		x_{114}	b_{107}	-1	$(-1,-1,1,0,-1)$
		x_{115}	b_{108}	-1	$(1,0,0,-1,-1)$
		x_{116}	b_{109}	-1	$(0,1,-1,0,1)$
		x_{117}	b_{110}	-1	$(0,0,-1,1,-1)$
		x_{118}	b_{111}	-1	$(-1,0,1,-1,-1)$
		x_{119}	b_{113}	-1	$(0,0,0,-1,1)$
		x_{120}	b_{114}	-1	$(0,1,-1,0,-1)$
		x_{121}	b_{116}	-1	$(0,-1,0,0,1)$
		x_{122}	b_{117}	-1	$(0,0,0,-1,-1)$
		x_{123}	b_{119}	-1	$(0,-1,0,0,-1)$
		x_{124}	b_{112}	-2	$(1,0,0,0,0)$
		x_{125}	b_{115}	-2	$(-1,0,1,0,0)$
		x_{126}	b_{118}	-2	$(0,1,-1,1,0)$
		x_{127}	$b_{120}+b_{123}$	-2	$(0,0,0,0,0)$
		x_{128}	$b_{120}-b_{123}$	-2	$(0,0,0,0,0)$
		x_{129}	b_{121}	-2	$(0,-1,0,1,0)$
		x_{130}	b_{122}	-2	$(0,1,0,-1,0)$
		x_{131}	b_{124}	-2	$(0,-1,1,-1,0)$
		x_{132}	b_{125}	-2	$(1,0,-1,0,0)$
		x_{133}	b_{126}	-2	$(-1,0,0,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{8,63}$, $F_{43,61}$, $F_{44,47}$, $F_{45,62}$, $F_{48,64}$, $F_{6,44}$, $F_{46,63}$,

$F_{5,47},\ F_{3,46},\ F_{12,42}$:

$$\begin{aligned}
[\Theta_8, \Theta_{63}] &= 72 - 4\Theta_{81} - 16\Theta_{80} - 12\Theta_{79} - 42\Theta_{78} - 6\Theta_{77} - 2\Theta_7 + \Theta_4 \\
&\quad +2\Theta_{78}\Theta_{81} + 6\Theta_{78}\Theta_{80} + 4\Theta_{78}\Theta_{79} + 2\Theta_{78}^2 + 2\Theta_{77}\Theta_{78} \\
&\quad -2\Theta_{59}\Theta_{75} - 2\Theta_{58}\Theta_{74} - 2\Theta_{52}\Theta_{68} - 2\Theta_{50}\Theta_{66} + 2\Theta_{49}\Theta_{65} \\
&\quad -2\Theta_{46}\Theta_{63} + 2\Theta_{45}\Theta_{62} \\
[\Theta_{43}, \Theta_{61}] &= \Theta_{77} \\
[\Theta_{44}, \Theta_{47}] &= \Theta_{80} - \Theta_{78} \\
[\Theta_{45}, \Theta_{62}] &= \Theta_{79} \\
[\Theta_{48}, \Theta_{64}] &= \Theta_{81} \\
[\Theta_6, \Theta_{44}] &= 36\Theta_{80} - 4\Theta_{79} - 42\Theta_{78} - 2\Theta_{77} - 2\Theta_7 + \Theta_4 - 2\Theta_{80}\Theta_{81} - 6\Theta_{80}^2 \\
&\quad -4\Theta_{79}\Theta_{80} + 2\Theta_{78}\Theta_{81} + 6\Theta_{78}\Theta_{80} + 4\Theta_{78}\Theta_{79} - 2\Theta_{77}\Theta_{80} \\
&\quad +2\Theta_{77}\Theta_{78} - 2\Theta_{57}\Theta_{73} + 2\Theta_{56}\Theta_{72} - 2\Theta_{55}\Theta_{71} + 2\Theta_{54}\Theta_{70} \\
&\quad -2\Theta_{52}\Theta_{68} + 2\Theta_{50}\Theta_{66} - 2\Theta_{44}\Theta_{47} \\
[\Theta_{46}, \Theta_{63}] &= -4 + \Theta_{80} + \Theta_{78} \\
[\Theta_5, \Theta_{47}] &= -6\Theta_{80} - 4\Theta_{79} - 2\Theta_{77} - 2\Theta_7 + \Theta_4 + 2\Theta_{57}\Theta_{73} - 2\Theta_{56}\Theta_{72} \\
&\quad +2\Theta_{55}\Theta_{71} - 2\Theta_{54}\Theta_{70} + 2\Theta_{52}\Theta_{68} - 2\Theta_{50}\Theta_{66} + 4\Theta_{44}\Theta_{47} \\
[\Theta_3, \Theta_{46}] &= -48 + 4\Theta_{81} + 10\Theta_{80} + 12\Theta_{79} + 28\Theta_{78} + 6\Theta_{77} + 2\Theta_7 - \Theta_4 \\
&\quad -2\Theta_{78}\Theta_{81} - 4\Theta_{78}\Theta_{80} - 4\Theta_{78}\Theta_{79} - 2\Theta_{77}\Theta_{78} - 2\Theta_{59}\Theta_{75} \\
&\quad -2\Theta_{58}\Theta_{74} + 2\Theta_{52}\Theta_{68} + 2\Theta_{50}\Theta_{66} + 2\Theta_{49}\Theta_{65} + 2\Theta_{45}\Theta_{62} \\
[\Theta_{12}, \Theta_{42}] &= -36 + 6\Theta_{81} + 26\Theta_{80} + 4\Theta_{79} - 20\Theta_{78} + 2\Theta_{77} - \Theta_7 - 2\Theta_{80}\Theta_{81} \\
&\quad -4\Theta_{80}^2 - 4\Theta_{79}\Theta_{80} + 2\Theta_{78}\Theta_{81} + 2\Theta_{78}\Theta_{80} + 4\Theta_{78}\Theta_{79} + 2\Theta_{78}^2 \\
&\quad -2\Theta_{77}\Theta_{80} + 2\Theta_{77}\Theta_{78} + 2\Theta_{60}\Theta_{76} + 2\Theta_{59}\Theta_{75} + 2\Theta_{58}\Theta_{74} \\
&\quad +2\Theta_{56}\Theta_{72} + 2\Theta_{54}\Theta_{70} + 2\Theta_{50}\Theta_{66} + 2\Theta_{48}\Theta_{64}.
\end{aligned}$$

We have one 1-dimensional representation:

$$\begin{aligned}\Theta_4 &\mapsto -36\\ \Theta_7 &\mapsto -24\\ \Theta_{77} &\mapsto 0\\ \Theta_{78} &\mapsto 2\\ \Theta_{79} &\mapsto 0\\ \Theta_{80} &\mapsto 2\\ \Theta_{81} &\mapsto 0,\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 3, so we can define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$ provided $\mathbb{k}$ has characteristic $p > 3$.

6.4.3 The orbit $(3A_1)'$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $(3A_1)'$. In Table 6.19 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{42} + b_{56} + b_{59}, 3b_{127} + 4b_{128} + 6b_{129} + 8b_{130} + 6b_{131} + 4b_{132} + 2b_{133}, b_{105} + b_{119} + b_{122}) = (x_4, 3x_{66} + 2x_{67} + 8x_{68} + 2x_{69} + 2x_{84} + 6x_{85} + xb_{86}, x_{129})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{45}, x_{46}, x_{48}, x_{49}, x_{50}, x_{51}, x_{59}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{102}, x_{103}, x_{104}, x_{105}, x_{106}, x_{107}, x_{108}, x_{109}, x_{110}, x_{111}, x_{112}, x_{113}, x_{114}, x_{115}, x_{116}, x_{117}, x_{118}, x_{119}, x_{120}, x_{121}, x_{122}, x_{123}, x_{124}, x_{126}, x_{129}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{66}, x_{67}, x_{68}, x_{69}\}$. We calculate $\kappa(e, f) = 108$.

Table 6.19: Basis for type E_7, orbit $(3A_1)'$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{62}	3	$(-1,0,0,0)$
		x_2	b_{63}	3	$(1,0,0,0)$
		x_3	b_{37}	2	$(0,-1,0,-1)$
		x_4	$b_{42}+b_{56}+b_{59}$	2	$(0,0,0,0)$
		x_5	b_{46}	2	$(0,1,-1,-1)$
		x_6	b_{47}	2	$(0,-1,0,1)$
		x_7	b_{50}	2	$(0,-1,1,-1)$
		x_8	b_{51}	2	$(0,0,-1,0)$
		x_9	b_{53}	2	$(0,1,0,-1)$
		x_{10}	b_{54}	2	$(0,-2,1,0)$
		x_{11}	b_{55}	2	$(0,1,-1,1)$
		x_{12}	b_{56}	2	$(0,0,0,0)$
		x_{13}	b_{57}	2	$(0,-1,1,1)$
		x_{14}	b_{58}	2	$(0,1,0,1)$
		x_{15}	b_{59}	2	$(0,0,0,0)$
		x_{16}	b_{60}	2	$(0,2,-1,0)$
		x_{17}	b_{61}	2	$(0,0,1,0)$
		x_{18}	b_3	1	$(-1,0,-1,0)$
		x_{19}	b_8	1	$(1,0,-1,0)$
		x_{20}	b_{10}	1	$(-1,-2,1,0)$
		x_{21}	b_{14}	1	$(1,-2,1,0)$
		x_{22}	$b_{15}-b_{24}$	1	$(-1,0,0,0)$
		x_{23}	b_{17}	1	$(-1,-1,0,-1)$
		x_{24}	$b_{20}-b_{27}$	1	$(1,0,0,0)$
		x_{25}	b_{21}	1	$(1,-1,0,-1)$
		x_{26}	b_{22}	1	$(-1,1,-1,-1)$
		x_{27}	$b_{24}+b_{45}$	1	$(-1,0,0,0)$

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Table 6.19 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{28}	b_{26}	1	$(1,1,-1,-1)$
		x_{29}	$b_{27}+b_{48}$	1	$(1,0,0,0)$
		x_{30}	b_{28}	1	$(-1,-1,1,-1)$
		x_{31}	b_{29}	1	$(-1,2,-1,0)$
		x_{32}	b_{31}	1	$(-1,-1,0,1)$
		x_{33}	b_{32}	1	$(1,-1,1,-1)$
		x_{34}	b_{33}	1	$(1,2,-1,0)$
		x_{35}	b_{34}	1	$(1,-1,0,1)$
		x_{36}	b_{35}	1	$(-1,0,1,0)$
		x_{37}	b_{36}	1	$(-1,1,-1,1)$
		x_{38}	b_{38}	1	$(1,0,1,0)$
		x_{39}	b_{39}	1	$(1,1,-1,1)$
		x_{40}	b_{40}	1	$(-1,1,0,-1)$
		x_{41}	b_{41}	1	$(-1,-1,1,1)$
		x_{42}	b_{43}	1	$(1,1,0,-1)$
		x_{43}	b_{44}	1	$(1,-1,1,1)$
		x_{44}	b_{49}	1	$(-1,1,0,1)$
		x_{45}	b_{52}	1	$(1,1,0,1)$
		x_{46}	b_1	0	$(2,0,0,0)$
		x_{47}	b_2+b_{12}	0	$(0,2,-1,0)$
		x_{48}	b_4	0	$(0,-2,2,0)$
		x_{49}	b_5-b_{88}	0	$(0,1,-1,-1)$
		x_{50}	b_6+b_{30}	0	$(0,1,0,1)$
		x_{51}	b_7-b_{79}	0	$(0,-1,0,1)$
		x_{52}	b_9-b_{18}	0	$(0,0,1,0)$
		x_{53}	$b_{11}+b_{82}$	0	$(0,-1,1,-1)$
		x_{54}	b_{13}	0	$(0,0,0,2)$
		x_{55}	$b_{16}-b_{70}$	0	$(0,1,0,-1)$

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Table 6.19 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{56}	$b_{19} + b_{74}$	0	$(0, 1, -1, 1)$
		x_{57}	b_{23}	0	$(0, 2, 0, 0)$
		x_{58}	$b_{25} - b_{68}$	0	$(0, -1, 1, 1)$
		x_{59}	b_{64}	0	$(-2, 0, 0, 0)$
		x_{60}	$b_{65} + b_{75}$	0	$(0, -2, 1, 0)$
		x_{61}	b_{67}	0	$(0, 2, -2, 0)$
		x_{62}	$b_{69} + b_{93}$	0	$(0, -1, 0, -1)$
		x_{63}	$b_{72} - b_{81}$	0	$(0, 0, -1, 0)$
		x_{64}	b_{76}	0	$(0, 0, 0, -2)$
		x_{65}	b_{86}	0	$(0, -2, 0, 0)$
		x_{66}	b_{127}	0	$(0, 0, 0, 0)$
		x_{67}	$b_{128} + b_{131} + b_{132}$	0	$(0, 0, 0, 0)$
		x_{68}	b_{130}	0	$(0, 0, 0, 0)$
		x_{69}	$b_{132} + b_{133}$	0	$(0, 0, 0, 0)$
		x_{70}	b_{15}	1	$(-1, 0, 0, 0)$
		x_{71}	b_{20}	1	$(1, 0, 0, 0)$
		x_{72}	b_{2}	0	$(0, 2, -1, 0)$
		x_{73}	b_{5}	0	$(0, 1, -1, -1)$
		x_{74}	b_{6}	0	$(0, 1, 0, 1)$
		x_{75}	b_{7}	0	$(0, -1, 0, 1)$
		x_{76}	b_{9}	0	$(0, 0, 1, 0)$
		x_{77}	b_{11}	0	$(0, -1, 1, -1)$
		x_{78}	b_{16}	0	$(0, 1, 0, -1)$
		x_{79}	b_{19}	0	$(0, 1, -1, 1)$
		x_{80}	b_{25}	0	$(0, -1, 1, 1)$
		x_{81}	b_{65}	0	$(0, -2, 1, 0)$
		x_{82}	b_{69}	0	$(0, -1, 0, -1)$
		x_{83}	b_{72}	0	$(0, 0, -1, 0)$

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Table 6.19 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{84}	b_{128}	0	$(0,0,0,0)$
		x_{85}	b_{129}	0	$(0,0,0,0)$
		x_{86}	b_{131}	0	$(0,0,0,0)$
		x_{87}	b_{66}	-1	$(1,0,1,0)$
		x_{88}	b_{71}	-1	$(-1,0,1,0)$
		x_{89}	b_{73}	-1	$(1,2,-1,0)$
		x_{90}	b_{77}	-1	$(-1,2,-1,0)$
		x_{91}	b_{78}	-1	$(1,0,0,0)$
		x_{92}	b_{80}	-1	$(1,1,0,1)$
		x_{93}	b_{84}	-1	$(-1,1,0,1)$
		x_{94}	b_{85}	-1	$(1,-1,1,1)$
		x_{95}	b_{87}	-1	$(1,0,0,0)$
		x_{96}	b_{89}	-1	$(-1,-1,1,1)$
		x_{97}	b_{91}	-1	$(1,1,-1,1)$
		x_{98}	b_{94}	-1	$(1,1,0,-1)$
		x_{99}	b_{95}	-1	$(-1,1,-1,1)$
		x_{100}	b_{103}	-1	$(1,-1,0,1)$
		x_{101}	b_{108}	-1	$(1,0,0,0)$
$\mathfrak{m}$		x_{102}	b_{83}	-1	$(-1,0,0,0)$
		x_{103}	b_{90}	-1	$(-1,0,0,0)$
		x_{104}	b_{92}	-1	$(1,-2,1,0)$
		x_{105}	b_{96}	-1	$(-1,-2,1,0)$
		x_{106}	b_{97}	-1	$(-1,1,0,-1)$
		x_{107}	b_{98}	-1	$(1,0,-1,0)$
		x_{108}	b_{99}	-1	$(1,-1,1,-1)$
		x_{109}	b_{101}	-1	$(-1,0,-1,0)$
		x_{110}	b_{102}	-1	$(-1,-1,1,-1)$
		x_{111}	b_{104}	-1	$(1,1,-1,-1)$

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Table 6.19 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{112}	b_{106}	-1	$(-1,-1,0,1)$
		x_{113}	b_{107}	-1	$(-1,1,-1,-1)$
		x_{114}	b_{111}	-1	$(-1,0,0,0)$
		x_{115}	b_{112}	-1	$(1,-1,0,-1)$
		x_{116}	b_{115}	-1	$(-1,-1,0,-1)$
		x_{117}	b_{100}	-2	$(0,1,0,1)$
		x_{118}	$b_{105}-b_{119}$	-2	$(0,0,0,0)$
		x_{119}	b_{109}	-2	$(0,-1,1,1)$
		x_{120}	b_{110}	-2	$(0,1,0,-1)$
		x_{121}	b_{113}	-2	$(0,1,-1,1)$
		x_{122}	b_{114}	-2	$(0,0,1,0)$
		x_{123}	b_{116}	-2	$(0,-1,0,1)$
		x_{124}	b_{117}	-2	$(0,2,-1,0)$
		x_{125}	b_{118}	-2	$(0,-1,1,-1)$
		x_{126}	$b_{119}-b_{122}$	-2	$(0,0,0,0)$
		x_{127}	b_{120}	-2	$(0,1,-1,-1)$
		x_{128}	b_{121}	-2	$(0,-1,0,-1)$
		x_{129}	$b_{105}+b_{119}+b_{122}$	-2	$(0,0,0,0)$
		x_{130}	b_{123}	-2	$(0,-2,1,0)$
		x_{131}	b_{124}	-2	$(0,0,-1,0)$
		x_{132}	b_{125}	-3	$(1,0,0,0)$
		x_{133}	b_{126}	-3	$(-1,0,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{46,59}$, $F_{48,61}$, $F_{49,58}$, $F_{50,62}$, $F_{51,55}$, $F_{3,50}$, $F_{9,51}$, $F_{13,49}$, $F_{23,45}$:

$$\begin{aligned}
[\Theta_{46}, \Theta_{59}] &= -\tfrac{5}{2} + \Theta_{66} \\
[\Theta_{48}, \Theta_{61}] &= \Theta_{68} \\
[\Theta_{49}, \Theta_{58}] &= -3 + \Theta_{69} + \Theta_{68} \\
[\Theta_{50}, \Theta_{62}] &= -6 + \Theta_{69} + \Theta_{68} + \Theta_{67} \\
[\Theta_{51}, \Theta_{55}] &= -\Theta_{69} + \Theta_{68} + \Theta_{67} \\
[\Theta_{3}, \Theta_{50}] &= -45 - 3\Theta_{69} + 3\Theta_{68} + 21\Theta_{67} + 9\Theta_{66} + \Theta_{15} + 2\Theta_{12} - \Theta_{4} \\
&\quad -6\Theta_{67}\Theta_{68} - 3\Theta_{66}\Theta_{67} + 3\Theta_{57}\Theta_{65} + 3\Theta_{54}\Theta_{64} + 3\Theta_{52}\Theta_{63} \\
&\quad +3\Theta_{47}\Theta_{60} \\
[\Theta_{9}, \Theta_{51}] &= 42\Theta_{68} - 3\Theta_{67} - \Theta_{15} - 2\Theta_{12} + \Theta_{4} - 9\Theta_{68}^2 - 3\Theta_{67}\Theta_{68} \\
&\quad -3\Theta_{66}\Theta_{68} - 3\Theta_{57}\Theta_{65} + 3\Theta_{54}\Theta_{64} - 3\Theta_{53}\Theta_{56} + 3\Theta_{51}\Theta_{55} \\
&\quad +3\Theta_{49}\Theta_{58} \\
[\Theta_{13}, \Theta_{49}] &= -93 + 54\Theta_{69} + 69\Theta_{68} + 3\Theta_{67} + 3\Theta_{66} - 2\Theta_{15} - \Theta_{12} + \Theta_{4} \\
&\quad -3\Theta_{69}^2 - 12\Theta_{68}\Theta_{69} - 9\Theta_{68}^2 - 3\Theta_{67}\Theta_{69} - 3\Theta_{67}\Theta_{68} - 3\Theta_{66}\Theta_{69} \\
&\quad -3\Theta_{66}\Theta_{68} - 3\Theta_{54}\Theta_{64} - 3\Theta_{52}\Theta_{63} - 3\Theta_{49}\Theta_{58} - 3\Theta_{48}\Theta_{61} \\
[\Theta_{23}, \Theta_{45}] &= -\tfrac{9}{2} + 33\Theta_{68} - \Theta_{15} - 3\Theta_{68}\Theta_{69} - 6\Theta_{68}^2 - 3\Theta_{67}\Theta_{68} - 3\Theta_{66}\Theta_{68} \\
&\quad +3\Theta_{57}\Theta_{65} + 3\Theta_{54}\Theta_{64} + \tfrac{3}{2}\Theta_{50}\Theta_{62} + 3\Theta_{46}\Theta_{59}.
\end{aligned}$$

We have one 1-dimensional representation:

$$\begin{aligned}
\Theta_{4} &\mapsto -\tfrac{81}{2} \\
\Theta_{12} &\mapsto -\tfrac{45}{2} \\
\Theta_{15} &\mapsto -\tfrac{9}{2} \\
\Theta_{66} &\mapsto \tfrac{5}{2} \\
\Theta_{67} &\mapsto 3 \\
\Theta_{68} &\mapsto 0 \\
\Theta_{69} &\mapsto 3,
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.4.4 The orbit $4A_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $4A_1$. In Table 6.20 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{45} + b_{47} + b_{52} + b_{53}, 3b_{127} + 5b_{128} + 6b_{129} + 9b_{130} + 7b_{131} + 5b_{132} + 3b_{133}, b_{108} + b_{110} + b_{115} + b_{116}) = (x_{12}, 2x_{61} + 5x_{62} + 9x_{63} + x_{82} + 5x_{83} - x_{84} + 3x_{85}, x_{117})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{45},\ x_{46},\ x_{48},\ x_{49},\ x_{50},\ x_{51},\ x_{59}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{102},\ x_{103},\ x_{104},\ x_{105},\ x_{106},\ x_{107},\ x_{108},\ x_{109},\ x_{110},\ x_{111},\ x_{112},\ x_{113},\ x_{114},\ x_{115},\ x_{116},\ x_{117},\ x_{118},\ x_{119},\ x_{120},\ x_{121},\ x_{122},\ x_{123},\ x_{124},\ x_{126},\ x_{129}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{61},\ x_{62},\ x_{63}\}$. We calculate $\kappa(e, f) = 144$.

Table 6.20: Basis for type E_7, orbit $4A_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{56}	3	$(0, -1, 0)$
		x_2	b_{58}	3	$(-1, 0, 0)$
		x_3	b_{60}	3	$(1, 1, -1)$
		x_4	b_{61}	3	$(-1, -1, 1)$
		x_5	b_{62}	3	$(0, 1, 0)$
		x_6	b_{63}	3	$(1, 0, 0)$
		x_7	b_{30}	2	$(-1, -1, 0)$
		x_8	b_{36}	2	$(0, 1, -1)$
		x_9	b_{39}	2	$(1, 0, -1)$
		x_{10}	b_{41}	2	$(-2, -1, 1)$
		x_{11}	b_{44}	2	$(-1, -2, 1)$
		x_{12}	$b_{45} + b_{47} + b_{52} + b_{53}$	2	$(0, 0, 0)$

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Table 6.20 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{13}	b_{47}	2	$(0,0,0)$
		x_{14}	b_{48}	2	$(1,-1,0)$
		x_{15}	b_{49}	2	$(-1,1,0)$
		x_{16}	b_{51}	2	$(2,1,-1)$
		x_{17}	b_{52}	2	$(0,0,0)$
		x_{18}	b_{53}	2	$(0,0,0)$
		x_{19}	b_{54}	2	$(0,-1,1)$
		x_{20}	b_{55}	2	$(1,2,-1)$
		x_{21}	b_{57}	2	$(-1,0,1)$
		x_{22}	b_{59}	2	$(1,1,0)$
		x_{23}	b_2	1	$(0,0,-1)$
		x_{24}	b_7-b_{16}	1	$(0,-1,0)$
		x_{25}	b_9	1	$(-2,-2,1)$
		x_{26}	$b_{13}+b_{15}$	1	$(-1,0,0)$
		x_{27}	$b_{15}-b_{23}$	1	$(-1,0,0)$
		x_{28}	$b_{16}+b_{20}$	1	$(0,-1,0)$
		x_{29}	$b_{19}-b_{22}$	1	$(1,1,-1)$
		x_{30}	$b_{22}-b_{33}$	1	$(1,1,-1)$
		x_{31}	$b_{25}+b_{28}$	1	$(-1,-1,1)$
		x_{32}	b_{26}	1	$(2,0,-1)$
		x_{33}	$b_{28}-b_{38}$	1	$(-1,-1,1)$
		x_{34}	b_{29}	1	$(0,2,-1)$
		x_{35}	$b_{31}+b_{40}$	1	$(0,1,0)$
		x_{36}	b_{32}	1	$(0,-2,1)$
		x_{37}	$b_{34}-b_{37}$	1	$(1,0,0)$
		x_{38}	b_{35}	1	$(-2,0,1)$
		x_{39}	$b_{37}+b_{43}$	1	$(1,0,0)$
		x_{40}	$b_{40}-b_{42}$	1	$(0,1,0)$

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Table 6.20 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{41}	b_{46}	1	$(2,2,-1)$
		x_{42}	b_{50}	1	$(0,0,1)$
		x_{43}	$b_1 + b_{69}$	0	$(1,-1,0)$
		x_{44}	$b_3 + b_{12}$	0	$(1,2,-1)$
		x_{45}	b_4	0	$(-2,-2,2)$
		x_{46}	$b_5 - b_8$	0	$(2,1,-1)$
		x_{47}	$b_6 + b_{64}$	0	$(-1,1,0)$
		x_{48}	$b_{10} - b_{18}$	0	$(-1,0,1)$
		x_{49}	$b_{11} + b_{14}$	0	$(0,-1,1)$
		x_{50}	$b_{17} - b_{27}$	0	$(1,1,0)$
		x_{51}	b_{21}	0	$(2,0,0)$
		x_{52}	b_{24}	0	$(0,2,0)$
		x_{53}	$b_{66} + b_{75}$	0	$(-1,-2,1)$
		x_{54}	b_{67}	0	$(2,2,-2)$
		x_{55}	$b_{68} - b_{71}$	0	$(-2,-1,1)$
		x_{56}	$b_{73} - b_{81}$	0	$(1,0,-1)$
		x_{57}	$b_{74} + b_{77}$	0	$(0,1,-1)$
		x_{58}	$b_{80} - b_{90}$	0	$(-1,-1,0)$
		x_{59}	b_{84}	0	$(-2,0,0)$
		x_{60}	b_{87}	0	$(0,-2,0)$
		x_{61}	$b_{127} + b_{129} + b_{131}$	0	$(0,0,0)$
		x_{62}	$b_{129} + b_{131} + b_{132}$	0	$(0,0,0)$
		x_{63}	b_{130}	0	$(0,0,0)$
		x_{64}	b_7	1	$(0,-1,0)$
		x_{65}	b_{13}	1	$(-1,0,0)$
		x_{66}	b_{19}	1	$(1,1,-1)$
		x_{67}	b_{25}	1	$(-1,-1,1)$
		x_{68}	b_{31}	1	$(0,1,0)$

Continued on next page

Table 6.20 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{69}	b_{34}	1	$(1,0,0)$
		x_{70}	b_{1}	0	$(1,-1,0)$
		x_{71}	b_{3}	0	$(1,2,-1)$
		x_{72}	b_{5}	0	$(2,1,-1)$
		x_{73}	b_{6}	0	$(-1,1,0)$
		x_{74}	b_{10}	0	$(-1,0,1)$
		x_{75}	b_{11}	0	$(0,-1,1)$
		x_{76}	b_{17}	0	$(1,1,0)$
		x_{77}	b_{66}	0	$(-1,-2,1)$
		x_{78}	b_{68}	0	$(-2,-1,1)$
		x_{79}	b_{73}	0	$(1,0,-1)$
		x_{80}	b_{74}	0	$(0,1,-1)$
		x_{81}	b_{80}	0	$(-1,-1,0)$
		x_{82}	b_{127}	0	$(0,0,0)$
		x_{83}	b_{128}	0	$(0,0,0)$
		x_{84}	b_{129}	0	$(0,0,0)$
		x_{85}	b_{133}	0	$(0,0,0)$
		x_{86}	b_{65}	-1	$(0,0,1)$
		x_{87}	b_{70}	-1	$(0,1,0)$
		x_{88}	b_{72}	-1	$(2,2,-1)$
		x_{89}	b_{76}	-1	$(1,0,0)$
		x_{90}	b_{78}	-1	$(1,0,0)$
		x_{91}	b_{79}	-1	$(0,1,0)$
		x_{92}	b_{83}	-1	$(0,1,0)$
		x_{93}	b_{86}	-1	$(1,0,0)$
		x_{94}	b_{88}	-1	$(1,1,-1)$
		x_{95}	b_{91}	-1	$(1,1,-1)$
		x_{96}	b_{95}	-1	$(0,2,-1)$

Continued on next page

Table 6.20 – continued from previous page

				n_i	β_i
		x_{97}	b_{98}	-1	$(2,0,-1)$
		x_{98}	b_{101}	-1	$(1,1,-1)$
$\mathfrak{m}$		x_{99}	b_{82}	-1	$(-1,-1,1)$
		x_{100}	b_{85}	-1	$(-1,-1,1)$
		x_{101}	b_{89}	-1	$(-2,0,1)$
		x_{102}	b_{92}	-1	$(0,-2,1)$
		x_{103}	b_{94}	-1	$(0,-1,0)$
		x_{104}	b_{96}	-1	$(-1,-1,1)$
		x_{105}	b_{97}	-1	$(-1,0,0)$
		x_{106}	b_{100}	-1	$(-1,0,0)$
		x_{107}	b_{103}	-1	$(0,-1,0)$
		x_{108}	b_{105}	-1	$(0,-1,0)$
		x_{109}	b_{106}	-1	$(-1,0,0)$
		x_{110}	b_{109}	-1	$(-2,-2,1)$
		x_{111}	b_{113}	-1	$(0,0,-1)$
		x_{112}	b_{93}	-2	$(1,1,0)$
		x_{113}	b_{99}	-2	$(0,-1,1)$
		x_{114}	b_{102}	-2	$(-1,0,1)$
		x_{115}	b_{104}	-2	$(2,1,-1)$
		x_{116}	b_{107}	-2	$(1,2,-1)$
		x_{117}	$b_{108}+b_{110}+b_{115}+b_{116}$	-2	$(0,0,0)$
		x_{118}	$-b_{108}+b_{110}$	-2	$(0,0,0)$
		x_{119}	b_{111}	-2	$(-1,1,0)$
		x_{120}	b_{112}	-2	$(1,-1,0)$
		x_{121}	b_{114}	-2	$(-2,-1,1)$
		x_{122}	$-b_{110}+b_{115}$	-2	$(0,0,0)$
		x_{123}	$-b_{110}+b_{116}$	-2	$(0,0,0)$
		x_{124}	b_{117}	-2	$(0,1,-1)$

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Table 6.20 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{125}	b_{118}	-2	$(-1,-2,1)$
		x_{126}	b_{120}	-2	$(1,0,-1)$
		x_{127}	b_{122}	-2	$(-1,-1,0)$
		x_{128}	b_{119}	-3	$(0,1,0)$
		x_{129}	b_{121}	-3	$(1,0,0)$
		x_{130}	b_{123}	-3	$(-1,-1,1)$
		x_{131}	b_{124}	-3	$(1,1,-1)$
		x_{132}	b_{125}	-3	$(0,-1,0)$
		x_{133}	b_{126}	-3	$(-1,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{43,47}$, $F_{44,53}$, $F_{45,54}$, $F_{51,59}$, $F_{14,47}$, $F_{27,37}$, $F_{20,53}$, $F_{26,37}$, $F_{2,37}$:

$$
\begin{aligned}
[\Theta_{43},\Theta_{47}] &= -\Theta_{62}+\Theta_{61}\\
[\Theta_{44},\Theta_{53}] &= -5+\Theta_{62}\\
[\Theta_{45},\Theta_{54}] &= \tfrac{5}{2}+\Theta_{63}\\
[\Theta_{51},\Theta_{59}] &= -\tfrac{5}{2}+\Theta_{63}+\Theta_{61}\\
[\Theta_{14},\Theta_{47}] &= -8\Theta_{63}-4\Theta_{62}-8\Theta_{61}-\Theta_{18}-2\Theta_{17}-\Theta_{13}+\Theta_{12}+4\Theta_{52}\Theta_{60}\\
&\quad -4\Theta_{51}\Theta_{59}-6\Theta_{43}\Theta_{47}\\
[\Theta_{27},\Theta_{37}] &= -72+52\Theta_{63}-4\Theta_{62}+10\Theta_{61}-\Theta_{18}+\Theta_{17}-\Theta_{13}-8\Theta_{63}^2\\
&\quad -2\Theta_{62}\Theta_{63}+2\Theta_{62}^2-4\Theta_{61}\Theta_{63}-2\Theta_{61}\Theta_{62}+4\Theta_{51}\Theta_{59}\\
&\quad +2\Theta_{50}\Theta_{58}+4\Theta_{48}\Theta_{56}+4\Theta_{44}\Theta_{53}+2\Theta_{43}\Theta_{47}\\
[\Theta_{20},\Theta_{53}] &= -120+22\Theta_{63}+20\Theta_{62}+\Theta_{17}-\Theta_{13}+4\Theta_{52}\Theta_{60}-4\Theta_{45}\Theta_{54}\\
&\quad +2\Theta_{44}\Theta_{53}
\end{aligned}
$$

$$\begin{aligned}
[\Theta_{26}, \Theta_{37}] &= -124 + 60\Theta_{63} + 34\Theta_{62} - 18\Theta_{61} - 2\Theta_{13} - 6\Theta_{63}^2 - 8\Theta_{62}\Theta_{63} \\
&\quad -2\Theta_{62}^2 + 4\Theta_{61}\Theta_{63} + 2\Theta_{61}^2 + 8\Theta_{51}\Theta_{59} + 2\Theta_{50}\Theta_{58} + 2\Theta_{43}\Theta_{47} \\
[\Theta_{2}, \Theta_{37}] &= -3624 + 1884\Theta_{63} + 2052\Theta_{62} - 1076\Theta_{61} + 28\Theta_{18} - 18\Theta_{17} \\
&\quad -64\Theta_{13} - 2\Theta_{12} - 232\Theta_{63}^2 - 848\Theta_{62}\Theta_{63} - 336\Theta_{62}^2 \\
&\quad +592\Theta_{61}\Theta_{63} + 160\Theta_{61}\Theta_{62} + 112\Theta_{61}^2 + 24\Theta_{52}\Theta_{60} + 240\Theta_{51}\Theta_{59} \\
&\quad +20\Theta_{50}\Theta_{58} + 40\Theta_{48}\Theta_{56} - 8\Theta_{46}\Theta_{55} - 16\Theta_{45}\Theta_{54} + 16\Theta_{44}\Theta_{53} \\
&\quad -12\Theta_{43}\Theta_{47} + 4\Theta_{34}\Theta_{36} - 4\Theta_{32}\Theta_{38} - 2\Theta_{27}\Theta_{37} - 4\Theta_{21}\Theta_{56} \\
&\quad -4\Theta_{20}\Theta_{53} - 6\Theta_{18}\Theta_{63} - 2\Theta_{18}\Theta_{62} - 2\Theta_{18}\Theta_{61} + 6\Theta_{17}\Theta_{63} \\
&\quad +2\Theta_{17}\Theta_{62} + 2\Theta_{17}\Theta_{61} - 2\Theta_{15}\Theta_{43} + 10\Theta_{13}\Theta_{63} + 10\Theta_{13}\Theta_{62} \\
&\quad -2\Theta_{13}\Theta_{61} + 2\Theta_{7}\Theta_{50} + 72\Theta_{62}\Theta_{63}^2 + 72\Theta_{62}^2\Theta_{63} + 16\Theta_{62}^3 \\
&\quad -72\Theta_{61}\Theta_{63}^2 - 48\Theta_{61}\Theta_{62}\Theta_{63} - 24\Theta_{61}^2\Theta_{63} - 16\Theta_{61}^2\Theta_{62} \\
&\quad -40\Theta_{51}\Theta_{59}\Theta_{63} - 40\Theta_{51}\Theta_{59}\Theta_{62} + 8\Theta_{51}\Theta_{59}\Theta_{61} - 8\Theta_{43}\Theta_{50}\Theta_{59}.
\end{aligned}$$

We have one 1-dimensional representation:

$$\begin{aligned}
\Theta_{12} &\mapsto -\tfrac{617}{4} \\
\Theta_{13} &\mapsto -\tfrac{363}{4} \\
\Theta_{17} &\mapsto -\tfrac{63}{4} \\
\Theta_{18} &\mapsto -72 \\
\Theta_{61} &\mapsto 5 \\
\Theta_{62} &\mapsto 5 \\
\Theta_{63} &\mapsto -\tfrac{5}{2},
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.4.5 The orbit $A_2 + 2A_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $A_2 + 2A_1$. In Table 6.21 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{41} + b_{42} + b_{43} + b_{51}, 4b_{127} + 6b_{128} + 8b_{129} + 12b_{130} + 9b_{131} + 6b_{132} + 3b_{133}, 2b_{104} + b_{105} + 2b_{106} + b_{114}) = (x_{18}, 6x_{50} + 3x_{51} + 4x_{77} + 8x_{78} + 12x_{79} + 6x_{80}, x_{120})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{42},\ x_{43},\ x_{44},\ x_{45},\ x_{46},\ x_{47},\ x_{48}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{93},\ x_{94},\ x_{95},\ x_{96},\ x_{97},\ x_{98},\ x_{99},\ x_{100},\ x_{101},\ x_{102},\ x_{103},\ x_{104},\ x_{105},\ x_{106},\ x_{107},\ x_{108},\ x_{110},\ x_{111},\ x_{112},\ x_{113},\ x_{115},\ x_{116},\ x_{117},\ x_{120}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{49},\ x_{50},\ x_{51}\}$. We calculate $\kappa(e, f) = 216$.

Table 6.21: Basis for type E_7, orbit $A_2 + 2A_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{61}	4	$(0,0,0)$
		x_2	b_{62}	4	$(-1,0,0)$
		x_3	b_{63}	4	$(1,0,0)$
		x_4	b_{50}	3	$(0,-1,-1)$
		x_5	b_{53}	3	$(0,1,-1)$
		x_6	b_{54}	3	$(1,-1,-1)$
		x_7	b_{56}	3	$(1,1,-1)$
		x_8	b_{57}	3	$(0,-1,1)$
		x_9	b_{58}	3	$(0,1,1)$
		x_{10}	b_{59}	3	$(-1,-1,1)$
		x_{11}	b_{60}	3	$(-1,1,1)$
		x_{12}	b_{28}	2	$(0,0,-2)$
		x_{13}	b_{32}	2	$(2,0,-2)$
		x_{14}	$b_{35} - b_{45}$	2	$(-1,0,0)$
		x_{15}	b_{37}	2	$(1,0,-2)$
		x_{16}	$b_{38} - b_{47}$	2	$(1,0,0)$

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Table 6.21 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{17}	b_{40}	2	$(-2,0,0)$
		x_{18}	$b_{41}+b_{42}+b_{43}+b_{51}$	2	$(0,0,0)$
		x_{19}	$b_{41}+b_{43}$	2	$(0,0,0)$
		x_{20}	b_{42}	2	$(0,0,0)$
		x_{21}	b_{44}	2	$(2,0,0)$
		x_{22}	$b_{45}-b_{46}$	2	$(-1,0,0)$
		x_{23}	$b_{47}-b_{48}$	2	$(1,0,0)$
		x_{24}	b_{49}	2	$(-2,0,2)$
		x_{25}	b_{52}	2	$(0,0,2)$
		x_{26}	b_{55}	2	$(-1,0,2)$
		x_{27}	b_4+b_{21}	1	$(1,-1,-1)$
		x_{28}	b_9+b_{26}	1	$(1,1,-1)$
		x_{29}	$b_{10}-b_{11}$	1	$(0,-1,-1)$
		x_{30}	b_{14}	1	$(2,-1,-1)$
		x_{31}	$b_{15}-b_{16}$	1	$(0,1,-1)$
		x_{32}	b_{17}	1	$(-1,-1,-1)$
		x_{33}	$b_{18}-b_{31}$	1	$(-1,-1,1)$
		x_{34}	b_{20}	1	$(2,1,-1)$
		x_{35}	b_{22}	1	$(-1,1,-1)$
		x_{36}	$b_{23}-b_{36}$	1	$(-1,1,1)$
		x_{37}	b_{24}	1	$(-2,-1,1)$
		x_{38}	$b_{25}+b_{27}$	1	$(0,-1,1)$
		x_{39}	b_{29}	1	$(-2,1,1)$
		x_{40}	$b_{30}+b_{33}$	1	$(0,1,1)$
		x_{41}	b_{34}	1	$(1,-1,1)$
		x_{42}	b_{39}	1	$(1,1,1)$
		x_{43}	b_2	0	$(0,2,0)$
		x_{44}	$b_3+\frac{1}{2}b_5+\frac{1}{2}b_{70}-\frac{1}{2}b_{71}$	0	$(-1,0,0)$

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Table 6.21 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{45}	$b_6 + b_{19}$	0	$(-1, 0, 2)$
		x_{46}	$b_7 - 2b_8 + b_{66} + b_{68}$	0	$(1, 0, 0)$
		x_{47}	b_{65}	0	$(0, -2, 0)$
		x_{48}	$b_{69} + b_{82}$	0	$(1, 0, -2)$
		x_{49}	$b_{127} - b_{131} - b_{132}$	0	$(0, 0, 0)$
		x_{50}	b_{128}	0	$(0, 0, 0)$
		x_{51}	$b_{131} + 2b_{132} + b_{133}$	0	$(0, 0, 0)$
		x_{52}	b_{35}	2	$(-1, 0, 0)$
		x_{53}	b_{38}	2	$(1, 0, 0)$
		x_{54}	b_{41}	2	$(0, 0, 0)$
		x_{55}	b_4	1	$(1, -1, -1)$
		x_{56}	b_9	1	$(1, 1, -1)$
		x_{57}	b_{10}	1	$(0, -1, -1)$
		x_{58}	b_{15}	1	$(0, 1, -1)$
		x_{59}	b_{18}	1	$(-1, -1, 1)$
		x_{60}	b_{23}	1	$(-1, 1, 1)$
		x_{61}	b_{25}	1	$(0, -1, 1)$
		x_{62}	b_{30}	1	$(0, 1, 1)$
		x_{63}	b_1	0	$(2, 0, 0)$
		x_{64}	b_3	0	$(-1, 0, 0)$
		x_{65}	b_5	0	$(-1, 0, 0)$
		x_{66}	b_6	0	$(-1, 0, 2)$
		x_{67}	b_7	0	$(1, 0, 0)$
		x_{68}	b_8	0	$(1, 0, 0)$
		x_{69}	b_{12}	0	$(-2, 0, 2)$
		x_{70}	b_{13}	0	$(0, 0, 2)$
		x_{71}	b_{64}	0	$(-2, 0, 0)$
		x_{72}	b_{66}	0	$(1, 0, 0)$

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Table 6.21 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{73}	b_{69}	0	$(1, 0, -2)$
		x_{74}	b_{70}	0	$(-1, 0, 0)$
		x_{75}	b_{75}	0	$(2, 0, -2)$
		x_{76}	b_{76}	0	$(0, 0, -2)$
		x_{77}	b_{127}	0	$(0, 0, 0)$
		x_{78}	b_{129}	0	$(0, 0, 0)$
		x_{79}	b_{130}	0	$(0, 0, 0)$
		x_{80}	b_{131}	0	$(0, 0, 0)$
		x_{81}	b_{67}	-1	$(-1, 1, 1)$
		x_{82}	b_{73}	-1	$(0, 1, 1)$
		x_{83}	b_{74}	-1	$(0, 1, 1)$
		x_{84}	b_{77}	-1	$(-2, 1, 1)$
		x_{85}	b_{78}	-1	$(0, -1, 1)$
		x_{86}	b_{79}	-1	$(0, -1, 1)$
		x_{87}	b_{80}	-1	$(1, 1, 1)$
		x_{88}	b_{81}	-1	$(1, 1, -1)$
		x_{89}	b_{84}	-1	$(-1, 1, 1)$
		x_{90}	b_{85}	-1	$(1, -1, 1)$
		x_{91}	b_{87}	-1	$(2, 1, -1)$
		x_{92}	b_{94}	-1	$(1, 1, -1)$
$\mathfrak{m}$		x_{93}	b_{72}	-1	$(-1, -1, 1)$
		x_{94}	b_{83}	-1	$(-2, -1, 1)$
		x_{95}	b_{86}	-1	$(1, -1, -1)$
		x_{96}	b_{88}	-1	$(0, 1, -1)$
		x_{97}	b_{89}	-1	$(-1, -1, 1)$
		x_{98}	b_{90}	-1	$(0, 1, -1)$
		x_{99}	b_{92}	-1	$(2, -1, -1)$
		x_{100}	b_{93}	-1	$(0, -1, -1)$

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Table 6.21 – continued from previous page

				n_i	β_i
$\mathfrak{m}$		x_{101}	b_{96}	-1	$(0,-1,-1)$
		x_{102}	b_{97}	-1	$(-1,1,-1)$
		x_{103}	b_{99}	-1	$(1,-1,-1)$
		x_{104}	b_{102}	-1	$(-1,-1,-1)$
		x_{105}	b_{91}	-2	$(0,0,2)$
		x_{106}	b_{95}	-2	$(-2,0,2)$
		x_{107}	b_{98}	-2	$(1,0,0)$
		x_{108}	b_{100}	-2	$(-1,0,2)$
		x_{109}	b_{101}	-2	$(-1,0,0)$
		x_{110}	b_{103}	-2	$(2,0,0)$
		x_{111}	$b_{104}-b_{105}$	-2	$(0,0,0)$
		x_{112}	$b_{105}-b_{106}$	-2	$(0,0,0)$
		x_{113}	$b_{106}-b_{114}$	-2	$(0,0,0)$
		x_{114}	b_{107}	-2	$(-2,0,0)$
		x_{115}	b_{108}	-2	$(1,0,0)$
		x_{116}	b_{109}	-2	$(1,0,0)$
		x_{117}	b_{110}	-2	$(-1,0,0)$
		x_{118}	b_{111}	-2	$(-1,0,0)$
		x_{119}	b_{112}	-2	$(2,0,-2)$
		x_{120}	$2b_{104}+b_{105}+2b_{106}+b_{114}$	-2	$(0,0,0)$
		x_{121}	b_{115}	-2	$(0,0,-2)$
		x_{122}	b_{118}	-2	$(1,0,-2)$
		x_{123}	b_{113}	-3	$(0,1,1)$
		x_{124}	b_{116}	-3	$(0,-1,1)$
		x_{125}	b_{117}	-3	$(-1,1,1)$
		x_{126}	b_{119}	-3	$(-1,-1,1)$
		x_{127}	b_{120}	-3	$(0,1,-1)$
		x_{128}	b_{121}	-3	$(0,-1,-1)$

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Table 6.21 – continued from previous page

				n_i	β_i
		x_{129}	b_{122}	-3	$(1, 1, -1)$
		x_{130}	b_{123}	-3	$(1, -1, -1)$
$\mathfrak{m}$		x_{131}	b_{124}	-4	$(0, 0, 0)$
		x_{132}	b_{125}	-4	$(1, 0, 0)$
		x_{133}	b_{126}	-4	$(-1, 0, 0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, we require the commutators $F_{44,46}$, $F_{45,48}$, $F_{43,47}$, $F_{15,45}$, $F_{14,46}$, $F_{32,42}$, $F_{22,46}$, $F_{16,44}$, $F_{23,44}$, $F_{26,48}$, $F_{2,46}$, $F_{3,44}$:

$$\begin{aligned}
[\Theta_{44}, \Theta_{46}] &= \tfrac{5}{2} - \tfrac{1}{2}\Theta_{51} - \Theta_{49} \\
[\Theta_{45}, \Theta_{48}] &= -3 + \Theta_{51} \\
[\Theta_{43}, \Theta_{47}] &= -3 + \Theta_{50} \\
[\Theta_{15}, \Theta_{45}] &= -54 + 6\Theta_{50} - 42\Theta_{49} - 2\Theta_{20} - \Theta_{19} + \Theta_{18} + 6\Theta_{49}\Theta_{50} \\
[\Theta_{14}, \Theta_{46}] &= -36 + 36\Theta_{51} + 6\Theta_{50} + 90\Theta_{49} + 4\Theta_{20} + 2\Theta_{19} - 2\Theta_{18} \\
&\quad -6\Theta_{50}\Theta_{51} + 6\Theta_{49}\Theta_{51} - 12\Theta_{49}\Theta_{50} + 12\Theta_{49}^2 - 12\Theta_{44}\Theta_{46} \\
[\Theta_{32}, \Theta_{42}] &= -\Theta_{20} - \Theta_{19} + \Theta_{18} - 6\Theta_{43}\Theta_{47} \\
[\Theta_{22}, \Theta_{46}] &= -90 + 18\Theta_{51} + 12\Theta_{50} + 24\Theta_{49} - 2\Theta_{20} - 4\Theta_{19} + 3\Theta_{18} \\
&\quad +6\Theta_{45}\Theta_{48} + 12\Theta_{44}\Theta_{46} \\
[\Theta_{16}, \Theta_{44}] &= -63 + 27\Theta_{51} + 9\Theta_{50} + 57\Theta_{49} + \Theta_{20} - \Theta_{19} + \tfrac{1}{2}\Theta_{18} - 3\Theta_{50}\Theta_{51} \\
&\quad +3\Theta_{49}\Theta_{51} - 6\Theta_{49}\Theta_{50} + 6\Theta_{49}^2 + 3\Theta_{45}\Theta_{48} \\
[\Theta_{23}, \Theta_{44}] &= -90 + 33\Theta_{51} + 9\Theta_{50} + 57\Theta_{49} + \Theta_{20} + 2\Theta_{19} - \tfrac{3}{2}\Theta_{18} - 3\Theta_{51}^2 \\
&\quad -3\Theta_{50}\Theta_{51} - 9\Theta_{49}\Theta_{51} - 6\Theta_{49}\Theta_{50} - 6\Theta_{49}^2 - 3\Theta_{45}\Theta_{48} \\
[\Theta_{26}, \Theta_{48}] &= 90 - 12\Theta_{51} - 6\Theta_{50} + 60\Theta_{49} + 2\Theta_{20} + \Theta_{19} - \Theta_{18} - 6\Theta_{49}\Theta_{51} \\
&\quad -6\Theta_{49}\Theta_{50} - 6\Theta_{45}\Theta_{48}
\end{aligned}$$

$$
\begin{aligned}
[\Theta_2, \Theta_{46}] &= 9360 - 5544\Theta_{51} - 2304\Theta_{50} - 10620\Theta_{49} - 204\Theta_{20} - 6\Theta_{19} \\
&\quad -6\Theta_{18} - \Theta_1 + 324\Theta_{51}^2 + 1260\Theta_{50}\Theta_{51} + 144\Theta_{50}^2 + 108\Theta_{49}\Theta_{51} \\
&\quad +2448\Theta_{49}\Theta_{50} - 1152\Theta_{49}^2 - 612\Theta_{45}\Theta_{48} - 72\Theta_{44}\Theta_{46} \\
&\quad -36\Theta_{43}\Theta_{47} - 12\Theta_{35}\Theta_{41} - 12\Theta_{34}\Theta_{37} + 12\Theta_{32}\Theta_{42} + 12\Theta_{30}\Theta_{39} \\
&\quad -6\Theta_{26}\Theta_{48} + 6\Theta_{20}\Theta_{51} + 24\Theta_{20}\Theta_{50} - 12\Theta_{20}\Theta_{49} - 6\Theta_{19}\Theta_{51} \\
&\quad -12\Theta_{19}\Theta_{49} + 6\Theta_{18}\Theta_{51} + 12\Theta_{18}\Theta_{49} - 12\Theta_{16}\Theta_{44} - 6\Theta_{15}\Theta_{45} \\
&\quad -6\Theta_{14}\Theta_{46} - 36\Theta_{50}\Theta_{51}^2 - 72\Theta_{50}^2\Theta_{51} + 36\Theta_{49}\Theta_{51}^2 - 144\Theta_{49}\Theta_{50}^2 \\
&\quad +72\Theta_{49}^2\Theta_{51} + 144\Theta_{49}^2\Theta_{50} + 36\Theta_{45}\Theta_{48}\Theta_{51} + 72\Theta_{45}\Theta_{48}\Theta_{50} \\
[\Theta_3, \Theta_{44}] &= -13374 + 4950\Theta_{51} + 2826\Theta_{50} + 8604\Theta_{49} + 135\Theta_{20} - 90\Theta_{19} \\
&\quad +18\Theta_{18} - \tfrac{1}{2}\Theta_1 - 198\Theta_{51}^2 - 864\Theta_{50}\Theta_{51} - 144\Theta_{50}^2 + 36\Theta_{49}\Theta_{51} \\
&\quad -1584\Theta_{49}\Theta_{50} + 828\Theta_{49}^2 + 396\Theta_{45}\Theta_{48} + 36\Theta_{44}\Theta_{46} - 18\Theta_{43}\Theta_{47} \\
&\quad +3\Theta_{35}\Theta_{41} - 6\Theta_{34}\Theta_{37} - 3\Theta_{32}\Theta_{42} - 3\Theta_{31}\Theta_{38} + 6\Theta_{30}\Theta_{39} \\
&\quad +3\Theta_{29}\Theta_{40} + 6\Theta_{28}\Theta_{33} - 6\Theta_{27}\Theta_{36} + 3\Theta_{26}\Theta_{48} - 6\Theta_{23}\Theta_{44} \\
&\quad -12\Theta_{20}\Theta_{50} + 12\Theta_{20}\Theta_{49} + 6\Theta_{19}\Theta_{51} + 6\Theta_{19}\Theta_{50} + 6\Theta_{19}\Theta_{49} \\
&\quad -3\Theta_{18}\Theta_{51} - 6\Theta_{18}\Theta_{49} - 6\Theta_{16}\Theta_{44} + 3\Theta_{15}\Theta_{45} + 18\Theta_{50}\Theta_{51}^2 \\
&\quad +36\Theta_{50}^2\Theta_{51} - 18\Theta_{49}\Theta_{51}^2 + 72\Theta_{49}\Theta_{50}^2 - 36\Theta_{49}^2\Theta_{51} - 72\Theta_{49}^2\Theta_{50} \\
&\quad -18\Theta_{45}\Theta_{48}\Theta_{51} - 36\Theta_{45}\Theta_{48}\Theta_{50}.
\end{aligned}
$$

We have one 1-dimensional representation:

$$
\begin{aligned}
\Theta_1 &\mapsto 576 \\
\Theta_{18} &\mapsto -96 \\
\Theta_{19} &\mapsto -36 \\
\Theta_{20} &\mapsto -60 \\
\Theta_{49} &\mapsto 1 \\
\Theta_{50} &\mapsto 3 \\
\Theta_{51} &\mapsto 3,
\end{aligned}
$$

with all other generators $\Theta_i \mapsto 0$. The denominators which appear in the presentation of $U(\mathfrak{g}, e)$ are all powers of 2, so we can define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ provided $\Bbbk$ has characteristic $p > 3$.

6.4.6 The orbit $2A_2 + A_1$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $2A_2 + A_1$. In Table 6.22 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{34} + b_{35} + b_{36} + b_{37} + b_{43}, 5b_{127} + 7b_{128} + 10b_{129} + 14b_{130} + 11b_{131} + 8b_{132} + 4b_{133}, 2b_{97} + 2b_{98} + 2b_{99} + b_{100} + 2b_{106}) = (x_{23}, -7x_{42} + 4x_{43} + 12x_{74} + 10x_{75} + 10x_{76} + 6x_{77} + 8x_{78}, x_{112})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{35},\ x_{37},\ x_{38},\ x_{39},\ x_{40},\ x_{41}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{89},\ x_{90},\ x_{91},\ x_{92},\ x_{93},\ x_{94},\ x_{95},\ x_{96},\ x_{97},\ x_{98},\ x_{99},\ x_{100},\ x_{101},\ x_{102},\ x_{103},\ x_{104},\ x_{105},\ x_{106},\ x_{107},\ x_{110},\ x_{112},\ x_{114}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{42},\ x_{43}\}$. We calculate $\kappa(e, f) = 324$.

Table 6.22: Basis for type E_7, orbit $2A_2 + A_1$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{62}	5	$(-1, 0)$
		x_2	b_{63}	5	$(1, 0)$
		x_3	b_{55}	4	$(0, -2)$
		x_4	b_{57}	4	$(0, 0)$
		x_5	b_{58}	4	$(2, 0)$
		x_6	b_{59}	4	$(-2, 0)$
		x_7	b_{60}	4	$(0, 0)$
		x_8	b_{61}	4	$(0, 2)$
		x_9	b_{42}	3	$(1, -2)$
		x_{10}	b_{46}	3	$(-1, -2)$
		x_{11}	$b_{47} + b_{52}$	3	$(1, 0)$
		x_{12}	$b_{49} - b_{50}$	3	$(-1, 0)$

Continued on next page

Table 6.22 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{13}	$b_{50} - b_{51}$	3	$(-1, 0)$
		x_{14}	$b_{52} - b_{53}$	3	$(1, 0)$
		x_{15}	b_{54}	3	$(-1, 2)$
		x_{16}	b_{56}	3	$(1, 2)$
		x_{17}	b_{24}	2	$(-2, -2)$
		x_{18}	$b_{27} - b_{29}$	2	$(0, -2)$
		x_{19}	$b_{31} - b_{40}$	2	$(-2, 0)$
		x_{20}	b_{33}	2	$(2, -2)$
		x_{21}	$b_{34} + b_{35}$	2	$(0, 0)$
		x_{22}	$b_{36} + b_{43}$	2	$(0, 0)$
		x_{23}	$b_{34} + b_{35} + b_{36} + b_{37} + b_{43}$	2	$(0, 0)$
		x_{24}	$b_{38} - b_{39}$	2	$(2, 0)$
		x_{25}	$b_{41} - b_{48}$	2	$(0, 2)$
		x_{26}	b_{44}	2	$(2, 2)$
		x_{27}	b_{45}	2	$(-2, 2)$
		x_{28}	$b_3 - b_{12}$	1	$(-1, -2)$
		x_{29}	$b_6 - b_8$	1	$(1, -2)$
		x_{30}	$b_{10} + \frac{1}{2}b_{18} + \frac{1}{2}b_{19} - \frac{1}{2}b_{21}$	1	$(-1, 0)$
		x_{31}	$b_{13} + 2b_{14} - b_{15} - b_{23}$	1	$(1, 0)$
		x_{32}	$b_{14} - b_{15} + b_{26}$	1	$(1, 0)$
		x_{33}	b_{17}	1	$(-3, 0)$
		x_{34}	$b_{18} + b_{19} + b_{21} - 2b_{22}$	1	$(-1, 0)$
		x_{35}	b_{20}	1	$(3, 0)$
		x_{36}	$b_{25} + b_{28}$	1	$(-1, 2)$
		x_{37}	$b_{30} - b_{32}$	1	$(1, 2)$
		x_{38}	$b_1 - b_2 + b_{68}$	0	$(2, 0)$
		x_{39}	$b_4 - b_7 - b_{16}$	0	$(0, 2)$
		x_{40}	$b_5 + b_{64} - b_{65}$	0	$(-2, 0)$

Continued on next page

Table 6.22 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{41}	$b_{67} - b_{70} - b_{79}$	0	$(0, -2)$
		x_{42}	$b_{127} + b_{128} - b_{131}$	0	$(0, 0)$
		x_{43}	$b_{128} + 2b_{130} + b_{131} + b_{133}$	0	$(0, 0)$
		x_{44}	b_{47}	3	$(1, 0)$
		x_{45}	b_{49}	3	$(-1, 0)$
		x_{46}	b_{27}	2	$(0, -2)$
		x_{47}	b_{31}	2	$(-2, 0)$
		x_{48}	b_{34}	2	$(0, 0)$
		x_{49}	b_{36}	2	$(0, 0)$
		x_{50}	b_{38}	2	$(2, 0)$
		x_{51}	b_{41}	2	$(0, 2)$
		x_{52}	b_3	1	$(-1, -2)$
		x_{53}	b_6	1	$(1, -2)$
		x_{54}	b_{10}	1	$(-1, 0)$
		x_{55}	b_{13}	1	$(1, 0)$
		x_{56}	b_{14}	1	$(1, 0)$
		x_{57}	b_{15}	1	$(1, 0)$
		x_{58}	b_{18}	1	$(-1, 0)$
		x_{59}	b_{19}	1	$(-1, 0)$
		x_{60}	b_{25}	1	$(-1, 2)$
		x_{61}	b_{30}	1	$(1, 2)$
		x_{62}	b_1	0	$(2, 0)$
		x_{63}	b_2	0	$(2, 0)$
		x_{64}	b_4	0	$(0, 2)$
		x_{65}	b_5	0	$(-2, 0)$
		x_{66}	b_7	0	$(0, 2)$
		x_{67}	b_9	0	$(2, 2)$
		x_{68}	b_{11}	0	$(-2, 2)$

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Table 6.22 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{69}	b_{64}	0	$(-2,0)$
		x_{70}	b_{67}	0	$(0,-2)$
		x_{71}	b_{70}	0	$(0,-2)$
		x_{72}	b_{72}	0	$(-2,-2)$
		x_{73}	b_{74}	0	$(2,-2)$
		x_{74}	b_{127}	0	$(0,0)$
		x_{75}	b_{128}	0	$(0,0)$
		x_{76}	b_{129}	0	$(0,0)$
		x_{77}	b_{130}	0	$(0,0)$
		x_{78}	b_{132}	0	$(0,0)$
		x_{79}	b_{66}	-1	$(1,2)$
		x_{80}	b_{69}	-1	$(-1,2)$
		x_{81}	b_{71}	-1	$(-1,2)$
		x_{82}	b_{73}	-1	$(1,0)$
		x_{83}	b_{75}	-1	$(1,2)$
		x_{84}	b_{80}	-1	$(3,0)$
		x_{85}	b_{81}	-1	$(1,0)$
		x_{86}	b_{82}	-1	$(1,0)$
		x_{87}	b_{84}	-1	$(1,0)$
		x_{88}	b_{85}	-1	$(1,0)$
$\mathfrak{m}$		x_{89}	b_{76}	-1	$(-1,0)$
		x_{90}	b_{77}	-1	$(-1,0)$
		x_{91}	b_{78}	-1	$(-1,0)$
		x_{92}	b_{83}	-1	$(-3,0)$
		x_{93}	b_{86}	-1	$(-1,0)$
		x_{94}	b_{88}	-1	$(1,-2)$
		x_{95}	b_{89}	-1	$(-1,0)$
		x_{96}	b_{91}	-1	$(1,-2)$

Continued on next page

Table 6.22 – continued from previous page

				n_i	β_i
		x_{97}	b_{93}	-1	$(-1,-2)$
		x_{98}	b_{95}	-1	$(-1,-2)$
		x_{99}	b_{87}	-2	$(2,2)$
		x_{100}	b_{90}	-2	$(0,2)$
		x_{101}	b_{92}	-2	$(0,2)$
		x_{102}	b_{94}	-2	$(2,0)$
		x_{103}	b_{96}	-2	$(-2,2)$
		x_{104}	$b_{97}-b_{98}$	-2	$(0,0)$
		x_{105}	$b_{98}-b_{99}$	-2	$(0,0)$
		x_{106}	$b_{99}-b_{100}$	-2	$(0,0)$
		x_{107}	$b_{100}-b_{106}$	-2	$(0,0)$
		x_{108}	b_{101}	-2	$(-2,0)$
		x_{109}	b_{102}	-2	$(-2,0)$
$\mathfrak{m}$		x_{110}	b_{103}	-2	$(2,0)$
		x_{111}	b_{104}	-2	$(0,-2)$
		x_{112}	$2b_{97}+2b_{98}+2b_{99}+b_{100}+2b_{106}$	-2	$(0,0)$
		x_{113}	b_{107}	-2	$(-2,-2)$
		x_{114}	b_{108}	-2	$(2,-2)$
		x_{115}	b_{111}	-2	$(0,-2)$
		x_{116}	b_{105}	-3	$(-1,2)$
		x_{117}	b_{109}	-3	$(1,2)$
		x_{118}	b_{110}	-3	$(-1,0)$
		x_{119}	b_{112}	-3	$(1,0)$
		x_{120}	b_{113}	-3	$(1,0)$
		x_{121}	b_{114}	-3	$(1,0)$
		x_{122}	b_{115}	-3	$(-1,0)$
		x_{123}	b_{116}	-3	$(-1,0)$
		x_{124}	b_{117}	-3	$(1,-2)$

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Table 6.22 – continued from previous page

				n_i	β_i
		x_{125}	b_{119}	-3	$(-1,-2)$
		x_{126}	b_{118}	-4	$(0,2)$
		x_{127}	b_{120}	-4	$(0,0)$
		x_{128}	b_{121}	-4	$(-2,0)$
$\mathfrak{m}$		x_{129}	b_{122}	-4	$(2,0)$
		x_{130}	b_{123}	-4	$(0,0)$
		x_{131}	b_{124}	-4	$(0,-2)$
		x_{132}	b_{125}	-5	$(1,0)$
		x_{133}	b_{126}	-5	$(-1,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g}, e)$, we require the commutators $F_{39,41}$, $F_{38,40}$, $F_{19,38}$, $F_{33,35}$, $F_{18,39}$, $F_{25,41}$, $F_{28,37}$, $F_{24,40}$, $F_{8,41}$, $F_{5,40}$, $F_{6,38}$, $F_{3,39}$, $F_{10,37}$:

$$\begin{aligned}
[\Theta_{39}, \Theta_{41}] &= -4 + \Theta_{43} \\
[\Theta_{38}, \Theta_{40}] &= -4 + \Theta_{42} \\
[\Theta_{19}, \Theta_{38}] &= 240 - 33\Theta_{43} - 108\Theta_{42} - \Theta_{22} + \Theta_{21} + 9\Theta_{42}\Theta_{43} - 18\Theta_{42}^2 \\
&\quad +9\Theta_{38}\Theta_{40} \\
[\Theta_{33}, \Theta_{35}] &= -56 - 54\Theta_{42} - \Theta_{23} + \Theta_{22} + \Theta_{21} + 3\Theta_{38}\Theta_{40} \\
[\Theta_{18}, \Theta_{39}] &= 144 - 54\Theta_{43} - 27\Theta_{42} - \Theta_{22} + \Theta_{21} - 9\Theta_{42}\Theta_{43} - 9\Theta_{42}^2 \\
&\quad -9\Theta_{38}\Theta_{40} \\
[\Theta_{25}, \Theta_{41}] &= 252 - 117\Theta_{43} - 63\Theta_{42} - \Theta_{22} + \Theta_{21} + 9\Theta_{43}^2 - 9\Theta_{42}^2 \\
&\quad -18\Theta_{39}\Theta_{41} - 9\Theta_{38}\Theta_{40} \\
[\Theta_{28}, \Theta_{37}] &= -144 + 18\Theta_{43} - 108\Theta_{42} + \Theta_{23} - 2\Theta_{22} - \Theta_{21} + 9\Theta_{42}\Theta_{43} \\
&\quad -9\Theta_{42}^2 + 3\Theta_{39}\Theta_{41} \\
[\Theta_{24}, \Theta_{40}] &= -336 + 3\Theta_{43} - 36\Theta_{42} - \Theta_{22} + \Theta_{21} - 27\Theta_{38}\Theta_{40}
\end{aligned}$$

$$
\begin{aligned}
[\Theta_8, \Theta_{41}] &= -10152 + 3267\Theta_{43} - 567\Theta_{42} + 9\Theta_{23} + 9\Theta_{22} - 207\Theta_{21} - \Theta_7 \\
&\quad -\Theta_4 - 243\Theta_{43}^2 + 243\Theta_{42}\Theta_{43} + 81\Theta_{42}^2 - 837\Theta_{39}\Theta_{41} \\
&\quad +1296\Theta_{38}\Theta_{40} - 9\Theta_{29}\Theta_{36} - 9\Theta_{28}\Theta_{37} - 9\Theta_{25}\Theta_{41} + 18\Theta_{21}\Theta_{43} \\
&\quad -9\Theta_{21}\Theta_{42} - 9\Theta_{19}\Theta_{38} + 81\Theta_{39}\Theta_{41}\Theta_{43} - 81\Theta_{39}\Theta_{41}\Theta_{42} \\
&\quad -81\Theta_{38}\Theta_{40}\Theta_{43} + 162\Theta_{38}\Theta_{40}\Theta_{42} \\
[\Theta_5, \Theta_{40}] &= -32724 + 1485\Theta_{43} - 5940\Theta_{42} + 3\Theta_{23} - 171\Theta_{22} + 135\Theta_{21} \\
&\quad -\Theta_7 + \Theta_4 + 135\Theta_{42}\Theta_{43} - 297\Theta_{42}^2 + 54\Theta_{39}\Theta_{41} - 2484\Theta_{38}\Theta_{40} \\
&\quad +9\Theta_{33}\Theta_{35} + 18\Theta_{30}\Theta_{31} - 9\Theta_{24}\Theta_{40} + 9\Theta_{22}\Theta_{43} - 9\Theta_{22}\Theta_{42} \\
&\quad -9\Theta_{21}\Theta_{43} + 9\Theta_{21}\Theta_{42} - 9\Theta_{18}\Theta_{39} + 81\Theta_{38}\Theta_{40}\Theta_{43} \\
&\quad -243\Theta_{38}\Theta_{40}\Theta_{42} \\
[\Theta_6, \Theta_{38}] &= 29160 - 6021\Theta_{43} - 6048\Theta_{42} - 3\Theta_{23} + \Theta_7 - \Theta_4 + 297\Theta_{43}^2 \\
&\quad +1350\Theta_{42}\Theta_{43} - 1404\Theta_{42}^2 + 1053\Theta_{39}\Theta_{41} + 2052\Theta_{38}\Theta_{40} \\
&\quad -9\Theta_{33}\Theta_{35} - 18\Theta_{30}\Theta_{31} + 9\Theta_{25}\Theta_{41} + 9\Theta_{24}\Theta_{40} + 9\Theta_{22}\Theta_{42} \\
&\quad +9\Theta_{19}\Theta_{38} - 81\Theta_{42}\Theta_{43}^2 + 162\Theta_{42}^2\Theta_{43} - 81\Theta_{39}\Theta_{41}\Theta_{43} \\
&\quad +162\Theta_{39}\Theta_{41}\Theta_{42} - 162\Theta_{38}\Theta_{40}\Theta_{43} + 162\Theta_{38}\Theta_{40}\Theta_{42} \\
[\Theta_3, \Theta_{39}] &= 16848 - 9072\Theta_{43} - 3078\Theta_{42} + 9\Theta_{23} - 135\Theta_{22} - 27\Theta_{21} - \Theta_7 \\
&\quad -\Theta_4 + 567\Theta_{43}^2 - 1296\Theta_{42}\Theta_{43} - 1377\Theta_{42}^2 + 459\Theta_{39}\Theta_{41} \\
&\quad -1134\Theta_{38}\Theta_{40} - 9\Theta_{29}\Theta_{36} - 9\Theta_{28}\Theta_{37} + 9\Theta_{24}\Theta_{40} + 9\Theta_{22}\Theta_{43} \\
&\quad -9\Theta_{22}\Theta_{42} - 18\Theta_{18}\Theta_{39} + 81\Theta_{42}\Theta_{43}^2 - 81\Theta_{42}^3 - 81\Theta_{39}\Theta_{41}\Theta_{43} \\
&\quad +81\Theta_{38}\Theta_{40}\Theta_{43} - 81\Theta_{38}\Theta_{40}\Theta_{42} \\
[\Theta_{10}, \Theta_{37}] &= -4860 + 1512\Theta_{43} - 1188\Theta_{42} - 27\Theta_{23} + 87\Theta_{22} + 39\Theta_{21} + \Theta_7 \\
&\quad -27\Theta_{43}^2 + 270\Theta_{42}\Theta_{43} - 27\Theta_{42}^2 + 216\Theta_{39}\Theta_{41} + 54\Theta_{38}\Theta_{40} \\
&\quad +9\Theta_{33}\Theta_{35} + \tfrac{9}{2}\Theta_{32}\Theta_{34} - \tfrac{9}{2}\Theta_{31}\Theta_{34} + 9\Theta_{30}\Theta_{32} - 9\Theta_{30}\Theta_{31} \\
&\quad +9\Theta_{28}\Theta_{37} + 9\Theta_{23}\Theta_{43} + 9\Theta_{23}\Theta_{42} - 18\Theta_{22}\Theta_{43} - 9\Theta_{22}\Theta_{42} \\
&\quad -9\Theta_{21}\Theta_{43} - 9\Theta_{21}\Theta_{42} + 6\Theta_{18}\Theta_{39} + 54\Theta_{39}\Theta_{41}\Theta_{42}.
\end{aligned}
$$

We have one 1-dimensional representation:

$$\begin{aligned} \Theta_4 &\mapsto 756 \\ \Theta_7 &\mapsto 34668 \\ \Theta_{21} &\mapsto -308 \\ \Theta_{22} &\mapsto -776 \\ \Theta_{23} &\mapsto -1356 \\ \Theta_{42} &\mapsto 4 \\ \Theta_{43} &\mapsto 4, \end{aligned}$$

with all other generators $\Theta_i \mapsto 0$. We know that we cannot define $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$ for $\Bbbk$ of characteristic 2 or 3, but without a presentation we cannot establish that these are the only primes which have to be excluded for the definition of $U(\mathfrak{g}_{\Bbbk}, e_{\Bbbk})$.

6.4.7 The orbit $(A_3 + A_1)'$

Here we consider the finite W-algebra associated to the orbit of $\mathfrak{g}$ with Bala–Carter label $(A_3 + A_1)'$. In Table 6.23 we give our choice of basis for $\mathfrak{g}$. We take our $\mathfrak{sl}_2$-triple to be $(e, h, f) = (b_{27} + b_{28} + b_{39} + b_{49}, 6b_{127} + 8b_{128} + 11b_{129} + 16b_{130} + 12b_{131} + 8b_{132} + 4b_{133}, 3b_{90} + 4b_{91} + 3b_{102} + b_{112}) = (x_{25}, -4x_{39} + 11x_{40} + 12x_{41} + 6x_{75} + 12x_{76} + 16x_{77} + 8x_{78}, x_{110})$. With this basis, a minimal generating set for $\mathfrak{g}^e$ is $\{x_{12},\ x_{24},\ x_{25},\ x_{32},\ x_{33},\ x_{34},\ x_{35},\ x_{36},\ x_{37},\ x_{38}\}$. A minimal generating set for $\mathfrak{m}$ is $\{x_{88},\ x_{89},\ x_{90},\ x_{91},\ x_{92},\ x_{93},\ x_{94},\ x_{95},\ x_{96},\ x_{97},\ x_{98},\ x_{99},\ x_{101},\ x_{102},\ x_{104},\ x_{105},\ x_{106},\ x_{108},\ x_{109},\ x_{110},\ x_{116}\}$. The subalgebra $\mathfrak{t}^e$ has basis $\{x_{39},\ x_{40},\ x_{41}\}$. We calculate $\kappa(e, f) = 396$.

Table 6.23: Basis for type E_7, orbit $(A_3 + A_1)'$.

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_1	b_{63}	6	$(0,0,0)$
		x_2	b_{61}	5	$(0,-1,0)$
		x_3	b_{62}	5	$(0,1,0)$
		x_4	$b_{50}+b_{56}$	4	$(0,0,0)$
		x_5	b_{53}	4	$(2,0,0)$
		x_6	b_{54}	4	$(-2,0,0)$
		x_7	b_{57}	4	$(-1,0,-1)$
		x_8	b_{58}	4	$(1,0,-1)$
		x_9	b_{59}	4	$(-1,0,1)$
		x_{10}	b_{60}	4	$(1,0,1)$
		x_{11}	$b_{32}-b_{52}$	3	$(0,-1,0)$
		x_{12}	$b_{37}-b_{55}$	3	$(0,1,0)$
		x_{13}	b_{38}	3	$(1,-1,-1)$
		x_{14}	b_{42}	3	$(1,1,-1)$
		x_{15}	b_{43}	3	$(1,-1,1)$
		x_{16}	b_{44}	3	$(-1,-1,-1)$
		x_{17}	b_{46}	3	$(1,1,1)$
		x_{18}	b_{47}	3	$(-1,1,-1)$
		x_{19}	b_{48}	3	$(-1,-1,1)$
		x_{20}	b_{51}	3	$(-1,1,1)$
		x_{21}	$b_{14}-b_{41}$	2	$(-1,0,-1)$
		x_{22}	$b_{20}+b_{35}$	2	$(1,0,-1)$
		x_{23}	$b_{21}-b_{45}$	2	$(-1,0,1)$
		x_{24}	$b_{26}+b_{40}$	2	$(1,0,1)$
		x_{25}	$b_{27}+b_{28}+b_{39}+b_{49}$	2	$(0,0,0)$
		x_{26}	b_{49}	2	$(0,0,0)$
		x_{27}	$b_{18}-b_{30}$	1	$(0,-1,0)$

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Table 6.23 – continued from previous page

				n_i	β_i
$\mathfrak{p}$	$\mathfrak{g}^e$	x_{28}	b_{23}	1	$(2,-1,0)$
		x_{29}	$b_{24}-b_{36}$	1	$(0,1,0)$
		x_{30}	b_{25}	1	$(-2,-1,0)$
		x_{31}	b_{29}	1	$(2,1,0)$
		x_{32}	b_{31}	1	$(-2,1,0)$
		x_{33}	b_2+b_{70}	0	$(2,0,0)$
		x_{34}	b_3	0	$(0,2,0)$
		x_{35}	b_5	0	$(0,0,2)$
		x_{36}	b_7+b_{65}	0	$(-2,0,0)$
		x_{37}	b_{66}	0	$(0,-2,0)$
		x_{38}	b_{68}	0	$(0,0,-2)$
		x_{39}	$b_{128}-b_{133}$	0	$(0,0,0)$
		x_{40}	b_{129}	0	$(0,0,0)$
		x_{41}	b_{131}	0	$(0,0,0)$
		x_{42}	b_{50}	4	$(0,0,0)$
		x_{43}	b_{32}	3	$(0,-1,0)$
		x_{44}	b_{37}	3	$(0,1,0)$
		x_{45}	b_{14}	2	$(-1,0,-1)$
		x_{46}	b_{20}	2	$(1,0,-1)$
		x_{47}	b_{21}	2	$(-1,0,1)$
		x_{48}	b_{26}	2	$(1,0,1)$
		x_{49}	b_{27}	2	$(0,0,0)$
		x_{50}	b_{28}	2	$(0,0,0)$
		x_{51}	b_{33}	2	$(2,0,0)$
		x_{52}	b_{34}	2	$(-2,0,0)$
		x_{53}	b_1	1	$(0,-1,0)$
		x_{54}	b_4	1	$(-1,-1,-1)$
		x_{55}	b_8	1	$(0,1,0)$

Continued on next page

Table 6.23 – continued from previous page

				n_i	β_i
$\mathfrak{p}$		x_{56}	b_9	1	$(1,-1,-1)$
		x_{57}	b_{10}	1	$(-1,1,-1)$
		x_{58}	b_{11}	1	$(-1,-1,1)$
		x_{59}	b_{15}	1	$(1,1,-1)$
		x_{60}	b_{16}	1	$(1,-1,1)$
		x_{61}	b_{17}	1	$(-1,1,1)$
		x_{62}	b_{18}	1	$(0,-1,0)$
		x_{63}	b_{22}	1	$(1,1,1)$
		x_{64}	b_{24}	1	$(0,1,0)$
		x_{65}	b_2	0	$(2,0,0)$
		x_{66}	b_6	0	$(1,0,-1)$
		x_{67}	b_7	0	$(-2,0,0)$
		x_{68}	b_{12}	0	$(1,0,1)$
		x_{69}	b_{13}	0	$(-1,0,-1)$
		x_{70}	b_{19}	0	$(-1,0,1)$
		x_{71}	b_{69}	0	$(-1,0,1)$
		x_{72}	b_{75}	0	$(-1,0,-1)$
		x_{73}	b_{76}	0	$(1,0,1)$
		x_{74}	b_{82}	0	$(1,0,-1)$
		x_{75}	b_{127}	0	$(0,0,0)$
		x_{76}	b_{128}	0	$(0,0,0)$
		x_{77}	b_{130}	0	$(0,0,0)$
		x_{78}	b_{132}	0	$(0,0,0)$
		x_{79}	b_{64}	-1	$(0,1,0)$
		x_{80}	b_{67}	-1	$(1,1,1)$
		x_{81}	b_{72}	-1	$(-1,1,1)$
		x_{82}	b_{73}	-1	$(1,-1,1)$
		x_{83}	b_{74}	-1	$(1,1,-1)$

Continued on next page

Table 6.23 – continued from previous page

				n_i	β_i
		x_{84}	b_{81}	-1	$(0,1,0)$
		x_{85}	b_{88}	-1	$(2,1,0)$
		x_{86}	b_{93}	-1	$(0,1,0)$
		x_{87}	b_{94}	-1	$(2,-1,0)$
$\mathfrak{m}$		x_{88}	b_{71}	-1	$(0,-1,0)$
		x_{89}	b_{78}	-1	$(-1,-1,1)$
		x_{90}	b_{79}	-1	$(-1,1,-1)$
		x_{91}	b_{80}	-1	$(1,-1,-1)$
		x_{92}	b_{85}	-1	$(-1,-1,-1)$
		x_{93}	b_{86}	-1	$(-2,1,0)$
		x_{94}	b_{87}	-1	$(0,-1,0)$
		x_{95}	b_{92}	-1	$(-2,-1,0)$
		x_{96}	b_{99}	-1	$(0,-1,0)$
		x_{97}	b_{77}	-2	$(1,0,1)$
		x_{98}	b_{83}	-2	$(-1,0,1)$
		x_{99}	b_{84}	-2	$(1,0,-1)$
		x_{100}	b_{89}	-2	$(-1,0,-1)$
		x_{101}	$b_{90}-b_{91}$	-2	$(0,0,0)$
		x_{102}	$b_{91}-b_{102}$	-2	$(0,0,0)$
		x_{103}	b_{96}	-2	$(-2,0,0)$
		x_{104}	b_{97}	-2	$(2,0,0)$
		x_{105}	b_{98}	-2	$(-1,0,1)$
		x_{106}	$b_{102}-b_{112}$	-2	$(0,0,0)$
		x_{107}	b_{103}	-2	$(-1,0,-1)$
		x_{108}	b_{104}	-2	$(1,0,1)$
		x_{109}	b_{108}	-2	$(1,0,-1)$
		x_{110}	$3b_{90}+4b_{91}+3b_{102}+b_{112}$	-2	$(0,0,0)$
		x_{111}	b_{95}	-3	$(0,1,0)$

Continued on next page

Table 6.23 – continued from previous page

				n_i	β_i
		x_{112}	b_{100}	-3	$(0,-1,0)$
		x_{113}	b_{101}	-3	$(-1,1,1)$
		x_{114}	b_{105}	-3	$(-1,-1,1)$
		x_{115}	b_{106}	-3	$(-1,1,-1)$
		x_{116}	b_{107}	-3	$(1,1,1)$
		x_{117}	b_{109}	-3	$(-1,-1,-1)$
		x_{118}	b_{110}	-3	$(1,-1,1)$
		x_{119}	b_{111}	-3	$(1,1,-1)$
		x_{120}	b_{114}	-3	$(1,-1,-1)$
		x_{121}	b_{115}	-3	$(0,1,0)$
$\mathfrak{m}$		x_{122}	b_{118}	-3	$(0,-1,0)$
		x_{123}	b_{113}	-4	$(0,0,0)$
		x_{124}	b_{116}	-4	$(-2,0,0)$
		x_{125}	b_{117}	-4	$(2,0,0)$
		x_{126}	b_{119}	-4	$(0,0,0)$
		x_{127}	b_{120}	-4	$(1,0,1)$
		x_{128}	b_{121}	-4	$(-1,0,1)$
		x_{129}	b_{122}	-4	$(1,0,-1)$
		x_{130}	b_{123}	-4	$(-1,0,-1)$
		x_{131}	b_{124}	-5	$(0,1,0)$
		x_{132}	b_{125}	-5	$(0,-1,0)$
		x_{133}	b_{126}	-6	$(0,0,0)$

By Theorem 5.3.5, to determine the 1-dimensional representations of $U(\mathfrak{g},e)$, we require the commutators $F_{33,36}$, $F_{34,37}$, $F_{35,38}$, $F_{28,32}$, $F_{25,41}$, $F_{25,39}$, $F_{25,40}$, $F_{5,36}$, $F_{25,26}$, $F_{6,33}$, $F_{21,24}$, $F_{12,27}$, $F_{4,25}$, $F_{7,24}$, $F_{11,12}$, $F_{1,25}$, $F_{2,12}$:

$$
\begin{aligned}
[\Theta_{35}, \Theta_{38}] &= -1 + \Theta_{41} \\
[\Theta_{33}, \Theta_{36}] &= -3 + \Theta_{39} \\
[\Theta_{34}, \Theta_{37}] &= -\tfrac{5}{2} + \Theta_{40} \\
[\Theta_{28}, \Theta_{32}] &= -88 + 77\Theta_{41} + 33\Theta_{40} - \tfrac{99}{2}\Theta_{39} - \Theta_{26} - 11\Theta_{41}^2 - 11\Theta_{40}\Theta_{41} \\
&\quad +11\Theta_{39}\Theta_{41} + 11\Theta_{34}\Theta_{37} + \tfrac{11}{2}\Theta_{33}\Theta_{36} \\
[\Theta_{25}, \Theta_{41}] &= 0 \\
[\Theta_{25}, \Theta_{39}] &= 0 \\
[\Theta_{25}, \Theta_{40}] &= 0 \\
[\Theta_{5}, \Theta_{36}] &= -\tfrac{63525}{2} + 8107\Theta_{41} + 8107\Theta_{40} - 484\Theta_{39} + 198\Theta_{26} - 187\Theta_{25} \\
&\quad -\Theta_4 - 484\Theta_{41}^2 - 1089\Theta_{40}\Theta_{41} - 484\Theta_{40}^2 + 121\Theta_{39}\Theta_{41} \\
&\quad +121\Theta_{35}\Theta_{38} + 121\Theta_{34}\Theta_{37} - 3025\Theta_{33}\Theta_{36} + 11\Theta_{30}\Theta_{31} \\
&\quad -11\Theta_{28}\Theta_{32} - 22\Theta_{26}\Theta_{41} - 22\Theta_{26}\Theta_{40} + 22\Theta_{25}\Theta_{41} + 22\Theta_{25}\Theta_{40} \\
&\quad +363\Theta_{33}\Theta_{36}\Theta_{41} + 363\Theta_{33}\Theta_{36}\Theta_{40} \\
[\Theta_{6}, \Theta_{33}] &= \tfrac{12705}{2} - 2057\Theta_{41} - 2420\Theta_{40} - 6292\Theta_{39} + 165\Theta_{26} - 154\Theta_{25} \\
&\quad -\Theta_4 + 242\Theta_{41}^2 + 363\Theta_{40}\Theta_{41} + 242\Theta_{40}^2 + 2420\Theta_{39}\Theta_{41} \\
&\quad +2420\Theta_{39}\Theta_{40} - 1936\Theta_{39}^2 + 121\Theta_{35}\Theta_{38} + 121\Theta_{34}\Theta_{37} \\
&\quad +242\Theta_{33}\Theta_{36} - 11\Theta_{30}\Theta_{31} + 11\Theta_{28}\Theta_{32} - 22\Theta_{26}\Theta_{41} - 22\Theta_{26}\Theta_{40} \\
&\quad +11\Theta_{26}\Theta_{39} + 22\Theta_{25}\Theta_{41} + 22\Theta_{25}\Theta_{40} - 11\Theta_{25}\Theta_{39} - 242\Theta_{39}\Theta_{41}^2 \\
&\quad -484\Theta_{39}\Theta_{40}\Theta_{41} - 242\Theta_{39}\Theta_{40}^2 + 363\Theta_{39}^2\Theta_{41} + 363\Theta_{39}^2\Theta_{40} \\
&\quad -121\Theta_{39}^3 \\
[\Theta_{21}, \Theta_{24}] &= -\tfrac{7381}{2} + 6655\Theta_{41} + 3751\Theta_{40} + 7381\Theta_{39} - 99\Theta_{26} + 143\Theta_{25} \\
&\quad +\Theta_4 - 1331\Theta_{41}^2 - 3146\Theta_{40}\Theta_{41} - 484\Theta_{40}^2 - 1815\Theta_{39}\Theta_{41} \\
&\quad -2662\Theta_{39}\Theta_{40} + 847\Theta_{39}^2 + 1210\Theta_{35}\Theta_{38} - 121\Theta_{34}\Theta_{37} \\
&\quad +2662\Theta_{33}\Theta_{36} - 11\Theta_{30}\Theta_{31} + 11\Theta_{28}\Theta_{32} - 11\Theta_{26}\Theta_{41} \\
&\quad +22\Theta_{26}\Theta_{40} - 22\Theta_{26}\Theta_{39} - 11\Theta_{25}\Theta_{41} - 22\Theta_{25}\Theta_{40} + 11\Theta_{25}\Theta_{39} \\
&\quad +363\Theta_{40}\Theta_{41}^2 + 242\Theta_{40}^2\Theta_{41} + 121\Theta_{39}\Theta_{41}^2 + 363\Theta_{39}\Theta_{40}\Theta_{41} \\
&\quad +242\Theta_{39}\Theta_{40}^2 - 121\Theta_{39}^2\Theta_{41} - 121\Theta_{39}^2\Theta_{40} - 242\Theta_{35}\Theta_{38}\Theta_{41}
\end{aligned}
$$

$$
\begin{aligned}
& -363\Theta_{35}\Theta_{38}\Theta_{39} - 605\Theta_{33}\Theta_{36}\Theta_{41} - 363\Theta_{33}\Theta_{36}\Theta_{40} \\
& +121\Theta_{33}\Theta_{36}\Theta_{39} \\
[\Theta_{12}, \Theta_{27}] \ = \ & \tfrac{18029}{4} - 19239\Theta_{41} - \tfrac{8591}{2}\Theta_{40} + \tfrac{32065}{4}\Theta_{39} + 440\Theta_{26} - \tfrac{341}{2}\Theta_{25} \\
& -\Theta_4 + \tfrac{10527}{2}\Theta_{41}^2 + 6413\Theta_{40}\Theta_{41} + 605\Theta_{40}^2 - 3267\Theta_{39}\Theta_{41} \\
& -2178\Theta_{39}\Theta_{40} + 363\Theta_{39}^2 + 121\Theta_{35}\Theta_{38} - \tfrac{5445}{2}\Theta_{34}\Theta_{37} \\
& -5324\Theta_{33}\Theta_{36} + 11\Theta_{30}\Theta_{31} + 11\Theta_{28}\Theta_{32} + 11\Theta_{27}\Theta_{29} \\
& -55\Theta_{26}\Theta_{41} - 66\Theta_{26}\Theta_{40} + 11\Theta_{26}\Theta_{39} + 22\Theta_{25}\Theta_{41} + 22\Theta_{25}\Theta_{40} \\
& -\tfrac{11}{2}\Theta_{25}\Theta_{39} - 363\Theta_{41}^3 - 847\Theta_{40}\Theta_{41}^2 \\
& -484\Theta_{40}^2\Theta_{41} + \tfrac{605}{2}\Theta_{39}\Theta_{41}^2 + 484\Theta_{39}\Theta_{40}\Theta_{41} + 121\Theta_{39}\Theta_{40}^2 \\
& -\tfrac{121}{2}\Theta_{39}^2\Theta_{41} - \tfrac{121}{2}\Theta_{39}^2\Theta_{40} + 363\Theta_{34}\Theta_{37}\Theta_{41} + 484\Theta_{34}\Theta_{37}\Theta_{40} \\
& -\tfrac{121}{2}\Theta_{34}\Theta_{37}\Theta_{39} + 726\Theta_{33}\Theta_{36}\Theta_{41} + 968\Theta_{33}\Theta_{36}\Theta_{40} \\
& -242\Theta_{33}\Theta_{36}\Theta_{39} \\
[\Theta_{25}, \Theta_{26}] \ = \ & 0 \\
[\Theta_7, \Theta_{24}] \ = \ & -\tfrac{1004905}{4} - 684134\Theta_{41} - \tfrac{464519}{2}\Theta_{40} - 511104\Theta_{39} + 12947\Theta_{26} \\
& -10527\Theta_{25} + 22\Theta_4 + \Theta_1 + 287496\Theta_{41}^2 + 548372\Theta_{40}\Theta_{41} \\
& +94501\Theta_{40}^2 + 98494\Theta_{39}\Theta_{41} + 230263\Theta_{39}\Theta_{40} - \tfrac{91839}{2}\Theta_{39}^2 \\
& -202312\Theta_{35}\Theta_{38} + \tfrac{9317}{2}\Theta_{34}\Theta_{37} - 516428\Theta_{33}\Theta_{36} + 1210\Theta_{30}\Theta_{31} \\
& -1573\Theta_{28}\Theta_{32} - 363\Theta_{26}\Theta_{41} - 5203\Theta_{26}\Theta_{40} + 2420\Theta_{26}\Theta_{39} \\
& -22\Theta_{26}^2 + 1694\Theta_{25}\Theta_{41} + 3146\Theta_{25}\Theta_{40} - 726\Theta_{25}\Theta_{39} \\
& +11\Theta_{25}\Theta_{26} + 11\Theta_{22}\Theta_{23} + 11\Theta_{21}\Theta_{24} + 11\Theta_5\Theta_{36} - 27951\Theta_{41}^3 \\
& -139755\Theta_{40}\Theta_{41}^2 - 103818\Theta_{40}^2\Theta_{41} - 7986\Theta_{40}^3 + 1331\Theta_{39}\Theta_{41}^2 \\
& -23958\Theta_{39}\Theta_{40}\Theta_{41} - 41261\Theta_{39}\Theta_{40}^2 + 5324\Theta_{39}^2\Theta_{41} \\
& +15972\Theta_{39}^2\Theta_{40} + 83853\Theta_{35}\Theta_{38}\Theta_{41} + 63888\Theta_{35}\Theta_{38}\Theta_{40} \\
& +1331\Theta_{35}\Theta_{38}\Theta_{39} - 2662\Theta_{34}\Theta_{37}\Theta_{40} + 2662\Theta_{34}\Theta_{37}\Theta_{39} \\
& +174361\Theta_{33}\Theta_{36}\Theta_{41} + 159720\Theta_{33}\Theta_{36}\Theta_{40} - \tfrac{113135}{2}\Theta_{33}\Theta_{36}\Theta_{39} \\
& +121\Theta_{31}\Theta_{32}\Theta_{37} - 363\Theta_{30}\Theta_{31}\Theta_{41} - 121\Theta_{30}\Theta_{31}\Theta_{40} \\
& +121\Theta_{29}\Theta_{30}\Theta_{33} + 363\Theta_{28}\Theta_{32}\Theta_{41} + 242\Theta_{28}\Theta_{32}\Theta_{40}
\end{aligned}
$$

$$
\begin{aligned}
& -121\Theta_{28}\Theta_{30}\Theta_{34} - 121\Theta_{27}\Theta_{32}\Theta_{33} - 363\Theta_{26}\Theta_{41}^2 \\
& +242\Theta_{26}\Theta_{40}\Theta_{41} + 484\Theta_{26}\Theta_{40}^2 - 121\Theta_{26}\Theta_{39}\Theta_{41} \\
& -363\Theta_{26}\Theta_{39}\Theta_{40} - 121\Theta_{26}\Theta_{35}\Theta_{38} + 121\Theta_{26}\Theta_{33}\Theta_{36} \\
& -242\Theta_{25}\Theta_{40}\Theta_{41} - 242\Theta_{25}\Theta_{40}^2 + 121\Theta_{25}\Theta_{39}\Theta_{40} \\
& -121\Theta_{25}\Theta_{35}\Theta_{38} - 121\Theta_{25}\Theta_{33}\Theta_{36} + 7986\Theta_{40}\Theta_{41}^3 \\
& +13310\Theta_{40}^2\Theta_{41}^2 + 5324\Theta_{40}^3\Theta_{41} - 1331\Theta_{39}\Theta_{40}\Theta_{41}^2 \\
& +2662\Theta_{39}\Theta_{40}^2\Theta_{41} + 2662\Theta_{39}\Theta_{40}^3 - 1331\Theta_{39}^2\Theta_{40}\Theta_{41} \\
& -1331\Theta_{39}^2\Theta_{40}^2 - 7986\Theta_{35}\Theta_{38}\Theta_{41}^2 - 13310\Theta_{35}\Theta_{38}\Theta_{40}\Theta_{41} \\
& -3993\Theta_{35}\Theta_{38}\Theta_{40}^2 + 1331\Theta_{35}\Theta_{38}\Theta_{39}\Theta_{41} - 2662\Theta_{35}\Theta_{38}\Theta_{39}\Theta_{40} \\
& +1331\Theta_{35}\Theta_{38}\Theta_{39}^2 + 3993\Theta_{35}^2\Theta_{38}^2 - 13310\Theta_{33}\Theta_{36}\Theta_{41}^2 \\
& -27951\Theta_{33}\Theta_{36}\Theta_{40}\Theta_{41} - 11979\Theta_{33}\Theta_{36}\Theta_{40}^2 \\
& +9317\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{41} + 7986\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{40} - 1331\Theta_{33}\Theta_{36}\Theta_{39}^2 \\
& +10648\Theta_{33}\Theta_{35}\Theta_{36}\Theta_{38} \\
[\Theta_4, \Theta_{25}] \;=\; & 0 \\
[\Theta_{11}, \Theta_{12}] \;=\; & \tfrac{10763797}{8} + \tfrac{2232087}{2}\Theta_{41} - \tfrac{775973}{4}\Theta_{40} + \tfrac{488477}{4}\Theta_{39} - 29403\Theta_{26} \\
& +16335\Theta_{25} - 77\Theta_4 - 2\Theta_1 - \tfrac{1468093}{2}\Theta_{41}^2 - 791945\Theta_{40}\Theta_{41} \\
& -\tfrac{105149}{2}\Theta_{40}^2 + \tfrac{320771}{2}\Theta_{39}\Theta_{41} + \tfrac{1331}{4}\Theta_{39}\Theta_{40} + \tfrac{1331}{4}\Theta_{39}^2 \\
& -57233\Theta_{35}\Theta_{38} + \tfrac{1497375}{4}\Theta_{34}\Theta_{37} + 795938\Theta_{33}\Theta_{36} \\
& +1089\Theta_{30}\Theta_{31} + 1089\Theta_{28}\Theta_{32} - \tfrac{847}{2}\Theta_{27}\Theta_{29} + 9075\Theta_{26}\Theta_{41} \\
& +9075\Theta_{26}\Theta_{40} - 1573\Theta_{26}\Theta_{39} - 22\Theta_{26}^2 - 4477\Theta_{25}\Theta_{41} \\
& -\tfrac{9801}{2}\Theta_{25}\Theta_{40} + \tfrac{1331}{2}\Theta_{25}\Theta_{39} + 22\Theta_{25}\Theta_{26} - \tfrac{11}{2}\Theta_{25}^2 - 11\Theta_{12}\Theta_{27} \\
& -22\Theta_{11}\Theta_{29} + 119790\Theta_{41}^3 + \tfrac{531069}{2}\Theta_{40}\Theta_{41}^2 + 149072\Theta_{40}^2\Theta_{41} \\
& +7986\Theta_{40}^3 - \tfrac{107811}{2}\Theta_{39}\Theta_{41}^2 - 61226\Theta_{39}\Theta_{40}\Theta_{41} - 3993\Theta_{39}\Theta_{40}^2 \\
& +\tfrac{9317}{2}\Theta_{39}^2\Theta_{41} + 7986\Theta_{35}\Theta_{38}\Theta_{41} + 7986\Theta_{35}\Theta_{38}\Theta_{40} \\
& -2662\Theta_{35}\Theta_{38}\Theta_{39} - 159720\Theta_{34}\Theta_{37}\Theta_{41} - 130438\Theta_{34}\Theta_{37}\Theta_{40} \\
& +\tfrac{59895}{2}\Theta_{34}\Theta_{37}\Theta_{39} - 287496\Theta_{33}\Theta_{36}\Theta_{41} - 258214\Theta_{33}\Theta_{36}\Theta_{40} \\
& +95832\Theta_{33}\Theta_{36}\Theta_{39} - 484\Theta_{31}\Theta_{32}\Theta_{37} - 242\Theta_{30}\Theta_{31}\Theta_{40}
\end{aligned}
$$

$$
\begin{aligned}
& -242\Theta_{28}\Theta_{32}\Theta_{40} + 484\Theta_{28}\Theta_{30}\Theta_{34} - 726\Theta_{26}\Theta_{41}^2 \\
& -1210\Theta_{26}\Theta_{40}\Theta_{41} - 726\Theta_{26}\Theta_{40}^2 + 242\Theta_{26}\Theta_{39}\Theta_{41} \\
& +121\Theta_{26}\Theta_{39}\Theta_{40} - 726\Theta_{26}\Theta_{34}\Theta_{37} + 363\Theta_{25}\Theta_{41}^2 \\
& +605\Theta_{25}\Theta_{40}\Theta_{41} + 363\Theta_{25}\Theta_{40}^2 - 121\Theta_{25}\Theta_{39}\Theta_{41} \\
& -\tfrac{121}{2}\Theta_{25}\Theta_{39}\Theta_{40} + 363\Theta_{25}\Theta_{34}\Theta_{37} - \tfrac{11979}{2}\Theta_{41}^4 - 19965\Theta_{40}\Theta_{41}^3 \\
& -22627\Theta_{40}^2\Theta_{41}^2 - 7986\Theta_{40}^3\Theta_{41} + 3993\Theta_{39}\Theta_{41}^3 + \tfrac{17303}{2}\Theta_{39}\Theta_{40}\Theta_{41}^2 \\
& +5324\Theta_{39}\Theta_{40}^2\Theta_{41} - \tfrac{1331}{2}\Theta_{39}^2\Theta_{41}^2 - \tfrac{1331}{2}\Theta_{39}^2\Theta_{40}\Theta_{41} \\
& +11979\Theta_{34}\Theta_{37}\Theta_{41}^2 + 27951\Theta_{34}\Theta_{37}\Theta_{40}\Theta_{41} \\
& +11979\Theta_{34}\Theta_{37}\Theta_{40}^2 - 3993\Theta_{34}\Theta_{37}\Theta_{39}\Theta_{41} \\
& -\tfrac{11979}{2}\Theta_{34}\Theta_{37}\Theta_{39}\Theta_{40} + \tfrac{1331}{2}\Theta_{34}\Theta_{37}\Theta_{39}^2 - \tfrac{11979}{2}\Theta_{34}^2\Theta_{37}^2 \\
& +23958\Theta_{33}\Theta_{36}\Theta_{41}^2 + 47916\Theta_{33}\Theta_{36}\Theta_{40}\Theta_{41} + 21296\Theta_{33}\Theta_{36}\Theta_{40}^2 \\
& -15972\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{41} - 15972\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{40} \\
& +2662\Theta_{33}\Theta_{36}\Theta_{39}^2 - 10648\Theta_{33}\Theta_{34}\Theta_{36}\Theta_{37}
\end{aligned}
$$

$$
\begin{aligned}
[\Theta_2, \Theta_{12}] \quad = \quad & \tfrac{1289096127}{8} + \tfrac{1189625173}{8}\Theta_{41} - \tfrac{131080873}{4}\Theta_{40} + \tfrac{138957731}{4}\Theta_{39} \\
& -\tfrac{13436445}{4}\Theta_{26} + \tfrac{8519731}{4}\Theta_{25} - \tfrac{52393}{4}\Theta_{4} - 308\Theta_{1} - \tfrac{467853155}{4}\Theta_{41}^2 \\
& -\tfrac{500444021}{4}\Theta_{40}\Theta_{41} - 5739272\Theta_{40}^2 + \tfrac{48037121}{4}\Theta_{39}\Theta_{41} \\
& -\tfrac{42092875}{4}\Theta_{39}\Theta_{40} + \tfrac{29852999}{4}\Theta_{39}^2 - \tfrac{35943655}{4}\Theta_{35}\Theta_{38} \\
& +\tfrac{219951743}{4}\Theta_{34}\Theta_{37} + \tfrac{370402659}{4}\Theta_{33}\Theta_{36} - \tfrac{656183}{4}\Theta_{30}\Theta_{31} \\
& -\tfrac{326095}{4}\Theta_{28}\Theta_{32} + \tfrac{41261}{2}\Theta_{27}\Theta_{29} + \tfrac{4327081}{4}\Theta_{26}\Theta_{41} + \tfrac{2040423}{2}\Theta_{26}\Theta_{40} \\
& -\tfrac{581647}{2}\Theta_{26}\Theta_{39} + 363\Theta_{26}^2 - \tfrac{2796431}{4}\Theta_{25}\Theta_{41} - 630894\Theta_{25}\Theta_{40} \\
& +\tfrac{328757}{2}\Theta_{25}\Theta_{39} - \tfrac{363}{2}\Theta_{25}\Theta_{26} + \tfrac{6171}{2}\Theta_{22}\Theta_{23} - 2904\Theta_{21}\Theta_{24} \\
& -11\Theta_{16}\Theta_{17} - 11\Theta_{15}\Theta_{18} + 11\Theta_{14}\Theta_{19} + 11\Theta_{13}\Theta_{20} - 242\Theta_{12}\Theta_{27} \\
& -605\Theta_{11}\Theta_{29} - 22\Theta_{11}\Theta_{12} + 11\Theta_{10}\Theta_{21} + 11\Theta_{9}\Theta_{22} - 11\Theta_{8}\Theta_{23} \\
& -11\Theta_{7}\Theta_{24} + 2420\Theta_{6}\Theta_{33} + 2541\Theta_{5}\Theta_{36} + \tfrac{5203}{2}\Theta_{4}\Theta_{41} \\
& +\tfrac{8107}{2}\Theta_{4}\Theta_{40} - 363\Theta_{4}\Theta_{39} - 11\Theta_{4}\Theta_{26} + \tfrac{11}{2}\Theta_{4}\Theta_{25} - 22\Theta_{2}\Theta_{29} \\
& +44\Theta_{1}\Theta_{41} + 44\Theta_{1}\Theta_{40} - 11\Theta_{1}\Theta_{39} + 25285007\Theta_{41}^3 \\
& +\tfrac{220420255}{4}\Theta_{40}\Theta_{41}^2 + 30863228\Theta_{40}^2\Theta_{41} + 1654433\Theta_{40}^3
\end{aligned}
$$

$$
\begin{aligned}
&-7803653\Theta_{39}\Theta_{41}^2 - \tfrac{14187129}{2}\Theta_{39}\Theta_{40}\Theta_{41} + 1024870\Theta_{39}\Theta_{40}^2\\
&-\tfrac{7627961}{4}\Theta_{39}^2\Theta_{41} - \tfrac{5021863}{2}\Theta_{39}^2\Theta_{40} + 644204\Theta_{39}^3\\
&+\tfrac{4348377}{2}\Theta_{35}\Theta_{38}\Theta_{41} + \tfrac{5578221}{2}\Theta_{35}\Theta_{38}\Theta_{40} - 1288408\Theta_{35}\Theta_{38}\Theta_{39}\\
&-\tfrac{115327157}{4}\Theta_{34}\Theta_{37}\Theta_{41} - 25167879\Theta_{34}\Theta_{37}\Theta_{40}\\
&+\tfrac{10087649}{2}\Theta_{34}\Theta_{37}\Theta_{39} - 43644821\Theta_{33}\Theta_{36}\Theta_{41}\\
&-\tfrac{77143429}{2}\Theta_{33}\Theta_{36}\Theta_{40} + \tfrac{41975747}{4}\Theta_{33}\Theta_{36}\Theta_{39} - 79860\Theta_{31}\Theta_{32}\Theta_{37}\\
&+\tfrac{155727}{2}\Theta_{30}\Theta_{31}\Theta_{41} + \tfrac{105149}{2}\Theta_{30}\Theta_{31}\Theta_{40} - 17303\Theta_{30}\Theta_{31}\Theta_{39}\\
&-\tfrac{3993}{2}\Theta_{29}\Theta_{30}\Theta_{33} + 1331\Theta_{29}^2\Theta_{37} + \tfrac{126445}{2}\Theta_{28}\Theta_{32}\Theta_{41}\\
&+\tfrac{75867}{2}\Theta_{28}\Theta_{32}\Theta_{40} - 38599\Theta_{28}\Theta_{32}\Theta_{39} + 27951\Theta_{28}\Theta_{30}\Theta_{34}\\
&+\tfrac{3993}{2}\Theta_{28}\Theta_{29}\Theta_{36} + 23958\Theta_{27}\Theta_{32}\Theta_{33} - 25289\Theta_{27}\Theta_{31}\Theta_{36}\\
&-\tfrac{14641}{2}\Theta_{27}\Theta_{29}\Theta_{41} - 7986\Theta_{27}\Theta_{29}\Theta_{40} + 1331\Theta_{27}\Theta_{29}\Theta_{39}\\
&+\tfrac{3993}{2}\Theta_{27}^2\Theta_{34} - 103818\Theta_{26}\Theta_{41}^2 - \tfrac{302137}{2}\Theta_{26}\Theta_{40}\Theta_{41}\\
&-78529\Theta_{26}\Theta_{40}^2 + \tfrac{99825}{2}\Theta_{26}\Theta_{39}\Theta_{41} + \tfrac{89177}{2}\Theta_{26}\Theta_{39}\Theta_{40}\\
&-7986\Theta_{26}\Theta_{39}^2 - 2662\Theta_{26}\Theta_{35}\Theta_{38} - \tfrac{278179}{2}\Theta_{26}\Theta_{34}\Theta_{37}\\
&-65219\Theta_{26}\Theta_{33}\Theta_{36} + 363\Theta_{26}\Theta_{30}\Theta_{31} + 363\Theta_{26}\Theta_{28}\Theta_{32}\\
&-121\Theta_{26}^2\Theta_{41} + \tfrac{142417}{2}\Theta_{25}\Theta_{41}^2 + 110473\Theta_{25}\Theta_{40}\Theta_{41}\\
&+46585\Theta_{25}\Theta_{40}^2 - 29282\Theta_{25}\Theta_{39}\Theta_{41} - 25289\Theta_{25}\Theta_{39}\Theta_{40}\\
&+3993\Theta_{25}\Theta_{39}^2 - \tfrac{3993}{2}\Theta_{25}\Theta_{35}\Theta_{38} + \tfrac{91839}{2}\Theta_{25}\Theta_{34}\Theta_{37}\\
&+3993\Theta_{25}\Theta_{33}\Theta_{36} - \tfrac{363}{2}\Theta_{25}\Theta_{30}\Theta_{31} - \tfrac{363}{2}\Theta_{25}\Theta_{28}\Theta_{32} + \tfrac{121}{2}\Theta_{25}\Theta_{26}\Theta_{41}\\
&-363\Theta_{22}\Theta_{23}\Theta_{41} - 484\Theta_{22}\Theta_{23}\Theta_{40} + 121\Theta_{22}\Theta_{23}\Theta_{39} + 363\Theta_{21}\Theta_{24}\Theta_{41}\\
&+484\Theta_{21}\Theta_{24}\Theta_{40} - 121\Theta_{21}\Theta_{24}\Theta_{39} - 121\Theta_{12}\Theta_{30}\Theta_{33} + 121\Theta_{12}\Theta_{28}\Theta_{36}\\
&-121\Theta_{12}\Theta_{27}\Theta_{41} - 121\Theta_{12}\Theta_{27}\Theta_{40} - 363\Theta_{6}\Theta_{33}\Theta_{41} - 484\Theta_{6}\Theta_{33}\Theta_{40}\\
&+121\Theta_{6}\Theta_{33}\Theta_{39} - 363\Theta_{5}\Theta_{36}\Theta_{41} - 484\Theta_{5}\Theta_{36}\Theta_{40} + 121\Theta_{5}\Theta_{36}\Theta_{39}\\
&-\tfrac{363}{2}\Theta_{4}\Theta_{41}^2 - 484\Theta_{4}\Theta_{40}\Theta_{41} - 363\Theta_{4}\Theta_{40}^2 + \tfrac{121}{2}\Theta_{4}\Theta_{39}\Theta_{41}\\
&+\tfrac{121}{2}\Theta_{4}\Theta_{39}\Theta_{40} - \tfrac{363}{2}\Theta_{4}\Theta_{34}\Theta_{37} - 2196150\Theta_{41}^4 - 7159449\Theta_{40}\Theta_{41}^3\\
&-\tfrac{15680511}{2}\Theta_{40}^2\Theta_{41}^2 - 2884277\Theta_{40}^3\Theta_{41} - 87846\Theta_{40}^4 + 1083434\Theta_{39}\Theta_{41}^3\\
&+\tfrac{4436223}{2}\Theta_{39}\Theta_{40}\Theta_{41}^2 + \tfrac{2298637}{2}\Theta_{39}\Theta_{40}^2\Theta_{41} - 43923\Theta_{39}\Theta_{40}^3 + 58564\Theta_{39}^2\Theta_{41}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{600281}{2}\Theta_{39}^2\Theta_{40}\Theta_{41}+219615\Theta_{39}^2\Theta_{40}^2-102487\Theta_{39}^3\Theta_{41}-117128\Theta_{39}^3\Theta_{40}\\
&+14641\Theta_{39}^4-\tfrac{248897}{2}\Theta_{35}\Theta_{38}\Theta_{41}^2-322102\Theta_{35}\Theta_{38}\Theta_{40}\Theta_{41}\\
&-204974\Theta_{35}\Theta_{38}\Theta_{40}^2+\tfrac{278179}{2}\Theta_{35}\Theta_{38}\Theta_{39}\Theta_{41}+\tfrac{336743}{2}\Theta_{35}\Theta_{38}\Theta_{39}\Theta_{40}\\
&-29282\Theta_{35}\Theta_{38}\Theta_{39}^2+73205\Theta_{35}^2\Theta_{38}^2+\tfrac{8360011}{2}\Theta_{34}\Theta_{37}\Theta_{41}^2\\
&+8389293\Theta_{34}\Theta_{37}\Theta_{40}\Theta_{41}+3718814\Theta_{34}\Theta_{37}\Theta_{40}^2-\tfrac{2679303}{2}\Theta_{34}\Theta_{37}\Theta_{39}\Theta_{41}\\
&-\tfrac{3089251}{2}\Theta_{34}\Theta_{37}\Theta_{39}\Theta_{40}+87846\Theta_{34}\Theta_{37}\Theta_{39}^2-\tfrac{131769}{2}\Theta_{34}\Theta_{35}\Theta_{37}\Theta_{38}\\
&-\tfrac{1595869}{2}\Theta_{34}^2\Theta_{37}^2+6149220\Theta_{33}\Theta_{36}\Theta_{41}^2+11756723\Theta_{33}\Theta_{36}\Theta_{40}\Theta_{41}\\
&+5285401\Theta_{33}\Theta_{36}\Theta_{40}^2-2884277\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{41}-2767149\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{40}\\
&+278179\Theta_{33}\Theta_{36}\Theta_{39}^2+117128\Theta_{33}\Theta_{35}\Theta_{36}\Theta_{38}-1200562\Theta_{33}\Theta_{34}\Theta_{36}\Theta_{37}\\
&+10648\Theta_{31}\Theta_{32}\Theta_{37}\Theta_{41}+10648\Theta_{31}\Theta_{32}\Theta_{37}\Theta_{40}-2662\Theta_{31}\Theta_{32}\Theta_{37}\Theta_{39}\\
&-\tfrac{11979}{2}\Theta_{30}\Theta_{31}\Theta_{41}^2-10648\Theta_{30}\Theta_{31}\Theta_{40}\Theta_{41}-3993\Theta_{30}\Theta_{31}\Theta_{40}^2\\
&+\tfrac{3993}{2}\Theta_{30}\Theta_{31}\Theta_{39}\Theta_{41}+\tfrac{3993}{2}\Theta_{30}\Theta_{31}\Theta_{39}\Theta_{40}+\tfrac{1331}{2}\Theta_{30}\Theta_{31}\Theta_{34}\Theta_{37}\\
&-\tfrac{11979}{2}\Theta_{28}\Theta_{32}\Theta_{41}^2-10648\Theta_{28}\Theta_{32}\Theta_{40}\Theta_{41}-3993\Theta_{28}\Theta_{32}\Theta_{40}^2\\
&+\tfrac{11979}{2}\Theta_{28}\Theta_{32}\Theta_{39}\Theta_{41}+\tfrac{14641}{2}\Theta_{28}\Theta_{32}\Theta_{39}\Theta_{40}-1331\Theta_{28}\Theta_{32}\Theta_{39}^2\\
&+\tfrac{1331}{2}\Theta_{28}\Theta_{32}\Theta_{34}\Theta_{37}-2662\Theta_{28}\Theta_{30}\Theta_{34}\Theta_{41}-2662\Theta_{28}\Theta_{30}\Theta_{34}\Theta_{40}\\
&-1331\Theta_{28}\Theta_{30}\Theta_{34}\Theta_{39}-3993\Theta_{27}\Theta_{32}\Theta_{33}\Theta_{41}-5324\Theta_{27}\Theta_{32}\Theta_{33}\Theta_{40}\\
&+1331\Theta_{27}\Theta_{32}\Theta_{33}\Theta_{39}+3993\Theta_{27}\Theta_{31}\Theta_{36}\Theta_{41}+5324\Theta_{27}\Theta_{31}\Theta_{36}\Theta_{40}\\
&-1331\Theta_{27}\Theta_{31}\Theta_{36}\Theta_{39}+2662\Theta_{27}\Theta_{30}\Theta_{33}\Theta_{34}-2662\Theta_{27}\Theta_{28}\Theta_{34}\Theta_{36}\\
&+\tfrac{3993}{2}\Theta_{26}\Theta_{41}^3+1331\Theta_{26}\Theta_{40}\Theta_{41}^2-1331\Theta_{26}\Theta_{40}^2\Theta_{41}-\tfrac{1331}{2}\Theta_{26}\Theta_{39}\Theta_{41}^2\\
&-\tfrac{1331}{2}\Theta_{26}\Theta_{39}\Theta_{40}\Theta_{41}+\tfrac{35937}{2}\Theta_{26}\Theta_{34}\Theta_{37}\Theta_{41}+18634\Theta_{26}\Theta_{34}\Theta_{37}\Theta_{40}\\
&-3993\Theta_{26}\Theta_{34}\Theta_{37}\Theta_{39}+7986\Theta_{26}\Theta_{33}\Theta_{36}\Theta_{41}+10648\Theta_{26}\Theta_{33}\Theta_{36}\Theta_{40}\\
&-2662\Theta_{26}\Theta_{33}\Theta_{36}\Theta_{39}-\tfrac{3993}{2}\Theta_{25}\Theta_{41}^3-\tfrac{6655}{2}\Theta_{25}\Theta_{40}\Theta_{41}^2-1331\Theta_{25}\Theta_{40}^2\Theta_{41}\\
&+\tfrac{1331}{2}\Theta_{25}\Theta_{39}\Theta_{41}^2+\tfrac{1331}{2}\Theta_{25}\Theta_{39}\Theta_{40}\Theta_{41}-\tfrac{11979}{2}\Theta_{25}\Theta_{34}\Theta_{37}\Theta_{41}\\
&-5324\Theta_{25}\Theta_{34}\Theta_{37}\Theta_{40}+1331\Theta_{25}\Theta_{34}\Theta_{37}\Theta_{39}+\tfrac{131769}{2}\Theta_{41}^5\\
&+\tfrac{570999}{2}\Theta_{40}\Theta_{41}^4+468512\Theta_{40}^2\Theta_{41}^3+336743\Theta_{40}^3\Theta_{41}^2\\
&+87846\Theta_{40}^4\Theta_{41}-43923\Theta_{39}\Theta_{41}^4-\tfrac{278179}{2}\Theta_{39}\Theta_{40}\Theta_{41}^3\\
&-\tfrac{307461}{2}\Theta_{39}\Theta_{40}^2\Theta_{41}^2-58564\Theta_{39}\Theta_{40}^3\Theta_{41}+\tfrac{14641}{2}\Theta_{39}^2\Theta_{41}^3
\end{aligned}
$$

$$
\begin{aligned}
& +14641\Theta_{39}^2\Theta_{40}\Theta_{41}^2 + \tfrac{14641}{2}\Theta_{39}^2\Theta_{40}^2\Theta_{41} - 175692\Theta_{34}\Theta_{37}\Theta_{41}^3 \\
& -\tfrac{1185921}{2}\Theta_{34}\Theta_{37}\Theta_{40}\Theta_{41}^2 - 585640\Theta_{34}\Theta_{37}\Theta_{40}^2\Theta_{41} \\
& -175692\Theta_{34}\Theta_{37}\Theta_{40}^3 + 73205\Theta_{34}\Theta_{37}\Theta_{39}\Theta_{41}^2 \\
& +204974\Theta_{34}\Theta_{37}\Theta_{39}\Theta_{40}\Theta_{41} + 117128\Theta_{34}\Theta_{37}\Theta_{39}\Theta_{40}^2 \\
& -\tfrac{14641}{2}\Theta_{34}\Theta_{37}\Theta_{39}^2\Theta_{41} - 14641\Theta_{34}\Theta_{37}\Theta_{39}^2\Theta_{40} \\
& +\tfrac{219615}{2}\Theta_{34}^2\Theta_{37}^2\Theta_{41} + 87846\Theta_{34}^2\Theta_{37}^2\Theta_{40} - 29282\Theta_{34}^2\Theta_{37}^2\Theta_{39} \\
& -263538\Theta_{33}\Theta_{36}\Theta_{41}^3 - 790614\Theta_{33}\Theta_{36}\Theta_{40}\Theta_{41}^2 \\
& -761332\Theta_{33}\Theta_{36}\Theta_{40}^2\Theta_{41} - 234256\Theta_{33}\Theta_{36}\Theta_{40}^3 \\
& +175692\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{41}^2 + 351384\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{40}\Theta_{41} \\
& +175692\Theta_{33}\Theta_{36}\Theta_{39}\Theta_{40}^2 - 29282\Theta_{33}\Theta_{36}\Theta_{39}^2\Theta_{41} \\
& -29282\Theta_{33}\Theta_{36}\Theta_{39}^2\Theta_{40} + 146410\Theta_{33}\Theta_{34}\Theta_{36}\Theta_{37}\Theta_{41} \\
& +117128\Theta_{33}\Theta_{34}\Theta_{36}\Theta_{37}\Theta_{40} - 29282\Theta_{33}\Theta_{34}\Theta_{36}\Theta_{37}\Theta_{39}
\end{aligned}
$$

$$
\begin{aligned}
[\Theta_1, \Theta_{25}] \quad = \quad & \tfrac{14641}{4} - \tfrac{102487}{2}\Theta_{41} + \tfrac{102487}{2}\Theta_{39} + \tfrac{1331}{2}\Theta_{26} + \tfrac{14641}{2}\Theta_{41}^2 \\
& +\tfrac{14641}{2}\Theta_{40}\Theta_{41} - \tfrac{14641}{2}\Theta_{39}\Theta_{41} - \tfrac{14641}{2}\Theta_{39}\Theta_{40} - \tfrac{14641}{2}\Theta_{34}\Theta_{37} \\
& -\tfrac{14641}{4}\Theta_{33}\Theta_{36} + \tfrac{1331}{2}\Theta_{30}\Theta_{31} + \tfrac{1331}{2}\Theta_{28}\Theta_{32} - \tfrac{1331}{2}\Theta_{31}\Theta_{32}\Theta_{37} \\
& -\tfrac{1331}{2}\Theta_{28}\Theta_{30}\Theta_{34}.
\end{aligned}
$$

We have two 1-dimensional representations:

$$
\begin{aligned}
\Theta_1 &\mapsto -756008 \\
\Theta_4 &\mapsto 16214 \\
\Theta_{25} &\mapsto -\tfrac{671}{2} \\
\Theta_{26} &\mapsto -\tfrac{165}{2} \\
\Theta_{39} &\mapsto 3 \\
\Theta_{40} &\mapsto \tfrac{5}{2} \\
\Theta_{41} &\mapsto 1,
\end{aligned}
$$

and

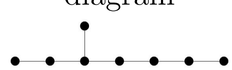
$$\begin{aligned}
\Theta_1 &\mapsto -\tfrac{3531143}{4} \\
\Theta_4 &\mapsto 18029 \\
\Theta_{25} &\mapsto -352 \\
\Theta_{26} &\mapsto -\tfrac{165}{2} \\
\Theta_{39} &\mapsto 3 \\
\Theta_{40} &\mapsto \tfrac{5}{2} \\
\Theta_{41} &\mapsto 1,
\end{aligned}$$

with all other generators $\Theta_i \mapsto 0$ for each of these. We know that we cannot define $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$ for $\mathbb{k}$ of characteristic $2, 3$ or 11, but without a presentation we cannot establish that these are the only primes which have to be excluded for the definition of $U(\mathfrak{g}_{\mathbb{k}}, e_{\mathbb{k}})$.

6.5 Type E_8

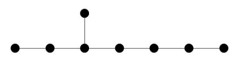
Due to computational limits, it has not been feasible to carry out the calculations above for each of the 17 non-zero rigid nilpotent orbits when $\mathfrak{g}$ is of type E_8, so we summarize the results obtained for the 14 accessible orbits.

Table 6.24: Results for type E_8.

Orbit	Dynkin diagram 	$\kappa(e,f)$	$\dim(\mathfrak{g}^e)$	$\dim(\mathfrak{t}^e)$	# 1-dim reps
A_1	0 0 0 0 0 0 0 1	60	190	7	1

Continued on next page

Table 6.24 – continued from previous page

Orbit	Dynkin diagram 	$\kappa(e,f)$	$\dim(\mathfrak{g}^e)$	$\dim(\mathfrak{t}^e)$	# 1-dim reps
$2A_1$	0 1 0 0 0 0 0 0	120	156	6	1
$3A_1$	0 0 0 0 0 0 1 0	180	136	5	1
$4A_1$	1 0 0 0 0 0 0 0	240	120	4	1
$A_2 + A_1$	0 1 0 0 0 0 0 1	300	112	5	1
$A_2 + 2A_1$	0 0 0 0 0 1 0 0	360	102	4	1
$A_2 + 3A_1$	0 0 1 0 0 0 0 0	420	94	3	1
$2A_2 + A_1$	0 1 0 0 0 0 1 0	540	86	3	1
$A_3 + A_1$	0 0 0 0 0 1 0 1	660	84	4	2
$2A_2 + 2A_1$	0 0 0 0 1 0 0 0	600	80	2	1
$A_3 + 2A_1$	0 0 1 0 0 0 0 1	720	76	3	1
$D_4(a_1) + A_1$	1 0 0 0 0 0 1 0	780	72	2	1
$A_3 + A_2 + A_1$	0 0 0 1 0 0 0 0	900	66	2	1
$2A_3$	0 1 0 0 1 0 0 0	1200	60	2	1

The 3 remaining orbits offer such computational difficulties largely due to the heights of $A_4 + A_3$, $D_5(a_1) + A_2$ and $A_5 + A_1$ being $9, 10$ and 10 respectively.

6.6 The number of 1-dimensional representations

From the above tables we see that for rigid e in exceptional $\mathfrak{g}$ (except possibly for type E_8 with e lying in one of the 3 remaining orbits), $U(\mathfrak{g}, e)$ admits 1 or 2 1-dimensional representations. A common feature of each of the orbits for which there is precisely one 1-dimensional representation is that we can choose a generating set of $\mathfrak{g}^e$ from the $\operatorname{ad} h$ weight spaces $\mathfrak{g}(0)$ and $\mathfrak{g}(1)$; in the (so far, 4) cases where there are two 1-dimensional representations any generating set of $\mathfrak{g}^e$ must contain an element of $\mathfrak{g}(3)$.

Of the remaining E_8 orbits, we can choose generating sets for $\mathfrak{g}^e$ as follows: for e in the orbit $A_4 + A_3$ we have that $\mathfrak{g}^e$ is generated by $\mathfrak{g}(0) \cup \mathfrak{g}(1)$; for e in the orbit $D_5(a_1) + A_2$ we have that $\mathfrak{g}^e$ is generated by $\mathfrak{g}(0) \cup \mathfrak{g}(1) \cup \mathfrak{g}(2) \cup \mathfrak{g}(3)$; and for e in the orbit $A_5 + A_1$ we have that $\mathfrak{g}^e$ is generated by $\cup_{i=0}^{4} \mathfrak{g}(i)$. It seems likely that we will find that in these cases there is more than one 1-dimensional representation.

Appendix A

Finding an $\mathfrak{sl}_2$-triple from a weighted Dynkin diagram using GAP4

For a simple Lie algebra `L` (with Chevalley basis `b`), GAP4 gives a corresponding root system, a set of positive roots, and the associated Cartan matrix. The following procedure takes this data, together with a weighted Dynkin diagram corresponding to a nilpotent orbit in `L`, and returns a list $[e,\ h,\ f]$ such that these elements of `L` nontrivially satisfy the $\mathfrak{sl}_2$ relations, where the coefficients are integers and the number of basis elements whose sum is e is minimal. If no such triple exists (i.e. the weighted Dynkin diagram does not correspond to a nilpotent orbit), the procedure returns `"fail"`.

The function `Sl2h(WDD)` returns the semisimple element of the desired $\mathfrak{sl}_2$-triple by solving a set of linear equations involving the Cartan matrix and the weighted Dynkin diagram `WDD`. The function `EValue(h,y)` returns the $\operatorname{ad} h$ eigenvalue of an element `y` in `L`; if `y` is not an eigenvector for $\operatorname{ad} h$ the function returns `"fail"`. The function `ESpace(h,u)` returns the list of basis elements of `L` whose $\operatorname{ad} h$ eigenvalue is `u`.

```
L:=SimpleLieAlgebra("type", rank, field);
b:=Basis(L);
n:=Dimension(L);

RSL:=RootSystem(L);
RSLpos:=PositiveRoots(RSL);
C:=CartanMatrix(RSL);
RankL:=Length(C);

Sl2h:=function(WDD)
  local CT, v, w, i;
  CT:=TransposedMat(C);
  v:=SolutionMat(CT,WDD);
  w:=List([1..RankL],i->b[2*Length(RSLpos)+i]);
  return v*w;
end;;

EValue:=function(h,y)
  local yx, a;
  yx:=ExtRepOfObj(y);
  a:=ExtRepOfObj(h*y);
  if a in VectorSpace(Rationals,[yx]) then
  return SolutionMat([yx],a)[1];
  else return "fail";
  fi;
end;;

ESpace:=function(h,u)
  local i;
  return Filtered(b,i->EValue(h,i)=u);
end;;
```

```
GetSl2:=function(WDD)
  local h, A, CA, B, CB, i, U, V, M, j, N, P, e, M1, f, v;
  h:=Sl2h(WDD);
  A:=ESpace(h,2);
  CA:=Combinations(A);
  Sort(CA,function(v,w) return Length(v)<=Length(w); end);
  CB:=List(CA,i->List(Positions(ExtRepOfObj(Sum(i)),1)
                                        +(n-RankL)/2,j->b[j]));
  for i in [2..Length(CA)] do
  U:=CA[i];
  V:=CB[i];
  M:=NullMat(Length(U),Length(V));
    for i in [1..Length(U)] do
      for j in [1..Length(V)] do
      M[i][j]:=U[i]*V[j];
      od;
    od;
  N:=TransposedMat(M);
  P:=List(N,i->Sum(i));
  if h in VectorSpace(Rationals,P) then
  e:=Sum(U);
  M1:=List(V,j->ExtRepOfObj(e*j));
  v:=SolutionMat(M1,ExtRepOfObj(h));
    if ForAll(v,k->k in Integers) then
    f:=v*V;
    return [e,h,f];
    fi;
  fi;
  od;
  return "fail";
end;;
```

Appendix B

A GAP4 procedure for creating a new basis for a Lie algebra

Our purpose is to specify a basis for our Lie algebra $\mathfrak{g}$ according to the requirements of Section 3.2. The input is the simple Lie algebra `L` (and `n` is defined to be the dimension of `L`) and a list `c` of `n` elements in terms of the inbuilt Chevalley basis corresponding to the new ordered basis of $\mathfrak{g}$.

This is carried out by means of a table `TSC` of structure constants. We also create the following: the associated universal enveloping algebra `Ug`; the set of `n` PBW generators `x` of `Ug` corresponding to the ordered basis `c` of $\mathfrak{g}$; and a polynomial ring `R` on the set `r` of n indeterminates. We declare `famUg` and `famR` to be the respective families of objects containing the elements of `Ug` and `R`.

```
NewSCTable:=function(c)
  local C, Y, R, TSC, M, N, i, j, k, A, u, V;
  C:=NullMat(n,n);
  Y:=[];
  V:=[];
  TSC:=EmptySCTable(n,0,"antisymmetric");
  M:=List(c,i->ExtRepOfObj(i));
  N:=Inverse(M);
```

```
  for j in [1..Length(c)] do
    for k in [j..n] do
    C[j][k]:=c[j]*c[k];
    od;
  od;
  for i in [1..n] do
    for j in [i+1..n] do
    if not c[i]*c[j]=Zero(L) then
    u:=ExtRepOfObj(c[i]*c[j]);
    A:=u*N;
      for k in [1..n] do
      V[2*k]:=k;
      V[2*k-1]:=A[k];
      SetEntrySCTable(TSC, i, j, V);
      od;
    fi;
    od;
  od;
  return TSC;
end;;

TSC:=NewSCTable(c);
g:=LieAlgebraByStructureConstants(Rationals,TSC);
c:=Basis(g);
n:=Length(c);
Ug:=UniversalEnvelopingAlgebra(g);
x:=GeneratorsOfAlgebraWithOne(Ug);
famUg:=FamilyObj(One(Ug));
R:=PolynomialRing(Rationals,n);
r:=IndeterminatesOfPolynomialRing(R);
famR:=FamilyObj(One(R));
```

Appendix C

Calculating in $U(\mathfrak{g}, e)$ using GAP4

We have elements `e`, `h` and `f` as elements of `g`. Elements of $U(\mathfrak{g}, e)$ are stored in GAP4 as elements of the polynomial ring `R`. Addition and scalar multiplication are carried out in `R`. The associative multiplication operation in $U(\mathfrak{g}, e)$ on two elements is carried out by the function `Mult(y,z)`, which converts each argument into a list of monomials in `Ug` (using the function `MonomialList`), then takes the sum of the term by term products in `Ug`, and factors out the left ideal I_χ by evaluating each basis element of $\mathfrak{m}$ at χ (the function `InQ(y)`). The function `MultList` carries out the multiplication operation on a list of elements of $U(\mathfrak{g}, e)$. The function `Com(y,z)` returns the commutator of two elements in $U(\mathfrak{g}, e)$ by similar means. The function `Kdeg(y)` returns the Kazhdan degree of an element `y` of `Ug`.

```
d:=Dimension(Centralizer(g,e));
q:=1/2*(n+d);
mBasis:=[q+1..n];
hDeg:=List([1..n],i->EValue(h,c[i]));
Kappa_ef:=Trace(AdjointMatrix(c,e)*AdjointMatrix(c,f));
chi:=List(mBasis,i->1/Kappa_ef*
              Trace(AdjointMatrix(c,e)*AdjointMatrix(c,c[i])));
```

```
CSA:=Filtered([1..n],i->h*c[i]=0*c[i] and
            e*c[i]=0*c[i] and
            ForAll(c,j->c[i]*j in VectorSpace(Rationals,[j])));
CSA_R:=List(CSA,i->r[i]);

InQ:=function(y)
  local i;
  return Value(y,List(mBasis,i->r[i]),chi)*One(R);
end;;

MonomialList:=function(y)
  local yx, i;
  yx:=ExtRepPolynomialRatFun(y);
  return List([1..Length(yx)/2],i->
                 [ObjByExtRep(famUg,[0,[yx[2*i-1],yx[2*i]]])]);
end;;

Mult:=function(y,z)
  local u, yList, zList, i, j, v;
  u:=Zero(R);
  yList:=MonomialList(y);
  zList:=MonomialList(z);
  for i in [1..Length(yList)] do
  v:=[];
    for j in [1..Length(zList)] do
    v[j]:=PolynomialByExtRep(famR,
                            ExtRepOfObj(yList[i]*zList[j])[2]);
    od;
  u:=u+Sum(v);
  od;
  return InQ(u);
```

```
end;;

MultList:=function(Y)
  local a, i ;
  a:=One(R);
  for i in [1..Length(Y)] do
  a:=Mult(a,Y[i]);
  od;
  return a;
end;;

Com:=function(y,z)
  local u, yList, zList, i, j, v;
  u:=Zero(R);
  yList:=MonomialList(y);
  zList:=MonomialList(z);
  for i in [1..Length(yList)] do
  v:=[];
    for j in [1..Length(zList)] do
    v[j]:=PolynomialByExtRep(famR,
          ExtRepOfObj(yList[i]*zList[j]-zList[j]*yList[i])[2]);
    od;
  u:=u+InQ(Sum(v));
  od;
  return u;
end;;

KList:=function(Y)
  local i, u;
  u:=0;
  for i in [1..Length(Y)/2] do
```

```
  u:=u+(hDeg[Y[2*i-1]]+2)*Y[2*i];
  od;
  return u;
end;;

KDeg:=function(y)
  local i, yx, pxf;
  if y = Zero(R) then
  return 0;
  else yx:=ExtRepPolynomialRatFun(y);
  return Maximum(List(Filtered(yx,i->IsList(i)),j->KList(j)));
  fi;
end;;

Weight:=function(A,y)
  local w, i;
  w:=[];
  for i in [1..Length(A)] do
  if Com(A[i],y)=Zero(R) then
  w[i]:=0;
  elif Com(A[i],y) in VectorSpace(Rationals,[y]) then
  w[i]:=ExtRepPolynomialRatFun(Com(A[i],y))[2]/
                                ExtRepPolynomialRatFun(y)[2];
  else return "fail";
  fi;
  od;
  return w;
end;;
```

Appendix D

Finding generators of $U(\mathfrak{g}, e)$ using GAP4

The following is to give the generators of $U(\mathfrak{g}, e)$ in accordance with Section 3.2. We have a list `mGens` whose elements are the indices of a set of generators of $\mathfrak{m}$. The first stage is to list all monomials which may appear in the generators along with their Kazhdan degrees and $\mathfrak{t}^e$-weights. This is given by the function `GetBigList(y)` where the argument `y` gives an upper bound on the Kazhdan degree. We then set the variable `BigList` to be this list for some choice of bound `y`. The function `MonList(y)` then filters this list to include only those monomials which may have a non-zero coefficient in the expression for the generator with leading term `y`.

The function `GetGen(z)` finds the coefficients for the monomials found by `MonList(r[z])`. For this we create a polynomial ring `S` over the rationals with the number of indeterminates equal to the number of monomials in the list `D:=MonList(r[z])` for which we need the coefficient. We then create a polynomial ring `Q` over `S` with `n` indeterminates. Taking the commutators of each generator of $\mathfrak{m}$ (the list `mGens`) with each monomial (in the list `D`), we get a list of elements of the iterated polynomial ring `Q`. For each monomial in `D` (and also the constants) we take the sum of the coefficients (elements of the polynomial ring `S`). Thus each element of the list `D` gives a linear polynomial in `S`. Setting the leading coefficient to be 1 (that is, evaluating at `S.1=One(S)`) and triangulizing the matrix of coefficients of polynomials (which has rank `Length(D)`-1) gives a list of coefficients for the required generator $\Theta_{\mathtt{z}}$.

```
HighIndex:=function(y)
local yx, u, i;
  yx:=ExtRepPolynomialRatFun(y);
  u:=Filtered(yx,i->IsList(i) and not i=[]);
  if u=[] then
  return 0;
  else
  return Maximum(List(u,i->i[Length(i)-1]));
  fi;
end;;

GetBigList:=function(y)
local U, u, B, C, A, v, i, j;
  U:=PolynomialRing(Rationals,q);
  u:=IndeterminatesOfPolynomialRing(U);
  B:=List([1..q],i->u[i]);
  C:=List([1..q],i->List([i..q],j->u[j]));
  A:=[B];
  v:=-1+y/Minimum(Filtered(hDeg,i-> i>0));
  for i in [1..v] do
  A[i+1]:=[];
    for j in [1..Length(A[i])] do
    Append(A[i+1],A[i][j]*Filtered(C[HighIndex(A[i][j])],
                                 k->KDeg(k)<=y-KDeg(A[i][j])));
    od;
  od;
  return(List(Flat(A),i->[i,Weight(CSA_R,i),KDeg(i)]));
end;;
```

```
BigList:=GetBigList(y);

MonList:=function(y)
  local A, i, w, v, B;
  w:=Weight(CSA_R,y);
  v:=Weight([r*ExtRepOfObj(h)],y)[1];
  A:=Filtered(BigList,i->i[2]=w and i[3]<=v+2);
  B:=List(A,i->[ExtRepPolynomialRatFun(i[1]),i[2],i[3]]);
  B:=Filtered(B,i->PolynomialByExtRep(famR,i[1])=y
                 or (i[1][1][Length(i[1][1])-1]>d
                   and not (Length(i[1][1])=2
                   and i[1][1][2]=1 and i[3]=v+2)));
  return List(B,i->PolynomialByExtRep(famR,i[1]));
end;;

ITER_POLY_WARN:=false;
GetGen:=function(z)
  local D, LD, S, famS, Q, famQ, F, i, j, k, a, w, wx, A, U, V,
                                                        M, gen, v;
  D:=MonList(r[z]);
  LD:=Length(D);
  A:=[];
  if LD=1 then
  return r[z];
  else
  S:=PolynomialRing(Rationals,LD);
  famS:=FamilyObj(One(S));
  Q:=PolynomialRing(S,n);
  famQ:=FamilyObj(One(Q));
  F:=List(mBasis,i->PolynomialByExtRep(famQ,[[i,1],One(S)]));
```

```
for i in mGens do
U:=Zero(Q);
  for j in [1..LD] do
  a:=ExtRepPolynomialRatFun(Com(r[i],D[j]));
  a:=List(a,i->i);
    for k in [2,4..Length(a)] do
    a[k]:=One(S)*a[k];
    od;
  U:=U+PolynomialByExtRep(famQ,a)*PolynomialByExtRep
                                                   (famS,[[j,1],1]);
  od;
  w:=Value(U,F,One(Q)*chi);
  wx:=ExtRepPolynomialRatFun(w);
    for k in [2,4..Length(wx)] do
    Add(A,Value(wx[k],[S.1],[One(S)]));
    od;
od;
V:=List(A,i->ExtRepPolynomialRatFun(i));
M:=NullMat(Length(V),LD);
gen:=D[1];
for i in [1..Length(V)] do
  for j in [2,4..Length(V[i])] do
  if V[i][j-1]=[] then
  M[i][LD]:=-V[i][j];
  else M[i][V[i][j-1][1]-1]:=V[i][j];
  fi;
  od;
od;
TriangulizeMat(M);
v:=List(M,i->i[Length(i)]);
for i in [2..LD] do
```

```
  gen:=gen+v[i-1]*D[i];
  od;
  return gen;
  fi;
end;;
```

Appendix E

Finding relations for $U(\mathfrak{g}, e)$ using GAP4

We have a list `t` giving generators $\Theta_1, \ldots, \Theta_d$ of generators of $U(\mathfrak{g}, e)$, stored as elements of the polynomial ring `R`. The following procedure takes 2 generators Θ_y and Θ_z and returns a polynomial F_{yz} such that $[\Theta_y, \Theta_z] = F_{yz}(\Theta_1, \ldots, \Theta_d)$.

The function `PolyCalc(p)` takes a polynomial in `R` and returns that polynomial evaluated at the generators of $U(\mathfrak{g}, e)$. The function `InCent(p)` takes an element `p` of $U(\mathfrak{g})$ and returns the image of `p` under the projection into $U(\mathfrak{g}^e)$, along with the terms whose sum is that polynomial, and the Kazhdan degrees of those terms. The function `GetRel(p)` takes an element `p` of $U(\mathfrak{g}, e)$ where `p` is obtained by taking the commutator of two of the generators, say `t[y]` and `t[z]`, and returns the desired polynomial F_{yz}.

```
PolyCalc:=function(p)
  local px, u, i, j, k, a, A;
  px:=ExtRepPolynomialRatFun(p);
  u:=List([1..Length(px)/2],i->[px[2*i-1],px[2*i]]);
  a:=Zero(R);
  for i in [1..Length(u)] do
  A:=[];
    for j in [1..Length(u[i][1])/2] do
```

```
      for k in [1..u[i][1][2*j]] do
      Add(A,t[u[i][1][2*j-1]]);
      od;
    od;
    a:=a+u[i][2]*MultList(A);
  od;
  return a;
end;;

InCent:=function(p)
  local u, i, v, pc, a, ax, Q, QK;
  u:=List([d+1..n],i->r[i]);
  v:=List(u,i->0);
  if not p=Zero(R) then
  a:=Value(p,u,v)*One(R);
  ax:=ExtRepPolynomialRatFun(a);
  Q:=List([1..Length(ax)/2],i->PolynomialByExtRep
                                  (famR,[ax[2*i-1],ax[2*i]]));
  QK:=List(Q,i->KDeg(i));
  return [a,Q,QK];
  fi;
  return [p,[p],[0]];
end;;

GetRel:=function(p)
  local U, pc, a, i, A;
  U:=[];
  if InQ(p)=Zero(R) then
  return Zero(R);
  else
  while not p=Zero(R) do
```

```
  pc:=InCent(p);
  a:=Maximum(pc[3]);
  A:=Filtered(pc[2],i->KDeg(i)=a);
  Append(U,A);
  p:=p-PolyCalc(Sum(A));
  od;
  return Sum(U);
  fi;
end;;
```

Appendix F

Sample code for type G_2, orbit $\tilde{A}_1$

Here we show how the above procedures are used in order to obtain a presentation of the finite W-algebra $U(\mathfrak{g}, e)$ associated to some element e in the nilpotent orbit $\tilde{A}_1$ of the complex simple Lie algebra of type G_2.

We first construct our Lie algebra and its Chevalley basis, and specify the ordered list of weights for the Dynkin diagram for the orbit in question:

```
L:=SimpleLieAlgebra("G",2,Rationals);
b:=Basis(L);
n:=Dimension(L);
WDD:=[1,0];
```

We read the content of Appendix A and find our $\mathfrak{sl}_2$-triple $e = b_4$, $h = 2b_{13} + 3b_{14}$, $f = b_{10}$:

```
sl2:=GetSl2([1,0]);
e:=sl2[1];
h:=sl2[2];
f:=sl2[3];
```

We give a new ordered basis for our Lie algebra:

```
c:=[b[6],b[5],b[4],b[2],b[8],b[14],
  b[1],b[3],b[13],b[9],
  b[7],b[10],b[11],b[12]];
```

We read the content of Appendix B and define our $\mathfrak{sl}_2$-triple in terms of the new basis:

```
e:=c[3];
f:=c[12];
h:=e*f;
```

We read the content of Appendix C. We find a generating set for the subalgebra $\mathfrak{m}$ - the set $\{c_{11}, c_{12}, c_{14}\}$ is minimal - and list the indices:

```
mGens:=[11, 12, 14];
```

We read the content of Appendix D up to the function `GetBigList(y)`, and we note that the highest Kazhdan degree of a generator is 5 (for Θ_1 and Θ_2), so we set `y:=5`, evaluate `BigList:=GetBigList(y)` and read Appendix D from the definition of the function `MonList` on. We calculate the 6 generators:

```
t:=List([1..d],i->GetGen(i));
```

We read the content of Appendix E and calculate all relations, stored as polynomials in the matrix `Rels`:

```
Rels:=NullMat(d,d);
for i in [1..d] do
  for j in [1..d] do
  Rels[i][j]:=GetRel(Com(t[i],t[j]));
  od;
od;
```

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