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Flux Compactifications of Type II String Theories Under Non-Perturbative Dualities

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Abstract

We consider string vacua formed by compactifying Type II string theories on toroidal orbifolds and generalised Calabi-Yau manifolds and their transformations under a set of non-perturbative dualities. The dualities are the Type IIA-IIB exchanging T duality, the self-symmetry of Type IIB S duality, the non-trivial combination of the two, U duality, and the generalisation of T duality to include Calabi-Yaus, mirror symmetry. The requirement of the effective theory superpotential being invariant under these dualities is used to justify additional fluxes which do not descend via compactification from the ten dimensional action, which form an $\mathcal{N}=2$ theory. Their non-geometric structures, Bianchi constraints and tadpoles are determined and then classified in terms of modular S duality induced multiplets. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold is used as an explicit example of the general methods, with $\mathcal{N}=1$ Type IIB non-geometric vacua which possess T and S duality invariance also constructed. These are then used to motivate the existence of an exchange between moduli spaces on self mirror dual manifolds with $\mathcal{N}=2$. Such an exchange is seen to result in flux structures which are schematically the same as the standard formulation but with inequivalent flux constraints.

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Preface

The work in this thesis is based on two closely related papers, which are given in reverse chronological order.

- *The Generalised Geometry of Type II Non-Geometric Fluxes Under T and S Dualities*, G. J. Weatherill [9].
- *Non-geometric flux vacua, S-duality and algebraic geometry*, A. Guarino and G. J. Weatherill [10].

The work of the first is found in Chapters 4, 5 and 7. The work of the second is found in Chapter 6. A third paper relating to the AdS/CFT correspondence is not included.

- *Holographic integral equations and walking technicolour*, R. Alvares, N. Evans, A. Gebauer and G. J. Weatherill [11].

"Nature uses only the longest threads to weave her patterns, so that each small piece of her fabric reveals the organisation of the entire tapestry."

- R. Feynman

Chapter 1

Introduction

1.1 Gravity and The Standard Model

The physics of the Twentieth Century has been dominated by two areas of research; the description of gravity by general relativity, dominant at cosmological scales, and the description of the three forces dominant at subatomic scales; electromagnetism, the weak force and the strong force, by quantum field theory.

Einstein's development of general relativity in 1915, building upon his work in special relativity in 1905, views gravity as the deformation of the geometry of space-time. The path traced out by an massive object is not viewed as it being 'tugged' by gravity but rather that it traces out a geodesic in space-time, the shortest path between its source and its destination. An infinitesimal space-time interval is defined by a generalisation of the Euclidean Pythagorian formula, $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$, with g_{ab} being the metric of the space-time. In four dimensional Euclidean space the metric is positive definite with $g_{ab} = \delta_{ab}$, the Kronecker Delta of signature $(++++)$, and in special relativity $g_{ab} = \eta_{ab}$, the Lorentzian metric of signature $(-+++)$.

General relativity linked the behaviour of space-time (and thus gravity) to the contents of the spacetime, such as energy and matter, via the Einstein field equations, written in terms of the metric dependent tensor $G_{ab}(g)$.

$$G_{ab}(g) \equiv R_{ab}(g) - \frac{1}{2}R(g)g_{ab} + \Lambda g_{ab} \qquad G_{ab} = 8\pi T_{ab} \tag{1.1.1}$$

The conservation of energy and momentum follow from the constraint $\nabla_b G^{ab} = \nabla_b T^{ab} = 0$. ∇_a is a covariant derivative whose connection is generally set to be the Levi-Cevita connection, making it 'compatible' with the metric, $\nabla_a g_{bc} = 0$. The Ricci tensor R_{ab} and scalar $R = g^{ab}R_{ab}$ are dependent on the second derivatives of the metric, being defined by the Riemann curvature tensor $R^a{}_{bcd}$, a measure of the non-commutativity of covariant derivatives.

$$[\, \nabla_a \,,\, \nabla_b \,]\, \xi^c = R^c_{dab}\, \xi^d \tag{1.1.2}$$

With the curvature tensor defined in terms of $[\, \nabla_a \,,\, \nabla_b \,]$ it contains first and second derivatives of the metric. Since the stress-energy tensor T_{ab} is defined by the matter content of space it follows that the Einstein field equations form a set of second order partial derivatial equations on the metric. Of particular phenomenological interest are those cases where $T_{ab} = 0$ as they represent a universe without matter. The Einstein field equations reduce to $G_{ab} = 0$ and from which it follows that $g^{ab}G_{ab} = 0$ and the Ricci scalar is dependent upon Λ, the cosmological constant.

$$0 = g^{ab}G_{ab} = -R + 4\Lambda \quad \Rightarrow \quad R = 4\Lambda \tag{1.1.3}$$

For the majority of the Twentieth Century it was believed that $\Lambda = 0$ but following observations of supernovae in the 1990s it was determined that $\Lambda > 0$ and the universe is de Sitter over large distances. Over cosmological distances the domination of gravity has made general relativity the defacto model for cosmologists but over shorter distances, particularly the subatomic, the effects of gravity are so weak that it is largely neglected and instead quantum field theory is used.

Maxwell's unification of electric and magnetic effects into a single formalism, electromagnetism, was the theoretical completion of the experimental work done by Faraday in the mid Nineteenth Century. Though initially done using the quaternions it was later reformulated into a gauge theory. A point particle has associated to it a vector field A_a and from which a field strength F_{ab} can be defined by $\partial_{[a}A_{b]}$. Such a quantity is invariant under transformations $A_a \to A_a + \partial_a \xi$ for some differentiable function $\xi(x^a)$ and any quantity built from F_{ab} will thus be gauge invariant. Maxwell's equations and thus electromagnetism follow from the Lagrangian density defined by F_{ab}.

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{e^2} F_{ab} F^{ab} \tag{1.1.4}$$

The covariant and contravariant indices of the F differ by the metric and so the extension of Maxwell's equations in flat space-time to curved space-time is forthcoming.

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{e^2} F_{ab} F_{cd} g^{ac} g^{bd} \tag{1.1.5}$$

Over short distances, in the laboratory, the approximation $g_{ab} \approx \eta_{ab}$ is sufficient. However, by the beginning of the Twentieth Century a number of phenomena had been observed which could not be explained by Maxwell's formulation of electromagnetism, in flat or curved space. One such phenomena was the structure of the atom. Rutherford deduced, by means of firing alpha particles at gold, that the atom had the majority of its mass concentrated in a small, positively charged, central region and Thompson had discovered that much lighter, negatively charged, particles flittered about the edge of the atom and could, in some cases, be stripped off. Such particles, the electrons, appeared to be circulating around the atom in curved paths but Maxwell's work predicted any accelerating charge radiates energy, yet atoms were stable. This problem was resolved by the work of people such as Planck, Heisenberg, Bohr, Schrodinger and Dirac, who deduced phenomena on sub-atomic scales did not exchange energy in continuous, infinitely divisible, portions but in multiples of quanta. The size of these quanta were determined by the properties of operators within the theory, namely the measure of the non-commutativity of the position $\widehat{x}$ and momentum $\widehat{p}$ operators in terms of a quantity h, Planck's Constant.

$$\left[\widehat{x}\,,\,\widehat{p}\right] = i\hbar \tag{1.1.6}$$

Classical predictions could be obtained by taking the $h \to 0$ limit and the precise value of h was determined by Planck's analysis of black body spectra. Initially this work was done without regard for special relativity but in the 1920s Dirac developed the special relativity extension of quantum mechanics, quantum field theory. A procedure was developed whereby a classical field theory could be 'quantised' by use of non-commutative operators. The position and momentum operators of non-relativistic quantum mechanics are conjugate variables and other conjugate field pairings obey the same general form.

$$\left[\phi(x,t)\,,\,\Phi(y,t)\right] = i\hbar\,\delta^{(3)}(x-y) \qquad \Phi(x,t) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{1.1.7}$$

$$\left\{\psi(x,t)\,,\,\Psi(y,t)\right\} = i\hbar\,\delta^{(3)}(x-y) \qquad \Psi(x,t) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}} \tag{1.1.8}$$

The ϕ are bosonic fields and ψ fermionic and thus obey commutation and anticommutation relations, respectively. This quantisation method applied to electromagnetism, to construct quantum electrodynamics, accounted for high precision experiments such as the $g-2$ factor of the electron or the Lamb shift in emission spectra. With the advent of high energy colliders distances smaller than the nucleus could be probed and it was discovered that additional forces, not just electromagnetism, operated over such scales. Nucleons were found to be composed of three quarks which are bound together by self interacting massless gauge bosons named gluons. Furthermore, nuclear beta decays were explained as one of these quarks emitting an additional kind of gauge boson, one with mass, which then promptly decayed into a pair of leptons. The framework used for electromagnetism was insufficient as it could not account for massive gauge bosons or self interacting massless gauge bosons. This was solved by generalising electromagnetism to other gauge connections. In electromagnetism this is $D_a = \partial_a + iA_a$ and thus $[D_a, D_b] = \partial_{[a}A_{b]} = F_{ab}$. This represents the simplest case of a more general construction of gauge theories, since it assumes $[A_a, A_b] = 0$. Upgrading the gauge connection to be Lie algebra valued[1] for some Lie algebra $\mathfrak{g}$, generated by T^τ, a non-abelian field strength is obtained.

$$\Big[D_a\,,\,D_b\Big] = \Big(\partial_{[a}A^\tau_{b]} + f^{\tau\sigma\rho}A^\sigma_a A^\rho_b\Big)T^\tau \equiv F^\tau_{ab}T^\tau \tag{1.1.9}$$

This approach follows much the same schematic methodology as the description of curvature of gravity. The general relativity description of gravity is formulated through the use of covariant derivatives whose connections are metric dependent while gauge theories are formulated by covariant derivatives whose connections are gauge field dependent. The strong force is described by the $\mathfrak{g} = \mathfrak{su}(3)$ case, known as quantum electrodynamics, and the weak force by $\mathfrak{g} = \mathfrak{su}(2)$ and the combination of these three into a single theory is now known as the standard model. The fact this method applies so

[1] In the case of electromagnetism it is Lie algebra valued but for $\mathfrak{u}(1)$ and thus has vanishing structure constant.

readily to the electromagnetic, weak and strong forces prompted a search for a single gauge group within which to unify the three forces. In the late 1960s and early 1970s partial success was had with the unification of the electromagnetic and weak forces into the electroweak force. Below approximately 90GeV the two forces are separate, with electromagnetism having a U(1) gauge symmetry. Above the unification scale the three weak bosons and the electromagnetic photon obtain an enhanced SU(2) $\times$ U(1) and form massless superpositions. This is known as the electroweak model. By computing the gauge coupling running for the electroweak and strong forces the energy scale at which their unification might occur can be estimated and is of the order 10^{16} GeV. Though far beyond any conceivable direct experimentation, such gauge unification groups as SU(5) have already been excluded from being the unification group of the electroweak and strong forces as it predicts too short a lifetime for the proton, due to massive gauge bosons mediating non-SM processes. All of these forces are 'renormalisable', in that they require only finitely many inputs to make viable physical predictions over all possible energy ranges. Non-renormalisable theories require infinitely many inputs to make physical predictions if they are to be applied at all possible energy scales and thus do not have useful predictive power. However, not all renormalisable theories are consistent and not all non-renormalisable theories are without use. As a model on its own, quantum electrodynamics has an inconsistency due to its Landau pole, the running coupling of the theory flows to an infinite value at a finite energy scale. By embedding it within the electroweak model the gauge unification at 90 GeV alters the gauge running and removes the Landau pole.

Despite the similarity between the formulation of gravity within general relativity and the field theories of the SM, as well as the success in unifying the gauge theories, there are a number of crucial differences between general relativity and the standard model. The most important of these is the non-renormalisability of gravity. This follows from power counting of the gravitational coupling in the Einstein-Hilbert action of general relativity.

$$\mathcal{L}_G = \frac{1}{G}\sqrt{g}R \tag{1.1.10}$$

With units of (length)^2 perturbation series in G are not valid at high energy if the quantisation process is done in the same manner as used for the fields of the standard model.

$$\Big[g_{ab}(x,t)\,,\,p_{cd}(y,t)\Big] = i\hbar\,\delta^{(3)}(x-y) \qquad\qquad p_{cd} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\tau g_{cd})} \tag{1.1.11}$$

This does not preclude the usefulness of general relativity as an effective theory, valid in a particular range of energies (or length scales), in a similar way to quantum electrodynamics being valid for low energy processes without regard to electroweak unification. Experiments probing the effect of gravity from cosmological distances of order 10^{26} metres down to 10^{-4} metres confirm the accuracy of general relativity. The lack of a quantised model of gravity, to explain the interaction of gravitational quanta, gravitons, with other particles on energy scales of 10^{18} GeV is a theoretical stumbling block. A number of different approaches have been considered for the development of a short range, high energy, gravitational theory whose large distance limit is general relativity, including loop quantum gravity [1], twistor theory [2], non-commutative geometry [3] and string theory, which we will consider in this thesis.

1.2 String Theory

Despite finding considerable applications in the realms of quantum gravity, string theory began as an attempt to understand the behaviour of quarks inside nucleons under the strong force, before and during the advent of quantum chromodynamics. Experiments had found that the strength of the coupling between two quarks increased with distance, in contrast to the more familiar inverse square law of classical electromagnetism and gravity. The behaviour of this coupling was found to be linear with distance before suddenly disappearing, with the quark pair splitting and pair producing new partners for themselves. This was noted to have the same qualitative behaviour as a string or spring stretching between the quarks before eventually snapping once too much energy was introduced into the system. The mass spectra of mesons and baryons were found to follow the Regge slope behaviour of strings under tension [4]. This and the fact that the symmetry between s and t channel meson interactions could be viewed as closed strings scattering off one another [5] led to the initial idea that fundamental processes might be described by extended objects. However, the flux tube interpretation of quark couples within quantum chromodynamics demonstrated itself to be a superior model of the strong force and string theory, as a model of the strong force, was no longer a mainstream topic of research.

Development in the examination of wave mode mechanics on strings, both open and closed, continued independently of the research into standard model processes during the early 1970s. The Lagrangian for the oscillations[2], $X^\mu(\sigma^\alpha)$, on a string with tension $\frac{1}{\alpha'}$ was well known from classical mechanics,

[2] $\sigma^\alpha = (\tau, \sigma)$, the two parameters for the worldsheet swept out by the string.

being the two dimensional extension of the concept of a worldline swept out by a particle, the Nambu-Goto action.

$$S = T \int d^2\sigma \sqrt{|\det(g_{\alpha\beta})|} = \frac{1}{\alpha'} \int d^2\sigma \sqrt{|\det(\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu)|} \quad (1.2.1)$$

The presence of a square root makes quantisation difficult and this was avoided by using the classically equivalent Polyakov action [6], which introduced a non-physical metric field h. Upon using the equation of motion for h to remove it from the integrand the Polyakov action reduced to the Nambu-Goto action. Additional advantages included manifest diffeomorphic, Poincare and Weyl conformal invariance which could be used to reduce the action to a particular case, with $h_{\alpha\beta} = \eta_{\alpha\beta}$.

$$S_X = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta} g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu = \frac{1}{2\pi\alpha'} \int d^2\sigma\; \partial^\alpha X^\mu \partial_\alpha X_\mu \quad (1.2.2)$$

Applying the same quantisation processes to these bosonic oscillations results in a first quantised theory where the modes on the string are quantised but the string is not.

$$\Big[X^\mu(\tau,\sigma)\,,\, \Pi^\nu(\tau,\sigma')\Big] = i\hbar\, \eta^{\mu\nu} \delta(\sigma - \sigma') \qquad \Pi_\mu(\tau,\sigma) \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \quad (1.2.3)$$

Following through the implications of such a quantisation procedure it was found that in order to prevent anomalies in the conformal symmetry present in the Polyakov action the number of space-time dimensions the oscillations exist within must be set to a particular value, twenty six [7]. Though a prediction unique compared to other quantum field theories, this is markedly different from the number of space-time dimensions observed in experiments, on any scale, presenting an aesthetic as well as technical problem for the theory. Further issues develop upon considering the tower of states formed by the quantisation procedure. A string without oscillations was regarded as the quantum mechanical ground state but it was found to be tachyonic, throwing the entire consistency of the theory into a questionable light. However, the next set of states in the mass tower, those which are massless, provided motivation for further work due to them containing a massless spin 2 particle which obeyed the Einstein field equations, the graviton. Not only did string theory stipulate the number of space-time dimensions, it required the existence of gravity also. The existence of the tachyonic ground state was remedied by the inclusion of fermionic modes on the string, which allowed for the existence of supersymmetry. While the number of space-time dimensions is also altered it still requires the existence of extra dimensions. A number of choices exist in how these fermionic modes can be added into the bosonic theory and of primary interest to this work are the Type II superstring theories formed of closed strings whose modes are supersymmetric.

1.3 Type II String Theories

The field theory of open strings with both fermionic ψ^μ and bosonic X^ν modes can be described by adding a fermionic set of terms to the preexisting bosonic ones. It is convenient to define the complexified coordinate $z = \sigma^1 + i\sigma^2$ and $\partial \equiv \partial_z$ and likewise for its conjugate.

$$S = S_X + S_\psi = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial^\alpha X^\mu \partial_\alpha X_\mu + \psi^\mu \overline{\partial}\psi_\mu + \widetilde{\psi}^\nu \partial \widetilde{\psi}_\nu\right)$$

Parametering the string length by $\sigma^1 \in [0,\pi]$ the construction of closed strings is obtained by stipulating the string end points to be at the same space-time point, $X^\mu(z) = X^\mu(z+\pi)$. The fermionic modes have two possible conditions which are consistent with Lorentz invariance.

$$\begin{array}{llll} \text{Ramond} & : \psi^\mu(z) = \psi^\mu(z+\pi) & \widetilde{\psi}^\mu(\overline{z}) = \widetilde{\psi}^\mu(\overline{z}+\pi) \\ \text{Neveu-Schwarz} & : \psi^\mu(z) = -\psi^\mu(z+\pi) & \widetilde{\psi}^\mu(\overline{z}) = -\widetilde{\psi}^\mu(\overline{z}+\pi) \end{array} \tag{1.3.1}$$

Sector	$SO(8)$ spin	Representation
NS_+	$\mathbf{8}_v$	vector
R_-	$\mathbf{8}_c$	spinor
R_+	$\mathbf{8}_s$	spinor

Table 1.1: Open string modes classified by SO(8) representations.

These can be summarised by doubling the range of σ and expressing the anti-analytic modes in terms of the analytic ones, $\widetilde{\psi}^{\mu}(\sigma,\tau) = \psi^{\mu}(2\pi - \sigma, \tau)$ which in complex worldsheet coordinates is $\widetilde{\psi}^{\mu}(\overline{z}) = \psi^{\mu}(2\pi - z)$.

$$\begin{aligned} \text{Ramond} \quad &: \psi^{\mu}(z) \ = \ \ \ \psi^{\mu}(z+\pi) \\ \text{Neveu-Schwarz} \ &: \psi^{\mu}(z) \ = \ -\psi^{\mu}(z+\pi) \end{aligned} \tag{1.3.2}$$

With the addition of fermionic modes it is possible to build up the tower of states formed of string oscillations in more ways than those of the bosonic construction. Constructions using only bosonic modes possess tachyonic states but with the inclusion of fermionic modes the stipulation of supersymmetry causes this state to be projected out by the GSO operator, causing the lightest superstring modes to be massless. The dimensionality of space-time is also altered by the fermionic modes, from 26 to 10. The massless field content obtained from the string oscillations are classified by SO(8) representations. This is the little group which leaves null and time-like oscillations unchanged as their removal is a requirement for gauge and conformal κ invariance on the string oscillations. The three possible SO(8) representations relate to the three independent choices of fermion field periodicity and the sign of the worldsheet fermionic counter $(-1)^F$. The closed string states are then constructed from the tensor product of two of these open string states, the left moving and the right moving. The GSO projection can be applied independently on these two sets of states and there are two possible choices for how this might be done; the projections are equivalent, Type IIB, or they are inequivalent, Type IIA. We follow the notation of [8].

$$\begin{aligned} \text{Type IIA} &: (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) \\ \text{Type IIB} &: (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) \end{aligned} \tag{1.3.3}$$

These representations define the massless field content of the two theories, with their group theoretic decompositions being classifiable in terms of bosonic and fermionic fields. The SO(8) representations expand into terms which fall into three generic categories; the NS-NS sector, the R-R sector and the mixed NS-R or R-NS sectors. Three of these four sectors are common to both theories, they only differ in the R-R sector.

$$\begin{aligned} (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) &= (\mathbf{8}_v \otimes \mathbf{8}_v) \oplus (\mathbf{8}_v \otimes \mathbf{8}_s) \oplus (\mathbf{8}_c \otimes \mathbf{8}_v) \oplus (\mathbf{8}_c \otimes \mathbf{8}_s) \\ &\qquad \text{NS-NS} \qquad\quad \text{NS-R} \qquad\quad \text{R-NS} \qquad\quad \text{R-R} \\ (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) &= (\mathbf{8}_v \otimes \mathbf{8}_v) \oplus (\mathbf{8}_v \otimes \mathbf{8}_s) \oplus (\mathbf{8}_s \otimes \mathbf{8}_v) \oplus (\mathbf{8}_s \otimes \mathbf{8}_s) \end{aligned} \tag{1.3.4}$$

The NS-NS sector defines a symmetric traceless rank 2 tensor $G_{\mu\nu}$, an anti-symmetric rank 2 tensor $B_{\mu\nu}$ and a scalar singlet Φ, all of which are bosonic.

$$\mathbf{8}_v \otimes \mathbf{8}_v = \Phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \tag{1.3.5}$$

These fields define the metric G, the potential associated to the string charge B and the dilaton Φ which determines the string coupling by $g_s = e^{-\Phi}$. The fields defined by mixed periodicity conditions are common to each theory, representing fermionic fields and due to the expansion of (1.3.4) there is a two-fold symmetry giving a pair of gravitini, which have equal chiralities in Type IIA but opposite chiralities in Type IIB.

$$\mathbf{8}_v \otimes \mathbf{8}_c = \mathbf{8}_s \oplus \mathbf{56}_c \qquad \mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_c \oplus \mathbf{56}_s$$

The remaining multiplets are the dilatini which have a spinorial SO(8) representation $\mathbf{8}_{c/s}$. With a pair of gravitini the theories are $\mathcal{N} = 2$ supersymmetric in ten dimensions and it is this which gives rise to their name Type II. By supersymmetry the remaining fields in either theory must be bosonic and we consider them in turn.

1.3.1 Type IIA String Theory

Type IIA has an R-R sector defined by $\mathbf{8}_c \otimes \mathbf{8}_s$ which decomposes in terms of $[n]$, the antisymmetric representations of SO(8) of rank n, following the notation of [8].

$$\mathbf{8}_c \otimes \mathbf{8}_s = [1] \oplus [3] = \mathbf{8}_v \oplus \mathbf{56}_t$$

The NS-NS sector included $B_{\mu\nu} \sim B_2$, the bi-vector potential associated to the string charge. The fields of the R-R sector define extended objects in the same manner; $[p+1]$ is the potential A_{p+1} with field strength $F_{p+2} = \mathrm{d}C_{p+1}$ (in the simplest cases) associated to an p dimensional object, the Dp-brane. As a result of the decomposition of the R-R sector it is observed that the Type IIA theory possesses Dp-branes for $p = 0, 2$ but this can be extended to include other even dimensional branes through the fact that in 10 dimensions the potential C_{p+1} can couple electrically to a Dp-brane but magnetically to a D$(6-p)$-brane via Hodge dual field strengths[3].

$$* \, : \, \text{D}p\text{-brane} \ni \mathrm{d}C_{p+1} \; = \; F_{p+2} \longleftrightarrow F'_{8-p} \; = \; \mathrm{d}C'_{7-p} \in \text{D}(6-p)\text{-brane}$$

Therefore in a ten dimensional space-time the fields coupling electrically to the D2 and D0-branes couple magnetically to D4 and D6 branes. The field strength associated to a D8-brane is non-dynamical, being dual to a scalar and arises in the context of massive Type IIA. More generally in an N dimensional space-time p-branes[4] are related to $(N-4-p)$-branes by this

[3] The formal definition of the Hodge dual will be given in the next section.

[4] We make the distinction between general 'branes' and D-branes deliberately as we shall later consider other kinds of dynamical extended objects.

kind of duality. Unfortunately the massless field content of the Type IIA string theory is not sufficient to construct the string theory action itself. However, it is possible to construct the $\alpha' \to 0$ effective theory of Type IIA, Type IIA supergravity, by dimensional reduction on the unique eleven dimensional supergravity. The nature of the spin of fields in supermultiplets is dependent upon the dimensionality of space-time and this is sufficiently stringent in eleven dimensions to preclude the kind of choice in field configurations which distinguish Type IIA and Type IIB. In eleven dimensional supergravity the bosonic fields of the theory are the metric G and the 3-form A_3, which defines a field strength $F_4 = dA_3$, and possess a total of 128 degrees of freedom. These have fermionic partners in the 128 dimensional $SO(9)$ vector-spinor gravitino and from these fields the bosonic action for eleven dimensional supergravity can be constructed.

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x\sqrt{-G}\left(\mathcal{R} - \frac{1}{2}|F_4|^2\right) - \frac{1}{3!}\int A_3 \wedge F_4 \wedge F_4$$

Since this action is defined by supersymmetry, Lorentz and gauge symmetries and does not involve α' in anyway it is not a string theory action. Nor can it be viewed as the $\alpha' \to 0$ limit of a string theory as it exists in more dimensions than any supersymmetric string theory does. However, it is the low energy limit of M theory which is totally constrained by symmetries and contains only two types of branes, the M2 and its eleven dimensional magnetic dual, the M5. Upon compactifying one of the spacial dimensions to a circle the eleven dimensional fields decompose into ten dimensional fields. All eleven dimensional p-forms contribute to their ten dimensional versions but also give rise to the fields associated to the stringy fields such as B_2. The metric decomposes via the Kaluza-Klein method, contributing a gauge field and a scalar field.

$$A_3^{(11)} \to A_3^{(10)}\,,\, B_2 \qquad\qquad G^{(11)} \to G^{(10)}\,,\, A_1^{(10)}\,,\, \Phi$$

Applying this decomposition and relabelling $A_n^{(10)} \to C_n$ the reduced ten dimensional action is obtained in three parts; pure NS-NS sector terms S_{NS}, pure R-R sector terms S_R and the Chern-Simons terms, which are not of pure flux sector due to contributions from both NS-NS and R-R fields.

$$\begin{aligned} S_{IIA} &= S_{NS} + S_R + S_{CS} \\ S_{NS} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G}\, e^{-2\Phi} \Big(\mathcal{R} + 4\nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{2}|H_3|^2 \Big) \\ S_R &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(|F_2|^2 + |\widetilde{F}_4|^2 \right) \\ S_{CS} &= -\frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \end{aligned}$$

This action contains a number of field strengths defined from the fields obtained by examining the massless states of the theory.

- NS-NS field strength $H_3 \equiv \mathrm{d}B_2$.
- R-R field strengths by $F_2 \equiv \mathrm{d}C_1$ and $\widetilde{F}_4 \equiv \mathrm{d}C_3 + H_3 \wedge C_1$.
- Chern-Simons contributing field strength $F_4 \equiv \mathrm{d}C_3$.

The three contributions to the Type IIA action are individually gauge invariant, though in the case of the Chern-Simons term gauge transformations alter the integrand by an exact form[5]. The potentials C_n and B_2 define two types sets of equations; the Euler-Lagrange equations due to the action and the Bianchi constraints associated to d, though only in the case of $\widetilde{F}_4$ is the Bianchi constraint non-trivial. The basis in which we expand the fields must satisfy these equations. This action and its field content is incomplete

[5]Their contribution to the Lagrangian is topological in nature as it depends on the structure of space-time.

if we wished to examine the full string theory but is sufficient for examining effective theories of Type IIA string theory.

1.3.2 Type IIB String Theory

The R-R sector of Type IIB is defined by $\mathbf{8}_c \otimes \mathbf{8}_c$ which also decomposes in terms of $[n]$, the antisymmetric representations of SO(8) of rank n but the representations are not entirely complete.

$$\mathbf{8}_c \otimes \mathbf{8}_c = [0] \oplus [2] \oplus [4]_+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+$$

Following the same method as the Type IIA case the [0] and [2] are the potentials C_0 and C_2 and they couple to Dp-branes. In the case of [2] the branes are one dimensional and known as D-strings and is noteworthy that in the same way as the fundamental strings (or F-strings) having the NS-NS [2] potential B_2. The magnetic dual of the D-strings are the D5-branes and the NS-NS fields also have this magnetic dualisation, giving an NS5-brane as the magnetic dual of the F-string. The brane upon which C_0 resides is neither extended in time nor space and so is an instanton and has D7-branes as its magnetic dual. The remaining case is [4], associated to the F_5 living on D3-branes. Due to the relationship between electric-magnetic relationship of $D_p - D_{6-p}$ branes the C_4 couples to D3-branes both electrically and magnetically and therefore there is a 5-form field strength, which is self dual $\widetilde{F}_5 = *\widetilde{F}_5$. This self duality reduces the number of degrees of freedom by half, $[4] = \mathbf{70} \to [4]_+ = \mathbf{35}_+$. A ten dimensional supergravity theory with this field content in its massless sector cannot be constructed by dimensional reduction and instead is obtained by requiring gauge, Lorentz and supersymmetry transformations invariance. The resultant action has the same three part decomposition as in Type IIA and the NS-NS sector is indeed the same.

$$
\begin{aligned}
S_{IIB} &= S_{NS} + S_R + S_{CS} \\
S_{NS} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G}\, e^{-2\Phi} \Big(\mathcal{R} + 4\nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{2}|H_3|^2 \Big) \\
S_R &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \Big(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2 \Big) \\
S_{CS} &= -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3
\end{aligned}
$$

The non-standard field strengths are not the same as the Type IIA case due to different R-R content.

- R-R field strength $\widetilde{F}_3 \equiv F_3 - C_0 \wedge H_3$.
- Self dual R-R field strength $\widetilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$.

The Euler-Lagrange equations follow in the same manner as the Type IIA case except for $\widetilde{F}_5$. There is no covariant way in which to write the Type IIB supergravity action such that $\widetilde{F}_5 = *\widetilde{F}_5$ follows from its equations of motion.

1.3.3 Flux Compactifications

The Type II superstring theories and their low energy supergravity limits exist in ten dimensional space-time which is not the observed dimensionality of space-time at large and to reconcile these two facts the extra six spacial dimensions are curled up into a small compact structure. With a radius of the order 10^{-35} metres the small nature of space makes the direct probing of string scale physics out of reach of any current, planned or even realistically possible particle colliders. However, the properties of the compact space have effects on observed phenomena in the external space via a four dimensional effective theory.

- Amount of supersymmetry and the scale at which it is broken.

- Vacuum potential determining the cosmological constant.
- Cosmic microwave background power spectrum.
- Inflation and reheating in the early universe.

Both Type II theories have $\mathcal{N} = 2$ in their uncompactified ten dimensional formulation but the number of gravitini in a compactified supersymmetric theory is dependent on the space-time configuration. Upon the naive compactification of the extra dimensions onto a six dimensional torus this increases to $\mathcal{N} = 8$. It is not possible for $\mathcal{N} > 1$ theories to have chiral fermions in their low energy standard model limit, even if the supersymmetry is spontaneously broken. As a result the internal space must have additional restrictions applied to it in order to break the compactified $\mathcal{N} = 8$ theory down to $\mathcal{N} = 1$.

With string theory naturally including the graviton, whose equations of motion are the Einstein field equations, the structure of the internal space is dynamical and must be consistent with such a relativistic point of view. The field content of the full ten dimensional string theory descends to the effective theory and plays a role in defining the structure of the internal space. The field configurations associated to charges living on strings or branes define 'fluxes', constant quantised fields which contribute to the properties of the internal space. An internal space whose non-trivial fluxes led to a stable space-time configuration defines a space-time vacuum state for the effective theory and the general construction is known as a flux compactification.

1.4 Thesis Overview

The subject of this thesis is to examine the kinds of fluxes which descend from the ten dimensional string actions upon compactification on a large class of compact spaces. Of primary interest will be the effect that stringy dualities have upon these fluxes and the kind of generalisations, beyond those configurations obtained purely by compactification, required of the fluxes in order to obtain a general $\mathcal{N} = 2$ superpotential. These equivalences between different stringy constructions, generally referred to as dualities, link the different ten dimensional Type II string theories, as well as the heterotic and Type I string theories and the eleven dimensional supergravity, together into a single framework. Though the dualities can be used to obtain any particular string theory from a given string construction we shall restrict ourselves to the Type II string theories. However, the Type I string theory is obtainable in a straight forward manner from a particular Type IIB construction but this will only be mentioned in passing.

We begin in Chapter 2 with an overview of compact spaces. We shall outline our differential geometry notation and basic results in exterior calculus and give motivation in terms of string phenomenology for the particular properties of the internal spaces we wish to consider and how to construct such spaces in terms of orbifolds and orientifolds. We shall define the parameter spaces associated with compact six dimensional spaces, outside of a string theory context, and then interpret them in terms of string vacua. Finally, the fluxes obtained by direct compactification of the ten dimensional actions are given and the contributions they make to the dynamics of the internal space via a superpotential stated. Chapter 3 outlines the dualities which arise in string theory which are not seen in standard quantum field theories, those of T, S and U duality. The T duality link between the Type II theories compactified on a torus is illustrated and the Calabi-Yau generalisation to mirror symmetry discussed. The weak-strong self duality of Type IIB, S duality, is illustrated on the level of the ten dimensional action as well as the effective theory's superpotential. These chapters are an overview of the background to flux compactifications and are provided to give the main work context. No claim of originality is made, other than slight modifications to notation commonly found in the literature where it is convenient.

In Chapter 4 the different Type II superpotentials are compared and the existence of additional fluxes due to the dualities motivated and the form of their contributions to the superpotential constructed. A number of different ways of representing the fluxes are discussed, as well as the differences obtained by considering only 'light' modes and their associated fluxes as compared to all possible modes. Having motivated the existence and the structure of duality induced fluxes Chapter 5 discusses their consistency constraints. These come in two general types, Bianchi constraints and tadpole constraints. The transformation properties of the fluxes under the dualities and coordinate redefinitions are considered, with the constraints being classified in terms of their $\mathrm{SL}(2,\mathbb{Z})$ multiplet representations. In the restricted case of the orientifolded Type IIB T duality induced fluxes are given a Lie algebra interpretation, with S duality viewed as a deformation of the algebra and the Bianchi constraints seen as integrability and cohomology conditions. The methods thus far outlined are applied to a Type IIB compactification on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold in Chapter 6, where Lie algebra methods are used to provide a set of general solutions to the T and then S duality induced constraints and example vacua are constructed. Finally we make note of a number of symmetries observed in the flux constructions of the Type II theories, including in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold, and in Chapter 7 we examine ways in which to construct superpotential-like expressions in a manner similar to that given in previous chapters but with the roles of the moduli exchanged. Motivation is given from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold and the inequivalence of the Bianchi constraints of such a construction demonstrated. The same T and S duality induced structures are observed in the Bianchi constraints, which are again classified by $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets.

In the summary an overview of main results and a brief discussion of possible future paths of research is given. The Appendix provides a short overview of algebraic geometry terminology, methods and results used throughout, as well as derivation of particular differential form identities used in the main body of work.

Chapter 2

Compact Dimensions

This chapter is to outline the notation used throughout for the properties of compact spaces, the properties such spaces are required to have in order to be candidates for a phenomenologically viable string compactification, how spaces with properties can be constructed and finally the dynamics of such spaces and the fields with live in them.

2.1 Properties of Internal Spaces

The notation and basic definitions are taken to follow in the main Refs. [12, 15] and in places Ref. [13], where all assumed results we make use of are proven. In order to be able to make use of a number of important or useful differential geometry results we immediately restrict our attention to compact orientated connected six dimensional spaces with all six dimensions being space-like and whose generic member we shall denote as $\mathcal{M}$. Orientation is required such that physical quantities like volume can be well defined, while connectedness is argued on the grounds of physicality. Compactness is not required in general for consistent compactifications, despite the name, as non-compact Calabi-Yau manifolds can be defined [14]. Such spaces are beyond the scope of this work but we will develop notation such that their definition can be stated in passing.

2.1.1 Differentiable Structure

The six dimensions of $\mathcal{M}$ are given the coordinates X^m and the bases of the tangent bundle $T\mathcal{M}$ and the cotangent bundle fibres $T^*\mathcal{M}$ are $\frac{\partial}{\partial X^m}$ and $\mathrm{d}X^m$ respectively. The space of p-forms constructed from the cotangent basis is denoted in the literature as $\Omega^p(\mathcal{M})$. Using the cotangent basis we can define a second set of 1-forms, η^m, which define a new basis for the fibre and exterior products thereof.

$$\eta^m = N^m_n(X^p)\,\mathrm{d}X^n \quad , \quad \eta^{m_1 \ldots m_p} = \eta^{m_1} \wedge \ldots \wedge \eta^{m_p} \tag{2.1.1}$$

These η^m are linear combinations of the $\mathrm{d}X^n$ where the linear combination can depend on the internal coordinates and the space of possible basis choices form the fibre of a frame bundle. The coordinate induced basis is obtained by the simplest choice for N^m_n but we will wish to distinguish between this basis and a different (but particular) choice of N^m_n. As such we shall depart slightly from the standard notation for p-forms so as to make it clear which basis we are using. To that end we consider a general element in $\Omega^p(\mathcal{M})$, λ.

$$\lambda = \frac{1}{p!}\lambda_{n_1 \ldots n_p}(X)\,\mathrm{d}X^{n_1 \ldots n_p} = \frac{1}{p!}\lambda'_{n_1 \ldots n_p}(X)\,\eta^{n_1 \ldots n_p} \in \Omega^p(\mathcal{M})$$

Of interest to the examination of fluxes will be the special case where the fluxes have support in constant sections of $\Omega^p(\mathcal{M})$, the p-form coefficients $\lambda_{n_1 \ldots n_p}$ are constant[1]. Since the coefficients transform under a change of basis this vector subspace of $\Omega^p(\mathcal{M})$ is dependent on the basis η^a and it is not

[1] In this context 'constant' is taken to mean independent of the coordinates of $\mathcal{M}$, dependency on other parameters is allowable.

automatic that such a particular set of 1-forms, $\{\eta^m_{(0)}\}$, can be globally defined. Those spaces for which there exists such a basis are parallelisable and restricting to constant coefficients requires us to be clear which cotangent basis we use, η rather than $\mathrm{d}X$. As such we we shall refer to the cotangent space as $T^*\mathcal{M}$ when using the $\mathrm{d}X$ basis for the fibres and E^* when using the η basis for the fibres. The space of forms will then be denoted as either $\Omega^p(T^*\mathcal{M})$ or $\Omega^*(\mathsf{E}^*)$ respectively. This deviation from standard notation is redundant for the space of forms with $\mathcal{M}$ dependent coefficients but not for the spaces obtained by the restriction to constant coefficients, $\Lambda^p(T^*\mathcal{M})$ and $\Lambda^p(\mathsf{E}^*)$ respectively. We can define a set of interior forms which form the basis of a new space, E, by their action on elements of E^* and are in some way their dual.

$$\iota_m(\eta^n) = \delta^n_m \quad , \quad \{\iota_m, \iota_n\} = 0 \quad , \quad \mathsf{E} = \langle \iota_m \rangle \tag{2.1.2}$$

The basis of E is anticommuting and the elements of $\Omega^p(\mathsf{E})$ follow the same structure as those of $\Omega^p(\mathsf{E}^*)$, which contains $\Lambda^p(\mathsf{E})$ in the same way.

$$\lambda^* = \frac{1}{p!}\widetilde{\lambda}^{n_1\ldots n_p}(X)\,\iota_{n_1\ldots n_p} \in \Omega^p(\mathsf{E})$$

The action of elements of $\Lambda^p(\mathsf{E})$ on elements of $\Lambda^p(\mathsf{E}^*)$ are defined by (2.1.2), which we demonstrate explicitly for $p = 2$.

$$\iota_{pq}(\eta^{ab}) = \iota_p(\delta^a_q\eta^b - \eta^a\delta^b_q) = \delta^a_q\delta^b_p - \delta^b_q\delta^a_p$$

Generalisations to $p > 2$ follow in a straightforward manner. The bases of E and E^* define an $\mathrm{O}(6,6)$ Clifford algebra via $\Gamma_m = \iota_m$ and $\Gamma^n = \eta^n\wedge$.

$$\{\Gamma_m, \Gamma_n\} = 0 \quad , \quad \{\Gamma^m, \Gamma^n\} = 0 \quad , \quad \{\Gamma_m, \Gamma^n\} = \delta^n_m \tag{2.1.3}$$

$\Omega^*(\mathsf{E}^*)$ is the ring of all $\Omega^p(\mathsf{E}^*)$ and we take its dual $\Omega^*(\mathsf{E})$ as the ring of all $\Omega^p(\mathsf{E})$, which can be split into even and odd subrings.

$$\Omega^*(\mathsf{E}^*) = \Omega^+(\mathsf{E}^*) \oplus \Omega^-(\mathsf{E}^*) = \left(\bigoplus_{n=0}^{3} \Omega^{2n}(\mathsf{E}^*)\right) \oplus \left(\bigoplus_{n=0}^{2} \Omega^{2n+1}(\mathsf{E}^*)\right)$$

The exterior derivative d is defined in terms of the X^m coordinates in $\Omega^*(\mathsf{E}^*)$.

$$\mathrm{d}(A(X^m)\,\mathrm{d}X^{m_1} \wedge \ldots \mathrm{d}X^{m_p}) = (\partial_n A)\,\mathrm{d}X^n \wedge \mathrm{d}X^{m_1} \wedge \ldots \mathrm{d}X^{m_p}$$

In terms of the $\Omega^p(\mathsf{E}^*)$ subspaces of $\Omega^*(\mathsf{E}^*)$ the exterior derivative has the action $\mathrm{d} : \Omega^p(\mathsf{E}^*) \to \Omega^{p+1}(\mathsf{E}^*)$ and due to the anticommuting nature of the $\mathrm{d}X^n$, in contrast to the commuting partial derivatives, it satisfies $\mathrm{d}^2 = 0$. The exterior derivative naturally defines two subsets of $\Omega^p(\mathsf{E}^*)$, $B^p(\mathsf{E}^*)$ and $Z^p(\mathsf{E}^*)$.

- Closed p-forms : $B^p(\mathsf{E}^*) = \Big\{\psi \in \Omega^p(\mathsf{E}^*) \quad \text{s.t.} \quad \mathrm{d}\psi = 0\Big\}$
- Exact p-forms : $Z^p(\mathsf{E}^*) = \Big\{\mathrm{d}\psi \in \Omega^p(\mathsf{E}^*) \quad \text{s.t.} \quad \psi \in \Omega^{p-1}(\mathsf{E}^*)\Big\}$

These two subspaces of $\Omega^p(\mathsf{E}^*)$ allow for the construction of an equivalence relation and from the resultant quotient space is defined $H^p(\mathsf{E}^*)$, the p'th cohomology of $\mathcal{M}$.

$$H^p(\mathsf{E}^*) \equiv B^p/Z^p \quad \Leftrightarrow \quad \big[\,\psi\,\big] \sim \big[\,\phi\,\big] \quad \text{iff} \quad \psi = \phi + \mathrm{d}\xi \quad \xi \in \Omega^{p-1}(\mathsf{E}^*)$$

Since the equivalence classes are well defined under exterior multiplication the ring $H^*(\mathsf{E}^*)$ is formed from $H^p(\mathsf{E}^*)$ in the same way that $\Omega^*(\mathsf{E}^*)$ is formed from the $\Omega^p(\mathsf{E}^*)$. The cohomologies are dual to sets of submanifolds of $\mathcal{M}$ known as chains. C_p is the set of p-dimensional chains and by defining the boundary operator d such that for $\gamma \in C_p$ the $p-1$ dimensional chain $\mathsf{d}\gamma$ is its boundary. γ is a cycle if $\mathsf{d}\gamma$ is empty and is itself a boundary if there is a $p+1$ dimensional chain β such that $\mathsf{d}\beta = \gamma$. It can be shown that $\mathsf{d}^2 = 0$, a boundary has no boundary and thus we have the same algebraic structure as the $H^*(\mathsf{E}^*)$.

- p-cycles : $B_p = \Big\{ \gamma \in C_p \quad \text{s.t.} \quad \mathsf{d}\gamma = 0 \Big\}$
- p-boundaries : $Z^p = \Big\{ \gamma \in C_p \quad \text{s.t.} \quad \gamma = \mathsf{d}\beta \, , \, \beta \in C_{p+1} \Big\}$

These two sets of cycles allow for the construction of an equivalence relation and from the resultant quotient space is defined $H_p(\mathcal{M})$, the p'th homology of $\mathcal{M}$.

$$H_p(\mathcal{M}) \equiv B_p/Z_p \quad \Leftrightarrow \quad \big[\, \gamma \,\big] \sim \big[\, \gamma' \,\big] \quad \text{iff} \quad \gamma = \gamma + \mathsf{d}\beta \quad \beta \in C_{p+1}$$

This $H_p \cong H^p$ vector space duality is determined through integration.

$$\big[\, \gamma \,\big] \cdot \big[\, \phi \,\big] \equiv \int_{[\gamma]} [\phi] \tag{2.1.4}$$

By the use of Stoke's theorem it follows that this inner product is independent of the representative element used and so is well defined. This relationship links the topological non-triviality of $\mathcal{M}$, as measured by the homology, with the algebraic structure of forms defined on $\mathcal{M}$, as measured by the cohomology. As a result the dimension of $H^p(\mathsf{E}^*)$ is b_p, the p'th Betti number. It is possible to make a second p-form equivalence class dual to $[\gamma]$, the Poincaré dual, which is the following $[\psi]$ equivalence class.

$$\big[\, \gamma \,\big] \cdot \big[\, \phi \,\big] = \int_{[\gamma]} [\phi] \equiv \int_{\mathcal{M}} [\phi] \wedge [\psi]$$

For a given $\mathcal{M}$ there are many possible metrics which can be defined on it and we denote the metric space obtained by a choice of the metric as $(\mathcal{M}, G)$. On $(\mathcal{M}, G)$ it is possible to construct an inner product between two p-forms in terms of their components by using the metric to define a contraction.

$$\langle\!\langle \, \psi, \phi \, \rangle\!\rangle \equiv \frac{1}{p!} \psi_{n_1 \ldots n_p} \phi^{n_1 \cdots n_p} = \frac{1}{p!} \psi_{n_1 \ldots n_p} g^{n_1 m_1} \ldots g^{n_p m_p} \phi_{m_1 \cdots m_p}$$

This provides a natural isomorphism $H^p(\mathsf{E}^*) \cong H^{6-p}(\mathsf{E}^*)$ by noting a p-form must be combined with a $(6-p)$-form to define an integral over $\mathcal{M}$ and it is this which defines the Hodge duality operator $\star$.

$$
\begin{aligned}
H^p \ni \phi &= \frac{1}{p!}\phi_{n_1\cdots n_p}\,\eta^{n_1\cdots n_p} \\
\Omega^p \ni \star\phi &= \frac{1}{(6-p)!}\frac{1}{p!}\epsilon_{n_{p+1}\ldots n_6}{}^{n_1\cdots n_p}\phi_{n_1\cdots n_p}\,\eta^{n_{p+1}\cdots n_6}
\end{aligned}
$$

The Hodge star defines an inner product on $H^p(\mathsf{E}^*)$, separately on each p, which is symmetric, a result easily seen from the component form of the inner product.

$$
\langle\!\langle\, \psi,\phi \,\rangle\!\rangle \equiv \int_{\mathcal{M}} \psi\wedge\star\phi = \langle\!\langle\, \phi,\psi \,\rangle\!\rangle \qquad \psi,\phi \in H^p(\mathsf{E}^*) \tag{2.1.5}
$$

As a result it follows from $H^p(\mathsf{E}^*) \cong H^{6-p}(\mathsf{E}^*)$ that $b_p = b_{6-p}$. The inner product and exterior derivative together define an adjoint derivative $\mathrm{d}^\dagger$ whose $\Omega^p(\mathsf{E}^*)$ action is $\mathrm{d}^\dagger : \Omega^p(\mathsf{E}^*) \to \Omega^{p-1}(\mathsf{E}^*)$, in contrast to the d action, and is not Leibnitz.

$$
\langle\!\langle\, \mathrm{d}\psi,\phi \,\rangle\!\rangle \equiv \langle\!\langle\, \psi,\mathrm{d}^\dagger\phi \,\rangle\!\rangle \qquad \psi \in H^{p-1}(\mathsf{E}^*)\,,\, \phi \in H^p(\mathsf{E}^*)
$$

The adjoint derivative can be expressed in terms of d and $\star$, though the specific proportionality factor depends on the signature of the bilinear form associated to the inner product. We will later consider an explicit case where the sign structure will be clarified but the general form is taken to be $\mathrm{d}^\dagger = \pm\star\,\mathrm{d}\,\star$. It follows from the definition of $\mathrm{d}^\dagger$ that $(\mathrm{d}^\dagger)^2 = 0$. Combinations of d and $\mathrm{d}^\dagger$ can be constructed that are endomorphisms on $\Omega^p(\mathsf{E}^*)$ and of particular note is the Laplacian.

$$
\Delta \equiv \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d} = (\mathrm{d}+\mathrm{d}^\dagger)^2 \equiv D_{\mathrm{d}}^2
$$

In the second equality we have used $\mathrm{d}^2 = (\mathrm{d}^\dagger)^2 = 0$ and $\mathrm{d}+\mathrm{d}^\dagger$ is a Dirac operator D_{d}. The harmonic p-forms, $\mathfrak{H}^p(\mathsf{E}^*)$, are the elements $\psi \in \Omega^p(\mathsf{E}^*)$

satisfying $\Delta\psi = 0$.

$$0 = \langle\!\langle\, (\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d})\psi, \psi \,\rangle\!\rangle = \langle\!\langle\, \mathrm{d}^\dagger\psi, \mathrm{d}^\dagger\psi \,\rangle\!\rangle + \langle\!\langle\, \mathrm{d}\psi, \mathrm{d}\psi \,\rangle\!\rangle \geq 0$$

Both terms are non-negative due to the positive definite nature of the inner product and therefore must separately vanish if $\Delta\psi = 0$ and it follows they are both closed, $\mathrm{d}\psi = 0$, and co-closed, $\mathrm{d}^\dagger\psi = 0$. Furthermore we have $D_{\mathrm{d}}\psi = 0$, which can also be seen by the fact D_{d} and Δ share zero eigenvalues. Though a cohomology class can be represented by infinitely many different closed p-forms there is a unique harmonic representative ξ and so the dimension of $\mathfrak{H}^p(\mathsf{E}^*)$ is b_p. As a result if $\phi \in \mathfrak{H}^p(\mathsf{E}^*) \subset H^p$ then $*\phi \in \mathfrak{H}^{6-p}(\mathsf{E}^*) \subset H^{6-p}$.

2.1.2 Complex Structure

A complex manifold is a real $2n$ dimensional manifold with additional structure to it. The coordinates can be taken to be $\{x^\mu, y^\nu\}$ for $\mu, \nu \in \{1, \ldots, n\}$ and these define the canonical tangent space basis $\{\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}\}$. This tangent space basis can be complexified, $T_p\mathcal{M} \to T_p\mathcal{M} \otimes \mathbb{C} \equiv T_p\mathcal{M}^{\mathbb{C}}$.

$$\begin{aligned} \tfrac{\partial}{\partial z^\mu} &= \tfrac{1}{2}\left(\tfrac{\partial}{\partial x^\mu} + i\tfrac{\partial}{\partial y^\mu}\right) \quad , \quad \mathrm{d}z^\mu = \mathrm{d}x^\mu + i\,\mathrm{d}y^\mu \\ \tfrac{\partial}{\partial \overline{z}^\mu} &= \tfrac{1}{2}\left(\tfrac{\partial}{\partial x^\mu} - i\tfrac{\partial}{\partial y^\mu}\right) \quad , \quad \mathrm{d}\overline{z}^\mu = \mathrm{d}x^\mu - i\,\mathrm{d}y^\mu \end{aligned}$$

If a real manifold $\mathcal{M}$ possesses an J_p at $p \in \mathcal{M}$ which satisfies $\mathsf{J}_p^2 = -\mathrm{id}_{T_p\mathcal{M}}$ on $T_p\mathcal{M}$ then it admits an almost complex structure.

$$\mathsf{J}_p\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial y^\mu} \quad , \quad \mathsf{J}_p\left(\frac{\partial}{\partial y^\mu}\right) = -\frac{\partial}{\partial x^\mu} \tag{2.1.6}$$

A local change of basis to complex coordinates has the effect of diagonalising J_p, with eigenvalues $\pm i$.

$$\left.\begin{aligned} \mathsf{J}\left(\tfrac{\partial}{\partial z^\mu}\right) &= i\tfrac{\partial}{\partial z^\mu} \\ \mathsf{J}\left(\tfrac{\partial}{\partial \overline{z}^\mu}\right) &= -i\tfrac{\partial}{\partial \overline{z}^\mu} \end{aligned}\right\} \quad \Rightarrow \quad \mathsf{J}_p = i\,\mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\mu} - i\,\mathrm{d}\overline{z}^{\overline{\nu}} \otimes \frac{\partial}{\partial \overline{z}^{\overline{\nu}}}$$

The almost complex structure is upgraded to complex structure if J_p is integrable or equivalently the Nijenhuis tensor disappears idenitically. In such cases the complex structure is globally defined, so J_p on $T_p\mathcal{M}^{\mathbb{C}}$ defines J on $T\mathcal{M}^{\mathbb{C}}$ and global coordinates on $\mathcal{M}$ can be defined.

$$z^{\mu} = x^{\mu} + iy^{\mu} \quad , \quad \overline{z}^{\overline{\mu}} \;=\; x^{\mu} - iy^{\mu}$$

However, this choice of complex structure is by no means unique and, like the metric on $\mathcal{M}$, since $\mathcal{M}$ can have many different complex structures we use $(\mathcal{M},\mathsf{J})$ to specify the complex structure on $\mathcal{M}$. In such cases it is possible to extend the differential geometry results we have considered thus far for $\mathcal{M}$ to $(\mathcal{M},\mathsf{J})$.

$$\mathrm{d}z^{\mu_1} \wedge \ldots \wedge \mathrm{d}z^{\mu_p} \wedge \mathrm{d}\overline{z}^{\nu_1} \wedge \ldots \wedge \mathrm{d}\overline{z}^{\nu_q} \in \Lambda^{p,q}(T^*M) \subset \Omega^{p,q}(\mathsf{E}^*)$$

The complexification can be absorbed into the E^* frame definition and we shall always take E^* to include this, neglecting the $\mathbb{C}$ label. Expressing such (p,q)-forms in terms of the real basis it is immediate that $\Omega^{p,q}(\mathsf{E}^*) \in \Omega^{p+q}(\mathsf{E}^*)$ and we have a decomposition of the set of real p-forms into complex subspaces.

$$\Omega^{p}(\mathsf{E}^*) = \bigoplus_{n=0}^{p} \Omega^{n,p-n}(\mathsf{E}^*)$$

The exterior derivative's action on $\Omega^{p,q}(\mathsf{E}^*)$ can be determined by using the real basis and it splits into two parts, the Dolbeault operators.

$$\mathrm{d} = \partial + \overline{\partial} \quad : \quad \Omega^{p,q}(\mathsf{E}^*) \to \Omega^{p+1,q}(\mathsf{E}^*) \oplus \Omega^{p,q+1}(\mathsf{E}^*)$$

By expanding $\mathrm{d}^2\lambda = 0$ in terms of these operators their properties can be seen by noting which $\Omega^{p,q}(\mathsf{E}^*)$ each term belongs to, if $\lambda \in \Omega^{r,s}(\mathsf{E}^*)$.

$$\begin{array}{cccccc} \mathrm{d}^2\lambda \;= & \partial^2\lambda & + & (\partial\overline{\partial} + \overline{\partial}\partial)\lambda & + & \overline{\partial}^2\lambda \\ & \in \Omega^{r+2,s}(\mathsf{E}^*) & & \in \Omega^{r+1,s+1}(\mathsf{E}^*) & & \in \Omega^{r,s+2}(\mathsf{E}^*) \end{array}$$

These terms vanish separately and so both operators are nilpotent and they anticommute, $\partial\overline{\partial}+\overline{\partial}\partial = 0$. Each operator defines a separate set of subspaces of $\Omega^{p,q}(\mathsf{E}^*)$ by those (p,q)-forms which are closed or exact under their action.

- $\overline{\partial}$ closed (p,q)-forms : $B^{p,q}_{\overline{\partial}} = \Big\{ \psi \in \Omega^{p,q}(\mathsf{E}^*) \quad \text{s.t.} \quad \overline{\partial}\psi = 0 \Big\}$
- $\overline{\partial}$ exact (p,q)-forms : $Z^{p,q}_{\overline{\partial}} = \Big\{ \overline{\partial}\varphi \in \Omega^{p,q}(\mathsf{E}^*) \quad \text{s.t.} \quad \varphi \in \Omega^{p,q-1}(\mathsf{E}^*) \Big\}$

The ∂ cases only differ by having $\psi \in \Omega^{p-1,q}(\mathsf{E}^*)$ and a pair of cohomology rings follow from these subspaces of $\Omega^*(\mathsf{E}^*)$.

$$H^{p,q}(\mathsf{E}^*)_{\overline{\partial}} \equiv B^{p,q}_{\overline{\partial}}/Z^{p,q}_{\overline{\partial}} \;\Leftrightarrow\; \big[\,\psi\,\big] \sim \big[\,\phi\,\big] \quad \text{iff} \quad \psi = \phi + \overline{\partial}\xi \quad \xi \in \Omega^{p,q-1}(\mathsf{E}^*)$$

A holomorphic p-form χ is an element of $\Omega^{p,0}(\mathsf{E}^*)$ satisfying $\overline{\partial}\chi = 0$ and conversely an anti-holomorphic q-form φ is an element of $\Omega^{0,q}(\mathsf{E}^*)$ satisfying $\partial\varphi = 0$. The dimension of $H^{p,q}(\mathsf{E}^*)$ is the Hodge number $h^{p,q}$ and because the $H^p(\mathsf{E}^*)$ can be written in terms of the $H^{n,p-n}(\mathsf{E}^*)$ in the same way as the $\Omega^p(\mathsf{E}^*)$ the $h^{p,q}$ are related to Betti numbers.

$$H^p(\mathsf{E}^*) = \bigoplus_{n=0}^{p} H^{n,p-n}(\mathsf{E}^*) \quad \Rightarrow \quad b_p = \sum_{n=0}^{p} h^{n,p-n}$$

The symmetries of the Hodge numbers can be expressed in a convenient manner by arranging them into the Hodge diamond, where topological symmetries of $\mathcal{M}$ lead to symmetries in the Hodge diamond's Hodge number components.

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{0,1} & & h^{1,0} & & \\
 & h^{0,2} & & h^{1,1} & & h^{2,0} & \\
h^{0,3} & & h^{1,2} & & h^{2,1} & & h^{3,0} \\
 & h^{1,3} & & h^{2,2} & & h^{3,1} & \\
 & & h^{2,3} & & h^{3,2} & & \\
 & & & h^{3,3} & & &
\end{array}$$

The number of independent Hodge numbers can be generically reduced by considering the properties we have thus far assumed about $\mathcal{M}$.

- Poincare duality between $H^{p}(\mathsf{E}^{*})$ and $H^{6-p}(\mathsf{E}^{*})$ implies $h^{r,s} = h^{3-s,3-r}$.
- Complex conjugation between $H^{r,s}(\mathsf{E}^{*})$ and $H^{s,r}(\mathsf{E}^{*})$ implies $h^{r,s} = h^{s,r}$.
- Compact, connected and simply connected implies $h^{0,0} = 1$, $h^{1,0} = 0$.

The first condition gives the diamond reflection symmetry about the central row while the second condition gives the diamond a reflection symmetry about the central column. Further conditions will arise due to additional symmetries on those $\mathcal{M}$ of primary interest to us but we defer that until a later section.

2.1.3 Kähler Manifolds

If $\mathcal{M}$ has a metric G and a complex structure J then G is Hermitian with respect to J if its arguments are invariant under J and the Kähler form J is defined by this Hermitian metric and its complex structure.

$$\begin{array}{lcl} \text{Hermitian metric} : G(v_1,v_2) & = & G(\mathsf{J}\cdot v_1, \mathsf{J}\cdot v_2) \\ \text{Kähler form} \qquad : J(v_1,v_2) & \equiv & G(\mathsf{J}\cdot v_1, v_2) \end{array} \quad , \quad v_i \in \mathsf{E}$$

As with the complex structure the metric is not automatically unique unless constrained by other conditions. Since G, J and J are all non-degenerate and interdependent only two of the three are needed to reconstruct the full set and so we denote a particular choice of complex structure and metric (and thus Kähler form) on $\mathcal{M}$ as $(\mathcal{M}, \mathsf{J}, G)$. By using the complexified coordinates the components of J can be written in terms of those of G.

$$\begin{aligned} G &= G_{m\overline{n}}\,\mathrm{d}z^m \otimes \mathrm{d}\overline{z}^n + G_{\overline{m}n}\,\mathrm{d}\overline{z}^m \otimes \mathrm{d}z^n \\ J &= iG_{m\overline{n}}\,\mathrm{d}z^m \otimes \mathrm{d}\overline{z}^n - iG_{\overline{m}n}\,\mathrm{d}\overline{z}^m \otimes \mathrm{d}z^n = iG_{m\overline{n}}\,\mathrm{d}z^m \wedge \mathrm{d}\overline{z}^n \end{aligned}$$

The components of G which are of pure form, G_{mn} and $G_{\overline{mn}}$, are zero due to the Hermiticity conditions. The Kähler form of a $2n$ dimensional space determines a nowhere vanishing $2n$ form by n exterior products of itself. By the nature of $2n$-forms $\mathrm{d}J^n = 0$ but this is not synonymous with $\mathrm{d}J = 0$. If this more stringent condition is satisfied then $(\mathcal{M}, \mathsf{J}, G)$ is Kähler. Using the definitions of (2.1.7) this can be reexpressed as constraints on the metric components.

$$\mathrm{d}J = 0 \quad \Leftrightarrow \quad G_{m\overline{n},p} = G_{p\overline{n},m} \quad , \quad G_{m\overline{n},\overline{p}} = G_{m\overline{p},\overline{n}}$$

These equations can be solved locally by defining the metric in terms of the Kähler potential K, which naturally incorporates a gauge freedom due to the (anti)holomorphic mixed derivatives.

$$G_{m\overline{n}} = \frac{\partial^2}{\partial z^m \partial \overline{z}^n} K = \frac{\partial^2}{\partial z^m \partial \overline{z}^n} \mathcal{K} \quad , \quad \mathcal{K} = K + f_1(z^m) + f_2(\overline{z}^n)$$

Though we have approached this definition from the point of view of G being the space-time metric the definitions apply for other kinds of manifolds whose coordinates are degrees of freedom other than space-time position, a point we will return to. The symmetries of a Kähler manifold are quite stringent and greatly reduce the independent components of such objects as the Christofell symbol in torsion-less $\mathcal{M}$ and the Riemann curvature [16].

2.2 String Compactifications

Having covered the basic properties of a large class of compact spaces we now consider what physical implications such spaces have if we partly compactify a ten dimensional string theory onto them.

2.2.1 Field Content

Under the compactification $M_{10} \to M_4 \ltimes \mathcal{M}$ the ten dimensional metric decomposes into a number of smaller fields, including a metric for the external space G_{KL} and a metric for the internal space G_{mn}, as was observed in the construction of Type IIA supergravity from its eleven dimensional parent. Restrictions due to supersymmetry and Lorentz invariance do not preclude G_{KL} being dependent on the coordinates of the internal space through a warp factor $A(X^p)$.

$$ds^2 = e^{2A(X^p)} G_{KL}(x^M) \mathrm{d}x^K \mathrm{d}x^L + G_{mn}(X^p) \mathrm{d}X^m \mathrm{d}X^n$$

We will only be considering those space-time configurations where the external space is entirely independent of the internal coordinates, so $A(X^p) = 0$. Assuming an external space that is Minkowski the nine dimensional Laplacian undergoes the same disjoint splitting as the metric, $\Delta \to \Delta_3 + \Delta_6$, and therefore the Klein Gordon equation for scalar fields is also altered.

$$\partial^a \partial_a + M^2 = -\partial_t^2 + \Delta + M^2 \to -\partial_t^2 + \Delta_3 + \Delta_6 + M^2$$

Since the Laplacian is positive definite the four dimensional theory views the mass of a field to be greater than or equal to the ten dimensional mass. To analyse this we follow the method of Kaluza-Klein reduction but for six periodic directions, rather than one. With that in mind we consider a Fourier decomposition of a general ten dimensional p-form χ into combinations of $(p-q)$-forms dependent on external coordinates x^K and q-forms dependent on the internal coordinates X^m, for $q = 0, \ldots, p \leq 6$.

$$\chi(x^K, X^m) = \sum_{q=0}^{p} \sum_{n} \varphi_{p-q,n}(x^K) \wedge \psi_{q,n}(X^m) \tag{2.2.1}$$

In the simplest cases, particularly toroidal compactifications, the resultant Kaluza-Klein field content has clear distinctions between the different levels in the mass tower. We take this tower to be parameterised by the n in $\psi_{q,n}$ and the length scale of the compact dimensions R such that the masses of $\psi_{q,n}$ are of order $\frac{n}{R}$ and the basis in which $\psi_{q,n}$ is expanded we denote by $\Delta^q_{(n)}$, which complex subspaces $\Delta^{p,q-p}_{(n)}$ as previously defined. Of note is the $n = 0$ case as toroidal Kaluza-Klein field decompositions include massless forms if $\mathfrak{H}^q(\mathsf{E}^*)$ is not empty and the lightest modes define the field content of the effective theory. Generically we shall denote the p-forms on $\mathcal{M}$ which are sufficiently light to enter into the effective theory as $\Delta^p(\mathsf{E}^*) \subset \Omega^p(\mathsf{E}^*)$. In cases where $\mathfrak{H}^p(\mathsf{E}^*)$ is not empty we have $\Delta^q_{(0)} \equiv \Delta^q(\mathsf{E}^*) = \mathfrak{H}^q(\mathsf{E}^*)$ and the field content of the effective theory in the non-compactified dimensions is clearly defined by the massless forms.

Compactification causes a similar decomposition of the fermionic sector of the Type II fields and of primary interest are the gravitini, as they quantify the amount of supersymmetry. In the same way as the bosonic fields a ten dimensional spinor splits into four and six dimensional sections, $\psi \to \eta_4 \otimes \xi_6$. The spinor in the six dimensional space, $\xi_6 = \xi$, transforms under the holonomy group of $\mathcal{M}$, $H \subseteq \mathrm{SO}(6)$. The spinorial transformations are obtained by noting that the Lie algebras of SO(6) and SU(4) are isomorphic and ξ transforms in terms of a $\mathfrak{su}(4)$ multiplet, specifically the fundamental $\mathbf{4}$. This four-fold multiplicity manifests itself in the four dimensional effective theory as a quadrupling of the space-time supersymmetry from the non-compactified case, thus providing the source of the $\mathcal{N} = 8$. If $\mathcal{N} = 2$ is to be obtained then the holonomy group of $\mathcal{M}$ must be restricted to one of the subgroups of $SO(6)$ which admits spinor singlets, thus preventing the four-fold increase in supersymmetry generators.

The fundamental **4** of SU(4) can be broken to the fundamental SU(3) triplet **3** and an SU(3) singlet **1** and the components of the ten dimensional gravitini which become associated to the internal space's gravitini belong to this singlet. Clearly this prevents the four-fold increase in supersymmetry generators and so the SU(3) holonomy can be regarded as breaking 3/4 of the supersymmetry. However, it does not break the $\mathcal{N} = 2$ down to the required $\mathcal{N} = 1$ but we shall first consider the specifics of SU(3) holonomy before progressing further.

2.2.2 SU(3) Holonomy and Calabi-Yaus

It is possible to deduce an important property of spaces with SU(3) holonomy by considering how vectors change under parallel transport in terms of the geometry of $(\mathcal{M}, \mathsf{J}, G)$. Given a path beginning at $x_0 \in \mathcal{M}$ defined by a small parallelogram whose sides are the vectors φ^ρ and $\overline{\chi}^{\overline{\sigma}}$ the linear transformation on a vector ζ^μ associated to the holonomy can be expressed as a function of the Riemann curvature tensor [12, 20].

$$\zeta^\mu \to \zeta'^\mu = \zeta^\mu + R^\mu{}_{\nu\rho\overline{\sigma}}\zeta^\nu\varphi^\rho\overline{\chi}^{\overline{\sigma}} = (\delta^\mu_\nu + R^\mu{}_{\nu\rho\overline{\sigma}}\varphi^\rho\overline{\chi}^{\overline{\sigma}})\zeta^\nu = h^\mu_\nu\zeta^\nu$$

A general 6 dimensional Kähler manifold has $h^\nu_\mu \in U(3)$ and thus $(h-\delta)^\nu_\mu \in \mathfrak{u}(3)$, which can be written as $\mathfrak{u}(3) = \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ near the identity element of U(3). The $\mathfrak{u}(1)$ term is responsible for the trace contributions and if $H \subseteq$ SU(3) then these $\mathfrak{u}(1)$ associated terms must be zero, the combination $R^\rho{}_{\mu\rho\overline{\sigma}} = R_{\rho\overline{\sigma}}$ of the curvature tensor components must vanish. As a result $(\mathcal{M}, \mathsf{J}, G)$ must be Ricci flat, as well as Kähler. This can be done in terms of spinors by using the fact that SU(3) holonomy requires the existence of a covariantly constant spinor $\xi_6 = \xi$, $\nabla\xi = 0$. It can be shown that the commutation relations of the covariant derivatives on spinors take a similar form to those on vectors, through the use of $\Gamma^{ab} = \gamma^{[a}\gamma^{b]}$, elements of the Clifford algebra of $\mathcal{M}$ [20, 21].

$$[\,\nabla_a\,,\,\nabla_b\,]\,\xi = \frac{1}{4}R_{abcd}\Gamma^{cd}\xi = 0 \quad\Rightarrow\quad R_{ab} = 0$$

A compact orientated space with no singularities and SU(n) holonomy is, by definition, a Calabi-Yau manifold. It implies the existence of a unique non-vanishing holomorphic n-form which in turn results in $h^{n,0} = 1$. These symmetries further reduce the independent Hodge numbers.

- Calabi-Yau symmetries $H^p(\mathsf{E}^*) \cong H^{n-p}(\mathsf{E}^*)$ implies $h^{p,0} = h^{n-p,0}$.

These symmetries are sufficiently stringent for $n = 1, 2$ to uniquely determine all Hodge numbers but for the case of primary interest to us, $n = 3$, there are two independent Hodge numbers. Despite the uniqueness of the Hodge diamonds in the $n = 1$ and $n = 2$ cases there are more than one compact Calabi-Yau manifolds of dimensions 2 or 4. The two dimensional torus $\mathbb{T}^2$ is Calabi-Yau and by the factorisation $\mathbb{T}^{n+m} = \mathbb{T}^n \times \mathbb{T}^m$ so are all $\mathbb{T}^{2n}$. Other generic examples are the submanifolds of the complex projective space $\mathbb{CP}^{n+1}$ defined by the roots of homogeneous polynomials of degree $n+2$. In the $n = 2$ case a K3 manifold is obtained. The non-unique specification of a manifold by its Hodge numbers continues in the $n = 3$ case and it is possible to have several distinct Calabi-Yau manifolds for a given pair of

Hodge numbers.

$$(\mathcal{M},\mathsf{J},G)\ \text{Calabi-Yau} \quad \Rightarrow \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

It is not currently known if there are infinitely many distinct Calabi-Yaus nor are many explicit cases known in terms of their metrics. Over and above their supersymmetry breaking properties they are of interest to string theorists because they admit sets of harmonic forms. Since all $\chi \in \mathfrak{H}^p(\mathsf{E}^*)$ satisfy $\mathrm{d}\chi = 0 = \mathrm{d}\star\chi$ this background is a good effective theory string background, providing massless modes on $\mathcal{M}$ as well as a basis which automatically satisfies the equations of motion and constraints on the fluxes which descend from the ten dimensional action. Furthermore such compactifications have important phenomenological implications. $R_{\mu\overline{\nu}} = 0$ is a solution to the Einstein field equations in the case of vanishing cosmological constant and zero stress-energy tensor. There is still the issue of breaking the $\mathcal{N} = 2$ to $\mathcal{N} = 1$ but this can be solved by considering a set of spaces closely related to Calabi-Yaus and other restricted holonomy spaces; orbifolds and orientifolds.

2.2.3 Orbifolds

The compactification of a Type II string theory down onto a six dimensional torus, $\mathbb{T}^6$, using Kaluza-Klein methods, has the disadvantage that it has too much supersymmetry but benefits from the properties of the torus, that it has a known Ricci flat metric. The metric is the Euclidean metric and is obtained by viewing the $\mathbb{T}^6$ as a quotient of $\mathbb{C}^3$ and an equivalence relation defined by discrete translations in $\mathbb{C}^3$ by six linearly independent vectors $\mathbf{R}^j$.

$$x \sim x + \sum_j n_j \mathbf{R}^j \qquad n_j \in \mathbb{Z}$$

The holonomy group can be altered by imposing additional discrete symmetries on the space by further use of equivalence relations to construct orbifolds [19]. Given a discrete group G with generators g_i the orbifold equivalence relation is defined by making all images of a point in the space equivalent, $x \sim g_i(x)$. The choice of G is restricted to those whose generators preserve the lattice structure by acting crystalographically and such groups take the general form $G = \mathbb{Z}_N$ or $G = \mathbb{Z}_N \times \mathbb{Z}_M$. Not all possible choices of abelian group act crystalographically and the complete classification of the properties of $\mathbb{T}^6$ defined orbifolds has been done [22, 23]. The standard way of expressing the action of these generators on the field content of the orbifold is to define their actions in terms of the E^* basis, giving the orbifold group's generators g_i a matrix representation, $(g_i)^m{}_n$.

$$\eta^m \to g^m{}_n \eta^n \quad \Rightarrow \quad \eta^{m_1 \ldots m_p} \to g^{m_1}{}_{n_1} \cdots g^{m_p}{}_{n_p} \eta^{n_1 \ldots n_p}$$

When the orbifold group has a fixed point the resultant quotient space possess a conic singularity. Closed loops around these singularities are not contractable and as a result modify the holonomy group of the orbifold. For the case of $G = \mathbb{Z}_N$ with generator $g = e^{2\pi i/N}$ the holonomy group formed from such loops is precisely $\mathbb{Z}_N$, while the group associated with loops not circumnavigating the singularity remains trivial. In general Calabi-Yaus the cohomologies are defined by the exterior derivative while the analogous forms in orbifolds are those which form a space invariant under the orbifold group [24]. More restrictive orbifold groups reduce the size of such sets and this allows for a much simpler description of the internal space. However, the price paid is the singular nature of the orbifold at a finite number of points. In certain cases it is possible to remove these singularities by replacing them with Eguchi-Hanson spaces [8], resulting in a space which is Calabi-Yau but it no longer possesses a Euclidean metric. Taking this process in reverse, orbifolds can be viewed as singular limits of Calabi-Yau manifolds, where all the Riemann curvature of the manifold is 'pushed' into a finite number of regions, which are then shrunk to become singularities.

2.2.4 Orientifolds

In Type II compactifications both orbifolds and Calabi-Yaus have $\mathcal{N} = 2$ supersymmetry in their effective theories and in order to obtain $\mathcal{N} = 1$ supersymmetry half of their supersymmetry generators need to be removed. This is acheived by imposing an additional $\mathbb{Z}_2$ constraint on the space, formed from a number of different string properties and operators.

- Worldsheet parity : $\mathsf{P}(\sigma) = 2\pi - \sigma$
- A $\mathbb{Z}_2$ grading by left or right fermion counters F_L and F_R : $(-1)^{F_{L/R}}$.
- Orientifold involution : $\widehat{\sigma}(\eta^m) = \widehat{\sigma}^m{}_n \eta^n$ with $\widehat{\sigma}^m{}_r \widehat{\sigma}^r{}_n = \delta^m_n$.

There is no freedom in the definition of P and $F_{L/R}$ but $\widehat{\sigma}$ is only constrained by the requirement it is an involution, that the Kähler 2-form is +1 eigenvalued eigenfunction of it and its action on Ω is one of three possible choices. These $\mathbb{Z}_2$ operators combine to form three possible orientifold projection generators $g_{\mathcal{O}}$. This projection has the effect on Type IIB of turning certain closed strings into open ones, whose end points lie on regions of spacetime which are invariant under $g_{\mathcal{O}}$, as stated in Table 2.1. Open string end points define D-branes in those theories lacking an orientifold projection and

		IIB$_1$	IIA	IIB$_2$
$\widehat{\sigma}(\Omega)$	:	Ω	$\overline{\Omega}$	$-\Omega$
$g_{\mathcal{O}}$	:	$(-1)^{F_L}\Omega\widehat{\sigma}$	$\Omega\widehat{\sigma}$	$\Omega\widehat{\sigma}$
O-planes	:	O5/O9	O6	O3/O7

Table 2.1: Possible orientifold actions and planes in Type II compactifications.

therefore we can associate with these string ends produced by the orientifold projection a set of extended objects known as O-planes. The dimensionality of an Op-plane is determined in the same manner as a Dp-brane but due to it being defined by a non-dynamical projection, as stated in Table 2.1, they are static. Despite O-planes being non-dynamical objects they carry tension and charge in the same way as D-branes. However, it is important to note that they carry negative tension, relative to a D-brane, which allows for contruction of stable brane configurations without the requirement of anti-branes, which break supersymmetry. The effect of these planes on the field content of the theory is obtained by applying a change of basis to $\Delta^p(\mathsf{E}^*)$ such that the basis elements are eigenforms of $g_{\mathcal{O}}$, whose eigenvalues are ±1 and the resultant forms with eigenvalue $+1$ (-1) are known as even (odd) forms. By definition the Kähler form is an eigenfunction of eigenvalue $+1$ but not all elements of the eigenbasis of $\Delta^2(\mathsf{E}^*)$ have eigenvalue $+1$ under the projection. The effect of this is the splitting of the $h^{1,1}$ dimensional basis of $\Delta^2(\mathsf{E}^*)$ into $h^{1,1}_+$ even forms and $h^{1,1}_-$ odd forms and only the even forms survive the projection. As a result the general Kähler form has some degrees of freedom removed but the rest are left unchanged. The number of degrees of freedom that are projected out is $h^{1,1} - h^{1,1}_+$ but it is not automatic that $h^{1,1} - h^{1,1}_+ = h^{1,1}_+$, in that there is not an equal splitting of the cohomology by the two eigenvalues of the projection. In some cases $h^{1,1} = h^{1,1}_{+}$ and the Kähler form is left unchanged by the orientifolding. This is independent of the property that half of the supersymmetry generators are removed by the projection, to give $\mathcal{N} = 1$.

2.3 Moduli Spaces

The notation of this section and all discussion of moduli spaces closely, but not exactly, follows that of Ref. [17, 74, 75]. We previously commented that it is currently unknown if there are infinitely many distinct Calabi-Yau but we did not clarify what 'distinct' means. Given a particular Calabi-Yau it is possible to smoothly deform it without breaking its SU(3) holonomy and while this means it is possible to construct infinitely many Calabi-Yaus, given one, they are not distinct in the topological sense. Two Calabi-Yaus with different Hodge numbers are considered distinct and if their Hodge numbers are equal then they are distinct if they cannot be smoothly deformed into one another. The fields which parameterise these deformations are scalars known as moduli and two non-equal but equivalent Calabi-Yaus are considered to be at different points in the moduli parameter space. The ways in which a Calabi-Yau can be deformed can be obtained by considering the freedom in the metric, $G_{mn} \to G_{mn} + \delta G_{mn}$, such that the properties of $(\mathcal{M}, \mathsf{J}, G)$, being Kähler and Ricci flats, are preserved.

$$R_{mn}(G_{rs}) = 0 \quad \to \quad R_{mn}(G_{rs} + \delta G_{rs}) = 0$$

Under a metric compatibility condition of ∇, $\nabla^{n}\delta G_{mn} = 0$, the Ricci flatness preservation condition becomes the Lichnerowicz equation.

$$\nabla^{m}\nabla_{m}\delta G_{pq} + 2R_{p\ q}^{\ r\ s}\delta G_{rs} = 0$$

In terms of the complexified coordinates $(z^\rho, \overline{z}^{\overline{\sigma}})$ this equation is separately true for those perturbation metric components which are of pure degree, $\delta G_{\rho\sigma}$, and those which are of mixed degree $\delta G_{\rho\overline{\sigma}}$. With those perturbations of pure degree we can associate a variation in Ω which belongs to $\Delta^{2,1}(\mathsf{E}^*)$ and for those of mixed degree a variation of J in $\Delta^{1,1}(\mathsf{E}^*)$.

$$\begin{aligned} \delta\Omega \;&\sim\; \Omega_{\rho\sigma}{}^{\overline{\tau}}\delta G_{\overline{\lambda}\overline{\tau}} \mathrm{d}z^\rho \wedge \mathrm{d}z^\sigma \wedge \mathrm{d}z^{\overline{\lambda}} \;\in\; \Delta^{2,1}(\mathsf{E}^*) \\ \delta J \;&\sim\; i\delta G_{\rho\overline{\sigma}} \mathrm{d}z^\rho \wedge \mathrm{d}z^{\overline{\sigma}} \;\in\; \Delta^{1,1}(\mathsf{E}^*) \end{aligned} \tag{2.3.1}$$

These different variations define a metric on the moduli space and an additional set of variations can be applied to the 2-form B associated to the fundamental strings.

$$\mathrm{d}s^2 = \frac{1}{2\mathcal{V}} \int_{\mathcal{M}} G^{\rho\overline{\sigma}} G^{\tau\overline{\lambda}} \Big[\delta G_{\rho\tau} \delta G_{\overline{\sigma}\overline{\lambda}} - \Big(\delta G_{\rho\overline{\lambda}} \delta G_{\tau\overline{\sigma}} - \delta B_{\rho\overline{\lambda}} \delta B_{\tau\overline{\sigma}} \Big) \Big] \sqrt{G}\, d^6 X \tag{2.3.2}$$

The quantity $\mathcal{V}$ is the volume of $\mathcal{M}$. The fact the interval decomposes into terms entirely dependent upon variations with pure indices and another dependent upon variations in mixed indices implies that locally the two moduli spaces do not mix.

$$\mathcal{M}_{\mathcal{U}} \otimes \mathcal{M}_{\mathcal{T}} \subset \mathcal{M}_{\mathcal{M}}$$

This motivates us to consider the different moduli variations separately. It is noteworthy that the number of independent ways in which such perturbations can occur are the moduli and of note is the fact that dimensionality of the moduli spaces are defined by the topological properties of $\mathcal{M}$. Since we are considering supersymmetric theories the moduli can be viewed as the scalar fields of supermultiplets which in the four dimensional effective theory are massless. The number of massless scalar fields is determined by the dimension of $\mathfrak{H}^*(\mathsf{E}^*)$ but they form different structures depending on which $\Delta^{p,q}(\mathsf{E}^*) \equiv \mathfrak{H}^{p,q}(\mathsf{E}^*)$ they are associated to.

The definitions of the special Kähler manifolds and their associated supermultiplets are given in Appendix A and also follow the methods and notation, for the most part, of Refs. [74, 75]. Those definitions are done in a general manner rather than considering explicit moduli spaces or holomorphic forms but for clarity and completeness we also consider them explicitly here. The inclusion of the dilaton alters the Kähler structure of one of the moduli spaces but we shall initially examine both moduli spaces from the point of view that they are not dilaton dependent.

2.3.1 The Complex Structure Moduli Space

The complex structure moduli are defined on $\Delta^3(\mathsf{E}^*)$ and since forms in such spaces anticommute there is a natural symplectic structure, which is seen in the two inner products of Section A. As a result we do not need not consider the choice in $\langle\ \rangle_\pm$ in our discussions.

Every Calabi-Yau has a unique, up to an overall factor, holomorphic 3-form $\Omega \in \Delta^{3,0}(\mathsf{E}^*)$. Since it is defined in terms of $\mathrm{d}z^\rho$ it is deformed by any changes to the complex structure of the space. Rather than work with the $\Omega^p(\mathsf{E}^*)$ defined components of (2.3.1) it is preferable to use $\Delta^3(\mathsf{E}^*)$ defined components. In order to define the moduli associated to the complex structure deformations we can define $h^{2,1}+1$ pairs of 3-cycles A^I, B_J, $I,J = 0,\cdots,h^{2,1}$ with a symplectic structure in their intersection numbers.

$$A^I \cap B_J = -B_J \cap A^I = \delta^I_J \qquad A^I \cap A^J = B_I \cap B_J = 0$$

The (α_I,β^J) basis of $\Delta^3(\mathsf{E}^*)$ is defined as the dual of the (A^I, B_J) homology 3-cycles.

$$\int_{A^J} \alpha_I = \int_{\mathcal{M}} \alpha_I \wedge \beta^J = \delta^I_J \qquad \int_{B_J} \beta^I = \int_{\mathcal{M}} \beta^I \wedge \alpha_J = -\delta^I_J \qquad (2.3.3)$$

These 3-forms are a natural basis in which to examine moduli dynamics and overall they possess a $\mathrm{Sp}(h^{2,1}+1)$ symmetry. The set of $h^{2,1}+1$ coordinates $\mathcal{U}_I$ are defined by the 3-cycles and the holomorphic 3-form Ω and the remaining 3-cycles define $\mathcal{U}^J$, a set of degree 2 homogeneous functions in the $\mathcal{U}_I$.

$$\mathcal{U}_I \equiv \int_{A^I} \Omega \quad , \quad \mathcal{U}^J \equiv \int_{B_J} \Omega \quad \Rightarrow \quad \Omega = \mathcal{U}_I \alpha_I - \mathcal{U}^J \beta^J$$

By definition Ω can be written in terms of the $\mathrm{d}z$ up to an overall factor $f(z^\rho)$ which is unconstrained, other than being holomorphic in the z^ρ, due to the projective definition of the moduli.

$$\Omega \equiv f(z^\rho)\,\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \tag{2.3.4}$$

Since the coordinates are defined as $U_i = \frac{\mathcal{U}_i}{\mathcal{U}_0}$ and the section of $\mathrm{d}s^2$ in (2.3.2) dependent on the $\mathcal{U}$ can be written as the second derivatives of a $\mathcal{U}$ dependent function $(K_{\mathcal{U}})_{I\overline{J}} = \partial_{\mathcal{U}_I}\partial_{\overline{\mathcal{U}}_{\overline{J}}} K_{\mathcal{U}}$, it follows that the complex structure moduli space is a local special Kähler manifold of dimension $h^{2,1}$. The Kähler potential $K_{\mathcal{U}}$ can be written succinctly using a Hitchin function, as defined in Appendix A, on Ω. The Kähler derivative can include $\mathcal{U}_0$ terms as indices can range over I, J, not just i, j, due to the fact $\delta\mathcal{U}_0 = 0$ and makes no contribution to the interval. All functions dependent upon $\mathcal{U}_0$ are evaluated at $\mathcal{U}_0 = 1$, after any $\mathcal{U}_0$ derivatives are taken.

$$K_{\mathcal{U}} = -\ln H(\Omega) = i \int_{\mathcal{M}} \Omega \wedge \overline{\Omega} \quad \Rightarrow \quad \mathrm{d}s^2 = 2(K_{\mathcal{U}})_{I\overline{J}}\, \delta\mathcal{U}_I\, \delta\overline{\mathcal{U}}_{\overline{J}} + \ldots$$

For future reference we also define a set of vector spaces in terms of the symplectic forms for future reference.

$$\begin{aligned} \Delta^3(\mathsf{E}^*) &= \quad \langle \alpha_0 \rangle \quad \oplus \quad \langle \alpha_i \rangle \quad \oplus \quad \langle \beta^j \rangle \quad \oplus \quad \langle \beta^0 \rangle \\ &\equiv \mathcal{H}^{3,0}(\mathsf{E}^*) \oplus \mathcal{H}^{2,1}(\mathsf{E}^*) \oplus \mathcal{H}^{1,2}(\mathsf{E}^*) \oplus \mathcal{H}^{0,3}(\mathsf{E}^*) \end{aligned}$$

2.3.2 Kähler Moduli

The second type of deformation are those to Kähler form J parameterised by $h^{1,1}$ real scalar coefficients of the $\Delta^{1,1}(\mathsf{E}^*)$ basis elements ω_a. However, by generalising this to include contributions from the 2-form B_2 the Kähler moduli are made complex, which is consistent with the $\mathrm{d}s^2$ of the moduli space in (2.3.2). The basis elements of $\Delta^{1,1}(\mathsf{E}^*)$ have a set of $h^{1,1}$ 2-cycles associated to them, $\mathcal{A}^a$ with $a, b = 1, \cdots, h^{1,1}$, which are dual to a set of 4-cycles $\mathcal{B}_b$ partners that define the set of $h^{1,1}$ basis elements for $\Delta^{2,2}(\mathsf{E}^*)$, $\widetilde{\omega}^b$. These two sets of forms have non-trivial intersection numbers ${g_a}^b$ and ${g^a}_b$.

$$\int_{\mathcal{A}^b} \omega_a = \int_{\mathcal{M}} \omega_a \wedge \widetilde{\omega}^b = {g_a}^b \qquad \int_{\mathcal{B}_b} \widetilde{\omega}^a = \int_{\mathcal{M}} \widetilde{\omega}^a \wedge \omega_b = {g^a}_b$$

We can expand this basis to include the $\Delta^{0,0}(\mathsf{E}^*)$ and $\Delta^{3,3}(\mathsf{E}^*)$ forms, $\omega_0 \equiv 1$ and $\widetilde{\omega}^0 \equiv \mathrm{vol}_6$. The 6-form $\widetilde{\omega}^0$ is associated with $\mathcal{B}_0 \equiv \mathcal{M}$ itself, which is the only 6-cycle if $\mathcal{M}$ is connected. In the case of ω_0 we have to associate it with a 0-cycle point $\mathcal{A}^0$, which can be any point other than singularities, should $\mathcal{M}$ contain any.

$$\int_{\mathcal{A}^0} \omega_0 = \int_{\mathcal{M}} \omega_0 \wedge \widetilde{\omega}^0 = 1 \qquad \int_{\mathcal{B}_0} \widetilde{\omega}^0 = \int_{\mathcal{M}} \widetilde{\omega}^0 \wedge \omega_0 = 1$$

An additional sets of intersection numbers are defined by expressing the elements of $\Delta^+(\mathsf{E}^*)$ in terms of the elements of $\Delta^2(\mathsf{E}^*)$.

$$\int_{\mathcal{M}} \omega_a \wedge \omega_b \wedge \omega_c = \kappa_{abc} \quad , \quad \widetilde{\omega}^a = f^{abc} \omega_b \wedge \omega_c \quad , \quad {g^a}_b = f^{acd} \kappa_{bcd} \tag{2.3.5}$$

Following the lead of the complex structure moduli space we will find it more convenient to work with a set of projective coordinates $\mathcal{T}_A$ from which the standard Kähler moduli can be reconstructed by $T_a \equiv \frac{\mathcal{T}_a}{\mathcal{T}_0}$. The intersection numbers of these basis can be simplified by constructing the Kähler moduli holomorphic form, obtained by exponentiating the complexified Kähler form, $e^{\mathcal{J}} \equiv \mho$ and defining $\mathcal{J}^{(n)} = \frac{1}{n!}\mathcal{J}^n \in \Delta^{n,n}(\mathsf{E}^*)$.

$$\begin{aligned} \mho &= \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{J}^n = \sum_{n=0}^{\infty} \mathcal{J}^{(n)} = \mathcal{J}^{(0)} + \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)} \\ &= \mathcal{T}_0\,\omega_0 + \mathcal{T}_a\,\omega_a + \frac{1}{2!}\mathcal{T}_a\mathcal{T}_b\,\omega_a\wedge\omega_b + \frac{1}{3!}\mathcal{T}_a\mathcal{T}_b\mathcal{T}_c\,\omega_a\wedge\omega_b\wedge\omega_c \end{aligned}$$

This expansion for $\mho$ can be put into a form similar to that of the Ω expansion in $\Delta^-(\mathsf{E}^*)$. The specific sign structure we choose to define the expansion of the holomorphic form in depends on the sign structure of the inner product we will make use of. The inner product discussed in Appendix A has the option of being symmetric or antisymmetric. If antisymmetric then the construction of the Kähler moduli space is as the complex structure space but $\mathcal{U} \to \mathcal{T}$, $\Omega \to \mho$ *etc.* However the literature that considers T and S duality transformations on the superpotential and fluxes on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold [52, 53, 54, 60, 61, 92], which will be our explicit example in later chapters, has the Kähler moduli sector defined with a symmetric inner product. We will keep our discussion as general as possible but will reduce to the symmetric case when considering explicit examples. As such the $\pm$ sign ambiguities are associated to the $\langle\ \rangle_{\pm}$ inner product.

$$\Omega = \mathcal{U}_0\alpha_0 + \mathcal{U}_i\alpha_i - \mathcal{U}^j\beta^j - \mathcal{U}^0\beta^0 \quad , \quad \mho = \mathcal{T}_0\,\omega_0 \pm \mathcal{T}_a\,\omega_a + \mathcal{T}^b\widetilde{\omega}^b \pm \mathcal{T}^0\widetilde{\omega}^0$$

By comparing the coefficients we can express $\mathcal{T}^a$ in terms of the derivatives of $\mathcal{T}^0$ and in doing so reduce the intersection numbers to the simplest case of ${g^b}_a \to \delta^b_a$.

$$\mathcal{T}^0 = \frac{1}{3!}\kappa_{abc}\mathcal{T}_a\mathcal{T}_b\mathcal{T}_c \quad \Rightarrow \quad \mathcal{T}^b{g^b}_a = \frac{1}{2!}\kappa_{abc}\mathcal{T}_b\mathcal{T}_c = \frac{\partial\mathcal{T}^0}{\partial\mathcal{T}_a}$$

If we set $\mathcal{T}^a = \frac{\partial\mathcal{T}^0}{\partial\mathcal{T}_a}$ the expansion of $\mho$ simplifies. However the $\langle\ \rangle_+$ inner product has a sign structure which does not admit a manifest special

Kähler construction. This is seen more explicitly by constructing the Kähler potential for the moduli space. For either inner product we require a reformulation of the Kähler moduli contributions to $\mathrm{d}s^2$ in (2.3.2) so as to be in the standard form.

$$\mathrm{d}s^2 = 2(K_{\mathcal{T}})_{A\overline{B}}\,\delta\mathcal{T}_A\,\delta\overline{\mathcal{T}}_{\overline{B}}$$

This $\mathcal{M}_{\mathcal{T}}$ Kähler potential is related to the volume of $\mathcal{M}$ by $K_{\mathcal{T}} = -\ln\mathcal{V}$.

$$\mathcal{V} = \int_{\mathcal{M}} \mathrm{vol}_6 = \frac{2^3}{3!}\int_{\mathcal{M}} J \wedge J \wedge J = \frac{2^3}{3!}\kappa_{abc}\tau_a\tau_b\tau_c$$

In Appendix A, specifically (A.1.10), it is demonstrated that such expressions arise in terms of the Hitchin function $H(e^{\psi})$ on both the symmetric and antisymmetric inner products. The symmetric inner product is dependent only on $\mathrm{Re}(\psi)$ and thus we define our Kähler form to be $\mathcal{J} = J+iB$. If we opt to make the special Kähler structure manifest we set $\mathcal{J} = -B+iJ$.

$$\begin{aligned} K_{\mathcal{T}} &= -\ln \int_{\mathcal{M}} \big\langle\, \mho, \overline{\mho}\,\big\rangle_+ &= -\ln H(\mho)_+ &= -\ln\Big(2\,\mathrm{Re}\Big(\mathcal{T}_B\overline{\mathcal{T}}^B\Big)\Big) \\ K_{\mathcal{T}} &= -\ln i \int_{\mathcal{M}} \big\langle\, \mho, \overline{\mho}\,\big\rangle_- &= -\ln H(\mho)_- &= -\ln\Big(2\,\mathrm{Im}\Big(\mathcal{T}_B\overline{\mathcal{T}}^B\Big)\Big) \end{aligned} \tag{2.3.6}$$

This confirms that we can use the symmetric inner product and retain the structure of a local Kähler manifold of dimension $h^{1,1}$ but it is associated to an $\mathrm{O}(h^{1,1}+1, h^{1,1}+1)$ vector bundle rather than a $\mathrm{Sp}(h^{1,1}+1)$ vector bundle. Other ways in which the analysis would change if we used the antisymmetric inner product on the Kähler moduli will be commented upon at the appropriate time. As with $\Delta^3(\mathsf{E}^*)$ the basis of $\Delta^+(\mathsf{E}^*)$ decomposes into subspaces of $\Delta^*(\mathsf{E}^*)$.

$$\begin{aligned} \Delta^+(\mathsf{E}^*) &= \langle\omega_0\rangle \oplus \langle\omega_a\rangle \oplus \langle\widetilde{\omega}^b\rangle \oplus \langle\widetilde{\omega}^0\rangle \\ &= \Delta^0(\mathsf{E}^*) \oplus \Delta^2(\mathsf{E}^*) \oplus \Delta^4(\mathsf{E}^*) \oplus \Delta^6(\mathsf{E}^*) \\ &\equiv \mathcal{H}^{0,0}(\mathsf{E}^*) \oplus \mathcal{H}^{1,1}(\mathsf{E}^*) \oplus \mathcal{H}^{2,2}(\mathsf{E}^*) \oplus \mathcal{H}^{3,3}(\mathsf{E}^*) \end{aligned}$$

2.3.3 The Dilaton

The moduli thus far considered are, for the most part, geometric in nature. The topological non-triviality of $\mathcal{M}$ determines the kinds of deformation that $(\mathcal{M}, \mathsf{J}, G)$ can undergo without breaking particular properties such as Ricci flatness. Aside from the inclusion of the 2-form field B_2 in order to complexify the Kähler moduli little reference to string theory has been made. However, there is an additional scalar field which is stringy in origin, not geometric, known as the dilaton and it too can be viewed in a string theory context as the scalar field in a supermultiplet. In a Kaluza Klein orientifold reduction of the $\mathcal{N} = 2$ theory there is no unique way of projecting down the $\mathcal{N} = 2$ degrees of freedom into the $\mathcal{N} = 1$ multiplets but the generic $\mathcal{N} = 1$ structures these degrees of freedom arrange themselves into is not orientifold dependent. The specific construction varies slightly between Type IIA and Type IIB and since Type IIB is the most straight forward due to its brane content we consider it explicitly.

Type IIB possesses R-R fields of the form C_{2n} and for the case $n = 0$ the field is a scalar. Such a field implies that the brane it electrically couples to is an instanton D(-1)-brane, a brane localised in both time and space. This scalar combines with the four dimensional dilaton which has descended from the ten dimensional action. Since they share their NS-NS sector the descent is the same in each Type II theory and as such we recall two terms from ten dimensional Type II supergravity action from the previous chapter.

$$S_{G,\Phi} = \int d^{10}x \, \sqrt{-G} \, e^{-2\Phi} \Big(\mathcal{R} + 4\nabla_\mu \Phi \nabla^\mu \Phi \Big) \tag{2.3.7}$$

Using the block diagonal splitting of the metric $G \to G_4 \oplus G_6$ to apply a Weyl rescaling to just G_4 the action can be reformulated [75] such that the kinetic terms of the scalar define a new dilaton, ϕ.

$$e^{-2\phi} = \frac{2^3}{3!}\int_{\mathcal{M}} e^{-2\Phi} J\wedge J\wedge J = \int_{\mathcal{M}} e^{-2\Phi}\mathrm{vol}_6 \tag{2.3.8}$$

For Φ constant on $\mathcal{M}$ this relationship reduces to $e^{\Phi} = \sqrt{\mathcal{V}}e^{\phi}$. In Type IIB these two scalars, C_0 and ϕ, combine to form a complex scalar which is associated to a hypermultiplet.

$$S = C_0 + ie^{\phi} \tag{2.3.9}$$

The four dimensional effective theory kinetic terms for the dilaton are obtained from the compactification, having originally been the kinetic terms for the ten dimensional dilaton Φ and due to being only a single field the construction of a Kähler potential for just the dilaton is straightforward.

$$\mathcal{L} \ni \frac{\partial_\mu S\overline{\partial^\mu S}}{2\,\mathrm{Im}(S)^2} \quad \Rightarrow \quad e^{-K_S} = \frac{1}{2i}(S-\overline{S}) \tag{2.3.10}$$

This expression suggests that the dilaton defines a third moduli space, $\mathcal{M}_S$, along side $\mathcal{M}_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{U}}$. However it is the case that the inclusion of the dilaton causes one of these two moduli spaces to become dilaton dependent, they combine to form a more elaborate moduli space. The total moduli space of the effective theory is reformulated into the local product of two special Kähler manifolds but such that the dilaton dependent one is embedded within a larger quaternionic manifold.

$$\mathcal{M}_{\mathcal{M}} = \mathcal{M}^K \otimes \mathcal{M}^Q$$

Which moduli are in which special Kähler manifold is dependent on which Type II theory is being considered, with $\mathcal{M}^Q = \mathcal{M}^Q(\mathcal{T},\phi)$ in Type IIB and $\mathcal{M}^Q = \mathcal{M}^Q(\mathcal{U},\phi)$ in Type IIA [74]. In Type IIA the C_3 R-R potential plays an analogous role to the C_0 of Type IIB just described and their effective theory holomorphic forms take schematically similar forms [53, 47, 56, 60, 75]. This is done generically, so as to apply to both Type IIA and Type IIB, for a finite dimensional set of forms in Appendix A.

$$\text{Type IIA} \,:\, \Omega \to \Omega_c \quad , \quad \text{Type IIB} \,:\, \mho \to \mho_c$$

It is also demonstrated that the holomorphic forms of $\mathcal{M}^Q$ can be used to construct a Kähler potential which factorises the dilaton dependence out. This factorisation gives rise to the dilaton kinetic contribution of (2.3.10) as well as the standard Kähler potential of the associated special Kähler moduli space $\mathcal{M}^{K'}$.

$$K(\mathcal{M}^Q) \sim K(\mathcal{M}^{K'}) + K(\mathcal{M}_S) \tag{2.3.11}$$

In preserving this $\mathcal{N} = 1$ structure the O-planes produced by the orientifold are linked to $\mathcal{M}^Q$ [75]. This can be seen by noting that O-planes of dimension p have support in the three external spacial directions and in $p - 3$ internal directions. For even $p \geq 3$ the affected cycles correspond to elements in $\Delta^+(\mathsf{E}^*)$ and therefore those moduli which combine with the dilaton to make $\mathcal{M}^Q$ are the Kähler moduli of $\mho \in \Delta^+(\mathsf{E}^*)$. Conversely, for odd $p \geq 3$ the complex structure moduli of $\Omega \in \Delta^-(\mathsf{E}^*)$ combine with the dilaton to make $\mathcal{M}^Q$. This can be seen explicitly in terms of the $\Delta^*(\mathsf{E}^*)$ defined fields in each Type II theory.

2.3.4 Twisted Moduli

There is an additional complication to the analysis of orbifolds brought about the discrete symmetry group. It is possible for open string states in the parent space to become closed string states upon orbifolding the space by $G = \langle g_i \rangle$, which wrap around any conic singularities in the orbifold.

$$g_i \, X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$$

These are the twisted strings and have associated to them a set of twisted moduli, the number of which are determined by the number and type of singularities in the orbifold and which are in turn defined by the orbiold group. As a result the number of twisted moduli are known for all viable string orbifolds [22, 23]. However, in cases where we examine orbifolds we will restrict our discussions to the untwisted moduli.

2.4 Vacua

The Lagrangian associated to the moduli is obtained by reducing the Type II supergravity actions to their four dimensional effective theories. For $\mathcal{N} \leq 2$ theories the moduli have a scalar potential[2] determined by two functions; the Kähler potential for all moduli spaces K and the superpotential W. The construction of these depends on the amount of supersymmetry preserved in the effective theory. In the $\mathcal{N} = 2$ case sets of Killing prepotentials [71, 72, 74] are used which arise by considering the variation of the gravitini. These are projected down into an $\mathcal{N} = 1$ theory by orientifolding but it is not a unique reduction, with many $\mathcal{N} = 1$ theories descending from the same $\mathcal{N} = 2$ theory. The orientifold projection effect is determined by the number of $+1$ eigen p-forms of the involution and this is related to breaking the SU(2) R symmetry of the $\mathcal{N} = 2$ theory to the $U(1)$ R symmetry of $\mathcal{N} = 1$ [74, 75]. However, we shall use the $\mathcal{N} = 1$ formulation [25] directly and instead consider the effect of the orientifold projection operators once a potential is constructed.

$$V = e^K \left(K^{M\overline{N}} D_M W \overline{D_N W} - 3|W|^2 \right) \tag{2.4.1}$$

[2] We do not include in our analysis D terms, so gauge kinetic functions are neglected.

The indices are such that they range over all moduli and the partial derivative has been generalised to the Kähler derivative D_M, defined by $D_M = \partial_M + \partial_M K$. This can be more succinctly expressed in terms of a different function $\mathcal{G} = K + \ln |W|^2$.

$$V = e^{\mathcal{G}}\Big(\mathcal{G}^{M\overline{N}}\partial_M \mathcal{G}\overline{\partial_N \mathcal{G}} - 3)$$

The fact $\mathcal{G}$ entirely determines V and vice versa leads to the existence of a gauge freedom between the Kähler potential and the superpotential. We previously noted that the Kähler potential has a gauge freedom in terms of the Kähler metric it defined because the metric is determined by the potential's second, mixed, derivatives. Such a transformation does not leave $\mathcal{G}$ invariant unless there is a corresponding change in the superpotential also.

$$\mathcal{G}(K, W) = \mathcal{G}(K + f + \overline{f}, e^{-f} W)$$

The Kähler potential K is determined from the Kähler potentials of the individual moduli spaces. We previously saw that the dilaton's kinetic term takes a simple form and a Kähler potential can be defined for it. As a result we can regard the total moduli space as locally a direct product of the individual moduli spaces.

$$\mathcal{M}_{\mathcal{M}} = \mathcal{M}_{\mathcal{U}} \otimes \mathcal{M}_{\mathcal{T}} \otimes \mathcal{M}_{S} \quad \Rightarrow \quad K = K_{\mathcal{U}} + K_{\mathcal{T}} + K_{S}$$

The moduli are constant in time if their values are associated to a local minimum of V and due to the singular nature of e^{K} for vanishing moduli values it is only possible to have local minima if the moduli have a non-zero VEVs. Such moduli values are, in principle, obtained by solving the equations of motion for the moduli but it is more natural to consider the turning points of the potential as a function of the moduli. For a given scalar potential the vacua are those values of moduli which solve $\partial_M V = 0$

and have $\partial_M \partial_{\overline{N}} V > 0$. This can be rephrased in terms of algebraic geometry [26, 27] by viewing the set of polynomials $\partial_M V$ as the generating functions of an ideal and any vacuum state will belong to the variety associated to this ideal.

$$\mathcal{I} = \langle \partial_{T_a} V \, , \, \partial_{U_i} V \, , \, \partial_S V \rangle \quad \leftrightarrow \quad \mathbf{V}_{\mathcal{I}} \subset \mathbb{C}^{\dim(\mathcal{M}_{\mathcal{M}})}$$

Not all points in $\mathbf{V}_{\mathcal{I}}$ correspond to stable vacua, unstable ones are included and the methods by which algebraic geometry separates these are given in Appendix C, along with an overview of relevant algebraic geometry terminology and methods. Phenomenological properties of the effective theory can be obtained from the vacuum expectation values of two sets of terms; the potential itself and the Kähler derivatives.

- Cosmological constant, $\Lambda \equiv \langle V \rangle$.
- Supersymmetry breaking scale defining F-term, $F_M \equiv \langle D_M W \rangle$.

The immediate implication of these two definitions is that if the theory has no supersymmetry breaking then the external space is Anti de Sitter (AdS) or Minkowski.

$$\forall \, M \quad F_M = 0 \quad \Rightarrow \quad \Lambda = \langle V \rangle = \langle -3e^{K} |W|^2 \rangle = \langle -3e^{\mathcal{G}} \rangle \leq 0$$

For supersymmetric vacua the equality, and therefore Minkowski space-time, occurs only if $\langle W \rangle = 0$. This in turn reduces the F-term expressions as the Kähler derivatives simplify down to partial derivatives. By considering the expansion in terms of K and W of $\partial_M V$ it can be seen that $\langle W \rangle = \langle \partial_M W \rangle = 0$ are sufficient for a stable vacuum and they represent considerably simpler equations than the first derivatives of the scalar potential.

$$\langle W \rangle = 0 = \langle \partial_M W \rangle \quad \Rightarrow \quad \langle \partial_M V \rangle = 0 = \langle V \rangle$$

Unbroken supersymmetry and non-positive cosmological constant are not phenomenological but because their associated constraints are simpler than the $\partial_M V$ expressions they have received a great deal more attention in model building. Regardless of which constraints are solved to construct a vacuum state the scalar potential must first be obtained, which amounts to finding K and W for a particular space. K is, up to gauge freedoms, completely determined through the properties of the moduli spaces of $\mathcal{M}$ but this is not so for the superpotential.

2.5 Fluxes and the Superpotential

Since we are not considering non-perturbative effects the scalar potential is a polynomial in the moduli and the coefficients are known as fluxes and since the Kähler potential is set by the internal space the flux dependence resides entirely in the superpotential. We previously discussed the effect the compactification of a ten dimensional theory onto a six dimensional internal space has on the field content and those components which exist in the internal space contribute to the equations of motion of the moduli as fluxes. A generic contribution to the superpotential due to an element of $\Omega^p(\mathsf{E}^*)$ is expressed in terms of an integral over $\mathcal{M}$, so any element of $\Omega^p(\mathsf{E}^*)$ must be paired with $\Omega^{6-p}(\mathsf{E}^*)$ to contribute. This differs from how the inner product (2.1.5) combines two forms of the same degree. Taking the Hitchin function's definition in terms of elements in $\Omega^{\pm}(\mathsf{E}^*)$ as a guide we instead consider elements in the subspaces $\Omega^{\pm}(\mathsf{E}^*) \subset \Omega^*(\mathsf{E}^*)$, or more specifically the light forms $\Delta^{\pm}(\mathsf{E}^*) \subset \Omega^{\pm}(\mathsf{E}^*)$.

$$W \ni \int_{\mathcal{M}} \left\langle \psi, \phi \right\rangle_s \qquad \left. \begin{array}{rcl} \psi & \in & \Delta^p(\mathsf{E}^*) \\ \phi & \in & \Delta^{6-p}(\mathsf{E}^*) \end{array} \right\} \in \Delta^{\pm}(\mathsf{E}^*)$$

The individual flux components defined by a $\Delta^{*}(\mathsf{E}^{*})$ expansion are constant but this is not automatically the case if the fluxes are written in terms of the $\Omega^{*}(\mathsf{E}^{*})$ basis. Only in parallelisable $\mathcal{M}$ is it possible to choose a E^{*} basis such that the $\Omega^{*}(\mathsf{E}^{*})$ defined flux components are constant over $\mathcal{M}$, thus having the fluxes belong to a particular $\Lambda^{p}(\mathsf{E}^{*}) \subset \Omega^{p}(\mathsf{E}^{*})$. This is discussed further in Appendix B.1.2.

The two Type II theories possess different field content on the ten dimensional level and this is carried through to the effective theory upon compactification. As a result the superpotential definitions differ in their R-R sectors but it is also true that they differ in their NS-NS sectors and so we shall consider each Type II theory in turn. Strictly speaking since we wish to ultimately examine $\mathcal{N} = 1$ compactifications we should be considering only those fluxes which survive the orientifold projection but at present we will neglect any constraints on the fluxes and talk about fluxes which might arise. Comprehensive review of flux compactifications, their phenomenology and their properties are given in [28, 29], as well as the phenomenology of including branes and orientifold planes in [30]. It is from these which we take our initial superpotential and tadpole constructions.

2.5.1 Type IIB Fluxes

The ten dimensional 3-form NS-NS flux strength descends to the effective theory and if we are restricted to the massless field content the H_3 defined on $\mathcal{M}$ is expanded in the $\Delta^{3}(\mathsf{E}^{*})$ basis.

$$H_3 \equiv h_I \alpha_I - h^J \beta^J$$

For $\mathcal{N}=1$ compactifications this contributes to the superpotential by combining with Ω in the form of the Gukov-Vafa-Witten equation [31], which stipulates the inclusion of the dilaton, and hence provides a non-zero potential for the complex structure moduli and the dilaton.

$$W \ni \int_{\mathcal{M}} \langle\, \Omega, -S\, H_3 \,\rangle_{\pm} = -S \left(h_I \mathcal{U}^I - h^J \mathcal{U}_J \right) \tag{2.5.1}$$

The GVW superpotential integrand is the combination of two 3-forms we can define it in terms of the generalised inner product (A.1.4) without having to set the sign structure as $\Delta^3(\mathsf{E}^*)$ is antisymmetric in each. A second 3-form descends from the ten dimensional theory, the R-R sector's F_3, which can also be written in the $\Delta^3(\mathsf{E}^*)$ basis.

$$F_3 \equiv f_I \alpha_I - f^J \beta^J$$

These fluxes do not couple to the dilaton but otherwise contribute to the superpotential in the same manner.

$$\begin{aligned} W &= \int_{\mathcal{M}} \langle\, \Omega, \left(F_3 - S\, H_3 \right) \rangle_{\pm} \\ &= \left(f_I \mathcal{U}^I - f^J \mathcal{U}_J \right) - S \left(h_I \mathcal{U}^I - h^J \mathcal{U}_J \right) \end{aligned} \tag{2.5.2}$$

If $\mathcal{M}$ is Calabi-Yau then $\Delta^1(\mathsf{E}^*)$ and $\Delta^5(\mathsf{E}^*)$ are empty and so we do not need to consider other contributions to the superpotential from fluxes in Type IIB.

2.5.2 Type IIA Fluxes

The Type IIA 3-form field strength descends to the effective theory in the same manner as the Type IIB case and thus combines with a holomorphic 3-form. In Type IIA the complex structure moduli combine with the dilaton to form $\mathcal{M}^Q$ and so the 3-form flux couples to the complexified holomorphic

3-form, $\Omega \to \Omega_c$. The complexified holomorphic form combines with the $\Delta^3(\mathsf{E}^*)$ flux to provide a non-zero potential for the complex structure moduli and the dilaton but not the same one obtained in the Type IIB case.

$$W \ni \int_{\mathcal{M}} \langle\, \Omega_c, H_3 \,\rangle_{\pm} = -S\mathcal{U}^0 h_0 + \mathcal{U}^i h_i + S\mathcal{U}_0 h^0 - \mathcal{U}_j h^j \tag{2.5.3}$$

The effective theory in Type IIA differs, at least in terms of flux structures, from Type IIB in a more manifest manner in its R-R sector. Type IIA does not contain a 3-form flux in the R-R sector, only field strengths of the form F_{2m}. The formal sum of these fluxes is written using the $\Delta^+(\mathsf{E}^*)$ basis and we again keep the sign choice manifest in the expansion.

$$F_{RR} = \sum_{n=0}^{3} F_{2n} = \mathfrak{f}_0\omega_0 \pm \mathfrak{f}_a\omega_a + \mathfrak{f}^b\widetilde{\omega}^b \pm \mathfrak{f}^0\widetilde{\omega}^0$$

In Type IIA the R-R fluxes obtained by compactification do not contribute a dilaton dependency as the Kähler moduli make up the local special Kähler manifold $\mathcal{M}^K$ and therefore these R-R fluxes contribute to the superpotential by combining with $\mho$.

$$W \ni \int_{\mathcal{M}} \langle\, \mho, F_{RR} \,\rangle_{\pm} = \mathfrak{f}_0\mathcal{T}^0 \pm \mathfrak{f}_A\mathcal{T}^A + \mathfrak{f}^B\mathcal{T}_B \pm \mathfrak{f}^B\mathcal{T}_B \tag{2.5.4}$$

2.5.3 Branes and Tadpoles

Thus far we have been considering the effective field theory obtained by compactifications of a Type II supergravity theory and have not considered quantum corrections to this. Additional constraints follow from the construction of tadpoles from fluxes and R-R potentials. Due to the relationship between dimensionality of potentials, the branes they live on and the definitions of the Type II theories, the expressions for the tadpole constraints will not be common between the two theories. However, the method of analysing them can be applied to either Type II theory and because of this we shall first consider the tadpoles of Type IIB. A much studied [32, 21] Type IIB tadpole is that which arises for F_5, whose equation of motion is already modified by the inclusion of fluxes, as given in Section 1.3.2. In the compactified effective theory the insertion of D3-branes which fill the external space-time contributes to the energy momentum tensor a charge density[3] ρ_3 which modifies the F_5 equation of motion.

$$\mathrm{d}\widetilde{F}_5 = H_3 \wedge F_3 + 2\kappa^2 T_3 \rho_3$$

Integration over the internal space causes the left hand side to vanish by virtue of it being exact and a relationship between the fluxes and the total charge Q_3 is obtained.

$$\frac{1}{2\kappa^2 T_3} \int_{\mathcal{M}} H_3 \wedge F_3 + Q_3 = 0$$

Both D3-branes and O3-planes can contribute to Q_3, the branes in a positive way and the O-planes in a negative way due to their negative tension, and this tadpole constraint on the fluxes can be expressed in terms of the number of three dimensional objects living in the external space-time.

$$\frac{1}{2\kappa^2 T_3} \int_{\mathcal{M}} H_3 \wedge F_3 + N_3 = 0 \qquad N_3 = N_{D3} - \frac{1}{2} N_{O3}$$

The general extension of this is that if a Dp-brane is allowed by the symmetries of a space and the Type II theory being considered then the C_{p+1} potential it couples to can contribute a tadpole constraint.

$$\int_{M^4 \times \mathcal{M}} \left(C_{p+1} \wedge X_{9-p} + 2\kappa^2 T \rho_p \right) = 0$$

Due to our assumption that the internal and external spaces are not interdependent any C_{p+1} for $p \geq 3$ factorises into a term relating to the external space and terms relating to the cycles the Dp-brane is wrapping,

[3] Delta functions associated to the point in each dimension $\mathcal{M}$ the D3 is located at [21] are surpressed.

$C_{p+1} \to \mathrm{vol}_4 \wedge \widetilde{C}_{p+1}$. If a set of branes are wrapping a q-cycle γ in $\mathcal{M}$ with total charge N_γ then X_{6-q} must have support in the dual cycle $\star\gamma$.

$$0 \;=\; \frac{1}{2\kappa^2 T}\int_{\star\gamma} X_{6-p} + N_\gamma$$

By considering how these terms descend from the full ten dimensional theory it follows that the X_{p+1} are $\mathrm{d}\widetilde{F}_p$, the contribution of brane sources alters the R-R sector's dynamics by modifying their closure properties.

2.6 Generalised Calabi-Yaus

We motivated our examination of fluxes via harmonic forms by using the properties of Calabi-Yaus; their Ricci flatness and symmetry breaking SU(3) holonomy group. Our entire analysis has been based on requiring the compactified Type II theory to admit a pair of SU(3) holonomy spinors, each associated to a gravitino. The Calabi-Yau admits a non-zero set of harmonic forms in $\Omega^{1,1}(\mathsf{E}^*)$ and $\Omega^{2,1}(\mathsf{E}^*)$ and the effective theory is dependent upon their associated moduli. In the fluxless case the space possesses a standard exterior derivative and we can use it to construct our $\mathfrak{H}^{p,q}(\mathsf{E}^*)$ and thus have $\Delta^{p,q}(\mathsf{E}^*) = \mathfrak{H}^{p,q}(\mathsf{E}^*)$. However, the inclusion of fluxes alters the compact space such that $\mathfrak{H}^{p,q}(\mathsf{E}^*)$ is empty or at the very least reduced in dimensionality and therefore $\Delta^{p,q}(\mathsf{E}^*)$ is not harmonic. The inclusion of non-zero fluxes feeds through into curvature of $(\mathcal{M},\mathsf{J},G)$ and thus modifies the holonomy group such that $(\mathcal{M},\mathsf{J},G)$ is no longer of SU(3) holonomy but instead can possess SU(3) structure.

2.6.1 Breaking SU(3) Holonomy

The breaking of SU(3) holonomy does not signal a removal of the SU(3) singlet the effective theory gravitini belong to nor a breaking of Ricci flatness. The stipulation of SU(3) holonomy is a sufficient but not necessary requirement for these conditions, it is a particular case of a more general set of spinor transformations which provide SU(3) singlets and Ricci flatness. To see this we consider the variations of $\mathcal{N} = 1$ supersymmetry spinorial fields of the gravitino ψ_M, the dilatino λ and the gluino χ^a at the vacuum in terms of a spinor η and the bosonic fields and couplings, following the notation of [20].

$$\begin{aligned}
\delta\psi_M &= \tfrac{1}{\kappa}\nabla_M\eta + \tfrac{\kappa}{32g^2\phi}\left(\Gamma_M{}^{NPQ} - 9\,\delta_M^N\Gamma^{PQ}\right)H_{NPQ}\,\eta &= 0 \\
\delta\chi^a &= -\tfrac{1}{4g\sqrt{\phi}}\Gamma^{MN}F^a_{MN}\,\eta &= 0 \\
\delta\lambda &= -\tfrac{1}{\sqrt{2}\phi}(\Gamma^M\partial_M\phi)\eta + \tfrac{\kappa}{8\sqrt{2}g^2\phi}\Gamma^{PQR}H_{PQR}\,\eta &= 0
\end{aligned}$$

Turning off H, thus making ϕ constant, reduces these to a pair of equations on η as the $\delta\lambda$ case becomes trivial.

$$\begin{aligned}
\delta\psi_M &= 0 \quad \Rightarrow \quad \nabla_M\eta &= 0 \\
\delta\chi^a &= 0 \quad \Rightarrow \quad \Gamma^{MN}F^a_{MN}\,\eta &= 0
\end{aligned}$$

The $\delta\psi_M$ condition is precisely the one which we have previously seen lead to the curvature constraint $R_{MNPQ}\Gamma^{PQ}\eta = 0$. Under the assumption the ten dimensional space-time splits into Minkowski space-time and $\mathcal{M}$ it follows that the R_{MNPQ} for $M, N, P, Q \in \{0, 1, 2, 3\}$ vanishes and thus $\nabla_M\eta \to \partial_M\eta$ for $M \leq 3$. Therefore the spinor is independent of the larger space-time and covariantly constant on $\mathcal{M}$ and $(\mathcal{M}, \mathsf{J}, G)$ has SU(3) holonomy as a result.

In deriving such a result we had to assume that the field strength H is turned off and the dilaton is constant. Both of these are not true in the general analysis of flux compactifications. Turning on H makes η no longer covariantly constant due to the $\delta\psi$ equation of motion, nor can the dilaton be constant due to the $\delta\lambda$ equation of motion. The inclusion of the second term in the $\delta\psi$ equation of motion can be seen to be schematically similar to the addition of a torsion term to ∇, provided H is constant.

$$\delta\psi_M = \left(\frac{1}{\kappa}\nabla_M + \frac{\kappa}{32g^2\phi}\left(\Gamma_M{}^{NPQ} - 9\,\delta_M^N\Gamma^{PQ}\right)H_{NPQ}\right)\eta \equiv \frac{1}{\kappa}\widehat{\nabla}_M\eta$$

The $\delta\lambda$ equation of motion can also be reduced to an expression on ϕ.

$$\Gamma^M\partial_M\phi - \frac{\kappa}{8g^2}\Gamma^{PQR}H_{PQR} = 0$$

The SU(3) holonomy case is now seen to be mearly a particular solution to the problem of the internal space admitting a single spinor to the effective theory for each spinor of the uncompactified theory. In fact, there is a much larger class of compactifications which could provide the same kind of phenomenology. With the Riemann curvature associated to ∇ no longer being exactly zero it is possible to build space-times which are curved, with de Sitter and Anti de Sitter being those of primary interest to cosmologists. Such spaces can be obtained in restricted cases, where the compactification is such that the metric splits into external and internal parts which are only linked via a warp factor dependent on the internal space.

$$ds^2 = e^{A(X)}g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{KL}(X)\mathrm{d}X^K\mathrm{d}X^L$$

However, we wish to phrase these generalisations in terms of our effective theory, the fluxes and the superpotential.

2.6.2 Effective Theory Light Fields

Although fluxes may deform the space such that they no longer admit harmonic forms the fluxless Calabi-Yau provides a convenient initial ansatz for what p-forms define a consistent basis for $\Delta^{p,q}(\mathsf{E}^*)$. This can be seen by considering a generic expansion of a field dependent on the coordinates of

$\mathcal{M}$ in terms of a Kaluza-Klein tower. Recalling the field decomposition of (2.2.1) we examine a p-form with only $\mathcal{M}$ dependence.

$$\begin{array}{rcccccccc} \Psi_q & = & \sum_n \psi_{q,n} & = & \psi_{q,0} & + & \psi_{q,1} & + & \dots \\ & & & & \in \Delta^q_{(0)} & & \in \Delta^q_{(1)} & & \\ \Delta\Psi_q & = & \sum_n \Delta\psi_{q,n} & = & \Delta\psi_{q,0} & + & \Delta\psi_{q,1} & + & \dots \end{array} \tag{2.6.1}$$

For a six dimensional torus the quantisation of the momentum of the fields on the circles gives a clear tower of states such that $\langle\!\langle\, \psi_{q,n}, \Delta\psi_{q,n} \,\rangle\!\rangle \sim \left(\frac{n}{R}\right)^2$. The massless nature of the $\psi_{q,0}$ state corresponds to it being harmonic and thus $\Delta^q_{(0)} \cong \mathfrak{H}^q(\mathsf{E}^*)$. These modes also satisfy the fields' equations of motion and Bianchi constraints and so provide a valid basis for the effective theory. With the inclusion of fluxes, which we generically denote as $\mathfrak{f}$, the bases $\Delta^{p,q}(\mathsf{E}^*)$ are not longer the $\mathfrak{H}^{p,q}(\mathsf{E}^*)$ due to the deformations of metric of $(\mathcal{M}, \mathsf{J}, G)$ via the stress-energy tensor. As a result the tower expansion of Ψ_q is altered and dependent on the fluxes.

$$\langle\!\langle\, \varphi, \Delta\varphi \,\rangle\!\rangle \equiv M^2_\varphi(\mathfrak{f}) \tag{2.6.2}$$

By definition, if $\varphi \in \mathfrak{H}^p(\mathsf{E}^*)$ then $M^2_\varphi = 0$. Generically we must allow for the possibility that $M^2_{\psi_{q,0}}$ could be sufficiently large to become comparable to the $M^2_{\psi_{q,1}}$ and in such a case the effective theory field content is no longer clear. However, for small values of the fluxes we can suppose that the deformation is sufficiently small so as to maintain the excitation splittings in the mass tower. We shall continue denote the corresponding space of p-forms spanned by these elements as $\Delta^p(\mathsf{E}^*)$ whose definition we can now state more formally.

$$\Delta^p(\mathsf{E}^*) \equiv \left\{\, \varphi \in \Omega^p(\mathsf{E}^*) \quad \text{s.t.} \quad M^2_\varphi(0) = 0 \,\right\} \tag{2.6.3}$$

In constructing a basis for $\Delta^p(\mathsf{E}^*)$ we require that supersymmetry can still be obtained, as it is the guiding principle for our examination of Calabi-Yaus. $\mathcal{N} = 2$ supersymmetry is associated to the existence of special Kähler moduli manifolds [74] and the definition of such manifolds given in Appendix A does not require the finite basis to be harmonic. Instead the special Kähler conditions were related to the intersection numbers defined by the basis elements (the ϖ used in Appendix A). Provided those are met it is possible to construct the associated holomorphic section. Therefore if we truncate [74] the Kaluza-Klein expansion to such a basis we allow for the possibility of preserving supersymmetry in some way. Further algebraic properties of this truncation, over and above this special Kähler preserving structure, such as being closed under the exterior derivative are outlined in Appendix B.1.1. By definition the harmonic forms on a Calabi-Yau satisfy all of these conditions and we make the assumption that the small deformations caused by the inclusion of fluxes do not reduce the associated intersection numbers to being degenerate.

$$g_I{}^J(\mathfrak{f}) = \int \alpha_I(\mathfrak{f}) \wedge \beta^J(\mathfrak{f}) \quad \Rightarrow \quad \det(g_I{}^J(\mathfrak{f})) \neq 0 \tag{2.6.4}$$

This suggests that an ansatz basis for the space obtained by the slight deformation of a Calabi-Yau is comprised of elements of $\Omega^*(\mathsf{E}^*)$ which become harmonic if the fluxes are set to zero.

$$\Delta^*(\mathsf{E}^*) = \langle \omega_A(\mathfrak{f}), \widetilde{\omega}^B(\mathfrak{f}), \alpha_I(\mathfrak{f}), \beta^J(\mathfrak{f}) \rangle \quad \Rightarrow \quad \Delta^*(\mathsf{E}^*)\Big|_{\mathfrak{f}=0} = \mathfrak{H}^*(\mathsf{E}^*) \tag{2.6.5}$$

The alternative way to consider the small flux limit is the large volume or complex structure limit and in these limits this basis ansatz is justified [79]. This generalisation beyond the Calabi-Yau case is known as general geometry [66, 67] and the resultant spaces are twisted and/or generalised Calabi-Yaus [28] whose holomorphic pure forms are no longer closed. The fluxes can be regarded as parameterising the deviation of the $\Delta^*(\mathsf{E}^*)$ elements from being closed under exterior differentiation [63, 64, 65, 74, 30]. We shall compare this definition of the fluxes to an alternative construction in terms of gauge Lie algebras in more detail later and in the explicit example of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold we will see that fluxes allow the construction of non-Minkowski vacua [68, 69, 93, 94]. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold has empty $\Delta^1(\mathsf{E}^*)$ and $\Delta^5(\mathsf{E}^*)$ and although a generic deformation of a Calabi-Yau would be expected to have non-empty $\Delta^1(\mathsf{E}^*)$ and $\Delta^5(\mathsf{E}^*)$ we shall restrict our considerations to truncated bases which do not contain such forms.

Summary

We have reviewed the three different types of moduli which arise in string compactifications; the complex structure moduli $\mathcal{U}$, the Kähler moduli $\mathcal{T}$ and the dilaton modulus S. In the case of the $\mathcal{U}$ and $\mathcal{T}$ they are associated to the topological non-triviality of $\mathcal{M}$, with a direct correspondence between the number of (2,1) and (1,1) cycles and the number of harmonic ways $(\mathcal{M}, \mathsf{J}, G)$ can be deformed while remaining a Calabi-Yau. The distinction in how the Type II theories construct their effective theory superpotentials has been stated, as well as the different ways in which the fluxes contribute to possible tadpoles due to branes. Irrespective of how the superpotential is written in terms of fluxes we have noted the constraints on the superpotential required for stable vacua, with the supersymmetric Minkowski case being of particular interest due to its greatly simplified nature when compared to the fully general approach. These simplified constraints will be used in our examination of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold. Finally we have taken into consideration the fact that turning on fluxes will deform the space and non-trivialise the question of what fields descend into the four dimensional effective theory. Having stated the fluxes and their superpotentials obtained from direct compactification of the ten dimensional string actions we now consider the dualities and symmetries these string actions possess.

Chapter 3

String Dualities

Symmetries are a fundamental concept within physics, such as defining conserved quantities via Noether's theorem or protecting gauge boson masses from renormalisation effects. Generally they are constructed by considering variations in fields $\xi \to \xi + \delta\xi$ which leave an action or equations of motion invariant and thus relate two physically equivalent constructions within the same theory. Dualities differ from this in that they are exact equivalences between two theories or constructions which have different equations of motion or actions. An example of this outside of string theory is Seiberg duality [33], which relates two different supersymmetric non-abelian gauge theories.

$$\mathcal{SD} \ : \ \mathcal{L}_1 = \mathcal{L}(\tau^2, F_{ab}) \quad \leftrightarrow \quad \mathcal{L}_2 = \mathcal{L}(-\tau^{-2}, \widetilde{F}_{ab})$$

The Lagrangian density $\mathcal{L}_1$ is that of the standard gauge theory, while $\mathcal{L}_2$ is its Seiberg dual. If the gauge coupling g is such that $\mathcal{L}_1$ is a weakly coupled theory then $\mathcal{L}_2$ must be strongly coupled due to its inverse relationship. The utility of such an equivalence arises by being able to convert strongly coupled problems into a weakly coupled regime, find a perturbative solution and then convert into a solution to the strongly coupled problem. Such a weak-strong duality in string theory, the AdS/CFT correspondence [34] based on the concept of holography [35], has received considerable interest because of its possible applications into understanding strongly coupled gauge theory [36]. In the original correspondence the $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory due to open strings ending on D3-branes have a dual description in the gravitational theory of the $\mathrm{AdS}_5 \times S^5$ space-time, following from the closed string description. Though $\mathcal{N} = 4$ SYM is unphysical due to conformality and excessive supersymmetry, modifications to the space-time feed back into the gauge theory, breaking the gauge theory to a less symmetric and thus more realistic one. Though a gravity dual to quantum chromodynamics is not currently known, or even known to exist, it has provided insight into confinement [37], hadronisation [38], flavour physics [39], technicolour induced Higgs mechanisms [11] and finite temperature physics [40].

In this chapter we will consider dualities which arise in a more direct fashion than the AdS/CFT correspondence, appearing at the level of the actions or mode expansion of the fields in the Type II theories. The first case we shall consider is T duality, which is the relationship between Type II theories when they are compactified on a toroidal space and is demonstrated on the level of mode expansion in the oscillation modes of the string. Toroidal spaces are not the only possible spaces upon which string theory can be compactified on, as commented in the previous chapter, and the extension of T duality to cover Calabi-Yau manifolds, mirror symmetry, is considered after T duality. The third duality is one whose existence follows from the action of Type IIB supergravity and is S duality, a strong-weak coupling equivalence between Type IIB formulations. This has applications to the AdS/CFT correspondence because Type IIB is the gravity dual of $\mathcal{N} = 4$ supersymmetry Yang-Mills theory. Finally we briefly cover a combination of T and S dualities known as U duality, which arises from the fact T duality makes Type II theories dual yet only one of them possesses S duality invariance.

3.1 T Duality

We first consider T duality between Type IIA and Type IIB. Though we are considering supersymmetric string theories and thus have fermionic modes in such constructions the motivation for the symmetry is done using bosonic modes. The effects of T duality on such things as branes occurs in bosonic string theory as well as supersymmetric string theories. However, since it is the fermionic sector which defines the two Type II theories T duality in the bosonic string theory is not a duality in the same way it is between supersymmetric string theories.

3.1.1 A Stringy Phenomenon

T duality is a fundamentally string phenomenon arising from the fact the string has length, unlike standard quantum field theories, which comes into play when considering compactified dimensions. To see this in ten dimensional space-time we can make a generic expansion of the bosonic modes on the string without taking the $\alpha' \to 0$ limit and using $z = \sigma^1 + i\sigma^2 = \sigma + i\tau$.

$$X^\mu(z,\bar{z}) = \frac{x^\mu}{2} + \frac{\widetilde{x}^\mu}{2} + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \widetilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \widetilde{\alpha}_0^\mu)\sigma + \dots \quad (3.1.1)$$

The string momentum p^μ can be determined in terms of the coefficient of τ.

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \widetilde{\alpha}_0^\mu)$$

Compactifying x^9 onto a circle of radius R quantised the momentum and the scalar field is no longer single valued in σ, making the field dependent on two integers; n and w.

$$p^9 = \frac{n}{R} \quad , \quad X^9(\sigma + 2w\pi, \tau) = X^9(\sigma, \tau) + 2\pi w\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \widetilde{\alpha}_0^\mu)$$

Expressing the momentum in terms of the zero modes provides a pair of simultaneous equations dependent on the two integers.

$$\begin{aligned} \frac{2n}{R}\sqrt{\frac{\alpha'}{2}} &= \alpha_0^9 + \widetilde{\alpha}_0^9 \qquad & \alpha_0^9 &= \left(\frac{n}{R} + \frac{wR}{\alpha'}\right) \\ wR\sqrt{\frac{2}{\alpha'}} &= \alpha_0^9 - \widetilde{\alpha}_0^9 \qquad & \widetilde{\alpha}_0^9 &= \left(\frac{n}{R} - \frac{wR}{\alpha'}\right) \end{aligned} \tag{3.1.2}$$

The interpretation of w is that it is the string winding number. An open string can wrap around a circular dimension but can be smoothly shrunk down to a length much smaller than R. This is not the case for closed strings, they can be viewed as open strings which have circumnavigated the circular dimension, before joining their ends and w is the number of times it has wrapped the dimension. $w < 0$ can be viewed in terms of an orientated string wrapped in the opposite direction to the $w > 0$ cases. As with any other relativistic theory the mass-energy formula of the string can be written in terms of the momentum by $p_\mu p^\mu = -M^2$ and the contribution due to the x^9 direction is of primary interest, all other terms are independent of n and w.

$$M^2 = \frac{n^2}{R^2} + \frac{w^2R^2}{(\alpha')^2} + \ldots \equiv M^2(n, w, R)$$

This mass formula has a symmetry between the two integers provided we also change the circumference of the circle.

$$M^2(n, w, R) = M^2\left(w, n, \frac{\alpha'}{R}\right)$$

We can define a T duality transformation $\mathbf{T}_9$ in the circular x^9 direction of radius R_9 in terms of these exchanges.

$$\mathbf{T}_9 \quad : \quad n_9 \leftrightarrow w_9 \quad , \quad R_9 \leftrightarrow \frac{\alpha'}{R_9} \tag{3.1.3}$$

There are additional implications of these transformations, as can be seen by noting that the zero modes of the field depend on the momentum and winding numbers.

$$\mathbf{T}_9 \quad : \quad \alpha_0^9 \to \alpha_0^9 \quad , \quad \widetilde{\alpha}_0^9 \to -\widetilde{\alpha}_0^9$$

The scalar field as a whole has this sign change in the right moving modes, the T duality transformation can be viewed as a parity operator in the those modes.

$$X^9(z,\bar{z}) = X^9(z) + X^9(\bar{z}) \to X^9(z) - X^9(\bar{z})$$

If the string theory is a superconformal one then in order to have invariance in such terms as $X^\mu \psi_\mu$ under a T duality in x^9 the change in sign in the bosonic term induces the same change in sign in the fermion fields. This parity-like change in sign alters the chirality of the theory. If the left and right moving modes have equal chirality before T duality in a single direction then afterwards they have different chiralities and vice versa. This difference in chirality is the distinguishing features in the boundary conditions of Type II definitions and so T duality exchanges Type IIA and Type IIB.

This can be further justified by the brane content of a Type II theory. If T duality exchanges Type IIA and Type IIB then it must alter the dimensional of the branes in those theories and to confirm this we consider the definition of the branes; they are the space-time regions on which an open string's end may end on and so the end points obey a set of Neumann or Dirchlet boundary conditions.

- Dirichlet condition : $\partial_\tau X^\mu = 0$
- Neumann condition : $\partial_\sigma X^\mu = 0$

Appling the transformations in (3.1.3) to the mode expansion of X^μ in (3.1.1) we can see the coefficients of τ and σ are exchanged and therefore T duality exchanges the boundary conditions of the end points. If the string could previously move in the x^μ compact dimension then its T dual cannot and vice versa and therefore a Dp-brane either increases or decreases in dimension by one. This is precisely the relationship required if T duality exchanges the brane content of the Type II theories. Though we have outlined the derivation of T duality in the perturbative regime of string theory this symmetry holds for all orders and non-perturbatively.

3.1.2 Background Fields : R-R Sector

The effective theory is defined in terms of background fields and in order to understand how T duality might affect the effective theory we address how the ten dimensional stringy fields behave under T duality. The simplest case is that of the R-R fluxes in either theory as we already have a geometric interpretation of how D-branes are affected by T duality and the R-R fluxes reside on these branes and to illustrate this we consider the Type IIA 1-form C_μ under a T duality in the x^9 direction. This singles out the C_9 component of the Type IIA field and it becomes the Type IIB 0-form C_0, while the remaining components of C_μ can be regarded as being the $C_{\mu 9}$ components of the Type IIB 2-form C_2. The inclusion of non-zero NS-NS background fields alters this relationship slightly but the general transformations are known as the Buscher Rules. We shall only be considering a simplified case, following the notation of [32] in both flux sectors.

$$\widetilde{F}_{\mu_1 \ldots \mu_{n-1} 9} = F_{\mu_1 \ldots \mu_{n-1}} \quad , \quad \widetilde{F}_{\mu_1 \ldots \mu_{n-1} \mu_n} = F_{\mu_1 \ldots \mu_n 9} \tag{3.1.4}$$

3.1.3 Background Fields : NS-NS Sector

The R-R sectors of the two Type II theories are different and we have seen how they are exchanged by a single T duality. The NS-NS sector is common between the two theories and therefore T dualities do not change the generic structure of these fields. These fields arise from the $\mathbf{8}_v \times \mathbf{8}_v$ SO(8) representations of the string polarisation modes and as a result the transformation rules for these fields can be written in a succinct manner in terms of $\xi_\mu \leftrightarrow \mathbf{8}_v$.

$$\begin{array}{ccccccccccc} \mathbf{8}_v \times \mathbf{8}_v & = & (\mathbf{35}+\mathbf{1}) & + & \mathbf{28} & = & \mathbf{35} & + & \mathbf{28} & + & \mathbf{1} \\ \xi_\mu \xi_\nu & = & g_{\mu\nu} & + & B_{\mu\nu} & = & G_{\mu\nu} & + & B_{\mu\nu} & + & \frac{1}{D}\delta_{\mu\nu}\mathrm{Tr}(g) \end{array}$$

The two rank two tensors define a traceless tensor $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$, which can be decomposed into G and B by their symmetric and antisymmetric index structure. We take the circular directions to be x^i for $i \in \{1, \ldots, n\}$, all of which we T dualise, the x^α to be the non-compact directions and the new fields are obtained from $\widetilde{E}$ by the same decomposition as used for E.

$$\widetilde{E}_{ij} = E^{ij} \quad , \quad \widetilde{E}_{\alpha j} = E_{\alpha k}E^{kj} \quad , \quad \widetilde{E}_{\alpha\beta} = E_{\alpha\beta} - E_{\alpha i}E^{ij}E_{j\beta} \qquad (3.1.5)$$

The dilaton changes under T duality and it too is dependent on E.

$$e^{2\widetilde{\Phi}} = e^{2\Phi}\det(E^{ij})$$

These transformations form a T duality symmetry group $\mathrm{O}(6,6)$ whose existence can be seen by reformulating the mass formula. In this we follow [21] and we take our indices to range over the compact directions $M, N = 1, \ldots, 6$. The winding modes around X^M are W^M and the momentum modes are K_M. The metric G and 2-form B define a 12×12 matrix $\mathcal{G}$.

$$\mathcal{G}^{-1} = \begin{pmatrix} 2(G - BG^{-1}B) & BG^{-1} \\ -G^{-1}B & \frac{1}{2}G^{-1} \end{pmatrix} \quad , \quad \mathcal{G} = \begin{pmatrix} \frac{1}{2}G^{-1} & -G^{-1}B \\ BG^{-1} & 2(G - BG^{-1}B) \end{pmatrix}$$

The matrix $\mathcal{G}^{-1}$ acts as the quadratic form for the mass formula of M_0.

$$\frac{1}{2}M_0^2 = \begin{pmatrix} W & K \end{pmatrix} \mathcal{G}^{-1} \begin{pmatrix} W \\ K \end{pmatrix}$$

This expression is invariant under two kinds of transformations, which form the generators of $\mathrm{O}(6,6,\mathbb{Z})$. The first is the generalisation of the $R \leftrightarrow \frac{1}{R}$, which is known to exchange the winding and momentum modes and the corresponding transformation on $\mathcal{G}$ is easily deduced.

$$W^M \leftrightarrow K_M \quad , \quad \mathcal{G} \leftrightarrow \mathcal{G}^{-1}$$

This is then extended by discrete shifts in the 2-form, which induce shifts in the momentum.

$$B_{MN} \to B_{MN} + \frac{1}{2}b_{MN} \quad , \quad W^M \to W^M \quad , \quad K_M \to K_M + b_{MN}W^N \tag{3.1.6}$$

These can be summarised in terms of $\mathrm{O}(6,6,\mathbb{Z})$ elements, which we denote generically by A.

$$\mathcal{G} \to A\,\mathcal{G}\,A^{\top} \quad , \quad \begin{pmatrix} W \\ K \end{pmatrix} \to A \begin{pmatrix} W \\ K \end{pmatrix}$$

This is the maximal symmetry group of the momentum and winding modes due to the requirement of level matching, which requires $K_M W^M$ be invariant. The $\mathrm{O}(6,6,\mathbb{Z})$ element A can be formed by two kinds of generators.

$$\text{inversion} \quad : \quad A = \begin{pmatrix} 0 & \mathbb{I}_6 \\ \mathbb{I}_6 & 0 \end{pmatrix} \qquad \text{shift} \quad : \quad A = \begin{pmatrix} \mathbb{I}_6 & 0 \\ b_{MN} & \mathbb{I}_6 \end{pmatrix}$$

3.2 Mirror Symmetry

In constructing T duality we assumed that the compact dimensions were circular, making the internal space toric. Since orbifolds are constructed from tori the application of T duality to them is straightforward but this is not so clear in the case of Calabi-Yaus as they are not generally toric. However, there is a conjectured symmetry which generalisas T duality to a larger class of spaces, including Calabi-Yaus, known as mirror symmetry. A technical of mirror symmetry is given in Refs. [13, 70] but we will follow the less technical approach of considering moduli symmetries, as outlined in Ref. [21].

T duality exchanges a Type IIA theory defined on $\mathcal{M}_1$ for a Type IIB theory defined on $\mathcal{M}_2$, where $\mathcal{M}_i$ are toroidal. Mirror symmetry is a conjectured extension of this such that $\mathcal{M}_i$ are not required to be toroidal. A number of important statements about the properties of the $\mathcal{M}_i$ can be made by considering the field content of the two Type II theories when compactified on a generic Calabi-Yau $\mathcal{M}$ with Hodge numbers $h^{p,q}$, as given in [21]. The massless field contents of each theory are given in Table 3.1 where the coordinates for the ten dimensions of $M_4 \times \mathcal{M}$ are $(x^a, z^\rho, \bar{z}^{\bar{\sigma}})$ and other than for the gravity multiplet fermions are not stated. Each Type II theory has a gravity multiplet, with a pair of gravitini for $\mathcal{N} = 2$, and a hypermultiplet. This hypermultiplet is the universal hypermultiplet which is responsible for the dilaton in each theory, as mentioned in Section 2.3.3, with the symmetry in the NS-NS and R-R fields evidence in the Type IIB case, a point we will return to shortly. The remaining fields form different multiplets in each theory, with the $h^{2,1}$ metric components of pure degree belonging to hypermultiplets in Type IIA but vector multiplets in Type IIB. Conversely, those metric components of mixed degree belong to vector multiplets in Type IIB but hypermultiplets in Type IIB. Overall there are $2h^{1,1} + 4(h^{2,1} + 1)$ massless scalars in Type IIA yet $2h^{2,1} + 4(h^{1,1} + 1)$ in Type IIB, again illustrating the $h^{1,1} \leftrightarrow h^{2,1}$ symmetry between the theories. Therefore, if Type IIA is compactified on $\mathcal{M}$ and Type IIB on $\mathcal{W}$ we have a relationship between the cohomologies of each compact space.

$$H^{p,q}(\mathcal{M}) = H^{3-p,q}(\mathcal{W})$$

Therefore, except in very special cases, Type IIA and Type IIB compactified string theories can only be dual to one another if they are defined on different spaces, as seen by obtaining the Hodge numbers of $\mathcal{W}$ from those of $\mathcal{M}$.

$$h^{1,1}(\mathcal{M}) = h^{2,1}(\mathcal{W}) \quad , \quad h^{2,1}(\mathcal{M}) = h^{1,1}(\mathcal{W})$$

With the supermultiplet field content defining the moduli of each theory by their scalar components we have a similar relationship between the moduli of each theory as determined for their cohomologies.

$$\mathcal{M}_{\mathcal{T}}(\mathcal{M}) = \mathcal{M}_{\mathcal{U}}(\mathcal{W}) \quad , \quad \mathcal{M}_{\mathcal{U}}(\mathcal{M}) = \mathcal{M}_{\mathcal{T}}(\mathcal{W}) \tag{3.2.1}$$

These are necessary conditions for the Type II compactified theories to be dual to one another but they are not sufficient. In the same way that T duality provides a bijection between the structures of Type IIA and Type IIB on different tori Type IIA on $\mathcal{M}$ is dual to Type IIB on $\mathcal{W}$ if their field contents are isomorphic to one another. It is important to note that because of the relationship (3.2.1) between moduli spaces the mirror dual theories will label their moduli in different manners. This was not seen in our review of T duality but is none-the-less also seen in T duality because on toroidal spaces mirror symmetry is equivalent to the combination of three distinct T dualities [41]. This is seen by noting how the orientifold action works in Table 2.1, where the O-planes of the different σ cannot be related by a single T duality due to their dimensionality. It has been studied explicitly for the

Type IIA			Type IIB		
#	Multiplet	Fields	#	Multiplet	Fields
1	gravity	$G_{ab}, \Psi_a, \widetilde{\Psi}_a, C_a$	1	gravity	$G_{ab}, \Psi_a, \widetilde{\Psi}_a, C_{a\tau\rho\sigma}$
$h^{1,1}$	vector	$C_{a\rho\bar{\sigma}}, G_{\rho\bar{\sigma}}, B_{\rho\bar{\sigma}}$	$h^{2,1}$	vector	$C_{a\tau\rho\bar{\sigma}}, G_{\rho\sigma}$
$h^{2,1}$	hyper	$C_{\tau\rho\bar{\sigma}}, G_{\rho\sigma}$	$h^{1,1}$	hyper	$C_{ab\rho\bar{\sigma}}, G_{\rho\bar{\sigma}}, B_{\rho\bar{\sigma}}, C_{\rho\bar{\sigma}}$
1	hyper	$\Phi, B_{ab}, C_{\tau\rho\sigma}$	1	hyper	$\Phi, B_{ab}, C_0, C_{ab}$

Table 3.1: Type II massless supermultiplets on a Calabi-Yau.

$\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold [10, 60, 61] and is a point we shall return to later. It is important to distinguish between two different but similar conjectures.

- Type IIA on a Calabi-Yau $\mathcal{M}$ is mirror dual to Type IIB on $\mathcal{W}$.
- Type IIB on a Calabi-Yau $\mathcal{W}$ is mirror dual to Type IIA on $\mathcal{M}$.

This difference arises from the the properties of Calabi-Yaus in terms of their Hodge numbers, $h^{1,1} \geq 1$ and $h^{2,1} \geq 0$. If $h^{2,1}(\mathcal{M}) = 0$ then $h^{1,1}(\mathcal{W}) = 0$ and thus $\mathcal{W}$ cannot be Calabi-Yau. As a result the conjecture is in reference to a slightly larger space of manifolds, tautologically defined as Calabi-Yaus and their mirror duals, but this is a technicality we will not address in any further detail.

In order to simplify our algebraic notation when comparing Type IIA theories on $\mathcal{M}$ to Type IIB theories on $\mathcal{W}$ we will only ever make reference to the Hodge numbers of $\mathcal{M}$, $h^{p,q} \equiv h^{p,q}(\mathcal{M})$. The Type IIB complex structure moduli space of $\mathcal{W}$ therefore has $h^{1,1}$ dimensions and moduli $\widetilde{\mathcal{U}}_A$, while the dimension of the Type IIB Kähler moduli space of $\mathcal{W}$ is $h^{2,1}$, with indices I, J ranging over 0 to $h^{2,1}$. This labelling convention is summarised in Table 3.2. It is important to note that we cannot automatically make the assumption that $\mathcal{U}_I = \widetilde{\mathcal{T}}_I$ as the relationship between the moduli of $\mathcal{M}$ and $\mathcal{W}$ depends on the specific mirror symmetry action.

If two effective theories compactified on $\mathcal{M}$ and $\mathcal{W}$ are mirror dual to one another they require equivalent superpotentials. In general this is not simply the relabelling the moduli in the manner of Table 3.2 because of quantum corrections. A superpotential on $\mathcal{M}$ linear in $\mathcal{T}$ need not map into a superpotential on $\mathcal{W}$ linear in $\widetilde{\mathcal{U}}$ unless the background is exact and needs no corrections, such as toroidal orbifolds. Although this will mean we can't use moduli dependency to compare two superpotentials defined on $\mathcal{M}$ and $\mathcal{W}$ it does not alter the fact that mirror dual superpotentials should have the same number of independent fluxes. This allows us to compare superpotentials on $\mathcal{M}$ and $\mathcal{W}$ without having to give too much attention to the explicit stringy origins of the individual fluxes themselves.

We will see that due to the manner in which the superpotential is dependent on the holomorphic forms Ω and $\mho$ it is possible to construct the most general superpotential without having to necessarily know the origin of all contributions. As such, the existence or not of particular moduli terms will be used as a guide in determining the structure of the induced fluxes if not their string theoretic origins. An important contribution to that approach is the fact that mirror symmetry exchanges the holomorphic forms $\Omega \leftrightarrow \mho$ as well as their dilaton complexified extensions $\Omega_c^{(\prime)} \leftrightarrow \mho_c^{(\prime)}$ [78, 79, 74, 64].

Type IIA on $\mathcal{M}$		Type IIB on $\mathcal{W}$
$\mathcal{U}_I\,,\mathcal{U}^J$	$I,J=0,\cdots,h^{2,1}$	$\widetilde{\mathcal{T}}_I\,,\widetilde{\mathcal{T}}^J$
$\mathcal{T}_A\,,\mathcal{T}^B$	$A,B=0,\cdots,h^{1,1}$	$\widetilde{\mathcal{U}}_A\,,\widetilde{\mathcal{U}}^B$

Table 3.2: The moduli of mirror pair $\mathcal{M}$ and $\mathcal{W}$ in terms of $h^{p,q}(\mathcal{M})$

3.3 S Duality

There is a second non-perturbative duality which arises in Type IIB string theory which we are able to examine independently of T duality, that of S duality. It is hinted at in Table 3.1, where the universal hypermultiplet of Type IIB contains a pair of scalars and a pair of 2-forms[1] and one member of each pair is associated to NS-NS fields and the other is associated to R-R fields. This symmetry is not unique to the compactified theory, it arises in the full ten dimensional supergravity action and is conjectured to be a symmetry of the full string theory.

3.3.1 Type IIB SL$(2,\mathbb{R})$ Invariance

It is not immediately clear from the Type IIB supergravity action there exists an SL$(2,\mathbb{R})$ symmetry in the theory. To make this symmetry manifest we must transform the Type IIB supergravity action into the Einstein Frame, where $(G_E)_{\mu\nu}=e^{-\frac{\Phi}{2}}G_{\mu\nu}$, and put certain fluxes into doublets.

$$M_{ij}=\frac{1}{\mathrm{Im}(S)}\begin{pmatrix}1 & -\mathrm{Re}(S)\\ -\mathrm{Re}(S) & |S|^2\end{pmatrix}\quad,\quad \mathsf{F}^i=\begin{pmatrix}F_3\\ H_3\end{pmatrix}$$

The rescaling of the metric decouples the dilaton from the Ricci scalar and in doing so has motivated the complexified dilaton definition given in (2.3.9).

[1]Technically when building the hypermultiplet B_{ab} and C_{ab} are regarded as scalars too but the salient point is that there is a pairing between NS-NS and R-R objects.

Substituting these definitions into the supergravity action and using $\mathcal{R}_E$ as the Ricci scalar associated to the new metric G_E we obtain a formulation which has the symmetry manifest.

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_E}\left(\mathcal{R}_E - \frac{\partial_\mu S\overline{\partial^\mu S}}{2(\mathrm{Im}(S))^2} - \frac{1}{2}\mathsf{F}^i\cdot M_{ij}\cdot\mathsf{F}^j - \frac{1}{4}|\widetilde{F}_5|^2\right) \\ &+ \frac{\epsilon_{ij}}{8\kappa_{10}^2}\int C_4\wedge\mathsf{F}^i\wedge\mathsf{F}^j \end{aligned} \tag{3.3.1}$$

With $\mathcal{R}_E$ independent of the dilaton all derivatives of S arise in the kinetic term. This kinetic term is of the same form as a two dimensional hyperbolic metric $\mathrm{d}s^2 \sim \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$, known to possess a modular invariance. This becomes a symmetry of the entire action if the F^i transform in such a way as to make the third term in the integral invariant, while the metric and 5-form are unchanged.

$$S \to \frac{aS+b}{cS+d} \quad , \quad \mathsf{F}\to L\cdot\mathsf{F} \quad \text{where} \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \tag{3.3.2}$$

This invariance in the supergravity action is not automatically an invariance in the string theory, we neglected the stringy contributions to the action when we took the supergravity $\alpha' \to 0$ limit and any quantisation requirements which follow from the fact the F^i are associated to charged objects. When we include such constraints the continuous symmetry is broken its maximal discrete subgroup, $\mathrm{SL}(2,\mathbb{Z})$.

3.3.2 Type IIB SL$(2,\mathbb{Z})$ Invariance

In the Type IIB superpotential there are two contributions, both of which are of the same rank and have the same number of coefficients. As a result it is possible to construct well defined linear combinations of the two flux multiplets. Both of these fluxes exist in the full ten dimensional supergravity action and they formed a doublet under the $\mathrm{SL}(2,\mathbb{R})$ transformations so we can consider the same in the superpotential. The continuous group can be seen to be broken to the discrete subgroup SL$(2,\mathbb{Z})$ by noting that the Dirac quantisation condition requires any flux formed by the redefinitions to be integers.

$$S \to \frac{aS+b}{cS+d} \quad , \quad \mathsf{F} \to L \cdot \mathsf{F} \quad \text{where} \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \quad (3.3.3)$$

This can be viewed in terms of stringy objects through the fact the 3-form fluxes couple to either F or D-strings. The F-string carries NS-NS charge with field strength H_3 and the D-string carries R-R charge with field strength F_3. Under a general SL$(2,\mathbb{Z})_S$ transformation linear combinations of these field strengths, pH_3+qF_3, are formed. The physical object this field strength couples to is the (p,q)-string, a bound state of p F-strings and q D-strings, though this interpretation is only strictly valid at weak or strong coupling where one of the two string types becomes massive compared to the other. For couplings which are neither weak or strong, $g_s \sim 1$, they form a single object which carries p lots of NS-NS charge and q lots of R-R charge. Their magnetic duals follow the same pattern, the D5 and NS5-branes form a bound state which is only viewable in terms of these constituents in the weak or strong coupling limit.

3.3.3 AdS/CFT Correspondence

The existence and behaviour of S duality in Type IIB string theory has a more well known formulation, via the use of the standard formulation of the AdS/CFT correspondence linking a gauge theory with a gravity theory.

$$\mathcal{N} = 4 \text{ SYM} \quad \leftrightarrow \quad \text{Type IIB in } \mathrm{AdS}_5 \times S^5$$

The correspondence is justified by equating structures from each side of the correspondence and thus there is an equivalent of the dilaton within the $\mathcal{N} = 4$ gauge theory. This is the complexified gauge coupling τ, written in terms of the standard gauge coupling g and θ, which parameterised a CP violating term in the Lagrangian, $\mathcal{L}_\theta$.

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \qquad \mathcal{L}_\theta = -\frac{\theta g^2}{32\pi^2}\mathrm{Tr}\Big(F_{\mu\nu}(\star F^{\mu\nu})\Big)$$

The parameter θ is dual to the Type IIB scalar C_0. C_0 arises from instanton charges and the same is true of θ, it is a topological quantity associated to the non-abelian gauge group of $F_{\mu\nu}$.

3.4 U Duality

There is a non-trivial extension of these dualities known as U duality which can be expressed entirely in terms of T and S duality transformations but represent a space of transformations G_U which are more than the disjoint sum of T duality transformations $G_T = \mathrm{O}(6,6,\mathbb{Z})$ and S duality transformations $G_S = \mathrm{SL}(2,\mathbb{Z})_S$. For compactification of Type II theories on T^6 the discrete group is denoted as $E_7(\mathbb{Z})$ [21].

$$G_T \times G_S = \mathrm{O}(6,6,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})_S \subset G_U \equiv E_7(\mathbb{Z})$$

The existence of an extended symmetry can be seen by noting that if T duality reduces the two Type II theories to being equivalent to one another on tori then the same structures should exist in each Type II theory. However, the fact Type IIA lacks the S duality symmetry appears to violate this equivalence. Never the less, modular transformations on the dilaton in Type IIA can be obtained by first T dualising into Type IIB, performing an S duality transformations and then applying the same T duality transformation to obtain the original Type IIA construction [21, 32]. Strongly coupled Type IIA theory can be viewed in a geometric manner as M theory compactified on S^1, which combines the $\mathrm{SL}(2,\mathbb{Z})_S$ symmetry of the dilaton into the M theory internal T^7 symmetry. The modular symmetry group of T^n is $\mathrm{SL}(n,\mathbb{Z})$ and as a result $\mathrm{SL}(7,\mathbb{Z}) \subset E_7(\mathbb{Z})$. U duality is then obtained by the M theory modular group and the Type II T duality group knitting together to give a symmetry group which provides both kinds of symmetry and is denoted as $E_7(\mathbb{Z})$. Its precise definition in terms of maximal non-compact subgroups of the continuous groups is given in Refs. [21, 32].

In terms of the superpotential defined effective theory the resultant transformations on the fluxes of Type IIA go beyond the fluxes forming doublet pairs of an NS-NS flux and an R-R flux, else it would be self S-dual naturally. To examine the specific effect this has on Type IIA fluxes we are first required to construct a T duality invariant theory, so that the relationship between Type IIA and Type IIB are known in terms of the fluxes, and then extend the Type IIB construction to include S duality. Since Type IIB is self S-dual the R-R sector has all the same structures and properties as the NS-NS sector but this is not true in the Type IIA case. Modular transformations in Type IIA result in highly non-trivial transformations in the Type IIA fluxes, there is no way to consider T duality seperately from $\mathrm{SL}(2,\mathbb{Z})_S$ transformations. The explicit difference between the construction of the two U duality invariant R-R sectors will be obtained in the next chapter.

3.5 The Web of Dualities

We have seen how Type IIA and Type IIB can be made dual to one another by compactifying them onto toroidal or Calabi-Yau internal spaces. We have also seen how Type IIB possesses a self duality. However, this is not the totality of links between different quantum field theories involving extended objects. The supersymmetric open string theory known as Type I can be constructed from Type IIB through the use of the O9-plane generating orientifolding discussed in the previous section. Such a theory includes open strings, whose end points transform under the SO(32) gauge group. This is not as general a duality as that between Type IIA and Type IIB and this follows by considering the amount of supersymmetry possible in each theory. Type I possesses $\mathcal{N} = 1$ supersymmetry, its name following the same convention as the Type II theories, and thus upon compactification to the same class of spaces as Type II theories it will possess half the amount of supersymmetry. This is resolved through the use of the orientifold projection in Type IIB and space-filling O9-planes and D9-branes give rise to the possibility of open strings which are able to move through all of space-time.

The Type I theory does not possess the self S duality of Type IIB but it does possess an S dual, that of a heterotic string theory with the same SO(32) gauge group, the HO string. This is not restricted to those Type I constructions obtained from some compactified Type IIB model, Type I and HO are S dual on any kind of space-time. However, when compactified on toroidal spaces T duality transformations can be applied to HO and it is transformed into the other heterotic string theory, whose gauge group is $E_8 \times E_8$, the HE string.

Despite their differences, Type IIA and HE can both be viewed as ten dimensional 'stringy' limits of an eleven dimensional theory built of two dimensional membranes, M theory. We have previously seen how the low energy limit of M theory in eleven dimensions, 11d supergravity, can be dimensionally reduced on S^1 to give the Type IIA supergravity action and the same process on the orbifold $S^1/\mathbb{Z}_2$ results in HE.

Further relationships between M and string theories exist when we consider the gravity/gauge duality of the AdS/CFT correspondence previously discussed in the context of Type IIB. A stack of N coincident D3-branes, carrying N lots of D3-brane charge or flux, leads to a background which gives $\text{AdS}_5 \times S^5$ space-time in the large N limit. A gauge theory is definable on the four dimensional boundary of the AdS_5 space and in the large N limit this is conformal. For D3-branes this is the well studied $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The familiarity of this gauge theory and the fact it is defined in four dimensional space-time makes it of central interest to the investigation of phenomenological strongly coupled gauge theories. More physically viable models are constructed by breaking the symmetry of the string theory construction, such as by the insertion of D7-branes which fed through into the gauge theory as a reduction in the supersymmetry. This is not unique to Type IIB but since in Type IIA p must be even it is not possible to have an AdS space whose boundary is four dimensional. However, D4-brane Type IIA constructions have received attention [42], with the extra dimension being compactified. This is not restricted to string theories, stacking large quantities of M2- or M5-branes in M theory leads to space-times which tend to $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ respectively [21]. All of these dualities and relations are summarised in Figure 3.1.

Summary

In this chapter we have reviewed a number of dualities which are inherent to Type II string theories. T duality links the two Type II string theories when compactified on toriodal internal spaces and as many T dualities can be applied as there are toroidal directions. Mirror symmetry generalised this to Calabi-Yau internal spaces, where the non-toroidal nature of the internal space does not make the same T duality transformations clear. Mirror symmetry is such that when the internal space is a toroidal one it becomes equivalent to simultaneous application of three distinct T dualities. On the level of the ten dimensional action Type IIB possesses a symmetry between its NS-NS sector and R-R sector in the form of the weak-strong S duality but is not shared with Type IIA due to their differing brane content. However, since T duality links Type IIA and Type IIB if both T and S dualities are used they combine to provide Type IIA with the same $\mathrm{SL}(2,\mathbb{Z})$ modular invariance of Type IIB, resulting in U duality. At present we have only considered these dualities on the level of the ten dimensional action or mode expansions. If the effective theories obtained by compactification are to have the same dualities then the superpotential must be invariant in the same way the original action is and it is the implications of such a requirement we consider next.

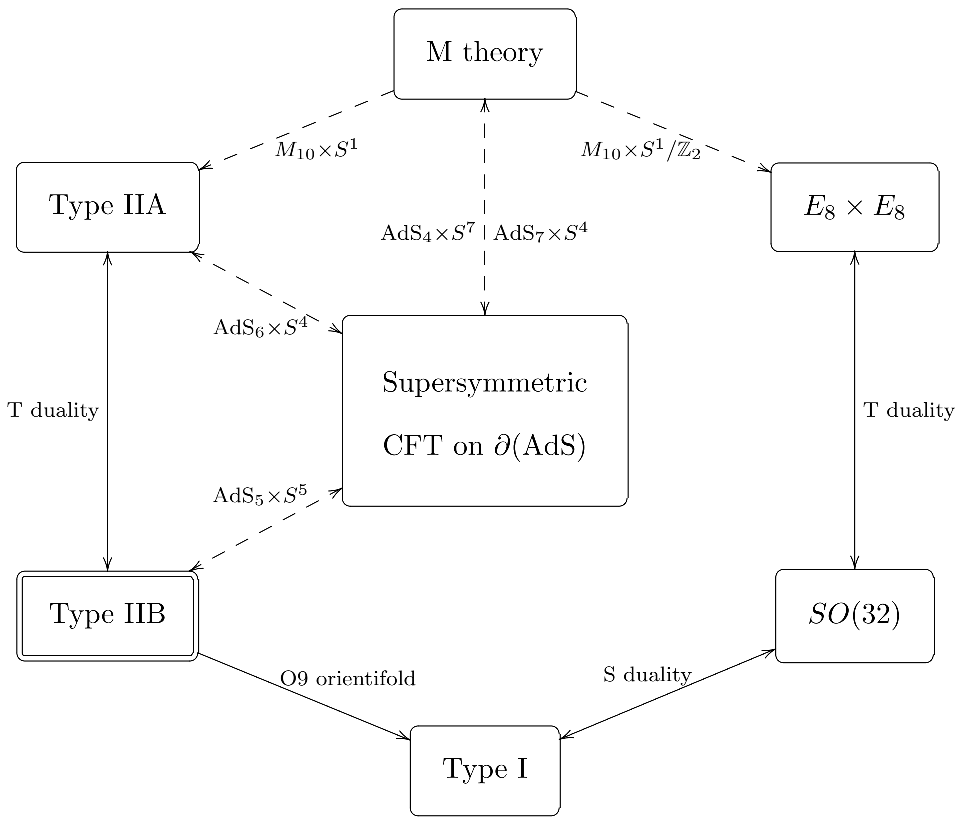

Figure 3.1: String and M theory dualities. Dashed lines require a particular spacetime topology. T duality requires compact spaces. Not all Type II AdS/CFT correspondences stated.

Chapter 4

Duality Induced Fluxes

The four dimensional effective theory is invariant under a duality if the Kähler functional $\mathcal{G} = K + \ln |W|^2$ is invariant, which occurs if the effect on the superpotential can be reduced to a gauge transformation $W \to e^{-f} W$. In the context of mirror symmetry the two flux sectors can be treated separately because the symmetry does not mix them. The approach we will take to obtaining a duality invariant Type II superpotential is to consider the set of fluxes known to exist in Type IIA NS-NS constructions and using arguments of symmetry obtain additional fluxes within that sector. These induce fluxes in the Type IIB NS-NS sector on the grounds of requiring the superpotentials to be the same, up to moduli relabellings. Given a full NS-NS sector in Type IIB S duality can then be used to induce the entire R-R sector of Type IIB and these finally induce the R-R sector of Type IIA by T duality. The methods used implicitly assume that $\mathcal{M}$ is a toroidal space and the fluxes have their components defined in terms of indices in the generalised frame bundle $\mathfrak{E} = \mathsf{E} \oplus \mathsf{E}^*$. This allows the simple application of T duality to the effective theory so that the form of induced fluxes can be summised. To make it clear when we are referring to parallelisable spaces we shall denote the vector space the fluxes belong to as $\Lambda^*(\mathsf{E}^*)$, rather than the more general $\Delta^*(\mathsf{E}^*) \subset \Omega^*(\mathsf{E}^*)$ of non-parallelisable spaces. Once we have obtained the general superpotential of each Type II theory we will be able to convert it into a formalism which can be applied to Calabi-Yaus and the action of mirror symmetry is clearer. Work in this chapter is found in [9].

4.1 The Type IIA NS-NS Flux Sector

We begin with the Type IIA NS-NS flux sector due to the manner in which the Kähler moduli dependence arises, an issue which will be of importance in the Type IIB case. As will shortly be demonstrated, the Type IIA superpotential can be dependent on all three moduli types by including only those NS-NS fluxes with geometric interpretations. It is this ability to provide a non-flat potential in all moduli which motivated considerable work in Type IIA constructions and phenomenology [43, 44, 45, 46, 47, 48]. However, a series of no-go theorems exist in Type IIA which preclude the construction of phenomenologically viable vacua given only such fluxes. Fortunately the Type IIA fluxes which arise beyond those obtained by compactification provide motivation for how T duality invariance between Type IIA and Type IIB can be acheived and these no-go theorems evaded. It is this we shall outline and examine now. We follow the construction of Type IIA orientifold theories in terms of $\mathfrak{E}$ flux components as done for the $\mathbb{Z}_4$ [56] and $\mathbb{Z}_2 \times \mathbb{Z}_2$ [60] orientifolds.

4.1.1 T Duality Induced Parallelisable Fluxes

The 3-form flux $\mathcal{H}_3$ of (2.5.3) is obtained by compactification of the full ten dimensional string action but the existence of an additional flux multiplet can be obtained by recalling the definition of the frame basis η^p and allowing N to be dependent on the coordinates of $\mathcal{M}$.

$$\mathrm{d}\eta^m = \mathrm{d}\Big(N^m_n(X^p)\mathrm{d}X^n\Big) = \left(\frac{\partial N^m_n}{\partial X^r}(N^{-1})^r_q(N^{-1})^n_p\right)\eta^{qp}$$

By allowing $\partial_{X^p}N^m_n \neq 0$ the 1-forms define a 'twisted' internal space. Given the index structure of the $\mathrm{d}\eta^m$ expression we can associate this twisting with a new flux f whose components are f^m_{pq}.

$$\frac{1}{2!}f^m_{pq} \equiv \frac{\partial N^m_n}{\partial X^r}(N^{-1})^r_{[q}(N^{-1})^n_{p]} \quad \Rightarrow \quad \mathrm{d}\eta^m = \frac{1}{2!}f^m_{pq}\eta^{qp}$$

This definition in terms of a geometric property, the exterior derivative's effect on tangent forms, results in f being known as a metric or geometric flux and belongs to the NS-NS sector. The existence of such fluxes has been known independently of T duality [49, 50] and contribute to the Type IIA superpotential in a natural way [44]. Given that the Type IIA NS-NS superpotential's integrand can be generally written as $\langle\, \Omega_c, G_3 \,\rangle$, where $G_3 \in \Lambda^3(\mathsf{E}^*)$, it follows that f can only contribute to the superpotential if a 3-form can be constructed from its components. The natural approach is to lower the raised subscript of f^m_{pq} and the metric with regards to such indices is the Kähler form $\mathcal{J}$.

$$\Lambda^3(\mathsf{E}^*) \ni (f\cdot\mathcal{J}) = \frac{1}{3!}(f\cdot\mathcal{J})_{pqr}\eta^{pqr} \quad \text{where} \quad (f\cdot\mathcal{J})_{pqr} \equiv f^m_{[pq}\mathcal{J}_{r]m}$$

The components of $\mathcal{J}$ are Kähler moduli dependent and so $f\cdot\mathcal{J}$ is linear in $\mathcal{T}$, which allows us to write its generic contribution to the superpotential.

$$W \ni \int_{\mathcal{M}} \langle\, \Omega_c, (f\cdot\mathcal{J}) \,\rangle \equiv \mathcal{T}_a\,\mathcal{P}^{(a)}_f(S,\mathcal{U})$$

The 3-form G_3 now has two contributions, $\mathcal{H}_3$ and f, and it can be written in a symmetric manner if we explicitly introduce $\mathcal{T}_0$ dependence to the $\mathcal{H}_3$

term by contracting[1] it with the $\mathcal{T}_0$ dependent $\mathcal{J}^{(0)} = \mathcal{T}_0\omega_0$. The result of this is that G_3 can be factorised into two parts; one of which is moduli independent and the other flux independent.

$$G_3 = \mathcal{H}_3 \cdot \mathcal{J}^{(0)} + f \cdot \mathcal{J}^{(1)} = \Big(\mathcal{H}_3 \cdot + f\, \cdot\Big)\Big(\mathcal{J}^{(0)} + \mathcal{J}^{(1)}\Big)$$

We can add additional terms to this and not alter the superpotential if those terms do not belong to the same $\Lambda^p(\mathsf{E}^*)$ as $\mathcal{J}^{(0)}$ or $\mathcal{J}^{(1)}$. A natural choice for such terms is seen by recalling that $\mho = e^{\mathcal{J}} = \sum \mathcal{J}^{(n)}$.

$$G_3 = \Big(\mathcal{H}_3 \cdot + f\, \cdot\Big)\Big(\mathcal{J}^{(0)} + \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)}\Big) = \Big(\mathcal{H}_3 \cdot + f\, \cdot\Big)(\mho)$$

These two terms allow G_3 to be expressed as a function of the Kähler moduli holomorphic form and if $\mathcal{T}^B$ contributions are to arise in the Type IIA NS-NS superpotential then the flux dependent factor of G_3 is not yet complete. The inclusion of extra terms can be further motivated by viewing the flux dependent factor of G_3 as a differential operator, a viewpoint already justified by the definition of f.

$$\mathcal{D} \equiv \mathrm{d} + \mathcal{H}_3 \sim \mathcal{H}_3 \cdot + f \cdot$$

This covariant derivative can have its action on elements of $\Lambda^*(\mathsf{E}^*)$ written in terms of the componenets of its constituent fluxes by expanding in terms of η^m and ι_n bases.

$$\mathcal{D} = \mathcal{H}_3 \cdot + f \cdot = \frac{1}{3!}\mathcal{H}_{mpq}\eta^{mpq} + \frac{1}{2!} f^m_{pq}\eta^{pq}\iota_m$$

Since $\mathcal{D}$ is an extension of the exterior derivative it is required to be nilpotent, $\mathrm{d}^2 \to \mathcal{D}^2 = 0$, and in writing this in terms of the components we observe

[1] Since $\mathcal{H}_3$ is already a 3-form in this case contraction is simply multiplication but we refer to it as such so as to fit in with other possible terms.

that the components satisfy constraints similar in form to those of a Lie algebra.

$$\mathcal{D}^2 = 0 \quad \Leftrightarrow \quad f^m_{[pq} f^s_{r]m} = 0 = \mathcal{H}_{m[pq} f^m_{rs]} = 0$$

The Lie algebra interpretation is obtained from the gauge sector of the ten dimensional string theory [51]. Upon compactification the theory has diffeomorphism generators Z_m for the metric G and gauge symmetry generators X^n for the 2-form potential B. In the absense of fluxes these generators form a $U(1)^{12}$ abelian algebra but this is made non-abelian by the inclusion of fluxes, tautologically so in the case of f. The frame bundle definition of the η^m leads to an explicit construction of vector fields Z_m and in the absense of $\mathcal{H}_3$ fluxes these form a six dimensional algebra.

$$\eta^m = N^m_n \mathrm{d}x^n \quad \Rightarrow \quad Z_m = (N^{-1})^n_m \partial_n \quad \Rightarrow \quad \big[Z_m, Z_n\big] = -f^p_{mn} Z_p \tag{4.1.1}$$

This relationship between the η^m and Z_m structures is the Cartan-Maurer equation. Including the gauge generators and turning on $\mathcal{H}_3$ extends this commutation relation into the twelve dimensional algebra, with Z_m no longer forming a subalgebra.

$$\begin{aligned} \big[\ Z_m\ ,\ Z_n\ \big] &= \mathcal{H}_{mnp}\ X^p \ -\ f^p_{mn}\ Z_p \\ \big[\ Z_m\ ,\ X^n\ \big] &= f^n_{mp}\ \ X^p \\ \big[\ X^m\ ,\ X^n\ \big] &= 0 \end{aligned} \tag{4.1.2}$$

In this formulation it is clear that additional terms can be included for a more general algebra [52]. This is further motivated by noting that the expansion of $\mathcal{D}$ in (4.1.1) includes both E^* and E elements and we would wish to reformulate them in terms of the generalised frame bundle $\mathfrak{E} = \mathsf{E} \oplus \mathsf{E}^*$.

$$\mathcal{H}_3 \in \Lambda^3(\mathsf{E}^*) \wedge \Lambda^0(\mathsf{E}) \quad , \quad f \in \Lambda^2(\mathsf{E}^*) \wedge \Lambda^1(\mathsf{E}) \tag{4.1.3}$$

Because of the fact their components are constant the fluxes belong to combinations of $\Lambda^3(\mathsf{E}^{(*)}) \subset \Omega^3(\mathsf{E}^{(*)})$. It is important to note that despite $\mathcal{H}_3$ acting more generally as $\mathcal{H}_3 : \Lambda^p(\mathsf{E}^*) \to \Lambda^{p+3}(\mathsf{E}^*)$ the fact $\mathcal{D}(\mho)$ couples to $\Omega \in \Lambda^3(\mathsf{E}^*)$ the only case of interest is the $p = 0$ one and this logic extends to f too. The inclusion of two further spaces makes the reformulation into the generalised frame basis particularly simply.

$$\Lambda^3(\mathsf{E} \oplus \mathsf{E}^*) = \bigoplus_{n=0}^{3} \Big(\Lambda^{3-n}(\mathsf{E}^*) \wedge \Lambda^n(\mathsf{E}) \Big) \tag{4.1.4}$$

This suggests that if it is possible to reformulate the effective theory in terms of $\mathfrak{E}$ we would expect two further sets of fluxes to exist. Two of the three gaps in the commutation relations of (4.1.2) have the same index structure, with two raised indices and one lowered. In the same way f appears twice these two gaps are filled by the same flux [52, 56].

$$\begin{aligned} [\, Z_m \,,\, Z_n \,] &= \mathcal{H}_{mnp}\, X^p - f^p_{mn}\, Z_p \\ [\, Z_m \,,\, X^n \,] &= f^n_{mp}\, X^p + Q^{np}_m\, Z_p \\ [\, X^m \,,\, X^n \,] &= Q^{mn}_p\, X^p - R^{mnp}\, Z_p \end{aligned} \tag{4.1.5}$$

Under $\mathbf{T}_a$, the T duality in the η^a direction, the algebra's generators are exchanged $X^a \leftrightarrow Z_a$. The resultant change in the index structure leads to a sequence of T duality induced fluxes starting from the NS-NS 3-form [52].

$$\mathcal{H}_{abc} \overset{\mathbf{T}_a}{\longleftrightarrow} f^a_{bc} \overset{\mathbf{T}_b}{\longleftrightarrow} Q^{ab}_c \overset{\mathbf{T}_c}{\longleftrightarrow} R^{abc} \tag{4.1.6}$$

These fluxes have not arisen by compactification or by twisting the E^* basis, they are non-geometric in nature [53, 54, 55]. The NS-NS Buscher rules of (3.1.5) for a T duality $\mathbf{T}_r$ allow for a decomposition of E_{pq} into the metric and 2-form if E_{pq} is independent of X^r. In cases where E is dependent on particular X^r the Buscher rules break down. Such an example on a three dimensional torus is explored in [54] in detail, one with $f^x_{yz} = N$.

$$\mathrm{d}s^2 = (\mathrm{d}x - Nz\mathrm{d}y)^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

The relation $(x, y, z) \sim (x+Ny, y, z+1)$ twists the space. Further T duality, in the y direction, gives a less pleasant metric.

$$\mathrm{d}s^2 = \frac{1}{1+N^2z^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2$$

There is no simple relation for $z \to z + 1$ now, the metric is not globally defined but provided the variation in z is confined to a small region this is not a problem and the metric is locally valid for Q_z^{xy}. T dualising $\mathbf{T}_z$ cannot be done under the Buscher rules and no $\mathrm{d}s^2$ can be defined, losing all notion of a geometry for the space [52]. The $\mathbf{T}_z$ image of Q_z^{xy} is R^{xyz} and so it, along with Q, is referred to as a non-geometric flux. We can revert back to the differential operator formalism by the stipulation that the Q and R contribute two terms to $\mathcal{D}$ such that the Bianchi constraints of $\mathcal{D}^2 = 0$ are equal to the Jacobi constraints of the gauge sector's algebra [56].

$$\begin{aligned} \mathcal{D} &= \mathcal{H}_3 \cdot + f \cdot + Q \cdot + R \cdot \\ &= \frac{1}{3!}\mathcal{H}_{mpq}\eta^{mpq} + \frac{1}{2!}f^m_{pq}\eta^{pq}\iota_m + \frac{1}{2!}Q^{mp}_q\eta^q\iota_{pm} + \frac{1}{3!}R^{mpq}\iota_{qpm} \end{aligned} \quad (4.1.7)$$

The factors of $p!$ are chosen to account for antisymmeterising indices and are such that they match the definitions of [56] when it comes to the fluxes acting on a general q-form A_q. This can be expressed in terms of the components of A_q, $A_q = \frac{1}{q!}A_{i_1\dots i_q}\eta^{i_1\dots i_q}$.

$$\begin{aligned}
0!\,(\mathcal{H}\cdot A)_{i_1\dots i_{q+3}} &= \begin{pmatrix} q+3 \\ 3 \end{pmatrix} \mathcal{H}_{[i_1 i_2 i_3}A_{i_4\dots i_{q+3}]} \\
1!\,(f\cdot A)_{i_1\dots i_{q+1}} &= \begin{pmatrix} q+1 \\ 2 \end{pmatrix} f^{j}_{[i_1 i_2}A_{|j|i_3\dots i_{q+1}]} \\
2!\,(Q\cdot A)_{i_1\dots i_{q-1}} &= \begin{pmatrix} q-1 \\ 1 \end{pmatrix} Q^{jk}_{[i_1}A_{|jk|i_2\dots i_{q-1}]} \\
3!\,(R\cdot A)_{i_1\dots i_{q-3}} &= \begin{pmatrix} q-3 \\ 0 \end{pmatrix} R^{jkl}A_{jkl[i_1\dots i_{q-3}]}
\end{aligned} \qquad (4.1.8)$$

The derivative equivalent of the T duality induced Lie algebra generator $X^a \leftrightarrow Z_a$ exchange is the exchange of the $\eta_a \in \mathsf{E}^*$ and $\iota_a \in \mathsf{E}$ basis elements. The fluxes are associated to different subspaces of $\Lambda^3(\mathfrak{E})$ in (4.1.4) and can be viewed in terms of the 'doubled geometry' [57, 58, 59] of $\mathfrak{E} = \mathsf{E} \oplus \mathsf{E}^*$. From this point of view T dualities alter which sections of the doubled frame bundle fibres the fluxes are associated to.

The two new terms contribute coefficients for $\mathcal{T}^B$ in the Type IIA NS-NS superpotential, which can be expressed in a simple manner.

$$W = \int_{\mathcal{M}} \langle\, \Omega_c, \mathcal{D}(\mho)\, \rangle \qquad (4.1.9)$$

To compress notation and to avoid confusion between the Type IIA fluxes and the Type IIB fluxes considered in later sections we relabel the terms of $\mathcal{D}$ in terms of flux dependent operators, $\mathcal{F}_m : \mathcal{J}^{(m)} \to \mathcal{F}_m \cdot \mathcal{J}^{(m)}$ where $\mathcal{F}_m \cdot \mathcal{J}^{(m)} \in \Lambda^3(\mathsf{E}^*)$.

$$\begin{aligned}
\mathcal{D} &= \mathcal{F}_0 + \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 \\
&= \frac{1}{3!}\mathcal{F}_{mpq}\eta^{mpq} + \frac{1}{2!}\mathcal{F}^{m}_{pq}\eta^{pq}\iota_m + \frac{1}{2!}\mathcal{F}^{mp}_{q}\eta^{q}\iota_{pm} + \frac{1}{3!}\mathcal{F}^{mpq}\iota_{pqm}
\end{aligned} \qquad (4.1.10)$$

We have dropped the subscripts in the $\mathcal{F}_n$ when considering components since their index structures are unambiguous.

4.1.2 T Duality Induced Generalised Fluxes

Constructing the fluxes in terms of $\Lambda^3(\mathfrak{E})$ is not always possible but due to the ease of applying T duality transformations it is convenient for deducing the existence of fluxes which do not descend from the ten dimensional action easily. Having motivated their existence we now wish to reformulate our methods so as to not require the $\Lambda^3(\mathfrak{E})$ notation. Much of the analysis is the same as the parallelisable case but with the relabelling $\Lambda^p(\mathsf{E}^*) \to \Delta^p(\mathsf{E}^*) \subset \Omega^p(\mathsf{E}^*)$ but there are a number of important differences which we shall examine.

If all of the complex structure coefficients in $\Omega_c \in \Delta^3(\mathsf{E}^*)$ are to contribute to the superpotential then potentially the $\mathcal{F}_n \cdot \mathcal{J}^{(n)}$ must have nonzero coefficients for any of the basis forms of $\Delta^3(\mathsf{E}^*)$. This provides us with a general action for the fluxes in terms of the light forms, as the $\mathcal{J}^{(n)}$ are expanded in the $\Delta^+(\mathsf{E}^*)$ basis.

$$\begin{array}{lll} \mathcal{F}_0 : \langle \omega_0 \rangle \to \langle \alpha_I, \beta^J \rangle & \quad & \mathcal{F}_3 : \langle \widetilde{\omega}^0 \rangle \to \langle \alpha_I, \beta^J \rangle \\ \mathcal{F}_1 : \langle \omega_a \rangle \to \langle \alpha_I, \beta^J \rangle & \quad & \mathcal{F}_2 : \langle \widetilde{\omega}^b \rangle \to \langle \alpha_I, \beta^J \rangle \end{array}$$

In this context we consider the fluxes simply as operators rather than defined in terms of $\mathfrak{E}$ elements. The components of the $\mathcal{F}_n$ defined in terms of the light form are defined by applying them to their associated $\Delta^{2n}(\mathsf{E}^*)$.

$$\begin{aligned} \mathcal{F}_0(\omega_0) &= (\mathcal{F}_0)_I \ \alpha_I - (\mathcal{F}_0)^J \ \beta^J \\ \mathcal{F}_1(\omega_a) &= (\mathcal{F}_1)_{(a)I} \ \alpha_I - (\mathcal{F}_1)_{(a)}{}^J \ \beta^J \\ \mathcal{F}_2(\widetilde{\omega}^b) &= (\mathcal{F}_2)^{(b)}{}_I \ \alpha_I - (\mathcal{F}_2)^{(a)J} \ \beta^J \\ \mathcal{F}_3(\widetilde{\omega}^0) &= (\mathcal{F}_3)_I \ \alpha_I - (\mathcal{F}_3)^J \ \beta^J \end{aligned} \tag{4.1.11}$$

There are $4(h^{2,1}+1)(h^{1,1}+1)$ fluxes that can contribute to the NS-NS sector's superpotential, illustrating the possibility of the symmetry between the moduli spaces by the $h^{2,1} \leftrightarrow h^{1,1}$ exchange. However this symmetry is not quite manifest. This can be seen by noting that the complex structure moduli indices take values in $\{0,\ldots,h^{2,1}\}$ while the Kähler moduli indices range only over $\{1,\ldots,h^{1,1}\}$ and we would therefore wish to reformulate our notation to have the Kähler moduli index vary over $\{0,1,\ldots,h^{1,1}\}$. This cannot be done simply by extending the Kähler indices to include the $A,B=0$ case due to the algebraic properties of the $\Delta^{+}(\mathsf{E}^{*})$ basis elements. This result is obtained by considering the action of $\mathcal{D}$ on $\Delta^{3}(\mathsf{E}^{*})$ elements, which we shall now construct.

The $\mathcal{F}_n$ couple to $\mathcal{J}^{(n)} \in \Delta^{2n}(\mathsf{E}^{*})$ and thus we defined their $\Delta^{*}(\mathsf{E}^{*})$ components in (4.1.11) accordingly. However, in parallelisable spaces the definition of the flux components given in (4.1.10) allows for $\mathcal{D}$ to be applied to the basis elements of $\Delta^{-}(\mathsf{E}^{*})$. Though this interpretation can not be used in non-parallelisable spaces we still expect the operators $\mathcal{F}_n$ to have some kind of well defined action on elements of $\Delta^{-}(\mathsf{E}^{*})$ which depends on the components in (4.1.11) but is not dependent on the inner product. We shall consider the four fluxes in turn, beginning with the simplest case of $\mathcal{F}_0$.

$$\mathcal{F}_0 \,:\, \Delta^{0,0}(\mathsf{E}^{*}) \to \Delta^{3}(\mathsf{E}^{*}) \quad : \quad \mathcal{F}_0(\omega_0) = (\mathcal{F}_0)_I\alpha_I - (\mathcal{F}_0)^J\beta^J \quad (4.1.12)$$

Due to the scalar nature of ω_0 the action of $\mathcal{F}_0$ on $\Delta^{0,0}(\mathsf{E}^{*})$ reduces to exterior multiplication by 3-forms in $\Delta^{3}(\mathsf{E}^{*})$. A particular term in the flux component expansions such as $\alpha_I\wedge$ acts as both $\alpha_I : \langle\omega_0\rangle \to \langle\alpha_I\rangle$ and $\alpha_I : \langle\beta^I\rangle \to \langle\widetilde{\omega}^0\rangle$ and as a result of this the action of $\mathcal{F}_0\wedge$ on elements of $\Delta^{3}(\mathsf{E}^{*})$ can be easily constructed by applying it to the basis elements.

$$\begin{aligned}\mathcal{F}_0(\omega_0)\wedge\beta^K &= (\mathcal{F}_0)_I\alpha_I\wedge\beta^K = (\mathcal{F}_0)_I\delta_I^K\widetilde{\omega}^0 = (\mathcal{F}_0)_I\widetilde{\omega}^0\\ \mathcal{F}_0(\omega_0)\wedge\alpha_K &= -(\mathcal{F}_0)^J\beta^J\wedge\alpha_K = (\mathcal{F}_0)^J\delta_K^J\widetilde{\omega}^0 = (\mathcal{F}_0)^J\widetilde{\omega}^0\end{aligned}$$

These two sets of coefficients play the roles of flux components in $\mathcal{F}_0$ but they can be expressed in two different ways, the first of which is (4.1.12).

$$\mathcal{F}_0\,:\,\Delta^3(\mathsf{E}^*)\to\Delta^{3,3}(\mathsf{E}^*)\quad\text{s.t.}\quad\begin{aligned}\mathcal{F}_0(\alpha_I) &= (\mathcal{F}_0)^J\widetilde{\omega}^0\\ \mathcal{F}_0(\beta^J) &= (\mathcal{F}_0)_I\widetilde{\omega}^0\end{aligned}\tag{4.1.13}$$

In this case we have made use of the fact $\widetilde{\omega}^0\sim\text{vol}_6=\alpha_I\wedge\beta^I=-\beta^I\wedge\alpha_I$ (no sum) and this same factorisation can be used for the $\mathcal{F}_3$ case.

$$\mathcal{F}_3\,:\,\Delta^{3,3}(\mathsf{E}^*)\to\Delta^3(\mathsf{E}^*)\quad:\quad\mathcal{F}_3(\widetilde{\omega}^0)=(\mathcal{F}_3)_I\alpha_I-(\mathcal{F}_3)^J\beta^J\tag{4.1.14}$$

The action of $\mathcal{F}_3$ can be viewed as the removal of one of the $\Delta^-(\mathsf{E}^*)$ terms from the factorisation of $\widetilde{\omega}^0\sim\text{vol}_6$ and thus the action of $\mathcal{F}_3$ on $\Delta^-(\mathsf{E}^*)$ is straightforward.

$$\mathcal{F}_3:\Delta^3(\mathsf{E}^*)\to\Delta^{0,0}(\mathsf{E}^*)\quad\text{s.t.}\quad\begin{aligned}\mathcal{F}_3(\alpha_I) &= -(\mathcal{F}_3)^I\omega_0\\ \mathcal{F}_3(\beta^J) &= -(\mathcal{F}_3)_J\omega_0\end{aligned}\tag{4.1.15}$$

In the cases of $\mathcal{F}_1$ and $\mathcal{F}_2$ this simple addition or removal of 3-forms does not occur and so we must use a different method. $\mathcal{F}_1$ arises from the non-closed nature of the basis forms and due to its geometric nature it can be expressed in terms of the exterior derivative d.

$$\mathcal{F}_1\,:\,\Delta^{1,1}(\mathsf{E}^*)\to\Delta^3(\mathsf{E}^*)\quad:\quad\mathcal{F}_1(\omega_a)\equiv\mathcal{F}_{(a)I}\alpha_I-\mathcal{F}_{(a)}{}^J\beta^J\cong\text{d}(\omega_a)$$

The two sets of coefficients can be extracted from this expression by integrating over the appropriate 3-cycles and then converting these to integrals over the entire space[2].

$$\int_{A^J} \mathrm{d}(\omega_a) = \int_{A^J} \mathcal{F}_{(a)I}\alpha_I \quad \Rightarrow \quad \int_{\mathcal{M}} \mathrm{d}(\omega_a)\wedge\beta^J = \int_{\mathcal{M}} \mathcal{F}_{(a)I}\alpha_I\wedge\beta^J$$

By Stokes theorem and $\Delta^5(\mathsf{E}^*)$ being empty [56] the left hand side integral with integrand $\mathrm{d}(\omega_a)\wedge\beta^J$ converts to an integral over $\mathcal{M}$ with integrand $-\omega_a\wedge\mathrm{d}(\beta^J)$ but this can be reexpressed as an integral over a 4-cycle, that which is associated to the ω_a 2-form. Thus the right hand side is related to the non-closure of β^J.

$$\mathcal{F}_{(a)J} = -\int_{\mathcal{M}} \omega_a\wedge\mathrm{d}(\beta^J) = \int_{\mathcal{M}} \mathrm{d}(-\beta^J)\wedge\omega_a = \int_{\mathcal{B}_a} \mathrm{d}(-\beta^J)$$

Therefore the action of $\mathcal{F}_1$ on the β^J, $\mathcal{F}_1(\beta^J)\sim\mathrm{d}(\beta^J)$, has a contribution in $\Delta^{2,2}(\mathsf{E}^*)$ of $-\mathcal{F}_{(b)J}\widetilde{\omega}^b$. Repeating this method but integrating over the B_I 3-cycle gives the contribution of the non-closure of α_I, $\mathrm{d}\alpha_I\sim\mathcal{F}_1(\alpha_I)$ in $\Delta^{2,2}(\mathsf{E}^*)$, ${\mathcal{F}_{(b)}}^I\widetilde{\omega}^b$. These two results allow us to explicitly state the action of $\mathcal{F}_1$ on $\Delta^-(\mathsf{E}^*)$ in terms of its action on $\Delta^+(\mathsf{E}^*)$.

$$\mathcal{F}_1 : \Delta^3(\mathsf{E}^*)\to\Delta^{2,2}(\mathsf{E}^*) \quad \text{s.t.} \quad \begin{array}{rcl} \mathcal{F}_1(\alpha_I) &=& -{(\mathcal{F}_1)_{(a)}}^I\widetilde{\omega}^a \\ \mathcal{F}_1(\beta^J) &=& -(\mathcal{F}_1)_{(a)J}\widetilde{\omega}^a \end{array}$$

The remaining case of $\mathcal{F}_2$ does not immediately lend itself to the same methodology since the schematic action of the flux is $\mathcal{F}_2 : \Delta^p(\mathsf{E}^*)\to\Delta^{p-1}(\mathsf{E}^*)$. This is in contrast to the behaviour of $\mathcal{F}_1$ and the exterior derivative d, $\mathcal{F}_1 : \Delta^p(\mathsf{E}^*)\to\Delta^{p+1}(\mathsf{E}^*)$, and is a reflection of its non-geometric nature.

$$\mathcal{F}_2 \,:\, \Delta^3(\mathsf{E}^*)\to\Delta^{1,1}(\mathsf{E}^*) \quad : \quad \mathcal{F}_2(\widetilde{\omega}^b) = {(\mathcal{F}_2)^{(b)}}_I\alpha_I - (\mathcal{F}_2)^{(a)J}\beta^J$$

A method which views the non-geometric flux as some kind of adjoint derivative is given in Appendix B.1.5 but the question of whether such a derivative

[2] As commented in [56] integration over $\mathcal{M}$ can be non-trivial if $\mathcal{M}$ is not a manifold, such as an orbifold or singular Calabi-Yau, but we neglect this technicality and will demonstrate the derived identities for the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold later.

interpretation is well defined is beyond the scope of this work. However, the result is consistent with the explicit $\Lambda^3(\mathfrak{E})$ construction on parallelisable spaces [56], a fact which will be seen in our analysis of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold in Chapter 6.

$$\mathcal{F}_2 \,:\, \Delta^3(\mathsf{E}^*) \to \Delta^{1,1}(\mathsf{E}^*) \quad \text{s.t.} \quad \begin{aligned} \mathcal{F}_2(\alpha_I) &= (\mathcal{F}_2)^{(b)J}\omega_b \\ \mathcal{F}_2(\beta^J) &= (\mathcal{F}_2)^{(b)}{}_I\omega_b \end{aligned} \tag{4.1.16}$$

Despite the derivation of the alternative actions of each of the $\mathcal{F}_n$ using a different method, they all share the feature that if a particular term $\mathfrak{f} \in \mathcal{F}_n$ has an action $\mathfrak{f} : \langle\xi\rangle \to \langle\zeta\rangle$ then it will also have an action $\mathfrak{f} : \langle\star\zeta\rangle \to \langle\star\xi\rangle$, where $\langle\chi\rangle$ is the space spanned by the form χ.

In considering (4.1.13 - 4.1.16) we observe that the sign structure of the flux actions on $\Delta^-(\mathsf{E}^*)$ basis differs in $A, B = 0$ case compared to the $A, B > 0$ cases. (4.1.15) and (4.1.16) have a factor of -1, while (4.1.13) and (4.1.16) do not. In order to obtain as symmetric an examination of the two moduli spaces we are therefore motivated to do a change of basis in the Kähler moduli space which addresses this sign structure. This change of basis is constrained by two requirements; the intersection numbers of the basis should not change and the dilaton dependence of holomorphic forms associated to $\mathcal{M}^Q$ is unchanged. Since the complex structure moduli on one Type II theory are related to the Kähler of the other Type II theory via mirror symmetry we redefine the complex structure also.

$$(\alpha_I, \beta^J, \mathcal{U}) \to (\mathfrak{a}_I, \mathfrak{b}^J, \mathfrak{U}) \quad , \quad (\omega_A, \widetilde{\omega}^B, \mathcal{T}) \to (\nu_A, \widetilde{\nu}^B, \mathfrak{T}) \tag{4.1.17}$$

There is no unique way to do these redefinitions but in Table 4.1 we choose the simplest one which will reduce later algebraic workings, makes symmetries clearer to see and and is such that the component expansion of $\mho$ still

Old	ω_0	ω_a	$\widetilde{\omega}^0$	$\widetilde{\omega}^b$	α_0	α_i	β^0	β^j
New	$\widetilde{\nu}^0$	ν_a	ν_0	$\widetilde{\nu}^b$	$-\mathfrak{b}^0$	$\mathfrak{a}_i$	$\mathfrak{a}_0$	$\mathfrak{b}^j$
Old	$\mathcal{T}_0$	$\mathcal{T}_a$	$\mathcal{T}^0$	$\mathcal{T}^b$	$\mathcal{U}_0$	$\mathcal{U}_i$	$\mathcal{U}^0$	$\mathcal{U}^j$
New	$\pm\mathfrak{T}^0$	$\mathfrak{T}^a$	$\pm\mathfrak{T}_0$	$\mathfrak{T}_b$	$\mathfrak{U}^0$	$\mathfrak{U}_i$	$-\mathfrak{U}_0$	$\mathfrak{U}^j$

Table 4.1: Redefined moduli and $\Delta^*(\mathsf{E}^*)$ basis elements for $\langle\ \rangle_\pm$ bracket.

takes the same form as that of Ω.

$$\Omega \to \mathfrak{U}_I \mathfrak{a}_I - \mathfrak{U}^J \mathfrak{b}^J \quad , \quad \mho \to \mathfrak{T}_A \nu_A \pm \mathfrak{T}^B \widetilde{\nu}^B \tag{4.1.18}$$

Given this new set of $\Delta^*(\mathsf{E}^*)$ basis elements we can define the action of $\mathcal{D}$ explicitly such that the index structure of the components takes a particular form.

$$\begin{array}{rcl} \mathcal{D}(\nu_A) &=& \mathcal{F}_{(A)I}\mathfrak{a}_I - \mathcal{F}_{(A)}{}^J\mathfrak{b}^J \\ \mathcal{D}(\widetilde{\nu}^B) &=& \mathcal{F}^{(B)}{}_I\mathfrak{a}_I - \mathcal{F}^{(B)J}\mathfrak{b}^J \end{array} \Leftrightarrow \begin{array}{rcl} \mathcal{D}(\mathfrak{a}_I) &=& \mathcal{F}^{(A)I}\nu_A - \mathcal{F}_{(B)}{}^I\widetilde{\nu}^B \\ \mathcal{D}(\mathfrak{b}^J) &=& \mathcal{F}^{(A)}{}_J\nu_A - \mathcal{F}_{(B)J}\widetilde{\nu}^B \end{array} \tag{4.1.19}$$

Before using this explicit action of $\mathcal{D}$ to construct the T duality and mirror symmetry invariant Type IIA NS-NS superpotential we consider a way of expressing the fluxes such that the action of their individual flux components is manifest and which makes the variance in the sign structure of the original $\Delta^+(\mathsf{E}^*)$ basis clearer.

4.1.3 Generalised Flux Operators

For parallelisable $\mathcal{M}$ we were able to define the $\Lambda^3(\mathfrak{E})$ components of fluxes in the manner of (4.1.10), without considering them to be acting on some holomorphic form or basis element of $\Delta^*(\mathsf{E}^*)$ as done in (4.1.8). The advantage of this was seen to be that the nilpotency condition $\mathcal{D}^2 = 0$ could be expressed in terms of the flux components without having to apply it to some general element of $\Omega^*(\mathsf{E}^*)$, as is done in Ref. [56]. We wish to extend this in some way to non-parallelisable $\mathcal{M}$. The motivation is discussed further in Appendix B.1.3, where an analogue of $\Lambda^3(\mathfrak{E})$ is defined, a number of important algebraic identities obtained and a demonstration that this reduces to the $\Lambda^3(\mathfrak{E})$ case when $\mathcal{M}$ is parallelisable. As a result we restrict the discussion here to simply examining some of the results.

The fluxes in parallelisable $\mathcal{M}$ can be regarded as elements of $\Lambda^3(\mathfrak{E})$, as given in (4.1.3) and (4.1.4). For non-parallelisable $\mathcal{M}$ we extend this to the light $\Delta^*(\mathfrak{E})$. We initially consider the $(\omega_A, \widetilde{\omega}^B)$ and (α_I, β^J) basis so that motivation for the change of basis can be demostrated more clearly.

$$\begin{aligned}
\mathcal{F}_0 &= (\mathcal{F}_0)_I \alpha_I \iota_{\omega_0} - (\mathcal{F}_0)^J \beta^J \iota_{\omega_0} \\
\mathcal{F}_1 &= (\mathcal{F}_1)_{(a)I} \alpha_I \iota_{\omega_a} - {(\mathcal{F}_1)_{(a)}}^J \beta^J \iota_{\omega_a} \\
\mathcal{F}_2 &= {(\mathcal{F}_2)^{(b)}}_I \alpha_I \iota_{\widetilde{\omega}^b} - (\mathcal{F}_2)^{(a)J} \beta^J \iota_{\widetilde{\omega}^b} \\
\mathcal{F}_3 &= (\mathcal{F}_3)_I \alpha_I \iota_{\widetilde{\omega}^0} - (\mathcal{F}_3)^J \beta^J \iota_{\widetilde{\omega}^0}
\end{aligned} \tag{4.1.20}$$

We then make use of the results given in (4.1.13-4.1.16) so that the action of the fluxes on elements of $\Delta^-(\mathsf{E}^*)$ is made manifest but at the expense of the action on $\Delta^+(\mathsf{E}^*)$ no longer being manifest. The way in which the sign structure of the individual flux components changes in this reformulation is clear to see in comparing the two expressions.

$$\begin{aligned}
\mathcal{F}_0 &= (\mathcal{F}_0)_I \widetilde{\omega}^0 \iota_{\beta^I} + (\mathcal{F}_0)^J \widetilde{\omega}^0 \iota_{\alpha_J} \\
\mathcal{F}_1 &= -(\mathcal{F}_1)_{(a)I} \widetilde{\omega}^a \iota_{\beta^I} - {(\mathcal{F}_1)_{(a)}}^J \widetilde{\omega}^a \iota_{\alpha_I} \\
\mathcal{F}_2 &= {(\mathcal{F}_2)^{(b)}}_I \omega_b \iota_{\beta^I} + (\mathcal{F}_2)^{(a)J} \omega_b \iota_{\alpha_J} \\
\mathcal{F}_3 &= -(\mathcal{F}_3)_I \omega_0 \iota_{\beta^I} - (\mathcal{F}_3)^J \omega_0 \iota_{\alpha_J}
\end{aligned} \tag{4.1.21}$$

The relationship between the two operator formulations is obtained to equating the two different operators which each flux component is associated to.

In doing this we obtain an equivalence between different elements of $\Delta^*(\mathfrak{E})$ such that the different sign structure of the Kähler index 0 cases compared to the $a \in \{1, \ldots, h^{1,1}\}$ cases are manifest.

$$
\begin{aligned}
\alpha_I \iota_{\omega_0} &\simeq \widetilde{\omega}^0 \iota_{\beta^I} & \beta^J \iota_{\omega_0} &\simeq -\widetilde{\omega}^0 \iota_{\alpha_J} \\
\alpha_I \iota_{\omega_a} &\simeq -\widetilde{\omega}^a \iota_{\beta^I} & \beta^J \iota_{\omega_a} &\simeq \widetilde{\omega}^a \iota_{\alpha_J} \\
\alpha_I \iota_{\widetilde{\omega}^a} &\simeq \omega_a \iota_{\beta^I} & \beta^J \iota_{\widetilde{\omega}^a} &\simeq -\omega_a \iota_{\alpha_J} \\
\alpha_I \iota_{\widetilde{\omega}^0} &\simeq -\omega_0 \iota_{\beta^I} & \beta^J \iota_{\widetilde{\omega}^0} &\simeq \omega_0 \iota_{\alpha_J}
\end{aligned}
\qquad (4.1.22)
$$

It is worth noting that this variation in the sign structure in the $\Delta^+(\mathsf{E}^*)$ basis is seen in the Mukai bracket definition stated in (A.1.4). We previously noted that the use of $\langle\ \rangle_-$ makes the special Kähler structure of $\mathcal{M}_{\mathcal{T}}$ manifest and later we will see how the choice of $\langle\ \rangle_-$ can simplify several more expressions. Using the new basis of $\Delta^*(\mathsf{E}^*)$ given in Table 4.1 we obtain a more streamlined set of operator equivalences.

$$
\begin{aligned}
\mathfrak{a}_I \iota_{\nu_A} &\simeq -\widetilde{\nu}^A \iota_{\mathfrak{b}^I} & \mathfrak{b}^J \iota_{\nu_A} &\simeq \widetilde{\nu}^A \iota_{\mathfrak{a}_J} \\
\mathfrak{a}_I \iota_{\widetilde{\nu}^B} &\simeq \nu_B \iota_{\mathfrak{b}^I} & \mathfrak{b}^J \iota_{\widetilde{\nu}^B} &\simeq -\nu_B \iota_{\mathfrak{a}_J}
\end{aligned}
\qquad (4.1.23)
$$

Generically expressions in both (4.1.22) and (4.1.23) have the same structure, which can be written in terms of the Hodge star $\star$.

$$\xi\, \iota_\varphi \cong \pm \star \varphi\, \iota_{\star\xi} \qquad (4.1.24)$$

The connection between the sign choice in this expression with the intersection numbers of $\langle\ \rangle_\pm$ can be made manifest through the use of an alternative to the Hodge star, $*$, defined using $\langle\ \rangle_\pm$ rather than simple exterior multiplication defined in Appendix B.1.6.

$$\xi\, \iota_\varphi \cong - * \varphi\, \iota_{*\xi} \qquad (4.1.25)$$

Since we have changed the bases of $\Delta^*(\mathsf{E}^*)$ the individual components of the fluxes are relabelled into the form given in (4.1.19) and we can therefore extract from (4.1.19) the individual fluxes without viewing them as acting on a particular element of $\Delta^*(\mathsf{E}^*)$.

$$\begin{aligned}
\mathcal{F}_3 &\equiv \mathcal{F}_{(0)I}\mathfrak{a}_I\iota_{\nu_0} &-& \mathcal{F}_{(0)}{}^J\mathfrak{b}^J\iota_{\nu_0} \\
\mathcal{F}_1 &\equiv \mathcal{F}_{(a)I}\mathfrak{a}_I\iota_{\nu_a} &-& \mathcal{F}_{(a)}{}^J\mathfrak{b}^J\iota_{\nu_a} \\
\mathcal{F}_0 &\equiv \mathcal{F}^{(0)}{}_I\mathfrak{a}_I\iota_{\widetilde{\nu}^0} &-& \mathcal{F}^{(0)J}\mathfrak{b}^J\iota_{\widetilde{\nu}^0} \\
\mathcal{F}_2 &\equiv \mathcal{F}^{(b)}{}_I\mathfrak{a}_I\iota_{\widetilde{\nu}^b} &-& \mathcal{F}^{(b)J}\mathfrak{b}^J\iota_{\widetilde{\nu}^b}
\end{aligned} \tag{4.1.26}$$

Given this set of components and elements of $\Delta^*(\mathfrak{E})$ we can construct the $\Delta^*(\mathsf{E}^*)$ version of the expansion of $\mathcal{D}$ in (4.1.10) except that we have two different but equivalent formulations, each associated to one of the two actions of $\mathcal{D}$ in (4.1.19).

$$\begin{aligned}
\mathcal{D} &= \Big(\mathcal{F}_{(A)I}\mathfrak{a}_I - \mathcal{F}_{(A)}{}^J\mathfrak{b}^J \Big)\iota_{\nu_A} + \Big(\mathcal{F}^{(B)}{}_I\mathfrak{a}_I - \mathcal{F}^{(B)J}\mathfrak{b}^J \Big)\iota_{\widetilde{\nu}^B} \\
&= \Big(-\mathcal{F}^{(A)I}\nu_A + \mathcal{F}_{(B)}{}^I\widetilde{\nu}^B \Big)\iota_{\mathfrak{a}_I} + \Big(-\mathcal{F}^{(A)}{}_J\nu_A + \mathcal{F}_{(B)J}\widetilde{\nu}^B \Big)\iota_{\mathfrak{b}^J}
\end{aligned} \tag{4.1.27}$$

The explicit dependency of the $\mathcal{F}_n$ on the components of (4.1.19) is not relevant to the majority of the analysis of the superpotential and its comparision to the Type IIB mirror. For such cases as the tadpoles, particularly in Type IIB, the dependency requires more specific attention but this is a point we shall return to later. Instead we focus our attention on the superpotential itself for the time being.

4.1.4 T Duality Invariant Superpotential

Given (4.1.19) or (4.1.27) the G_3 component of the Type IIA NS-NS superpotential can be written explicitly in terms of its moduli, though a sign choice is inherited from (4.1.18).

$$\begin{aligned}
G_3 = \mathcal{D}(\mho) &= \mathfrak{T}_A \Big(\mathcal{F}_{(A)I}\mathfrak{a}_I - \mathcal{F}_{(A)}{}^J\mathfrak{b}^J \Big) \pm \mathfrak{T}^B \Big(\mathcal{F}^{(B)}{}_I\mathfrak{a}_I - \mathcal{F}^{(B)J}\mathfrak{b}^J \Big) \\
&= \mathfrak{a}_I \Big(\mathfrak{T}_A\mathcal{F}_{(A)I} \pm \mathfrak{T}^B\mathcal{F}^{(B)}{}_I \Big) - \mathfrak{b}^J \Big(\mathfrak{T}_A\mathcal{F}_{(A)}{}^J \pm \mathfrak{T}^B\mathcal{F}^{(B)J} \Big)
\end{aligned}$$

This couples to the $\mathcal{M}^Q$ moduli space holomorphic form Ω_c via the integrand $\langle \Omega_c, G_3 \rangle_{\pm}$ and performing the integral over $\mathcal{M}$ provides us with the polynomial expression for W.

$$\begin{aligned} W \;=\; & \quad S\Big(\mathfrak{T}_A \mathcal{F}_{(A)}{}^{0} \pm \mathfrak{T}^B \mathcal{F}^{(B)0}\Big)\mathfrak{U}_0 \;-\; \Big(\mathfrak{T}_A \mathcal{F}_{(A)}{}^{i} \pm \mathfrak{T}^B \mathcal{F}^{(B)i}\Big)\mathfrak{U}_i \\ & - \; S\Big(\mathfrak{T}_A \mathcal{F}_{(A)0} \pm \mathfrak{T}^B \mathcal{F}^{(B)}{}_{0}\Big)\mathfrak{U}^0 \;+\; \Big(\mathfrak{T}_A \mathcal{F}_{(A)j} \pm \mathfrak{T}^B \mathcal{F}^{(B)}{}_{j}\Big)\mathfrak{U}^j \end{aligned} \tag{4.1.28}$$

An important point of note is the effect an orientifold projection has on this superpotential. Thus far we have not considered constraints or restrictions on the fluxes but rather what contributions to the superpotential might exist in principle. In the case of the orientifold projection in Type IIA the complexified holomorphic 3-form is split into those terms which are even under the projection and those which are odd. A particularly simple choice is the projection with the $\mathfrak{a}_I$ being even and the $\mathfrak{b}^J$ odd and we therefore obtain an $\mathcal{N}=1$ Type IIA superpotential by setting $\mathfrak{U}^J = 0$ in the above expression. This is not a restriction we will consider in general as the inclusion of all possible fluxes will be seen to make particular symmetries of the superpotential clearer but it is of critical importance in how we relate the Type IIA NS-NS superpotential to its Type IIB counterpart.

4.1.5 Scalar Product Representations

A short overview of the basic definitions and derivations of particular results which we make use of in this section is given in Section B.2. Restricting our attention entirely to the bases of the $\Delta^*(\mathsf{E}^*)$ we are able to examine the fluxes and superpotentials in terms of matrices. With $\Delta^1(\mathsf{E}^*)$ and $\Delta^5(\mathsf{E}^*)$ being empty we can construct an exact sequence for $\mathcal{D}$ in terms of $\Delta^3(\mathsf{E}^*)$ and $\Delta^+(\mathsf{E}^*)$.

$$\cdots \xrightarrow{\mathcal{D}} \Delta^3(\mathsf{E}^*) \xrightarrow{\mathcal{D}} \Delta^+(\mathsf{E}^*) \xrightarrow{\mathcal{D}} \Delta^3(\mathsf{E}^*) \xrightarrow{\mathcal{D}} \cdots$$

These subspaces of $\Delta^*(\mathsf{E}^*)$ can be described using a pair of vectors whose entries are defined by the basis elements of the $\Delta^*(\mathsf{E}^*)$.

$$\mathbf{e}_{(\mathfrak{a})} \equiv \begin{pmatrix} \mathfrak{a}_0 & \mathfrak{a}_i & \mathfrak{b}^0 & \mathfrak{b}^j \end{pmatrix} \quad , \quad \mathbf{e}_{(\nu)} \equiv \begin{pmatrix} \nu_0 & \nu_a & \widetilde{\nu}^0 & \widetilde{\nu}^b \end{pmatrix} \tag{4.1.29}$$

With the entries of these vectors forming the basis for any harmonic differential form in $\mathcal{M}$ we can express the $\mathcal{D}$ image of any given form as a linear combination of other $\Delta^*(\mathsf{E}^*)$ basis elements, thus giving matrix representations to $\mathcal{D}$ [64, 28]. For later convenience we choose to write them in such a manner that factors of h matrices are explicit.

$$\mathcal{D}\begin{pmatrix} \mathbf{e}_{(\nu)} \\ \mathbf{e}_{(\mathfrak{a})} \end{pmatrix} = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \begin{pmatrix} h_\nu & 0 \\ 0 & h_{\mathfrak{a}} \end{pmatrix} \begin{pmatrix} \mathbf{e}_{(\nu)} \\ \mathbf{e}_{(\mathfrak{a})} \end{pmatrix}$$

To combine the two moduli spaces into a single description we define a $2h^{2,1} + 2h^{1,1} + 4$ dimensional vector of p-forms $\mathbf{e}$ by combining $\mathbf{e}_{(\mathfrak{a})}$ and $\mathbf{e}_{(\nu)}$ and the moduli vectors combine in the same manner.

$$\begin{aligned} \mathbf{e} &\equiv \left(\begin{array}{cc} \mathbf{e}_{(\nu)} & \mathbf{e}_{(\mathfrak{a})} \end{array} \right) \equiv \left(\begin{array}{cccccccc} \nu_0 & \nu_a & \widetilde{\nu}^0 & \widetilde{\nu}^b & \mathfrak{a}_0 & \mathfrak{a}_i & \mathfrak{b}^0 & \mathfrak{b}^j \end{array} \right) \\ \underline{\Phi}^\top &\equiv \left(\begin{array}{cc} \underline{\mathfrak{T}}^\top & \underline{\mathfrak{U}}^\top \end{array} \right) \equiv \left(\begin{array}{cccccccc} \mathfrak{T}_0 & \mathfrak{T}_a & \mathfrak{T}^0 & \mathfrak{T}^b & \mathfrak{U}_0 & \mathfrak{U}_i & \mathfrak{U}^0 & \mathfrak{U}^j \end{array} \right) \end{aligned} \tag{4.1.30}$$

The action of the derivative thus defines a matrix $\underline{\underline{\mathcal{D}}}$ on this basis.

$$\mathcal{D}(\mathbf{e}) = \underline{\underline{\mathcal{D}}} \cdot h \cdot \mathbf{e} \quad \Rightarrow \quad \underline{\underline{\mathcal{D}}} \equiv \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \tag{4.1.31}$$

The entries of M and N can be obtained from (4.1.19), with the entries of M defining the entries of N or vice versa.

$$M \cdot h_{\mathfrak{a}} = \begin{pmatrix} \mathcal{F}_{(A)I} & -\mathcal{F}_{(A)}{}^{J} \\ \mathcal{F}^{(B)}{}_{I} & -\mathcal{F}^{(B)J} \end{pmatrix} \quad , \quad N \cdot h_\nu = \begin{pmatrix} \mathcal{F}^{(A)I} & -\mathcal{F}_{(B)}{}^{I} \\ \mathcal{F}^{(A)}{}_{J} & -\mathcal{F}_{(B)J} \end{pmatrix} \tag{4.1.32}$$

We have abused notation slightly since such expressions as $\mathcal{F}_{(A)I}$ represent a matrix of components. Strictly speaking if the entries of N are defined by the entries of M then we should denote its components as transpositions but instead we rely on index structure to define summations of components. However, since this schematic relationship between the two actions of a derivative on the $\Delta^{\pm}(\mathsf{E}^{*})$ is independent of which Type II theory, we find it useful to express this relationship in a way which doesn't use the Type IIA NS-NS specific fluxes of (4.1.32) but instead generic matrices m_i

$$M \cdot h_{\mathfrak{a}} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \quad \Rightarrow \quad N \cdot h_{\nu} = \begin{pmatrix} -m_4^{\top} & m_2^{\top} \\ m_3^{\top} & -m_1^{\top} \end{pmatrix} \tag{4.1.33}$$

Inverting this such that we define the entries of N rather than those of M.

$$N \cdot h_{\nu} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \quad \Rightarrow \quad M \cdot h_{\mathfrak{a}} = \begin{pmatrix} -m_4^{\top} & m_2^{\top} \\ m_3^{\top} & -m_1^{\top} \end{pmatrix} \tag{4.1.34}$$

In this formulation it is clear that the map $\mathfrak{Ad} : M \cdot h_{\mathfrak{a}} \to N \cdot h_{\nu}$ is an involution. We can obtain this relationship between M and N in terms of bilinear forms through the use of (4.1.23) or (4.1.32).

$$N \cdot h_{\nu} = g_{\mathfrak{a}} \cdot (M \cdot h_{\mathfrak{a}})^{\top} \cdot \mathsf{g}_{\mathfrak{a}} \quad \Rightarrow \quad N = \Sigma_{\mathfrak{a}} \cdot M^{\top} \cdot \mathsf{g}_{\mathfrak{a}}^{\top} \cdot h_{\nu} \tag{4.1.35}$$

Though the relationship between M and N is the result of algebraic identities the bilinear forms are dependent on the inner product and so the formulation of the relationship in (4.1.35) in terms of bilinear forms is also dependent on this choice. If we decide on which inner product to use then (4.1.35) can be simplified.

$$\begin{aligned} \langle\ \rangle \to \langle\ \rangle_{-} \quad &\Rightarrow \quad N = \Sigma_{\mathfrak{a}} \cdot M^{\top} \cdot \Sigma_{\mathfrak{a}}^{\top} \\ \langle\ \rangle \to \langle\ \rangle_{+} \quad &\Rightarrow \quad N = \Sigma_{\mathfrak{a}} \cdot M^{\top} \cdot \mathsf{g}_{\mathfrak{a}}^{\top} \end{aligned} \tag{4.1.36}$$

To keep our analysis as general as possible we refrain from setting the inner product at this point. Before considering how the dilaton couples to particular fluxes via $\Omega \to \Omega_c$ we examine a toy model of a superpotential which has no dilaton dependence in that we view both moduli spaces as the standard $\mathcal{M}^K$ manifolds described in Section 2.3. As a result this toy model has a superpotential integrand is of the form $\left\langle \Omega, \mathcal{D}(\mho) \right\rangle_{\pm}$. For comparision we also consider the superpotential which would be obtained from the integrand $\left\langle \mho, \mathcal{D}(\Omega) \right\rangle_{\pm}$ as this will arise later. Given any form $\chi \in \Delta^*(\mathsf{E}^*)$ the associated vector is defined via the general factorisation $\chi = \underline{\chi}^\top \cdot h \cdot \mathbf{e}$ and we can define our moduli by $\underline{\mathfrak{T}} \equiv \underline{\mho}$ and $\underline{\mathfrak{U}} \equiv \underline{\Omega}$.

$$\begin{aligned}
\Omega \;&=\; \mathfrak{U}_I \mathfrak{a}_I - \mathfrak{U}^J \mathfrak{b}^J \;\;=\; \underline{\Omega}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \;=\; \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \\
\mho \;&=\; \mathfrak{T}_A \nu_A \pm \mathfrak{T}^B \widetilde{\nu}^B \;=\; \underline{\mho}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)} \;=\; \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)}
\end{aligned}$$

To construct the superpotential, or expressions like it, we need the action of the derivatives upon these holomorphic forms and to that end we consider the two alternative actions of $\mathcal{D}$ on the basis elements of $\Delta^+(\mathsf{E}^*)$ and $\Delta^-(\mathsf{E}^*)$.

$$\mathcal{D} \cdot \mathbf{e}_{(\nu)} \equiv M \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \quad , \quad \mathcal{D} \cdot \mathbf{e}_{(\mathfrak{a})} \equiv N \cdot h_\nu \cdot \mathbf{e}_{(\nu)}$$

These two expressions have followed from the two alternative ways of expressing the derivative's action on the $\Delta^*(\mathsf{E}^*)$ given in (4.1.27). The two formulations of $\mathcal{D}$ on $\Delta^*(\mathfrak{E})$ can be written in terms of the flux matrices of (4.1.30) and the dual of the $\mathbf{e}$ sub-vectors.

$$\mathcal{D} = \mathbf{e}_{(\mathfrak{a})}^\top \cdot h_{\mathfrak{a}}^\top \cdot M^\top \cdot \iota_{\mathbf{e}_{(\nu)}} = \mathbf{e}_{(\nu)}^\top \cdot h_\nu^\top \cdot N^\top \cdot \iota_{\mathbf{e}_{(\mathfrak{a})}}$$

The transpositions are done so as to make it clear the ι do not act on the $\mathbf{e}$ p-forms. Since $\mathcal{D}$, or any other derivative, must act on some q-form to construct a scalar product expression this transposition can be undone once such an expression is formed. To this end we shall write the holomorphic forms in terms of their vector factorisations and using the above expressions for the images of $\Delta^+(\mathsf{E}^*)$ and $\Delta^-(\mathsf{E}^*)$ basis elements under $\mathcal{D}$ construct the vectors associated to $\mathcal{D}(\mho)$ and $\mathcal{D}(\Omega)$.

$$\begin{aligned}
\mathcal{D}(\mho) &= \mathcal{D}(\underline{\mathfrak{T}}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)}) & \mathcal{D}(\Omega) &= \mathcal{D}(\underline{\mathfrak{U}}^\top \cdot h_\mathfrak{a} \cdot \mathbf{e}_{(\mathfrak{a})}) \\
&= \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot \mathcal{D}(\mathbf{e}_{(\nu)}) & &= \underline{\mathfrak{U}}^\top \cdot h_\mathfrak{a} \cdot \mathcal{D}(\mathbf{e}_{(\mathfrak{a})}) \\
&= \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot M \cdot h_\mathfrak{a} \cdot \mathbf{e}_{(\mathfrak{a})} & &= \underline{\mathfrak{U}}^\top \cdot h_\mathfrak{a} \cdot N \cdot h_\nu \cdot \mathbf{e}_{(\nu)} \\
&= \underline{\mathcal{D}(\mho)}^\top \cdot h_\mathfrak{a} \cdot \mathbf{e}_{(\mathfrak{a})} & &= \underline{\mathcal{D}(\Omega)}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)}
\end{aligned}$$

With these expressions we construct two expressions whose form is schematically similar to dilaton dependent superpotentials, which we ultimately wish to examine.

$$\begin{aligned}
\int_\mathcal{M} \big\langle\, \Omega, \mathcal{D}(\mho) \,\big\rangle_\pm &= g\Big(\Omega, \mathcal{D}(\mho)\Big) = \underline{\mathcal{D}(\mho)}^\top \cdot g_\mathfrak{a} \cdot \underline{\Omega} = \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot M \cdot g_\mathfrak{a} \cdot \underline{\mathfrak{U}} \\
\int_\mathcal{M} \big\langle\, \mho, \mathcal{D}(\Omega) \,\big\rangle_\pm &= g\Big(\mho, \mathcal{D}(\Omega)\Big) = \underline{\mathcal{D}(\Omega)}^\top \cdot g_\nu \cdot \underline{\mho} = \underline{\mathfrak{U}}^\top \cdot h_\mathfrak{a} \cdot N \cdot g_\nu \cdot \underline{\mathfrak{T}}
\end{aligned} \tag{4.1.37}$$

The integrands are similar to one another and this can be examined further by using (4.1.35) to write N in terms of M. This relationship depends on the choice of $\langle\ \rangle_\pm$ via the identities given in Appendix B.2 and $g_\nu^\top = \pm g_\nu$. As such we consider $\langle\ \rangle_+$ first.

$$\begin{aligned}
\int_\mathcal{M} \big\langle\, \mho, \mathcal{D}(\Omega) \,\big\rangle_+ &= \underline{\mathfrak{U}}^\top \cdot h_\mathfrak{a} \cdot \big(\Sigma_\mathfrak{a} \cdot M^\top \cdot \mathsf{g}_\mathfrak{a}^\top \cdot h_\nu\big) \cdot g_\nu \cdot \underline{\mathfrak{T}} \\
&= \underline{\mathfrak{U}}^\top \cdot g_\mathfrak{a} \cdot M^\top \cdot \mathsf{g}_\mathfrak{a}^\top \cdot \Sigma_\nu \cdot \underline{\mathfrak{T}} \\
&= \underline{\mathfrak{T}}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot M \cdot g_\mathfrak{a} \cdot \underline{\mathfrak{U}}
\end{aligned} \tag{4.1.38}$$

We have used the fact the Type IIA Σ_ν is equal to the Type IIB $\Sigma_\mathfrak{a}$. Comparing this with (4.1.37) we see that the flux entries of M must satisfy $h_\nu \cdot M = \mathsf{h}_\mathfrak{a} \cdot M$ if they are to be equal. In terms of the fluxes this is equivalent to $\mathcal{F}_0 = 0 = \mathcal{F}_2$ but no restrictions on $\mathcal{F}_1$ or $\mathcal{F}_3$. The fact $\mathcal{F}_1$ can be non-zero follows from the graded Leibnitz property of $\mathrm{d} \sim \mathcal{F}_1$.

$$\int \mathrm{d}(\Omega \wedge \mho) = \int \mho \wedge \mathrm{d}(\Omega) - \int \Omega \wedge \mathrm{d}(\mho) \tag{4.1.39}$$

Since we are considering only $\mathcal{F}_1 \neq 0$ we have $\mathrm{d}\Omega \in \Delta^4(\mathsf{E}^*)$ and the $\langle\ \rangle_\pm$ of (A.1.4) is such that $\psi_2 \wedge \varphi_4 = \big\langle\, \psi_2, \varphi_4 \,\big\rangle_\pm$ and likewise for $\Omega \wedge \mathrm{d}(\mho) =$

$\langle\, \Omega, \mathrm{d}(\mho) \,\rangle_{\pm}$. With the left hand side of the above expression being zero we obtain the result for the $\mathcal{D} \to \mathrm{d} \sim \mathcal{F}_1$ simplified case regardless of the components of $\mathcal{F}_1$. The $\langle\ \rangle_{-}$ case allows us to make more use of the identities in Appendix B.2, namely $\mathsf{g}_{\mathfrak{a}} = g_{\nu}$ and that the bilinear forms of each $\Delta^{\pm}(\mathsf{E}^{*})$ basis are equal, $h_{\mathfrak{a}} = h_{\nu}$ *etc.*

$$\begin{aligned}\int_{\mathcal{M}} \langle\, \mho, \mathcal{D}(\Omega) \,\rangle_{-} &= \underline{\mathfrak{U}}^{\top} \cdot h_{\mathfrak{a}} \cdot \left(\Sigma_{\mathfrak{a}} \cdot M^{\top} \cdot g_{\nu}^{\top} \cdot h_{\nu} \right) \cdot g_{\nu} \cdot \underline{\mathfrak{T}} \\ &= \underline{\mathfrak{T}}^{\top} \cdot h_{\nu} \cdot M \cdot g_{\mathfrak{a}} \cdot \underline{\mathfrak{U}} \end{aligned} \tag{4.1.40}$$

The different sign structure of $\langle\ \rangle_{-}$ results in $h_{\mathfrak{a}} = h_{\nu}$ and (4.1.39) vanishes for general $\mathcal{D}$. Taking into account the antisymmetric nature of $\langle\ \rangle_{-}$ the arguments can be exchanged in one of the expressions and since this result is not dependent on the $\underline{\mathfrak{T}}$ and $\underline{\mathfrak{U}}$ vectors associated to the holomorphic forms it is true for any $\psi \in \Delta^{+}(\mathsf{E}^{*})$ and $\varphi \in \Delta^{3}(\mathsf{E}^{*})$ and it follows that $\mathcal{D}$ is anti self adjoint on $\langle\ \rangle_{-}$ for any combination of fluxes.

$$\int_{\mathcal{M}} \langle\, \psi, \mathcal{D}(\varphi) \,\rangle_{-} = -\int_{\mathcal{M}} \langle\, \mathcal{D}(\psi), \varphi \,\rangle_{-} \tag{4.1.41}$$

Although we have constructed this result by working purely on the level of the superpotential and the properties of the $\Delta^{*}(\mathsf{E}^{*})$ basis elements it is possible to construct the same result for non-zero $\mathcal{F}_0$ and $\mathcal{F}_1$ by a direct compactification of the ten dimensional string action [74, 75, 64, 63, 44, 29, 28] but it is difficult to construct non-geometric fluxes using such methods.

It is worth noting that for either inner product the sum of these expressions can be expressed in a very natural way in terms of $\underline{\Phi}$, $\underline{\underline{\mathcal{D}}}$ and the bilinear forms defined on $\mathbf{e}$, putting the two moduli spaces into a single expression.

This follows from the identity $\mho + \Omega = \underline{\Phi}^{\top} \cdot h \cdot \mathbf{e}$.

$$\begin{aligned} \int \left\langle \mho, \mathcal{D}(\Omega) \right\rangle_{\pm} + \int \left\langle \Omega, \mathcal{D}(\mho) \right\rangle_{\pm} &= g_{\nu}(\mho, \mathcal{D}(\Omega)) + g_{\mathfrak{a}}(\Omega, \mathcal{D}(\mho)) \\ &= g(\mho + \Omega, \mathcal{D}(\Omega) + \mathcal{D}(\mho)) \\ &= \underline{\Phi}^{\top} \cdot h \cdot \underline{\underline{\mathcal{D}}} \cdot g \cdot \underline{\Phi} \end{aligned} \qquad (4.1.42)$$

This superpotential-like expression treats the two moduli spaces in exactly the same manner and in the case of $\langle \; \rangle_{-}$ they are in fact equal. However this symmetry is broken when we consider the complexified holomorphic forms. We shall return to this result later when considering a particular set of internal spaces where such a symmetric formalism is possible even for the complexified holomorphic forms.

To examine precisely how the inclusion of dilaton couplings in the complexified holomorphic forms breaks this symmetry we recall the matrices associated to the holomorphic forms in the $\mathbf{e}_{(\mathfrak{a})}$ and $\mathbf{e}_{(\nu)}$ bases.

$$\Omega \;=\; \underline{\Omega}^{\top} \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \quad , \quad \mho \;=\; \underline{\mho}^{\top} \cdot h_{\nu} \cdot \mathbf{e}_{(\nu)}$$

With the Type IIA fluxes defined as $\mathcal{D}$ images of $\Delta^{+}(\mathsf{E}^{*})$ elements we consider the complexification of Ω, with the $\mho$ cases following the same general method.

$$\Omega_c = \begin{pmatrix} \mathfrak{U}_0 \\ \mathfrak{U}_i \\ \mathfrak{U}^0 \\ \mathfrak{U}^j \end{pmatrix}^{\top} \begin{pmatrix} -S\mathbb{I}_1 & & & \\ & \mathbb{I}_{h^{2,1}} & & \\ & & +S\mathbb{I}_1 & \\ & & & -\mathbb{I}_{h^{2,1}} \end{pmatrix} \begin{pmatrix} \mathfrak{a}_0 \\ \mathfrak{a}_i \\ \mathfrak{b}^0 \\ \mathfrak{b}^j \end{pmatrix} \qquad (4.1.43)$$

This can be factorised so that the complexification is due to a single[3] matrix,

[3]Infact there are two complexified holomorphic forms, Ω_c and Ω'_c and so we distinguish their complexification matrices with a prime, $\mathbb{C}$ and $\mathbb{C}'$ respectively.

$\mathbb{C}$, which modifies the original expressions for the holomorphic forms.

$$\Omega = \underline{\mathfrak{U}}^{\top} \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \quad \rightarrow \quad \Omega_c \equiv \underline{\mathfrak{U}}^{\top} \cdot \mathbb{C} \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})}$$

The matrix expression for $\mathbb{C}$ can be easily read off from the definition of Ω_c, factorising out the $h_{\mathfrak{a}}$ term from the matrix expression of (4.1.43). The $\mathbb{C}'$ case for Ω'_c follows in the same way.

$$\mathbb{C} = \begin{pmatrix} -S\mathbb{I}_1 & & & \\ & \mathbb{I}_{h^{2,1}} & & \\ & & -S\mathbb{I}_1 & \\ & & & \mathbb{I}_{h^{2,1}} \end{pmatrix} \quad , \quad \mathbb{C}' = \begin{pmatrix} \mathbb{I}_1 & & & \\ & -S\mathbb{I}_{h^{2,1}} & & \\ & & \mathbb{I}_1 & \\ & & & -S\mathbb{I}_{h^{2,1}} \end{pmatrix}$$

It will be useful for examining S duality later to express $\mathbb{C}$ and $\mathbb{C}'$ as linear combinations of a set of projection operators. The projection operators are such that they separate out the $\Delta^{3,0}(\mathsf{E}^*)$ and $\Delta^{0,3}(\mathsf{E}^*)$ basis elements from the $\Delta^{2,1}(\mathsf{E}^*)$ and $\Delta^{1,2}(\mathsf{E}^*)$ bases and are built from $\mathrm{SO}(n,m)$ metrics with signature $(+,\cdots,-,\cdots)$, which we shall denote as $\eta_{(n,m)}$.

$$\begin{aligned} \mathcal{A}_n \;&\equiv\; \mathbb{I}_2 \otimes \tfrac{1}{2}\Big(\eta_{(n+1,0)} - \eta_{(1,n)}\Big) \;=\; \mathbb{I}_2 \otimes \begin{pmatrix} 0 & \\ & \mathbb{I}_n \end{pmatrix} \;=\; \mathbb{I}_2 \otimes \mathfrak{A}_n \\ \mathcal{B}_n \;&\equiv\; \mathbb{I}_2 \otimes \tfrac{1}{2}\Big(\eta_{(n+1,0)} + \eta_{(1,n)}\Big) \;=\; \mathbb{I}_2 \otimes \begin{pmatrix} 1 & \\ & 0_n \end{pmatrix} \;=\; \mathbb{I}_2 \otimes \mathfrak{B}_n \end{aligned} \tag{4.1.44}$$

Of note are the following set of identities for combining $\mathfrak{A}_m$ and $\mathfrak{B}_n$ and since the dimensionalities of the matrices are unambigious we suppress the indices.

$$\mathfrak{A} \cdot \mathfrak{A} = \mathfrak{A} \quad , \quad \mathfrak{B} \cdot \mathfrak{B} = \mathfrak{B} \quad , \quad \mathfrak{A} \cdot \mathfrak{B} = 0 = \mathfrak{B} \cdot \mathfrak{A} \tag{4.1.45}$$

The $\mathcal{A}$ and $\mathcal{B}$ inherit the same set of identities due to their definitions in terms of $\mathfrak{A}$ and $\mathfrak{B}$ and it is these matrices which define the two complex structure complexified holomorphic forms.

$$\mathbb{C} = \mathcal{A}_{h^{2,1}} - S\mathcal{B}_{h^{2,1}} \qquad \mathbb{C}' = \mathcal{B}_{h^{2,1}} - S\mathcal{A}_{h^{2,1}}$$

With complexification having the effect[4] of $h_{\mathfrak{a}} \to \mathbb{C}^{(\prime)} \cdot h_{\mathfrak{a}}$ on the expansion of Ω the inner product expression for the T duality induced superpotential is obtained by altering the toy model expression previously found in (4.1.37).

$$\int_{\mathcal{M}} \left\langle\, \Omega_c, \mathcal{D}(\mho) \,\right\rangle_{\pm} = \underline{\mathfrak{T}}^{\top} \cdot h_{\nu} \cdot M \cdot g_{\mathfrak{a}} \cdot \mathbb{C} \cdot \underline{\mathfrak{U}} = \int_{\mathcal{M}} \left\langle\, \mho, \mathcal{D}(\Omega_c) \,\right\rangle_{-} \quad (4.1.46)$$

For the choice of $\langle \; \rangle \to \langle \; \rangle_{-}$ we have been able to move $\mathcal{D}$ to the other argument of the inner product using (4.1.41). This change has the advantage that it more closely resembles the Type IIB NS-NS superpotential, which we will construct next. Unfortunately the flux index structure does not have the manifest form required for the Lie algebra interpretation, which we will examine further in the next chapter.

4.2 The Type IIB Superpotential

Having constructed a Type IIA NS-NS superpotential for $\mathcal{M}$ on the grounds of the completion of the twelve dimensional algebra generated by X and Z we can construct the corresponding Type IIB NS-NS superpotential for $\mathcal{W}$ by the use of mirror symmetry. Given the self S duality nature of Type IIB the R-R sector can then be obtained by performing a modular transformation on the dilaton and considering resultant terms in the superpotential. We will not yet concern ourselves with the explicit relationship between the fluxes of each Type II superpotential, only the schematic form of the superpotentials. To that end we recall that in our analysis of the algebraic properties of the

[4]As $[\mathbb{C}, h_{\mathfrak{a}}] = 0$ this could alternatively be written as $h_{\mathfrak{a}} \cdot \mathbb{C}$ and likewise for other complexification matrices.

Type IIA on $\mathcal{M}$		Type IIB on $\mathcal{W}$
$\mathfrak{U}_I\,,\,\mathfrak{U}^J$	$I,J=0,\cdots,h^{2,1}$	$\mathsf{T}_I\,,\,\mathsf{T}^J$
$\mathfrak{T}_A\,,\,\mathfrak{T}^B$	$A,B=0,\cdots,h^{1,1}$	$\mathsf{U}_A\,,\,\mathsf{U}^B$

Table 4.2: The new moduli of mirror pair $\mathcal{M}$ and $\mathcal{W}$.

basis elements of $\Delta^*(\mathsf{E}^*)$ in $\mathcal{M}$ we were forced to redefine our basis elements and moduli used to expand the holomorphic forms of each moduli space, partly through invoking mirror symmetry. The moduli of $\mathcal{M}$ transformed as $(\mathcal{T},\mathcal{U}) \to (\mathfrak{T},\mathfrak{U})$ and we represent the moduli redefinitions of $\mathcal{W}$ in a similar manner, given in Table 4.2 which follows Table 3.2 in format. We use the same notation for the bases of $\Delta^3(\mathsf{E}^*)$ and $\Delta^+(\mathsf{E}^*)$ but the index labels are exchanged so as to illustrate the different dimensions of the moduli spaces.

4.2.1 The Type IIB NS-NS Flux Sector

As in the Type IIA case the Type IIB superpotential, in the absence of dualities, is determined by the 3-form flux H_3 which couples to the non-complexified holomorphic 3-form but none-the-less the superpotential has an overall factor of S.

$$W = \int_{\mathcal{W}} \langle\, \Omega, -S\,H_3\,\rangle_{\pm}$$

For parallelisable $\mathcal{W}$ the inclusion of geometric fluxes is obtained in the same manner as in Type IIA on $\mathcal{M}$ and by extension of the twelve dimensional Lie algebra globally and locally non-geometric fluxes are also induced [52, 53, 54].

$$\begin{aligned}
[\; Z_m \;,\; Z_n \;] &= H_{mnp}\; X^p \;-\; \mathsf{f}^q_{mn}\; Z_q \\
[\; Z_m \;,\; X^n \;] &= \mathsf{f}^n_{mp}\; X^p \;+\; \mathsf{Q}^{nq}_m\; Z_q \\
[\; X^m \;,\; X^n \;] &= \mathsf{Q}^{mn}_p\; X^p \;-\; \mathsf{R}^{mnq}\; Z_q
\end{aligned} \tag{4.2.1}$$

Given these fluxes possess the same index structure as the Type IIA fluxes to contribute to the superpotential via exterior multiplication with Ω they couple to elements of $\Delta^{+}(\mathsf{E}^{*})$. In Type IIA on $\mathcal{M}$ the moduli which coupled to the dilaton were $\mathfrak{U}_0$ and $\mathfrak{U}^0$ via $\Omega \to \Omega_c$ and $\mathcal{M}^Q = \mathcal{M}^Q(\mathfrak{U}, S)$. By mirror symmetry the corresponding moduli in Type IIB on $\mathcal{W}$ are T_0 and T^0 via $\mho \to \mho_c$ as $\mathcal{M}^Q = \mathcal{M}^Q(\mathsf{T}, S)$. Therefore H and R couple to the dilaton in their entirety and hence we denote their relabelled forms with a hat for later algebraic convenience.

$$
\begin{aligned}
\left[\; Z_m \;,\; Z_n \;\right] &= \widehat{\mathsf{F}}_{mnp}\; X^p \;-\; \mathsf{F}^{q}_{mn}\;\; Z_q \\
\left[\; Z_m \;,\; X^n \;\right] &= \mathsf{F}^{n}_{mp}\;\; X^p \;+\; \mathsf{F}^{nq}_{m}\;\; Z_q \\
\left[\; X^m \;,\; X^n \;\right] &= \mathsf{F}^{mn}_{p}\;\; X^p \;-\; \widehat{\mathsf{F}}^{mnq}\; Z_q
\end{aligned}
\tag{4.2.2}
$$

Following the same method as Type IIA we can construct a flux dependent operator[5] $\mathbf{G}$ such that its Bianchi constraints, $\mathbf{G}^2 = 0$, are synonymous with the Jacobi constraints of the Lie algebra.

$$
\begin{aligned}
\mathbf{G} &= \widehat{\mathsf{F}}_0 \cdot + \mathsf{F}_1 \cdot + \mathsf{F}_2 \cdot + \widehat{\mathsf{F}}_3 \cdot \\
&= \frac{1}{3!}\widehat{\mathsf{F}}_{mpq}\eta^{mpq} + \frac{1}{2!}\mathsf{F}^{m}_{pq}\eta^{pq}\iota_m + \frac{1}{2!}\mathsf{F}^{mp}_{q}\eta^{q}\iota_{pm} + \frac{1}{3!}\widehat{\mathsf{F}}^{mpq}\iota_{qpm}
\end{aligned}
\tag{4.2.3}
$$

The scalar product matrix expressions for this operator are defined by its action on the $\Delta^{*}(\mathsf{E}^{*})$ bases, though we need only consider the action on $\Delta^{3}(\mathsf{E}^{*})$ at present.

$$
\mathbf{G} \cdot \mathbf{f}_{(\nu)} \equiv \mathsf{G} \cdot \mathsf{h}_{\mathfrak{a}} \cdot \mathbf{f}_{(\mathfrak{a})} \quad , \quad \mathsf{G} \cdot \mathsf{h}_{\mathfrak{a}} = \begin{pmatrix} \mathcal{G}_{(I)A} & -\mathcal{G}_{(I)}{}^{B} \\ \mathcal{G}^{(J)}{}_{A} & -\mathcal{G}^{(J)B} \end{pmatrix} \tag{4.2.4}
$$

Given this matrix definition we could construct a scalar product expression for the superpotential whose integrand is $\left\langle \Omega, \mathbf{G}(\mho_c) \right\rangle_{\pm}$ but this is not the

[5]We consider $\mathbf{G}$ rather than some derivative as in the Type IIA case for reasons explained shortly.

Type IIB superpotential. It is known that the non-geometric flux F_2 contributes a linear Kähler moduli dependency to the superpotential by coupling to $\mathcal{J}_c = -\mathcal{T}_i \widetilde{\omega}^i$ [52, 53, 60]. This is in contrast to $\mathcal{F}_2$ in Type IIA, which gave quadratic Kähler dependence by coupling to $\mathcal{J}^{(2)}$. Therefore the integrand of the superpotential cannot be written as $\mathbf{G}(\mho_c)$, the fluxes of F_1 and F_2 couple to the Kähler moduli in the wrong manner. To rectify this in a way which leaves the Bianchi constraints invariant we consider two holomorphic forms, $\check{\Omega}$ and $\check{\mho}$, which are modifications of the standard expressions over and above simple relabellings.

$$\begin{aligned} \mho &= \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathbf{f}_{(\mathfrak{a})} \;\rightarrow\; \underline{\mathsf{T}}^\top \cdot \mathsf{L}^\top \cdot \mathsf{h}_\nu \cdot \mathbf{f}_{(\nu)} = \check{\mho} \\ \Omega &= \underline{\mathsf{U}}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \mathbf{f}_{(\nu)} \;\rightarrow\; \underline{\mathsf{U}}^\top \cdot \mathsf{K}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \mathbf{f}_{(\mathfrak{a})} = \check{\Omega} \end{aligned} \tag{4.2.5}$$

We include the possibility of the complex structure moduli being altered in a similar way because it allows us to consider a more general formulation and similar structures will be considered later. It is also convenient to define $\check{\mho}$ in terms of $\check{\mathcal{J}}^{(n)} \in \Delta^{2n}(\mathsf{E}^*)$, where $\check{\mathcal{J}}^{(2)} = \mathcal{J}_c$. We include $n = 0, 3$ even though their moduli dependence are not an issue.

$$\begin{aligned} \check{\mho} &= \check{\mathcal{J}}^{(0)} + \check{\mathcal{J}}^{(1)} + \check{\mathcal{J}}^{(2)} + \check{\mathcal{J}}^{(3)} \\ \mathbf{G}(\check{\mho}) &= \widehat{\mathsf{F}}_0 \cdot \check{\mathcal{J}}^{(0)} + \mathsf{F}_1 \cdot \check{\mathcal{J}}^{(1)} + \mathsf{F}_2 \cdot \check{\mathcal{J}}^{(2)} + \widehat{\mathsf{F}}_3 \cdot \check{\mathcal{J}}^{(3)} \end{aligned} \tag{4.2.6}$$

Given these modified holomorphic forms and $\mathbf{G}$ we are able to construct a scalar product which is a generalisation of the usual Type IIB superpotential, due to L and K, and which included the dilaton complexification $\mho \rightarrow \mho_c$ using the matrix C.

$$W = \int_{\mathcal{W}} \big\langle\, \check{\Omega}, \mathbf{G}(\check{\mho}_c) \,\big\rangle_\pm = \underline{\mathsf{T}}^\top \cdot \mathsf{L}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \mathsf{G} \cdot \mathsf{g}_\mathfrak{a} \cdot \mathsf{K} \cdot \underline{\mathsf{U}} \tag{4.2.7}$$

To put this into the standard superpotential scalar product format of the expressions in (4.1.38) we must move h_ν and C through $\mathsf{L}^\top$. With regards to C we make the assumption that L commutes with C, which will be justified later, because of its dilaton dependence which we don't wish to move into the fluxes in a non-trivial manner. In the case of h_ν we use the adjoint operator defined in (B.2.3) and $\mathsf{h}^2 = \mathbb{I}$ such that $\mathsf{L}^\top \cdot \mathsf{h}_\nu = \mathsf{h}_\nu \cdot \mathrm{Ad}_{\mathsf{h}_\nu}(\mathsf{L}^\top)$. We do likewise with K and $\mathsf{g}_\mathfrak{a}$.

$$\begin{aligned} W &= \int_{\mathcal{W}} \left\langle \check{\Omega}, \mathbf{G}(\check{\mho}_c) \right\rangle_\pm \\ &= \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \Big(\mathrm{Ad}_{\mathsf{h}_\nu}(\mathsf{L}^\top) \cdot \mathsf{G} \cdot \mathrm{Ad}_{\mathsf{g}_\mathfrak{a}}(\mathsf{K}) \Big) \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathsf{U}} \\ &\equiv \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \mathsf{M} \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathsf{U}} \\ &\equiv \int_{\mathcal{W}} \left\langle \Omega, \mathsf{D}(\mho_c) \right\rangle_\pm \end{aligned} \tag{4.2.8}$$

The flux matrix M is defined in the same manner as M and has partner N.

$$\mathsf{D} \cdot \mathbf{f}_{(\nu)} \equiv \mathsf{M} \cdot \mathsf{h}_\mathfrak{a} \cdot \mathbf{f}_{(\mathfrak{a})} \quad , \quad \mathsf{D} \cdot \mathbf{f}_{(\mathfrak{a})} \equiv \mathsf{N} \cdot \mathsf{h}_\nu \cdot \mathbf{f}_{(\nu)}$$

The entries of M determine those of N and so we can define the entries of both in the same manner as (4.1.32), but noting the change in index ranges.

$$\mathsf{M} \cdot \mathsf{h}_\mathfrak{a} = \begin{pmatrix} \mathsf{F}_{(I)A} & -\mathsf{F}_{(I)}{}^{B} \\ \mathsf{F}^{(J)}{}_{A} & -\mathsf{F}^{(J)B} \end{pmatrix} \quad , \quad \mathsf{N} \cdot \mathsf{h}_\nu = \begin{pmatrix} \mathsf{F}^{(I)A} & -\mathsf{F}_{(J)}{}^{A} \\ \mathsf{F}^{(I)}{}_{B} & -\mathsf{F}_{(J)B} \end{pmatrix} \tag{4.2.9}$$

We can express this interdependence between M and N in the same manner as their $\mathcal{D}$ counterparts in (4.1.35), that of a matrix equation.

$$N = \Sigma \cdot M^\top \cdot \mathsf{g}_\mathfrak{a}^\top \cdot h_\nu \quad , \quad \mathsf{N} = \Sigma \cdot \mathsf{M}^\top \cdot g_\mathfrak{a}^\top \cdot \mathsf{h}_\nu \tag{4.2.10}$$

These determine the action of D on the light $\Delta^*(\mathsf{E}^*)$ basis.

$$\begin{aligned} \mathsf{D}(\nu_I) &= \mathsf{F}_{(I)A}\mathfrak{a}_A - \mathsf{F}_{(I)}{}^{B}\mathfrak{b}^B \\ \mathsf{D}(\widetilde{\nu}^J) &= \mathsf{F}^{(J)}{}_{A}\mathfrak{a}_A - \mathsf{F}^{(J)B}\mathfrak{b}^B \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \mathsf{D}(\mathfrak{a}_A) &= \mathsf{F}^{(I)A}\nu_I - \mathsf{F}_{(J)}{}^{A}\widetilde{\nu}^J \\ \mathsf{D}(\mathfrak{b}^B) &= \mathsf{F}^{(I)}{}_{B}\nu_I - \mathsf{F}_{(J)B}\widetilde{\nu}^J \end{aligned} \tag{4.2.11}$$

The two ways the superpotential is written in (4.2.8) provides us the link between the entries of M defined above and the G matrix which is simplified by the identity $\mathrm{Ad}_y(x^{-1}) = \mathrm{Ad}_y(x)^{-1}$ and though not true in general the nature of the intersection numbers is such that $\mathrm{Ad}_y(x^\top) = \mathrm{Ad}_y(x)^\top$ if y is one of the bilinear forms.

$$\mathsf{M} \equiv \mathrm{Ad}_{\mathsf{h}_\nu}(\mathsf{L}^\top)\cdot\mathsf{G}\cdot\mathrm{Ad}_{\mathsf{g}_\mathfrak{a}}(\mathsf{K}) \quad , \quad \mathsf{G} \equiv \mathrm{Ad}_{\mathsf{h}_\nu}(\mathsf{L}^{\top 1})^{-1}\cdot\mathsf{M}\cdot\mathrm{Ad}_{\mathsf{g}_\mathfrak{a}}(\mathsf{K})^{-1} \tag{4.2.12}$$

The specific form of K and L is a matter of preference but they are constrained by a number of requirements. They are defined such that the Bianchi constraints of $\mathbf{G}$ and D are equivalent but in order to alter dilaton complexification they cannot mix T_0 and T^0 with T_i and T^j, these two sets of Kähler moduli must be transformed separately from one another. These matrices will be discussed in more detail when we consider the Type IIA R-R sector and the constraints when we consider Bianchi constraints. At present we only require the additional fact that L commutes with the dilaton complexification matrix. Due to the freedom in the choice of L and K we cannot express the entries of the Type IIB fluxes in terms of these components in the same manner as the Type IIA fluxes of (4.1.26). However, we can still associate particular flux components with which of the Type IIB fluxes they contribute to, even if we cannot explicitly state how they contribute, by moduli coefficients.

To examine this further we define a new set of fluxes akin to (4.2.6) such that we can discuss the individual contributions to the superpotential when using the standard holomorphic forms.

$$\begin{aligned} \mathsf{D}(\mho) &= (\star\widehat{\mathsf{F}}_0)\cdot\mathcal{J}^{(0)} + (\star\mathsf{F}_1)\cdot\mathcal{J}^{(1)} + (\star\mathsf{F}_2)\cdot\mathcal{J}^{(2)} + (\star\widehat{\mathsf{F}}_3)\cdot\mathcal{J}^{(3)} \\ \mathbf{G}(\check{\mho}) &= \widehat{\mathsf{F}}_0\cdot\check{\mathcal{J}}^{(0)} + \mathsf{F}_2\cdot\check{\mathcal{J}}^{(2)} + \mathsf{F}_1\cdot\check{\mathcal{J}}^{(1)} + \widehat{\mathsf{F}}_3\cdot\check{\mathcal{J}}^{(3)} \end{aligned} \tag{4.2.13}$$

Given this definition of the new fluxes and the action of D in (4.2.11) we can give a component definition to each of the fluxes and determine which flux components of (4.2.11) relate to which $\widehat{\mathcal{F}}_m$ and $\mathcal{F}_n$.

$$\begin{aligned}
\star\widehat{\mathsf{F}}_0 &\equiv \mathsf{F}^{(0)}{}_A\mathfrak{a}_A\iota_{\widetilde{\nu}^0} &-& \mathsf{F}^{(0)B}\mathfrak{b}^B\iota_{\widetilde{\nu}^0} \sim \widehat{\mathsf{F}}_0 \\
\star\mathsf{F}_1 &\equiv \mathsf{F}_{(i)A}\mathfrak{a}_A\iota_{\nu_i} &-& \mathsf{F}_{(a)}{}^B\mathfrak{b}^B\iota_{\nu_i} \sim \mathsf{F}_2 \\
\star\widehat{\mathsf{F}}_3 &\equiv \mathsf{F}_{(0)A}\mathfrak{a}_A\iota_{\nu_0} &-& \mathsf{F}_{(0)}{}^B\mathfrak{b}^B\iota_{\nu_0} \sim \widehat{\mathsf{F}}_3 \\
\star\mathsf{F}_2 &\equiv \mathsf{F}^{(j)}{}_A\mathfrak{a}_A\iota_{\widetilde{\nu}^j} &-& \mathsf{F}^{(j)B}\mathfrak{b}^B\iota_{\widetilde{\nu}^j} \sim \mathsf{F}_1
\end{aligned} \tag{4.2.14}$$

This index structure for the components of the fluxes is completely the reverse of the index structures of (4.1.20). The Kähler moduli coupling affects the F_1 and F_2 components while our redefinition of the $\Delta^+(\mathsf{E}^*)$ basis affected the $\widehat{\mathsf{F}}_0$ and $\widehat{\mathsf{F}}_3$ components. The choice of a different basis such that $\omega_a \leftrightarrow \widetilde{\omega}^a$ rather then $\omega_0 \leftrightarrow \widetilde{\omega}^0$ would have countered this reversal but would lead to less pleasant results later.

The components of F_n and those of $\star\mathsf{F}_n$ are generally different but due to the manner in which we have defined K and L to mix the Kähler moduli they do couple to the dilaton in the same way. In Type IIB we are able to drop K from our considerations because it can be absorbed into the holomorphic form via a symplectic transformation. The fact K has this property will be demonstrated to follow from the requirement $\mathbf{G}$ and D define the same Bianchi constraints in the next chapter. This is also the case for Type IIA, we need not consider $\Omega \to \check{\Omega}$ as it can be transformed away. This seems at odds with the fact mirror symmetry should map the modified Type IIB $\check{\mho}_c$ into a modified Type IIA $\check{\Omega}_c$ so exchanges of moduli should apply to both Type II constructions. However, the exchange comes from the different manner in which the Type II theories label their complex structure moduli. This will be explored further in our analysis of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold as it is self mirror dual [64, 61] and yet the $\mathfrak{E}$ defined expressions for the $\Delta^-(\mathsf{E}^*)$ basis are dependent on which Type II theory is being considered [60, 92].

We shall keep an explicit dependence in our more general analysis in order to illustrate additional symmetries in the superpotential and we return to this in Chapter 6.

4.2.2 The Type IIB R-R Flux Sector

The R-R sector of Type IIB follows from the NS-NS sector by S duality transformations. We have previously discussed these transformations in the context of those fluxes which are obtained by compactification of the ten dimensional action, $H_3 = \widehat{\mathsf{F}}_0$ and $F_3 \equiv \mathsf{F}_0$. These two fluxes transform as a doublet and their D- and F-string charge quantisation conditions break the $\mathrm{SL}(2,\mathbb{R})$ continuous symmetry of (3.3.3) to $\mathrm{SL}(2,\mathbb{Z})_S$.

$$S \to \frac{n_1 S + n_2}{n_3 S + n_4} \quad \Leftrightarrow \quad \begin{pmatrix} \mathsf{F}_0 \\ \widehat{\mathsf{F}}_0 \end{pmatrix} \to \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \begin{pmatrix} \mathsf{F}_0 \\ \widehat{\mathsf{F}}_0 \end{pmatrix} \tag{4.2.15}$$

This symmetry is evident in the Type IIB superpotential (2.5.2), the fluxes and the dilaton transform in precisely the same manner if the superpotential is to be invariant, up to a gauge freedom.

$$\mathsf{F}_0 - S\widehat{\mathsf{F}}_0 \to \frac{1}{n_3 S + n_4}\left(n_1 n_4 - n_2 n_3\right)\left(\mathsf{F}_0 - S\widehat{\mathsf{F}}_0\right)$$

This doublet structure holds true for the other NS-NS fluxes, each of which inducing a partner in the R-R sector. Given the identical index structure an R-R partner to the Lie algebra (4.2.2) can be constructed from generators $\mathbf{X}$ and $\mathbf{Z}$, which are the magnetic duals of the X and Z [60].

$$\begin{aligned} \left[\ \mathbf{Z}_m\ ,\ \mathbf{Z}_n\ \right] &= \mathsf{F}_{mnp}\ \mathbf{X}^p - \widehat{\mathsf{F}}^p_{mn}\ \mathbf{Z}_p \\ \left[\ \mathbf{Z}_m\ ,\ \mathbf{X}^n\ \right] &= \widehat{\mathsf{F}}^n_{mp}\ \mathbf{X}^p + \widehat{\mathsf{F}}^{np}_m\ \mathbf{Z}_p \\ \left[\ \mathbf{X}^m\ ,\ \mathbf{X}^n\ \right] &= \widehat{\mathsf{F}}^{mn}_p\ \mathbf{X}^p - \mathsf{F}^{mnp}\ \mathbf{Z}_p \end{aligned} \tag{4.2.16}$$

These can be seen to follow from $\mho_c$, as all dilaton dependence in the NS-NS superpotential arises through it. Under the modular inversion $S \to$

$-\frac{1}{S}$ the NS-NS fluxes are all exchanged to their R-R counterparts and the corresponding dilaton dependent Kähler holomorphic form is $\mho'_c$ defined in Appendix A.2. By considering this transformation on (4.2.13) we obtain the new fluxes.

$$\mathsf{D}'(\mho'_c) = (\star \mathsf{F}_0)\cdot \mathcal{J}^{(0)} - S\,(\star\widehat{\mathsf{F}}_1)\cdot\mathcal{J}^{(1)} - S\,(\star\widehat{\mathsf{F}}_2)\cdot\mathcal{J}^{(2)} + (\star\mathsf{F}_3)\cdot\mathcal{J}^{(3)} \quad (4.2.17)$$

The components of the R-R fluxes we shall denote in much the same manner as (4.2.11), but with a hat, and the associated derivative as D'.

$$\begin{array}{rclcrcl} \mathsf{D}'(\nu_I) &=& \widehat{\mathsf{F}}_{(I)A}\mathfrak{a}_A - \widehat{\mathsf{F}}_{(I)}{}^B\mathfrak{b}^B & \Leftrightarrow & \mathsf{D}'(\mathfrak{a}_A) &=& \widehat{\mathsf{F}}^{(I)A}\nu_I - \widehat{\mathsf{F}}_{(J)}{}^A\widetilde{\nu}^J \\ \mathsf{D}'(\widetilde{\nu}^J) &=& \widehat{\mathsf{F}}^{(J)}{}_A\mathfrak{a}_A - \widehat{\mathsf{F}}^{(J)B}\mathfrak{b}^B & & \mathsf{D}'(\mathfrak{b}^B) &=& \widehat{\mathsf{F}}^{(I)}{}_B\nu_I - \widehat{\mathsf{F}}_{(J)B}\widetilde{\nu}^J \end{array} \quad (4.2.18)$$

The relationship between these components and those of the R-R version of $\mathbf{G}$, $\mathbf{G}'$, is precisely the same as the NS-NS case.

4.2.3 T and S Duality Invariant Superpotential

With the R-R superpotential being of the same schematic form as the NS-NS superpotential the full T and S duality invariant Type IIB superpotential can be succinctly stated in terms of an integral or a scalar product.

$$\int_{\mathcal{W}} \big\langle\, \Omega, \Big(\mathsf{D}(\mho_c) + \mathsf{D}'(\mho'_c)\Big)\,\big\rangle_{\pm} = \underline{\mathsf{T}}^{\top}\cdot \mathsf{h}_{\nu}\cdot \Big(\mathsf{C}\cdot\mathsf{M} + \mathsf{C}'\cdot\mathsf{M}'\Big)\cdot \mathsf{g}_{\mathfrak{a}}\cdot\underline{\mathsf{U}} \quad (4.2.19)$$

To express this in terms of the components of M and M' we first construct the expressions for $\mathsf{D}(\mho_c)$ and $\mathsf{D}'(\mho'_c)$, taking into account the sign choice in $\mho$ and its complexifications. We collect the coefficients of $\mathfrak{a}_A$ and $\mathfrak{b}^B$ for convenience when performing the superpotential integral.

$$\begin{aligned} \mathsf{D}(\mho_c) =& \Big(-S\mathsf{T}_0\mathsf{F}_{(0)A} + \mathsf{T}_i\mathsf{F}_{(i)A} - S(\pm\mathsf{T}^0)\mathsf{F}^{(0)}{}_A + (\pm\mathsf{T}^j)\mathsf{F}^{(j)}{}_A\Big)\mathfrak{a}_A \\ &-\Big(-S\mathsf{T}_0\mathsf{F}_{(0)}{}^B + \mathsf{T}_i\mathsf{F}_{(i)}{}^B - S(\pm\mathsf{T}^0)\mathsf{F}^{(0)B} + (\pm\mathsf{T}^j)\mathsf{F}^{(j)B}\Big)\mathfrak{b}^B \\ \mathsf{D}'(\mho'_c) =& \Big(\mathsf{T}_0\widehat{\mathsf{F}}_{(0)A} - S\mathsf{T}_i\widehat{\mathsf{F}}_{(i)A} + (\pm\mathsf{T}^0)\widehat{\mathsf{F}}^{(0)}{}_A - S(\pm\mathsf{T}^j)\widehat{\mathsf{F}}^{(j)}{}_A\Big)\mathfrak{a}_A \\ &-\Big(\mathsf{T}_0\widehat{\mathsf{F}}_{(0)}{}^B - S\mathsf{T}_i\widehat{\mathsf{F}}_{(i)}{}^B + (\pm\mathsf{T}^0)\widehat{\mathsf{F}}^{(0)B} - S(\pm\mathsf{T}^j)\widehat{\mathsf{F}}^{(j)B}\Big)\mathfrak{b}^B \end{aligned}$$

Exterior pre-multiplication by Ω and performing the integral over $\mathcal{W}$ provides us with the polynomial expression for the superpotential, which we split into the NS-NS and R-R parts and collect the coefficients of T so as to simplify the sign structure for $\langle \ \rangle_{\pm}$.

$$\begin{aligned} W_{NS} = & \quad S\Big(\mathsf{U}_B\mathsf{F}_{(0)}{}^{B} - \mathsf{U}^A\mathsf{F}_{(0)A}\Big)\mathsf{T}_0 + \Big(\mathsf{U}^A\mathsf{F}_{(i)A} - \mathsf{U}_B\mathsf{F}_{(i)}{}^{B}\Big)\mathsf{T}_i \\ & \pm S\Big(\mathsf{U}_B\mathsf{F}^{(0)B} - \mathsf{U}^A\mathsf{F}^{(0)}{}_{A}\Big)\mathsf{T}^0 \pm \Big(\mathsf{U}^A\mathsf{F}^{(0)}{}_{A} - \mathsf{U}_B\mathsf{F}^{(j)B}\Big)\mathsf{T}^j \\ W_R = & - \Big(\mathsf{U}_B\widehat{\mathsf{F}}_{(0)}{}^{B} - \mathsf{U}^A\widehat{\mathsf{F}}_{(0)A}\Big)\mathsf{T}_0 + S\Big(\mathsf{U}_B\widehat{\mathsf{F}}_{(i)}{}^{B} - \mathsf{U}^A\widehat{\mathsf{F}}_{(i)A}\Big)\mathsf{T}_i \\ & \pm \Big(\mathsf{U}^A\widehat{\mathsf{F}}^{(0)}{}_{A} - \mathsf{U}_B\widehat{\mathsf{F}}^{(0)B}\Big)\mathsf{T}^0 \pm S\Big(\mathsf{U}_B\widehat{\mathsf{F}}^{(j)B} - \mathsf{U}^A\widehat{\mathsf{F}}^{(0)}{}_{A}\Big)\mathsf{T}^j \end{aligned} \tag{4.2.20}$$

4.3 The Type IIA R-R Flux Sector

4.3.1 U Duality Induced Fluxes

Type IIA does not possess the same $\mathrm{SL}(2,\mathbb{Z})_S$ self-duality as Type IIB and even allowing for the possibility that Type IIA supergravity might have symmetries Type IIA string theory does not, it is clear from the Type IIA superpotential in (2.5.4) that the flux structure of the two sectors are not schematically the same. To examine the R-R sector of Type IIA we view it as the T dual of the Type IIB sector. Before we considered S duality in Type IIB we obtained the R-R sectors of each theory from the SO(8) representations of string oscillations and these were seen to couple to branes. Under T duality the brane content of theory is exchanged due to T duality's effects on the boundary conditions and hence the R-R sector of the two theories are T duality invariant without the inclusion of any further fluxes [52, 60]. This can be seen in a straight forward manner by recalling from Section 2.5 the R-R fluxes of each Type II theory which descend from the ten dimensional actions.

$$\begin{aligned} F_{RR} &\equiv \mathfrak{f}_A \nu_A \pm \mathfrak{f}^B \widetilde{\nu}^B &\Rightarrow& \quad \int_{\mathcal{M}} \langle \mho, F_{RR} \rangle_{\pm} = \mathfrak{f}_A \mathfrak{T}^A \pm \mathfrak{f}^B \mathfrak{T}_B \\ F_3 &\equiv \mathfrak{f}_A \alpha_A - \mathfrak{f}^B \beta^B &\Rightarrow& \quad \int_{\mathcal{W}} \langle \Omega, F_3 \rangle_{\pm} = f_A \mathsf{U}^A - f^B \mathsf{U}_B \end{aligned} \tag{4.3.1}$$

With the inclusion of S duality the Type IIB side is greatly extended by the NS-NS sector inducing the entire R-R sector and these feed through to the Type IIA R-R side by T duality. In Section 3.2 we commented that mirror dual superpotentials need not have the same moduli dependency due to quantum corrections, a Type IIA superpotential linear in $\mathcal{T}$ might be dual to a Type IIB superpotential quadratic in U. However the number of independent fluxes should be equal and thus we can infer the existent of further fluxes in the Type IIA R-R sector because of the larger number of independent fluxes in the Type IIB R-R sector. Despite not being usable in determing mirror paired superpotentials moduli dependency still provides a guide in how to reformulate the superpotential integrands of each Type II construction. As such, the lack of any complex structure or Kähler moduli dependency in the first and second expressions, respectively, in (4.3.1) prompts us to make the dependency on the projective coordinates $\mathcal{U}_0 = \mathfrak{U}^0$ and $\widetilde{\mathcal{T}}_0 = \mathsf{T}^0$ explicit.

$$\int_{\mathcal{M}} \langle \mho, F_{RR} \rangle_{\pm} = \mathfrak{U}^0 (\mathfrak{f}_A \mathfrak{T}^A \pm \mathfrak{f}^B \mathfrak{T}_B) \quad , \quad \int_{\mathcal{W}} \langle \Omega, F_3 \rangle_{\pm} = \mathsf{T}^0 (f_A \mathsf{U}^A - f^B \mathsf{U}_B)$$

Though we have not stated it this relationship makes the assumption that the three T dualities or mirror symmetry used is such that the Type IIA $\mathfrak{f}_0$ coefficient of ν_0 in F_{RR} is mapped to the coefficient of $\mathfrak{a}_0$ in F_3. This is seen more clearly if we revert back to the original $\Delta^*(\mathsf{E}^*)$ basis and note that there are two ways to map between Type IIA and Type IIB which preserve the schematic dilaton dependence of the $\mathcal{M}^Q$ holomorphic form. For the

$\mathcal{N}=1$ construction this is akin to the choice of the orientifold projection.

$$\mathfrak{M} : \alpha_0 \in \text{IIA} \quad \to \quad \left\{ \begin{array}{lll} \omega_0 \in \text{IIB} & : & \text{IIA/O6} \to \text{IIB/O3} \\ \widetilde{\omega}^0 \in \text{IIB} & : & \text{IIA/O6} \to \text{IIB/O9} \end{array} \right. \tag{4.3.2}$$

As previously commented in regards to mirror symmetry this is not always the case due to quantum corrections because it does not always preserve the moduli polynomial order of the superpotential. For toroidal, and thus parallelisable, spaces this is not an issue as they receive no such corrections. As such we can define the directions which are transformed by stipulating the $\Delta^+(\mathsf{E}^*)(\mathcal{M})$ and $\Delta^3(\mathsf{E}^*)(\mathcal{W})$ bases or we can define our basis of $\Delta^3(\mathsf{E}^*)(\mathcal{W})$ by stipulating $\Delta^+(\mathsf{E}^*)(\mathcal{M})$ and the directions T duality or mirror symmetry are applied to. In the latter case for parallelisable spaces the R-R Buscher rules of (3.1.4) are used. We examine this further in Section 6.2.3 for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold whose orbifold symmetries make for greatly simplified algebra.

In the Type IIA NS-NS flux sector and both flux sectors of Type IIB the Kähler moduli are obtained by the fluxes contracting with $\mathcal{J}^{(n)}$ terms in $\mho$, or its dilaton complexifications, but this cannot be the case for the Type IIA R-R superpotential. Instead the inclusion of $\mathcal{U}_0 = \mathfrak{U}^0$ suggests that F_{RR} is formed by a flux acting upon the first term in the expansion of Ω, $\mathcal{U}_0\alpha_0$, which is the last term in the new basis, $\mathfrak{U}^0\mathfrak{b}^0$. Using this as motivation we define a new expansion for Ω akin to that of $\mho$'s expansion in the $\mathcal{J}^{(n)}$.

$$\begin{aligned} \Omega &= \mathfrak{J}^{(0)} + \mathfrak{J}^{(1)} + \mathfrak{J}^{(2)} + \mathfrak{J}^{(3)} \\ &= \mathcal{U}_0\alpha_0 + \mathcal{U}_i\alpha_i - \mathcal{U}^j\beta^j - \mathcal{U}^0\beta^0 \\ &= -\mathfrak{U}^0\mathfrak{b}^0 + \mathfrak{U}_i\mathfrak{a}_i - \mathcal{U}^j\mathfrak{b}^j + \mathfrak{U}_0\mathfrak{a}_0 \end{aligned} \tag{4.3.3}$$

If F_{RR} is to have $\mathcal{U}_0 = \mathcal{U}^0$ dependence then it must be formed from an operator acting on $\mathfrak{J}^{(0)}$ and so we have a contribution to the R-R derivative

$\mathcal{D}''$.

$$\mathcal{U}_0 F_{RR} \equiv \mathcal{D}' \cdot \mathfrak{J}^{(0)} = \mathcal{D}' \cdot (\mathcal{U}_0 \alpha_0) = \mathcal{D}' \cdot (-\mathfrak{U}^0 \mathfrak{b}^0)$$

In Type IIB the R-R fluxes coupled to a second dilaton dependent holomorphic form $\mho'_c$ and thus $\mathcal{D}'$ couples to a second dilaton dependent holomorphic 3-form Ω'_c. Their schematic structure is discussed in Appendix A.2.2, where they are generically denoted as Φ'_c. Just as we denoted those Type IIB fluxes which couple to the dilaton with a hat we are able do the same with the Type IIA case as both have contributions of the form $D(\Phi'_c)$.

$$\mathcal{D}' = \mathfrak{F}_0 \cdot + \widehat{\mathfrak{F}}_1 \cdot + \widehat{\mathfrak{F}}_2 \cdot + \mathfrak{F}_3 \cdot \tag{4.3.4}$$

Due to the fact $\mathcal{D}' : \Delta^3(\mathsf{E}^*) \to \Delta^+(\mathsf{E}^*)$ the individual fluxes do not have a simple index structure as that seen in (4.1.4).

$$\mathcal{F}_n \in \Lambda^{3-n}(\mathsf{E}^*) \wedge \Lambda^n(\mathsf{E}) \not\ni \mathfrak{F}_n \tag{4.3.5}$$

As a result we cannot give the same $\Lambda^3(\mathfrak{E})$ component structures to the individual terms in $\mathcal{D}'$.

$$\mathcal{D}' \neq \frac{1}{3!}\mathfrak{F}_{mpq}\eta^{mpq} + \frac{1}{2!}\widehat{\mathfrak{F}}^m_{pq}\eta^{pq}\iota_m + \frac{1}{2!}\widehat{\mathfrak{F}}^{mp}_q\eta^q\iota_{pm} + \frac{1}{3!}\mathfrak{F}^{mqp}\iota_{pqm} \tag{4.3.6}$$

This is the disadvantage illuded to in (4.1.46) when we change which holomorphic form the derivative acted on. As a result of this fact, that we define the components of the fluxes of $\mathcal{D}'$ by the action of the derivative on the $\Delta^3(\mathsf{E}^*)$ bases, we have a sign ambiguity steming from how we define the components of fluxes with regards to $\langle\ \rangle_\pm$ and this filters through to the action of $\mathcal{D}'$ on $\Delta^+(\mathsf{E}^*)$.

$$\begin{array}{rcl} \mathcal{D}'(\mathfrak{a}_I) &=& \widehat{\mathfrak{F}}_{(I)A}\nu_A \pm \widehat{\mathfrak{F}}_{(I)}{}^B\widetilde{\nu}^B \\ \mathcal{D}'(\mathfrak{b}^J) &=& \widehat{\mathfrak{F}}^{(J)}{}_A\nu_A \pm \widehat{\mathfrak{F}}^{(J)B}\widetilde{\nu}^B \end{array} \Leftrightarrow \begin{array}{rcl} \mathcal{D}'(\nu_A) &=& \mp\widehat{\mathfrak{F}}^{(I)A}\mathfrak{a}_I \pm \widehat{\mathfrak{F}}_{(J)}{}^A\mathfrak{b}^J \\ \mathcal{D}'(\widetilde{\nu}^B) &=& \widehat{\mathfrak{F}}^{(I)}{}_B\mathfrak{a}_I - \widehat{\mathfrak{F}}_{(J)B}\mathfrak{b}^J \end{array} \tag{4.3.7}$$

The flux matrices associated with these actions are defined in the standard manner, $\mathcal{D}'(\mathbf{e}) = \underline{\underline{\mathcal{D}'}} \cdot h \cdot \mathbf{e}$, where the entries of $\underline{\underline{\mathcal{D}'}}$ are defined as in (4.1.31) but with primes.

$$N' \cdot h_\nu = \begin{pmatrix} \widehat{\mathfrak{F}}_{(I)A} & \pm\widehat{\mathfrak{F}}_{(I)}{}^{B} \\ \widehat{\mathfrak{F}}^{(J)}{}_{A} & \pm\widehat{\mathfrak{F}}^{(J)B} \end{pmatrix} \quad , \quad M' \cdot h_{\mathfrak{a}} = \begin{pmatrix} \mp\widehat{\mathfrak{F}}^{(I)A} & \pm\widehat{\mathfrak{F}}_{(J)}{}^{A} \\ \widehat{\mathfrak{F}}^{(I)}{}_{B} & -\widehat{\mathfrak{F}}_{(J)B} \end{pmatrix} \tag{4.3.8}$$

These flux matrices are related to one another in the same way as the NS-NS flux matrices in (4.1.34) and (4.1.35) and we surpress transpositions, trusting in the index constractions to prevent ambiguity. We define the Type IIA R-R superpotential's integrand to be of the same form as the Type IIB R-R superpotential but with the holomorphic forms exchanged, as suggested by (4.3.1).

$$W_R = \int_{\mathcal{M}} \left\langle\, \mho, \mathcal{D}'(\Omega'_c) \,\right\rangle_{\pm} = \underline{\mathfrak{U}}^{\top} \cdot h_{\mathfrak{a}} \cdot \mathbb{C}' \cdot M' \cdot g_\nu \cdot \underline{\mathfrak{T}} \tag{4.3.9}$$

In the consideration of flux constraints in the next chapter we will find it convenient to regroup these fluxes such that the R-R sector is written in the same manner as the NS-NS sector. To that end we redefine the components of M' so as to match the structure of the components of M.

$$M' \cdot h_{\mathfrak{a}} = \begin{pmatrix} \mp\widehat{\mathfrak{F}}^{(I)A} & \pm\widehat{\mathfrak{F}}_{(J)}{}^{A} \\ \widehat{\mathfrak{F}}^{(I)}{}_{B} & -\widehat{\mathfrak{F}}_{(J)B} \end{pmatrix} \equiv \begin{pmatrix} \widehat{\mathcal{F}}_{(A)I} & -\widehat{\mathcal{F}}_{(A)}{}^{J} \\ \widehat{\mathcal{F}}^{(B)}{}_{I} & -\widehat{\mathcal{F}}^{(B)J} \end{pmatrix} \tag{4.3.10}$$

This reformulation allows us to use the NS-NS sector's results simply by applying the relabelling $\mathcal{F} \leftrightarrow \widehat{\mathcal{F}}$ on all flux expressions. Furthermore on parallelisable spaces we are more easily able to express $\mathcal{D}'$ in terms of flux multiplets defined by their mutual $\mathfrak{E} = \mathsf{E} \oplus \mathsf{E}^*$ index structures.

$$\begin{aligned} \mathcal{D}' &= \widehat{\mathcal{F}}_0 + \widehat{\mathcal{F}}_1 + \widehat{\mathcal{F}}_2 + \widehat{\mathcal{F}}_3 = \mathbf{e}^{\top}_{(\mathfrak{a})} \cdot h^{\top}_{\mathfrak{a}} \cdot M'^{\top} \cdot \iota_{\mathbf{e}_{(\nu)}} \\ &= \frac{1}{3!}\widehat{\mathcal{F}}_{mpq}\eta^{mpq} + \frac{1}{2!}\widehat{\mathcal{F}}^{m}_{pq}\eta^{pq}\iota_m + \frac{1}{2!}\widehat{\mathcal{F}}^{mp}_{q}\eta^{q}\iota_{pm} + \frac{1}{3!}\widehat{\mathcal{F}}^{mpq}\iota_{pqm} \end{aligned} \tag{4.3.11}$$

4.3.2 U Duality Invariant Superpotential

Using the integral definition of the Type IIA R-R superpotential given in (4.3.9) we can obtain the polynomial form by using the flux components defined in (4.3.8) to construct $\mathcal{D}'(\Omega'_c)$ for the inner product $\langle\ \rangle_{\pm}$.

$$\begin{aligned}\mathcal{D}'(\Omega'_c) = & \left(\mathfrak{U}_0\widehat{\mathfrak{F}}_{(0)A} - S\,\mathfrak{U}_i\widehat{\mathfrak{F}}_{(i)A} - \mathfrak{U}^0\widehat{\mathfrak{F}}^{(0)}{}_A + S\,\mathfrak{U}^j\widehat{\mathfrak{F}}^{(j)}{}_A\right)\nu_A \\ & \pm\left(\mathfrak{U}_0\widehat{\mathfrak{F}}_{(0)}{}^B - S\,\mathfrak{U}_i\widehat{\mathfrak{F}}_{(i)}{}^B - \mathfrak{U}^0\widehat{\mathfrak{F}}^{(0)B} + S\,\mathfrak{U}^j\widehat{\mathfrak{F}}^{(j)B}\right)\widetilde{\nu}^B\end{aligned}$$

Combining this with $\mho$ via $\left\langle \mho, \mathcal{D}'(\Omega'_c)\right\rangle_{\pm}$ and performing the integral over $\mathcal{W}$ provides us with the polynomial expression for the superpotential and we collect the coefficients of $\mathfrak{T}$ so as to make for an easy comparison with (4.1.28).

$$\begin{aligned}W_R = & \left(\mathfrak{T}^A\widehat{\mathfrak{F}}_{(0)A} \pm \mathfrak{T}_B\widehat{\mathfrak{F}}_{(0)}{}^B\right)\mathfrak{U}_0 - S\left(\mathfrak{T}^A\widehat{\mathfrak{F}}_{(i)A} \pm \mathfrak{T}_B\widehat{\mathfrak{F}}_{(i)}{}^B\right)\mathfrak{U}_i \\ & - \left(\mathfrak{T}^A\widehat{\mathfrak{F}}^{(0)}{}_A \pm \mathfrak{T}_B\widehat{\mathfrak{F}}^{(0)B}\right)\mathfrak{U}^0 + S\left(\mathfrak{T}^A\widehat{\mathfrak{F}}^{(0)}{}_A \pm \mathfrak{T}_B\widehat{\mathfrak{F}}^{(j)B}\right)\mathfrak{U}^j\end{aligned} \tag{4.3.12}$$

Although this superpotential is of the same order in each modulus as (4.1.28) the structure of the flux components are different. Instead the structure is closer to the Type IIB superpotential (4.2.20) and so we now turn to equating the Type II superpotentials in each flux sector to examine this further.

4.4 Type II Flux Interdependency

We have used S, T and U duality to motivate the existence and structure of four derivatives associated to the flux sectors of each Type II theory but thus far we have labelled the components of the derivatives independently. In our discussion of this we will find it convenient to refer to the various superpotential constructions of each flux sector in each Type II construction in terms of their moduli dependence. As a result we elaborate slightly on the superpotential expressions, begining with the NS-NS case.

$$\begin{aligned} W^{\rm NS}_{\rm IIA}(\mathfrak{U}_I,\mathfrak{T}_A) &= \int_{\mathcal{M}} \big\langle\, \Omega_c(\mathfrak{U},S), \mathcal{D}(\mho)(\mathfrak{T})\,\big\rangle_{\pm} \\ W^{\rm NS}_{\rm IIB}(\mathsf{T}_I,\mathsf{U}_A) &= \int_{\mathcal{W}} \big\langle\, \Omega(\mathsf{U}), \mathsf{D}(\mho_c)(\mathsf{T},S)\,\big\rangle_{\pm} \end{aligned} \tag{4.4.1}$$

Since we wish to consider both $\langle\ \rangle_{\pm}$ we have not altered the Type IIA case through the result of (4.1.41) on $\langle\ \rangle_{-}$ so as to bring it inline with the other flux sector superpotentials where the derivative acts on the $\mathcal{M}^Q$ holomorphic form.

$$\begin{aligned} W^{\rm RR}_{\rm IIA}(\mathfrak{U}_I,\mathfrak{T}_A) &= \int_{\mathcal{M}} \big\langle\, \mho(\mathfrak{T}), \mathcal{D}'(\Omega'_c)(\mathfrak{U},S)\,\big\rangle_{\pm} \\ W^{\rm RR}_{\rm IIB}(\mathsf{T}_I,\mathsf{U}_A) &= \int_{\mathcal{W}} \big\langle\, \Omega(\mathsf{U}), \mathsf{D}'(\mho'_c)(\mathsf{T},S)\,\big\rangle_{\pm} \end{aligned} \tag{4.4.2}$$

If the Type II superpotentials are to be equivalent then the fluxes of D define, and are defined by, the fluxes of $\mathcal{D}$ and likewise for D' and $\mathcal{D}'$. To obtain the explicit expressions we use two methods;

- Compare polynomial coefficients in the superpotentials.
- Compare scalar products in terms of the flux matrices.

The latter method is considerably more compact in its algebraic methodology and allows us to consider quantum corrections to the moduli equivalences. The former method cannot accomodate quantum corrections in a convenient manner but for cases where no such corrections exist, such as the $\mathbb{Z}_2\times\mathbb{Z}_2$ orientifold case we examine later, it is more convenient for explicit calculations.

4.4.1 Moduli Equivalences

It is important to note that this comparision of mirror dual superpotentials is dependent upon how we choose to relate the moduli of $\mathcal{M}$ to those of $\mathcal{W}$. In the simplest cases of $(\mathcal{M},\mathcal{W})$ pairs, such as toroidal compactifications, they take the form of exact solutions to the background equations of motion. As a result of this the superpotentials associated to their moduli receive no quantum corrections and they match order by order in the moduli once the relabelling has been accounted for.

$$\mathfrak{U}_I \equiv \varsigma_I \equiv \mathsf{T}_I \quad , \quad \mathfrak{T}_A \equiv \varrho_A \equiv \mathsf{U}_A \tag{4.4.3}$$

We have used new moduli labels ς and ϱ so as to avoid possible confusion in labelling a superpotential on $\mathcal{M}$ with moduli defined in $\mathcal{W}$ or vice versa. This can be stated in a straightforward manner using the expressions in (4.4.1) and (4.4.2).

$$W^{\rm NS}_{\rm IIA}(\varsigma_I, \varrho_A) = W^{\rm NS}_{\rm IIB}(\varsigma_I, \varrho_A) \quad , \quad W^{\rm RR}_{\rm IIA}(\varsigma_I, \varrho_A) = W^{\rm RR}_{\rm IIB}(\varsigma_I, \varrho_A) \tag{4.4.4}$$

In the majority of cases this moduli equivalence is not the case and mirror dual superpotentials need not have manifest matching moduli dependency and thus comparing superpotentials under the relabelling $\mathfrak{T} \leftrightarrow \mathsf{U}$ and $\mathfrak{U} \leftrightarrow \mathsf{T}$ is not in general appropriate. However, in our discussion of the Type IIB superpotential we had to take into account the fact that the fluxes did not couple to the Kähler moduli in the same manner as their $\mathfrak{E}$ index structure might have suggested. This caused us to effectively redefine the Type IIB moduli in (4.2.5) via $\underline{\mathsf{T}} \to \underline{\mathsf{T}}' = \mathsf{L} \cdot \underline{\mathsf{T}}$ and $\underline{\mathsf{U}} \to \underline{\mathsf{U}}' = \mathsf{K} \cdot \underline{\mathsf{U}}$. Refactorising $\mathsf{D}(\mho_c)$ such that the new $\mho_c$ depended on T' and U', $\check{\mho}_c$, provided a new flux matrix given in (4.2.12). Quantum corrections can be viewed in the same manner, modifying the moduli definitions and we use Q and $\widetilde{\mathsf{Q}}$ to distinguish from the differently sourced K and L alterations.

$$\begin{array}{ll} \text{Exact} & : \underline{\mathfrak{U}} \to \underline{\varsigma} \ , \ \underline{\mathfrak{T}} \to \underline{\varrho} \ , \ \underline{\mathsf{T}} \to \underline{\varsigma} \ , \ \underline{\mathsf{U}} \to \underline{\varrho} \\ \text{Quantum} & : \underline{\mathfrak{U}} \to \underline{\varsigma} \ , \ \underline{\mathfrak{T}} \to \underline{\varrho} \ , \ \underline{\mathsf{T}} \to \mathsf{Q} \cdot \underline{\varsigma} \ , \ \underline{\mathsf{U}} \to \widetilde{\mathsf{Q}} \cdot \underline{\varrho} \end{array} \tag{4.4.5}$$

Strictly speaking the modifications due to quantum corrections occur in one moduli space of each Type II theory but we can absorb the effects on the

Type IIA case into the new moduli (ς, ϱ). As a result of this we can use the Type IIB results of (4.2.12). Rather than refactorising $\mathsf{D}(\mho_c) = \mathbf{G}(\check{\mho}_c)$ we alter the moduli $\underline{\mathsf{T}} \to \underline{\mathsf{Q}} \cdot \underline{\mathsf{T}}$ and $\underline{\mathsf{U}} \to \widetilde{\mathsf{Q}} \cdot \underline{\mathsf{U}}$ and then refactorise the 3-form to $\mathbf{G}(\mho_c)$. This absorbs the quantum corrections to the moduli equivalences into the flux matrices, so that if we replace the flux matrices of D with those of $\mathbf{G}$ we can then use the exact moduli equivalences of (4.4.5). To demonstrate this explicitly we consider the different flux sectors and the different inner products in turn.

4.4.2 NS-NS Fluxes

To investigate the different ways of associating the moduli of each Type II theory we first consider the flux matrix dependent scalar product expressions. The simplest case is the $\langle\ \rangle_-$ inner product as it allows us to make use of (4.1.41) and the Type II NS-NS scalar product expressions of (4.4.1) have the same schematic forms, (4.1.37) and (4.2.8).

$$\begin{aligned} W^{\rm NS}_{\rm IIA}(\mathfrak{U},\mathfrak{T}) \;&=\; \int_{\mathcal{M}} \big\langle\, \mho, \mathcal{D}(\Omega_c)\,\big\rangle_- \;=\; \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbb{C} \cdot N \cdot g_\nu \cdot \underline{\mathfrak{T}} \\ W^{\rm NS}_{\rm IIB}(\mathsf{T},\mathsf{U}) \;&=\; \int_{\mathcal{W}} \big\langle\, \Omega, \mathsf{D}(\mho_c)\,\big\rangle_- \;=\; \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \mathsf{M} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \underline{\mathsf{U}} \end{aligned} \tag{4.4.6}$$

The two complexification matrices are equal due to the relationship between the Hodge numbers of $\mathcal{M}$ and $\mathcal{W}$, $\mathbb{C} = \mathsf{C}$, and it is straightforward to see that as a result they do not factor into the issue of how to relate the fluxes of each theory and we ignore them from this point onwards. Under the exact moduli equivalence of (4.4.5) and the fact the choice of $\langle\ \rangle_-$ has $h = \mathsf{h}$ and $g = \mathsf{g}$ for both $\Delta^{\pm}(\mathsf{E}^*)$ bases it follows that the flux matrices are equal.

$$W^{\rm NS}_{\rm IIA}(\varsigma,\varrho) = W^{\rm NS}_{\rm IIB}(\varsigma,\varrho) \quad \Rightarrow \quad N = \mathsf{M} \quad \Rightarrow \quad M = \mathsf{N} \tag{4.4.7}$$

Since $\langle\ \rangle_+$ does not admit the anti-self adjoint property of the derivative seen for $\langle\ \rangle_-$ in (4.1.41) we cannot use the same expression for $W^{\rm NS}_{\rm IIA}(\mathfrak{U},\mathfrak{T})$

as given in (4.4.6) and instead recall the standard definition of the Type IIA NS-NS superpotential.

$$\begin{aligned} W^{\text{NS}}_{\text{IIA}}(\mathfrak{U},\mathfrak{T}) &= \int_{\mathcal{M}} \big\langle\, \Omega_c, \mathcal{D}(\mho)\,\big\rangle_+ = \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot M \cdot \mathbb{C} \cdot g_{\mathfrak{a}} \cdot \underline{\mathfrak{U}} \\ W^{\text{NS}}_{\text{IIB}}(\mathsf{T},\mathsf{U}) &= \int_{\mathcal{W}} \big\langle\, \Omega, \mathsf{D}(\mho_c)\,\big\rangle_+ = \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \mathsf{M} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \underline{\mathsf{U}} \end{aligned} \tag{4.4.8}$$

This choice of ordering in the arguments of $\langle\ \rangle_+$ is consistent with the $\langle\ \rangle_-$ case. Equating the scalar product expressions of (4.4.8) is done by setting the equivalence in the moduli of $(\mathfrak{U},\mathfrak{T}) \leftrightarrow (\mathsf{T},\mathsf{U})$ and we must transpose one of the two expressions as a result.

$$\begin{aligned} \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot M \cdot \mathbb{C} \cdot g_{\mathfrak{a}} \cdot \underline{\mathfrak{U}} &= \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \mathsf{M} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \underline{\mathsf{U}} \\ \Rightarrow \quad h_\nu \cdot M \cdot g_{\mathfrak{a}} &= \Big(\mathsf{h}_\nu \cdot \mathsf{M} \cdot \mathsf{g}_{\mathfrak{a}}\Big)^\top \end{aligned} \tag{4.4.9}$$

Despite $\mathcal{D}$ not acting on the dilaton dependent holomorphic form, in contrast to D, the dilaton complexification matrix can be neglected since the important fact is that mirror dual moduli couple to the dilaton in the same manner. This equation reduces to $\mathsf{M} = g_{\mathfrak{a}} \cdot M^\top \cdot \mathsf{g}_{\mathfrak{a}}$ using the identities of the bilinear forms but the flux matrix expression $M^\top$ arises in the definition of N in terms of M, providing us with simpler relationships between the Type II fluxes.

$$\mathsf{M} = -h_{\mathfrak{a}} \cdot N \cdot h_\nu \quad , \quad M = -\mathsf{h}_{\mathfrak{a}} \cdot \mathsf{N} \cdot \mathsf{h}_\nu \tag{4.4.10}$$

We have explicitly included the factors of h_ν and h_ν for future comparision with the R-R sector results and again note that these relationships represent the action of an involution, as expected for mirror symmetry. If we include quantum corrections then we alter the way in which the moduli are related by modifying the Type IIB side.

$$W^{\text{NS}}_{\text{IIB}}(\mathsf{Q}\cdot\varsigma, \widetilde{\mathsf{Q}}\cdot\varrho) = \underline{\varsigma}^\top \cdot \mathsf{Q}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \mathsf{M} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \widetilde{\mathsf{Q}} \cdot \underline{\varrho} \tag{4.4.11}$$

Type IIA	$\mathcal{F}_{(A)I}$	$\mathcal{F}_{(A)}{}^{J}$	$\mathcal{F}^{(A)}{}_{I}$	$\mathcal{F}^{(A)I}$
Type IIB	$\mp\mathsf{F}^{(I)A}$	$\mathsf{F}_{(I)}{}^{A}$	$\mathsf{F}^{(I)}{}_{A}$	$\mp\mathsf{F}_{(I)A}$

Table 4.3: Type II NS-NS fluxes mirror equivalences defined by $\langle\ \rangle_{\pm}$.

Comparing this with the standard Type IIA superpotential, reordering the matrices to the standard form, as in (4.2.8), and dropping the complexification matrices we obtain the more general expression linking N and M and by symmetry the relationship between M and N follows.

$$N = \mathrm{Ad}_{\mathsf{h}_\nu}(\mathsf{Q}^\top)\cdot\mathsf{M}\cdot\mathrm{Ad}_{\mathfrak{g}_a}(\widetilde{\mathsf{Q}}) \quad\Rightarrow\quad M = \mathrm{Ad}_{\mathsf{h}_\nu}(\mathsf{Q}^\top)\cdot\mathsf{N}\cdot\mathrm{Ad}_{\mathfrak{g}_a}(\widetilde{\mathsf{Q}})$$

Without knowing the specific form of Q and $\widetilde{\mathsf{Q}}$ we cannot express these in terms of flux components and so we consider only the exact case. We can construct the flux component expressions either by inserting the component definitions of the flux matrices into the above relations or compare the superpotential polynomials in (4.1.28) and (4.2.20). In Table 4.3 we summarise the relationship between the flux components of $\mathcal{D}$ and D and note that the fact the sign structure is not changed by this equivalence the sign structure of $\mathcal{D}$ and D expressions of (4.4.12) and (4.4.13) are unchanged. However, despite the result $N = \mathsf{M}$ and $M = \mathsf{N}$ the index structures are different. This is the result of the fact we defined by set of components by the action of the derivatives on the $\Delta^+(\mathsf{E}^*)$ basis, in contrast to the implication of mirror symmetry where if one is defined on $\Delta^+(\mathsf{E}^*)$ then the other should be defined on $\Delta^-(\mathsf{E}^*)$. As a result we defined the components of M and M, which aren't mirror related. Never the less it is possible to state the actions of $\mathcal{D}$ and D in terms of one anothers components for $\langle\ \rangle_\pm$ in a simple way.

- Type IIA in terms of Type IIB on $\langle\ \rangle_\pm$.

$$\begin{aligned}
\mathsf{D}(\nu_I) &= \mathsf{F}_{(I)A}\mathfrak{a}_A - \mathsf{F}_{(I)}{}^B\mathfrak{b}^B & \mathcal{D}(\mathfrak{a}_I) &= \mp\mathsf{F}_{(I)A}\nu_A - \mathsf{F}_{(I)}{}^B\widetilde{\nu}^B \\
\mathsf{D}(\widetilde{\nu}^J) &= \mathsf{F}^{(J)}{}_A\mathfrak{a}_A - \mathsf{F}^{(J)B}\mathfrak{b}^B \quad\Rightarrow & \mathcal{D}(\mathfrak{b}^J) &= \quad \mathsf{F}^{(J)}{}_A\nu_A \pm \mathsf{F}^{(J)B}\widetilde{\nu}^B \\
\mathsf{D}(\mathfrak{a}_A) &= \mathsf{F}^{(I)A}\nu_I - \mathsf{F}_{(J)}{}^A\widetilde{\nu}^J & \mathcal{D}(\nu_A) &= \mp\mathsf{F}^{(I)A}\mathfrak{a}_I - \mathsf{F}_{(J)}{}^A\mathfrak{b}^J \\
\mathsf{D}(\mathfrak{b}^B) &= \mathsf{F}^{(I)}{}_B\nu_I - \mathsf{F}_{(J)B}\widetilde{\nu}^J & \mathcal{D}(\widetilde{\nu}^B) &= \quad \mathsf{F}^{(I)}{}_B\mathfrak{a}_I \pm \mathsf{F}_{(J)B}\mathfrak{b}^J
\end{aligned} \tag{4.4.12}$$

- Type IIB in terms of Type IIA on $\langle\ \rangle_\pm$.

$$\begin{aligned}
\mathcal{D}(\nu_A) &= \mathcal{F}_{(A)I}\mathfrak{a}_I - \mathcal{F}_{(A)}{}^J\mathfrak{b}^J & \mathsf{D}(\mathfrak{a}_A) &= \mp\mathcal{F}_{(A)I}\nu_I - \mathcal{F}_{(A)}{}^J\widetilde{\nu}^J \\
\mathcal{D}(\widetilde{\nu}^B) &= \mathcal{F}^{(B)}{}_I\mathfrak{a}_I - \mathcal{F}^{(B)J}\mathfrak{b}^J \quad\Rightarrow & \mathsf{D}(\mathfrak{b}^B) &= \mathcal{F}^{(B)}{}_I\nu_I \pm \mathcal{F}^{(B)J}\widetilde{\nu}^J \\
\mathcal{D}(\mathfrak{a}_I) &= \mathcal{F}^{A(I)}\nu_A - \mathcal{F}_{(B)}{}^I\widetilde{\nu}^B & \mathsf{D}(\nu_I) &= \mp\mathcal{F}^{A(I)}\mathfrak{a}_A - \mathcal{F}_{(B)}{}^I\mathfrak{b}^B \\
\mathcal{D}(\mathfrak{b}^J) &= \mathcal{F}^{(A)}{}_J\nu_A - \mathcal{F}^{(B)J}\widetilde{\nu}^B & \mathsf{D}(\widetilde{\nu}^J) &= \mathcal{F}^{(A)}{}_J\mathfrak{a}_A \pm \mathcal{F}^{(B)J}\mathfrak{b}^B
\end{aligned} \tag{4.4.13}$$

For the case of $\langle\ \rangle_-$ these relations make the action of mirror symmetry particularly simple, as it reduces to relabellings of the $\Delta^\pm(\mathsf{E}^*)$ bases between $\mathcal{M}$ and $\mathcal{W}$ and their associated moduli.

$$\begin{array}{ccc}
\text{Type IIA on } \mathcal{M} & & \text{Type IIB on } \mathcal{W} \\
\mho \in \mathcal{M}^K(\nu_A, \widetilde{\nu}^B, \mathfrak{T}_A) & \longleftrightarrow & \Omega \in \mathcal{M}^K(\mathfrak{a}_A, \mathfrak{b}^B, \mathsf{U}_A) \\
\Omega_c \in \mathcal{M}^Q(\mathfrak{a}_I, \mathfrak{b}^J, \mathfrak{U}_I, S) & & \mho_c \in \mathcal{M}^Q(\nu_I, \widetilde{\nu}^J, \mathsf{T}_I, S)
\end{array} \tag{4.4.14}$$

More specifically we define the mirror map $\mathfrak{M}$ for $\langle\ \rangle_-$ by these relationship, due to the fact we have labelled the basis elements of the $\Delta^*(\mathsf{E}^*)$ on $\mathcal{M}$ and $\mathcal{W}$ in the same way.

$$\mathfrak{M} \quad : \quad \begin{array}{ccc} (\nu_N, \widetilde{\nu}^M) & \leftrightarrow & (\mathfrak{a}_N, \mathfrak{b}^M) \\ \text{Kähler} & \leftrightarrow & \text{Com. Str.} \end{array} \tag{4.4.15}$$

We have labelled the flux components of $\mathcal{D}$ and D in different ways but we now see that if $\langle\ \rangle_\pm \to \langle\ \rangle_-$ then $\mathcal{F} = \mathsf{F}$ and we only really have one derivative for the Type II NS-NS sector, thus recovering the results of Refs. [74, 75].

4.4.3 R-R Fluxes

We restrict our considerations to the exact moduli equivalence, neglecting quantum corrections. Unlike the NS-NS case we do not need to use the adjoint properties of $\mathcal{D}$ with respect to $\langle\ \rangle_-$ as the fluxes take a more manifestly mirror symmetric structure due to their D-brane definitions and we need only to state the superpotentials once.

$$\begin{aligned} W^{\rm RR}_{\rm IIA}(\mathfrak{U},\mathfrak{T}) &= \int_{\mathcal{M}} \left\langle \mho, \mathcal{D}'(\Omega'_c) \right\rangle_{\pm} = \underline{\mathfrak{U}}^{\top} \cdot \mathsf{h}_{\mathfrak{a}} \cdot \mathbb{C}' \cdot M' \cdot \mathsf{g}_{\nu} \cdot \underline{\mathfrak{T}} \\ W^{\rm RR}_{\rm IIB}(\mathsf{T},\mathsf{U}) &= \int_{\mathcal{W}} \left\langle \Omega, \mathsf{D}'(\mho'_c) \right\rangle_{\pm} = \underline{\mathsf{T}}^{\top} \cdot \mathsf{h}_{\nu} \cdot \mathsf{C}' \cdot \mathsf{M}' \cdot \mathsf{g}_{\mathfrak{a}} \cdot \underline{\mathsf{U}} \end{aligned} \tag{4.4.16}$$

The R-R sector's scalar product expressions are more easily equated than the NS-NS sector case by the fact that transposition is not required and as in the NS-NS case the dilaton dependency is unimportant because $\mathbb{C}' = \mathsf{C}'$.

$$\begin{aligned} \underline{\mathfrak{U}}^{\top} \cdot h_{\mathfrak{a}} \cdot \mathbb{C}' \cdot N' \cdot g_{\nu} \cdot \underline{\mathfrak{T}} &= \underline{\mathsf{T}}^{\top} \cdot \mathsf{h}_{\nu} \cdot \widetilde{\mathsf{C}'} \cdot \mathsf{M}' \cdot \mathsf{g}_{\mathfrak{a}} \cdot \underline{\mathsf{U}} \\ \Rightarrow \quad h_{\mathfrak{a}} \cdot N' \cdot g_{\nu} &= \mathsf{h}_{\nu} \cdot \mathsf{M}' \cdot \mathsf{g}_{\mathfrak{a}} \end{aligned} \tag{4.4.17}$$

As with the NS-NS case the inner product $\langle\ \rangle_-$ sets the bilinear forms of $\Delta^{\pm}(\mathsf{E}^*)$ to be the same structure and the same result as the NS-NS sector is obtained.

$$W^{\rm RR}_{\rm IIA}(\varsigma,\varrho) = W^{\rm RR}_{\rm IIB}(\varsigma,\varrho) \quad \Rightarrow \quad N' = \mathsf{M}' \quad \Rightarrow \quad M' = \mathsf{N}' \tag{4.4.18}$$

For $\langle\ \rangle_+$ we cannot make this simplification but can use $h_{\nu} = \mathbb{I}$ and that $g_{\nu} = \Sigma$ to simplify down (4.4.17) and then use flux matrix identities.

$$\mathsf{M}' = h_{\mathfrak{a}} \cdot N' \cdot \mathsf{h}_{\mathfrak{a}} \quad , \quad \mathsf{N}' = -h_{\nu} \cdot M' \cdot \mathsf{h}_{\nu} \tag{4.4.19}$$

Comparing coefficients of (4.3.12) and (4.2.20) and using the same moduli equating as in the NS-NS sector we obtain the flux relations given in Table 4.4. With these results we can state the defining action of the derivatives $\mathcal{D}'$ and D' in (4.3.7) and (4.2.18) in terms of only one set of fluxes.

Type IIA	$\widehat{\mathfrak{F}}_{(I)A}$	$\widehat{\mathfrak{F}}_{(I)}{}^{A}$	$\widehat{\mathfrak{F}}^{(I)}{}_{A}$	$\widehat{\mathfrak{F}}^{(I)A}$
Type IIB	$\widehat{\mathsf{F}}_{(I)A}$	$\mp\widehat{\mathsf{F}}_{(I)}{}^{A}$	$\mp\widehat{\mathsf{F}}^{(I)}{}_{A}$	$\widehat{\mathsf{F}}^{(I)A}$

Table 4.4: Type II R-R fluxes mirror equivalences defined by $\langle\ \rangle_{\pm}$.

- Type IIA in terms of Type IIB on $\langle\ \rangle_{\pm}$.

$$\begin{aligned}
\mathsf{D}'(\nu_I) &= \widehat{\mathsf{F}}_{(I)A}\mathfrak{a}_A - \widehat{\mathsf{F}}_{(I)}{}^{B}\mathfrak{b}^B & & \mathcal{D}'(\mathfrak{a}_I) = \ \ \widehat{\mathsf{F}}_{(I)A}\nu_A - \widehat{\mathsf{F}}_{(I)}{}^{B}\widetilde{\nu}^B \\
\mathsf{D}'(\widetilde{\nu}^J) &= \widehat{\mathsf{F}}^{(J)}{}_{A}\mathfrak{a}_A - \widehat{\mathsf{F}}^{(J)B}\mathfrak{b}^B & \Rightarrow & \mathcal{D}'(\mathfrak{b}^J) = \mp\widehat{\mathsf{F}}^{(J)}{}_{A}\nu_A \pm \widehat{\mathsf{F}}^{(J)B}\widetilde{\nu}^B \\
\mathsf{D}'(\mathfrak{a}_A) &= \widehat{\mathsf{F}}^{(I)A}\nu_I - \widehat{\mathsf{F}}_{(J)}{}^{A}\widetilde{\nu}^J & & \mathcal{D}'(\nu_A) = \mp\widehat{\mathsf{F}}^{(I)A}\mathfrak{a}_I - \widehat{\mathsf{F}}_{(J)}{}^{A}\mathfrak{b}^J \\
\mathsf{D}'(\mathfrak{b}^B) &= \widehat{\mathsf{F}}^{(I)}{}_{B}\nu_I - \widehat{\mathsf{F}}_{(J)B}\widetilde{\nu}^J & & \mathcal{D}'(\widetilde{\nu}^B) = \mp\widehat{\mathsf{F}}^{(I)}{}_{B}\mathfrak{a}_I - \widehat{\mathsf{F}}_{(J)B}\mathfrak{b}^J
\end{aligned} \tag{4.4.20}$$

- Type IIB in terms of Type IIA on $\langle\ \rangle_{\pm}$.

$$\begin{aligned}
\mathcal{D}'(\mathfrak{a}_I) &= \ \ \widehat{\mathfrak{F}}_{(I)A}\nu_A \pm \widehat{\mathfrak{F}}_{(I)}{}^{B}\widetilde{\nu}^B & & \mathsf{D}'(\nu_I) = \ \ \widehat{\mathfrak{F}}_{(I)A}\mathfrak{a}_I \pm \widehat{\mathfrak{F}}_{(I)}{}^{B}\mathfrak{b}^J \\
\mathcal{D}'(\mathfrak{b}^J) &= \ \ \widehat{\mathfrak{F}}^{(J)}{}_{A}\nu_A \pm \widehat{\mathfrak{F}}^{(J)B}\widetilde{\nu}^B & \Rightarrow & \mathsf{D}'(\widetilde{\nu}^J) = \pm\widehat{\mathfrak{F}}^{(J)}{}_{A}\mathfrak{a}_I \pm \widehat{\mathfrak{F}}^{(J)B}\mathfrak{b}^J \\
\mathcal{D}'(\nu_A) &= \mp\widehat{\mathfrak{F}}^{(I)A}\mathfrak{a}_I \pm \widehat{\mathfrak{F}}_{(J)}{}^{A}\mathfrak{b}^J & & \mathsf{D}'(\mathfrak{a}_A) = \mp\widehat{\mathfrak{F}}^{(I)A}\nu_I \pm \widehat{\mathfrak{F}}_{(J)}{}^{A}\widetilde{\nu}^J \\
\mathcal{D}'(\widetilde{\nu}^B) &= \ \ \widehat{\mathfrak{F}}^{(I)}{}_{B}\mathfrak{a}_I - \widehat{\mathfrak{F}}_{(J)B}\mathfrak{b}^J & & \mathsf{D}'(\mathfrak{b}^B) = \ \ \widehat{\mathfrak{F}}^{(I)}{}_{B}\nu_I - \widehat{\mathfrak{F}}_{(J)B}\widetilde{\nu}^J
\end{aligned} \tag{4.4.21}$$

Similar $\pm$ ambiguity in the relationships defined by $\langle\ \rangle_{\pm}$ defined fluxes have arisen as in the NS-NS case but due to the different way in which the R-R sector fluxes of Type IIA are defined, by the action of $\mathcal{D}'$ on $\Delta^{-}(\mathsf{E}^*)$, the specific location of the $\pm$ signs are different when compared to the NS-NS case. Never the less, the $\langle\ \rangle_{\pm} \to \langle\ \rangle_{-}$ case again results in the equivalences of (4.4.15) and we have only one Type II R-R derivative.

4.4.4 Flux Induced Moduli Masses

On Calabi-Yaus the existence of harmonic forms allowed for the effective theory to be easily defined and the (co)-closed nature of harmonic forms lead to the fields of the effective theory readily satisfying the equations of motion and constraints associated to them. With the inclusion of fluxes the basis elements of $\Delta^*(\mathsf{E}^*)$ are no longer closed and thus no longer harmonic. As an example we consider the Type IIA NS-NS derivative $\mathcal{D}$ and its Laplacian, as the other derivatives follow in the same manner, and since the inner product being considered is the natural one on $\Omega^*(\mathsf{E}^*)$ the results are independent of the $\langle\ \rangle_{\pm}$ choice or quantum corrections to moduli equivalences.

With the inclusion of fluxes the Laplacian defined contribution to the masses of fields in the effective theory becomes non-zero, with the fluxes essentially parameterising this quantity and we wish to construct this dependence explicitly. We previously outlined in Section 2.6.2 that our choice of $\Delta^*(\mathsf{E}^*)$ basis elements is such that they reduce to harmonic forms on a Calabi-Yau in the case of all fluxes being set to zero, the condition of (2.6.3). We have also constructed flux dependent extensions of the exterior derivative as well as their action on the various different $\Delta^*(\mathsf{E}^*)$ basis elements and found that the extension of Calabi-Yaus to generalised Calabi-Yaus is often a matter of $\mathrm{d} \to \mathcal{D}$. This suggests that the explicit form of the flux dependent Laplacian associated to $\mathcal{D}$ is obtained by this substitution.

$$\Delta_{\mathrm{d}} \equiv \mathrm{d}\,\mathrm{d}^{\dagger} + \mathrm{d}^{\dagger}\,\mathrm{d} \quad \to \quad \Delta_{\mathcal{D}} \equiv \mathcal{D}\,\mathcal{D}^{\dagger} + \mathcal{D}^{\dagger}\,\mathcal{D} \tag{4.4.22}$$

To obtain this in terms of the flux matrices of $\mathcal{D}$ we consider its action on $\phi \in \Delta^+(\mathsf{E}^*)$ and $\chi \in \Delta^3(\mathsf{E}^*)$.

$$\begin{aligned} \mathcal{D}(\phi) &= \underline{\phi}^{\top} \cdot h_{\nu} \cdot M \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \qquad & \mathcal{D}(\chi) &= \underline{\chi}^{\top} \cdot h_{\mathfrak{a}} \cdot N \cdot h_{\nu} \cdot \mathbf{e}_{(\nu)} \\ \mathcal{D}^{\dagger}(\phi) &= \underline{\phi}^{\top} \cdot h_{\nu} \cdot M^{\dagger} \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \qquad & \mathcal{D}^{\dagger}(\chi) &= \underline{\chi}^{\top} \cdot h_{\mathfrak{a}} \cdot N^{\dagger} \cdot h_{\nu} \cdot \mathbf{e}_{(\nu)} \end{aligned}$$

It should be noted that $M^{\dagger}$ and $N^{\dagger}$ are not the hermitian conjugates of M and N but the flux matrices associated to $\mathcal{D}^{\dagger}$ instead of $\mathcal{D}$. The adjoint actions are found by using the definition of the adjoint derivative.

$$\underline{\phi}^\top \cdot h_\nu \cdot M \cdot \underline{\chi} \;=\; \langle\!\langle\, \chi, \mathcal{D}(\phi) \,\rangle\!\rangle \;\equiv\; \langle\!\langle\, \phi, \mathcal{D}^\dagger(\chi) \,\rangle\!\rangle \;=\; \underline{\chi}^\top \cdot h_{\mathfrak{a}} \cdot N^\dagger \cdot \underline{\phi}$$
$$\underline{\chi}^\top \cdot h_{\mathfrak{a}} \cdot N \cdot \underline{\phi} \;=\; \langle\!\langle\, \phi, \mathcal{D}(\chi) \,\rangle\!\rangle \;\equiv\; \langle\!\langle\, \chi, \mathcal{D}^\dagger(\phi) \,\rangle\!\rangle \;=\; \underline{\phi}^\top \cdot h_\nu \cdot M^\dagger \cdot \underline{\chi}$$

Comparing coefficients in each expression we obtain the flux matrices which represent the adjoint action of the standard flux matrices.

$$N^\dagger = h_{\mathfrak{a}} \cdot M^\top \cdot h_\nu \quad , \quad M^\dagger = h_\nu \cdot N^\top \cdot h_{\mathfrak{a}}$$

From this the two terms of the Laplacian can be constructed for both $\Delta^3(\mathsf{E}^*)$ and $\Delta^+(\mathsf{E}^*)$.

$$\mathcal{D}^\dagger\mathcal{D}(\phi) \;=\; \underline{\phi}^\top \cdot h_\nu \cdot M \cdot h_{\mathfrak{a}} \cdot N^\dagger \cdot h_\nu \cdot \mathbf{e}_{(\nu)} \;=\; \underline{\phi}^\top \cdot h_\nu \cdot M \cdot M^\top \cdot \mathbf{e}_{(\nu)}$$
$$\mathcal{D}\mathcal{D}^\dagger(\phi) \;=\; \underline{\phi}^\top \cdot h_\nu \cdot M^\dagger \cdot h_{\mathfrak{a}} \cdot N \cdot h_\nu \cdot \mathbf{e}_{(\nu)} \;=\; \underline{\phi}^\top \cdot N^\top \cdot N \cdot h_\nu \cdot \mathbf{e}_{(\nu)}$$
$$\mathcal{D}^\dagger\mathcal{D}(\chi) \;=\; \underline{\chi}^\top \cdot h_{\mathfrak{a}} \cdot N \cdot h_\nu \cdot M^\dagger \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \;=\; \underline{\chi}^\top \cdot h_{\mathfrak{a}} \cdot N \cdot N^\top \cdot \mathbf{e}_{(\mathfrak{a})}$$
$$\mathcal{D}\mathcal{D}^\dagger(\chi) \;=\; \underline{\chi}^\top \cdot h_{\mathfrak{a}} \cdot N^\dagger \cdot h_\nu \cdot M \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \;=\; \underline{\chi}^\top \cdot M^\top \cdot M \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})}$$

Using (4.1.35) the Laplacian for each flux sector can be written entirely in terms of the $\Delta^+(\mathsf{E}^*) \to \Delta^3(\mathsf{E}^*)$ defining fluxes.

$$\begin{aligned}
\langle\!\langle\, \phi, (\mathcal{D}\mathcal{D}^\dagger + \mathcal{D}^\dagger\mathcal{D})(\phi) \,\rangle\!\rangle \;&=\; \underline{\phi}^\top \cdot h_\nu \cdot \Big(M \cdot M^\top + N^\top \cdot N\Big) \cdot \underline{\phi} \\
&=\; \underline{\phi}^\top \cdot h_\nu \cdot \Big(M \cdot M^\top + \mathsf{g}_{\mathfrak{a}} \cdot M \cdot M^\top \cdot \mathsf{g}_{\mathfrak{a}}^\top\Big) \cdot \underline{\phi} \\
\langle\!\langle\, \chi, (\mathcal{D}\mathcal{D}^\dagger + \mathcal{D}^\dagger\mathcal{D})(\chi) \,\rangle\!\rangle \;&=\; \underline{\chi}^\top \cdot h_{\mathfrak{a}} \cdot \Big(M^\top \cdot M + N \cdot N^\top\Big) \cdot \underline{\chi} \\
&=\; \underline{\chi}^\top \cdot h_{\mathfrak{a}} \cdot \Big(M^\top \cdot M + \mathsf{g}_\nu \cdot M^\top \cdot M \cdot \mathsf{g}_\nu^\top\Big) \cdot \underline{\chi}
\end{aligned} \tag{4.4.23}$$

These two disjoint sections determine the properties of the Laplacian of $\mathcal{D}$ over the entire $\Delta^*(\mathsf{E}^*)$ of $\mathcal{M}$ and from which we can define a set of matrix expressions whose eigenvalues are the masses of the $\Delta^*(\mathsf{E}^*)$ basis element.

$$\left.\begin{aligned}
\underline{\underline{\Delta_{\mathcal{D}}^{(\nu)}}} \;&\equiv\; h_\nu \cdot \Big(M \cdot M^\top + \mathsf{g}_{\mathfrak{a}} \cdot M \cdot M^\top \cdot \mathsf{g}_{\mathfrak{a}}^\top\Big) \\
\underline{\underline{\Delta_{\mathcal{D}}^{(\mathfrak{a})}}} \;&\equiv\; h_{\mathfrak{a}} \cdot \Big(M^\top \cdot M + \mathsf{g}_\nu \cdot M^\top \cdot M \cdot \mathsf{g}_\nu^\top\Big)
\end{aligned}\right\} \quad \underline{\underline{\Delta_{\mathcal{D}}}} \equiv \underline{\underline{\Delta_{\mathcal{D}}^{(\nu)}}} \oplus \underline{\underline{\Delta_{\mathcal{D}}^{(\mathfrak{a})}}}$$

The cases of $\Delta_{\mathcal{D}'}$ and the Type IIB derivatives follow in the same manner and if $\langle\;\rangle_\pm \to \langle\;\rangle_-$ then we manifestly have that $\underline{\underline{\Delta_{\mathcal{D}}^{(\nu)}}} \in$ IIA is equal to

$\underline{\underline{\Delta_{\mathcal{D}}^{(\mathfrak{a})}}} \in$ IIB and likewise $\underline{\underline{\Delta_{\mathcal{D}}^{(\nu)}}} \in$ IIB is equal to $\underline{\underline{\Delta_{\mathcal{D}}^{(\mathfrak{a})}}} \in$ IIA. This is further complicated in the case of Type IIB as they mix non-trivially under S duality but we shall not consider that here.

In constructing these expressions we have made the assumption that the flux induced Laplacian in (4.4.22) is indeed how the Laplacian generalises with the inclusion of fluxes. To prove this we would have to do a full Kaluza-Klein reduction of the ten dimensional Laplacian in line with the space-time decomposition $M \to M_4 \times \mathcal{M}$ and the inclusion of non-geometric fluxes renders such an approach extremely difficult as the origins of non-geometric fluxes from a ten dimensional point of view are unknown. We shall instead consider the simplest non-trivial case, the inclusion of non-zero 3-form fluxes, which we denote by the standard H such that $\mathcal{D} = \mathrm{d} + H\wedge$, and all $\Delta^*(\mathsf{E}^*)$ elements are otherwise closed and co-closed. Under this assumption and the fact $\Delta^3(\mathsf{E}^*) \wedge \Delta^2(\mathsf{E}^*) = 0 = \Delta^3(\mathsf{E}^*) \wedge \Delta^4(\mathsf{E}^*)$ we need only consider $\Delta^0(\mathsf{E}^*)$, $\Delta^3(\mathsf{E}^*)$ and $\Delta^6(\mathsf{E}^*)$. We denote the individual terms of $\mathcal{D}^\dagger$ as $\mathcal{D}^\dagger = \mathrm{d}^\dagger + H^\dagger$ and note that $H^\dagger : \Delta^p(\mathsf{E}^*) \to \Delta^{p-3}(\mathsf{E}^*)$ by definition.

For $1 \in \Delta^0(\mathsf{E}^*)$ it follows that $\mathcal{D}^\dagger(1) = 0$ and thus the Laplacian reduces to $\mathcal{D}^\dagger\mathcal{D}(1)$. Since this is a 0-form we can consider its inner product with 1 without loss of generality and make use of the definition of an adjoint operator.

$$\mathcal{D}^\dagger\mathcal{D}(1) = \left\langle\!\!\left\langle\, 1, \mathcal{D}^\dagger\mathcal{D}(1) \,\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle\, \mathcal{D}(1), \mathcal{D}(1) \,\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle\, H, H \,\right\rangle\!\!\right\rangle$$

As expected this quantity is related to the fluxes of H and is only zero if H is turned off. From this the $\Delta^6(\mathsf{E}^*)$ case follows since the Laplacian commutes with the Hodge star and $\mathrm{vol}_6 = \star 1$.

$$\Delta_{\mathcal{D}}(\mathrm{vol}_6) = \star\Delta_{\mathcal{D}}(1) = \langle\!\langle\, H, H \,\rangle\!\rangle \mathrm{vol}_6$$

This is useful for the $\Delta^3(\mathsf{E}^*)$ case as it defines the $\mathcal{D}^\dagger$ action on vol_6. With $\mathcal{D}(\mathrm{vol}_6) = 0$ the Laplacian reduces to $\mathcal{D}\mathcal{D}^\dagger(\mathrm{vol}_6)$, an element of $\Delta^6(\mathsf{E}^*)$ and we can use the inner product without loss of generality again via $\mathcal{D}\mathcal{D}^\dagger(\mathrm{vol}_6) = \mathrm{vol}_6 \langle\!\langle\, \mathrm{vol}_6, \mathcal{D}\mathcal{D}^\dagger(\mathrm{vol}_6) \,\rangle\!\rangle$.

$$\langle\!\langle\, H, H \,\rangle\!\rangle = \langle\!\langle\, \mathrm{vol}_6, \mathcal{D}\mathcal{D}^\dagger(\mathrm{vol}_6) \,\rangle\!\rangle = \langle\!\langle\, \mathcal{D}^\dagger(\mathrm{vol}_6), \mathcal{D}^\dagger(\mathrm{vol}_6) \,\rangle\!\rangle$$

For $\Delta^3(\mathsf{E}^*)$ it follows from the definition of the adjoint and $H^\dagger : \Delta^3(\mathsf{E}^*) \to \Delta^0(\mathsf{E}^*)$ that $H^\dagger(\mathfrak{a}_I) = H_I$ and $H^\dagger(\mathfrak{b}^J) = -H^J$.

$$\begin{aligned}
\langle\!\langle\, \mathfrak{a}_I, \Delta\mathfrak{a}_J \,\rangle\!\rangle &= H^I H^J + H_I H_J \\
\langle\!\langle\, \mathfrak{b}^I, \Delta\mathfrak{a}_J \,\rangle\!\rangle &= H_I H^J - H^I H_J \\
\langle\!\langle\, \mathfrak{a}_I, \Delta\mathfrak{b}^J \,\rangle\!\rangle &= H^I H_J - H_I H^J \\
\langle\!\langle\, \mathfrak{b}^I, \Delta\mathfrak{b}^J \,\rangle\!\rangle &= H_I H_J + H^I H^J
\end{aligned}$$

All of these expressions vanish if and only if $H = 0$. Of particular interest is the case of M_H^2.

$$\langle\!\langle\, H, \Delta_{\mathcal{D}} H \,\rangle\!\rangle = \langle\!\langle\, \mathcal{D}(H), \mathcal{D}(H) \,\rangle\!\rangle + \langle\!\langle\, \mathcal{D}^\dagger(H), \mathcal{D}^\dagger(H) \,\rangle\!\rangle$$

With $\mathcal{D}(H) = 0$ the first term vanishes and the second term can be evaluated by noting $H^\dagger(H) \in \Delta^0(\mathsf{E}^*)$ and thus it follows that $H^\dagger(H) = \langle\!\langle\, H, H \,\rangle\!\rangle$.

$$\langle\!\langle\, H, \Delta_{\mathcal{D}} H \,\rangle\!\rangle = \langle\!\langle\, \mathcal{D}^\dagger(H), \mathcal{D}^\dagger(H) \,\rangle\!\rangle = \langle\!\langle\, H, H \,\rangle\!\rangle^2$$

We can express these three results in the same way by normalising.

$$\frac{\langle\!\langle\, \xi, \Delta_{\mathcal{D}}\xi \,\rangle\!\rangle}{\langle\!\langle\, \xi, \xi \,\rangle\!\rangle} = \langle\!\langle\, H, H \,\rangle\!\rangle \qquad \xi \in \{1, \mathrm{vol}_6, H\} \tag{4.4.24}$$

By using the component definitions of the flux matrices of $\mathcal{D}$ in (4.1.19) we obtain the same expressions, thus demonstrating the validity of (4.4.22) for the simplest non-trivial case.

Summary

In this chapter we have examined the effect of a number of dualities on the superpotentials of Type II theories and in doing so have deduced the existence of fluxes which do not appear naturally in the full ten dimensional string actions. These fluxes allow for the possible stablisation of all moduli types by giving a generic superpotential dependent on all moduli types simultaneously without the requirement for non-perturbative effects. The number of fluxes and their contribution to the various superpotentials was most conveniently expressed by using as our basis the elements of the $\Delta^p(\mathsf{E}^*)$ light forms defined on the internal space in the absence of fluxes. The inclusion of fluxes required the use of generalised geometry, within which the basis elements are no longer closed and thus the moduli obtain masses, for which analytic expressions in terms of the fluxes were obtained.

Thus far we have only concerned ourselves with the construction of the most general superpotentials under T, S and U duality transformations. Having done so we must consider the constraints upon such fluxes and it is to this we now turn.

Chapter 5

Flux Constraints

Since the Type IIA and Type IIB theories are dual to one another a set of fluxes which satisfy the constraints in one Type II theory should map to a set of fluxes in its mirror partner which satisfy their constraints. We shall consider the NS-NS sector of each theory first, where only T duality needs be considered, and then using it as a guide we then extend our considerations to the R-R sector induced by U duality, followed by the S duality mixing of the two sectors.

5.1 T Duality Bianchi Constraints

We have formulated our analysis of the flux components in two ways; the parallelisable $\Lambda^p(\mathsf{E}^*)$ components of (4.1.6) and the light form $\Delta^*(\mathsf{E}^*)$ components. Each of these approaches have advantages and disadvantages. The parallelisable construction is extremely restrictive to which $\mathcal{M}$ it can be applied to but the fluxes can be acted on both $\Lambda^{\pm}(\mathsf{E}^*)$ without having to be reformulated. The constraints on the fluxes also have a Lie algebra formulation whose structure we will make considerable use of and which include contributions which are missed if we restrict ourselves to the lightest forms.

An additional disadvantage of the $\Delta^*(\mathsf{E}^*)$ construction is that the fluxes have to be reformulated to act on the different $\Delta^{\pm}(\mathsf{E}^*)$ bases. This is required in order to determine the nilpotency constraints and even then the nilpotency constraints on the $\Delta^*(\mathsf{E}^*)$ forms do not provide sufficient conditions for nilpotency on $\Lambda^*(\mathsf{E}^*)$, as we will see both in general and specifically for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold later.

5.1.1 Type IIA NS-NS Flux Sector

We recall for $\mathcal{M}$ a parallelisable space the Type IIA NS-NS Lie algebra, which we shall refer to as $\mathcal{L}(\mathcal{D})$, in terms of the $\mathcal{F}$ fluxes [56, 60].

$$\begin{aligned}
\left[\ Z_m\ ,\ Z_n\ \right] &= \mathcal{F}_{mnp}\ X^p\ -\ \mathcal{F}^{p}_{mn}\ Z_p \\
\left[\ Z_m\ ,\ X^n\ \right] &= \mathcal{F}^{n}_{mp}\ X^p\ +\ \mathcal{F}^{np}_{m}\ Z_p \\
\left[\ X^m\ ,\ X^n\ \right] &= \mathcal{F}^{mn}_{p}\ X^p\ -\ \mathcal{F}^{mnp}\ Z_p
\end{aligned}$$

These fluxes define terms in the covariant derivative $\mathcal{D}$ of (4.1.10) and are such that its Bianchi constraints are equivalent to the Jacobi constraints of this algebra [56]. As such we shall refer to the algebra as $\mathcal{L}(\mathcal{D})$. Constructing the Jacobi constraints of this algebra and considering the coefficients of the X and Z provides five schematically different expressions [54, 56, 61].

$$\begin{aligned}
0 &= \mathcal{F}_{e[ab}\mathcal{F}^{e}_{cd]} \\
0 &= \mathcal{F}^{d}_{e[a}\mathcal{F}^{e}_{bc]}\ +\ \mathcal{F}_{e[ab}\mathcal{F}^{de}_{c]} \\
0 &= \mathcal{F}^{[ab]}_{e}\mathcal{F}^{e}_{[cd]}\ -\ 4\mathcal{F}^{[a}_{e[c}\mathcal{F}^{b]e}_{d]}\ +\ \mathcal{F}_{e[cd]}\mathcal{F}^{[ab]e} \\
0 &= \mathcal{F}^{e[a}_{d}\mathcal{F}^{bc]}_{e}\ +\ \mathcal{F}^{e[ab}\mathcal{F}^{c]}_{de} \\
0 &= \mathcal{F}^{e[ab}\mathcal{F}^{cd]}_{e}
\end{aligned} \tag{5.1.1}$$

The constraints of (5.1.1) can be taken to be the generating functions of an ideal, which we shall denote as $\langle\mathcal{L}(\mathcal{D})\rangle$. These conditions are not dependent on our choice of $\langle\ \rangle_{\pm}$ as we are not defining the components to match some kind of intersection number structure on the basis of forms. This is not the case for the derivative flux components of (4.1.19) defined on the $\Delta^*(\mathsf{E}^*)$ but the nilpotency conditions on such components are still constructed by combining the two actions of $\mathcal{D}$ [64]. Dependent upon which light subspace $\Delta^{\pm}(\mathsf{E}^*)$ is being considered different moduli space indices are summed; for $\mathcal{D}$ acting on $\Delta^{\pm}(\mathsf{E}^*)$ the contracted indices relate to $\Delta^{\mp}(\mathsf{E}^*)$. In the case of $\Delta^{+}(\mathsf{E}^*)$ it is the complex structure indices.

$$\begin{aligned}
\mathcal{D}^2(\nu_A) &= \Big(\mathcal{F}_{(A)I}\mathcal{F}^{(B)I} - \mathcal{F}_{(A)}{}^{J}\mathcal{F}^{(B)}{}_{J} \Big)\nu_B + \Big(\mathcal{F}_{(A)}{}^{J}\mathcal{F}_{(B)J} - \mathcal{F}_{(A)I}\mathcal{F}_{(B)}{}^{I} \Big)\widetilde{\nu}^B \\
\mathcal{D}^2(\widetilde{\nu}^B) &= \Big(\mathcal{F}^{(B)}{}_{I}\mathcal{F}^{(A)I} - \mathcal{F}^{(B)J}\mathcal{F}^{(A)}{}_{J} \Big)\nu_A + \Big(\mathcal{F}^{(B)J}\mathcal{F}_{(A)J} - \mathcal{F}^{(B)}{}_{I}\mathcal{F}_{(A)}{}^{I} \Big)\widetilde{\nu}^A
\end{aligned} \tag{5.1.2}$$

Conversely, in the case of $\Delta^{+}(\mathsf{E}^*)$ it is the Kähler indices.

$$\begin{aligned}
\mathcal{D}^2(\mathfrak{a}_I) &= \Big(\mathcal{F}^{(A)I}\mathcal{F}_{(A)J} - \mathcal{F}_{(B)}{}^{I}\mathcal{F}^{(B)}{}_{J} \Big)\mathfrak{a}_J + \Big(\mathcal{F}_{(B)}{}^{I}\mathcal{F}^{(B)J} - \mathcal{F}^{(B)I}\mathcal{F}_{(B)}{}^{J} \Big)\mathfrak{b}^J \\
\mathcal{D}^2(\mathfrak{b}^J) &= \Big(\mathcal{F}^{(A)}{}_{J}\mathcal{F}_{(A)I} - \mathcal{F}_{(B)J}\mathcal{F}^{(B)}{}_{I} \Big)\mathfrak{a}_I + \Big(\mathcal{F}_{(B)J}\mathcal{F}^{(B)I} - \mathcal{F}^{(A)}{}_{J}\mathcal{F}_{(A)}{}^{I} \Big)\mathfrak{b}^I
\end{aligned} \tag{5.1.3}$$

It is worth commenting that due to the sympletic construction of these expressions they are invariant under redefinitions of the $(\mathfrak{a}_I, \mathfrak{b}^J)$ basis which leave the intersection numbers of $\Delta^3(\mathsf{E}^*)$ invariant. This illustrates that the (α_I, β^J) basis could just as easily have been used but we include such a redefinition by following mirror symmetry on the change of basis of $\Delta^{+}(\mathsf{E}^*)$. These constraints can be written in a more compact form in terms of the flux matrices M and N by noting how they define the exact sequence associated to $\mathcal{D}$.

$$\cdots \xrightarrow{M\cdot h_{\mathfrak{a}}} \Delta^3(\mathsf{E}^*) \xrightarrow{N\cdot h_{\nu}} \Delta^3(\mathsf{E}^*) \xrightarrow{M\cdot h_{\mathfrak{a}}} \Delta^3(\mathsf{E}^*) \xrightarrow{N\cdot h_{\nu}} \cdots$$

The Bianchi constraints follow by constructing the matrix expression for $\mathcal{D}^2$ in terms of $\underline{\underline{\mathcal{D}}}$ and the h bilinear forms.

$$\mathcal{D}^2(\mathbf{e}) \;=\; \underline{\underline{\mathcal{D}}}\cdot h\cdot \underline{\underline{\mathcal{D}}}\cdot h\cdot \mathbf{e} \;=\; \begin{pmatrix} M\cdot h_{\mathfrak{a}}\cdot N\cdot h_{\nu} & 0 \\ 0 & N\cdot h_{\nu}\cdot M\cdot h_{\mathfrak{a}} \end{pmatrix} \begin{pmatrix} \mathbf{e}_{(\nu)} \\ \mathbf{e}_{(\mathfrak{a})} \end{pmatrix} \tag{5.1.4}$$

The flux polynomials associated to the Type IIA NS-NS flux sector nilpotency $\mathcal{D}^2 = 0$ define a new ideal, which we shall denote as $\langle \mathcal{D}^2 \rangle$, by having each of the components of $\underline{\underline{\mathcal{D}}} \cdot h \cdot \underline{\underline{\mathcal{D}}} \cdot h$ as a generating function for the ideal.

$$\langle \mathcal{D}^2 \rangle \;=\; \langle \;\; M \cdot h_{\mathfrak{a}} \cdot N \cdot h_{\nu} \;\;,\;\; N \cdot h_{\nu} \cdot M \cdot h_{\mathfrak{a}} \;\; \rangle \tag{5.1.5}$$

Since the $h_{\mathfrak{a}}$ and h_{ν} are non-degenerate those on the 'outside' of the matrix expressions can be neglected without changing the ideal the matrix expressions define. Since $\Delta^*(\mathsf{E}^*)$ is a truncated basis for Kaluza-Klein reductions we would not expect the constraints we construct using such a basis to be sufficient for the full untruncated theory. The truncation is not an issue for parallelisable spaces as the parallelisability is sufficiently restrictive on the flux components in its own right, $\Omega^*(T^*\mathcal{M}) \to \Lambda^*(\mathsf{E}^*)$. However, as we will discuss shortly, the constraints in these two formulations are not equal, $\langle \mathcal{D}^2 \rangle \neq \langle \mathcal{L}(\mathcal{D}) \rangle$, but we would expect that they reduce to one another on the $\Delta^*(\mathsf{E}^*)$ of a parallelisable $\mathcal{M}$. This can be expressed using the operator $\mathfrak{L} : \Lambda^*(\mathsf{E}^*) \to \Delta^*(\mathsf{E}^*)$, which projects down from the space of parallelised p-forms to the light form truncated basis.

$$\langle \mathcal{D}^2 \rangle = \mathfrak{L}\left(\langle \mathcal{L}(\mathcal{D}) \rangle \right) \tag{5.1.6}$$

5.1.2 Type IIB NS-NS Flux Sector

The derivation of the T duality induced fluxes of Type IIB followed the same method as the Type IIA fluxes; for parallelisable $\mathcal{W}$ we use the completion of a Lie algebra's commutation relations, resulting in the algebra (4.2.2) which we recall here, to deduce additional fluxes [52, 61].

$$\begin{aligned}
\left[\; Z_m \;\;,\;\; Z_n \; \right] &= \widehat{\mathsf{F}}_{mnp} \; X^p \;-\; \mathsf{F}^p_{mn} \;\; Z_p \\
\left[\; Z_m \;\;,\;\; X^n \; \right] &= \mathsf{F}^n_{mp} \;\; X^p \;+\; \mathsf{F}^{np}_{m} \;\; Z_p \\
\left[\; X^m \;\;,\;\; X^n \; \right] &= \mathsf{F}^{mn}_{p} \;\; X^p \;-\; \widehat{\mathsf{F}}^{mnp} \; Z_p
\end{aligned}$$

The fluxes define terms in the flux operator $\mathbf{G}$ of (4.2.3) in the same way as $\mathcal{D}$ was defined from the fluxes of the Type IIA NS-NS algebra and so this algebra we denote as $\mathcal{L}(\mathbf{G})$. The Jacobi constraints of this algebra, $\langle\mathcal{L}(\mathbf{G})\rangle$ take the same schematic form as the Type IIA case, $\langle\mathcal{L}(\mathcal{D})\rangle$.

$$\begin{aligned}
0 &= \widehat{\mathsf{F}}_{e[ab}\mathsf{F}^{e}_{cd]} \\
0 &= \mathsf{F}^{d}_{e[a}\mathsf{F}^{e}_{bc]} \quad + \quad \widehat{\mathsf{F}}_{e[ab}\mathsf{F}^{de}_{c]} \\
0 &= \mathsf{F}^{[ab]}_{e}\mathsf{F}^{e}_{[cd]} \; - \; 4\mathsf{F}^{[a}_{e[c}\mathsf{F}^{b]e}_{d]} \; + \; \widehat{\mathsf{F}}_{e[cd]}\widehat{\mathsf{F}}^{[ab]e} \\
0 &= \mathsf{F}^{e[a}_{d}\mathsf{F}^{bc]}_{e} \quad + \quad \widehat{\mathsf{F}}^{e[ab}\mathsf{F}^{c]}_{de} \\
0 &= \widehat{\mathsf{F}}^{e[ab}\mathsf{F}^{cd]}_{e}
\end{aligned} \tag{5.1.7}$$

It is a noteworthy aside that if we were to apply an orientifold projection which removed either the pair $(\mathsf{F}_1, \widehat{\mathsf{F}}_3)$, to give Type IIB/O3, or the pair $(\widehat{\mathsf{F}}_0, \mathsf{F}_2)$, to give Type IIB/O9 the algebra and its Bianchi constraints reduce in such a manner to admit a six dimensional subalgebra. In the original construction of the Type IIA f in (4.1.1) this was explicitly seen for the algebra generated by the Z. In Type IIB/O3 the subalgebra is generated by the X with structure constant F_2 and in Type IIB/O9 F_1 is the structure constant for generators Z. The Type IIA orientifold projection does not admit a manifest subalgebra as the orientifold action acts on the holomorphic form of $\mathcal{M}^Q$ and in Type IIA the fluxes are not defined as acting on this holomorphic form. We shall return to this point later but until then we do not consider orientifold projections any further.

Though the T duality induced fluxes follow by completion of $\mathcal{L}(\mathbf{G})$ they do not couple to the individual terms in $\mho_c$ in the same nature way as the fluxes of $\mathcal{D}$ do to $\mho$. We wish to work with Type IIB superpotentials defined in terms of the standard holomorphic forms and thus D rather than $\mathbf{G}$. To do this in a consistent manner we require that any analysis of the fluxes and their constraints in terms of D is equivalent to the same analysis in terms of $\mathbf{G}$, namely that $\langle \mathbf{G}^2 \rangle = \langle \mathsf{D}^2 \rangle$ and for a parallelisable $\mathcal{W}$ $\langle \mathcal{L}(\mathbf{G}) \rangle = \langle \mathcal{L}(\mathsf{D}) \rangle$. When we first defined $\mathbf{G}$ in order to examine its flux components' relationship to D we were forced to use the $\Delta^*(\mathsf{E}^*)$ bases and so we do the same here. The Bianchi constraints of D, $\langle \mathsf{D}^2 \rangle$, are of the same form as $\langle \mathcal{D}^2 \rangle$ in terms of flux matrices and we can use (4.2.10) to write N in terms of M. The immediate corrollary of this result is that we can express $\langle \mathbf{G}^2 \rangle$ in terms of G.

$$\begin{aligned} \langle \mathsf{D}^2 \rangle &= \langle \ \mathsf{M} \cdot \mathsf{h}_{\mathfrak{a}} \cdot \mathsf{N} \ , \ \mathsf{N} \cdot \mathsf{h}_{\nu} \cdot \mathsf{M} \ \rangle \\ &= \langle \ \mathsf{M} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \mathsf{M}^\top \ , \ \mathsf{M}^\top \cdot g_{\mathfrak{a}} \cdot \mathsf{M} \ \rangle \\ \langle \mathbf{G}^2 \rangle &= \langle \ \mathsf{G} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \mathsf{G}^\top \ , \ \mathsf{G}^\top \cdot g_{\mathfrak{a}} \cdot \mathsf{G} \ \rangle \end{aligned} \tag{5.1.8}$$

We can then compare these constraints by using (4.2.12) to convert the generating functions of $\langle \mathsf{D}^2 \rangle$ into being dependent on G. We again neglect any pre- or post-multiplication by non-degenerate matrices.

$$\langle \mathsf{D}^2 \rangle \rightarrow \langle \ \mathsf{G} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \mathsf{K} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \mathsf{K}^\top \cdot \mathsf{g}_{\mathfrak{a}} \cdot \mathsf{G}^\top \ , \ \mathsf{G}^\top \cdot \mathsf{L} \cdot g_{\mathfrak{a}} \cdot \mathsf{L}^\top \cdot \mathsf{G} \ \rangle \tag{5.1.9}$$

It therefore follows that the cohomology restricted Bianchi constraints of D and $\mathbf{G}$ are equivalent if L and K are symplectic matrices, though we have the option of an additional overall factor of -1.

$$\mathsf{K} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \mathsf{K}^\top = \pm \mathsf{g}_{\mathfrak{a}} \quad , \quad \mathsf{L} \cdot g_{\mathfrak{a}} \cdot \mathsf{L}^\top = \pm g_{\mathfrak{a}} \tag{5.1.10}$$

Given the overall factor of -1 is physically irrelevant we set the sign to be $+$ in the above expressions. There are restrictions on K and L over and above these expressions. We previously required L to commute with the complexification matrices and to not mix the Kähler moduli other than to exchange the T_i and T^j in some manner. We consider a specific case; on the grounds of treating the two moduli types in the same manner we restrict K

to be of a similar form and we can make a more explicit ansatz for the two matrices.

$$\mathsf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathsf{L}_1 \\ 0 & 0 & 1 & 0 \\ 0 & \mathsf{L}_2 & 0 & 0 \end{pmatrix} \quad , \quad \mathsf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathsf{K}_1 \\ 0 & 0 & 1 & 0 \\ 0 & \mathsf{K}_2 & 0 & 0 \end{pmatrix} \tag{5.1.11}$$

The sympletic constraints now reduce to skew-orthogonality of the submatrixes, $\mathsf{L}_2 \cdot \mathsf{L}_1^\top = -\mathbb{I}$ and $\mathsf{K}_2 \cdot \mathsf{K}_1^\top = -\mathbb{I}$. The simplest specific solution is $\mathsf{L}_1 = -\mathsf{L}_2 = \mathbb{I}$ which has the effect $\mathcal{J}^{(2)} = \pm\mathcal{T}^j\widetilde{\omega}^j \to \mp\mathcal{T}_j\widetilde{\omega}^j = \mathcal{J}_c$, the standard definition used for the Kähler 4-form [60, 61, 93]. Provided these conditions are met we can be sure that when restricted to the light modes the Bianchi constraints of D are equivalent to those of $\mathbf{G}$, $\langle \mathsf{D}^2 \rangle = \langle \mathbf{G}^2 \rangle$. These constraints can be stated explictly using (4.2.11) to obtain the Type IIB versions of (5.1.12) and (5.1.13). As in the Type IIA case different moduli space indices are summed dependent upon which $\Delta^{\pm}(\mathsf{E}^*)$ light form subspace the nilpotency is being considered on. In the case of $\Delta^{+}(\mathsf{E}^*)$ it is the complex structure indices.

$$\begin{aligned} \mathsf{D}^2(\nu_I) &= \Big(\mathsf{F}_{(I)A}\mathsf{F}^{(J)A} - \mathsf{F}_{(I)}{}^{B}\mathsf{F}^{(J)}{}_{B} \Big)\nu_J + \Big(\mathsf{F}_{(I)}{}^{B}\mathsf{F}_{(J)B} - \mathsf{F}_{(I)A}\mathsf{F}_{(J)}{}^{A} \Big)\widetilde{\nu}^J \\ \mathsf{D}^2(\widetilde{\nu}^J) &= \Big(\mathsf{F}^{(J)}{}_{A}\mathsf{F}^{(I)A} - \mathsf{F}^{(J)B}\mathsf{F}^{(I)}{}_{B} \Big)\nu_I + \Big(\mathsf{F}^{(J)B}\mathsf{F}_{(I)B} - \mathsf{F}^{(J)}{}_{A}\mathsf{F}_{(I)}{}^{A} \Big)\widetilde{\nu}^I \end{aligned} \tag{5.1.12}$$

Conversely, in the case of $\Delta^{3}(\mathsf{E}^*)$ it is the Kähler indices.

$$\begin{aligned} \mathsf{D}^2(\mathfrak{a}_A) &= \Big(\mathsf{F}_{(J)}{}^{A}\mathsf{F}^{(J)}{}_{B} - \mathsf{F}^{(I)A}\mathsf{F}_{(I)B} \Big)\mathfrak{a}_B + \Big(\mathsf{F}^{(J)A}\mathsf{F}_{(J)}{}^{B} - \mathsf{F}_{(J)}{}^{A}\mathsf{F}^{(J)B} \Big)\mathfrak{b}^B \\ \mathsf{D}^2(\mathfrak{b}^B) &= \Big(\mathsf{F}_{(J)B}\mathsf{F}^{(J)}{}_{A} - \mathsf{F}^{(I)}{}_{B}\mathsf{F}_{(I)A} \Big)\mathfrak{a}_A + \Big(\mathsf{F}^{(I)}{}_{B}\mathsf{F}_{(I)}{}^{A} - \mathsf{F}_{(J)B}\mathsf{F}^{(J)A} \Big)\mathfrak{b}^A \end{aligned} \tag{5.1.13}$$

We now consider this relationship for the parallelisable $\mathcal{W}$, where we can define the flux expansion of D in terms of the $\star\mathsf{F}$ and $\star\widehat{\mathsf{F}}$ in (4.2.13).

$$\begin{aligned} \mathsf{D} &= \mathbf{f}_{(\mathfrak{a})}^\top \cdot \mathsf{h}_{\mathfrak{a}}^\top \cdot \mathsf{M}^\top \cdot \iota_{\mathbf{f}_{(\nu)}} \qquad (5.1.14) \\ &= \frac{1}{3!}(\star\widehat{\mathsf{F}})_{mpq}\eta^{mpq} + \frac{1}{2!}(\star\mathsf{F})^{m}_{pq}\eta^{pq}\iota_m + \frac{1}{2!}(\star\mathsf{F})^{mp}_{q}\eta^{q}\iota_{pm} + \frac{1}{3!}(\star\widehat{\mathsf{F}})^{mqp}\iota_{pqm} \end{aligned}$$

As with the $\mathcal{D}$ in Type IIA and the natural flux operator $\mathbf{G}$ the Bianchi constraints of this derivative can be reformulated into Jacobi constraints of a Lie algebra $\mathcal{L}(\mathsf{D})$, the specific form of which follows previous examples but the generators are not those seen in $\mathcal{L}(\mathbf{G})$.

$$\begin{aligned} \left[\ \mathbb{Z}_m\ ,\ \mathbb{Z}_n\ \right] &= \ \ (\star\widehat{\mathsf{F}})_{mnp}\ \mathbb{X}^p + (\star\mathsf{F})^p_{mn}\ \mathbb{Z}_p \\ \left[\ \mathbb{Z}_m\ ,\ \mathbb{X}^n\ \right] &= -\ (\star\mathsf{F})^n_{mp}\ \mathbb{X}^p + (\star\mathsf{F})^{np}_m\ \mathbb{Z}_p \\ \left[\ \mathbb{X}^m\ ,\ \mathbb{X}^n\ \right] &= \ \ (\star\mathsf{F})^{mn}_p\ \mathbb{X}^p + (\star\widehat{\mathsf{F}})^{mnp}\ \mathbb{Z}_p \end{aligned} \tag{5.1.15}$$

The Jacobi constraints of $\mathcal{L}(\mathsf{D})$ take the same schematic form as the previous cases.

$$\begin{aligned} 0 &= (\star\widehat{\mathsf{F}})_{e[ab}(\star\mathsf{F})^e_{cd]} \\ 0 &= (\star\mathsf{F})^d_{e[a}(\star\mathsf{F})^e_{bc]} + (\star\widehat{\mathsf{F}})_{e[ab}(\star\mathsf{F})^{de}_{c]} \\ 0 &= (\star\mathsf{F})^{[ab]}_e(\star\mathsf{F})^e_{[cd]} - 4(\star\mathsf{F})^{[a}_{e[c}(\star\mathsf{F})^{b]e}_{d]} + (\star\widehat{\mathsf{F}})_{e[cd]}(\star\widehat{\mathsf{F}})^{[ab]e} \\ 0 &= (\star\mathsf{F})^{e[a}_d(\star\mathsf{F})^{bc]}_e + (\star\widehat{\mathsf{F}})^{e[ab}(\star\mathsf{F})^{c]}_{de} \\ 0 &= (\star\widehat{\mathsf{F}})^{e[ab}(\star\mathsf{F})^{cd]}_e \end{aligned} \tag{5.1.16}$$

Through the use of matrix expressions for the $\Delta^*(\mathsf{E}^*)$ definitions of $\mathbf{G}$ and D we have explicitly demonstrated the equivalence $\langle\mathbf{G}^2\rangle = \langle\mathsf{D}^2\rangle$ on a general $\mathcal{W}$. If $\mathcal{W}$ is parallelisable then we would also require that this equivalence lifts to the $\Lambda^*(\mathsf{E}^*)$.

$$\langle\mathbf{G}^2\rangle = \langle\mathsf{D}^2\rangle \quad \Rightarrow \quad \mathfrak{L}^{-1}(\langle\mathbf{G}^2\rangle) = \mathfrak{L}^{-1}(\langle\mathsf{D}^2\rangle)$$

It is not manifest that this is the case, the relationship between $\mathbf{G}$ and D is dependent on the intersection numbers of $\Delta^*(\mathsf{E}^*)$ basis elements and which are structures not seen in the $\Lambda^*(\mathsf{E}^*)$ construction. Justifying this for a general $\mathcal{W}$ and determining possible constraints on K and L is beyond the scope of this work. Instead we examine the parallelisable equivalent of the $\mathrm{Sp}(n)$ and $\mathrm{O}(m,m)$ invariances associated to the intersection numbers of the $\Delta^{\pm}(\mathsf{E}^*)$ bases.

5.1.3 $\mathbf{GL}(6,\mathbb{Z}) \subset \mathbf{O}(6,6)$ Covariant Bianchi Constraints

The Bianchi constraints defined in terms of $\Lambda^p(\mathsf{E}^*)$ components for the $\langle \mathcal{L}(\,\cdot\,)\rangle$ are defined in terms of contracted E and E^* indices. As a specific case we consider $\langle \mathcal{L}(\mathcal{D})\rangle$ defined in (5.1.1) and the fluxes $\mathcal{F}_n$ of Type IIA. These fluxes are defined by the η^p and ι_q and we consider a constant[1] transformation on this pair of dual bases.

$$\eta^p \to \widetilde{\eta}^p = \mathsf{P}^p{}_q \eta^q \quad \Leftrightarrow \quad \iota_q \to \widetilde{\iota}_q = (\mathsf{P}^{-\top})_q{}^p \iota_p \tag{5.1.17}$$

We have denoted the inverse of $\mathsf{P}^\top$ as $\mathsf{P}^{-\top}$ and shall use $(\mathsf{P}^{-\top})_q{}^p = (\mathsf{P}^{-1})^p{}_q$. This choice of transformation is motivated by the knowledge that the modular symmetry group of a six dimensional torus is $\mathrm{SL}(6,\mathbb{Z})$ and we wish to explicitly extract this symmetry from our construction of fluxes on a parallelisable space. Since we are taking the bases to transform in opposite ways P must be non-singular. Such a transformation on the bases induces a transformation on the fluxes and we consider two example cases, $\mathcal{F}_0$ and $\mathcal{F}_1$.

$$\begin{aligned}
\mathcal{F}_0 &= \tfrac{1}{3!}\mathcal{F}_{pqr}\eta^{pqr} = \tfrac{1}{3!}\widetilde{\mathcal{F}}_{pqr}\widetilde{\eta}^{pqr} &\Rightarrow\quad \mathcal{F}_{pqr} &= \widetilde{\mathcal{F}}_{tuv}\mathsf{P}^t{}_p\mathsf{P}^u{}_q\mathsf{P}^v{}_r \\
\mathcal{F}_1 &= \tfrac{1}{2!}\mathcal{F}^p_{qr}\eta^{qr}\iota_p = \tfrac{1}{2!}\widetilde{\mathcal{F}}^p_{qr}\widetilde{\eta}^{qr}\widetilde{\iota}_p &\Rightarrow\quad \mathcal{F}^p_{qr} &= (\mathsf{P}^{-1})^p{}_t\widetilde{\mathcal{F}}^t_{uv}\mathsf{P}^u{}_q\mathsf{P}^v{}_r
\end{aligned}$$

Some of the constraints on the $\mathcal{F}_n$ include the contraction between $\mathcal{F}_0$ and $\mathcal{F}_1$, $\mathcal{F}_{e[ab}\mathcal{F}^e_{cd]}$, and we consider the effect the change of basis has on this expression.

$$\mathcal{F}_{e[ab}\mathcal{F}^e_{cd]} \to \widetilde{\mathcal{F}}_{e[ab}\widetilde{\mathcal{F}}^e_{cd]} = (\mathsf{P}^{-1})^e{}_F\mathcal{F}_{EAB}\mathcal{F}^F_{CD}\mathsf{P}^E{}_e\mathsf{P}^A{}_{[a}\mathsf{P}^B{}_b\mathsf{P}^C{}_c\mathsf{P}^D{}_{d]}$$

The contracted indices transform in opposite ways and since P is non-singular the free indices are transformed in a non-degenerate way. The fact

[1] We do not consider $\mathcal{M}$ dependent transformations so as to not affect the nature of $\mathrm{d}\eta$.

they are transformed in non-degenerate ways means that the constraints on the fluxes are unchanged.

$$\langle \mathcal{F}_{e[ab}\mathcal{F}^{e}_{cd]} \rangle = \langle \widetilde{\mathcal{F}}_{e[ab}\widetilde{\mathcal{F}}^{e}_{cd]} \rangle \tag{5.1.18}$$

This extends to the other expressions in (5.1.1) and the ideals are indeed tensorial as they are coordinate independent. The P are further restricted, over and above being non-singular, by the symmetries of the compact space. In the case of orbifolds P must be invariant under the generators of the orbifold group and we shall examine this explicitly for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold [10, 92, 93, 94].

This symmetry is in fact a special case of the much larger set of transformations which leave parallelisable flux constructions invariant. The bases of E^* and E define the Clifford algebra of (2.1.3) whose bilinear form is the Kronecker delta. The contractions between different parallelisable flux components are determined by this Kroncker delta and thus (5.1.17) is seen to be embedded within $\mathrm{O}(6,6)$ [81], which is associated to the T duality group [50, 58, 66]. It is clear that general $\mathrm{O}(6,6)$ transformations on $\mathfrak{E} = \mathsf{E} \oplus \mathsf{E}^*$ would mix the different flux multiplets of (4.1.7), thus inducing the sequence in (4.1.6). The subgroups which leave the flux multiplets unchanged are thus which do not mix E and E^* and so define a pair of $\mathrm{O}(6)$ subgroups [51, 52, 56, 57].

We therefore have two formulations of the effective theory on a parallelisable $\mathcal{M}$ which manifest different kinds of symmetry. The $\Lambda^*(\mathsf{E}^*)$ formulation makes the $\mathrm{GL}(6,\mathbb{Z})$ invariance manifest while the $\Delta^*(\mathsf{E}^*)$ has the symmetries of the intersection numbers manifest. These types of transformations are disjoint, a constant transformation in the $\Lambda^*(\mathsf{E}^*)$ defining basis of E^* does not alter the intersection numbers or vice versa.

$$\begin{array}{lllll} \alpha_I &=& \frac{1}{3!}(\alpha_I)_{pqr}\eta^{pqr} \;\rightarrow\; \frac{1}{3!}(\widetilde{\alpha}_I)_{pqr}\widetilde{\eta}^{pqr} &=& \widetilde{\alpha}_I \\ \beta^J &=& \frac{1}{3!}(\beta^J)_{pqr}\eta^{pqr} \;\rightarrow\; \frac{1}{3!}(\widetilde{\beta}^J)_{pqr}\widetilde{\eta}^{pqr} &=& \widetilde{\beta}^J \end{array} \quad \text{s.t.} \quad \langle\, \alpha_I, \beta^J \,\rangle = \langle\, \widetilde{\alpha}_I, \widetilde{\beta}^J \,\rangle$$

This is particularly simple for the case of $\mathsf{P} \in \mathrm{SL}(6,\mathbb{Z})$ as vol_6 transforms as $\mathrm{vol}_6 \rightarrow (\det \mathsf{P})\mathrm{vol}_6$ and the $\Lambda^3(\mathsf{E}^*)$ formulation of the $\Delta^-(\mathsf{E}^*)$ basis is unchanged and we therefore have seen the modular group symmetry of a six dimensional parallelisable space.

$$\mathsf{P} \in \mathrm{SL}(6,\mathbb{Z}) \quad \Rightarrow \quad (\alpha_I)_{pqr} = (\widetilde{\alpha}_I)_{pqr} \quad , \quad (\beta^J)_{pqr} = (\widetilde{\beta}^J)_{pqr}$$

This construction is only possible on parallelisable $\mathcal{M}$ and so we return to the $\Delta^*(\mathsf{E}^*)$ defined components and their constraints.

5.1.4 Equivalent Type II Bianchi Constraints

Using (4.4.7) for $\langle\;\rangle_\pm \rightarrow \langle\;\rangle_-$ and (4.4.10) for $\langle\;\rangle_\pm \rightarrow \langle\;\rangle_+$ we can convert one set of Type II NS-NS expressions into its T duality partner and thus illustrate their equivalence [54] for both $\langle\;\rangle_\pm$ inner products. In the case of (4.4.7) for $\langle\;\rangle_-$ it is trivial that they are equivalent as the flux matrices and flux components are equal. For $\langle\;\rangle_+$ this is not the case but the constraints are still equivalent as the flux matrix expressions differ only by non-degenerate bilinear forms.

$$\begin{array}{l} \langle\;\rangle_- \;:\; \langle \mathsf{D}^2 \rangle \;=\; \left\langle \begin{array}{c} \mathsf{N}\cdot\mathsf{h}_\nu\cdot\mathsf{M} \\ \mathsf{M}\cdot\mathsf{h}_\mathfrak{a}\cdot\mathsf{N} \end{array} \right\rangle \rightarrow \left\langle \begin{array}{c} M\cdot h_\mathfrak{a}\cdot N \\ N\cdot h_\nu\cdot M \end{array} \right\rangle \;=\; \langle \mathcal{D}^2 \rangle \\ \langle\;\rangle_+ \;:\; \langle \mathsf{D}^2 \rangle \;=\; \left\langle \begin{array}{c} \mathsf{N}\cdot\mathsf{h}_\nu\cdot\mathsf{M} \\ \mathsf{M}\cdot\mathsf{h}_\mathfrak{a}\cdot\mathsf{N} \end{array} \right\rangle \rightarrow \left\langle \begin{array}{c} \mathsf{h}_\mathfrak{a}\cdot(M\cdot h_\mathfrak{a}\cdot N) \\ h_\mathfrak{a}\cdot(N\cdot h_\nu\cdot M) \end{array} \right\rangle \;=\; \langle \mathcal{D}^2 \rangle \end{array} \tag{5.1.19}$$

These equivalent constraints can be further examined by using (4.2.10) and (4.4.10) to write each of the flux matrix combinations in (5.1.19) in terms of a single flux matrix. Given each derivative has two flux matrices there are four choices for each combination.

$$\begin{aligned}\langle M\cdot h_{\mathfrak{a}}\cdot N\rangle &= \left\{\begin{array}{cc}\langle N^{\top}\cdot g_{\mathfrak{a}}\cdot N\rangle & \langle M\cdot g_{\mathfrak{a}}\cdot M^{\top}\rangle \\ \langle \mathsf{M}^{\top}\cdot g_{\mathfrak{a}}\cdot \mathsf{M}\rangle & \langle \mathsf{N}\cdot g_{\mathfrak{a}}\cdot \mathsf{N}^{\top}\rangle\end{array}\right\} = \langle \mathsf{N}\cdot \mathsf{h}_{\nu}\cdot \mathsf{M}\rangle \\ \langle N\cdot h_{\nu}\cdot M\rangle &= \left\{\begin{array}{cc}\langle M^{\top}\cdot \mathsf{g}_{\mathfrak{a}}\cdot M\rangle & \langle N\cdot \mathsf{g}_{\mathfrak{a}}\cdot N^{\top}\rangle \\ \langle \mathsf{N}^{\top}\cdot \mathsf{g}_{\mathfrak{a}}\cdot \mathsf{N}\rangle & \langle \mathsf{M}\cdot \mathsf{g}_{\mathfrak{a}}\cdot \mathsf{M}^{\top}\rangle\end{array}\right\} = \langle \mathsf{M}\cdot \mathsf{h}_{\mathfrak{a}}\cdot \mathsf{N}\rangle\end{aligned} \tag{5.1.20}$$

These results are independent of our choice of $\langle\;\rangle_{\pm}$, in both cases the nilpotency conditions become quadratic in one flux matrix, with a symplectic bilinear form between them. This form of the constraints suggests they can be rephrased to be in terms of an integrand defined by a pair of exact forms. Precisely how this is done depends on our choice of $\langle\;\rangle_{\pm}$. To examine this we use a pair of vectors $\underline{\chi}$ and $\underline{\varphi}$ of dimension $2h^{1,1}+2$ and a pair of vectors $\underline{\phi}$ and $\underline{\psi}$ of dimension $2h^{2,1}+2$, which then allow us to define sets of forms in either Type II theory.

$$\text{IIA}\quad\left\{\begin{array}{ccc}\chi \equiv \underline{\chi}^{\top}\cdot h_{\nu}\cdot \mathbf{e}_{(\nu)} & \leftrightarrow & \widetilde{\chi} \equiv \underline{\chi}^{\top}\cdot \mathsf{h}_{\mathfrak{a}}\cdot \mathbf{f}_{(\mathfrak{a})} \\ \phi \equiv \underline{\phi}^{\top}\cdot h_{\mathfrak{a}}\cdot \mathbf{e}_{(\mathfrak{a})} & \leftrightarrow & \widetilde{\phi} \equiv \underline{\phi}^{\top}\cdot \mathsf{h}_{\nu}\cdot \mathbf{f}_{(\nu)}\end{array}\right\}\quad\text{IIB}$$

In the case of $\langle\;\rangle_{-}$ the anti self adjoint nature of the derivatives immediately imply the result for both sets of expressions in (5.1.20).

$$\begin{aligned}\langle M\cdot h_{\mathfrak{a}}\cdot N\rangle &= \left\{\begin{array}{c}\int_{\mathcal{M}}\langle\,\chi,\mathcal{D}^2(\varphi)\,\rangle_{-} = \int_{\mathcal{M}}\langle\,\mathcal{D}(\varphi),\mathcal{D}(\chi)\,\rangle_{-} \\ \int_{\mathcal{W}}\langle\,\widetilde{\chi},\mathsf{D}^2(\widetilde{\varphi})\,\rangle_{-} = \int_{\mathcal{W}}\langle\,\mathsf{D}(\widetilde{\varphi}),\mathsf{D}(\widetilde{\chi})\,\rangle_{-}\end{array}\right\} = \langle M\cdot g_{\mathfrak{a}}\cdot M^{\top}\rangle \\ \langle N\cdot h_{\nu}\cdot M\rangle &= \left\{\begin{array}{c}\int_{\mathcal{M}}\langle\,\phi,\mathcal{D}^2(\psi)\,\rangle_{-} = \int_{\mathcal{M}}\langle\,\mathcal{D}(\psi),\mathcal{D}(\phi)\,\rangle_{-} \\ \int_{\mathcal{W}}\langle\,\widetilde{\phi},\mathsf{D}^2(\widetilde{\psi})\,\rangle_{-} = \int_{\mathcal{W}}\langle\,\mathsf{D}(\widetilde{\psi}),\mathsf{D}(\widetilde{\phi})\,\rangle_{-}\end{array}\right\} = \langle M^{\top}\cdot \mathsf{g}_{\mathfrak{a}}\cdot M\rangle\end{aligned}$$

The fact that $g_{\nu}=\mathsf{g}_{\mathfrak{a}}$ and $\mathsf{g}_{\nu}=g_{\mathfrak{a}}$ for $\langle\;\rangle_{-}$ allows us to formulate both sets of expressions as integrals over either $\mathcal{M}$ or $\mathcal{W}$. This is not the case for $\langle\;\rangle_{+}$ as $g_{\nu}\neq\mathsf{g}_{\mathfrak{a}}$ and $\mathsf{g}_{\nu}\neq g_{\mathfrak{a}}$. Instead we note that the first set of expressions in

(5.1.20) are defined with $g_{\mathfrak{a}}$ and the second set with $\mathsf{g}_{\mathfrak{a}}$ and by constructing the flux matrix expressions for combinations of 3-forms on $\mathcal{M}$ and $\mathcal{W}$ we obtain the required result.

$$\begin{aligned} \int_{\mathcal{M}} \big\langle\, \mathcal{D}(\chi), \mathcal{D}(\varphi)\,\big\rangle_{+} \;&=\; g_{\nu}\Big(\chi, \mathcal{D}\big(\mathcal{D}(\varphi)\big)\Big) \;=\; g_{\mathfrak{a}}\Big(\mathcal{D}(\chi), \mathcal{D}(\varphi)\Big) \;=\; 0 \\ \int_{\mathcal{W}} \big\langle\, \mathsf{D}(\widetilde{\phi}), \mathsf{D}(\widetilde{\psi})\,\big\rangle_{+} \;&=\; \mathsf{g}_{\nu}\Big(\widetilde{\phi}, \mathsf{D}\big(\mathsf{D}(\widetilde{\psi})\big)\Big) \;=\; \mathsf{g}_{\mathfrak{a}}\Big(\mathsf{D}(\widetilde{\phi}), \mathsf{D}(\widetilde{\psi})\Big) \;=\; 0 \end{aligned} \tag{5.1.21}$$

We can express these results for both inner products $\langle\ \rangle_{\pm}$ in a single way.

$$\langle \mathcal{D}^2 \rangle \;=\; \Big\langle g_{\mathfrak{a}}\Big(\mathcal{D}(\,\cdot\,), \mathcal{D}(\,\cdot\,)\Big), \mathsf{g}_{\mathfrak{a}}\Big(\mathsf{D}(\,\cdot\,), \mathsf{D}(\,\cdot\,)\Big)\Big\rangle \;=\; \langle \mathsf{D}^2 \rangle \tag{5.1.22}$$

5.2 T Duality Tadpole Constraints

We previously considered how the inclusion of branes and other extended objects can alter the dynamics of R-R fluxes living on those extended objects, giving rise to tadpole constraints. At present we are restricting our attention to only those fluxes induced by T duality and so the R-R sector has only $F_3 = \mathsf{F}_0$ in Type IIB and $F_{RR} = \mathfrak{F}_0$ in Type IIA [54, 56, 60, 61] but these couple to the geometric and non-geometric fluxes induced by T duality [82, 83].

5.2.1 Type IIB

Recalling the example of D3-branes extended through the external space-time we note how the flux dependent expression can be written as an exact form.

$$N_3 + \int_{\mathcal{W}} H_3 \wedge F_3 = N_3 + \int_{\mathcal{W}} \mathrm{d}\widetilde{F}_5^{(0)} = 0 \tag{5.2.1}$$

Here we are taking $\widetilde{F}_5^{(0)}$ to be the field strength in the case where there are no branes. Although we have made the explicit assumption that F_5 is trivial in $\mathcal{W}$ we can none-the-less express the $H_3 \wedge F_3$ expression as an exact form by using D and for later convenience we revert to our F, $\widehat{\mathsf{F}}$ notation.

$$\int_{\mathcal{W}} H_3 \wedge F_3 \to \int_{\mathcal{W}} \widehat{\mathsf{F}}_0 \wedge \mathsf{F}_0 = \int_{\mathcal{W}} \mathsf{D}(\mathsf{F}_3) \tag{5.2.2}$$

This can be taken a step further by noting that F_0 can be also written as an exact form in D', $\mathsf{F}_0 = \mathsf{D}'(\widetilde{\nu}^0)$, and that we can project out the coefficient of a 6-form using ι_{ν_0} we have a differential expression for N_3.

$$\widehat{\mathsf{F}}_0 \wedge \mathsf{F}_0 \to \mathsf{D}\mathsf{D}'(\widetilde{\nu}^0) \quad \Rightarrow \quad N_3 + \iota_{\nu_0}\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0) = 0 \tag{5.2.3}$$

This expression motivates further analysis of quadratic derivatives; the $\mathsf{D}\mathsf{D}'$ expressions take a form similar to those expressions already obtained for D^2 and D'^2 in the context of Bianchi constraints from T duality. In the case where F_0 is the only R-R flux the only non-zero expressions follow from $\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0)$.

$$\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0) = \Big(\widehat{\mathsf{F}}^{(0)B}\mathsf{F}^{(J)}{}_{B} - \widehat{\mathsf{F}}^{(0)}{}_{A}\mathsf{F}^{(J)A}\Big)\nu_J + \Big(\widehat{\mathsf{F}}^{(0)}{}_{A}\mathsf{F}_{(J)}{}^{A} - \widehat{\mathsf{F}}^{(0)B}\mathsf{F}_{(J)B}\Big)\widetilde{\nu}^J \tag{5.2.4}$$

With ι_{ν_0} projecting out the coefficient of ν_0 we obtain the flux expression for the tadpole contributions living on D3-branes which couple to the C_4.

$$\iota_{\nu_0}\Big(\widehat{\mathsf{F}}_0 \cdot \mathsf{F}_0\Big) = \iota_{\nu_0}\Big(\widehat{\mathsf{F}}_0 \wedge \mathsf{F}_0\Big) = \iota_{\nu_0}\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0) = \widehat{\mathsf{F}}^{(0)B}\mathsf{F}^{(0)}{}_{B} - \widehat{\mathsf{F}}^{(0)}{}_{A}\mathsf{F}^{(0)A} \tag{5.2.5}$$

The next set of Type IIB tadpoles follow from the D5-branes coupling electrically to C_6. Due to the fact the Type IIB fluxes contribute to the superpotential in a way different from their index structure, when compared to Type IIA, we cannot automatically assume that the tadpoles due to the C_6 will be obtained by projecting out the 4-forms of $\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0)$. Instead we restrict our attention to parallelisable $\mathcal{W}$ so as to revert back to the p-form component formulation and consider how the C_6 tadpole would be constructed in terms of such components.

$$\mathsf{F}_1 \,:\, \mathsf{F}_0 \to \mathsf{F}_1 \cdot \mathsf{F}_0 \;\propto\; \mathsf{F}^{m}_{pq}\mathsf{F}_{rsm}\eta^{pqrs} \in \Lambda^4(\mathsf{E}^*) \tag{5.2.6}$$

In the absence of additional fluxes this is the only way to construct a 4-form from a combination of an NS-NS flux and an R-R flux. Despite not being able to explicitly express F_1 in terms of the components of M, we have previously determined which components of M F_1 is dependent upon and thus we can use the parallelisable $\Lambda^*(\mathsf{E}^*)$ case to determine the general $\Omega^*(\mathsf{E}^*)$ one. Such fluxes arise in (5.2.4) in the coefficients of ν_j terms.

$$\iota_{\nu_j}\mathsf{DD}'(\widetilde{\nu}^0) = \widehat{\mathsf{F}}^{(0)B}\mathsf{F}^{(j)}{}_B - \widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}^{(j)A} \tag{5.2.7}$$

As a result of this non-standard Kähler moduli coupling we would expect to construct the standard tadpoles by using $\mathbf{G}$ and $\mathbf{G}'$, with the C_6 tadpoles being the coefficients $\iota_{\widetilde{\nu}^j}\mathbf{GG}'(\widetilde{\nu}^0)$. In our examination of Bianchi constraints we commented that the construction of the p-form components of $\mathbf{G}$ requires explicit choices for the L and K matrices of (4.2.8). However, we also demonstrated that the $\Delta^*(\mathsf{E}^*)$ formulation does not need the explicit form of K and L, they only needed to be symplectic and commute with the complexification matrices $\mathbb{C}^{(\prime)}$. Their symplectic nature results in the analysis of DD' Bianchi expressions being equivalent to an analysis of $\mathbf{GG}'$ Bianchi expressions. For the time being we assume this carries through into the tadpoles, we shall prove this assumption shortly for general U duality constructions. For the T duality only case it can be seen explicitly by considering (5.2.7). K defined a symplectic transformation on the $\Delta^3(\mathsf{E}^*)$ basis and the expression is manifestly invariant under such a transformation. L exchanges $\widetilde{\nu}^i$ dependence for ν_j dependence and this is the difference in how the C_6 tadpole arises in $\mathsf{DD}'(\widetilde{\nu}^0)$ compared to $\mathbf{GG}'(\widetilde{\nu}^0)$. As such we shall use D and D' rather than $\mathbf{G}$ and $\mathbf{G}'$ and use the same method as (5.2.6) to guide how we relate flux polynomials to the C_p tadpoles. Given an R-R potential C_p the symmetries of the space-time require it be of the form $C_p \sim \mathrm{vol}_4 \wedge \xi$ for $\xi \in \Delta^{p-4}(\mathsf{E}^*)$.

n	C_n	Cycles	Fluxes		Tadpole contribution
4	$\mathrm{vol}_4 \wedge \widetilde{\nu}^0$	$\mathcal{A}^0$	$\iota_{\nu_0}\mathsf{DD}'(\widetilde{\nu}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)B}\mathsf{F}^{(0)}{}_B - \widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}^{(0)A}$
6	$\mathrm{vol}_4 \wedge \nu_i$	$\mathcal{A}^i$	$\iota_{\nu_i}\mathsf{DD}'(\widetilde{\nu}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}^{(j)A} - \widehat{\mathsf{F}}^{(0)B}\mathsf{F}^{(j)}{}_B$
8	$\mathrm{vol}_4 \wedge \widetilde{\nu}^j$	$\mathcal{B}_j$	$\iota_{\widetilde{\nu}^j}\mathsf{DD}'(\widetilde{\nu}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)B}\mathsf{F}_{(i)B} - \widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}_{(i)}{}^A$
10	$\mathrm{vol}_4 \wedge \nu_0$	$\mathcal{B}_0$	$\iota_{\widetilde{\nu}^0}\mathsf{DD}'(\widetilde{\nu}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}_{(0)}{}^A - \widehat{\mathsf{F}}^{(0)B}\mathsf{F}_{(0)B}$

Table 5.1: Type IIB T duality tadpoles flux polynomials

Naively we would expect this to couple to $\star\xi\,\iota_{(\star\xi)}\Big(\mathsf{DD}'(\widetilde{\nu}^0)\Big)$ but we have now seen this is not the case for all fluxes, instead we expect it to couple to $\star\xi\,\iota_{(\star\xi)}\Big(\mathbf{GG}'(\widetilde{\nu}^0)\Big)$. With L relating these two constructions we can construct the schematic tadpole couplines for the derivatives.

$$\mathrm{tad}\left(C_{p+4} \sim \mathrm{vol}_4 \wedge \xi_p\right) \propto \iota_{(\star\mathsf{L}\cdot\xi)}\Big(\mathbf{GG}'(\widetilde{\nu}^0)\Big) \tag{5.2.8}$$

The case for F_2 follows in the same manner, with $\mathsf{F}_2 \cdot \mathsf{F}_0$ coupling to $C_8 \sim \mathrm{vol}_4 \wedge \widetilde{\omega}^i$, which has support on D7-branes wrapping 4-cycles in $\mathcal{W}$ but the relevant flux polynomials arising in $\mathsf{DD}'(\widetilde{\nu}^0)$ not as the coefficients of $\star\widetilde{\omega}^i$ but as the coefficients of $\star(\mathsf{L}\cdot\widetilde{\omega})^i \sim \widetilde{\omega}^i = \widetilde{\nu}^i$.

$$\iota_{\widetilde{\nu}^i}\mathsf{DD}'(\widetilde{\nu}^0) \;=\; \widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}_{(i)}{}^A - \widehat{\mathsf{F}}^{(0)B}\mathsf{F}_{(i)B} \tag{5.2.9}$$

The final case is C_{10}, found by projecting out the $\widetilde{\nu}^0$ component of $\mathsf{DD}'(\widetilde{\nu}^0)$.

$$\iota_{\widetilde{\nu}^0}\mathsf{DD}'(\widetilde{\nu}^0) = \widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}_{(0)}{}^A - \widehat{\mathsf{F}}^{(0)B}\mathsf{F}_{(0)B} \tag{5.2.10}$$

All of the Type IIB results are summarised in Table 5.1.

5.2.2 Type IIA

The Type IIA side has only one kind of tadpole, that which arises from C_7 on D6-branes, due to our stipulation that $\Delta^1(\mathsf{E}^*)$ and $\Delta^5(\mathsf{E}^*)$ are empty

p	$C_7^{(p)}$	Cycles	Fluxes		Tadpole contribution
0	$\text{vol}_4 \wedge \mathfrak{b}^0$	A^0	$\iota_{\mathfrak{a}_0}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathfrak{F}}^{(0)}{}_A\mathcal{F}_{(A)0} \pm \widehat{\mathfrak{F}}^{(0)B}\mathcal{F}^{(B)}{}_0$
1	$\text{vol}_4 \wedge \mathfrak{a}_i$	A^i	$\iota_{\mathfrak{b}^i}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathfrak{F}}^{(0)}{}_A\mathcal{F}_{(A)}{}^i \pm \widehat{\mathfrak{F}}^{(0)B}\mathcal{F}^{(B)i}$
2	$\text{vol}_4 \wedge \mathfrak{b}^j$	B_j	$\iota_{\mathfrak{a}_j}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathfrak{F}}^{(0)}{}_A\mathcal{F}_{(A)j} \pm \widehat{\mathfrak{F}}^{(0)B}\mathcal{F}^{(B)}{}_j$
3	$\text{vol}_4 \wedge \mathfrak{a}_0$	B_0	$\iota_{\mathfrak{b}^0}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathfrak{F}}^{(0)}{}_A\mathcal{F}_{(A)}{}^0 \pm \widehat{\mathfrak{F}}^{(0)B}\mathcal{F}^{(B)0}$

Table 5.2: Type IIA flux dependent Type IIA T duality tadpoles.

[60]. However, this does not result in Type IIA having a more trivial tadpole sector than its Type IIB counterpart. As in the Type IIB case the tadpoles are determined by the non-closure of the R-R fluxes which in the Type IIA case is F_{RR} and like $F_3 = \mathsf{F}_0$ it can be written as an exact form, $F_{RR} = \mathfrak{F}_0 \cdot \mathfrak{b}^0$. This, coupled with the T duality induced $\mathrm{d} \to \mathcal{D}$ leads to the same kind of expressions as the Type IIB case, $\mathrm{d}\mathcal{F}_{RR} \to \mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$ but we include the sign choice of $\langle\ \rangle_\pm$ as this is relevant to the definition of flux components of $\mathcal{D}'$ acting on $\Delta^-(\mathsf{E}^*)$.

$$\begin{aligned} \mathcal{D}\mathcal{D}'(\mathfrak{b}^0) &= \mathcal{D}\Big(\widehat{\mathfrak{F}}^{(0)}{}_A\nu_A \pm \widehat{\mathfrak{F}}^{(0)B}\widetilde{\nu}^B\Big) \\ &= \Big(\widehat{\mathfrak{F}}^{(0)}{}_A\mathcal{F}_{(A)I} \pm \widehat{\mathfrak{F}}^{(0)B}\mathcal{F}^{(B)}{}_I\Big)\mathfrak{a}_I - \Big(\widehat{\mathfrak{F}}^{(0)}{}_A\mathcal{F}_{(A)}{}^J \pm \widehat{\mathfrak{F}}^{(0)B}\mathcal{F}^{(B)J}\Big)\mathfrak{b}^J \end{aligned} \tag{5.2.11}$$

Since we are considering both inner products we have different index structures of the flux components of $\mathcal{D}$ and $\mathcal{D}'$. Following the same methodology as the Type IIB case $\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$ can be split into four parts, each associated to the cycles A^0, A^i, B_0, B_j which are symplectic in their intersection numbers, unlike the Type IIB tadpole cycles. As a result we obtain the Type IIA version of Table 5.1, Table 5.2. The conversion to different flux component definitions can be done using the results of Section 4.3 such that $\mathfrak{F} \to \widehat{\mathcal{F}}$ or vice versa. Since we will ultimately wish to compare the tadpoles of each

Type II theory, after the inclusion of $\mathrm{SL}(2,\mathbb{Z})_S$ transformations, we can use the results of (4.4.20) and (4.4.12) to express the Type IIA tadpole in terms Type IIB fluxes.

$$\mathcal{D}\mathcal{D}'(\mathfrak{b}^0) = \left(\widehat{\mathsf{F}}^{(0)}{}_A \mathsf{F}^{(J)A} \pm \widehat{\mathsf{F}}^{(0)A} \mathsf{F}^{(J)}{}_A \right) \mathfrak{a}_J \pm \left(\widehat{\mathsf{F}}^{(0)}{}_B \mathsf{F}_{(J)}{}^B \pm \widehat{\mathsf{F}}^{(0)A} \mathsf{F}_{(J)A} \right) \mathfrak{b}^J$$

Splitting this expression using the same cycle decomposition we obtain Table 5.3; a version of Table 5.2 but with the fluxes now being the Type IIB ones. Comparing the flux polynomials in Table 5.3 with those of Table 5.1 we see that for $\langle\ \rangle_{\pm} \to \langle\ \rangle_{-}$ the same polynomials, up to overall factors of ± 1, are obtained. This is to be expected given the action of T duality or the mirror map $\mathfrak{M}$ on the brane content of each construction. We previously noted in Section 4.3 that for spaces which do not receive quantum corrections the $\mathfrak{M}$ action is set by the definitions of the $\Delta^*(\mathsf{E}^*)$ bases of $\mathcal{M}$ and $\mathcal{W}$. This was explicitly stated in terms of the original $\Delta^*(\mathsf{E}^*)$ basis in (4.3.2) and we convert that into the new basis, though only consider one of the two maps.

$$\mathfrak{M}(\mathfrak{b}^0) \propto \widetilde{\nu}^0 \quad \Rightarrow \quad \mathfrak{M}(\mathfrak{a}_0) \propto \nu_0 \tag{5.2.12}$$

This is indeed the structure seen in the tadpole constraints of Tables 5.1 and 5.3. The C_4 expression of Table 5.1 and the $C_7^{(0)}$ expression of Table 5.3 are equal, the D6s wrapping the A^0 cycle of $\mathcal{W}$ are mirrored to D3s which have no support in $\mathcal{M}$. Conversely the D6s wrapping the dual B^0 cycle of $\mathcal{W}$ are mapped to D9s which fill the entirity of $\mathcal{M}$. We also observe that the effect of the Type IIB non-standard Kähler moduli coupling has been mirrored in the Type IIA construction. The $C_7^{(1)}$ and $C_7^{(2)}$ flux polynomials of Table 5.3 should be exchanged if the Type IIA tadpoles are to be manifestly the mirror of the Type IIB tadpole expressions. This is related to the different ways in which Type IIA and Type IIB define their complex structure moduli. This exchange can be done by a symplectic transformation, the Type IIA version

p	$C_7^{(p)}$	Cycles	Fluxes		Tadpole contribution
0	$\mathrm{vol}_4 \wedge \mathfrak{b}^0$	A^0	$\iota_{\mathfrak{a}_0}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}^{(0)A} \pm \widehat{\mathsf{F}}^{(0)A}\mathsf{F}^{(0)}{}_A$
1	$\mathrm{vol}_4 \wedge \mathfrak{a}_i$	A^i	$\iota_{\mathfrak{b}^i}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)}{}_B\mathsf{F}_{(i)}{}^B \pm \widehat{\mathsf{F}}^{(0)A}\mathsf{F}_{(i)A}$
2	$\mathrm{vol}_4 \wedge \mathfrak{b}^j$	B_j	$\iota_{\mathfrak{a}_j}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)}{}_A\mathsf{F}^{(j)A} \pm \widehat{\mathsf{F}}^{(0)A}\mathsf{F}^{(j)}{}_A$
3	$\mathrm{vol}_4 \wedge \mathfrak{a}_0$	B_0	$\iota_{\mathfrak{b}^0}\mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$	$\propto$	$\widehat{\mathsf{F}}^{(0)}{}_B\mathsf{F}_{(0)}{}^B \pm \widehat{\mathsf{F}}^{(0)A}\mathsf{F}_{(0)A}$

Table 5.3: Type IIB flux dependent Type IIA T duality tadpoles.

of K, and thus the motivation for a transformation of the $\Delta^3(\mathsf{E}^*)$ basis is not immediately apparent. This has been motivated by moduli dependency but for parallelisable spaces it can be done explicitly in terms of flux components using (3.1.4), a result we will see in our analysis of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold. Until then we shall not concern ourselves too much with which cycles the tadpoles are associated to, only that tadpole polynomials of the forms given in Tables 5.1 and 5.3 exist. With this basic framework constructed for a T duality invariant Type II effective theory we can now extend our analysis to the more general case of including S duality and its non-trivial union with T duality, U duality.

5.3 U Duality

Due to Type IIB being self S-dual its R-R sector can be examined in precisely the same way as its NS-NS sector, irrespective of which $\langle\ \rangle_\pm$ we consider. A Type IIB theory with only R-R fluxes can be analysed in the same way as a Type IIB theory with only NS-NS fluxes, though when both flux types are non-zero these constraints extend due to non-trivial mixing. This allows us to examine U duality in Type IIB in progressive steps [32], which is not possible in Type IIA for $\langle\ \rangle_+$, and so we begin our analysis extending the T duality invariant constraints of Type IIB by performing modular transformations on the dilaton. The examination of Type IIB will then guide our analysis of Type IIA and its full U duality invariance. We have already seen that for the T duality only case the Bianchi constraints on $\Lambda^*(\mathsf{E}^*)$ and $\Delta^*(\mathsf{E}^*)$ have entirely different structures and this is particularly important for the inclusion of S duality. We shall begin with the restricted parallelisable case but some of the analysis will be carried over into the more general $\Delta^*(\mathsf{E}^*)$ case. A guiding principle in our analysis will be to construct $\mathrm{SL}(2,\mathbb{Z})$ multiplets from the constraints under the two $\mathrm{SL}(2,\mathbb{Z})$ generators; the dilaton inversion $S \to -\frac{1}{S}$ and a real shift $S \to S+1$ and it is this which is common between the $\Lambda^*(\mathsf{E}^*)$ and $\Delta^*(\mathsf{E}^*)$ constructions.

5.3.1 Parallelised Type IIB S Duality Bianchi Constraints

The fluxes must satisfy the constraints following from the $\mathrm{SL}(2,\mathbb{Z})_S$ image of (5.1.15). We previously used the modular inversion $S \to -\frac{1}{S}$ to obtain the pure R-R sector from the NS-NS sector and the R-R Lie algebra (4.2.2). The Jacobi constraints for the R-R sector follow the same schematic form as the NS-NS sector [60].

$$\begin{aligned}
0 &= \mathsf{F}_{e[ab}\widehat{\mathsf{F}}^{e}_{\ cd]} \\
0 &= \widehat{\mathsf{F}}^{d}_{\ e[a}\widehat{\mathsf{F}}^{e}_{\ bc]} \quad + \quad \mathsf{F}_{e[ab}\widehat{\mathsf{F}}^{de}_{\ \ c]} \\
0 &= \widehat{\mathsf{F}}^{[ab]}_{e}\widehat{\mathsf{F}}^{e}_{\ [cd]} \quad - \quad 4\widehat{\mathsf{F}}^{[a}_{\ e[c}\widehat{\mathsf{F}}^{b]e}_{\ \ d]} \quad + \quad \mathsf{F}_{e[cd]}\widehat{\mathsf{F}}^{[ab]e} \\
0 &= \widehat{\mathsf{F}}^{e[a}_{d}\widehat{\mathsf{F}}^{bc]}_{e} \quad + \quad \mathsf{F}^{e[ab}\widehat{\mathsf{F}}^{c]}_{de} \\
0 &= \mathsf{F}^{e[ab}\widehat{\mathsf{F}}^{cd]}_{e}
\end{aligned} \qquad (5.3.1)$$

The R-R fluxes satisfying (5.3.1) and NS-NS fluxes (5.1.7) is not sufficient for full S duality invariance. As yet we have not considered the effect the second $\mathrm{SL}(2,\mathbb{Z})$ generator corresponding to $S \to S+1$ has on the constraints; the two flux sectors become mixed. Furthermore, we cannot simply consider all possible $\mathrm{SL}(2,\mathbb{Z})_S$ images of (5.1.7) and declare them to be the necessary and sufficient conditions for S duality invariance, there is a subtly in how the expressions arrange themselves into $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets. This can be seen by considering the possible combinations of fluxes which arise in Bianchi constraints. The F do not couple to the dilaton, while the $\widehat{\mathsf{F}}$ do. Therefore a pair of fluxes, of the kind seen in (5.3.1) or (5.1.7), can either couple to the dilaton in the same way or in a different way to one another. The transformation on the dilaton is taken to be that of (4.2.15) and we define $n_{ij} \equiv n_i n_j$, noting that $n_{14} - n_{23} = 1$. For the purposes of clarity we shall neglect the index structure of the expressions since we are only interested in the schematic form of the induced constraints and thus is applicable to non-parallelisable spaces too.

$$\begin{aligned}
\widehat{\mathsf{F}}\cdot\mathsf{F} &\to n_{31}\mathsf{F}\cdot\mathsf{F} + n_{32}\mathsf{F}\cdot\widehat{\mathsf{F}} + n_{41}\widehat{\mathsf{F}}\cdot\mathsf{F} + n_{42}\widehat{\mathsf{F}}\cdot\widehat{\mathsf{F}} \\
\mathsf{F}\cdot\widehat{\mathsf{F}} &\to n_{13}\mathsf{F}\cdot\mathsf{F} + n_{14}\mathsf{F}\cdot\widehat{\mathsf{F}} + n_{23}\widehat{\mathsf{F}}\cdot\mathsf{F} + n_{24}\widehat{\mathsf{F}}\cdot\widehat{\mathsf{F}} \\
\mathsf{F}\cdot\mathsf{F} &\to n_{11}\mathsf{F}\cdot\mathsf{F} + n_{12}\mathsf{F}\cdot\widehat{\mathsf{F}} + n_{21}\widehat{\mathsf{F}}\cdot\mathsf{F} + n_{22}\widehat{\mathsf{F}}\cdot\widehat{\mathsf{F}} \\
\widehat{\mathsf{F}}\cdot\widehat{\mathsf{F}} &\to n_{33}\mathsf{F}\cdot\mathsf{F} + n_{34}\mathsf{F}\cdot\widehat{\mathsf{F}} + n_{43}\widehat{\mathsf{F}}\cdot\mathsf{F} + n_{44}\widehat{\mathsf{F}}\cdot\widehat{\mathsf{F}}
\end{aligned} \tag{5.3.2}$$

By comparing coefficients it follows that these four quadratic flux combinations arrange themselves into a singlet and a triplet [60, 10].

$$\mathbf{1} = \langle \widehat{\mathsf{F}}\cdot\mathsf{F} - \mathsf{F}\cdot\widehat{\mathsf{F}} \rangle \quad , \quad \mathbf{3} = \langle \mathsf{F}\cdot\mathsf{F} \, , \, \widehat{\mathsf{F}}\cdot\widehat{\mathsf{F}} \, , \, \widehat{\mathsf{F}}\cdot\mathsf{F} + \mathsf{F}\cdot\widehat{\mathsf{F}} \rangle \tag{5.3.3}$$

Applying this transformation to (5.3.1) or (5.1.7) does not provide much of an insight into the structures induced by S duality over and above T duality. Instead we restrict our examination to one of the orientifolded constructions, which we take to be one with O3-planes such that all basis elements of $\Delta^2(\mathsf{E}^*)$ be eigenforms of eigenvalue $+1$ under the orientifold involution, $h^{1,1}_{+} = h^{1,1}$. We make this choice such that only $H_3 \cong \widehat{\mathsf{F}}_0$, $Q \cong \mathsf{F}_2$

and their R-R partners $F_3 \cong \mathsf{F}_0$ and $P \cong \widehat{\mathsf{F}}_2$ are non-zero, rather than allowing potentially all eight Type IIB flux multiplets to have some non-zero components. This is sufficiently restricted to examine the $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets in more detail and we explicitly state the $\Lambda^*(\mathsf{E}^*)$ indices.

$$\begin{aligned} \mathbf{1} &: \left\langle \widehat{\mathsf{F}}_{e[ab}\mathsf{F}^{ed}_{c]} - \mathsf{F}_{e[ab}\widehat{\mathsf{F}}^{ed}_{c]} \right\rangle \\ \mathbf{3} &: \left\langle \mathsf{F}^{e[a}_{d}\mathsf{F}^{bc]}_{e} \, , \, \widehat{\mathsf{F}}^{e[a}_{d}\widehat{\mathsf{F}}^{bc]}_{e} \, , \, \mathsf{F}^{e[a}_{d}\widehat{\mathsf{F}}^{bc]}_{e} + \widehat{\mathsf{F}}^{e[a}_{d}\mathsf{F}^{bc]}_{e} \right\rangle \end{aligned} \tag{5.3.4}$$

The motivation for our choice of orientifold projections is now seen to follow from the fact that now the NS-NS non-geometric flux F_2 playing the role of a structure constant for a six dimensional subalgebra generated by the X. Its S duality induced R-R partner, obtained under $S \to -\frac{1}{S}$, also has this property. Naively we might have expected the associated sets of constraints for F_2 and $\widehat{\mathsf{F}}_2$ to be satisfied separately if both T and S duality invariance were enforced but instead we find they form an $\mathrm{SL}(2,\mathbb{Z})_S$ singlet. The triplet also illustrates a structure which results from the mixing of the two flux sectors. This is extended by S duality such that both F_2 and its R-R partner $\widehat{\mathsf{F}}_2$ define separate six dimensional subalgebras, generated by X and its magnetic dual $\mathbf{X}$, but the third member of the triplet results in the algebras being interdependent. This non-trivial mixing can be viewed in terms of deformed Lie algebras and integrability conditions. To examine this further we start with a general Lie algebra $\mathcal{L}$ defined by its brackets[2].

$$[X^a, X^b] = C^{ab}_c \, X^c \tag{5.3.5}$$

These relations define an algebra if and only if the Jacobi identity on C^{ab}_c is fulfilled, namely $C^{[ab}_e \, C^{c]e}_d = 0$. Deformations to these commutation relations can be written in terms of an element φ of the second cohomology class of the

[2]We define generators with an upper index in analogy with the orientifolded U duality induced problem $[\mathbb{X}^a, \mathbb{X}^b] = Q^{ab}_c \, \mathbb{X}^c$ we are dealing with.

algebra, $H^2(\mathcal{L},\mathcal{L})$, with the 2-cocycles $\varphi \in H^2(\mathcal{L},\mathcal{L})$ being closed under the action of an exterior derivation d without being coboundaries. The closure of φ is not a trivial matter, being dependent upon both the co-cycle φ itself and the properties of the generators of $\mathcal{L}$.

$$\begin{aligned} 0 &= d\varphi(X^a, X^b, X^c) \\ &\equiv [X^a, \varphi(X^b, X^c)] + [X^c, \varphi(X^a, X^b)] + [X^b, \varphi(X^c, X^a)] + \\ &+ \varphi(X^a, [X^b, X^c]) + \varphi(X^c, [X^a, X^b]) + \varphi(X^b, [X^c, X^a]) \end{aligned} \tag{5.3.6}$$

The fact φ is not exact allows for the construction of a deformed Lie bracket for the X without the deformation being trivial.

$$[X^a, X^b]_\varphi = C^{ab}_c \, X^c + \varphi(X^a, X^b) \; . \tag{5.3.7}$$

This linear deformation $\mathcal{L} + \varphi$ is not automatically a Lie algebra, which we will denote as $\mathcal{L}_\varphi$. In order for φ to define a deformation of $\mathcal{L}$ that is also a Lie algebra an additional integrability condition has to be imposed.

$$\varphi(\varphi(X^a, X^b), X^c) + \varphi(\varphi(X^c, X^a), X^b) + \varphi(\varphi(X^b, X^c), X^a) = 0 \; . \tag{5.3.8}$$

If both the cohomology and the integrability conditions are fulfilled then the new structure constant of $\mathcal{L}_\varphi$ will automatically satisfy its Jacobi identity. These results can be put into the context of T and S duality induced Lie algebras by making a particular choice for the form of φ, $\varphi(X^a, X^b) := \alpha^{ab}_c \, X^c$ with $\alpha^{ab}_c = -\alpha^{ba}_c$. Then the cohomology conditions of (5.3.6) and the integrability conditions of (5.3.8) take on familiar forms.

$$\begin{aligned} (5.3.6) \quad &\Rightarrow \quad \alpha^{[ab}_e \, \alpha^{c]e}_d &= 0 \\ (5.3.8) \quad &\Rightarrow \quad C^{[ab}_e \, \alpha^{c]e}_d + \alpha^{[ab}_e \, C^{c]e}_d &= 0 \end{aligned} \tag{5.3.9}$$

Comparing these deformed $\mathcal{L}$ with (5.3.4) we are led to identifying $C^{ab}_c = \mathsf{F}^{ab}_c$ and $\alpha^{ab}_c = \widehat{\mathsf{F}}^{ab}_c$ or vice versa. Non-geometric NS-NS flux F_2 defines the six dimensional subalgebra while its R-R partner $\widehat{\mathsf{F}}_2$ flux defines a deformation.

From this point of view, the $\mathsf{F}_2 \cdot \mathsf{F}_2 = 0$ and $\widehat{\mathsf{F}}_2 \cdot \mathsf{F}_2 + \mathsf{F}_2 \cdot \widehat{\mathsf{F}}_2 = 0$ additional constraints are simply the integrability and cohomology conditions. The T-dual limit is trivially recovered when the deformation vanishes, ie. $\widehat{\mathsf{F}}_2 = 0$, and just the original condition $\mathsf{F}_2 \cdot \mathsf{F}_2 = 0$ remains unchanged. A second, less trivial, method of $\mathcal{L}$ and $\mathcal{L}_\varphi$ being isomorphic is the nullity of $H^2(\mathcal{L}, \mathcal{L})$, $\mathcal{L}$ is then known as stable or rigid. At a later point we will examine an explicit example of this, along with other non-stable and non-isomorphic examples, for the case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold. This ability to view the additional fluxes induced by U duality as deformations of a T duality invariant Type IIB theory allows a great deal of the machinery associated to Lie algebra to be used in our analysis, a result we will make use of in the next chapter.

In order to construct these examples of new flux structures we have had to orientifold out half of the fluxes, an approach which required us to make specific choices about which fluxes to remove. In order to examine the S duality induced fluxes as generally as possible we must work in the $\Delta^*(\mathsf{E}^*)$ construction. However, before doing that for the Bianchi constraints we first turn to tadpole constraints as the $\Lambda^*(\mathsf{E}^*)$ and $\Delta^*(\mathsf{E}^*)$ overlap in their results for tadpoles. This was discussed previously for T duality induced tadpole constraints due to the way the extended objects wrapped cycles in $\mathcal{W}$.

5.3.2 Type IIB S Duality Tadpole Constraints

The simplest tadpole is that associated to C_4, as it does not require the consideration of T duality induced fluxes and can be written in a number of ways due to being expressed in terms of 3-forms.

$$\widehat{\mathsf{F}}_0 \wedge \mathsf{F}_0 = \left\{ \begin{array}{rcr} \mathsf{D}(\mathsf{F}_0) & = & \mathsf{D}\mathsf{D}'(\widetilde{\nu}^0) \\ -\mathsf{D}'(\widehat{\mathsf{F}}_0) & = & -\mathsf{D}'\mathsf{D}(\widetilde{\nu}^0) \end{array} \right\} = \frac{1}{2}(\mathsf{D}\mathsf{D}' - \mathsf{D}'\mathsf{D})(\widetilde{\nu}^0) \quad (5.3.10)$$

The flux component version of this tadpole follows from the known actions of D and D'. We have previously stated the $\mathsf{DD}'(\nu_0)$ terms but for completeness we state all DD' terms, as well as the $\mathsf{D}'\mathsf{D}$ expressions in $\Delta^+(\mathsf{E}^*)$.

$$
\begin{aligned}
\mathsf{DD}'(\nu_I) &= \Big(\widehat{\mathsf{F}}_{(I)A}\mathsf{F}^{(J)A} - \widehat{\mathsf{F}}_{(I)}{}^{B}\mathsf{F}^{(J)}{}_{B}\Big)\nu_J + \Big(\widehat{\mathsf{F}}_{(I)}{}^{B}\mathsf{F}_{(J)B} - \widehat{\mathsf{F}}_{(I)A}\mathsf{F}_{(J)}{}^{A}\Big)\widetilde{\nu}^J \\
\mathsf{DD}'(\widetilde{\nu}^J) &= \Big(\widehat{\mathsf{F}}^{(J)}{}_{A}\mathsf{F}^{(I)A} - \widehat{\mathsf{F}}^{(J)B}\mathsf{F}^{(I)}{}_{B}\Big)\nu_I + \Big(\widehat{\mathsf{F}}^{(J)B}\mathsf{F}_{(I)B} - \widehat{\mathsf{F}}^{(J)}{}_{A}\mathsf{F}_{(I)}{}^{A}\Big)\widetilde{\nu}^I \\
\mathsf{D}'\mathsf{D}(\nu_I) &= \Big(\mathsf{F}_{(I)A}\widehat{\mathsf{F}}^{(J)A} - \mathsf{F}_{(I)}{}^{B}\widehat{\mathsf{F}}^{(J)}{}_{B}\Big)\nu_J + \Big(\mathsf{F}_{(I)}{}^{B}\widehat{\mathsf{F}}_{(J)B} - \mathsf{F}_{(I)A}\widehat{\mathsf{F}}_{(J)}{}^{A}\Big)\widetilde{\nu}^J \\
\mathsf{D}'\mathsf{D}(\widetilde{\nu}^J) &= \Big(\mathsf{F}^{(J)}{}_{A}\widehat{\mathsf{F}}^{(I)A} - \mathsf{F}^{(J)B}\widehat{\mathsf{F}}^{(I)}{}_{B}\Big)\nu_I + \Big(\mathsf{F}^{(J)B}\widehat{\mathsf{F}}_{(I)B} - \mathsf{F}^{(J)}{}_{A}\widehat{\mathsf{F}}_{(I)}{}^{A}\Big)\widetilde{\nu}^I
\end{aligned}
\tag{5.3.11}
$$

These expressions are independent of our choice of $\langle\ \rangle_\pm$, as in the T duality only case since they are determined entirely by (4.2.11). Relabelling the derivatives as $\mathsf{D} = \mathsf{D}_1$ and $\mathsf{D}' = \mathsf{D}_2$ we can note a number of algeraic identities linking the coefficients of these quadratic derivatives on elements of $\Delta^+(\mathsf{E}^*)$.

$$
\begin{aligned}
\iota_{\widetilde{\nu}^I}\mathsf{D}_n\mathsf{D}_m(\nu_J) &= -\iota_{\widetilde{\nu}^J}\mathsf{D}_m\mathsf{D}_n(\nu_I) \\
\iota_{\widetilde{\nu}^I}\mathsf{D}_n\mathsf{D}_m(\widetilde{\nu}^J) &= +\iota_{\nu_J}\mathsf{D}_m\mathsf{D}_n(\nu_I) \\
\iota_{\nu_I}\mathsf{D}_n\mathsf{D}_m(\widetilde{\nu}^J) &= -\iota_{\nu_J}\mathsf{D}_m\mathsf{D}_n(\widetilde{\nu}^I)
\end{aligned}
\tag{5.3.12}
$$

All tadpoles in the T duality only case follow from acting DD' on $\widetilde{\nu}^0$, as the only R-R fluxes required for T duality invariance are those associated to $\mathsf{D}'(\widetilde{\nu}^0)$. As a result of the inclusion of other fluxes due to S duality the action of D' on other elements of $\Delta^+(\mathsf{E}^*)$ is no longer zero and additional tadpole expressions can be constructed. These tadpoles no longer take the form of an NS-NS flux acting on the R-R 3-form, the initial R-R contributions are the 3-forms of the NS-NS sector superpotential $\mathsf{D}'(\mho'_c)$. Such fluxes are induced by S duality and so we should not consider tadpole constraints without regard for the $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets they form. As a result we cannot regard each coefficient of (5.3.11) as a separate tadpole constraint, they will form triplets or singlets by the same reasoning as the Bianchi constraints. The C_4 tadpole once again illustrates this in the simplest manner by virtue of the identities stated in (5.3.10). Since the modular inversion $S \to -\frac{1}{S}$ exchanges the flux sectors we would expect there to be a symmetry between D and D' in the multiplets and this is seen in the derivative expressions for the C_4 tadpole. Under an $\mathrm{GL}(2,\mathbb{Z})_S$ transformation Γ on the $(\mathsf{F}_0, \widehat{\mathsf{F}}_0)$ doublet the 6-form combination $\mathsf{F}_0 \wedge \widehat{\mathsf{F}}_0$ is taken to $|\Gamma|\, \mathsf{F}_0 \wedge \widehat{\mathsf{F}}_0$ and thus for $\Gamma \in \mathrm{SL}(2,\mathbb{Z})_S$ this tadpole is a singlet. A tadpole triplet arises from the $\mathrm{SL}(2,\mathbb{Z})_S$ images of a T duality induced tadpole formed from two fluxes which couple to the dilaton in the same manner, as was noted schematically in (5.3.2). To illustrate this we consider $\mathsf{F}_1 \cdot \mathsf{F}_0 \sim \iota_{\widetilde{\nu}^i}\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0)$. Its modular inversion partner is $\widehat{\mathsf{F}}_1 \cdot \widehat{\mathsf{F}}_0 \sim \iota_{\widetilde{\nu}^i}\mathsf{D}'\mathsf{D}(\widetilde{\nu}^0)$ and there is an additional member of the triplet which mixes the sectors further, $\mathsf{F}_1 \cdot \widehat{\mathsf{F}}_0 + \widehat{\mathsf{F}}_1 \cdot \mathsf{F}_0$. Given how F_1 and $\widehat{\mathsf{F}}_0$ both arise in D and likewise for $\widehat{\mathsf{F}}_1$ and F_0 in D' the derivative construction can be easily deduced and we can write the tadpole flux triplet in terms of quadratic derivatives.

$$\left\langle \begin{array}{c} \mathsf{F}_1 \cdot \mathsf{F}_0 \\ \widehat{\mathsf{F}}_1 \cdot \widehat{\mathsf{F}}_0 \\ \widehat{\mathsf{F}}_1 \cdot \mathsf{F}_0 + \mathsf{F}_1 \cdot \widehat{\mathsf{F}}_0 \end{array} \right\rangle = \left\langle \begin{array}{c} \iota_{\widetilde{\nu}^i}\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0) \\ \iota_{\widetilde{\nu}^i}\mathsf{D}'\mathsf{D}(\widetilde{\nu}^0) \\ \iota_{\widetilde{\nu}^i}(\mathsf{D}\mathsf{D} + \mathsf{D}'\mathsf{D}')(\widetilde{\nu}^0) \end{array} \right\rangle \tag{5.3.13}$$

It is clear that there is a fourth expression which can be constructed from quadratic pairings of the derivatives and which will take the same form as C_4 singlet, except the projection operator used is $\iota_{\widetilde{\nu}^i}$ rather than ι_{ν_0}, $\widehat{\mathsf{F}}_1 \cdot \mathsf{F}_0 - \mathsf{F}_1 \cdot \widehat{\mathsf{F}}_0 \sim \iota_{\widetilde{\nu}^i}(\mathsf{D}\mathsf{D} - \mathsf{D}'\mathsf{D}')(\widetilde{\nu}^0)$. This is precisely the $\Delta^*(\mathsf{E}^*)$ defined Bianchi constraint singlet we have previously seen in (5.3.4). This illustrates how the $\Delta^*(\mathsf{E}^*)$ is able to examine both of the T and S duality induced Bianchi and tadpole constraints simultaneously, with the tadpoles being those duality Bianchi constraints not equated to zero due to local sources. However, in order to see this we have had to consider the explicit case of the fluxes F_1, $\widehat{\mathsf{F}}_3$ and their S duality partners so as to construct their $\mathrm{SL}(2,\mathbb{Z})_S$

transformation properties. To generalise this method to all Type IIB fluxes and other constraints[3] we consider $\mathrm{SL}(2,\mathbb{Z})_S$ on the flux matrices and how this would affect the flux matrix constructions of the $\mathsf{D}_n\mathsf{D}_m$ expressions.

5.3.3 Generalised Type IIB U Duality Constraints

If Type IIB is self S dual than an $\mathrm{SL}(2,\mathbb{Z})_S$ transformation must leave the Kähler functional $\mathcal{G} = K + \ln|W|^2$ invariant and as such a transformation on the superpotential must reduce to being an overall gaugeable factor. In order to examine the flux structures induced by both T and S dualities we make use of the flux matrices associated to D and D' as such a formulation is independent of which specific components are non-zero. This has the advantage that orientifolds can be neglected during our analysis and it allows for the method to be generalised to a U duality invariant Type IIA construction. Under U duality all components of the flux matrices associated to $\mathcal{D}$ and $\mathcal{D}'$ can potentially be non-zero and the fact that for the choice of $\langle\ \rangle_{\pm} \to \langle\ \rangle_{+}$ the fluxes of Type IIA don't form $\mathrm{SL}(2,\mathbb{Z})_S$ doublets is irrelevant. In order to examine how the flux matrices of Type IIB transform to provide S duality invariant we define two matrices, one for each flux sector.

$$\underset{\sim}{\mathsf{F}} = \mathsf{A}\cdot\mathsf{M} + \mathsf{B}\cdot\mathsf{M}' \qquad \underset{\sim}{\widehat{\mathsf{F}}} = \mathsf{A}\cdot\mathsf{M}' + \mathsf{B}\cdot\mathsf{M} \tag{5.3.14}$$

The A and B follow the same properties as the $\mathcal{A}$ and $\mathcal{B}$ of (4.1.44) but with notation chosen to match the Type IIB matrices. These combinations are chosen such that the Type IIB superpotential can be written with its dilaton dependence manifest.

$$W = \underline{\mathsf{T}}^{\top}\cdot\mathsf{h}_{\nu}\cdot\Big(\mathsf{C}\cdot\mathsf{M} + \mathsf{C}'\cdot\mathsf{M}'\Big)\cdot\mathsf{g}_{\mathfrak{a}}\cdot\underline{\mathsf{U}} = \underline{\mathsf{T}}^{\top}\cdot\mathsf{h}_{\nu}\cdot\Big(\underset{\sim}{\mathsf{F}} - S\underset{\sim}{\widehat{\mathsf{F}}}\Big)\cdot\mathsf{g}_{\mathfrak{a}}\cdot\underline{\mathsf{U}}$$

[3]The tadpoles motivated an examination of the $\Delta^{+}(\mathsf{E}^{*})$ expressions but Bianchi constraints exist for $\Delta^{3}(\mathsf{E}^{*})$ too and thus tadpole-like expressions can be constructed despite their physical interpretation not being forthcoming.

If the superpotential is to be invariant then $\underset{\sim}{\mathsf{F}} - S\widehat{\underset{\sim}{\mathsf{F}}}$ must transform in the same manner that F_0 and $\widehat{\mathsf{F}}_0$ do in the original Type IIB action of (3.3.2) and so their S duality transformation properties are known. Through the use of the identities involving A and B this is easily inverted to express M and M' in terms of $\underset{\sim}{\mathsf{F}}$ and $\widehat{\underset{\sim}{\mathsf{F}}}$.

$$\begin{pmatrix} \underset{\sim}{\mathsf{F}} \\ \widehat{\underset{\sim}{\mathsf{F}}} \end{pmatrix} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{B} & \mathsf{A} \end{pmatrix} \begin{pmatrix} \mathsf{M} \\ \mathsf{M}' \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \mathsf{M} \\ \mathsf{M}' \end{pmatrix} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{B} & \mathsf{A} \end{pmatrix} \begin{pmatrix} \underset{\sim}{\mathsf{F}} \\ \widehat{\underset{\sim}{\mathsf{F}}} \end{pmatrix} \tag{5.3.15}$$

Using the relationship between $(\underset{\sim}{\mathsf{F}}, \widehat{\underset{\sim}{\mathsf{F}}})$ and $(\mathsf{M}, \mathsf{M}')$ and the known transformation properties of $(\underset{\sim}{\mathsf{F}}, \widehat{\underset{\sim}{\mathsf{F}}})$ we obtain S duality transformation properties of the flux matrices defining the actions of D and D' on $\Delta^+(\mathsf{E}^*)$.

$$\begin{pmatrix} \mathsf{M} \\ \mathsf{M}' \end{pmatrix} \to \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{B} & \mathsf{A} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{B} & \mathsf{A} \end{pmatrix} \begin{pmatrix} \mathsf{M} \\ \mathsf{M}' \end{pmatrix} \tag{5.3.16}$$

$$= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \mathsf{A} + \begin{pmatrix} d & c \\ b & a \end{pmatrix} \otimes \mathsf{B} \right] \begin{pmatrix} \mathsf{M} \\ \mathsf{M}' \end{pmatrix}$$

$$= \begin{pmatrix} a\mathsf{A} + d\mathsf{B} & b\mathsf{A} + c\mathsf{B} \\ c\mathsf{A} + b\mathsf{B} & d\mathsf{A} + a\mathsf{B} \end{pmatrix} \begin{pmatrix} \mathsf{M} \\ \mathsf{M}' \end{pmatrix} \tag{5.3.17}$$

If M transforms as $\mathsf{M} \to m \cdot \mathsf{M}$, where m is a matrix that commutes with both $g_{\mathfrak{a}}$ and g_{ν} then the corresponding transformation on N is $\mathsf{N} \to \mathsf{N} \cdot m^{\top}$. Both A and B satisfy this and are also symmetric, allowing us to express the $\mathrm{SL}(2,\mathbb{Z})_S$ transformations on each flux matrix in similar ways.

$$\begin{pmatrix} \mathsf{N} & \mathsf{N}' \end{pmatrix} \to \begin{pmatrix} \mathsf{N} & \mathsf{N}' \end{pmatrix} \left(\Gamma_S^{\top} \otimes \mathsf{A} + \left(\sigma \cdot \Gamma_S^{\top} \cdot \sigma \right) \otimes \mathsf{B} \right)$$

$$= \begin{pmatrix} \mathsf{N} & \mathsf{N}' \end{pmatrix} \begin{pmatrix} a\mathsf{A} + d\mathsf{B} & c\mathsf{A} + b\mathsf{B} \\ b\mathsf{A} + c\mathsf{B} & d\mathsf{A} + a\mathsf{B} \end{pmatrix} \tag{5.3.18}$$

Given these actions of $\mathrm{SL}(2,\mathbb{Z})_S$ it is noteworthy that due to their linear independence, relationship $\mathsf{A} + \mathsf{B} = \mathbb{I}$ and projection-like multiplicative action we can use them to decompose any matrix into four disjoint submatrices.

$$\begin{aligned} X &= (\mathsf{A}+\mathsf{B})\cdot X\cdot(\mathsf{A}+\mathsf{B}) \\ &= \mathsf{A}\cdot X\cdot\mathsf{A}+\mathsf{A}\cdot X\cdot\mathsf{B}+\mathsf{B}\cdot X\cdot\mathsf{A}+\mathsf{B}\cdot X\cdot\mathsf{B} \end{aligned} \tag{5.3.19}$$

This decomposition is worthy of further examination due to how it relates to $\mathrm{SL}(2,\mathbb{Z})_S$ transformations. The simplest way to examine them is to note that both A and B are square matrices of dimensions $2h^{2,1}+2$ and so in Type IIB can be coupled to the Kähler moduli T. $\mathsf{B}\cdot\mathsf{T}$ contains only the T_0 and T^0 moduli, while $\mathsf{A}\cdot\mathsf{T}$ contains only the T_i and T^j moduli. This can be seen from the way in which the complexification matrix associated to $\mho_c$ can be expressed in terms of $\mathsf{A}-S\mathsf{B}$. Reinterpreting this in terms of fluxes B couples to the F_n, $\widehat{\mathsf{F}}_m$ for $n,m=0,3$ and A couples to the fluxes for $n,m=1,2$.

There are four ways to form expressions which are quadratic in the derivatives D and D' and each of these provide a pair of flux matrix dependent expressions. To begin we consider the NS-NS sector constraints $\mathsf{M}\cdot\mathsf{h}_{\mathfrak{a}}\cdot\mathsf{N}$ and the $\mathrm{SL}(2,\mathbb{Z})_S$ image we arrange in accordance with the A, B inspired decomposition of (5.3.19) and for less cluttered notation use $\diamond\equiv\cdot\mathsf{h}_{\mathfrak{a}}\cdot$ and $\triangleleft\equiv\cdot\mathsf{h}_{\nu}\cdot$.

$$\mathsf{M}\cdot\mathsf{h}_{\mathfrak{a}}\cdot\mathsf{N}\;\rightarrow\;\begin{pmatrix} \mathsf{A}\cdot(a\mathsf{M}+b\mathsf{M}')\diamond(a\mathsf{N}+b\mathsf{N}')\cdot\mathsf{A} \\ +\;\mathsf{A}\cdot(a\mathsf{M}+b\mathsf{M}')\diamond(d\mathsf{N}+c\mathsf{N}')\cdot\mathsf{B} \\ +\;\mathsf{B}\cdot(d\mathsf{M}+c\mathsf{M}')\diamond(a\mathsf{N}+b\mathsf{N}')\cdot\mathsf{A} \\ +\;\mathsf{B}\cdot(d\mathsf{M}+c\mathsf{M}')\diamond(d\mathsf{N}+c\mathsf{N}')\cdot\mathsf{B} \end{pmatrix} \tag{5.3.20}$$

The corresponding $\mathrm{SL}(2,\mathbb{Z})_S$ image of $\mathsf{M}'\diamond\mathsf{N}'$, $\mathsf{M}\diamond\mathsf{N}'$ and $\mathsf{M}'\diamond\mathsf{N}$ follow the same schematic structure. Two transformations of particular note are $\Gamma_S : S\rightarrow S$ and $\Gamma_S : S\rightarrow-\frac{1}{S}$. The former implies that the NS-NS derivative is still nilpotent and the latter implies that the R-R derivative is also nilpotent.

$$\Gamma_1(\mathsf{M}\diamond\mathsf{N}) \;=\; (\mathsf{A}-\mathsf{B})\cdot\mathsf{M}'\diamond\mathsf{N}'\cdot(\mathsf{A}-\mathsf{B})$$

Though the modular inversion has not mapped the NS-NS constraint matrix exactly into the R-R version, due to the change of sign on B, the linear independence of the submatrices make this irrelevant. With these two conditions, (5.3.20) can be reduced down to only terms which mix the two sectors.

$$\mathsf{M}\cdot\mathsf{h}_{\mathfrak{a}}\cdot\mathsf{N} \;\to\; \begin{pmatrix} \mathsf{A}\cdot\Big(\; ab\,\mathsf{M}\diamond\mathsf{N}' \;+\; ab\,\mathsf{M}'\diamond\mathsf{N}\;\Big)\cdot\mathsf{A} \\ +\;\mathsf{A}\cdot\Big(\; ac\,\mathsf{M}\diamond\mathsf{N}' \;+\; bd\,\mathsf{M}'\diamond\mathsf{N}\;\Big)\cdot\mathsf{B} \\ +\;\mathsf{B}\cdot\Big(\; bd\,\mathsf{M}\diamond\mathsf{N}' \;+\; ac\,\mathsf{M}'\diamond\mathsf{N}\;\Big)\cdot\mathsf{A} \\ +\;\mathsf{B}\cdot\Big(\; cd\,\mathsf{M}\diamond\mathsf{N}' \;+\; cd\,\mathsf{M}'\diamond\mathsf{N}\;\Big)\cdot\mathsf{B} \end{pmatrix} \tag{5.3.21}$$

Comparing this expression with the similarly reduced form of $\mathsf{M}'\diamond\mathsf{N}'$ we find that in each case the $\mathrm{SL}(2,\mathbb{Z})_S$ integers factorise out as overall factors in both the $\mathsf{A}\cdot X\cdot\mathsf{A}$ and $\mathsf{B}\cdot X\cdot\mathsf{B}$ terms. Using this pre- and post-multiplication by A or B we can project out particular parts of (5.3.20) to form $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets.

$$\begin{aligned} \mathbf{3}_{\mathsf{AA}} &\equiv \langle\; \mathsf{A}\cdot\mathsf{M}\diamond\mathsf{N}\cdot\mathsf{A} \;\;,\;\; \mathsf{A}\cdot\mathsf{M}'\diamond\mathsf{N}'\cdot\mathsf{A} \;\;,\;\; \mathsf{A}\cdot(\mathsf{M}'\diamond\mathsf{N}+\mathsf{M}\diamond\mathsf{N}')\cdot\mathsf{A}\;\rangle \\ \mathbf{3}_{\mathsf{BB}} &\equiv \langle\; \mathsf{B}\cdot\mathsf{M}\diamond\mathsf{N}\cdot\mathsf{B} \;\;,\;\; \mathsf{B}\cdot\mathsf{M}'\diamond\mathsf{N}'\cdot\mathsf{B} \;\;,\;\; \mathsf{B}\cdot(\mathsf{M}'\diamond\mathsf{N}+\mathsf{M}\diamond\mathsf{N}')\cdot\mathsf{B}\;\rangle \end{aligned} \tag{5.3.22}$$

With A projecting out the $n,m=1,2$ cases of F_n and $\widehat{\mathsf{F}}_m$ we can see that the triplet of (5.3.4) is of this form. The components of (5.3.20) yet to be put into a multiplet are the $\mathsf{B}\cdot X\cdot\mathsf{A}$ and $\mathsf{B}\cdot X\cdot\mathsf{A}$ terms and by considering the $\mathrm{SL}(2,\mathbb{Z})_S$ integers for these parts we can construct another pair of triplets associated to these components.

$$\begin{aligned} \mathbf{3}_{\mathsf{AB}} &\equiv \langle\; \mathsf{A}\cdot\mathsf{M}'\diamond\mathsf{N}\cdot\mathsf{B} \;\;,\;\; \mathsf{A}\cdot\mathsf{M}\diamond\mathsf{N}'\cdot\mathsf{B} \;\;,\;\; \mathsf{A}\cdot(\mathsf{M}\diamond\mathsf{N}+\mathsf{M}'\diamond\mathsf{N}')\cdot\mathsf{B}\;\rangle \\ \mathbf{3}_{\mathsf{BA}} &\equiv \langle\; \mathsf{B}\cdot\mathsf{M}'\diamond\mathsf{N}\cdot\mathsf{A} \;\;,\;\; \mathsf{B}\cdot\mathsf{M}\diamond\mathsf{N}'\cdot\mathsf{A} \;\;,\;\; \mathsf{B}\cdot(\mathsf{M}\diamond\mathsf{N}+\mathsf{M}'\diamond\mathsf{N}')\cdot\mathsf{A}\;\rangle \end{aligned} \tag{5.3.23}$$

The triplet of (5.3.13) corresponds to terms in $\mathbf{3}_{\mathsf{AB}}$. By considering the triplets of (5.3.22) and (5.3.23) we can straightforwardly construct the four singlets associated to the terms of (5.3.19).

$$\begin{aligned}\mathbf{1}_{\mathsf{AA}} &\equiv \langle\ \mathsf{A}\cdot(\mathsf{M}\diamond\mathsf{N}'-\mathsf{M}'\diamond\mathsf{N})\cdot\mathsf{A}\ \rangle\\ \mathbf{1}_{\mathsf{BB}} &\equiv \langle\ \mathsf{B}\cdot(\mathsf{M}\diamond\mathsf{N}'-\mathsf{M}'\diamond\mathsf{N})\cdot\mathsf{B}\ \rangle\\ \mathbf{1}_{\mathsf{AB}} &\equiv \langle\ \mathsf{A}\cdot(\mathsf{M}\diamond\mathsf{N}-\mathsf{M}'\diamond\mathsf{N}')\cdot\mathsf{B}\ \rangle\\ \mathbf{1}_{\mathsf{BA}} &\equiv \langle\ \mathsf{B}\cdot(\mathsf{M}\diamond\mathsf{N}-\mathsf{M}'\diamond\mathsf{N}')\cdot\mathsf{A}\ \rangle\end{aligned} \tag{5.3.24}$$

With B projecting out the $n,m=0,3$ cases the C_4 tadpole singlet (5.3.10) is of the form $\mathbf{1}_{\mathsf{BB}}$. The triplet associated with the C_4 tadpole is trivial since $\mathsf{F}_0\wedge\mathsf{F}_0=0=\widehat{\mathsf{F}}_0\wedge\widehat{\mathsf{F}}_0$ and the remaining term $\widehat{\mathsf{F}}_0\wedge\mathsf{F}_0+\mathsf{F}_0\wedge\widehat{\mathsf{F}}_0=0$, all by antisymmetry. The triplet $\mathbf{3}_{\mathsf{BB}}$ is not trivial though, as it can include F^{abc} and F_{abc} combinations of terms. Given the four possible combinations of the flux matrices and the four terms arising from the A, B induced decomposition the multiplets of (5.3.22-5.3.24) make up all possible $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets. As we have seen for a few explicit cases, not all of these expressions are Bianchi constraints, some of them are tadpoles conditions and thus non-zero in general. Only those expressions which are an $\mathrm{SL}(2,\mathbb{Z})_S$ image of a T duality Bianchi constraint are S duality constraints and not all expressions defining the multiplets of (5.3.22-5.3.24) are of this form. The Bianchi constraints in the T duality only case of the form we have thus far considered are $\mathsf{M}\diamond\mathsf{N}$ and this can be decomposed using (5.3.19).

$$\begin{array}{ccccccccc}\mathsf{M}\diamond\mathsf{N} &=& \mathsf{A}\cdot\mathsf{M}\diamond\mathsf{N}\cdot\mathsf{A} &+& \mathsf{B}\cdot\mathsf{M}\diamond\mathsf{N}\cdot\mathsf{B} &+& \mathsf{A}\cdot\mathsf{M}\diamond\mathsf{N}\cdot\mathsf{B} &+& \mathsf{B}\cdot\mathsf{M}\diamond\mathsf{N}\cdot\mathsf{A}\\ && \in\mathbf{3}_{\mathsf{AA}} && \in\mathbf{3}_{\mathsf{BB}} && \in\mathbf{1}_{\mathsf{AB}} && \in\mathbf{1}_{\mathsf{BA}}\end{array} \tag{5.3.25}$$

In the absence of R-R fluxes the S duality Bianchi constraints must reduce to the T duality constraints and the decomposition (5.3.25) provides us with the $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets which are Bianchi constraints. The remaining multiplets are not restricted to being zero and we can conclude they are tadpole constraints. However, in some cases their physical interpretation is not clear, they are simply the $\mathrm{SL}(2,\mathbb{Z})_S$ multiplet which is the complement of the Bianchi constraints. To use a more explicit example we consider the Bianchi triplet of Type IIB/O3 in (5.3.4). The combinations of two non-geometric fluxes takes a p-form to a $(p-2)$-form and if constructed explicitly it can be seen that the terms arise from derivative actions on η^{pqrs}. The singlet complement's flux polynomials can be written in the same manner, $\mathsf{F}_2 \cdot \widehat{\mathsf{F}}_2 - \widehat{\mathsf{F}}_2 \cdot \mathsf{F}_2$, but must therefore be constructed in the same manner, from $\iota_{\nu_i}(\mathsf{DD}' - \mathsf{D}'\mathsf{D})(\widetilde{\nu}^j)$. Thus far the only tadpoles we have considered are those formed by the derivatives acting on $\widetilde{\nu}^0$. However, we did obtain the algebraic identities of (5.3.12) and this suggested acting the derivatives on other basis elements of $\Delta^+(\mathsf{E}^*)$ to form new expressions. Having constructed the $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets and determined what flux expressions are or are not zero we now have additional justification for expanding our set of tadpole conditions. The identities of (5.3.12) allow us to restrict our considerations of the multiplets because they imply $\mathbf{n}_{\mathsf{AB}} = \mathbf{n}_{\mathsf{BA}}$. This is an issue we will return to when we consider Type IIA tadpole conditions when U duality fluxes are included. These results are summarised in Table 5.4.

We now turn our attention to the action of the derivatives of general form $\mathcal{D}^2 : \Delta^+(\mathsf{E}^*) \to \Delta^+(\mathsf{E}^*)$. Unlike the quadratic action on $\Delta^3(\mathsf{E}^*)$ the A and B are 'internal' to the flux matrix expressions, rather than projecting out linearly independent sections of the constraints. A result of this is that the induced transformations are not as straightforward and as in the $\mathsf{M} \diamond \mathsf{N}$ case we find it convenient to express the flux matrix nilpotency expressions as scalar products.

Bianchi	Tadpole
$\mathbf{3}_{\mathsf{AA}}$	$\mathbf{1}_{\mathsf{AA}}$
$\mathbf{3}_{\mathsf{BB}}$	$\mathbf{1}_{\mathsf{BB}}$
$\mathbf{1}_{\mathsf{AB}} = \mathbf{1}_{\mathsf{BA}}$	$\mathbf{3}_{\mathsf{BA}} = \mathbf{3}_{\mathsf{AB}}$

(5.3.26)

Table 5.4: Type of flux constraints classified into $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets.

$$\begin{aligned} \mathsf{N}\cdot\mathsf{h}_\nu\cdot\mathsf{M} &= \mathsf{N}\triangleleft\mathsf{M} = \begin{pmatrix}\mathsf{N} & \mathsf{N}'\end{pmatrix}\begin{pmatrix}\mathbb{I} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathsf{h}_\nu & 0\\ 0 & \mathsf{h}_\nu\end{pmatrix}\begin{pmatrix}\mathsf{M}\\ \mathsf{M}'\end{pmatrix} \\ \mathsf{N}'\cdot\mathsf{h}_\nu\cdot\mathsf{M}' &= \mathsf{N}'\triangleleft\mathsf{M}' = \begin{pmatrix}\mathsf{N} & \mathsf{N}'\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \mathbb{I}\end{pmatrix}\begin{pmatrix}\mathsf{h}_\nu & 0\\ 0 & \mathsf{h}_\nu\end{pmatrix}\begin{pmatrix}\mathsf{M}\\ \mathsf{M}'\end{pmatrix} \end{aligned} \quad (5.3.27)$$

For a general combination of pairs of flux matrices we can define a related quadratic form, $\mathcal{X}$, and we will consider how this transforms under $\mathrm{SL}(2,\mathbb{Z})_S$.

$$p\,\mathsf{N}\triangleleft\mathsf{M} + q\,\mathsf{N}\triangleleft\mathsf{M}' + r\,\mathsf{N}'\triangleleft\mathsf{M} + s\,\mathsf{N}'\triangleleft\mathsf{M}' \equiv \begin{pmatrix}\mathsf{N} & \mathsf{N}'\end{pmatrix}(\mathcal{X}\otimes\mathsf{h}_\nu)\begin{pmatrix}\mathsf{M}\\ \mathsf{M}'\end{pmatrix} \quad (5.3.28)$$

The overall factor of the h_ν bilinear form reduces $\mathcal{X}$ to being a 2×2 matrix and it is convenient to factorise the $\mathcal{X}$ dependent term so as to use the identity $\mathbb{I} = \mathsf{A}+\mathsf{B}$.

$$\mathcal{X} \equiv \begin{pmatrix} p & q\\ r & s\end{pmatrix} \quad , \quad \mathcal{X}\otimes\mathsf{h}_\nu = \mathcal{X}\otimes\Big((\mathsf{A}+\mathsf{B})\cdot\mathsf{h}_\nu\Big) = \Big(\mathbb{I}_2\otimes\mathsf{h}_\nu\Big)\cdot\Big(\mathcal{X}\otimes(\mathsf{A}+\mathsf{B})\Big)$$

We can construct the transformation properties of the $\mathcal{X}$ dependent factor by using (5.3.16) and (5.3.18) and by their orthogonality, $\mathsf{A}\cdot\mathsf{B}=0$, the A and B terms decouple. Since the h bilinear form commutes with A and B we can construct the $\mathrm{SL}(2,\mathbb{Z})_S$ multiplet structure by considering only the

$\mathcal{X}$ dependent factor.

$$\mathcal{X} \otimes \mathbb{I} \;\rightarrow\; \Big(\Gamma_S^\top \otimes \mathsf{A} + \big(\sigma \cdot \Gamma_S^\top \cdot \sigma\big) \otimes \mathsf{B}\Big) \begin{pmatrix} p & q \\ r & s \end{pmatrix} \Big(\Gamma_S \otimes \mathsf{A} + \big(\sigma \cdot \Gamma_S \cdot \sigma\big) \otimes \mathsf{B}\Big)$$

$$= \underbrace{\left(\Gamma_S^\top \begin{pmatrix} p & q \\ r & s \end{pmatrix} \Gamma_S\right)}_{\Xi_\mathsf{A}} \otimes \mathsf{A} + \underbrace{\left((\sigma \cdot \Gamma_S^\top \cdot \sigma) \begin{pmatrix} p & q \\ r & s \end{pmatrix} (\sigma \cdot \Gamma_S \cdot \sigma)\right)}_{\Xi_\mathsf{B}} \otimes \mathsf{B} \quad (5.3.29)$$

Proceeding as before we wish to obtain an $\mathrm{SL}(2,\mathbb{Z})_S$ triplet by considering the image of the T duality constraints and also combination of terms which form a singlet. Due to the splitting of $\mathcal{X}$ by A and B the equations which are satisfied by a singlet reduce to $\Xi_\mathsf{A} = \mathcal{X} = \Xi_\mathsf{B}$. By using the fact that any element of $\mathrm{SL}(2,\mathbb{R})$ is a symplectic matrix and both Ξ_A and Ξ_B are of the form $m^\top \cdot \mathcal{X} \cdot m$ it follows that if $\mathcal{X}$ is the canonical symplectic form then the equations $\Xi_\mathsf{A} = \mathcal{X} = \Xi_\mathsf{B}$ are automatically satisfied for any $\mathrm{SL}(2,\mathbb{Z})_S$ transformation and we obtain a singlet.

$$\Gamma_S\Big(\mathsf{N} \triangleleft \mathsf{M}' - \mathsf{N}' \triangleleft \mathsf{M}\Big) = \mathsf{N} \triangleleft \mathsf{M}' - \mathsf{N}' \triangleleft \mathsf{M}$$

This can be taken a step further by noting that due to the linear independence of A and B $\mathcal{X}$ can be written as two independent terms, which transform separately.

$$\mathcal{X} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \mathsf{A} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \otimes \mathsf{B} \equiv \mathcal{X}_\mathsf{A} \otimes \mathsf{A} + \mathcal{X}_\mathsf{B} \otimes \mathsf{B} \quad (5.3.30)$$

With the decomposition of the $\mathrm{SL}(2,\mathbb{Z})_S$ image of $\mathcal{X}$ in (5.3.29) we have that Ξ_A depends on the a_i only and Ξ_B depends on the b_j only. Therefore the necessary and sufficient conditions for a singlet become the pair of conditions $\Xi_\mathsf{A} = \mathcal{X}_\mathsf{A}$, $\Xi_\mathsf{B} = \mathcal{X}_\mathsf{B}$ and we can construct two separate non-trivial singlets by setting one of $\mathcal{X}_\mathsf{A}$ or $\mathcal{X}_\mathsf{B}$ to zero and the other to the canonical sympletic form.

$$\mathbf{1}_\mathsf{A} \equiv \langle\, \mathsf{N} \triangleleft \mathsf{A} \cdot \mathsf{M}' - \mathsf{N}' \triangleleft \mathsf{A} \cdot \mathsf{M} \,\rangle \quad , \quad \mathbf{1}_\mathsf{B} \equiv \langle\, \mathsf{N} \triangleleft \mathsf{B} \cdot \mathsf{M}' - \mathsf{N}' \triangleleft \mathsf{B} \cdot \mathsf{M} \,\rangle \quad (5.3.31)$$

For the triplet we begin with the known NS-NS sector T duality constraint $\mathsf{N} \triangleleft \mathsf{M} = 0$ and consider its images under particular elements of $\mathrm{SL}(2,\mathbb{Z})_S$, which in the case of $\Gamma : S \to S$ and $\Gamma : S \to -\frac{1}{S}$ we obtain the T duality constraints of both the NS-NS sector and the R-R sector.

$$\Gamma_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad : \quad \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \to \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \tag{5.3.32}$$

Therefore we have that given S duality $\mathsf{N} \triangleleft \mathsf{M} = 0$ implies $\mathsf{N}' \triangleleft \mathsf{M}' = 0$. The other generator of $\mathrm{SL}(2,\mathbb{Z})_S$, $S \to S+1$, on a general linear combination of these two terms leads to different transformations in the A and B terms.

$$\Gamma_S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad : \quad \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \to \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} + p \begin{pmatrix} 0 & \mathsf{A} \\ \mathsf{A} & \mathsf{A} \end{pmatrix} + q \begin{pmatrix} \mathsf{B} & \mathsf{B} \\ \mathsf{B} & 0 \end{pmatrix} \tag{5.3.33}$$

Although setting the expressions associated with the second and third terms in the above expression is a necessary condition for joint T and S duality invariance, it is not sufficient. This can be seen by considering another $\mathrm{SL}(2,\mathbb{Z})_S$ transformation, that which is associated with the negative integer shift, $S \to S-1$.

$$\Gamma_S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad : \quad \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \to \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} + p \begin{pmatrix} 0 & -\mathsf{A} \\ -\mathsf{A} & \mathsf{A} \end{pmatrix} + q \begin{pmatrix} \mathsf{B} & -\mathsf{B} \\ -\mathsf{B} & 0 \end{pmatrix} \tag{5.3.34}$$

Given (5.3.32) is zero the requirement that both (5.3.33) and (5.3.34) are also zero leads to stronger constraints, $\mathsf{N} \triangleleft \mathsf{M} = 0$ is true by virtue of the $\mathbb{I} = \mathsf{A} + \mathsf{B}$ decomposition terms both vanishing separately. Apriori we could not assume that the A and B related terms form two separate, independent, systems but given the singlet structures we would expect the triplets to follow with the same splittings.

$$\begin{aligned} \mathbf{3}_{\mathsf{A}} &\equiv \langle\ \mathsf{N} \triangleleft \mathsf{A} \cdot \mathsf{M} \ ,\ \mathsf{N}' \triangleleft \mathsf{A} \cdot \mathsf{M}' \ ,\ \mathsf{N} \triangleleft \mathsf{A} \cdot \mathsf{M}' + \mathsf{N}' \cdot \mathsf{A} \cdot \mathsf{M}\ \rangle \\ \mathbf{3}_{\mathsf{B}} &\equiv \langle\ \mathsf{N} \triangleleft \mathsf{B} \cdot \mathsf{M} \ ,\ \mathsf{N}' \triangleleft \mathsf{B} \cdot \mathsf{M}' \ ,\ \mathsf{N} \triangleleft \mathsf{B} \cdot \mathsf{M}' + \mathsf{N}' \cdot \mathsf{B} \cdot \mathsf{M}\ \rangle \end{aligned} \tag{5.3.35}$$

As with the quadratic derivative actions of the form $\mathcal{D}^2 : \Delta^+(\mathsf{E}^*) \to \Delta^+(\mathsf{E}^*)$ it is possible that not all of these expressions automatically give Bianchi constraints. However, unlike the previous case, we cannot express these flux matrix expressions in terms of the natural Type IIB flux multiplets since they are not defined by derivative actions on $\Delta^3(\mathsf{E}^*)$. The $\mathbf{3}$ triplets contain expressions which arise in the T duality only case while the **1** singlets do not. Neither can be Type IIB tadpoles though, as they are expanded in the $\Delta^3(\mathsf{E}^*)$ basis and so can only couple to the Type IIA C_7 form. However, the use of flux matrices to examine the effect of $\mathrm{SL}(2,\mathbb{Z})_S$ transformations on the superpotential is independent of how we might label the flux multiplets of the particular Type II theory they are defined in. Hence $\mathrm{SL}(2,\mathbb{Z})_S$ transformations in Type IIA will result in its flux matrices forming the same set of multiplets. Therefore while the **1**s we have constructed here do not represent tadpole constraints in Type IIB they would in Type IIA and conversely the tadpoles found in the $\Delta^+(\mathsf{E}^*)$ case would not be Type IIA tadpoles. This can be further illustrated and examined by converting our analysis of Bianchi contraints from being in terms of flux matrices to being in terms of exact forms, as we previously considered in the context of Type IIA Bianchi constraints.

5.3.4 Generalised Type IIA U Duality Bianchi Constraints

The entirety of our analysis of the light mode truncated Type IIB structure induced by U duality has been done using the flux matrices associated to D and D'. This was independent of our choice for the inner product $\langle\ \rangle_\pm$ as both flux sectors had their flux multiplets defined by the derivative actions on the $\Delta^+(\mathsf{E}^*)$ forms. This is not the case in Type IIA, the flux sectors define their flux multiplets on different $\Delta^\pm(\mathsf{E}^*)$. If the Type IIB results are to be carried over into Type IIA we would need to modify one of the flux sectors so that both are defined by derivative actions on the same $\Delta^\pm(\mathsf{E}^*)$. For the choice $\langle\ \rangle_\pm \to \langle\ \rangle_-$ this is trivial to do as we have already seen that the derivatives become anti self adjoint on the $\Delta^\pm(\mathsf{E}^*)$ and the flux matrices of each Type II theory are equal. The remaining case to consider is the choice $\langle\ \rangle_\pm \to \langle\ \rangle_+$ whose superpotential construction we must modify slightly so as to be relatable to the Type IIB case. Type IIA is not self S dual because of the different ways in which the flux sectors define their individual fluxes. If Type IIA is to have the same method of analysis then its superpotential must be expressible in the same form as the Type IIB superpotential of (5.3.15).

$$\begin{aligned} W &= \int_{\mathcal{M}} \big\langle \Omega_c, \mathcal{D}(\mho) \big\rangle_+ + \int_{\mathcal{M}} \big\langle \mho, \mathcal{D}'(\Omega'_c) \big\rangle_+ \\ &= \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot M \cdot g_{\mathfrak{a}} \cdot \mathbb{C} \cdot \underline{\mathfrak{U}} + \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbb{C}' \cdot N' \cdot g_\nu \cdot \underline{\mathfrak{T}} \end{aligned} \tag{5.3.36}$$

We have two choices in how to reformulate the superpotential; alter the NS-NS sector to match the R-R sector or alter the R-R sector to match the NS-NS sector. The NS-NS sector's formulation has the advantage that on parallelisable $\mathcal{M}$ its fluxes can be used to define a Lie algebra but the reformulation of the fluxes given in (4.3.11) allows the R-R sector to be put into the same Lie algebra context. If the latter method is chosen then the Type IIA and Type IIB superpotentials then match one another's schematic form up to simple things such as moduli relabelling. This was seen in Section 4.4, where we derived the flux interdependencies in each flux sector. To that end we define a new derivative $\mathbb{D}$ in terms of $\mathcal{D}$.

$$\int_{\mathcal{M}} \big\langle \Omega_c, \mathcal{D}(\mho) \big\rangle_+ \equiv \int_{\mathcal{M}} \big\langle \mho, \mathbb{D}(\Omega_c) \big\rangle_+ \tag{5.3.37}$$

Given this definition $\mathbb{D}$ is some kind of adjoint to $\mathcal{D}$ with respect to $\big\langle\ \big\rangle_+$, though given the (anti)symmetric nature of $(\Delta^-(\mathsf{E}^*))$ $\Delta^+(\mathsf{E}^*)$ it is not an adjoint in the standard definition but we refer to it as such for convenience. Using the flux matrix expressions of (4.1.37) and (4.1.37) we can express the flux matrices of $\mathbb{D}$ in terms of those of $\mathcal{D}$ and make use of the fact the dilaton contribution can be neglected since it does not alter the flux matrices.

$$\begin{aligned} \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot M \cdot g_{\mathfrak{a}} \cdot \underline{\mathfrak{U}} &= \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbb{N} \cdot g_\nu \cdot \underline{\mathfrak{T}} \\ \Rightarrow \quad \mathbb{N} &= N \cdot \mathsf{h}_{\mathfrak{a}} \\ \Rightarrow \quad \mathbb{M} &= -\mathsf{h}_{\mathfrak{a}} \cdot M \end{aligned} \tag{5.3.38}$$

We have used the simplification that the h bilinear form associated to $\Delta^+(\mathsf{E}^*)$ is the identity. It immediately follows from these expressions that $\Delta^*(\mathsf{E}^*)$ Bianchi constraints of $\mathbb{D}$ are equivalent to those of $\mathcal{D}$. The Type IIA superpotential as a whole can now be written in a way which matches (5.3.15).

$$W = \int_{\mathcal{M}} \mho \wedge \Big(\mathbb{D}(\Omega_c) + \mathcal{D}'(\Omega'_c) \Big) = \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \Big(\mathbb{C} \cdot \mathbb{N} + \mathbb{C}' \cdot N' \Big) \cdot g_\nu \cdot \underline{\mathfrak{T}} \quad (5.3.39)$$

Invariance under $\mathrm{SL}(2,\mathbb{Z})_S$ transformations can now be examined in precisely the same manner as the Type IIB case, except with $(\mathsf{M},\mathsf{M}') \to (\mathbb{N}, N')$ and $(\mathsf{N},\mathsf{N}') \to (\mathbb{M}, M')$. The transformation rules of the Type IIB flux matrices can then be applied and then (5.3.38) used to convert back into the correct Type IIA fluxes. As a result it possesses the same multiplet structure but due to the way in which we have had to change the formulation of one of the two flux sectors it is not possible to express them in terms of nature Type IIA fluxes in the same way as was done in Type IIB.

5.3.5 Generalised Type IIA U Duality Tadpole Constraints

The light form defined Bianchi constraints of Type IIA and Type IIB are each defined on both $\Delta^{\pm}(\mathsf{E}^*)$ bases. This is not the case for tadpoles due to the differing brane content. The tadpoles of Type IIB are defined on the $\Delta^{+}(\mathsf{E}^*)$ while those of Type IIA are defined on $\Delta^{-}(\mathsf{E}^*)$ and we can justify the existence of the additional Type IIB tadpoles previously discussed through the examination of their Type IIA mirrors. The D6-branes and possible O6-planes provide support to the C_7 field which couples to 3-forms in the internal space. In the absence of U duality induced fluxes the tadpole conditions are, up to proportionality factors, measured by $\mathcal{D}(F_{RR})$. In our analysis of the Type IIA flux sector we noted that F_{RR} could be written as an exact derivative in terms of $\mathcal{D}'$ and so the tadpole term can be written as a quadratic derivative.

$$F_{RR} \equiv \mathfrak{F}_0 \cdot \alpha_0 = \mathcal{D}'(\alpha_0) = \mathcal{D}'(\mathfrak{b}^0) \quad \Rightarrow \quad \mathcal{D}(F_{RR}) = \mathcal{D}\mathcal{D}'(\mathfrak{b}^0) \qquad (5.3.40)$$

The fact F_{RR} could be written in this way provided a natural extension of the sector to include other $\mathfrak{F}$ fluxes. Given the way in which the Type IIB fluxes couple differently to the Kähler moduli than expected from their parallelisable component structure or the Type IIA manner we would expect its mirror to be a non-standard coupling of the complex structure moduli of Type IIA. As such we would expect the $\mathfrak{U}_i$ and $\mathfrak{U}^j$ of $\mathcal{M}$ to be exchanged in the same way T_i and T^j of $\mathcal{W}$. This is not relevant to the construction of tadpole constraints, as they follow from the definitions of $\mathcal{D}$ and $\mathcal{D}'$, other than to the label we assign to the $\mathfrak{F}_n$.

$$\begin{aligned}\mathcal{D}(\mathfrak{F}_0 \cdot \alpha_0) = \mathcal{D}\mathcal{D}'(\alpha_0) \propto \mathcal{D}\mathcal{D}'(\mathfrak{b}^0) \quad , \quad \mathcal{D}(\widehat{\mathfrak{F}}_1 \cdot \alpha_i) = \mathcal{D}\mathcal{D}'(\alpha_i) \propto \mathcal{D}\mathcal{D}'(\mathfrak{a}_i) \\ \mathcal{D}(\mathfrak{F}_3 \cdot \beta^0) = \mathcal{D}\mathcal{D}'(\beta^0) \propto \mathcal{D}\mathcal{D}'(\mathfrak{a}_0) \quad , \quad \mathcal{D}(\widehat{\mathfrak{F}}_2 \cdot \beta^j) = \mathcal{D}\mathcal{D}'(\beta^j) \propto \mathcal{D}\mathcal{D}'(\mathfrak{b}^j)\end{aligned} \tag{5.3.41}$$

As a result we are able to form additional 3-forms which have the schematic form of being a $\mathcal{D}$ derivative of a $\mathcal{D}'$ exact form. Since the $\Delta^*(\mathsf{E}^*)$ defined components of $\mathcal{D}$ and $\mathcal{D}'$ are defined in different manners we use the mirror symmetry equivalences to reexpress them to be in terms of Type IIB fluxes on $\langle\ \rangle_\pm$.

$$\begin{aligned}\mathcal{D}\mathcal{D}'(\mathfrak{a}_I) &= \mp \left(\widehat{\mathsf{F}}_{(I)A}\mathsf{F}^{(J)A} \pm \widehat{\mathsf{F}}_{(I)}{}^{B}\mathsf{F}^{(J)}{}_{B} \right)\mathfrak{a}_J - \left(\widehat{\mathsf{F}}_{(I)A}\mathsf{F}_{(J)}{}^{A} \pm \widehat{\mathsf{F}}_{(I)}{}^{B}\mathsf{F}_{(J)B} \right)\mathfrak{b}^J \\ \mathcal{D}\mathcal{D}'(\mathfrak{b}^I) &= \left(\widehat{\mathsf{F}}^{(I)}{}_{A}\mathsf{F}^{(J)A} \pm \widehat{\mathsf{F}}^{(I)A}\mathsf{F}^{(J)}{}_{A} \right)\mathfrak{a}_J \pm \left(\widehat{\mathsf{F}}^{(I)}{}_{B}\mathsf{F}_{(J)}{}^{B} \pm \widehat{\mathsf{F}}^{(I)A}\mathsf{F}_{(J)A} \right)\mathfrak{b}^J\end{aligned} \tag{5.3.42}$$

All of these expressions are constructed from applying $\mathcal{D}\mathcal{D}'$ to 3-forms and since they are also expanded in terms of the $\Delta^3(\mathsf{E}^*)$ basis it follows that all of these expressions can contribute tadpoles. The fluxes of F_{RR} have a straightforward definition in terms of the D-brane content of Type IIA but the remaining cases, $\mathcal{D}\mathcal{D}'(\mathfrak{a}_I)$ and $\mathcal{D}\mathcal{D}'(\mathfrak{b}^j)$, do not. The F_{RR} tadpole contributions can be decomposed as $\mathfrak{a}_I \wedge \mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$ and $\mathfrak{b}^J \wedge \mathcal{D}\mathcal{D}'(\mathfrak{b}^0)$ and the physical interpretation is the D6-brane wrapping particular 3-cycles in $\mathcal{W}$. In order to justify considering these expressions as new tadpole contributions we also construct the equivalent expressions for $\mathcal{D}'\mathcal{D}$.

$$\begin{aligned}\mathcal{D}'\mathcal{D}(\mathfrak{a}_I) &= \left(\mathsf{F}_{(I)A}\widehat{\mathsf{F}}^{(J)A} \pm \mathsf{F}_{(I)}{}^{B}\widehat{\mathsf{F}}^{(J)}{}_{B} \right)\mathfrak{a}_J \pm \left(\mathsf{F}_{(I)A}\widehat{\mathsf{F}}_{(J)}{}^{A} \pm \mathsf{F}_{(I)}{}^{B}\widehat{\mathsf{F}}_{(J)B} \right)\mathfrak{b}^J \\ \mathcal{D}'\mathcal{D}(\mathfrak{b}^I) &= -\left(\mathsf{F}^{(I)A}\widehat{\mathsf{F}}^{(J)}{}_{A} \pm \mathsf{F}^{(I)}{}_{A}\widehat{\mathsf{F}}^{(J)A} \right)\mathfrak{a}_J - \left(\mathsf{F}^{(I)}{}_{B}\widehat{\mathsf{F}}_{(J)}{}^{B} \pm \mathsf{F}^{(I)A}\widehat{\mathsf{F}}_{(J)A} \right)\mathfrak{b}^J\end{aligned} \tag{5.3.43}$$

By considering (5.3.42) and (5.3.43) we observe a number of identities relating the individual coefficients which allows us to link this standard tadpole to the new expressions. We use the relabellings $\mathcal{D} = \mathcal{D}_1$ and $\mathcal{D}' = \mathcal{D}_2$ for simplicity.

$$\begin{aligned} \iota_{\mathfrak{b}^I}\mathcal{D}_n\mathcal{D}_m(\mathfrak{a}_J) &= -\iota_{\mathfrak{b}^J}\mathcal{D}_m\mathcal{D}_n(\mathfrak{a}_I) \\ \iota_{\mathfrak{b}^I}\mathcal{D}_n\mathcal{D}_m(\mathfrak{b}^J) &= +\iota_{\mathfrak{a}_J}\mathcal{D}_m\mathcal{D}_n(\mathfrak{a}_I) \\ \iota_{\mathfrak{a}_I}\mathcal{D}_n\mathcal{D}_m(\mathfrak{b}^J) &= -\iota_{\mathfrak{a}_J}\mathcal{D}_m\mathcal{D}_n(\mathfrak{b}^I) \end{aligned} \tag{5.3.44}$$

Using these and the fact that on $\Delta^3(\mathsf{E}^*)$ we have $1 = \mathfrak{a}_I\iota_{\mathfrak{a}_I} + \mathfrak{b}^J\iota_{\mathfrak{b}^J}$ we can reexpress the F_{RR} tadpole in a new way.

$$\begin{aligned} \mathcal{D}(F_{RR}) \;=\; \mathcal{D}\mathcal{D}'(\mathfrak{a}_0) &= \mathfrak{a}_I\iota_{\mathfrak{a}_I}\mathcal{D}\mathcal{D}'(\mathfrak{a}_0) \;+\; \mathfrak{b}^J\iota_{\mathfrak{b}^J}\mathcal{D}\mathcal{D}'(\mathfrak{a}_0) \\ &= \mathfrak{a}_I\iota_{\mathfrak{b}^0}\mathcal{D}'\mathcal{D}(\mathfrak{b}^I) \;-\; \mathfrak{b}^J\iota_{\mathfrak{a}_0}\mathcal{D}'\mathcal{D}(\mathfrak{b}^J) \end{aligned} \tag{5.3.45}$$

Unlike the Type IIB case the tadpoles of Type IIA are of pure form since they are expanded in terms of elements of $\Delta^3(\mathsf{E}^*) \subset \Delta^-(\mathsf{E}^*)$ as we have restricted our analysis to SU(3) $\subset$ SU(3) $\times$ SU(3) structure. With elements of $\Delta^3(\mathsf{E}^*)$ being interchangable via symplectic transformations and the identities of (5.3.44) we can conclude that all coefficients in (5.3.42) and (5.3.43) are tadpole constraints. Their T or mirror duality images in Type IIB are therefore also viable tadpole constraints. However, in both cases the physical interpretations of the R-R fluxes beyond those obtained by compactification of the ten dimensional actions are unclear.

Summary

In this chapter we have formulated a number of ways of expressing constraints on the fluxes of the internal space. The parallelisable case for the NS-NS flux sectors of Type IIA and Type IIB and the R-R sector of Type IIB admitted a set of Jacobi constraints from their Lie algebra interpretation and we made note of the $\mathrm{GL}(6,\mathbb{Z}) \subset \mathrm{O}(6,6)$ invariance the Lie algebras and the resultant superpotential contributions possess. We addressed the issues which arise from the non-standard way in which the Type IIB Kähler moduli couple to the fluxes and found that provided the transformations $\mho \to \check{\mho}$ and $\Omega \to \check{\Omega}$ were symplectic then the Bianchi constraints associated to the fluxes are unchanged. By representing the derivatives in terms of flux matrices we found that the $\langle\ \rangle_+$ inner product does not alter the equivalence between the Type II flux constraints and allows the Type IIA R-R sector to be examined in the same manner as the other Type II flux sectors. With the interpretation of the R-R fluxes as derivatives, in the same manner as the NS-NS case, we reformulated the tadpole conditions in terms of quadratic derivatives and explicitly demonstrated the relationship between the Type II tadpoles known to exist due to their brane construction.

With the inclusion of S duality transformations the Type IIB NS-NS Lie algebra formed by the fluxes on parallelisable $\mathcal{W}$ extended to include R-R fluxes which also altered the Jacobi contraints forming $\mathrm{SL}(2,\mathbb{Z})$ multiplets. The R-R fluxes were then interpreted in terms of deformations to the original T duality only case, with the R-R only constraints defining cohomology conditions and the mixing of flux sectors defining integrability conditions. We then examined S duality from the point of view of requiring the superpotential to be invariant and thus constructed the way in which the flux matrices of the Type IIB derivatives transformed under $\mathrm{SL}(2,\mathbb{Z})_S$. These transformations were then applied to the Bianchi conditions formed by quadratic combinations of derivatives and their associated flux matrices. As in the parallelisable case we classified the results by $\mathrm{SL}(2,\mathbb{Z})$ multiplets and in doing so found that not all multiplets represented Bianchi constraints. We conjectured that those multiplets which were not Bianchi constraints were instead tadpole constraints. It was not immediately clear that this was the case because of the way in which the flux dependent expressions depended on $\Delta^+(\mathsf{E}^*)$ basis elements, which obscured the physical interpretation of some of these expressions. The mirror dual case of Type IIA tadpoles made it clear that such expressions were tadpoles due to the way in which Type IIA tadpoles are defined purely on $\Delta^3(\mathsf{E}^*)$. As a result we could construct explicit flux dependent polynomials which represent tadpole conditions even though their string theoretic origins were not known.

Chapter 6

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orientifold

Thus far we have considered a general class of spaces upon which to compactify Type II string theories, making particular restrictions on the space as and when required. In order to illustrate some of the methods and results obtained thus far we now consider a specific internal space upon which to construct a Type II string compactification, that of the parallelisable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold, $\mathcal{M} \equiv \mathcal{M}_{\mathbb{Z}_2^2}$. This space has received considerable interest in the context of flux compactifications, serving as the canonical example in the context of NS-NS non-geometric fluxes due to T duality [52, 53, 54], R-R non-geometric fluxes due to T and S duality [60], F theory compactifications [61], and SU(3) structure [64]. These works also resulted in it being the compact space whose T and S duality Bianchi constraints have been extensively studied and solved [92, 10] and the existence of phenomenological vacua obtained or no-go theorems stated [93, 94]. The work in this chapter is found in Ref. [10], which follows on from the work of Ref. [92].

6.1 Orientifold Construction

We begin with the basic construction of the $\Delta^p(\mathsf{E}^*)$ bases and the moduli of each Type II theory, allowing us to give an explicit $\Lambda^p(\mathsf{E}^*)$ expression for each basis element of $\Delta^p(\mathsf{E}^*)$. Our notation follows, for the most part, Refs. [92, 10].

6.1.1 Orbifold Group

We first construct the bases for the truncated basis of $\Delta^p(\mathsf{E}^*)$ in terms of the $\Lambda^p(\mathsf{E}^*)$ elements. In the case of orbifolds this can be done explictly once the action of the orbifold group's generators are stated. These generators are most easily stated in terms of the η^a and we define the generator actions purely in terms of parity changes.

$$\begin{aligned} \theta_1 &: (\ \eta^1\ ,\ \eta^2\ ,\ \eta^3\ ,\ \eta^4\ ,\ \eta^5\ ,\ \eta^6\ ) \to (\ \eta^1\ ,\ \eta^2\ ,\ -\eta^3\ ,\ -\eta^4\ ,\ -\eta^5\ ,\ -\eta^6\ ) \\ \theta_2 &: (\ \eta^1\ ,\ \eta^2\ ,\ \eta^3\ ,\ \eta^4\ ,\ \eta^5\ ,\ \eta^6\ ) \to (\ -\eta^1\ ,\ -\eta^2\ ,\ \eta^3\ ,\ \eta^4\ ,\ -\eta^5\ ,\ -\eta^6\ ) \end{aligned} \tag{6.1.1}$$

The orbifold group generated by θ_i includes an additional non-trivial term constructed by the combination of these two generators, $\theta_3 = \theta_1\theta_2$, and which has similar action on the η.

$$\theta_3 : (\ \eta^1\ ,\ \eta^2\ ,\ \eta^3\ ,\ \eta^4\ ,\ \eta^5\ ,\ \eta^6\ ) \to (\ -\eta^1\ ,\ -\eta^2\ ,\ -\eta^3\ ,\ -\eta^4\ ,\ \eta^5\ ,\ \eta^6\ ) \tag{6.1.2}$$

The group is manifestly invariant under a three-fold permutation symmetry in the η.

$$(\eta^1,\eta^2) \to (\eta^3,\eta^4) \to (\eta^5,\eta^6) \to (\eta^1,\eta^2) \tag{6.1.3}$$

As a result this orbifold group leads to the 6 dimensional torus factorising into three two dimensional sub-tori if we define our bases appropriately.

$$\begin{array}{ccccccc} \mathbb{T}^6 & = & \mathbb{T}^2_1 & \times & \mathbb{T}^2_2 & \times & \mathbb{T}^2_3\ . \\ & & (\eta^1\ ,\ \eta^2) & & (\eta^3\ ,\ \eta^4) & & (\eta^5\ ,\ \eta^6) \end{array}$$

The actions of the orbifold group on $\Lambda^3(\mathsf{E}^*)$ are such that only those 3-forms which have an index on each sub-torus survive the orbifolding, of which there are eight.

$$\eta^{135}\ ,\ \eta^{235}\ ,\ \eta^{145}\ ,\ \eta^{136}\ ,\ \eta^{246}\ ,\ \eta^{146}\ ,\ \eta^{236}\ ,\ \eta^{245} \tag{6.1.4}$$

$\Delta^+(\mathsf{E}^*)$	ω_0	ω_1	ω_2	ω_3	$\widetilde{\omega}^1$	$\widetilde{\omega}^2$	$\widetilde{\omega}^3$	$\widetilde{\omega}^0$
$\widetilde{\Lambda}^3(\mathsf{E}^*)$	1	η^{12}	η^{34}	η^{56}	η^{3456}	η^{1256}	η^{1234}	η^{123456}
$\Delta^+(\mathsf{E}^*)$	ν_0	ν_1	ν_2	ν_3	$\widetilde{\nu}^1$	$\widetilde{\nu}^2$	$\widetilde{\nu}^3$	$\widetilde{\nu}^0$
$\widetilde{\Lambda}^3(\mathsf{E}^*)$	η^{123456}	η^{12}	η^{34}	η^{56}	η^{3456}	η^{1256}	η^{1234}	1

Table 6.1: $(\omega_A, \widetilde{\omega}^B)$ and $(\nu_A, \widetilde{\nu}^B)$ bases of the $\Delta^p(\mathsf{E}^*)$ in terms of $\Lambda^p(\mathsf{E}^*)$.

In the case of the elements of $\Lambda^{2n}(\mathsf{E}^*)$ only those with indices on n and only n sub-tori will survive.

$$1 \ , \ \eta^{12} \ , \ \eta^{34} \ , \ \eta^{56} \ , \ \eta^{3456} \ , \ \eta^{5612} \ , \ \eta^{1234} \ , \ \eta^{123456} \tag{6.1.5}$$

In order to associated these with the (α_I, β^J) and $(\omega_A, \widetilde{\omega}^B)$ bases we need to consider the holomorphic forms and their moduli but before that we define the various orientifold projections for the space.

6.1.2 Type II Kähler Structure

For the Kähler moduli the definition is $\mathcal{J} = \mathcal{J}^{(1)} = T_a \omega_a$ and due to its linear properties in the moduli this is simply the sum of the three Kähler forms of each sub-torus.

$$\mathcal{J} = T_1\omega_1 + T_2\omega_2 + T_3\omega_3 = T_1\eta^{12} + T_2\eta^{34} + T_3\eta^{56}$$

This provides us with the definitions of the ω_a and the remaining $\Delta^+(\mathsf{E}^*)$ basis elements follow from expanding out the definition $\mho = e^{\mathcal{J}}$.

$$\begin{aligned} \mho &= \frac{1}{0!}1 + \frac{1}{1!}T_a\omega_a + \frac{1}{2!}T_aT_b\,\omega_a \wedge \omega_b + \frac{1}{3!}T_aT_bT_c\,\omega_a \wedge \omega_b \wedge \omega_c \\ &\equiv \omega_0 + T_a\omega_a + \frac{T_1T_2T_3}{T_b}\widetilde{\omega}^b + T_1T_2T_3\widetilde{\omega}^0 \end{aligned}$$

6.1.3 Type II Complex Structures and the Orientifold Group

The moduli associated to a two dimensional torus are well known; each torus has a Kähler modulus and a complex structure modulus. The definition of the complex structure modulus for such a torus can be defined easily in terms of the η by defining complexified tangent forms and the contribution to the period matrix τ.

$$\mathbb{T}^2 = \langle \eta^1, \eta^2 \rangle \quad \Rightarrow \quad \eta_{\mathbb{C}} = \eta^1 + i\tau\, \eta^2$$

This triplet factorisation of the six dimensional torus induces a particularly simple expression for Ω as each $\mathrm{d}z$ refers to a different sub-torus. Given the fact we will be considering frame bundles, rather than tangent bundles, we make the replacement of $\mathrm{d}z^i \to \eta^i_{\mathbb{C}}$ in Ω such that $\Omega = \eta^1_{\mathbb{C}} \wedge \eta^2_{\mathbb{C}} \wedge \eta^3_{\mathbb{C}}$.

$$\begin{aligned} \Omega &= (\eta^1 + i\tau_1\eta^2) \wedge (\eta^3 + i\tau_2\eta^4) \wedge (\eta^5 + i\tau_3\eta^6) \\ &\equiv \begin{array}{c} \eta^{135} \;+\; i\tau_1\eta^{235} \;+\; i\tau_2\eta^{145} \;+\; i\tau_3\eta^{136} \\ -i\tau_1\tau_2\tau_3\eta^{246} \;-\; \tau_2\tau_3\eta^{146} \;-\; \tau_1\tau_3\eta^{236} \;-\; \tau_1\tau_2\eta^{245} \end{array} \end{aligned} \tag{6.1.6}$$

The complex structure definitions in terms these τ_i is different in each Type II construction and we follow the definitions of Ref. [60]. In order to define the complex structure moduli we need to consider the effects of the orientifold projections in each Type II theory. In Type IIB we have two choices and we shall only explicitly consider the projection which results in O3-planes, $\sigma_B(\Omega) = -\Omega$. Such a projection is expressible in the same manner as the orbifold group's actions, in terms of the η.

$$\sigma_B : (\ \eta^1\ ,\ \eta^2\ ,\ \eta^3\ ,\ \eta^4\ ,\ \eta^5\ ,\ \eta^6\ ) \to (\ -\eta^1\ ,\ -\eta^2\ ,\ -\eta^3\ ,\ -\eta^4\ ,\ -\eta^5\ ,\ -\eta^6\ )$$

In Type IIA the O6-planes follow from the requirement that $\sigma_A(\Omega) = \overline{\Omega}$. Given the definition of Ω in terms of the τ_i it follows that such a requirement determines σ_A entirely by requiring a sign change on the η^{2n}.

$$\sigma_A : (\ \eta^1\ ,\ \eta^2\ ,\ \eta^3\ ,\ \eta^4\ ,\ \eta^5\ ,\ \eta^6\ ) \to (\ +\eta^1\ ,\ -\eta^2\ ,\ +\eta^3\ ,\ -\eta^4\ ,\ +\eta^5\ ,\ -\eta^6\ )$$

All terms in the Ω of Type IIB survive the projection, instead it is the terms coupling to $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(3)}$ which are projected out, and thus the complex structure moduli are obtained simply by $\tau_j = iU_j$ and we obtain the $\Delta^3(\mathsf{E}^*)$ basis given in Table 6.2. The Type IIA case is less straight forward and we make use of the orientifold projection which removes those 3-forms in (6.1.6) which have an odd number of η^{2m} in them. The complex structure moduli are then defined such that the remaining terms in Ω take the canonical form linear in the moduli. Since η^{135} is independent of τ_i it remains independent of the complex structure moduli and the $\tau_i\tau_j\eta^{pqr} \propto \frac{U_1U_2U_3}{U_iU_j}\alpha_k$ where $i,j,k = 1,2,3$ but are not equal to one another.

$$\begin{aligned}\Omega \;\to\; & \eta^{135} \;-\; \tau_2\tau_3\eta^{146} \;-\; \tau_1\tau_3\eta^{236} \;-\; \tau_1\tau_2\eta^{245} \\ = \;\; & \alpha_0 \;+\; U_1\alpha_1 \;+\; U_2\alpha_2 \;+\; U_3\alpha_3\end{aligned}$$

This expansion motivated the definition of the $\Delta^3(\mathsf{E}^*)$ basis given in Table 6.3. Comparing such a basis with that given in Table 6.2 it is clear that they are not the same, there is a sympletic transformation relating $h^{2,1}$ of the $h^{2,1}+1$ symplectic pairs.

$$\text{IIB} \;\ni\; (\alpha_i, \beta^j) \;\to\; (-\beta^i, \alpha_j) \;\in\; \text{IIA}$$

This is the source of the $\mho \to \check{\mho}$ redefinition of the Type IIB Kähler moduli coupling, via mirror symmetry. The Type II theories define their complex structure moduli in different manners but in Type IIA it can be absorbed into a sympletric transformation $\Omega \to \check{\Omega}$. The mirror of this in Type IIB cannot be so easily absorbed as the inherently different structure of the ω_i and $\widetilde{\omega}^j$ is manifest. The Kähler moduli are defined in the same manner in each theory, in contrast to the complex structure moduli. Rather than work with two different symplectic bases we can use a single one and define the Ω of each theory differently. Given its simple definition in terms of the $\eta^i_{\mathbb{C}}$ we choose to use the Type IIB basis of Table

$\Delta^3(\mathsf{E}^*)$	α_0	α_1	α_2	α_3	β^0	β^1	β^2	β^3
$\Lambda^3(\mathsf{E}^*)$	η^{135}	η^{235}	η^{145}	η^{136}	$-\eta^{246}$	η^{146}	η^{236}	η^{245}
$\Delta^3(\mathsf{E}^*)$	$\mathfrak{a}_0$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_3$	$\mathfrak{b}^0$	$\mathfrak{b}^1$	$\mathfrak{b}^2$	$\mathfrak{b}^3$
$\Lambda^3(\mathsf{E}^*)$	$-\eta^{246}$	η^{235}	η^{145}	η^{136}	$-\eta^{135}$	η^{146}	η^{236}	η^{245}

Table 6.2: Type IIB (α_A, β^B) and $(\mathfrak{a}_A, \mathfrak{b}^B)$ bases of the $\Delta^p(\mathsf{E}^*)$ in terms of $\Lambda^p(\mathsf{E}^*)$.

$\Delta^3(\mathsf{E}^*)$	α_0	α_1	α_2	α_3	β^0	β^1	β^2	β^3
$\Lambda^3(\mathsf{E}^*)$	η^{135}	η^{146}	η^{236}	η^{245}	$-\eta^{246}$	$-\eta^{235}$	$-\eta^{145}$	$-\eta^{136}$
$\Delta^3(\mathsf{E}^*)$	$\mathfrak{a}_0$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_3$	$\mathfrak{b}^0$	$\mathfrak{b}^1$	$\mathfrak{b}^2$	$\mathfrak{b}^3$
$\Lambda^3(\mathsf{E}^*)$	η^{246}	η^{146}	η^{236}	η^{245}	$-\eta^{135}$	$-\eta^{235}$	$-\eta^{145}$	$-\eta^{136}$

Table 6.3: Type IIA (α_I, β^J) and $(\mathfrak{a}_I, \mathfrak{b}^J)$ bases of the $\Delta^p(\mathsf{E}^*)$ in terms of $\Lambda^p(\mathsf{E}^*)$.

6.2 and define the two holomorphic forms appropriately.

$$\text{IIB} : \mathcal{U}_0\alpha_0 + \mathcal{U}_a\alpha_a - \mathcal{U}^b\beta^b - \mathcal{U}^0\beta^0 = \mathsf{U}_0\mathfrak{a}_0 + \mathsf{U}_a\mathfrak{a}_a - \mathsf{U}^b\mathfrak{b}^b - \mathsf{U}^0\mathfrak{b}^0$$
$$\text{IIA} : \mathcal{U}_0\alpha_0 - \mathcal{U}_a\beta^a - \mathcal{U}^b\alpha_b - \mathcal{U}^0\beta^0 = \mathsf{U}_0\alpha_0 + \mathsf{U}_a\alpha_a - \mathsf{U}^b\beta^b - \mathsf{U}^0\beta^0$$

With this construction we make explicit the mirror symmetry between the Type IIB $\mho$ and the Type IIA $\check{\Omega}$, both involving a symplectic transformation and a change of basis. Due to the fact the orientifold is self mirror dual $h^{1,1} = h^{2,1}$ and the symplectic transformations are one and the same.

$$\text{IIB} \ni \mho = \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathbf{f}_{(\nu)} \rightarrow \underline{\mathsf{T}}^\top \cdot \mathsf{L}^\top \cdot \mathsf{h}_\nu \cdot \mathbf{e}_{(\nu)} = \check{\mho}$$
$$\text{IIA} \ni \Omega = \underline{\mathcal{U}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \rightarrow \underline{\mathcal{U}}^\top \cdot \mathsf{L}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{f}_{(\mathfrak{a})} = \check{\Omega}$$

6.1.4 Basis Identities

With the natural bases of the $\Delta^p(\mathsf{E}^*)$ defined we can construct the alternative bases $(\mathfrak{a}_I, \mathfrak{b}^J)$ and $(\nu_A, \widetilde{\nu}^B)$ easily. The new moduli follow the relationships stated in (4.1.18) and (4.1.18). This section is done in the Type IIB bases, the Type IIA follow the same results and will not be stated explicitly.

Since we have an $\Lambda^p(\mathsf{E}^*)$ expression for each element of $\Delta^p(\mathsf{E}^*)$ we can explicitly demonstrate the relationships between $\Delta^*(\mathfrak{E})$ operators in (4.1.23). There are four different cases relating to those expressions which act as $\Lambda^p(\mathsf{E}^*) \to \Lambda^{p+n}(\mathsf{E}^*)$ for $n = \pm 3, \pm 1$. The $n = \pm 3$ cases were derived without having to consider integrals, due to the fact the volume form splits into a pair of 3-forms. The $n = \pm 1$ cases were less trivial and so we consider a representative example from each case in terms of the basis elements of Table 6.2. In the case of the terms responsible for the sequence $\Delta^{1,1}(\mathsf{E}^*) \to \Delta^3(\mathsf{E}^*) \to \Delta^{2,2}(\mathsf{E}^*)$ we consider the operators which act as $\omega_1 \to \alpha_1$ and $\omega_1 \to \beta^1$.

$$\begin{array}{rcrclcrcl} \eta^{35}\iota_1 & : & \omega_1 & = & \eta^{12} & \to & \eta^{235} & = & \alpha_1 \\ -\eta^{46}\iota_2 & : & \omega_1 & = & \eta^{12} & \to & \eta^{146} & = & \beta^1 \end{array}$$

In general for terms whose action on $\Delta^+(\mathsf{E}^*)$ is[1] $\omega_i \to \alpha_A$ the corresponding action on $\Delta^3(\mathsf{E}^*)$ will be to map β^I to some element in $\Delta^{2,2}(\mathsf{E}^*)$, as stated in (4.1.24). Hence we apply $-\eta^{46}\iota_1$ to α_1 and $-\eta^{35}\iota_2$ to β^1.

$$\begin{array}{rcrclcrcl} \eta^{35}\iota_1 & : & \beta^1 & = & \eta^{146} & \to & -\eta^{3456} & = & -\widetilde{\omega}^1 \\ -\eta^{46}\iota_2 & : & \alpha_1 & = & \eta^{235} & \to & \eta^{3456} & = & \widetilde{\omega}^1 \end{array}$$

We therefore have two ways of expressing these $\eta^{ab}\iota_c$ operators in terms of the $\Delta^*(\mathsf{E}^*)$ bases and Tables 6.1 and 6.2 allow for conversion into the alternative basis.

$$\begin{array}{rclclclcl} \eta^{35}\iota_1 & = & \alpha_1\iota_{\omega_1} & = & -\widetilde{\omega}^1\iota_{\beta^1} & = & \mathfrak{a}_1\iota_{\nu_1} & = & -\widetilde{\nu}^1\iota_{\mathfrak{b}^1} \\ -\eta^{46}\iota_2 & = & \beta^1\iota_{\omega_1} & = & \widetilde{\omega}^1\iota_{\alpha_1} & = & \mathfrak{b}^1\iota_{\nu_1} & = & \widetilde{\nu}^1\iota_{\mathfrak{a}_1} \end{array}$$

The other cases for $\mathfrak{a}_A$ and $\mathfrak{b}^B$ follow the same pattern, it is straightforward to see that acting the operators onto other elements of $\Delta^{1,1}(\mathsf{E}^*)$ and $\Delta^3(\mathsf{E}^*)$ are zero

[1] Since we are considering the Type IIB construction the $\Delta^*(\mathsf{E}^*)$ indices are labelled in the same way as done previously for general $\mathcal{W}$.

and so we obtain half of the results given in (4.1.23). The second half we obtain by considering those terms responsible for $\Delta^{2,2}(\mathsf{E}^*) \to \Delta^3(\mathsf{E}^*) \to \Delta^{1,1}(\mathsf{E}^*)$ and apply a pair of them to $\widetilde{\omega}^1$.

$$\begin{array}{rcrclcrcl} -\eta^1\iota_5\iota_3 & : & \widetilde{\omega}^1 & = & \eta^{3456} & \to & \eta^{146} & = & \beta^1 \\ \eta^2\iota_6\iota_4 & : & \widetilde{\omega}^1 & = & \eta^{3456} & \to & \eta^{235} & = & \alpha_1 \end{array}$$

As in the previous case if an operator maps $\widetilde{\omega}^1$ to ξ then we consider its action on $*\xi$ also.

$$\begin{array}{rcrclcrcl} -\eta^1\iota_5\iota_3 & : & \alpha_1 & = & \eta^{235} & \to & -\eta^{12} & = & -\omega_1 \\ \eta^2\iota_6\iota_4 & : & \beta^1 & = & \eta^{146} & \to & \eta^{12} & = & \omega_1 \end{array}$$

We therefore have two ways of expressing these $\eta^a\iota_{cb}$ operators in terms of the $\Delta^*(\mathsf{E}^*)$ bases and Table 6.1 allows for conversion into the alternative basis.

$$\begin{array}{rcrcrcrcl} -\eta^1\iota_5\iota_3 & = & \beta^1\iota_{\widetilde{\omega}^1} & = & -\omega_1\iota_{\alpha_1} & = & \mathfrak{a}_1\iota_{\nu_1} & = & \widetilde{\nu}^1\iota_{\mathfrak{b}^1} \\ \eta^2\iota_6\iota_4 & = & \alpha_1\iota_{\widetilde{\omega}^1} & = & \omega_1\iota_{\beta^1} & = & -\mathfrak{b}^1\iota_{\nu_1} & = & \widetilde{\nu}^1\iota_{\mathfrak{a}_1} \end{array}$$

The other cases for $\mathfrak{a}_i$ and $\mathfrak{b}^j$ follow the same pattern and give the second half of the results in (4.1.23).

6.2 Fluxes

We constructed the fluxes of Type IIB on $\mathcal{W}$ by viewing $\mathcal{W}$ as the mirror dual of $\mathcal{M}$. In this section we shall define the Type IIA fluxes on $\mathcal{M} = \mathcal{M}_{\mathbb{Z}_2^2}$ and the Type IIB fluxes on $\mathcal{W} = \mathcal{M}_{\mathbb{Z}_2^2}$ but we have to take into account the difference in the definition of the complex structure moduli and thus different symplectic bases.

6.2.1 Type IIA NS-NS Fluxes

The Type IIA NS-NS fluxes are the simplest to structure due to their index structure and the way they couple naturally to the Kähler moduli, unlike the

Type IIB case. We will not use them in our analysis of flux vacua as the methods to be discussed apply to Type IIB but we state the fluxes for completeness. The four fluxes have different index structures and so we consider them in turn. In the cases of $\mathcal{F}_1$ and $\mathcal{F}_2$ we shall explicitly give the derivation of some of the fluxes as the remainder will follow the same pattern. Since the components $\mathcal{F}_0$ follow by a standard expansion in the $\Delta^3(\mathsf{E}^*)$ basis we turn to $\mathcal{F}_1$ and consider which components couple to $\mathcal{T}_1$. Such dependency only arises from the $\mathcal{T}_1\omega_1$ term in $\mathcal{J} = \mathcal{T}_a\omega_a$ and using Table 6.1 we expand out $\mathcal{F}_1(\eta^{12})$. Unless otherwise stated all indices are space-time indices, not moduli space indices though there is no ambiguity as the index structure context is straightforward.

$$\frac{1}{2!}\mathcal{F}^c_{ab}\eta^{ab}\iota_c(\eta^{12}) = \frac{1}{2!}\mathcal{F}^c_{ab}\left(\delta^1_c\eta^{2ab} - \delta^2_c\eta^{1ab}\right) = \frac{1}{2!}\mathcal{F}^1_{ab}\eta^{2ab} - \frac{1}{2!}\mathcal{F}^2_{ab}\eta^{1ab}$$

Each contraction in the final expression is expanded to four terms and due to the fact ω_a has two indices on the same torus while any $\eta^{ijk} \in \Delta^3(\mathsf{E}^*)$ has an index on each torus the components of $\mathcal{F}_1$ are only non-zero if the indices are on three different tori too.

$$\begin{aligned}
\tfrac{1}{2!}\mathcal{F}^c_{ab}\eta^{ab}\iota_c(\eta^{12}) &= \mathcal{F}^1_{35}\eta^{235} + \mathcal{F}^1_{36}\eta^{236} + \mathcal{F}^1_{45}\eta^{245} + \mathcal{F}^1_{46}\eta^{246} \\
&\quad - \mathcal{F}^2_{35}\eta^{135} - \mathcal{F}^2_{36}\eta^{136} - \mathcal{F}^2_{45}\eta^{145} - \mathcal{F}^2_{46}\eta^{146} \\
\mathcal{F}_{(1)I}\mathfrak{a}_I - \mathcal{F}_{(1)}{}^J\mathfrak{b}^J &= -\mathcal{F}^1_{35}\mathfrak{b}^1 + \mathcal{F}^1_{36}\mathfrak{a}_2 + \mathcal{F}^1_{45}\mathfrak{a}_3 + \mathcal{F}^1_{46}\mathfrak{a}_0 \\
&\quad - \mathcal{F}^2_{35}\mathfrak{b}^0 + \mathcal{F}^2_{36}\mathfrak{b}^3 + \mathcal{F}^2_{45}\mathfrak{b}^2 - \mathcal{F}^2_{46}\mathfrak{a}_1
\end{aligned} \tag{6.2.1}$$

Comparing coefficients we obtain the $\Delta^*(\mathsf{E}^*)$ flux components in terms of the parallelised $\Lambda^p(\mathsf{E}^*)$ components. The cases for $\mathcal{T}_2$ and $\mathcal{T}_3$ follow in the same manner and we obtain the second section of Table 6.4. Repeating this method for the $\mathcal{F}_2$ case we consider the coefficients of $\mathcal{T}^1$ and so act $\mathcal{F}_2$ on $\widetilde{\omega}^1 = \eta^{3456}$.

$$\frac{1}{2!}\mathcal{F}^{bc}_a\eta^a\iota_{cb}(\eta^{3456}) = \frac{1}{2!}\mathcal{F}^{bc}_a\eta^a\iota_c(\delta^3_b\eta^{456} - \delta^4_b\eta^{356} + \delta^5_b\eta^{346} - \delta^6_b\eta^{345})$$

Applying ι_c to each of these four terms yields three terms each but if the remaining $\eta^{ab} \in \Delta^2(\mathsf{E}^*)$ have a and b indices on the same sub-torus then they are projected out by the orbifold group. Of the three terms only two of them survive orbifolding and $\mathcal{F}_2(\widetilde{\omega}^1)$ is expanded into eight terms which pair up due to the antisymmetric properties of $\mathcal{F}^{ab}_c$.

$$\begin{aligned}
\tfrac{1}{2!}\mathcal{F}^{bc}_a\eta^a\iota_{cb}(\eta^{3456}) &= -\mathcal{F}^{35}_a\eta^{a46} + \mathcal{F}^{36}_a\eta^{a45} + \mathcal{F}^{45}_a\eta^{a36} - \mathcal{F}^{46}_a\eta^{a35} \\
&= -\mathcal{F}^{35}_1\eta^{146} + \mathcal{F}^{36}_2\eta^{245} + \mathcal{F}^{45}_2\eta^{236} - \mathcal{F}^{46}_1\eta^{135} \\
&\quad -\mathcal{F}^{35}_2\eta^{246} + \mathcal{F}^{36}_1\eta^{145} + \mathcal{F}^{45}_1\eta^{136} - \mathcal{F}^{46}_2\eta^{235} \qquad (6.2.2) \\
\mathcal{F}^{(1)}{}_I\mathfrak{a}_I - \mathcal{F}^{(1)J}\mathfrak{b}^J &= -\mathcal{F}^{35}_1\mathfrak{a}_1 + \mathcal{F}^{36}_2\mathfrak{a}_3 + \mathcal{F}^{45}_2\mathfrak{a}_2 - \mathcal{F}^{46}_1\mathfrak{b}^0 \\
&\quad -\mathcal{F}^{35}_2\mathfrak{a}_0 - \mathcal{F}^{36}_1\mathfrak{b}^2 - \mathcal{F}^{45}_1\mathfrak{b}^3 + \mathcal{F}^{46}_2\mathfrak{b}^1
\end{aligned}$$

The final case, $\mathcal{F}_3(\widetilde{\nu}^0)$, is similar to the $\mathcal{F}_0$ case due to the factorisation of the volume form into a pair of symplectic 3-forms. Combined the results from all of these expansions of $\Delta^p(\mathsf{E}^*)$ in terms of $\Lambda^p(\mathsf{E}^*)$ we obtain Table 6.4.

6.2.2 Type IIB Fluxes

The Type IIB NS-NS sector is similar in structure to the Type IIA case but, due to the manner in which the fluxes couple to the Kähler moduli in a different way, we cannot use precisely the same method as the Type IIA case. In order to relate the $\Delta^*(\mathsf{E}^*)$ defined components and the $\Lambda^*(\mathsf{E}^*)$ components we need to make an explicit choice of the matrices K and L. In Type IIB/O3 the non-geometric flux F_2 couples to the Kähler moduli linearly by contracting with $\mathcal{J}_c = -\mathcal{T}_i\widetilde{\omega}^i = -\mathcal{T}_i\widetilde{\nu}^i$. We use this to motivate our reformulation of the Kähler holomorphic form and we set $\check{\Omega} = \Omega$ since the Type IIB superpotential is defined naturally in terms of Ω. Since $\mathbf{G}(\check{\mho}_c)$ and $\mathsf{D}(\mho_c)$ couple to the same complex moduli dependent holomorphic form we can neglect Ω in our analysis. As a result we can make use of projection operators which project out coefficients of $\mathfrak{a}_A$ and the coefficients of $\mathfrak{b}^B$.

$$\iota_{\mathfrak{b}^B}\mathbf{G}(\check{\mho}_c) = \iota_{\mathfrak{b}^B}\mathsf{D}(\mho_c) \qquad , \qquad \iota_{\mathfrak{a}_A}\mathbf{G}(\check{\mho}_c) = \iota_{\mathfrak{a}_A}\mathsf{D}(\mho_c)$$

	$\mathcal{F}_{(0)0}$	$\mathcal{F}_{(0)1}$	$\mathcal{F}_{(0)2}$	$\mathcal{F}_{(0)3}$	$\mathcal{F}_{(0)}{}^{0}$	$\mathcal{F}_{(0)}{}^{1}$	$\mathcal{F}_{(0)}{}^{2}$	$\mathcal{F}_{(0)}{}^{3}$	
$\mathcal{F}_{(0)I}$	$-\mathcal{F}^{135}$	$+\mathcal{F}^{235}$	$+\mathcal{F}^{145}$	$+\mathcal{F}^{136}$	$-\mathcal{F}^{246}$	$-\mathcal{F}^{146}$	$-\mathcal{F}^{236}$	$-\mathcal{F}^{245}$	$\mathcal{F}_{(0)}{}^{J}$

	$\mathcal{F}_{(a)0}$	$\mathcal{F}_{(a)1}$	$\mathcal{F}_{(a)2}$	$\mathcal{F}_{(a)3}$	$\mathcal{F}_{(a)}{}^{0}$	$\mathcal{F}_{(a)}{}^{1}$	$\mathcal{F}_{(a)}{}^{2}$	$\mathcal{F}_{(a)}{}^{3}$	
$\mathcal{F}_{(1)I}$	$+\mathcal{F}^{1}_{46}$	$-\mathcal{F}^{2}_{46}$	$+\mathcal{F}^{1}_{36}$	$+\mathcal{F}^{1}_{45}$	$-\mathcal{F}^{2}_{35}$	$+\mathcal{F}^{1}_{35}$	$-\mathcal{F}^{2}_{45}$	$-\mathcal{F}^{2}_{36}$	$\mathcal{F}_{(1)}{}^{J}$
$\mathcal{F}_{(2)I}$	$+\mathcal{F}^{3}_{62}$	$+\mathcal{F}^{3}_{61}$	$-\mathcal{F}^{4}_{62}$	$+\mathcal{F}^{3}_{52}$	$-\mathcal{F}^{4}_{51}$	$-\mathcal{F}^{4}_{52}$	$+\mathcal{F}^{3}_{51}$	$-\mathcal{F}^{4}_{61}$	$\mathcal{F}_{(2)}{}^{J}$
$\mathcal{F}_{(3)I}$	$+\mathcal{F}^{5}_{24}$	$+\mathcal{F}^{5}_{14}$	$+\mathcal{F}^{5}_{23}$	$-\mathcal{F}^{6}_{24}$	$-\mathcal{F}^{6}_{13}$	$-\mathcal{F}^{6}_{23}$	$-\mathcal{F}^{6}_{14}$	$+\mathcal{F}^{5}_{13}$	$\mathcal{F}_{(3)}{}^{J}$

	$\mathcal{F}^{(0)}{}_{0}$	$\mathcal{F}^{(0)}{}_{1}$	$\mathcal{F}^{(0)}{}_{2}$	$\mathcal{F}^{(0)}{}_{3}$	$\mathcal{F}^{(0)0}$	$\mathcal{F}^{(0)1}$	$\mathcal{F}^{(0)2}$	$\mathcal{F}^{(0)3}$	
$\mathcal{F}^{(0)}{}_{I}$	$+\mathcal{F}_{246}$	$+\mathcal{F}_{146}$	$+\mathcal{F}_{236}$	$+\mathcal{F}_{245}$	$-\mathcal{F}_{135}$	$+\mathcal{F}_{235}$	$+\mathcal{F}_{145}$	$+\mathcal{F}_{136}$	$\mathcal{F}^{(0)J}$

	$\mathcal{F}^{(a)}{}_{0}$	$\mathcal{F}^{(a)}{}_{1}$	$\mathcal{F}^{(a)}{}_{2}$	$\mathcal{F}^{(a)}{}_{3}$	$\mathcal{F}^{(a)0}$	$\mathcal{F}^{(a)1}$	$\mathcal{F}^{(a)2}$	$\mathcal{F}^{(a)3}$	
$\mathcal{F}^{(1)}{}_{I}$	$-\mathcal{F}^{35}_{2}$	$-\mathcal{F}^{35}_{1}$	$+\mathcal{F}^{45}_{2}$	$+\mathcal{F}^{36}_{2}$	$+\mathcal{F}^{46}_{1}$	$-\mathcal{F}^{46}_{2}$	$+\mathcal{F}^{36}_{1}$	$+\mathcal{F}^{45}_{1}$	$\mathcal{F}^{(1)J}$
$\mathcal{F}^{(2)}{}_{I}$	$-\mathcal{F}^{51}_{4}$	$+\mathcal{F}^{52}_{4}$	$-\mathcal{F}^{51}_{3}$	$+\mathcal{F}^{61}_{4}$	$+\mathcal{F}^{62}_{3}$	$+\mathcal{F}^{61}_{3}$	$-\mathcal{F}^{62}_{4}$	$+\mathcal{F}^{52}_{3}$	$\mathcal{F}^{(2)J}$
$\mathcal{F}^{(3)}{}_{I}$	$-\mathcal{F}^{13}_{6}$	$+\mathcal{F}^{23}_{6}$	$+\mathcal{F}^{14}_{6}$	$-\mathcal{F}^{13}_{5}$	$+\mathcal{F}^{24}_{5}$	$+\mathcal{F}^{14}_{5}$	$+\mathcal{F}^{23}_{5}$	$-\mathcal{F}^{24}_{6}$	$\mathcal{F}^{(3)J}$

Table 6.4: Component labels for Type IIA NS-NS fluxes.

The Kähler decomposition is achieved by considering only one $\mathcal{J}^{(n)} \in \mho_c$ and the simplest case is the 3-form flux and so we turn to that first, reducing the above expression to the terms involving $\mathcal{J}^{(0)}$ only.

$$\iota_{\mathfrak{b}^B}\mathbf{G}(\omega_0) = \iota_{\mathfrak{b}^B}\mathsf{D}(\widetilde{\nu}^0) \qquad , \qquad \iota_{\mathfrak{a}_A}\mathbf{G}(\omega_0) = \iota_{\mathfrak{a}_A}\mathsf{D}(\widetilde{\nu}^0)$$

To obtain the explicit expressions for each side of these expressions we expand out $\widehat{\mathsf{F}}_0$ in terms of elements in $\Delta^3(\mathsf{E}^*)$ but written as elements of $\Lambda^3(\mathsf{E}^*)$.

$$\begin{aligned} \tfrac{1}{3!}\widehat{\mathsf{F}}_{pqr}\eta^{pqr} \;=\; & \widehat{\mathsf{F}}_{135}\eta^{135} + \widehat{\mathsf{F}}_{235}\eta^{235} + \widehat{\mathsf{F}}_{145}\eta^{145} + \widehat{\mathsf{F}}_{136}\eta^{136} \\ & \widehat{\mathsf{F}}_{246}\eta^{246} + \widehat{\mathsf{F}}_{146}\eta^{146} + \widehat{\mathsf{F}}_{236}\eta^{236} + \widehat{\mathsf{F}}_{245}\eta^{245} \end{aligned} \tag{6.2.3}$$

Relabelling the η^{pqr} into the $(\mathfrak{a}_A, \mathfrak{b}^B)$ basis we can use the operators $\mathfrak{a}_A\iota_{\mathfrak{a}_A}$ and $\mathfrak{b}^B\iota_{\mathfrak{b}^B}$ to project out half of the terms in the expansion.

$$\begin{aligned} \mathfrak{a}_A\iota_{\mathfrak{a}_A}\widehat{\mathsf{F}}_0(\omega_0) &= +\widehat{\mathsf{F}}_{246}\mathfrak{a}_0 + \widehat{\mathsf{F}}_{235}\mathfrak{a}_1 + \widehat{\mathsf{F}}_{145}\mathfrak{a}_2 + \widehat{\mathsf{F}}_{136}\mathfrak{a}_3 \\ \mathfrak{b}^B\iota_{\mathfrak{b}^B}\widehat{\mathsf{F}}_0(\omega_0) &= +\widehat{\mathsf{F}}_{135}\mathfrak{b}^0 + \widehat{\mathsf{F}}_{146}\mathfrak{b}^1 + \widehat{\mathsf{F}}_{236}\mathfrak{b}^2 + \widehat{\mathsf{F}}_{245}\mathfrak{b}^3 \end{aligned}$$

Doing likewise for the expansion of $\mathsf{D}(\widetilde{\nu}^0)$ in the $(\mathfrak{a}_A, \mathfrak{b}^B)$ basis we can then compare coefficients.

$$\begin{aligned} \mathfrak{a}_A\iota_{\mathfrak{a}_A}\mathsf{D}(\widetilde{\nu}^0) &= +\mathsf{F}^{(0)}{}_0\mathfrak{a}_0 + \mathsf{F}^{(0)}{}_1\mathfrak{a}_1 + \mathsf{F}^{(0)}{}_2\mathfrak{a}_2 + \mathsf{F}^{(0)}{}_3\mathfrak{a}_3 \\ \mathfrak{b}^B\iota_{\mathfrak{b}^B}\mathsf{D}(\widetilde{\nu}^0) &= -\mathsf{F}^{(0)0}\mathfrak{b}^0 - \mathsf{F}^{(0)1}\mathfrak{b}^1 - \mathsf{F}^{(0)2}\mathfrak{b}^2 - \mathsf{F}^{(0)3}\mathfrak{b}^3 \end{aligned}$$

We repeat this method for the $\mathcal{T}_i$ for the case of $i = 1$ but due to the fact $\mathcal{J}_c = -\mathcal{T}_i\widetilde{\omega}^i$ we have a slightly different set of equations for comparing coefficients since $-\mathcal{T}_i\widetilde{\omega}^i \in \check{\mho}$ and $\mathcal{T}_i\omega_i = \mathcal{T}_i\nu_i \in \mho$, which contributes an overall factor of -1.

$$\iota_{\mathfrak{b}^B}\mathbf{G}(\widetilde{\omega}^i) = -\iota_{\mathfrak{b}^B}\mathsf{D}(\nu_i) \qquad , \qquad \iota_{\mathfrak{a}_A}\mathbf{G}(\widetilde{\omega}^i) = -\iota_{\mathfrak{a}_A}\mathsf{D}(\nu_i)$$

To obtain the $\Lambda^p(\mathsf{E}^*)$ components of F_2 we can use the expansion of $\mathcal{F}_2(\widetilde{\omega}^1)$ in (6.2.2) once we make the appropriate relabellings.

$$\begin{aligned} \tfrac{1}{2!}\mathsf{F}^{bc}_a\eta^a\iota_{cb}(\eta^{3456}) &= -\mathsf{F}^{35}_a\eta^{a46} + \mathsf{F}^{36}_a\eta^{a45} + \mathsf{F}^{45}_a\eta^{a36} - \mathsf{F}^{46}_a\eta^{a35} \\ &= -\mathsf{F}^{35}_1\eta^{146} + \mathsf{F}^{36}_2\eta^{245} + \mathsf{F}^{45}_2\eta^{236} - \mathsf{F}^{46}_1\eta^{135} \\ &\quad -\mathsf{F}^{35}_2\eta^{246} + \mathsf{F}^{36}_1\eta^{145} + \mathsf{F}^{45}_1\eta^{136} - \mathsf{F}^{46}_2\eta^{235} \end{aligned} \tag{6.2.4}$$

Relabelling the η^{pqr} into the $(\mathfrak{a}_A, \mathfrak{b}^B)$ basis we can use the operators $\mathfrak{a}_A \iota_{\mathfrak{a}_A}$ and $\mathfrak{b}^B \iota_{\mathfrak{b}^B}$ to project out half of the terms in the expansion.

$$\begin{aligned} \mathfrak{a}_A \iota_{\mathfrak{a}_A} \mathsf{F}_2(\widetilde{\omega}^1) &= -\mathsf{F}_2^{35}\mathfrak{a}_0 \;-\; \mathsf{F}_2^{46}\mathfrak{a}_1 \;+\; \mathsf{F}_1^{36}\mathfrak{a}_2 \;+\; \mathsf{F}_1^{45}\mathfrak{a}_3 \\ \mathfrak{b}^B \iota_{\mathfrak{b}^B} \mathsf{F}_2(\widetilde{\omega}^1) &= -\mathsf{F}_1^{46}\mathfrak{b}^0 \;-\; \mathsf{F}_1^{35}\mathfrak{b}^1 \;+\; \mathsf{F}_2^{45}\mathfrak{b}^2 \;+\; \mathsf{F}_2^{36}\mathfrak{b}^3 \end{aligned}$$

Doing likewise for the expansion of $\mathsf{D}(\widetilde{\nu}^1)$ in the $(\mathfrak{a}_A, \mathfrak{b}^B)$ basis and including the factor of -1 we can then compare coefficients.

$$\begin{aligned} -\mathfrak{a}_A \iota_{\mathfrak{a}_A} \mathsf{D}(\widetilde{\nu}^1) &= -\mathsf{F}^{(1)}{}_0\mathfrak{a}_0 \;-\; \mathsf{F}^{(1)}{}_2\mathfrak{a}_2 \;-\; \mathsf{F}^{(1)}{}_3\mathfrak{a}_3 \;-\; \mathsf{F}^{(1)}{}_1\mathfrak{a}_1 \\ -\mathfrak{b}^B \iota_{\mathfrak{b}^B} \mathsf{D}(\widetilde{\nu}^1) &= +\mathsf{F}^{(1)1}\mathfrak{b}^1 \;+\; \mathsf{F}^{(1)3}\mathfrak{b}^3 \;+\; \mathsf{F}^{(1)2}\mathfrak{b}^2 \;+\; \mathsf{F}^{(1)0}\mathfrak{b}^0 \end{aligned}$$

All of these flux components are given in Table 6.5, along with the corresponding F_1 and $\widehat{\mathsf{F}}_3$ components. The R-R sector of the Type IIB theory takes precisely the same form and are obtained by exchanging $\mathsf{F} \leftrightarrow \widehat{\mathsf{F}}$ in all expressions.

6.2.3 Type IIA R-R Fluxes

The Type IIA R-R sector is constructable as the mirror dual of the Type IIB R-R sector and the transformation properties of $F_{RR} \leftrightarrow F_3$ are known in terms of their $\Lambda^p(\mathsf{E}^*)$ components.

$$\begin{aligned} F_{RR} = \;& F_0\eta^0 + F_{12}\eta^{12} + F_{34}\eta^{34} + F_{56}\eta^{56} \\ & + F_{3456}\eta^{3456} + F_{5612}\eta^{5612} + F_{1234}\eta^{1234} + F_{123456}\eta^{123456} \\ F_3 = \;& F_{135}\eta^{135} + F_{235}\eta^{235} + F_{145}\eta^{145} + F_{136}\eta^{136} \\ & + F_{146}\eta^{146} + F_{236}\eta^{236} + F_{245}\eta^{245} + F_{246}\eta^{246} \end{aligned}$$

The T dualities, and the equivalent mirror symmetry, which relate these two expressions are obtained by comparing the two expansions of the holomorphic forms, the Ω of Type IIB and the $\mho$ of Type IIA, specifically focusing on the $\mathcal{U}_0$ term of Ω which is mapped to the $\widetilde{\mathcal{T}}_0$ term of $\mho$. Since we have selected the same bases for $\Delta^3(\mathsf{E}^*)$ in Type IIA as in Type IIB and likewise for $\Delta^+(\mathsf{E}^*)$ we have implicitly determined the T duality transformations to take F_0 to F_{135},

	$\mathsf{F}_{(0)0}$	$\mathsf{F}_{(0)1}$	$\mathsf{F}_{(0)2}$	$\mathsf{F}_{(0)3}$	$\mathsf{F}_{(0)}{}^{0}$	$\mathsf{F}_{(0)}{}^{1}$	$\mathsf{F}_{(0)}{}^{2}$	$\mathsf{F}_{(0)}{}^{3}$	
$\mathsf{F}_{(0)A}$	$-\widehat{\mathsf{F}}^{135}$	$-\widehat{\mathsf{F}}^{146}$	$-\widehat{\mathsf{F}}^{236}$	$-\widehat{\mathsf{F}}^{245}$	$-\widehat{\mathsf{F}}^{246}$	$-\widehat{\mathsf{F}}^{235}$	$-\widehat{\mathsf{F}}^{145}$	$-\widehat{\mathsf{F}}^{136}$	$\mathsf{F}_{(0)}{}^{B}$

	$\mathsf{F}_{(i)0}$	$\mathsf{F}_{(i)1}$	$\mathsf{F}_{(i)2}$	$\mathsf{F}_{(i)3}$	$\mathsf{F}_{(i)}{}^{0}$	$\mathsf{F}_{(i)}{}^{1}$	$\mathsf{F}_{(i)}{}^{2}$	$\mathsf{F}_{(i)}{}^{3}$	
$\mathsf{F}_{(1)A}$	$+\mathsf{F}^{35}_{2}$	$+\mathsf{F}^{46}_{2}$	$-\mathsf{F}^{36}_{1}$	$-\mathsf{F}^{45}_{1}$	$-\mathsf{F}^{46}_{1}$	$-\mathsf{F}^{35}_{1}$	$+\mathsf{F}^{45}_{2}$	$+\mathsf{F}^{36}_{2}$	$\mathsf{F}_{(1)}{}^{B}$
$\mathsf{F}_{(2)A}$	$+\mathsf{F}^{51}_{4}$	$-\mathsf{F}^{61}_{3}$	$+\mathsf{F}^{62}_{4}$	$-\mathsf{F}^{52}_{3}$	$-\mathsf{F}^{62}_{3}$	$+\mathsf{F}^{52}_{4}$	$-\mathsf{F}^{51}_{3}$	$+\mathsf{F}^{61}_{4}$	$\mathsf{F}^{(2)B}$
$\mathsf{F}_{(3)A}$	$+\mathsf{F}^{13}_{6}$	$-\mathsf{F}^{14}_{5}$	$-\mathsf{F}^{23}_{5}$	$+\mathsf{F}^{24}_{6}$	$-\mathsf{F}^{24}_{5}$	$+\mathsf{F}^{23}_{6}$	$+\mathsf{F}^{14}_{6}$	$-\mathsf{F}^{13}_{5}$	$\mathsf{F}_{(3)}{}^{B}$

	$\mathsf{F}^{(0)}{}_{0}$	$\mathsf{F}^{(0)}{}_{1}$	$\mathsf{F}^{(0)}{}_{2}$	$\mathsf{F}^{(0)}{}_{3}$	$\mathsf{F}^{(0)0}$	$\mathsf{F}^{(0)1}$	$\mathsf{F}^{(0)2}$	$\mathsf{F}^{(0)3}$	
$\mathsf{F}^{(0)}{}_{A}$	$+\widehat{\mathsf{F}}_{246}$	$+\widehat{\mathsf{F}}_{235}$	$+\widehat{\mathsf{F}}_{145}$	$+\widehat{\mathsf{F}}_{136}$	$-\widehat{\mathsf{F}}_{135}$	$-\widehat{\mathsf{F}}_{146}$	$-\widehat{\mathsf{F}}_{236}$	$-\widehat{\mathsf{F}}_{245}$	$\mathsf{F}^{(0)B}$

	$\mathsf{F}^{(i)}{}_{0}$	$\mathsf{F}^{(i)}{}_{1}$	$\mathsf{F}^{(i)}{}_{2}$	$\mathsf{F}^{(i)}{}_{3}$	$\mathsf{F}^{(i)0}$	$\mathsf{F}^{(i)1}$	$\mathsf{F}^{(i)2}$	$\mathsf{F}^{(i)3}$	
$\mathsf{F}^{(1)}{}_{A}$	$+\mathsf{F}^{1}_{46}$	$+\mathsf{F}^{1}_{35}$	$-\mathsf{F}^{2}_{45}$	$-\mathsf{F}^{2}_{36}$	$+\mathsf{F}^{2}_{35}$	$+\mathsf{F}^{2}_{46}$	$-\mathsf{F}^{1}_{36}$	$-\mathsf{F}^{1}_{45}$	$\mathsf{F}^{(1)B}$
$\mathsf{F}^{(2)}{}_{A}$	$+\mathsf{F}^{3}_{62}$	$-\mathsf{F}^{4}_{52}$	$+\mathsf{F}^{3}_{51}$	$-\mathsf{F}^{4}_{61}$	$+\mathsf{F}^{4}_{51}$	$-\mathsf{F}^{3}_{61}$	$+\mathsf{F}^{4}_{62}$	$-\mathsf{F}^{3}_{52}$	$\mathsf{F}_{(2)}{}^{B}$
$\mathsf{F}^{(3)}{}_{A}$	$+\mathsf{F}^{5}_{24}$	$-\mathsf{F}^{6}_{23}$	$-\mathsf{F}^{6}_{14}$	$+\mathsf{F}^{5}_{13}$	$+\mathsf{F}^{6}_{13}$	$-\mathsf{F}^{5}_{14}$	$-\mathsf{F}^{5}_{23}$	$+\mathsf{F}^{6}_{24}$	$\mathsf{F}^{(3)B}$

Table 6.5: Component labels for Type IIB NS-NS fluxes.

which is achieved by T dualising in the η^5, η^3 and η^1 directions, in that order, $\mathbf{T}_1 \circ \mathbf{T}_3 \circ \mathbf{T}_5 \circ \equiv \mathbf{T}_{135} \circ$. We consider this explicitly on the Type IIA F_{RR}.

$$\begin{aligned}
\mathbf{T}_5 \circ F_{RR} &= F_5\eta^5 + F_{125}\eta^{125} + F_{345}\eta^{345} + F_6\eta^6 \\
&\quad + F_{346}\eta^{346} + F_{612}\eta^{612} + F_{12345}\eta^{12345} + F_{12346}\eta^{12346} \\
\mathbf{T}_{35} \circ F_{RR} &= F_{35}\eta^{35} + F_{1235}\eta^{1235} + F_{45}\eta^{45} + F_{36}\eta^{36} \\
&\quad + F_{46}\eta^{46} + F_{3612}\eta^{3612} + F_{1245}\eta^{1245} + F_{1246}\eta^{1246} \\
\mathbf{T}_{135} \circ F_{RR} &= F_{135}\eta^{135} + F_{235}\eta^{235} + F_{145}\eta^{145} + F_{136}\eta^{136} \\
&\quad + F_{146}\eta^{146} + F_{236}\eta^{236} + F_{245}\eta^{245} + F_{246}\eta^{246}
\end{aligned}$$

Thus we obtain the result that $\mathbf{T}_{135} : F_{RR} \to F_3$, as we implicitly assumed in our choice of bases. Recalling that $F_{RR} \equiv \mathfrak{F}_0(\alpha_0) = \mathfrak{F}_0(\mathfrak{b}^0)$, with $\mathfrak{b}^0$ defined in Table 6.3, we can construct the $\Delta^*(\mathsf{E}^*)$ defined components, though we have the sign ambiguity due to a choice in $\langle\ \rangle_\pm$.

$$\begin{aligned}
F_{RR} &= + F_0\eta^0 + F_{12}\eta^{12} + F_{34}\eta^{34} + F_{56}\eta^{56} \\
&\quad + F_{3456}\eta^{3456} + F_{5612}\eta^{5612} + F_{1234}\eta^{1234} + F_{123456}\eta^{123456} \\
\mathfrak{F}_0 &= \pm \mathfrak{F}^{(0)0}\widetilde{\nu}^0\iota_{\mathfrak{b}^0} + \mathfrak{F}^{(0)}{}_1\nu_1\iota_{\mathfrak{b}^0} + \mathfrak{F}^{(0)}{}_2\nu_2\iota_{\mathfrak{b}^0} + \mathfrak{F}^{(0)}{}_3\nu_3\iota_{\mathfrak{b}^0} \\
&\quad \pm \mathfrak{F}^{(0)1}\widetilde{\nu}^1\iota_{\mathfrak{b}^0} \pm \mathfrak{F}^{(0)2}\widetilde{\nu}^2\iota_{\mathfrak{b}^0} \pm \mathfrak{F}^{(0)3}\widetilde{\nu}^3\iota_{\mathfrak{b}^0} + \mathfrak{F}^{(0)}{}_0\nu_0\iota_{\mathfrak{b}^0}
\end{aligned}$$

The $\mathfrak{F}$ are defined by the action of $\mathcal{D}'$ on the $\Delta^3(\mathsf{E}^*)$ which prevents the $\mathfrak{F}_n$ defining the same Lie algebra structure constants as the fluxes of the other derivatives. Since the action of a derivative on $\Delta^+(\mathsf{E}^*)$ defines its action on $\Delta^3(\mathsf{E}^*)$, and vice versa, we can convert the components of F_{RR} into the $\widehat{\mathcal{F}}$ fluxes of (4.3.10).

$$\begin{aligned}
\mathfrak{F}_0 &= F_0\iota_{531} - F_{12}\eta^2\iota_{53} - F_{34}\eta^4\iota_{15} - F_{56}\eta^6\iota_{31} \\
&\quad - F_{3456}\eta^{46}\iota_1 - F_{5612}\eta^{62}\iota_3 - F_{1234}\eta^{24}\iota_5 + F_{123456}\eta^{246} \\
&\equiv \widehat{\mathcal{F}}^{135}\iota_{531} + \widehat{\mathcal{F}}^{35}_2\eta^2\iota_{53} + \widehat{\mathcal{F}}^{51}_4\eta^4\iota_{15} + \widehat{\mathcal{F}}^{13}_6\eta^6\iota_{31} \\
&\quad + \widehat{\mathcal{F}}^1_{46}\eta^{46}\iota_1 + \widehat{\mathcal{F}}^3_{62}\eta^{62}\iota_3 + \widehat{\mathcal{F}}^5_{24}\eta^{24}\iota_5 + \widehat{\mathcal{F}}_{246}\eta^{246} \\
&\equiv \mp\widehat{\mathcal{F}}_{(0)0}\mathfrak{a}_0\iota_{\nu_0} + \widehat{\mathcal{F}}^{(1)}{}_0\mathfrak{a}_0\iota_{\widetilde{\nu}^1} + \widehat{\mathcal{F}}^{(2)}{}_0\mathfrak{a}_0\iota_{\widetilde{\nu}^2} + \widehat{\mathcal{F}}^{(3)}{}_0\mathfrak{a}_0\iota_{\widetilde{\nu}^3} \\
&\quad \mp\widehat{\mathcal{F}}_{(1)0}\mathfrak{a}_0\iota_{\nu_1} \mp \widehat{\mathcal{F}}_{(2)0}\mathfrak{a}_0\iota_{\nu_2} \mp \widehat{\mathcal{F}}_{(3)0}\mathfrak{a}_0\iota_{\nu_3} + \widehat{\mathcal{F}}^{(0)}{}_0\mathfrak{a}_0\iota_{\widetilde{\nu}^0}
\end{aligned}$$

Comparing coefficients in these different expansions we obtain the relationships stated in Table 6.6. The remaining Type IIA R-R fluxes induced by U duality we shall not explciitly construct as we shall restrict our attention to the Type IIB formulation of the $\mathcal{M}_{\mathbb{Z}_2^2}$ superpotential.

	${\mathfrak{F}^{(0)}}_0$	${\mathfrak{F}^{(0)}}_1$	${\mathfrak{F}^{(0)}}_2$	${\mathfrak{F}^{(0)}}_3$	$\mathfrak{F}^{(0)0}$	$\mathfrak{F}^{(0)1}$	$\mathfrak{F}^{(0)2}$	$\mathfrak{F}^{(0)3}$	
${\mathfrak{F}^{(0)}}_A$	$+\widehat{\mathcal{F}}_{246}$	$+\widehat{\mathcal{F}}_2^{35}$	$+\widehat{\mathcal{F}}_4^{51}$	$+\widehat{\mathcal{F}}_6^{13}$	$\pm\widehat{\mathcal{F}}^{135}$	$\pm\widehat{\mathcal{F}}_{46}^1$	$\pm\widehat{\mathcal{F}}_{62}^3$	$\pm\widehat{\mathcal{F}}_{24}^5$	$\mathfrak{F}^{(0)B}$
	${+\widehat{\mathcal{F}}^{(0)}}_0$	${+\widehat{\mathcal{F}}^{(1)}}_0$	${+\widehat{\mathcal{F}}^{(2)}}_0$	${+\widehat{\mathcal{F}}^{(3)}}_0$	$-\widehat{\mathcal{F}}_{(0)0}$	$-\widehat{\mathcal{F}}_{(1)0}$	$-\widehat{\mathcal{F}}_{(2)0}$	$-\widehat{\mathcal{F}}_{(3)0}$	

Table 6.6: Component labels for the fluxes of F_{RR}

6.3 Type IIB/O3 Construction

6.3.1 Fluxes

We restrict our considerations to the IIB orientifold with O3/O7-planes, reducing the algebra to one involving only half the fluxes. However, in order to satisfy S duality invariance we also require the R-R partner to this algebra obtained by the modular inversion $S \to -\frac{1}{S}$.

$$\begin{aligned} [\; Z_m \;,\; Z_n \;] &= H_{mnp} \; X^p & \qquad [\; \mathbf{Z}_m \;,\; \mathbf{Z}_n \;] &= F_{mnp} \; \mathbf{X}^p \\ [\; Z_m \;,\; X^n \;] &= Q_m^{np} \quad Z_p \quad , & [\; \mathbf{Z}_m \;,\; \mathbf{X}^n \;] &= P_m^{np} \quad \mathbf{Z}_p \\ [\; X^m \;,\; X^n \;] &= Q_p^{mn} \quad X^p & [\; \mathbf{X}^m \;,\; \mathbf{X}^n \;] &= P_p^{mn} \quad \mathbf{X}^p \end{aligned} \tag{6.3.1}$$

The non-geometric fluxes contribute to the superpotential by coupling to the Kähler 4-form $\mathcal{J}_c = -\widetilde{\mathcal{T}}_i \widetilde{\omega}^i = -\mathsf{T}_i \widetilde{\nu}^i$ and thus the fluxes define a superpotential linear in the dilaton and Kähler moduli but cubic in the complex structure moduli.

$$W = \int_{\mathcal{M}_{\mathbb{Z}_2^2}} \big\langle\, \Omega, F_3 - S\, H_3 + (Q - S\, P) \cdot \mathcal{J}_c \,\big\rangle_{\pm}$$

The generic contributions to the superpotential of a 3-form flux or a non-geometric flux can be written in terms of the $\Lambda^p(\mathsf{E}^*)$ components, which in the case of the

H_3	${\mathsf F^{(0)}}_0$	${\mathsf F^{(0)}}_1$	${\mathsf F^{(0)}}_2$	${\mathsf F^{(0)}}_3$	$\mathsf F^{(0)0}$	$\mathsf F^{(0)1}$	$\mathsf F^{(0)2}$	$\mathsf F^{(0)3}$	
${\mathsf F^{(0)}}_A$	$\mathsf b_0$	$\mathsf b_2^{(1)}$	$\mathsf b_2^{(2)}$	$\mathsf b_2^{(3)}$	$-\mathsf b_3$	$-\mathsf b_1^{(1)}$	$-\mathsf b_1^{(2)}$	$-\mathsf b_1^{(3)}$	$\mathsf F^{(0)B}$
F_3	${\widehat{\mathsf F}^{(0)}}_0$	${\widehat{\mathsf F}^{(0)}}_1$	${\widehat{\mathsf F}^{(0)}}_2$	${\widehat{\mathsf F}^{(0)}}_3$	$\widehat{\mathsf F}^{(0)0}$	$\widehat{\mathsf F}^{(0)1}$	$\widehat{\mathsf F}^{(0)2}$	$\widehat{\mathsf F}^{(0)3}$	
${\widehat{\mathsf F}^{(0)}}_A$	$\mathsf a_0$	$\mathsf a_2^{(1)}$	$\mathsf a_2^{(2)}$	$\mathsf a_2^{(3)}$	$-\mathsf a_3$	$-\mathsf a_1^{(1)}$	$-\mathsf a_1^{(2)}$	$-\mathsf a_1^{(3)}$	$\widehat{\mathsf F}^{(0)B}$

Table 6.7: $\Lambda^p(\mathsf E^*)$ components of F_3 and H_3 in Type IIB

non-geometric fluxes is non-trivial. Following the contraction definitions given in (4.1.8) we can expand $Q \cdot \mathcal{J}_c$.

$$Q \cdot \mathcal{J}_c = \frac{1}{3!}(Q \cdot \mathcal{J}_c)_{abc}\eta^{abc} \qquad (Q \cdot \mathcal{J}_c)_{abc} = \frac{1}{2!}Q^{de}_{[a}(\mathcal{J}_c)_{bc]de}$$

Both the non-geometric fluxes and the 3-form fluxes can be expanded in terms of the symplectic basis but rather than use the $\Delta^p(\mathsf E^*)$ defined components it is more convenient to define a new notation for the fluxes, those given in Tables 6.7 and 6.8.

$$\begin{aligned} H_3 &= \mathsf F_{(0)0}\,\mathfrak a_0 + \mathsf F_{(0)a}\,\mathfrak a_a - {\mathsf F_{(0)}}^b\,\mathfrak b^b - {\mathsf F_{(0)}}^0\,\mathfrak b^0 \\ &= \mathsf b_0\,\mathfrak a_0 + \mathsf b_2^{(a)}\,\mathfrak a_a + \mathsf b_1^{(b)}\,\mathfrak b^b + \mathsf b_3\,\mathfrak b^0 \end{aligned} \tag{6.3.2}$$

We defined the components of $\mathsf D$ such that the component expansion of $\mathsf D(\mathcal{J}^{(2)})$ takes a standard form and we can use Tables 6.5 and 6.8, which combine to give Table 6.9, to expand it in terms of the $\mathsf c_i^{(j)}$.

$$\begin{aligned} Q \cdot \mathcal{J}_c = \mathsf D(\mathcal{J}^{(2)}) &= T_i\Big({\mathsf F^{(i)}}_0\,\mathfrak a_0 + {\mathsf F^{(i)}}_a\,\mathfrak a_a - \mathsf F^{(i)b}\,\mathfrak b^b - \mathsf F^{(i)0}\,\mathfrak b^0 \Big) \\ &= T_i\Big(\mathsf c_0^{(i)}\,\mathfrak a_0 + \underline{\underline{\mathsf c}}_2^{(ia)}\,\mathfrak a_a + \underline{\underline{\mathsf c}}_1^{(ib)}\,\mathfrak b^b + \mathsf c_3^{(i)}\,\mathfrak b^0 \Big) \end{aligned} \tag{6.3.3}$$

The $\underline{\underline{\mathsf c}}_1$ and $\underline{\underline{\mathsf c}}_2$ are non-geometric flux matrices which make up part of the Q section of Table 6.9. In line with [92] we will use Greek indices α, β, γ for horizontal "$-$" x-like directions (η^1, η^3, η^5) and Latin indices i, j, k for vertical "|" y-like directions

(η^2, η^4, η^6) in the 2-tori.

$$\underline{\underline{\mathsf{c}}}_1 = \begin{pmatrix} \widetilde{\mathsf{c}}_1^{(1)} & -\widehat{\mathsf{c}}_1^{(3)} & -\check{\mathsf{c}}_1^{(2)} \\ -\check{\mathsf{c}}_1^{(3)} & \widetilde{\mathsf{c}}_1^{(2)} & -\widehat{\mathsf{c}}_1^{(1)} \\ -\widehat{\mathsf{c}}_1^{(2)} & -\check{\mathsf{c}}_1^{(1)} & \widetilde{\mathsf{c}}_1^{(3)} \end{pmatrix} \quad , \quad \underline{\underline{\mathsf{c}}}_2 = \begin{pmatrix} \widetilde{\mathsf{c}}_2^{(1)} & -\widehat{\mathsf{c}}_2^{(3)} & -\check{\mathsf{c}}_2^{(2)} \\ -\check{\mathsf{c}}_2^{(3)} & \widetilde{\mathsf{c}}_2^{(2)} & -\widehat{\mathsf{c}}_2^{(1)} \\ -\widehat{\mathsf{c}}_2^{(2)} & -\check{\mathsf{c}}_2^{(1)} & \widetilde{\mathsf{c}}_2^{(3)} \end{pmatrix} \tag{6.3.4}$$

The R-R partner of Q, P, is expanded in the same manner.

$$\begin{aligned} P \cdot \mathcal{J}_c \;=\; \mathsf{D}'(\mathcal{J}^{(2)}) \;&=\; T_i\Big(\;\widehat{\mathsf{F}}^{(i)}{}_0\, \mathfrak{a}_0 \;+\; \widehat{\mathsf{F}}^{(i)}{}_a\, \mathfrak{a}_a \;-\; \widehat{\mathsf{F}}^{(i)b}\, \mathfrak{b}^b \;-\; \widehat{\mathsf{F}}^{(i)0}\, \mathfrak{b}^0 \;\Big) \\ &=\; T_i\Big(\;\mathsf{d}_0^{(i)}\, \mathfrak{a}_0 \;+\; \underline{\underline{\mathsf{d}}}_2^{(ia)}\, \mathfrak{a}_a \;+\; \underline{\underline{\mathsf{d}}}_1^{(ib)}\, \mathfrak{b}^b \;+\; \mathsf{d}_3^{(i)}\, \mathfrak{b}^0 \;\Big) \end{aligned} \tag{6.3.5}$$

As with Q the $\underline{\underline{\mathsf{d}}}_i$ are defined in terms of the fluxes of D'.

$$\underline{\underline{\mathsf{d}}}_1 = \begin{pmatrix} \widetilde{\mathsf{d}}_1^{(1)} & -\widehat{\mathsf{d}}_1^{(3)} & -\check{\mathsf{d}}_1^{(2)} \\ -\check{\mathsf{d}}_1^{(3)} & \widetilde{\mathsf{d}}_1^{(2)} & -\widehat{\mathsf{d}}_1^{(1)} \\ -\widehat{\mathsf{d}}_1^{(2)} & -\check{\mathsf{d}}_1^{(1)} & \widetilde{\mathsf{d}}_1^{(3)} \end{pmatrix} \quad , \quad \underline{\underline{\mathsf{d}}}_2 = \begin{pmatrix} \widetilde{\mathsf{d}}_2^{(1)} & -\widehat{\mathsf{d}}_2^{(3)} & -\check{\mathsf{d}}_2^{(2)} \\ -\check{\mathsf{d}}_2^{(3)} & \widetilde{\mathsf{d}}_2^{(2)} & -\widehat{\mathsf{d}}_2^{(1)} \\ -\widehat{\mathsf{d}}_2^{(2)} & -\check{\mathsf{d}}_2^{(1)} & \widetilde{\mathsf{d}}_2^{(3)} \end{pmatrix} \tag{6.3.6}$$

Type	Components	Fluxes
$Q_{-}^{--} \equiv Q_{\alpha}^{\beta\gamma}$	$Q_1^{35}\,,\, Q_3^{51}\,,\, Q_5^{13}$	$\widetilde{\mathsf{c}}_1^{(1)}\,,\, \widetilde{\mathsf{c}}_1^{(2)}\,,\, \widetilde{\mathsf{c}}_1^{(3)}$
$Q_{\mid}^{\mid -} \equiv Q_{k}^{i\beta}$	$Q_4^{61}\,,\, Q_6^{23}\,,\, Q_2^{45}$	$\widehat{\mathsf{c}}_1^{(1)}\,,\, \widehat{\mathsf{c}}_1^{(2)}\,,\, \widehat{\mathsf{c}}_1^{(3)}$
$Q_{\mid}^{- \mid} \equiv Q_{k}^{\alpha j}$	$Q_6^{14}\,,\, Q_2^{36}\,,\, Q_4^{52}$	$\check{\mathsf{c}}_1^{(1)}\,,\, \check{\mathsf{c}}_1^{(2)}\,,\, \check{\mathsf{c}}_1^{(3)}$
$Q_{\mid}^{--} \equiv Q_{k}^{\alpha\beta}$	$Q_2^{35}\,,\, Q_4^{51}\,,\, Q_6^{13}$	$\mathsf{c}_0^{(1)}\,,\, \mathsf{c}_0^{(2)}\,,\, \mathsf{c}_0^{(3)}$
$Q_{-}^{\mid\mid} \equiv Q_{\gamma}^{ij}$	$Q_1^{46}\,,\, Q_3^{62}\,,\, Q_5^{24}$	$\mathsf{c}_3^{(1)}\,,\, \mathsf{c}_3^{(2)}\,,\, \mathsf{c}_3^{(3)}$
$Q_{-}^{\mid -} \equiv Q_{\gamma}^{i\beta}$	$Q_5^{23}\,,\, Q_1^{45}\,,\, Q_3^{61}$	$\check{\mathsf{c}}_2^{(1)}\,,\, \check{\mathsf{c}}_2^{(2)}\,,\, \check{\mathsf{c}}_2^{(3)}$
$Q_{-}^{- \mid} \equiv Q_{\beta}^{\gamma i}$	$Q_3^{52}\,,\, Q_5^{14}\,,\, Q_1^{36}$	$\widehat{\mathsf{c}}_2^{(1)}\,,\, \widehat{\mathsf{c}}_2^{(2)}\,,\, \widehat{\mathsf{c}}_2^{(3)}$
$Q_{\mid}^{\mid\mid} \equiv Q_{k}^{ij}$	$Q_2^{46}\,,\, Q_4^{62}\,,\, Q_6^{24}$	$\widetilde{\mathsf{c}}_2^{(1)}\,,\, \widetilde{\mathsf{c}}_2^{(2)}\,,\, \widetilde{\mathsf{c}}_2^{(3)}$

Table 6.8: $\Lambda^p(\mathsf{E}^*)$ components of Q. Components of P obtained by $c \to d$.

	$\mathsf{F}_{(i)0}$	$\mathsf{F}_{(i)1}$	$\mathsf{F}_{(i)2}$	$\mathsf{F}_{(i)3}$	$\mathsf{F}_{(i)}{}^{0}$	$\mathsf{F}_{(i)}{}^{1}$	$\mathsf{F}_{(i)}{}^{2}$	$\mathsf{F}_{(i)}{}^{3}$	
$\mathsf{F}_{(1)A}$	$+\mathsf{c}_0^{(1)}$	$+\widetilde{\mathsf{c}}_2^{(1)}$	$-\widehat{\mathsf{c}}_2^{(3)}$	$-\check{\mathsf{c}}_2^{(2)}$	$-\mathsf{c}_3^{(1)}$	$-\widetilde{\mathsf{c}}_1^{(1)}$	$+\widehat{\mathsf{c}}_1^{(3)}$	$+\check{\mathsf{c}}_1^{(2)}$	$\mathsf{F}_{(1)}{}^{B}$
$\mathsf{F}_{(2)A}$	$+\mathsf{c}_0^{(2)}$	$-\check{\mathsf{c}}_2^{(3)}$	$+\widetilde{\mathsf{c}}_2^{(2)}$	$-\widehat{\mathsf{c}}_2^{(1)}$	$-\mathsf{c}_3^{(2)}$	$+\check{\mathsf{c}}_1^{(3)}$	$-\widetilde{\mathsf{c}}_1^{(2)}$	$+\widehat{\mathsf{c}}_1^{(1)}$	$\mathsf{F}_{(2)}{}^{B}$
$\mathsf{F}_{(3)A}$	$+\mathsf{c}_0^{(3)}$	$-\widehat{\mathsf{c}}_2^{(2)}$	$-\check{\mathsf{c}}_2^{(1)}$	$+\widetilde{\mathsf{c}}_2^{(3)}$	$-\mathsf{c}_3^{(3)}$	$+\widehat{\mathsf{c}}_1^{(2)}$	$+\check{\mathsf{c}}_1^{(1)}$	$-\widetilde{\mathsf{c}}_1^{(3)}$	$\mathsf{F}_{(3)}{}^{B}$

Table 6.9: Component labels for Type IIB Q.

(α_A, β^B)	$\mathcal{U}_0$	$\mathcal{U}_a$	$\mathcal{U}^b$	$\mathcal{U}^0$
	1	U_a	$-\frac{U_1U_2U_3}{U_b}$	$U_1U_2U_3$
$(\mathfrak{a}_A, \mathfrak{b}^B)$	$-\mathsf{U}^0$	U_a	U^b	U_0

Table 6.10: Different Type IIB $\Delta^3(\mathsf{E}^*)$ complex structure moduli representations

6.3.2 Superpotential

The complex structure holomorphic form can be written in terms of the U_a explicitly and thus we have the U in terms of the U.

$$\begin{aligned} \Omega &= \alpha_0 + U_a\alpha_a - \left(-\frac{U_1U_2U_3}{U_b}\right)\beta^b - U_1U_2U_3\beta^0 \\ &= \mathfrak{b}^0 + U_a\mathfrak{a}_a - \left(-\frac{U_1U_2U_3}{U_b}\right)\mathfrak{b}^b + U_1U_2U_3\mathfrak{a}_0 \\ &= \mathsf{U}_0\mathfrak{a}_0 + \mathsf{U}_a\mathfrak{a}_a - \mathsf{U}^b\mathfrak{b}^b - \mathsf{U}^0\mathfrak{b}^0 \end{aligned} \tag{6.3.7}$$

Integrating the superpotential integrand over $\mathcal{M}_{\mathbb{Z}_2^2}$ we obtain the polynomial form, dependent on the U_a, T_i and S.

$$W = P_1(U_a) - S\,P_2(U_a) + T_iP_3^{(i)}(U_a) - S\,T_jP_4^{(j)}(U_a) \tag{6.3.8}$$

We recall our generic component expansion of a 3-form flux and a non-geometric flux contribution to the superpotential.

$$\begin{aligned} P_0 &= T_0\left(\mathsf{U}^0\widehat{\mathsf{F}}^{(0)}{}_0 + \mathsf{U}^b\widehat{\mathsf{F}}^{(0)}{}_b - \mathsf{U}_0\widehat{\mathsf{F}}^{(0)0} - \mathsf{U}_a\widehat{\mathsf{F}}^{(0)a}\right) \\ P_3 &= T_i\left(\mathsf{U}^0\widehat{\mathsf{F}}_{(i)0} + \mathsf{U}^b\widehat{\mathsf{F}}_{(i)b} - \mathsf{U}_0\widehat{\mathsf{F}}_{(i)}{}^0 - \mathsf{U}_a\widehat{\mathsf{F}}_{(i)}{}^a\right) \end{aligned}$$

Using the expansions of (6.3.2), (6.3.3), (6.3.5) and (6.3.7) we can give an explicit expansion for each of the flux induced polynomials.

$$\begin{aligned} P_1(U_a) &= -\ \mathsf{a}_0 \ +\ \mathsf{a}_1^{(a)} U_a \ -\ \mathsf{a}_2^{(b)} \tfrac{U_1 U_2 U_3}{U_b} \ +\ \mathsf{a}_3 U_1 U_2 U_3 \\ P_2(U_a) &= -\ \mathsf{b}_0 \ +\ \mathsf{b}_1^{(a)} U_a \ -\ \mathsf{b}_2^{(b)} \tfrac{U_1 U_2 U_3}{U_b} \ +\ \mathsf{b}_3 U_1 U_2 U_3 \end{aligned}$$

The non-geometric cases follow the same schematic format but with the additional Kähler moduli index.

$$\begin{aligned} P_3^{(i)}(U_a) &= -\ \mathsf{c}_0^{(i)} \ +\ \underline{\underline{\mathsf{c}}}_1^{(ia)} U_a \ -\ \underline{\underline{\mathsf{c}}}_2^{(ib)} \tfrac{U_1 U_2 U_3}{U_b} \ +\ \mathsf{c}_3^{(i)} U_1 U_2 U_3 \\ P_4^{(i)}(U_a) &= -\ \mathsf{d}_0^{(i)} \ +\ \underline{\underline{\mathsf{d}}}_1^{(ia)} U_a \ -\ \underline{\underline{\mathsf{d}}}_2^{(ib)} \tfrac{U_1 U_2 U_3}{U_b} \ +\ \mathsf{d}_3^{(i)} U_1 U_2 U_3 \end{aligned}$$

6.3.3 The Isotropic Ansatz

The number of independent moduli and fluxes is expressible in terms of the Hodge numbers; there are $h^{1,1} + h^{2,1} + 1$ closed string complex moduli and $8(h^{1,1}+1)(h^{2,1}+1)$ independent fluxes. The orientifold projection reduces this by projecting out half the fluxes and having $h^{1,1} \to h^{1,1}_{+}$, which in the case of $\mathcal{M}_{\mathbb{Z}_2^2}$ is $h^{1,1}_{+} = h^{1,1}$ so this does not reduce the fluxes further. In total the Type IIB/O3 orientifold has 7 complex moduli and 64 independent fluxes. The orientifold can be simplified because of its factorisation into a triplet of two dimensional sub-tori. Each possesses a Kähler moduli and a complex structure moduli, independently, but by setting them all to be equal the orientifold reduces in complexity. This is equivalent to requiring a permutation symmetry in the sub-tori, as noted in the action of the orbifold group generators in (6.1.3). The resultant isotropic orientifold possesses a moduli of each type, thus three complex moduli obtained from the anisotropic case by the simplification $T_i \to T$, $U_a \to U$. However, the number of independent fluxes is not obtained from the formula $8(h^{1,1}+1)(h^{2,1}+1)$ by setting $h^{1,1} \to 1$ and $h^{2,1} \to 1$. Despite it being a triplet copy of a two dimensional torus it is possible for the isotropic orientifold to have more structure than can be found in a two dimensional torus. The 3-form fluxes have eight independent components in the anisotropic case and setting $U_a \to U$ reduces this to four. This

H_3	$\mathsf{F}^{(0)}{}_0$	$\mathsf{F}^{(0)}{}_a$	$\mathsf{F}^{(0)0}$	$\mathsf{F}^{(0)b}$	$\widehat{\mathsf{F}}^{(0)}{}_0$	$\widehat{\mathsf{F}}^{(0)}{}_a$	$\widehat{\mathsf{F}}^{(0)0}$	$\widehat{\mathsf{F}}^{(0)b}$	F_3
	b_0	b_2	$-\mathsf{b}_3$	$-\mathsf{b}_1$	a_0	a_2	$-\mathsf{a}_3$	$-\mathsf{a}_1$	

Table 6.11: $\Delta^p(\mathsf{E}^*)$ components of isotropic F_3 and H_3 in Type IIB

is given explicitly in Table 6.11. For the non-geometric fluxes we might naively expect $U_a \to U$ and $T_i \to T$ to set the components $F^{(i)a}$ to be equal to one another but this is not the case. The restricted non-geometric components are given in Table 6.12. A further reduction in flux entries is discussed in [53, 92]. We are considering real integer flux entries and in order to have $\widetilde{\mathsf{c}}_i, \widetilde{\mathsf{d}}_i \in \mathbb{R}$ for $i \in \{1,2\}$, we have to equate $\widehat{\mathsf{c}}_i = \check{\mathsf{c}}_i \equiv \mathsf{c}_i$ and $\widehat{\mathsf{d}}_i = \check{\mathsf{d}}_i \equiv \mathsf{d}_i$. This reduces the number of independent fluxes in Q to only six and are given in Table 6.12. The superpotential's polynomial form reduces in the non-geometric sector to a triplet copy of a single expression and the U_a and T_i indices are dropped.

$$W = P_1(U) + S\,P_2(U) + 3T\,P_3(U) + 3S\,T\,P_4(U) \tag{6.3.9}$$

The explicit expansion for the individual polynomials are obtained by applying the aforementioned simplification of the fluxes.

$$\begin{aligned} P_1(U) &= -\ \mathsf{a}_0 \ +\ 3\mathsf{a}_1 U \ -\ 3\mathsf{a}_2 U^2 \ +\ \mathsf{a}_3 U^3 \\ P_2(U) &= -\ \mathsf{b}_0 \ +\ 3\mathsf{b}_1 U \ -\ 3\mathsf{b}_2 U^2 \ +\ \mathsf{b}_3 U^3 \end{aligned}$$

The non-geometric cases follow the same schematic format but the matrix summation reduces to a single coefficent.

$$\begin{aligned} P_3(U) &= -\ \mathsf{c}_0 \ +\ (\widetilde{\mathsf{c}}_1 - 2\mathsf{c}_1)U \ -\ (\widetilde{\mathsf{c}}_2 - 2\mathsf{c}_2)U^2 \ +\ \mathsf{c}_3 U^3 \\ P_4(U) &= -\ \mathsf{d}_0 \ +\ (\widetilde{\mathsf{d}}_1 - 2\mathsf{d}_1)U \ -\ (\widetilde{\mathsf{d}}_2 - 2\mathsf{d}_2)U^2 \ +\ \mathsf{d}_3 U^3 \end{aligned}$$

6.3.4 Kähler Potential

The Kähler potential in the anisotropic case illustrates the cyclic permutation symmetry in the T_i and U_a moduli and on restriction to the isotropic case the

	$\mathsf{F}_{(i)0}$	$\mathsf{F}_{(i)1}$	$\mathsf{F}_{(i)2}$	$\mathsf{F}_{(i)3}$	$\mathsf{F}_{(i)}{}^{0}$	$\mathsf{F}_{(i)}{}^{1}$	$\mathsf{F}_{(i)}{}^{2}$	$\mathsf{F}_{(i)}{}^{3}$	
$\mathsf{F}_{(1)A}$	$+\mathsf{c}_0$	$+\widetilde{\mathsf{c}}_2$	$-\mathsf{c}_2$	$-\mathsf{c}_2$	$-\mathsf{c}_3$	$-\widetilde{\mathsf{c}}_1$	$+\mathsf{c}_1$	$+\mathsf{c}_1$	$\mathsf{F}_{(1)}{}^{B}$
$\mathsf{F}_{(2)A}$	$+\mathsf{c}_0$	$-\mathsf{c}_2$	$+\widetilde{\mathsf{c}}_2$	$-\mathsf{c}_2$	$-\mathsf{c}_3$	$+\mathsf{c}_1$	$-\widetilde{\mathsf{c}}_1$	$+\mathsf{c}_1$	$\mathsf{F}_{(2)}{}^{B}$
$\mathsf{F}_{(3)A}$	$+\mathsf{c}_0$	$-\mathsf{c}_2$	$-\mathsf{c}_2$	$+\widetilde{\mathsf{c}}_2$	$-\mathsf{c}_3$	$+\mathsf{c}_1$	$+\mathsf{c}_1$	$-\widetilde{\mathsf{c}}_1$	$\mathsf{F}_{(3)}{}^{B}$

Table 6.12: Component labels for isotropic Type IIB Q.

summations reduce to overall factors.

$$\begin{aligned} K \;&=\; -\sum_{i=1}^{3} \ln\Big(- i(T_i - \bar{T}_i)\Big) - \ln\Big(- i(S - \bar{S})\Big) - \sum_{a=1}^{3} \ln\Big(- i(U_a - \bar{U}_a)\Big) \\ &\rightarrow -3\,\ln\Big(- i(T - \bar{T})\Big) - \ln\Big(- i(S - \bar{S})\Big) - 3\,\ln\Big(- i(U - \bar{U})\Big) \end{aligned}$$

6.4 Flux Constraints and Solutions

The constraints on fluxes fall into two categories; the Bianchi constraints and the tadpoles. In each case we have seen how they form S duality multiplets in Type IIB but initially we shall restrict ourselves to the T duality only case, which is obtained from the full U duality case by setting $P = 0$. The methods developed in the T duality only case will become crucial to exploring the algebraic structure once S-dual P flux has been included.

6.4.1 T Duality Non-Geometric Fluxes

In the T duality only Type IIB/O3 case the Bianchi constraints come in two expressions, one of which represents the constraints of the six dimensional subalgebra.

$$Q_{e}^{[ab}Q_{d}^{c]e} = 0 \quad , \quad Q_{[a}^{ed}H_{bc]d} = 0 \tag{6.4.1}$$

We will refer to these as $QQ = 0$ and $QH = 0$ from this point onwards. More generally the $A \cdot B$ contraction will be used to refer to $\Delta^{p}(\mathsf{E}^{*})$ component contractions and AB to be $\Lambda^p(\mathsf{E}^*)$ contractions. First we focus on Q^{ab}_c, which has the additional properties $Q^{ab}_b = 0$ and $Q^{ab}_c = -Q^{ba}_c$ and is playing the role of a structure constant in a 6 dimensional X^a gauge subalgebra of (6.3.1).

$$QQ = 0 \quad \Leftrightarrow \quad \begin{aligned} \mathsf{c}_0\,(\mathsf{c}_2 - \widetilde{\mathsf{c}}_2) + \mathsf{c}_1\,(\mathsf{c}_1 - \widetilde{\mathsf{c}}_1) &= 0 \\ \mathsf{c}_2\,(\mathsf{c}_2 - \widetilde{\mathsf{c}}_2) + \mathsf{c}_3\,(\mathsf{c}_1 - \widetilde{\mathsf{c}}_1) &= 0 \\ \mathsf{c}_0\mathsf{c}_3 - \mathsf{c}_1\mathsf{c}_2 &= 0 \end{aligned} \tag{6.4.2}$$

The three polynomials are the generating functions of the ideal $\langle QQ\rangle$. Due to the way in which covariant and contravariant $\Lambda^p(\mathsf{E}^*)$ indices contract in QQ the ideal $\langle QQ\rangle$ is invariant under a coordinate transformation, a point we noted in Section 5.1.3. Since Q satisfies all the conditions required to be a Lie algebra structure constant it must be isomorphic to a known[2] Lie algebra, where the isomorphism is a valid change of basis on the X^a generators. The independent components of Q are determined by the symmetries of the compact space upon which they are defined, a result which is particularly simple for this orientifold once we make the restriction to isotropy. Due to isotropic orbifold symmetries the 6 dimensional tangent forms basis η^a must be split into two three dimensional systems, $\eta^a \to (\xi^I, \widetilde{\xi}^I)$, which are invariant under the isotropic constraint $\xi^1 \to \xi^2 \to \xi^3 \to \xi^1$ and similarly for $\widetilde{\xi}^I$. This can be rephrased in terms of generator structure constants by use of the fact ϵ_{ijk} is the only isotropic rank 3 tensor, up to proportionality factors. There are only 5 isotropic non-trivial Lie algebras with such generators, $\mathfrak{so}(4) \sim \mathfrak{su}(2)^2$, $\mathfrak{so}(3,1)$, $\mathfrak{su}(2)+\mathfrak{u}(1)^3$, $\mathfrak{iso}(3)$ and $\mathfrak{nil}$[3]. We do not consider the abelian $\mathfrak{u}(1)^6$ since it is equivalent to a trivial $Q=0$ background. All these algebras are quasi-classical Lie algebras, ie. they have a bi-invariant non-degenerate metric built from their quadratic Casimir operator. In the redefined 1-forms $(\xi^I, \widetilde{\xi}^J)$ basis these algebras have the canonical forms shown in Table 6.13. The isotropic nature of the structure constants is particularly clear for the

[2] all non-semi-simple 6 dimensional Lie algebras are known [96].

[3] where $\mathfrak{nil} \equiv L_{6,26}$ in [96].

$\mathfrak{so}(4) \sim \mathfrak{su}(2)^2$ case as $\mathfrak{su}(2)$ is the algebra with structure constant ϵ.

$$\begin{array}{lll} [\ \sigma_i\ ,\ \sigma_j\ ] = \lambda_\sigma \epsilon_{ijk} \sigma_k\ ,\ \mathfrak{su}(2) = \mathcal{L}(\sigma) \\ [\ \tau_i\ ,\ \tau_j\ ] = \lambda_\tau \epsilon_{ijk} \tau_k\ ,\ \mathfrak{su}(2) = \mathcal{L}(\tau) \end{array} \quad , \quad \mathfrak{so}(4) = \mathcal{L}(\sigma) \oplus \mathcal{L}(\tau)$$

The method of finding a parametrised solution consists, first of all, of selecting

Algebra	$d\xi^I$	$d\widetilde{\xi}^I$
$\mathfrak{so}(4) \sim \mathfrak{su}(2)^2$	$\xi^J \wedge \xi^K$	$\widetilde{\xi}^J \wedge \widetilde{\xi}^K$
$\mathfrak{so}(3,1)$	$\xi^J \wedge \xi^K - \widetilde{\xi}^J \wedge \widetilde{\xi}^K$	$\xi^J \wedge \widetilde{\xi}^K$
$\mathfrak{su}(2) + \mathfrak{u}(1)^3$	$\xi^I \wedge \xi^K$	0
$\mathfrak{iso}(3)$	$\xi^J \wedge \xi^K$	$\xi^J \wedge \xi^K + \xi^J \wedge \widetilde{\xi}^K$
$\mathfrak{nil}$	0	$\xi^J \wedge \xi^K$

Table 6.13: Canonic non-geometric Q algebras.

one of Lie algebras in Table 6.13, $\mathfrak{g}_Q$, and constructing its canonical structure constant g^K_{IJ}. This can be directly read from the table. Since Q is defined in the non-manifestly canonical 1-form basis of η^a, there is a coordinate transformation relating both bases, $M^{-1} : (\eta^a) \rightarrow (\xi^I, \widetilde{\xi}^I)$. Instead of working in the 1-form basis, we will move to its dual generators basis with the transformation $M : (X^a) \rightarrow (E^I, \widetilde{E}^I)$ and so the structure constants transform into the canonical form.

$$M^I_a\, M^J_b\, Q^{ab}_c\, (M^{-1})^c_K = g^{IJ}_K \tag{6.4.3}$$

The transformation matrix M must satisfy the isotropy symmetry and so we have $M = \mathbb{I}_3 \otimes M_2$, where the four parameter matrix $M_2 \in SL(2,\mathbb{R})$ acts equally in each two dimensional sub-torus.

$$\begin{pmatrix} E^I \\ \widetilde{E}^I \end{pmatrix} = \frac{1}{|\Gamma_M|^2} \begin{pmatrix} -\alpha & \beta \\ -\gamma & \delta \end{pmatrix} \begin{pmatrix} X^{2I-1} \\ X^{2I} \end{pmatrix} \tag{6.4.4}$$

Here $|\Gamma_M| = \alpha\delta - \beta\gamma$, and it must be that $|\Gamma_M| \neq 0$. In the following we will refer to the $(\alpha, \beta, \gamma, \delta)$ parameters as the modular parameters, following the terminology of [92]. Transformation (6.4.3) is rearranged to $M_a^I\, M_b^J\, Q_c^{ab}\, (M^{-1})_K^c - g_K^{IJ} = 0$, so that we have an array of expressions, all of which are equated to zero. These equations can be solved uniquely for the fluxes c in terms of the modular parameters for each of the Lie algebras via algebraic geometry methods.

- Semisimple $\mathfrak{so}(4)$

$$\begin{aligned} \mathsf{c}_0 &= \beta\,\delta\,(\beta+\delta) \quad ; \quad \mathsf{c}_3 = -\,\alpha\,\gamma\,(\alpha+\gamma) \quad , \\ \mathsf{c}_1 &= \beta\,\delta\,(\alpha+\gamma) \quad ; \quad \mathsf{c}_2 = -\,\alpha\,\gamma\,(\beta+\delta) \quad , \\ \widetilde{\mathsf{c}}_2 &= \gamma^2\,\beta + \alpha^2\,\delta \quad ; \quad \widetilde{\mathsf{c}}_1 = -\,(\gamma\,\beta^2 + \alpha\,\delta^2) \quad . \end{aligned} \tag{6.4.5}$$

- Semisimple $\mathfrak{so}(3,1)$

$$\begin{aligned} \mathsf{c}_0 &= -\beta\left(\beta^2+\delta^2\right) \quad ; \quad \mathsf{c}_3 = \alpha\left(\alpha^2+\gamma^2\right) \quad , \\ \mathsf{c}_1 &= -\alpha\left(\beta^2+\delta^2\right) \quad ; \quad \mathsf{c}_2 = \beta\left(\alpha^2+\gamma^2\right) \quad , \\ \widetilde{\mathsf{c}}_2 &= -\beta\,(\alpha^2-\gamma^2) - 2\,\gamma\,\delta\,\alpha \quad ; \quad \widetilde{\mathsf{c}}_1 = \alpha\big(\beta^2-\delta^2\big) + 2\,\beta\,\gamma\,\delta \quad . \end{aligned} \tag{6.4.6}$$

- Non semisimple (ie. direct sum) $\mathfrak{su}(2) + \mathfrak{u}(1)^3$

$$\begin{aligned} \mathsf{c}_0 &= \beta\,\delta^2 \quad ; \quad \mathsf{c}_3 = -\alpha\,\gamma^2 \quad , \\ \mathsf{c}_1 &= \beta\,\delta\,\gamma \quad ; \quad \mathsf{c}_2 = -\alpha\,\gamma\,\delta \quad , \\ \widetilde{\mathsf{c}}_2 &= \gamma^2\,\beta \quad ; \quad \widetilde{\mathsf{c}}_1 = -\alpha\,\delta^2 \quad . \end{aligned} \tag{6.4.7}$$

- Non solvable (ie. semidirect sum) $\mathfrak{iso}(3)$

$$\begin{aligned} \mathsf{c}_0 &= -\delta^2\,(\beta-\delta) \quad ; \quad \mathsf{c}_3 = \gamma^2\,(\alpha-\gamma) \quad , \\ \mathsf{c}_1 &= -\delta^2\,(\alpha-\gamma) \quad ; \quad \mathsf{c}_2 = \gamma^2\,(\beta-\delta) \quad , \\ \widetilde{\mathsf{c}}_2 &= \gamma^2\,(\beta+\delta) - 2\,\gamma\,\delta\,\alpha \quad ; \quad \widetilde{\mathsf{c}}_1 = -\delta^2\,(\alpha+\gamma) + 2\,\gamma\,\delta\,\beta \quad . \end{aligned} \tag{6.4.8}$$

- Solvable (ie. nilpotent) $\mathfrak{nil}$

$$\begin{aligned} \mathsf{c}_0 &= \delta^3 \quad ; \quad \mathsf{c}_3 = -\gamma^3 \quad , \\ \mathsf{c}_1 &= \delta^2\,\gamma \quad ; \quad \mathsf{c}_2 = -\delta\,\gamma^2 \quad , \\ \widetilde{\mathsf{c}}_2 &= \delta\,\gamma^2 \quad ; \quad \widetilde{\mathsf{c}}_1 = -\delta^2\,\gamma \quad . \end{aligned} \tag{6.4.9}$$

It is straightforward to check that these flux configurations satisfy (6.4.2). The entries in M are not restricted to being integers. Starting with a configuration such that $\mathsf{c}_i \in \mathbb{Z}$, since the c_i are cubic in modular parameters, we see that $M' = \sqrt[3]{n}M$ with $n \in \mathbb{Z}$ still gives us $\mathsf{c}_i' = n\,\mathsf{c}_i \in \mathbb{Z}$. It is also useful to note that the flux induced cubic polynomials $P_3(U)$, as well as their roots structure in terms of the redefined complex structure $\mathcal{Z} = \frac{\alpha U + \beta}{\gamma U + \delta}$, depend crucially on the Q subalgebra behind the fluxes (see Table 6.14). We define the following 2-dimensional vectors in such a

Q-subalgebra	$\mathcal{P}_3(\mathcal{Z}) \equiv \frac{P_3(U)}{3\,(\gamma U+\delta)^3}$	Modular roots
$\mathfrak{so}(4)$	$\mathcal{Z}(\mathcal{Z}+1)$	$\mathcal{Z}=0\ ,\ \infty\ ,\ -1$
$\mathfrak{so}(3,1)$	$-\mathcal{Z}(\mathcal{Z}^2+1)$	$\mathcal{Z}=0\ ,\ +i\ ,\ -i$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathcal{Z}$	$\mathcal{Z}=0\ ,\ \infty$ (double)
$\mathfrak{iso}(3)$	$1-\mathcal{Z}$	$\mathcal{Z}=\infty$ (double) , $+1$
$\mathfrak{nil}$	1	$\mathcal{Z}_\infty=\infty$ (triple)

Table 6.14: Q-subalgebras and their flux induced polynomials.

way that they carry the information about the value of the roots once they are contracted with $\begin{pmatrix} U \\ 1 \end{pmatrix}$.

$$\begin{array}{llcll} \mathcal{Z}_0 = & (\alpha,\beta) & , & \mathcal{Z}_\infty = & (\gamma,\delta) \\ \mathcal{Z}_{-1} = & (\alpha+\gamma,\beta+\delta) & , & \mathcal{Z}_{+1} = & (\alpha-\gamma,\beta-\delta) \\ \mathcal{Z}_{+i} = i & \left(\sqrt{\alpha^2+\gamma^2}, \frac{(\alpha\beta+\gamma\delta)+i|\Gamma_M|}{\sqrt{\alpha^2+\gamma^2}}\right) & , & \mathcal{Z}_{-i} = i & \left(\sqrt{\alpha^2+\gamma^2}, \frac{(\alpha\beta+\gamma\delta)-i|\Gamma_M|}{\sqrt{\alpha^2+\gamma^2}}\right) \end{array} \tag{6.4.10}$$

Then the flux induced polynomial $P_3(U)$ for each gauge subalgebra can be easily reconstructed from its root structure as

$$P_3(U) = 3 \prod_{r=roots} \mathcal{Z}_r \begin{pmatrix} U \\ 1 \end{pmatrix} , \tag{6.4.11}$$

with $r \equiv 0, \infty, -1, +1, +i, -i$ according with the modular roots, as it is shown in Table 6.14. As an example, we reconstruct the cubic $P_3(U)$ for the subalgebra

$\mathfrak{so}(4)$. In this case, (6.4.11) reads

$$\begin{aligned} P_3(U) &= 3\,\mathcal{Z}_0 \begin{pmatrix} U \\ 1 \end{pmatrix} \cdot \mathcal{Z}_\infty \begin{pmatrix} U \\ 1 \end{pmatrix} \cdot \mathcal{Z}_{-1} \begin{pmatrix} U \\ 1 \end{pmatrix} = \\ &= 3\,(\alpha\,U + \beta)\,(\gamma\,U + \delta)\;\;[(\alpha+\gamma)\,U + (\beta+\delta)] = \\ &= 3\,(\gamma\,U + \delta)^3\,\mathcal{Z}\,(\mathcal{Z}+1)\;. \end{aligned}$$

We note that $\mathfrak{so}(3,1)$ is unique in the above results, in that it generates a polynomial whose roots are certain to be complex, given the real and non-zero nature of Γ_M.

6.4.2 U Duality Non-Geometric Fluxes

We have seen how S duality transformations deform both the Bianchi and tadpole constraints and we shall first focus on the S-dualization of T-dual Bianchi identities (6.4.1) and then consider the new constraints coming from the S-dualization of tadpoles. Applying the S-duality transformation to the non-geometric Q flux of the form given in (5.3.2) the $QQ = 0$ Bianchi identity in (6.4.1) gives rise to an $\mathrm{SL}(2,\mathbb{Z})_S$ triplet of constraints on Q and P we stated in terms of F_2 and $\widehat{\mathsf{F}}_2$ in (5.3.4).

$$Q_d^{[ab}Q_e^{c]d} = 0 \quad , \quad P_d^{[ab}P_e^{c]d} = 0 \quad , \quad Q_d^{[ab}P_e^{c]d} + P_d^{[ab}Q_e^{c]d} = 0 \qquad (6.4.12)$$

As before, we denote these schematically as $QQ = 0$, $PP = 0$ and $QP + PQ = 0$. The first expression in terms of fluxes is (6.4.2) and the second expression is obtained from (6.4.2) by $Q \to P$, $\mathsf{c} \to \mathsf{d}$. The third element of the triplet gives the mixing between Q and P fluxes.

$$\begin{aligned} \mathsf{c}_3\mathsf{d}_0 - \mathsf{c}_2\mathsf{d}_1 - \mathsf{c}_1\mathsf{d}_2 + \mathsf{c}_0\mathsf{d}_3 &= 0\;, \\ \mathsf{c}_1(\mathsf{d}_1 - \widetilde{\mathsf{d}}_1) + \mathsf{c}_0(\mathsf{d}_2 - \widetilde{\mathsf{d}}_2) + \mathsf{d}_0(\mathsf{c}_2 - \widetilde{\mathsf{c}}_2) + \mathsf{d}_1(\mathsf{c}_1 - \widetilde{\mathsf{c}}_1) &= 0\;, \qquad (6.4.13) \\ \mathsf{c}_3(\mathsf{d}_1 - \widetilde{\mathsf{d}}_1) + \mathsf{c}_2(\mathsf{d}_2 - \widetilde{\mathsf{d}}_2) + \mathsf{d}_2(\mathsf{c}_2 - \widetilde{\mathsf{c}}_2) + \mathsf{d}_3(\mathsf{c}_1 - \widetilde{\mathsf{c}}_1) &= 0\;. \end{aligned}$$

Using coordinate transformations, it is possible to solve the first two constraints in (6.4.12). This is achieved in the same manner as the previous section except we now pick two algebras, $\mathfrak{g}_Q$ and $\mathfrak{g}_P$, and equating their canonical structure constants, g and h respectively, to the transformed Q and P. Due to the piecewise structure of the transformations, we can apply independent coordinate transformations on the Q and P fluxes, M_Q and M_P, and each flux gives an equation of the form (6.4.3).

$$M_Q\, M_Q\, Q\, M_Q^{-1} = g_Q \quad , \quad M_P\, M_P\, P\, M_P^{-1} = g_P$$

As with the transformation on Q it is convenient to give specific modular parameters to M_Q and M_P.

$$\Gamma_Q = \begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha_p & \beta_p \\ \gamma_p & \delta_p \end{pmatrix} \tag{6.4.14}$$

We solve these expressions in the same manner as in the previous section and get parametrizations $Q = Q(\alpha_q, \ldots, \delta_q)$ and $P = P(\alpha_p, \ldots, \delta_p)$. Recalling (6.4.14), we can now define the two modular variables $\mathcal{Z}_Q$ and $\mathcal{Z}_P$ which are the S duality extension of the $\mathcal{Z}$ previously defined.

$$\mathcal{Z}_Q = \frac{\alpha_q\, U + \beta_q}{\gamma_q\, U + \delta_q} \quad , \quad \mathcal{Z}_P = \frac{\alpha_p\, U + \beta_p}{\gamma_p\, U + \delta_p} \tag{6.4.15}$$

Expressing the superpotential polynomials due to Q and P in terms of these, we have $\mathcal{P}_3(\mathcal{Z}_Q) \equiv P_3(U)/3(\gamma_q\, U + \delta_q)^3$ and $\mathcal{P}_4(\mathcal{Z}_P) \equiv -P_4(U)/3(\gamma_p\, U + \delta_p)^3$, where the polynomials relating to $\mathfrak{g}_Q$ and $\mathfrak{g}_P$ can be simply read off from Table 6.14, upon replacing $\mathcal{Z}$ by $\mathcal{Z}_Q$ and $\mathcal{Z}_P$ respectively. Making the restriction $M_Q = M_P$ would be sufficient to solve the first two of the three constraints, those viewable as integrability conditions, of (6.4.12) simultaneously. However, the third element of the triplet, the cohomology condition between $\mathcal{L}_Q$ and $\mathcal{L}_P$, requires two different sets of modular parameters. If $\mathcal{L}_Q = \mathcal{L}_P$ then the cohomology conditions reduce to the integrability conditons, the third set of constraints are satisfied if and only if the other two sets are. For cases where $\mathcal{L}_Q \neq \mathcal{L}_P$ it is not automatic that $QP+PQ=0$. Transforming Q and P in different ways reduces the constraints of $QP+PQ=0$ to being constraints on the modular parameters of (6.4.14). Furthermore, since different algebras lead to different parametrizations, each unique pairing of algebras $(\mathfrak{g}_Q, \mathfrak{g}_P)$ leads to a different set of constraints. The integrability conditions of Q and P are both solved through the use of prime decomposition and the same is true for the cohomology conditions and so we split $\langle QP+PQ \rangle$ into its prime ideals, J_i.

$$\langle\, QP+PQ \,\rangle = J_1 \cap \ldots \cap J_n$$

An ideal automatically has at least one prime ideal but in the case of some of the $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairings, we find as many as three prime ideals of varying complexity. These relate the Γ_Q and Γ_P modular matrices and so restrict the tranformations which are needed to bring the Q and P fluxes (understood as structure constants) to their canonical form. For the purpose of illustration we consider the example $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$ and $\mathfrak{g}_P = \mathfrak{so}(4)$ and read off the modular parameterisations from (6.4.7) and (6.4.5)

- Q flux fixing the T-dual gauge subalgebra to be $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$.

$$\begin{aligned} \mathsf{c}_0 &= \beta_q\, \delta_q^2 \quad &;& \quad \mathsf{c}_3 = -\alpha_q\, \gamma_q^2 \ , \\ \mathsf{c}_1 &= \beta_q\, \delta_q\, \gamma_q \quad &;& \quad \mathsf{c}_2 = -\alpha_q\, \gamma_q\, \delta_q \ , \\ \widetilde{\mathsf{c}}_2 &= \gamma_q^2\, \beta_q \quad &;& \quad \widetilde{\mathsf{c}}_1 = -\alpha_q\, \delta_q^2 \ . \end{aligned} \tag{6.4.16}$$

- P flux fixing the original T-dual gauge subalgebra to be deformed by $\mathfrak{g}_P = \mathfrak{so}(4)$,

$$\begin{aligned} \mathsf{d}_0 &= \beta_p\, \delta_p\, (\beta_p + \delta_p) \quad &;& \quad \mathsf{d}_3 = -\,\alpha_p\, \gamma_p\, (\alpha_p + \gamma_p) \quad , \\ \mathsf{d}_1 &= \beta_p\, \delta_p\, (\alpha_p + \gamma_p) \quad &;& \quad \mathsf{d}_2 = -\,\alpha_p\, \gamma_p\, (\beta_p + \delta_p) \quad , \\ \widetilde{\mathsf{d}}_2 &= \gamma_p^2\, \beta_p + \alpha_p^2\, \delta_p \quad &;& \quad \widetilde{\mathsf{d}}_1 = -\,(\gamma_p\, \beta_p^2 + \alpha_p\, \delta_p^2) \ . \end{aligned} \tag{6.4.17}$$

This leads to a $\langle QP+PQ\rangle$ cohomology condition ideal which has three prime ideals in its decomposition,

$$\begin{aligned} J_1 &= \langle\ \alpha_q\beta_p - \beta_q\alpha_p\ ,\ \gamma_q\delta_p - \delta_q\gamma_p\ \rangle\ , \\ J_2 &= \langle\ \alpha_q\delta_p - \beta_q\gamma_p\ \ ,\ \gamma_q\beta_p - \delta_q\alpha_p\ \rangle\ , \\ J_3 &= \langle\ \gamma_q(\beta_p+\delta_p) - \delta_q(\alpha_p+\gamma_p)\ \rangle\ . \end{aligned} \tag{6.4.18}$$

These constraints can be rewritten in terms of entries in 2 dimensional vectors.

$$\mathbf{a} = \begin{pmatrix} \mathsf{a}_1 \\ \mathsf{a}_2 \end{pmatrix}\ ,\ \mathbf{b} = \begin{pmatrix} \mathsf{b}_1 \\ \mathsf{b}_2 \end{pmatrix} \quad \Rightarrow \quad \mathsf{a}_1\mathsf{b}_2 - \mathsf{a}_2\mathsf{b}_1 = 0 \quad \Leftrightarrow \quad \mathbf{a}\times\mathbf{b} = 0$$

If two vectors satisfy $\mathbf{a}\times\mathbf{b}=0$ then they are parallel, which we denote by $\mathbf{a}\parallel\mathbf{b}$. With this notation and using the vectors given in (6.4.10), the cohomology conditions can be reexpressed.

$$\begin{aligned} J_1 &= \langle\ \mathcal{Z}_0^Q\times\mathcal{Z}_0^P\ ,\ \mathcal{Z}_\infty^Q\times\mathcal{Z}_\infty^P\ \rangle \longleftrightarrow \mathcal{Z}_0^Q\parallel\mathcal{Z}_0^P\ ,\ \mathcal{Z}_\infty^Q\parallel\mathcal{Z}_\infty^P\ , \\ J_2 &= \langle\ \mathcal{Z}_0^Q\times\mathcal{Z}_\infty^P\ ,\ \mathcal{Z}_\infty^Q\times\mathcal{Z}_0^P\ \rangle \longleftrightarrow \mathcal{Z}_0^Q\parallel\mathcal{Z}_\infty^P\ ,\ \mathcal{Z}_\infty^Q\parallel\mathcal{Z}_0^P\ , \\ J_3 &= \qquad \langle\ \mathcal{Z}_\infty^Q\times\mathcal{Z}_{-1}^P\ \rangle \qquad \longleftrightarrow \qquad \mathcal{Z}_\infty^Q\parallel\mathcal{Z}_{-1}^P\ , \end{aligned}$$

In each case the prime ideal's generating functions can be rewritten as a vanishing cross product. Infact, this happens for all prime ideals of all possible pairings $(\mathfrak{g}_Q,\mathfrak{g}_P)$. Therefore, the prime ideals of $\langle QP+PQ\rangle$ can be viewed as geometric constraints on the position of the vectors representing the roots of the cubic polynomials $P_3(U)$ and $P_4(U)$. Specifically, when the polynomials themselves are computed, this is equivalent to $P_3(U)$ and $P_4(U)$ sharing some roots. It is worth note that the $J_1=0$ and $J_2=0$ solutions also imply the piecewise vanishing $QP=PQ=0$, unlike $J_3=0$. Moreover, $J_1=0$ can be translated into $\mathcal{Z}_P\propto\mathcal{Z}_Q$ while $J_2=0$ implies $\mathcal{Z}_P\propto\Gamma_S\mathcal{Z}_Q$, where Γ_S is the inversion generator of $SL(2,\mathbb{Z})_S$. The full list of the vector alignments arising from the different prime ideals of the cohomology condition are given in Table 6.15 for each algebra pairing $(\mathfrak{g}_Q,\mathfrak{g}_P)$. Most of these solutions (those labelled by $(*)$) disappear under the more restrictive condition $QP = PQ = 0$, or equivalently, not all the pairings are allowed in a system where the full set of $SL(2,\mathbb{Z})^7$ dualities is used. Apart from each algebra being deformed by itself, there are the following possibilities in an $SL(2,\mathbb{Z})^7$-dual setup: $\mathfrak{so}(4)$ can be deformed by $\mathfrak{su}(2)+\mathfrak{u}(1)^3$; $\mathfrak{su}(2)+\mathfrak{u}(1)^3$ can be deformed by $\mathfrak{so}(4)$ and by $\mathfrak{nil}$; $\mathfrak{iso}(3)$ can be deformed by $\mathfrak{nil}$ and $\mathfrak{nil}$ can be deformed by $\mathfrak{su}(2)+\mathfrak{u}(1)^3$ and by $\mathfrak{iso}(3)$.

	$\mathfrak{g}_P$				
$\mathfrak{g}_Q$	$\mathfrak{so}(4)$	$\mathfrak{so}(3,1)$	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{iso}(3)$	$\mathfrak{nil}$
$\mathfrak{so}(4)$	$[0\vert 0]$, $[\infty\vert\infty]$ $[0\vert\infty]$, $[\infty\vert 0]$ $[-1\vert -1]^*$	$[-1\vert 0]^*$	$[0\vert 0]$, $[\infty\vert\infty]$ $[0\vert\infty]$, $[\infty\vert 0]$ $[-1\vert\infty]^*$	$[-1\vert +1]^*$	$[-1\vert\infty]^*$
$\mathfrak{so}(3,1)$	$[0\vert 1]^*$	$[+i\vert +i]$, $[-i\vert -i]$ $[+i\vert -i]$, $[-i\vert +i]$ $[0\vert 0]^*$	$[0\vert\infty]^*$	$[0\vert +1]^*$	$[0\vert\infty]^*$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$[0\vert 0]$, $[\infty\vert\infty]$ $[0\vert\infty]$, $[\infty\vert 0]$ $[\infty\vert -1]^*$	$[\infty\vert 0]^*$	$[0\vert\infty]$, $[\infty\vert 0]$ $[\infty\vert\infty]$	$[\infty\vert +1]^*$	$[\infty\vert\infty]$
$\mathfrak{iso}(3)$	$[+1\vert -1]^*$	$[+1\vert 0]^*$	$[+1\vert\infty]^*$	$[\infty\vert\infty]$ $[+1\vert +1]^*$	$[\infty\vert\infty]$ $[+1\vert\infty]^*$
$\mathfrak{nil}$	$[\infty\vert -1]^*$	$[\infty\vert 0]^*$	$[\infty\vert\infty]$	$[\infty\vert\infty]$ $[\infty\vert +1]^*$	$[\infty\vert\infty]$

Table 6.15: Cohomology condition in terms of the root alignments where $[x|y] \equiv \mathcal{Z}_x^Q \parallel \mathcal{Z}_y^P$. The branches labelled by $*$ disappear under the more restrictive condition $QP = PQ = 0$. Under the inversion $S \to -1/S$ transformation, the algebras $\mathfrak{g}_Q$ and $\mathfrak{g}_P$ are exchanged resulting in the symmetry of this table.

6.4.3 Tadpoles Cancellation Conditions

In this IIB orientifold Bianchi identities for R-R fluxes can be rephrased as tadpole cancellation conditions for the R-R 4-form C_4 and 8-form C_8 which couple to the O3/O7-planes sources allowed by the symmetry $\theta_I \sigma_B$. The C_4 tadpole follows from $H \wedge F = \widehat{\mathsf{F}}_0 \wedge \mathsf{F}_0$, which can be expanded in terms of the $\Lambda^p(\mathsf{E}^*)$ components, $\widehat{\mathsf{F}}_0 \wedge \mathsf{F}_0 \propto \mathsf{F}_{abc}\widehat{\mathsf{F}}_{def}\varepsilon^{abcdef}\mathrm{vol}_6$. We can also express the tadpole in terms of the $\Delta^*(\mathsf{E}^*)$ defined components, it is the ν_0 coefficient in (5.2.4).

$$0 = N_3 + \iota_{\nu_0}\mathsf{D}\mathsf{D}'(\widetilde{\nu}^0) = N_3 + \widehat{\mathsf{F}}^{(0)}{}_A \mathsf{F}^{(0)A} - \widehat{\mathsf{F}}^{(0)B}\mathsf{F}^{(0)}{}_B$$

Using Tables 6.5 and 6.7 we can express these in terms of the a_i and b_j and by noting that the total orientifold charge is -32, due to 64 O3-planes located at the fixed points of the $\mathbb{Z}_2^3$ orientifold involution, and D3-branes having charge +1 can be added then $N_3 = 32 - N_{D3}$.

$$\begin{aligned} N_{D3} - 32 &= \widehat{\mathsf{F}}^{(0)}{}_0\mathsf{F}^{(0)0} + \widehat{\mathsf{F}}^{(0)}{}_a\mathsf{F}^{(0)a} - \widehat{\mathsf{F}}^{(0)0}\mathsf{F}^{(0)}{}_0 - \widehat{\mathsf{F}}^{(0)b}\mathsf{F}^{(0)}{}_b \\ &= -\mathsf{a}_0\mathsf{b}_3 - \mathsf{a}_2^{(a)}\mathsf{b}_1^{(a)} + \mathsf{a}_3\mathsf{b}_0 + \mathsf{a}_1^{(b)}\mathsf{b}_2^{(b)} \end{aligned}$$

Upon the restriction to isotropy the summations over the $h^{1,1} = 3$ Kähler indices reduce to $h^{1,1}$ equal expressions.

$$\begin{aligned} N_{D3} - 32 &= \widehat{\mathsf{F}}^{(0)}{}_0\mathsf{F}^{(0)0} + \widehat{\mathsf{F}}^{(0)}{}_a\mathsf{F}^{(0)a} - \widehat{\mathsf{F}}^{(0)0}\mathsf{F}^{(0)}{}_0 - \widehat{\mathsf{F}}^{(0)b}\mathsf{F}^{(0)}{}_b \\ & \quad -\mathsf{a}_0\mathsf{b}_3 - 3\mathsf{a}_2\mathsf{b}_1 + \mathsf{a}_3\mathsf{b}_0 + 3\mathsf{a}_1\mathsf{b}_2 \end{aligned}$$

In the T duality only case the C_8 tadpole coupling to the D7-branes and O7-planes is defined by $Q \cdot F$.

$$-\int_{M_4 \times \mathcal{M}_{\mathbb{Z}_2^2}} C_8 \wedge (Q \cdot F) \tag{6.4.19}$$

$Q \cdot F$ is expanded in terms of the $\Delta^2(\mathsf{E}^*)$ basis but due to the manner we defined the components of D and D' and the way in which the fluxes couple to the Kähler moduli in Type IIB the relevant polynomials in (5.2.4) are the coefficients of $\widetilde{\nu}^i$.

Given we have an explicit $\Lambda^p(\mathsf{E}^*)$ basis for the $\Delta^p(\mathsf{E}^*)$ elements we can demonstrate their equivalence, taking the example of the coefficient of $\eta^{12} = \omega_1 = \nu_1$ in $Q \cdot F$.

$$
\begin{aligned}
(Q\cdot F)_{12} \;=\; \tfrac{1}{2}Q^{ij}_{[1}F_{2]ij} \;=&\quad Q^{35}_1 F_{235} \;+\; Q^{36}_1 F_{236} \;+\; Q^{45}_1 F_{245} \;+\; Q^{46}_1 F_{246} \\
&- \; Q^{35}_2 F_{135} \;-\; Q^{36}_2 F_{136} \;-\; Q^{45}_2 F_{145} \;-\; Q^{46}_2 F_{146} \\
=&\; - \; \mathsf{F}_{(1)}{}^{1}\widehat{\mathsf{F}}^{(0)}{}_{1} \;+\; \mathsf{F}_{(1)2}\widehat{\mathsf{F}}^{(0)2} \;+\; \mathsf{F}_{(1)3}\widehat{\mathsf{F}}^{(0)3} \;-\; \mathsf{F}_{(1)}{}^{0}\widehat{\mathsf{F}}^{(0)}{}_{0} \\
&+ \; \mathsf{F}_{(1)0}\widehat{\mathsf{F}}^{(0)0} \;-\; \mathsf{F}_{(1)}{}^{3}\widehat{\mathsf{F}}^{(0)}{}_{3} \;-\; \mathsf{F}_{(1)}{}^{2}\widehat{\mathsf{F}}^{(0)}{}_{2} \;+\; \mathsf{F}_{(1)1}\widehat{\mathsf{F}}^{(0)1}
\end{aligned}
$$

Collecting these into summations over Kähler indices we recover the coefficient of $\widetilde{\nu}^1$ given in (5.2.4) and Table 5.1.

$$
(Q\cdot F)_{12} = \widehat{\mathsf{F}}^{(0)A}\mathsf{F}_{(1)A} - \widehat{\mathsf{F}}^{(0)}{}_{B}\mathsf{F}_{(1)}{}^{B}
$$

Denoting the tadpole contribution due to the D7/O7s wrapping the cycle dual to $\widetilde{\omega}^i$ as $(Q\cdot F)_i$ and defined $N_{7_I} = -32 + N_{D7_I}$, where N_{D7_i} is the total number of D7-branes wrapping the i^{th} 4-cycle dual to the two-torus $\mathbb{T}^2_i$, we can express their components in a number of ways.

$$
\begin{aligned}
(Q\cdot F)_i \;&=\; \widehat{\mathsf{F}}^{(0)0}\mathsf{F}_{(i)0} \;+\; \widehat{\mathsf{F}}^{(0)A}\mathsf{F}_{(i)A} \;-\; \widehat{\mathsf{F}}^{(0)}{}_{0}\mathsf{F}_{(i)}{}^{0} \;-\; \widehat{\mathsf{F}}^{(0)}{}_{b}\mathsf{F}_{(i)}{}^{b} \\
N_{7_i} \;&\equiv\; -\;\mathsf{c}^{(i)}_0\mathsf{a}_3 \;-\; \underline{\underline{\mathsf{c}}}^{(ib)}_2\mathsf{a}^{(b)}_1 \;+\; \underline{\underline{\mathsf{c}}}^{(ia)}_1\mathsf{a}^{(a)}_2 \;+\; \mathsf{c}^{(i)}_3\mathsf{a}_0
\end{aligned}
\qquad (6.4.20)
$$

Going to the isotropic case, this set of conditions reduces to a single expression.

$$
N_{7_i} \;=\; N_7 \;\equiv\; \mathsf{a}_0\mathsf{c}_3 \;+\; \mathsf{a}_1(2\mathsf{c}_2 - \widetilde{c}_2) \;-\; \mathsf{a}_2(2\mathsf{c}_1 - \widetilde{c}_1) \;-\; \mathsf{a}_3\mathsf{c}_0 \qquad (6.4.21)
$$

Under S duality these constraints form one part of the $\mathrm{SL}(2,\mathbb{Z})_S$ triplet $\mathbf{3}_{\mathsf{AB}}$.

$$
\int C_8\wedge(Q\cdot F_3) \quad , \quad \int \widetilde{C}_8\wedge(P\cdot H_3) \quad , \quad \int C'_8\wedge(Q\cdot H_3 + P\cdot F_3)
$$

Their component expansions take the same schematic form as the T duality only case and so can be obtained from (6.4.20) by the appropriate relabelling of the fluxes.

$$\begin{aligned}
(P\cdot H)_i &= \quad \mathsf{F}^{(0)0}\widehat{\mathsf{F}}_{(i)0} + \mathsf{F}^{(0)A}\widehat{\mathsf{F}}_{(i)A} - \mathsf{F}^{(0)}{}_0\widehat{\mathsf{F}}_{(i)}{}^0 - \mathsf{F}^{(0)}{}_b\widehat{\mathsf{F}}_{(i)}{}^b \\
&= - \quad \mathsf{d}_0^{(i)}\mathsf{b}_3 - \underline{\underline{\mathsf{d}}}_2^{(ib)}\mathsf{b}_1^{(b)} + \underline{\underline{\mathsf{d}}}_1^{(ia)}\mathsf{b}_2^{(a)} + \mathsf{d}_3^{(i)}\mathsf{b}_0 \\
(Q\cdot H)_i &= \quad \mathsf{F}^{(0)0}\mathsf{F}_{(i)0} + \mathsf{F}^{(0)A}\mathsf{F}_{(i)A} - \mathsf{F}^{(0)}{}_0\mathsf{F}_{(i)}{}^0 - \mathsf{F}^{(0)}{}_b\mathsf{F}_{(i)}{}^b \\
&= - \quad \mathsf{c}_0^{(i)}\mathsf{b}_3 - \underline{\underline{\mathsf{c}}}_2^{(ib)}\mathsf{b}_1^{(b)} + \underline{\underline{\mathsf{c}}}_1^{(ia)}\mathsf{b}_2^{(a)} + \mathsf{c}_3^{(i)}\mathsf{b}_0 \\
(P\cdot F_3)_i &= \quad \widehat{\mathsf{F}}^{(0)0}\widehat{\mathsf{F}}_{(i)0} + \widehat{\mathsf{F}}^{(0)A}\widehat{\mathsf{F}}_{(i)A} - \widehat{\mathsf{F}}^{(0)}{}_0\widehat{\mathsf{F}}_{(i)}{}^0 - \widehat{\mathsf{F}}^{(0)}{}_b\widehat{\mathsf{F}}_{(i)}{}^b \\
&= - \quad \mathsf{d}_0^{(i)}\mathsf{a}_3 - \underline{\underline{\mathsf{d}}}_2^{(ib)}\mathsf{a}_1^{(b)} + \underline{\underline{\mathsf{d}}}_1^{(ia)}\mathsf{a}_2^{(a)} + \mathsf{d}_3^{(i)}\mathsf{a}_0
\end{aligned}$$

The new 2-form tadpole cancellation conditions for $\widetilde{C}_8$ and C'_8 follow from these expansions.

$$\widetilde{N}_{7_i} = (P\cdot H)_i \quad , \quad N'_{7_i} = (Q\cdot H + P\cdot F)_i \tag{6.4.22}$$

As in the T duality case we have defined $\widetilde{N}_{7_i} = 32 - N_{NS7_i}$ and $N'_{7_i} = 32 - N_{I7_i}$, where N_{NS7_i} and N_{I7_i} are the number of NS7-branes and I7-branes which can also be added to the system, wrapping the i^{th} 4-cycle dual to the two-torus $\mathbb{T}_i^2$. Restricting ourselves to the isotropic fluxes the three expressions for each tadpole become equal and we have only one polynomial per member of the $\mathrm{SL}(2,\mathbb{Z})_S$ tadpole triplet.

$$\begin{aligned}
\widetilde{N}_7 &= \mathsf{b}_0\,\mathsf{d}_3 + \mathsf{b}_1\,(2\,\mathsf{d}_2 - \widetilde{\mathsf{d}}_2) - \mathsf{b}_2\,(2\,\mathsf{d}_1 - \widetilde{\mathsf{d}}_1) - \mathsf{b}_3\,\mathsf{d}_0 \\
N'_7 &= \mathsf{b}_0\,\mathsf{c}_3 + \mathsf{b}_1\,(2\,\mathsf{c}_2 - \widetilde{\mathsf{c}}_2) - \mathsf{b}_2\,(2\,\mathsf{c}_1 - \widetilde{\mathsf{c}}_1) - \mathsf{b}_3\,\mathsf{c}_0 \\
&\quad + \mathsf{a}_0\,\mathsf{d}_3 + \mathsf{a}_1\,(2\,\mathsf{d}_2 - \widetilde{\mathsf{d}}_2) - \mathsf{a}_2\,(2\,\mathsf{d}_1 - \widetilde{\mathsf{d}}_1) - \mathsf{a}_3\,\mathsf{d}_0
\end{aligned} \tag{6.4.23}$$

A further simplification can be made, as noted in [60], which is an important result when considering some of the Bianchi constraints.

$$QH_3 = 0 \Rightarrow Q\cdot H = 0 \qquad PF_3 = 0 \Rightarrow P\cdot F_3 = 0 \tag{6.4.24}$$

6.4.4 3-Form Backgrounds

In the T duality only case the remaining Bianchi constraints, once $QQ = 0$ is solved, are $QH = 0$. When written in terms of the fluxes of Tables 6.7 and 6.8 it is seen that the constraints can be viewed as a non-geometric flux defined linear transformation on the NS-NS flux vector space $(\mathsf{b}_0, \mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3) \equiv \underline{\mathsf{b}}$.

$$\begin{aligned} -\mathsf{c}_2\,\mathsf{b}_0 + (\mathsf{c}_1 - \widetilde{\mathsf{c}}_1)\,\mathsf{b}_1 + \mathsf{c}_0\,\mathsf{b}_2 &= 0 \\ -\mathsf{c}_2\,\mathsf{b}_1 + (\mathsf{c}_1 - \widetilde{\mathsf{c}}_1)\,\mathsf{b}_2 + \mathsf{c}_0\,\mathsf{b}_3 &= 0 \\ -\mathsf{c}_3\,\mathsf{b}_0 - (\mathsf{c}_2 - \widetilde{\mathsf{c}}_2)\,\mathsf{b}_1 + \mathsf{c}_1\,\mathsf{b}_2 &= 0 \\ -\mathsf{c}_3\,\mathsf{b}_1 - (\mathsf{c}_2 - \widetilde{\mathsf{c}}_2)\,\mathsf{b}_2 + \mathsf{c}_1\,\mathsf{b}_3 &= 0 \end{aligned} \tag{6.4.25}$$

The problem is then reduced to computing the 2 dimensional nullspace of this linear system Φ_Q.

$$\begin{pmatrix} -\mathsf{c}_2 & +\mathsf{c}_1 - \widetilde{\mathsf{c}}_1 & +\mathsf{c}_0 & 0 \\ 0 & -\mathsf{c}_2 & +\mathsf{c}_1 - \widetilde{\mathsf{c}}_1 & +\mathsf{c}_0 \\ -\mathsf{c}_3 & -\mathsf{c}_2 + \widetilde{\mathsf{c}}_2 & +\mathsf{c}_1 & 0 \\ 0 & -\mathsf{c}_3 & -\mathsf{c}_2 + \widetilde{\mathsf{c}}_2 & +\mathsf{c}_1 \end{pmatrix} \begin{pmatrix} \mathsf{b}_0 \\ \mathsf{b}_1 \\ \mathsf{b}_2 \\ \mathsf{b}_3 \end{pmatrix} \equiv \Phi_Q \cdot \underline{\mathsf{b}} = 0 \tag{6.4.26}$$

S duality extends QH to the $\mathrm{SL}(2,\mathbb{Z})_S$ singlet $QH - PF$, where each pairing has the same index structure.

$$Q^{ab}_{[c} H_{de]b} = 0 \quad \rightarrow \quad Q^{ab}_{[c} H_{de]b} - P^{ab}_{[c} F_{de]b} = 0 \tag{6.4.27}$$

In terms of individual fluxes PF is obtained by relabelling (6.4.26) by $\mathsf{c} \to \mathsf{d}$ and $\mathsf{b} \to \mathsf{a}$ and $QH - PF$ is the difference of these expressions.

$$\begin{aligned} -\mathsf{c}_2\,b_0 + (\mathsf{c}_1 - \widetilde{c}_1)\,b_1 + \mathsf{c}_0\,b_2 \;+\; \mathsf{d}_2\,a_0 - (\mathsf{d}_1 - \widetilde{d}_1)\,a_1 - \mathsf{d}_0\,a_2 &= 0 \\ -\mathsf{c}_2\,b_1 + (\mathsf{c}_1 - \widetilde{c}_1)\,b_2 + \mathsf{c}_0\,b_3 \;+\; \mathsf{d}_2\,a_1 - (\mathsf{d}_1 - \widetilde{d}_1)\,a_2 - \mathsf{d}_0\,a_3 &= 0 \\ -\mathsf{c}_3\,b_0 - (\mathsf{c}_2 - \widetilde{c}_2)\,b_1 + \mathsf{c}_1\,b_2 \;+\; \mathsf{d}_3\,a_0 + (\mathsf{d}_2 - \widetilde{d}_2)\,a_1 - \mathsf{d}_1\,a_2 &= 0 \\ -\mathsf{c}_3\,b_1 - (\mathsf{c}_2 - \widetilde{c}_2)\,b_2 + \mathsf{c}_1\,b_3 \;+\; \mathsf{d}_3\,a_1 + (\mathsf{d}_2 - \widetilde{d}_2)\,a_2 - \mathsf{d}_1\,a_3 &= 0 \end{aligned} \tag{6.4.28}$$

These constraints can be rewritten in terms of matrices and flux vectors in the same way as the T duality only case.

$$\Phi_Q \cdot \underline{\mathsf{b}} - \Phi_P \cdot \underline{\mathsf{a}} = 0 \quad \Rightarrow \quad (\Phi_Q)_i^{\ j} \mathsf{b}_j = (\Phi_P)_i^{\ j} \mathsf{a}_j \tag{6.4.29}$$

In the case of the T duality only non-geometric fluxes we noted in Table 6.14 that the polynomial contribution to the superpotental takes a particularly simple form if we redefine our complex structure moduli $U \to \mathcal{Z}$ and factorise out $(\gamma U+\delta)^3$. S duality results in two different modular parameter dependent complex structure moduli, which define $\mathcal{P}_3$ and $\mathcal{P}_4$, and the same is true for the H and F induced polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$.

$$\begin{aligned} P_2(U) &= \mathsf{b}_i U^i = (\gamma_q U + \delta_q)^3 \, \mathcal{P}_2(\mathcal{Z}_Q) = \epsilon_i \, \mathcal{Z}_Q^i \\ P_1(U) &= \mathsf{a}_i U^i = (\gamma_p U + \delta_p)^3 \, \mathcal{P}_1(\mathcal{Z}_P) = \rho_j \, \mathcal{Z}_P^j \end{aligned} \tag{6.4.30}$$

The flux vectors $\underline{\mathsf{a}}$ and $\underline{\mathsf{b}}$ are related to $\underline{\epsilon}$ and $\underline{\rho}$ by these reformulations, which can be represented as linear transformations.

$$\begin{pmatrix} \mathsf{b}_0 \\ \mathsf{b}_1 \\ \mathsf{b}_2 \\ \mathsf{b}_3 \end{pmatrix} = \begin{pmatrix} -\beta_q^3 & -\beta_q\delta_q^2 & -\beta_q^2\delta_q & -\delta_q^3 \\ \alpha_q\beta_q^2 & \frac{1}{3}\delta_q\left(2\beta_q\gamma_q+\alpha_q\delta_q\right) & \frac{1}{3}\beta_q\left(\beta_q\gamma_q+2\alpha_q\delta_q\right) & \gamma_q\delta_q^2 \\ -\alpha_q^2\beta_q & -\frac{1}{3}\gamma_q\left(\beta_q\gamma_q+2\alpha_q\delta_q\right) & -\frac{1}{3}\alpha_q\left(2\beta_q\gamma_q+\alpha_q\delta_q\right) & -\gamma_q^2\delta_q \\ \alpha_q^3 & \alpha_q\gamma_q^2 & \alpha_q^2\gamma_q & \gamma_q^3 \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

We shall schematically denote this as $\underline{\mathsf{b}} = \mathsf{P}_{\mathsf{b}} \cdot \underline{\epsilon}$. The equivalent transformation for the R-R flux F $\underline{\mathsf{a}} = \mathsf{P}_{\mathsf{a}} \cdot \underline{\rho}$ is obtained by replacing the subscript $q \to p$ and $\epsilon_i \to \rho_i$. These parameterisations are well defined because their Jacobians have determinants $-|\Gamma_Q|^6/9$ and $-|\Gamma_P|^6/9$ so they never vanish, provided the isomorphisms used for bringing non-geometric fluxes to their canonical form are not singular. These transformations alter (6.4.29) from depending on a_i and b_j.

$$\begin{aligned} (\Phi_Q)_i^{\ j}\mathsf{b}_j - (\Phi_P)_i^{\ j}\mathsf{a}_j &= (\Phi_Q)_i^{\ j}(\mathsf{P}_\mathsf{b})_j^{\ k}\epsilon_k - (\Phi_P)_i^{\ j}(\mathsf{P}_\mathsf{a})_j^{\ k}\rho_k \\ &= (\tilde{\Phi}_Q)_i^{\ k}\epsilon_k - (\tilde{\Phi}_P)_i^{\ k}\rho_k \end{aligned} \tag{6.4.31}$$

Both $\tilde{\Phi}_Q$ and $\tilde{\Phi}_P$ are linear transformations and therefore the solutions space of (6.4.31) can be obtained from the intersection of their images.

$$\mathfrak{I}_{QP} \equiv \mathfrak{Im}(\tilde{\Phi}_Q) \cap \mathfrak{Im}(\tilde{\Phi}_P) \tag{6.4.32}$$

The parameters ϵ_i and ρ_j belong to the $\tilde{\Phi}_Q$ and $\tilde{\Phi}_P$ antimages of $\mathfrak{I}_{QP}$ respectively.

$$\underline{\epsilon} \in \tilde{\Phi}_Q^{-1}(\mathfrak{I}_{QP}) \quad , \quad \underline{\rho} \in \tilde{\Phi}_P^{-1}(\mathfrak{I}_{QP}) \tag{6.4.33}$$

Therefore we denote a geometric background for the H and F fluxes solving (6.4.31), by a pair of vectors $(\underline{\epsilon}, \underline{\rho})$ satisfying (6.4.33). The main features of this background, such as its dimension or its flux-induced C_8' tadpole, are severely restricted by the non-geometric background we have previously imposed. Furthermore, we are able to distinguish between two non-geometric flux setups by seeing whether or not $\mathfrak{I}_{QP}$ becomes trivial.

- Non-geometric type A setup:

$$\mathfrak{I}_{QP} = \{\mathbf{0}\} \ , \tag{6.4.34}$$

  A non-geometric background satisfying this fixes the geometric background to be $\underline{\epsilon} \in \ker(\tilde{\Phi}_Q)$ $(QH = 0)$ and $\underline{\rho} \in \ker(\tilde{\Phi}_P)$ $(PF_3 = 0)$. These constraints can be viewed as the pure T duality constraint and its the modular inversion $S \to -\frac{1}{S}$ image. It has dimension 4 and does not generate a flux-induced C_8' tadpole due to (6.4.24).

$$N_7' = 0 \qquad (\text{type A}). \tag{6.4.35}$$

- Non-geometric type B setup:

$$\mathfrak{I}_{QP} \neq \{\mathbf{0}\} \ , \tag{6.4.36}$$

  A non-geometric background satisfying results in a less restricted geometric one, of dimension 6, that can generate a flux-induced C_8' tadpole. This can always be written as

$$N_7' = \Delta_Q \ |\Gamma_Q|^3 + \Delta_P \ |\Gamma_P|^3 \qquad (\text{type B}) \tag{6.4.37}$$

  with Δ_Q and Δ_P depending on ϵ_i and ρ_i respectively[4] and vanishing in the special case of $\underline{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\underline{\rho} \in \ker(\tilde{\Phi}_P)$.

[4] $\ker(\tilde{\Phi}_Q)$, $\ker(\tilde{\Phi}_P)$, Δ_Q and Δ_P differ for each pairing $(\mathfrak{g}_Q, \mathfrak{g}_P)$, being easily computed in each case.

$\mathfrak{g}_Q$ original	$\mathfrak{g}_P$ deformation				
	$\mathfrak{so}(4)$	$\mathfrak{so}(3,1)$	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{iso}(3)$	$\mathfrak{nil}$
$\mathfrak{so}(4)$	$\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{-1}$	$\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{0}$	$\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{\infty}$	$\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{+1}$	$\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{\infty}$
$\mathfrak{so}(3,1)$	$\mathcal{Z}^Q_{0} \parallel \mathcal{Z}^P_{-1}$	$\mathcal{Z}^Q_{0} \parallel \mathcal{Z}^P_{0}$	$\mathcal{Z}^Q_{0} \parallel \mathcal{Z}^P_{\infty}$	$\mathcal{Z}^Q_{0} \parallel \mathcal{Z}^P_{+1}$	$\mathcal{Z}^Q_{0} \parallel \mathcal{Z}^P_{\infty}$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{-1}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{0}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{\infty}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{+1}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{\infty}$
$\mathfrak{iso}(3)$	$\mathcal{Z}^Q_{+1} \parallel \mathcal{Z}^P_{-1}$	$\mathcal{Z}^Q_{+1} \parallel \mathcal{Z}^P_{0}$	$\mathcal{Z}^Q_{+1} \parallel \mathcal{Z}^P_{\infty}$	$\mathcal{Z}^Q_{+1} \parallel \mathcal{Z}^P_{+1}$	$\mathcal{Z}^Q_{+1} \parallel \mathcal{Z}^P_{\infty}$
$\mathfrak{nil}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{-1}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{0}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{\infty}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{+1}$	$\mathcal{Z}^Q_{\infty} \parallel \mathcal{Z}^P_{\infty}$

Table 6.16: Roots alignment in non-geometric type B setups.

Whether a background is Type A or Type B is determined entirely by the choice of root alignment for the solution to the cohomology deformation conditions of the non-geometric fluxes stated in Table 6.15. Those root alignments shown in Table 6.16 are those which lead to Type B backgrounds, otherwise it is type A. To illustrate this we consider an example where $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$ and $\mathfrak{g}_P = \mathfrak{so}(4)$. Solving the cohomology condition through the $\mathcal{Z}^Q_{\infty} || \mathcal{Z}^P_{-1}$ branch of Table 6.15 leaves us with a non-geometric type B setup. The $\ker(\tilde{\Phi}_Q)$ is expanded by (ϵ_0, ϵ_3) while that of $\tilde{\Phi}(\mathfrak{g}_P)$ is expanded by (ρ_0, ρ_3) for this pairing. In this case, the NS-NS and R-R fluxes account for six degrees of freedom and generate a flux-induced C_8' tadpole given by (6.4.37) with $\Delta_Q = \epsilon_2/3$ and $\Delta_P = (\rho_2 - \rho_1)/3$.

The geometric H_3 and F_3 backgrounds determine the flux-induced $\mathcal{P}_2(\mathcal{Z}_Q)$ and $\mathcal{P}_1(\mathcal{Z}_P)$ polynomials in the superpotential. Fixing a non-geometric type A setup, $\mathcal{P}_2(\mathcal{Z}_Q)$ is shown in Table 6.17 for each $\mathfrak{g}_Q$ algebra. The equivalent expression for the polynomial $\mathcal{P}_1(\mathcal{Z}_P)$, resulting from the $\mathfrak{g}_P$ algebra, is obtained upon replacing $\epsilon_i \leftrightarrow \rho_i$ and $\mathcal{Z}_Q \leftrightarrow \mathcal{Z}_P$. This is also a solution (the simplest one) in a non-geometric type B setup for which there is no flux-induced C_8' tadpole. However, a more complicated geometric background can be switched on, in which a tadpole is generated.

$\mathfrak{g}_Q$	$\mathcal{P}_2(\mathcal{Z}_Q) \equiv \frac{P_2(U)}{(\gamma_q\, U+\delta_q)^3}$
$\mathfrak{so}(4)$	$\epsilon_3\, \mathcal{Z}_Q^3 + \epsilon_0$
$\mathfrak{so}(3,1)$	$\epsilon_3\, \mathcal{Z}_Q^3 - 3\,\epsilon_0\, \mathcal{Z}_Q^2 - 3\,\epsilon_3\, \mathcal{Z}_Q + \epsilon_0$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\epsilon_3\, \mathcal{Z}_Q^3 + \epsilon_0$
$\mathfrak{iso}(3)$	$\epsilon_1\, \mathcal{Z}_Q + \epsilon_0$
$\mathfrak{nil}$	$\epsilon_1\, \mathcal{Z}_Q + \epsilon_0$

Table 6.17: NS-NS flux-induced polynomials in the non-geometric type A setup.

6.5 Supersymmetric Solutions

Having constructed parameterisations for the Type IIB/O3 superpotential's individual polynomials we now consider the construction of specific vacua for the isotropic $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold where all three moduli types are stablised.

6.5.1 Analytic Methods

Using the results of the previous section we can write down the most general superpotential for $\mathcal{M}_{\mathbb{Z}_2^2}$.

$$\begin{aligned} W &= -(\gamma_p\, U + \delta_p)^3 \left[\Big(\sum_{i=0}^{3} \rho_i\, \mathcal{Z}_P^i \Big) + 3\, T\, S\, \mathcal{P}_4(\mathcal{Z}_P) \right] + \\ &+ \; (\gamma_q\, U + \delta_q)^3 \left[S \Big(\sum_{i=0}^{3} \epsilon_i\, \mathcal{Z}_Q^i \Big) + 3\, T\, \mathcal{P}_3(\mathcal{Z}_Q) \right] \end{aligned} \tag{6.5.1}$$

The specific forms $\mathcal{P}_3(\mathcal{Z}_Q)$, $\mathcal{P}_4(\mathcal{Z}_P)$ are taken from Table 6.14 according with a fixed pairing $(\mathfrak{g}_Q, \mathfrak{g}_P)$ and $\mathcal{Z}_Q$ and $\mathcal{Z}_P$ are the modular variables from (6.4.15). In general, $\mathcal{Z}_Q \neq \mathcal{Z}_P$, and we will have to deal with two modular variables instead of just one, $\mathcal{Z}$. Each pairing $(\mathfrak{g}_Q, \mathfrak{g}_P)$ gives rise to a specific superpotential due to the relationship between the root structure of a polynomial and its associated algebra. For simplicity we shall consider supersymmetric vacua. A supersymmetric vacuum implies the vanishing of the F-terms, which are the Kähler derivatives of the superpotential for each moduli.

$$\begin{aligned} F_T &= \partial_T W + \frac{3iW}{2\,\mathrm{Im}(T)} = 0 \\ F_S &= \partial_S W + \frac{iW}{2\,\mathrm{Im}(S)} = 0 \\ F_U &= \partial_U W + \frac{3iW}{2\,\mathrm{Im}(U)} = 0 \end{aligned} \tag{6.5.2}$$

The vanishing of the Kähler derivatives results in either Minkowski or AdS_4 solutions because the potential (2.4.1) at the minimum is given by $V_0 = -3e^{K_0}|W_0|^2 \leq 0$. Restricting our search to Minkowski solutions, i.e. $V_0 = 0$, simplifies the F Flat conditions further.

$$\partial_S W = \partial_T W = \partial_U W = W = 0 \tag{6.5.3}$$

The fact that the general expression for the superpotential is linear in S and T allows us to fix their values generically.

$$\begin{aligned} S_0 &= -\frac{P_3(U_0)}{P_4(U_0)} = \left(\frac{\gamma_q\, U + \delta_q}{\gamma_p\, U + \delta_p}\right)^3 \left.\frac{\mathcal{P}_3(\mathcal{Z}_Q)}{\mathcal{P}_4(\mathcal{Z}_P)}\right|_{U_0} \;, \\ T_0 &= -\frac{P_2(U_0)}{P_4(U_0)} = \left(\frac{\gamma_q\, U + \delta_q}{\gamma_p\, U + \delta_p}\right)^3 \left.\frac{\sum_{i=0}^{3} \epsilon_i\, \mathcal{Z}_Q^i}{\mathcal{P}_4(\mathcal{Z}_P)}\right|_{U_0} \;, \end{aligned} \tag{6.5.4}$$

We have denoted the VEVs of the moduli with a subscript 0, $\langle S\rangle = S_0$, etc. These values are subject to physical considerations;

- $\mathrm{Im}(S_0)$ must be positive because it is the inverse of the string coupling constant g_s.

- $\mathrm{Im}(T_0) = e^{-\phi}A$ where A is the area of a 2-dimensional subtorus, so it also has to be positive.

- For the modular variables $\mathcal{Z}_Q$ and $\mathcal{Z}_P$ at the minimum, it happens that $\mathrm{Im}(\mathcal{Z}_Q) = \mathrm{Im}(U_0)\frac{|\Gamma_Q|}{|\gamma_q U_0+\delta_q|^2}$ and $\mathrm{Im}(\mathcal{Z}_P) = \mathrm{Im}(U_0)\frac{|\Gamma_P|}{|\gamma_p U_0+\delta_p|^2}$. Therefore, necessarily $\mathrm{Im}(\mathcal{Z}_Q) \neq 0$ and $\mathrm{Im}(\mathcal{Z}_P) \neq 0$ because for $\mathrm{Im}(U_0) = 0$ the internal space is degenerate. Without loss of generality, we choose $\mathrm{Im}(U_0) > 0$.

- For the effective supergravity to be a reliable approximation to string theory, $g_s = \frac{1}{\mathrm{Im}(S_0)}$ has to be small to exclude non-perturbative string effects and large internal volume $V_{int} = \left(\frac{\mathrm{Im}(T_0)}{\mathrm{Im}(S_0)}\right)^{3/2}$ is also required to neglect corrections in α'.

The remaining $W = 0$ and $\partial_U W = 0$ conditions can be rewritten, using (6.5.4), as a pair of equations, provided[5] $P_4(U_0) \neq 0$.

$$\begin{aligned} E(U_0) \ &= \ P_1(U_0)\, P_4(U_0) - P_2(U_0)\, P_3(U_0) \ = \ 0 \\ E'(U_0) \ & \qquad\qquad\qquad\qquad\qquad\quad\ = \ 0 \end{aligned}$$

The prime denotes differentiation with respect to U and, therefore, $E(U)$ has a double root. The root must, given our definition for the Kähler potential, be complex and therefore $E(U)$ contains a double copy of complex conjugate pairs, accounting for 4 of its 6 roots. Therefore, we have the following factorisation property of $E(U)$, with $\tilde{E}(U) \equiv (g_2\, U^2 + g_1\, U + g_0)^2$ accounting for the double root that becomes complex iff $g_1^2 - 4\, g_2\, g_0 < 0$.

$$E(U) = (f_2\, U^2 + f_1\, U + f_0)\, \tilde{E}(U) \tag{6.5.5}$$

Information about the nature of the six roots of $E(U)$ can be immediately obtained from the generic superpotential polynomials once a $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairing is chosen and the full set of Bianchi identities, ie. integrability, cohomology and singlet Bianchi constraints, are applied. Four cases are automatically discarded because their $E(U)$ possesses at least four real roots, so they can never have a double complex root for the Minkowski vacua to be physically viable, i.e. $\text{Im}(U_0) \neq 0$. The number of real roots for each $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairing is summarized[6] in Table 6.18. A priori, all branches with $E(U)$ having a number of real roots less than three could accommodate supersymmetric Minkowski solutions. This is a necessary but not sufficient condition for the existence of Minkowski vacua because for $E(U)$ to split into the form (6.5.5), additional constraints on H and F_3 fluxes are needed. Therefore, several branches in Table 6.18 will exclude Minkowski vacua, even

[5] This has to be the case for $\text{Im}(U_0) \neq 0$ in all $\mathfrak{g}_P$ but $\mathfrak{g}_P = \mathfrak{so}(3,1)$ that has complex roots $\mathcal{Z}_P = \pm i$. For this singular case, $P_4(U_0) = 0$ implies $P_i(U_0) = 0$ for $i = 1, 2, 3, 4$ as can be seen from (6.5.3). Then S and T can not be simultaneously stabilized in a supersymmetric Minkowski vacuum.

[6] Entries in Table 6.18 are in one to one correspondence with entries in Table 6.15.

though they have a sufficient number of complex roots and we will provide an example of this. Despite this, several results can be read from Table 6.18 :

- There are no supersymmetric Minkowski solutions in the $(\mathfrak{nil}, \mathfrak{nil})$ case because all $E(U)$ roots become real for this pairing.

- For supersymmetric Minkowski solutions to exist in the pairings of non-geometric algebras $(\mathfrak{iso}(3), \mathfrak{iso}(3))$, $(\mathfrak{iso}(3), \mathfrak{nil})$ and $(\mathfrak{nil}, \mathfrak{iso}(3))$, it is necessary to have non-geometric type B setups (see Table 6.16), generating an eventually non vanishing flux-induced C_8' tadpole.

- The rest of the pairings are richer and supersymmetric Minkowski solutions could, in principle, exist in all branches that solve the cohomology condition (see Table 6.15).

6.5.2 Example $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ and $\mathfrak{so}(4)$ Vacua

For our first example, we shall continue to investigate the case $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$ deformed by $\mathfrak{g}_P = \mathfrak{so}(4)$, in order to show how simple supersymmetric solutions can be easily obtained using these methods. For the sake of simplicity, we will look for H and F_3 fluxes backgrounds with $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$, so $N_7' = 0$ but the net charges N_7 and $\tilde{N}_7$ are considered as free variables. In these solutions, $\mathcal{P}_2(\mathcal{Z}_Q)$ and $\mathcal{P}_1(\mathcal{Z}_P)$ can be obtained from Table 6.17 leaving us with a set $(\epsilon_0, \epsilon_3\,;\,\rho_0, \rho_3)$ of free parameters in the superpotential defining the geometric H and F_3 background fluxes. Taking the relevant polynomials from Tables 6.14 and 6.17 we can construct the generic superpotential.

$$\begin{aligned} W \;&=\; -(\gamma_p\, U + \delta_p)^3 \left[(\rho_3\, \mathcal{Z}_P^3 + \rho_0) + 3\, T\, S\, \mathcal{Z}_P(\mathcal{Z}_P + 1)\right] + \\ &+\; (\gamma_q\, U + \delta_q)^3 \left[S\, (\epsilon_3\, \mathcal{Z}_Q^3 + \epsilon_0) + 3\, T\, \mathcal{Z}_Q\right] \end{aligned} \tag{6.5.6}$$

$\mathfrak{g}_Q$ original	$\mathfrak{g}_P$ deformation $\mathfrak{so}(4)$	$\mathfrak{so}(3,1)$	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{iso}(3)$	$\mathfrak{nil}$
$\mathfrak{so}(4)$	2 2 1	1	2 2 1	1	1
$\mathfrak{so}(3,1)$	1	2 2 1	1	1	1
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	2 2 1	1	2 2	1	2
$\mathfrak{iso}(3)$	1	1	1	**4** 1	**4** 1
$\mathfrak{nil}$	1	1	2	**4** 1	**6**

Table 6.18: Number of real roots of $E(U)$ defined in Table 6.15 after imposing the full set of Bianchi constraints.

The tadpole cancellation conditions can be expressed in terms of the roots and $A_{ij} = -\rho_i\,\epsilon_j$.

$$\begin{aligned}
N_3 &= +A_{33}\,(\mathcal{Z}_0^Q \times \mathcal{Z}_0^P)^3 + A_{30}\,(\mathcal{Z}_\infty^Q \times \mathcal{Z}_0^P)^3 \\
&\quad +A_{03}\,(\mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P)^3 + A_{00}\,(\mathcal{Z}_\infty^Q \times \mathcal{Z}_\infty^P)^3 \\
N_7 &= \rho_3\,(\mathcal{Z}_0^Q \times \mathcal{Z}_0^P)\,(\mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P)^2 + \rho_0\,(\mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P)\,(\mathcal{Z}_\infty^Q \times \mathcal{Z}_\infty^P)^2 \qquad (6.5.7) \\
\tilde{N}_7 &= -\epsilon_3\,(\mathcal{Z}_0^Q \times \mathcal{Z}_0^P)\,(\mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P)\,(\mathcal{Z}_0^Q \times \mathcal{Z}_{-1}^P) \\
&\quad -\epsilon_0\,(\mathcal{Z}_\infty^Q \times \mathcal{Z}_0^P)\,(\mathcal{Z}_\infty^Q \times \mathcal{Z}_\infty^P)\,(\mathcal{Z}_\infty^Q \times \mathcal{Z}_{-1}^P)
\end{aligned}$$

We now impose the constraints from one of the prime ideals of the cohomology condition, of which there are three to choose for this pairing, as shown in Table 6.15 and explicitly stated in (6.4.19). The case $J_1 = 0$ is automatically fulfilled with an embedding $\Gamma_P = \Gamma_Q \equiv \Gamma$, or equivalently $\mathcal{Z}_P = \mathcal{Z}_Q \equiv \mathcal{Z}$, while the $J_2 = 0$ results are equivalent to this after applying a T-duality induced modular transformation $\mathcal{Z} \to -1/\mathcal{Z}$. The case $J_3 = 0$ is a little bit different from the previous ones, it cannot be transformed into $J_{1,2} = 0$ and so the resultant solutions are distinct from those of the first two branches. We will solve for each of the three branches and clarify their relation to the existence of both AdS_4 and Minkowski vacua.

Simple type A AdS_4 solutions

Imposing $J_1 = 0$, we fix the modular embeddings to be equal to one another.

$$\Gamma_P = \Gamma_Q \equiv \Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} , \qquad (6.5.8)$$

The fact the modular parameters of each non-geometric flux are the same allows for the superpotential to be written entirely in terms of the modular complex structure parameter $\mathcal{Z}$.

$$\frac{W}{(\gamma\,U + \delta)^3} = -(\rho_3\,\mathcal{Z}^3 + \rho_0) + S\,(\epsilon_3\,\mathcal{Z}^3 + \epsilon_0) + 3\,T\,\mathcal{Z} - 3\,T\,S\,\mathcal{Z}\,(\mathcal{Z} + 1)$$

This overall factor can be removed by a corresponding transformation in the Kähler potential. The superpotential is now a function of $\mathcal{Z}$ and by replacing $U \to \mathcal{Z}$ in the Kähler potential we obtain the same Kähler functional $\mathcal{G} = K + \ln |W|^2 = \mathcal{K} + \ln |\mathcal{W}|^2$.

$$
\begin{aligned}
\mathcal{K} &= -3 \ln\Big(-i(T-\bar{T})\Big) - \ln\Big(-i(S-\bar{S})\Big) - 3 \ln\Big(-i(\mathcal{Z}-\bar{\mathcal{Z}})\Big) \\
\mathcal{W} &= |\Gamma|^{3/2}\left[-(\rho_3 \mathcal{Z}^3 + \rho_0) + S\,(\epsilon_3 \mathcal{Z}^3 + \epsilon_0) + 3T\mathcal{Z} - 3TS\mathcal{Z}\,(\mathcal{Z}+1)\right]
\end{aligned}
$$

The tadpole cancellation conditions (6.5.7) simplify when written in terms of the ϵ_i and ρ_j.

$$
\begin{aligned}
N_3 &= |\Gamma|^3 (A_{03} - A_{30}) \;=\; |\Gamma|^3(\epsilon_0\,\rho_3 - \epsilon_3\,\rho_0) \\
N_7 &= \tilde{N}_7 = 0
\end{aligned}
\tag{6.5.9}
$$

It is worth noting that, by simply imposing the embedding (6.5.8), it becomes impossible to have non-geometric type B solutions, as we can see from Table 6.16. The alignment $\mathcal{Z}_\infty || \mathcal{Z}_{-1}$ results in $|\Gamma| = 0$ and the isomorphism is no longer valid. As a result, whenever we impose (6.5.8), automatically $\epsilon_1 = \epsilon_2 = \rho_1 = \rho_2 = 0$ and then $N_7' = N_7 = \tilde{N}_7 = 0$. It can also be proven that this system does not possess Minkowski vacua. To do this, we compute restrictions on H and F_3 fluxes needed for the polynomial $E(U)$ to be factorized as (6.5.5). From Table 6.18 we know that $E(U)$ has at least two real roots. Factorising out and dropping these real roots, $E(U) \to \tilde{E}(U)$, it can be shown that for $\tilde{E}(U)$ to possess a double complex root, the H and F_3 background fluxes must satisfy a pair of equations.

$$
\begin{aligned}
8\,\epsilon_0\,\rho_3 + (\epsilon_3 - 9\,\rho_3)\,\rho_0 &= 0 \\
(\epsilon_3 - \rho_3)^3 - 8\,\rho_3^2\,\rho_0 &= 0
\end{aligned}
$$

For complex roots we require $g_1^2 - 4\,g_2\,g_0 = 12\left(\rho_3^{1/3}\,\rho_0^{1/3}\right)^2 \geq 0$, fixing all six roots of $E(U)$ to be real and producing non physical vacua, i.e. $\mathrm{Im}(U_0) = 0$. However, we find that supersymmetric AdS$_4$ vacua can exist without introducing localized sources. This result is only possible by the inclusion of S duality. To illustrate this we fix $\epsilon_3 = \rho_3 = 0$ so as to have $N_3 = 0$ and set $\rho_0 = 2\,\epsilon_0$. Solving the F-flat conditions (6.5.2) we obtain stablised moduli for each moduli type.

$$\begin{aligned} \mathcal{Z}_0 &= -1.0434 + 0.4758\,i \\ S_0 &= -2.3802 + 4.1685\,i \\ \epsilon_0^{-1}\,T_0 &= -0.4022 + 1.1483\,i \end{aligned}$$

The moduli values and our choices of the fluxes determine the vacuum energy $V_0\,\epsilon_0/|\Gamma|^3 = -2.3958$ and without local sources, $N_3 = N_7 = \tilde{N}_7 = N_7' = 0$. These are not the physical moduli, we must convert $\mathcal{Z}$ into the original complex structure U by $U_0 = \Gamma^{-1}\,\mathcal{Z}_0$ with Γ the modular matrix given in (6.5.8). Restricting ourselves to the particular case of $\beta = \gamma = 0$, this solution corresponds to $\mathsf{a}_0 = -2\,\epsilon_0\,\delta^3$, $\mathsf{b}_0 = -\epsilon_0\,\delta^3$, $\widetilde{\mathsf{c}}_1 = \widetilde{\mathsf{d}}_1 = -\alpha\,\delta^2$ and $\widetilde{\mathsf{d}}_2 = \alpha^2\,\delta$. Large positive values of the ϵ_0 parameter translate into large geometric fluxes and reduce the effects of corrections in α'.

Simple type B Minkowski solutions

Now we explore the case $J_3 = 0$, or equivalently $\mathcal{Z}_\infty^Q || \mathcal{Z}_{-1}^P$. We cannot make the same choice of modular parameters but we can choose a particular case where the Γ matrices are different but dependent on the same parameters.

$$\Gamma_Q = \begin{pmatrix} \alpha & -\delta \\ \alpha & \delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$$

This results in a two dimensional family of non-geometric type B fluxes and substituting directly in (6.5.4) we obtain algebraic expressions for two of the moduli.

$$T_0 = \frac{1}{3\,\alpha\,\delta}\,\frac{\epsilon_3\,(\alpha U_0 - \delta)^3 + \epsilon_0\,(\delta + \alpha U_0)^3}{U_0\,(\delta + \alpha U_0)} \quad , \quad S_0 = \frac{\alpha\,U_0}{\delta} - \frac{\delta}{\alpha U_0} \qquad (6.5.10)$$

These parameters are further restricted by the requirement that the NS-NS H and R-R F_3 backgrounds lead to polynomial $E(U)$ being factorizable as (6.5.5). From Table 6.18, this $E(U)$ has at least one real root. Factorising out this real root,

$E(U) \to (f_1\, U + f_0)\, \tilde{E}(U)$, this imposes a series of restrictions on the modular fluxes.

$$\rho_0 = 0 \quad , \quad \epsilon_0 = -\epsilon_3 = \frac{\rho_3}{8} \quad , \quad f_1 = g_1 = 0 \quad , \quad \frac{g_0}{g_2} = \left(\frac{\delta}{\alpha}\right)^2$$

These automatically satisfy $g_1^2 - 4\, g_2\, g_0 < 0$, producing physical vacua $U_0 = i\left(\frac{\delta}{\alpha}\right)$. Substituting directly in (6.5.10), the moduli get stabilized.

$$U_0 = \left(\frac{\delta}{\alpha}\right) i \qquad , \qquad S_0 = 2\, i \qquad , \qquad T_0 = \frac{\rho_3}{12}\,(1+i)$$

This family is physical for $\rho_3 > 0$ and $|\Gamma_P| > 0$. These, together with Minkowski conditions such that $|\Gamma_P| = \alpha\, \delta$, determine the contributions from local sources to be positive, $N_3 > 0$, $N_7 > 0$ and $\tilde{N}_7 > 0$.

$$N_3 = \frac{\rho_3}{4} \quad , \quad N_7 = \rho_3 \quad , \quad \tilde{N}_7 = |\Gamma_P|^3\, \frac{\rho_3^2}{4}$$

In terms of the original fluxes, this solution corresponds to $\mathsf{c}_3 = -\alpha^3$, $\mathsf{c}_2 = \widetilde{\mathsf{c}}_2 = -\widetilde{\mathsf{d}}_2 = -\alpha^2\, \delta$, $\mathsf{c}_1 = \widetilde{\mathsf{c}}_1 = \widetilde{\mathsf{d}}_1 = -\alpha\, \delta^2$ and $\mathsf{c}_0 = -\delta^3$ for non-geometric fluxes; $\mathsf{b}_0 = -\delta^3\, \frac{\rho_3}{4}$ and $\mathsf{b}_2 = -\alpha^2\, \delta\, \frac{\rho_3}{4}$ for the NS-NS flux; and $\mathsf{a}_3 = \alpha^3\, \rho_3$ for the R-R flux. Again, large values of the ρ_3 parameter translate into large geometric fluxes and reduce the effects of corrections in α'. However, this also increases the number of localized sources and therefore their backreaction, which we are not taking into account.

6.5.3 General Type B Minkowski Vacua

In our previous example, we gave simple Minkowski solutions with all moduli stabilized in a physical vacuum with a vanishing flux-induced C_8' tadpole, ie. $N_7' = 0$. Now, we provide Minkowski solutions with $N_7' \neq 0$. Our main goal in this work has been to develop a systematic method to compute supersymmetric Minkowski vacua based on different $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairings which fulfil all algebraic constraints. To show how these methods work, we conclude by presenting several simple non-geometric type B examples involving all the six dimensional Lie algebras compatible with the orbifold symmetries. Besides finding analytic VEVs for the moduli, we also relate them to the net charge of localized sources which can exist, as well as some features of such vacua.

1: Vacua with unstabilized complex structure modulus

We wish to construct a simple family of Minkowski solutions with a vanishing flux-induced C_8' tadpole for which all the moduli but the complex structure modulus are fixed by the fluxes. These solutions were previously found in [91] and we now clarify their flux structure. We choose to fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(4)$ and $\mathfrak{g}_P = \mathfrak{iso}(3)$ respectively[7] under modular embeddings of a particular kind.

$$\Gamma_Q = \begin{pmatrix} \alpha_q & 0 \\ 0 & \delta_q \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha_p & \beta_p \\ 0 & \delta_p \end{pmatrix} \, . \tag{6.5.11}$$

Our choices of $(\mathfrak{g}_Q, \mathfrak{g}_P)$ have a unique cohomology condition branch $\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{+1}$ in Table 6.15, making it a type B setup, and which is satisfied if the modular matrices (6.5.11) are such that $\alpha_q = \lambda\, \alpha_p$ and $\delta_q = \lambda\, (\beta_p - \delta_p)$. Taking for simplicity $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$ results in $\epsilon_1 = \epsilon_2 = \rho_2 = \rho_3 = 0$. Moreover, we will also fix $\epsilon_3 = 0$ and therefore, substituting into (6.5.4), we obtain analytic expressions for S and T.

$$S_0 = -\lambda^3 \left(\frac{\alpha_p}{\delta_p^2}\right) (\beta_p - \delta_p)\, U_0 \quad , \quad T_0 = -\frac{\lambda^3\, \epsilon_0\, (\beta_p - \delta_p)^3}{3\, \delta_p^2\, (\alpha_p\, U_0 + (\beta_p - \delta_p))} \tag{6.5.12}$$

Upon substituting these moduli VEVs into the superpotential we have a superpotential with linear complex structure dependence.

$$W(U_0) = -\left(\frac{\alpha_p}{\delta_p^2}\right) \left(\lambda^6\, (\beta_p - \delta_p)^4\, \epsilon_0 + \delta_p^4\, \rho_1\right) U_0 - \delta_p^2 \left(\delta_p\, \rho_0 + \beta_p\, \rho_1\right) \tag{6.5.13}$$

[7] In this case, $\Delta_Q = (\epsilon_2 - \epsilon_1)/3$ and $\Delta_P = -\rho_3 - \rho_2/3$.

For Minkowski solutions to exist $\partial_U W = W = 0$. Moreover, because of $\alpha_p \, \delta_p \neq 0$, else $|\Gamma_P| = 0$, Minkowski vacua with complex structure modulus unstabilized do exist provided we satisfy a pair of equations.

$$\begin{aligned} \lambda^6 \, (\beta_p - \delta_p)^4 \, \epsilon_0 + \delta_p^4 \, \rho_1 &= 0 \\ \delta_p \, \rho_0 + \beta_p \, \rho_1 &= 0 \end{aligned}$$

Under these restrictions for ρ_0 and ρ_1, the tadpole cancellation conditions simplify to having only one non-zero set of local sources.

$$\begin{aligned} N_3 &= \tilde{N}_7 = N_7' = 0 \\ N_7 &= \frac{\lambda^9 \, \epsilon_0}{3} \left(\frac{\alpha_p^3}{\delta_p^2} \right) (\beta_p - \delta_p)^5 \end{aligned}$$

From the S and T stabilization (6.5.12), taking a physical vacuum with $\mathrm{Im}(U_0) > 0$ implies a pair of inequality on the fluxes for $\mathrm{Im}(S_0) > 0$ and $\mathrm{Im}(T_0) > 0$.

$$\begin{aligned} \lambda \, \alpha_p \, (\beta_p - \delta_p) &< 0 \\ \lambda \, \alpha_p \, (\beta_p - \delta_p) \, \epsilon_0 &> 0 \end{aligned}$$

It therefore follows that $\epsilon_0 < 0$ else the vacuum is not physical. The sign determinations require $N_7 > 0$ and so D7-branes are needed, several of which were presented in [91]. Large values of $|\lambda|$ and $|\epsilon_0|$ favour the SUGRA approximation, ie. $g_s \propto 1/|\lambda|^3$ and $V_{int} \propto |\epsilon_0|^{3/2}$, for a fixed Γ_P modular matrix and a given VEV for the complex structure modulus, U_0.

2: Vacua with a geometric/non-geometric flux hierarchy

In this example we wish to constuct a family of solutions with additional structure due to localized sources, analogous to [95]. This time we fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(4)$ and $\mathfrak{g}_P = \mathfrak{so}(4)$ respectively[8]. Just to illustrate some vacua with this algebraic structure, we set the modular embeddings to be less trivial than previous cases, with $\alpha \, \delta \neq 0$ and $\lambda \, (1 - \lambda^2) \neq 0$

[8] In this case $\Delta_Q = (\epsilon_2 - \epsilon_1)/3$ and $\Delta_P = (\rho_2 - \rho_1)/3$.

for the isomorphism to be well defined.

$$\Gamma_Q = \begin{pmatrix} \alpha & \delta \\ -\lambda\,\alpha & \lambda\,\delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} (1-\lambda)\,\alpha & 0 \\ 0 & (1+\lambda)\,\delta \end{pmatrix} \, , \tag{6.5.14}$$

The cohomology condition has three branches and the embeddings (6.5.14) satisfy $\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{-1}$, giving a type B setup. For simplicity we will fix again $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$ and so $\epsilon_1 = \epsilon_2 = \rho_1 = \rho_2 = 0$, resulting in $N_7' = 0$. Under this flux setup, $E(U)$ has 1 real root and we find that $E(U)$ can be factorized as (6.5.5) given the fluxes satisfy a number of relations.

$$\begin{aligned} \epsilon_3 &= \tfrac{1-\lambda^2}{8\lambda}\Big(\,(\lambda-1)^3\rho_3 + (\lambda+1)^3\rho_0\,\Big) \\ \epsilon_0 &= \tfrac{1-\lambda^2}{8\lambda^4}\Big(\,(\lambda-1)^3\rho_3 - (\lambda+1)^3\rho_0\,\Big) \end{aligned}$$

These flux choices determine the factorisation of the $E(U)$ polynomial.

$$g_1 = 0 \quad , \quad \frac{g_0}{g_2} = \left(\frac{\delta}{\alpha}\right)^2 \quad , \quad \frac{f_0}{f_1} = -\left(\frac{\delta}{\alpha}\right)\frac{(\lambda-1)^3\rho_3}{(\lambda+1)^3\rho_0}$$

Since $g_1^2 - 4\,g_2\,g_0 < 0$, these are physical vacua with $U_0 = i\left(\frac{\delta}{\alpha}\right)$. From (6.5.4), S and T get stabilized.

$$\begin{aligned} S_0 &= \left(\tfrac{2\lambda}{\lambda^2-1}\right) i \\ T_0 &= \tfrac{\lambda^2-1}{12\lambda(\lambda^2+1)}\Big(\,\tfrac{(\lambda+1)^4}{\lambda^2-1}\,\rho_0 - \tfrac{(\lambda-1)^4}{\lambda^2-1}\,\rho_3 \,+\, i\,\big(\,(\lambda-1)^2\,\rho_3 + (\lambda+1)^2\rho_0\,\big)\,\Big) \end{aligned}$$

The resultant tadpole conditions for these vacua are all determined analytically.

$$\begin{aligned} N_3 &= \tfrac{|\Gamma_Q|^3}{2\lambda^2}(\lambda^2-1)\,\big(\,(\lambda-1)^6\,\tilde{\rho}_3^2 + (\lambda+1)^6\,\tilde{\rho}_0^2\,\big) \\ N_7 &= \tfrac{|\Gamma_Q|^3}{2\lambda}(\lambda^2-1)\,\big(\,(\lambda-1)^2\,\tilde{\rho}_3 + (\lambda+1)^2\,\tilde{\rho}_0\,\big) \\ \tilde{N}_7 &= \tfrac{|\Gamma_Q|^3}{8\lambda^3}\,\big(\lambda^2-1\big)^3\,\big(\,(\lambda-1)^2\,\tilde{\rho}_3 + (\lambda+1)^2\,\tilde{\rho}_0\,\big) \end{aligned}$$

We have redefined the modular fluxes as $\rho_3 = 4\lambda\tilde{\rho}_3$ and $\rho_0 = 4\lambda\tilde{\rho}_0$. Then $N_3 > 0$, $N_7 > 0$ and $\tilde{N}_7 > 0$ is necessary for vacua to be physical[9]. In terms of the original fluxes, this solution corresponds to $\mathsf{c}_3 = -\alpha^3\,\lambda\,(\lambda-1)$, $\mathsf{c}_2 = \tilde{\mathsf{c}}_2 = \alpha^2\,\delta\,\lambda\,(\lambda+1)$, $\mathsf{c}_1 = \tilde{\mathsf{c}}_1 = -\alpha\,\delta^2\,\lambda\,(\lambda-1)$, $\mathsf{c}_0 = \delta^3\,\lambda\,(\lambda+1)$ and $\tilde{\mathsf{d}}_1 = \alpha\,\delta^2\,(\lambda^2-1)\,(\lambda+1)$,

[9]Fixing $|\Gamma_Q| > 0$ implies $\lambda > 0$ for $\mathrm{Im}(U_0) > 0$, $(\lambda^2-1) > 0$ for $\mathrm{Im}(S_0) > 0$ and $(\lambda-1)^2\,\tilde{\rho}_3 + (\lambda+1)^2\,\tilde{\rho}_0 > 0$ for $\mathrm{Im}(T_0 > 0$. This fixes the net charge of the tadpoles.

$\widetilde{\mathsf{d}}_2 = \alpha^2\,\delta\,(\lambda^2-1)\,(\lambda-1)$ for non-geometric fluxes; $\mathsf{b}_0 = \delta^3\,(\lambda^2-1)\,(\lambda-1)^3\,\tilde{\rho}_3$, $\mathsf{b}_1 = -\alpha\,\delta^2\,(\lambda^2-1)\,(\lambda+1)^3\,\tilde{\rho}_0$, $\mathsf{b}_2 = \alpha^2\,\delta\,(\lambda^2-1)\,(\lambda-1)^3\,\tilde{\rho}_3$ and $\mathsf{b}_3 = -\alpha^3\,(\lambda^2-1)\,(\lambda+1)^3\,\tilde{\rho}_0$ for NS-NS flux and $\mathsf{a}_0 = -4\,\delta^3\,\lambda\,(\lambda+1)^3\,\tilde{\rho}_0$ and $\mathsf{a}_3 = -4\,\alpha^3\,\lambda\,(\lambda-1)^3\,\tilde{\rho}_3$ for R-R flux.

By considering the fluxes' dependency on the parameter λ, we note that generically a hierarchy between geometric F_3, H and non-geometric Q, P fluxes occurs, in which the geometric fluxes, i.e. $\mathsf{a}_i \propto \lambda^4$, $\mathsf{b}_j \propto \lambda^5$ are large compared to the non-geometric fluxes, i.e. $\mathsf{c}_i \propto \lambda^2$, $\mathsf{d}_j \propto \lambda^3$, given $\lambda > 1$ for $\mathrm{Im}(S_0) > 0$. However, there is a critical value $\lambda_0 = 1 + \sqrt{2}$ for which $g_s \geq 1$ if $\lambda \geq \lambda_0$. Hence, there is a narrow range, $1 < \lambda < \lambda_0$, for which non perturbative string effects can be neglected, ie. $\lambda = 2$ implies $g_s = 3/4$. Finally, large values of the $\tilde{\rho}_0$ and $\tilde{\rho}_3$ parameters favour a large internal volume needed to disregard corrections in α'.

3: Vacua with a non-vanishing flux-induced C_8' tadpole

We now consider a simple family of solutions with a non vanishing flux-induced C_8' tadpole for which all moduli get stabilized. Let us fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(3,1)$ and $\mathfrak{g}_P = \mathfrak{so}(4)$ respectively[10]. Examples belonging to this pairing were also found in [60]. For simplicity, we fix the modular embeddings to be of a restricted type.

$$\Gamma_Q = \begin{pmatrix} \alpha & \delta \\ \alpha & -\delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \; . \tag{6.5.15}$$

The cohomology condition for this pairing has an unique branch $\mathcal{Z}_0^Q \parallel \mathcal{Z}_{-1}^P$. It is a non-geometric type B setups and therefore has a potentially non vanishing flux-induced C_8' tadpole. The modular embeddings (6.5.15) belong to this branch.

[10] In this case $\Delta_Q = -\epsilon_2/3 - \epsilon_0 = -\epsilon_0'/24$ and $\Delta_P = (\rho_2 - \rho_1)/3$.

For convenience we also redefine our H flux parameters.

$$\begin{pmatrix} \epsilon_3' \\ \epsilon_1' \\ \epsilon_2' \\ \epsilon_0' \end{pmatrix} = 8 \begin{pmatrix} 3 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \epsilon_3 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_0 \end{pmatrix} . \tag{6.5.16}$$

Solutions with NS-NS and R-R fluxes for which $\vec{\epsilon} \notin \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \notin \ker(\tilde{\Phi}_P)$ can be given parametrically in terms of (κ_1, κ_2) parameters.

$$\epsilon_3' = \kappa_1 + \kappa_2 \quad , \quad \epsilon_0' = \kappa_1 - \kappa_2 \quad , \quad \rho_1 = \kappa_2 \quad , \quad \rho_2 = \kappa_1$$

The remaining parameters $(\epsilon_1', \epsilon_2')$, expanding the $\ker(\tilde{\Phi}_Q)$, and (ρ_0, ρ_3), expanding the $\ker(\tilde{\Phi}_P)$, are completely free. For simplicity, we will deal just with a non vanishing κ_2 parameter plus the fluxes ρ_0 and ρ_3. All the Bianchi identities are, by construction, satisfied. In general, $E(U)$ has 1 real root for this algebra pairing, but under this specific flux configuration it has two real roots. Factorising out these real roots, $E(U) \to \tilde{E}(U)$, and requiring it to factorise as (6.5.5) we find analytic expressions for the coefficients.

$$f_1 = g_1 = \rho_0 = 0 \quad , \quad \rho_3 = \frac{4}{3}\kappa_2 \quad , \quad \frac{g_0}{g_2} = \left(\frac{\delta}{\sqrt{2}\alpha} \right)^2 \quad , \quad f_0\, g_2^2 = -16\, \alpha^4\, \delta\, \kappa_2$$

These values give $g_1^2 - 4\, g_2\, g_0 < 0$, producing physical vacua with $U_0 = i \left(\frac{\delta}{\sqrt{2}\alpha} \right)$. Using (6.5.4), the remaining moduli are stablised to analytic values.

$$U_0 = (\frac{\delta}{\sqrt{2}\alpha})\, i \quad , \quad S_0 = \sqrt{2}\, i \quad , \quad T_0 = -\frac{\kappa_2}{27}\, (1 + \sqrt{2}\, i)$$

These are physical for $\kappa_2 < 0$ and $|\Gamma_P| > 0$. The tadpole conditions for these vacua are determined such that $N_3 > 0$, $N_7 < 0$ and $N_7' > 0$ and $|\Gamma_P| = \alpha\, \delta$.

$$\tilde{N}_7 = 0 \quad , \quad N_3 \; = \; \frac{\kappa_2}{15}\, N_7 = -\frac{\kappa_2}{3}\, N_7' = \frac{2}{9}\, |\Gamma_P|^3\, \kappa_2^2$$

In terms of the original fluxes, this solution corresponds to $\mathsf{c}_3 = 2\,\alpha^3$, $\mathsf{c}_2 = \widetilde{\mathsf{c}}_2 = 2\,\widetilde{\mathsf{d}}_2 = 2\,\alpha^2\,\delta$, $\mathsf{c}_1 = \widetilde{\mathsf{c}}_1 = 2\,\widetilde{\mathsf{d}}_1 = -2\,\alpha\,\delta^2$ and $\mathsf{c}_0 = -2\,\delta^3$ for non-geometric fluxes;

$\mathsf{b}_0 = -\frac{\kappa_2}{6}\,\delta^3$ for NS-NS flux and $\mathsf{a}_3 = \frac{4}{3}\,\kappa_2\,\alpha^3$, $\mathsf{a}_1 = \frac{1}{3}\,\kappa_2\,\alpha\,\delta^2$ for R-R flux. The string coupling constant turns out to be $g_s = 1/\sqrt{2}$ and corrections in α' can be neglected taking large values for $|\kappa_2|$. This also increases the number of localized sources cancelling the flux-induced tadpoles.

4: Vacua with a non defined flux-induced C_8 tadpole sign

Finally, and for the sake of completeness, we fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(4)$ and $\mathfrak{g}_P = \mathfrak{nil}$ respectively[11] and fix the modular embeddings to be dependent on only two parameters.

$$\Gamma_Q = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha & -\delta \\ \alpha & \delta \end{pmatrix} \tag{6.5.17}$$

For the isomorphism to be well defined we require $\alpha\,\delta \neq 0$. In this case, we obtained a single cohomology condition, $\mathcal{Z}^Q_{-1} \parallel \mathcal{Z}^P_{\infty}$ which is satisfied by (6.5.17) and is again a type B setup. Once more, solutions with NS-NS and R-R fluxes for which $\vec{\epsilon} \notin \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \notin \ker(\tilde{\Phi}_P)$ can be given parametrically.

$$\epsilon_1 = -4\,(\kappa_1 - 3\,\kappa_2) \quad , \quad \epsilon_2 = -4\,(\kappa_1 + 3\,\kappa_2) \quad , \quad \rho_3 = \kappa_2 \quad , \quad \rho_2 = \kappa_1$$

The parameters (ϵ_0, ϵ_3) expanding the $\ker(\tilde{\Phi}_Q)$ and (ρ_0, ρ_1) expanding the $\ker(\tilde{\Phi}_P)$, are completely free. For this pairing, $E(U)$ has 1 real root and we find that $E(U)$ can be factorized as (6.5.5).

$$\begin{array}{rclcrcl} g_1 & = & 0 & , & f_1 & = & 0 \\ \epsilon_3 & = & \frac{2B^2}{A} - 2B + 4A & , & \epsilon_0 & = & -4A \\ \frac{g_0}{g_2} & = & \left(\frac{\delta}{\alpha}\right)^2 \frac{A}{B} & , & f_0 g_0^2 & = & -2A\delta^5 \\ \kappa_1 & = & \frac{1}{4}(B - 5A) & , & \kappa_2 & = & \frac{B-A}{4} \end{array}$$

We have used $A = \rho_1 - \rho_0$ and $B = \rho_1 - 5\rho_0$. Then $g_1^2 - 4\,g_2\,g_0 < 0$ provided $AB > 0$ and there are physical vacua with $U_0 = i\left(\frac{\delta}{\alpha}\right)\left(\frac{\sqrt{A}}{\sqrt{B}}\right)$. From (6.5.4), S and

[11] In this case $\Delta_Q = (\epsilon_2 - \epsilon_1)/3$ and $\Delta_P = -\rho_3$.

T get analytically stablised.

$$S_0 = \frac{\sqrt{A}\sqrt{B}}{(A+B)^2}\Big(2\sqrt{A}\sqrt{B} + i\,(B-A)\Big) \quad , \quad T_0 = \frac{4A}{3(A+B)}\Big(A + i\,\sqrt{A}\sqrt{B}\Big)$$

The resultant tadpole conditions for these vacua are such that $N_3 > 0$, N_7 has no defined sign, $\tilde{N}_7 < 0$ and $N'_7 < 0$ is required for physical vacua[12].

$$\begin{aligned} N_3 &= \tfrac{16}{3}|\Gamma_Q|^3\left((B-A)^2 + AB\right) \\ N_7 &= -\tfrac{2}{3}|\Gamma_Q|^3(B-2A) \\ \tilde{N}_7 &= -|\Gamma_Q|^3\tfrac{2(A+B)^2}{A} \\ N'_7 &= -4|\Gamma_Q|^3(B-A) \end{aligned}$$

In terms of the original fluxes, this solution corresponds to $\mathsf{d}_3 = -\alpha^3$, $-\mathsf{d}_2 = \widetilde{\mathsf{d}}_2 = \widetilde{\mathsf{c}}_2 = \alpha^2\,\delta$, $\mathsf{d}_1 = -\widetilde{\mathsf{d}}_1 = -\widetilde{\mathsf{c}}_1 = \alpha\,\delta^2$ and $\mathsf{d}_0 = \delta^3$ for non-geometric fluxes; $\mathsf{b}_0 = 4\,\delta^3\,A$, $\mathsf{b}_1 = \frac{2}{3}\,\alpha\,\delta^2\,(A+B)$, $\mathsf{b}_2 = \frac{4}{3}\,\alpha^2\,\delta\,(B-2A)$ and $\mathsf{b}_3 = 2\,\alpha^3\left(\frac{(B-A)^2}{A} + (A+B)\right)$ for NS-NS flux and $\mathsf{a}_0 = 2\,\delta^3\,A$, $\mathsf{a}_2 = \frac{2}{3}\,\alpha^2\,\delta\,(B-2A)$ for R-R flux. This family of solutions gives rise to $g_s > 1$ for $A, B > 0$ and then non perturbative string effects can not be neglected.

Summary

In this chapter we have considered the explicit case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold, which possesses the properties of the spaces we have been considering more generally in previous chapters. The duality induced fluxes of both flux sectors in Type IIB and the NS-NS sector of Type IIA were constructed in terms of their $\Lambda^p(\mathsf{E}^*)$ defined components and also the SU(3) structure $\Delta^p(\mathsf{E}^*)$ defined components and seen to match the structures derived in general previously. We focused on the Type IIB $\mathcal{N} = 1$ theory constructed by using the orientifold projection which

[12]Fixing $|\Gamma_Q| > 0$, then $A, B > 0$ for $\mathrm{Im}(T_0 > 0$ and $(B-A) > 0$ for $\mathrm{Im}(S_0 > 0$. This fixes the net charge of tadpoles but N_7 depends on the sign of $(B-2A)$, with $N_7 > 0$ for $(B-2A) < 0$ and $N_7 < 0$ for $(B-2A) > 0$.

constructs O3- and O7-planes and allows D3- and D7-branes and simplified by isotropy. The twelve dimensional Lie algebra interpretation of the fluxes reduces to having a six dimensional subalgebra dependent entirely on a single kind of flux, the non-geometric Q. As a result of this, its $\mathrm{GL}(6,\mathbb{Z})$ invariance in its $\Lambda^p(\mathsf{E}^*)$ defined components and the full classification of six dimensional Lie algebra we were able to solve the Bianchi constraints in full generality. The S duality extension of this was treated in the same way, with Q and P having their Bianchi constraints solved by different isomorphisms and their mnixed integrability conditions reduced to constraints on the isomorphisms. All of the non-geometric constraints; Jacobi, algebra deformations and integrability conditions, were solved through algebraic geometry methods which gave proof of a full classification of all possible solutions. The remaining Bianchi constraints were examined by the use of linear transformations dependent on fluxes and their relationship with the tadpoles explicitly observed. Finally we used these solutions to construct examples of vacua with interesting phenomenology; Minkowski vacua, vacua with broken supersymmetry, vacua with heirachy and vacua with vanishing tadpoles.

We have observed a number of interesting properties for this internal space. Its orbifold symmetries make it a triplet of two dimensional tori and it is known that the two dimensional torus is self mirror dual, under mirror symmetry its moduli exchange but it remains a two dimensional torus. The T duality constraints defined in $\Lambda^p(\mathsf{E}^*)$ components have $\mathrm{GL}(6,\mathbb{Z})$ invariance which, due to orbifold symmetry, leads to a modular symmetry in the complex structure moduli and the inclusion of S duality gives modular symmetry in the dilaton. The inherent symmetry between the complex structure and Kähler moduli of the two dimensional sub-torus has not appeared but would be expected and it is to this which we now turn.

Chapter 7

Symmetries in Moduli Space

Thus far we have observed a great deal of symmetry in how the different moduli spaces of $\mathcal{W}$ (and $\mathcal{M}$) can be described, though distinct differences exist. The largest one is the Kähler structure of the moduli spaces, the moduli dependence of $\mathcal{M}^{K}$ and $\mathcal{M}^{Q}$ is dependent upon which Type II theory we are considering, as given in Table 7.1. In this chapter we will consider the implications of making our descriptions of the moduli spaces as symmetric as possible in a given Type II construction, which we will take to be Type IIB for reasons which we will discuss shortly. More specifically, we reformulate the superpotentials and fluxes such that the roles of the moduli spaces are exchanged without having to apply a mirror transformation. Since this implies the two moduli spaces of the Type IIB theory on $\mathcal{W}$ are equivalent we would not expect it to be possible for all $\mathcal{W}$, only a particular set of spaces. The work in this chapter is found in [9].

7.1 The Motivation

To provide a motivation for this hypothesis we consider the results just obtained for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold. The invariance of the non-geometric flux constraints under coordinate transformations leads to a particular reparameterisation invari-

$\text{IIA} \in \mathcal{M}$	$\text{IIB} \in \mathcal{W}$
$\mho \in \mathcal{M}^K(\mathfrak{T})$	$\Omega \in \mathcal{M}^K(\mathsf{U})$
$\Omega_c^{(\prime)} \in \mathcal{M}^Q(\mathfrak{U}, S)$	$\mho_c^{(\prime)} \mathcal{M}^Q(\mathsf{T}, S)$

Table 7.1: Holomorphic forms of Type II moduli spaces.

ance in the complex structure moduli and due to the specific structure of $\mathcal{M}_{\mathbb{Z}_2^2}$ the invariance is precisely an $\text{SL}(2,\mathbb{Z})$ invariance in the U_a moduli. This was seen to be the restriction of the $\text{GL}(6,\mathbb{Z})$ invariance of Section 5.1.3 by the orbifold symmetries. Equivalently this can be seen to follow from the factorisation of $\mathcal{M}_{\mathbb{Z}_2^2}$ in terms of three two dimensional sub-tori. As a result of this the moduli of $\mathcal{M}_{\mathbb{Z}_2^2}$ pair off, with (T_a, U_a) being those moduli which describe the a'th sub-torus. It is noted in [54] that upon the dimensional reduction of [49] the kinetic terms of the Kähler and complex structure moduli of a two dimensional torus take on the same form, which is also equal to the form of the dilaton kinetic term of Type IIB as given in (3.3.1).

$$\frac{\partial_\mu S \overline{\partial^\mu S}}{2(\text{Im}(S))^2} \quad , \quad \frac{\partial_\mu T \overline{\partial^\mu T}}{2(\text{Im}(T))^2} \quad , \quad \frac{\partial_\mu U \overline{\partial^\mu U}}{2(\text{Im}(U))^2} \tag{7.1.1}$$

These kinetic terms have Kähler potentials defined by Hitchin functions [73, 74, 75], one for each holomorphic form in Table 7.1. However it is not clear in (7.1.1) which moduli type the dilaton couples to, both Type IIA and Type IIB lead to the same kinetic terms. This is to be expected given the structure of $\mathcal{M}_{\mathbb{Z}_2^2}$; being the combination of three two dimensional tori. Two dimensional tori have a pair of moduli, one complex structure and one Kähler, and these are exchanged under mirror transformations yet the underlying space is still a two dimensional torus. The fact that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold can be written in terms of two dimensional tori means it inherits some of the properties of such tori and one such symmetry is the modular symmetry of the Kähler moduli. Presupposing such modular

$\mathrm{SL}(2,\mathbb{Z})$ invariance in each of the seven moduli of the anisotropic orientifold has been investigated in [61] and while it does not result in the same constraints as T duality the methodology of their analysis is qualitatively the same. Such $\mathrm{SL}(2,\mathbb{Z})^7$ invariance is a top down approach and in the previous chapter we obtained a bottom up derivation of complex structure modular invariance and by the inclusion of S duality we also obtained dilaton modular invariance. We have not yet seen a bottom up construction of the $\mathrm{SL}(2,\mathbb{Z})_{\mathcal{T}}$ symmetries for $\mathcal{M}_{\mathbb{Z}_2^2}$. To illustrate this for the $\mathcal{M}_{\mathbb{Z}_2^2}$ more explicitly we recall its general polynomial form, where the moduli are grouped in terms of their Kähler moduli dependence.

$$\begin{aligned} W &= \int_{\mathcal{W}} \big\langle\, \Omega, \Big(\mathsf{D}(\mho_c) + \mathsf{D}'(\mho'_c)\Big) \,\big\rangle_{\pm} \\ &= \left(\begin{array}{c} \mathcal{T}_0\Big(\mathcal{P}_0(\mathcal{U}) - S\,\widehat{\mathcal{P}}_0(\mathcal{U})\Big) \;+\; \mathcal{T}_a\Big(\mathcal{P}_1^{(a)}(\mathcal{U}) - S\,\widehat{\mathcal{P}}_1^{(a)}(\mathcal{U})\Big) \;+ \\ \pm\; \mathcal{T}^0\Big(\mathcal{P}_3(\mathcal{U}) - S\,\widehat{\mathcal{P}}_3(\mathcal{U})\Big) \;\pm\; \mathcal{T}^b\Big(\mathcal{P}_2^{(b)}(\mathcal{U}) - S\,\widehat{\mathcal{P}}_2^{(b)}(\mathcal{U})\Big) \end{array} \right) \end{aligned} \qquad (7.1.2)$$

To obtain the results for the Kähler moduli we have already seen for the complex structure moduli we are motivated to exchange the roles of the two type of moduli. To illustrate this on the $\mathcal{M}_{\mathbb{Z}_2^2}$ we construct a superpotential whose complex structure and Kähler moduli play the opposing roles to the superpotential (7.1.2). This is done by simply rearranging (7.1.2), rather than use T duality or mirror symmetry to alter the superpotential.

$$W \to \left(\begin{array}{c} \mathcal{U}_0\Big(\mathfrak{P}_0(\mathcal{T}) - S\,\widehat{\mathfrak{P}}_0(\mathcal{T})\Big) \;+\; \mathcal{U}_a\Big(\mathfrak{P}_1^{(a)}(\mathcal{T}) - S\,\widehat{\mathfrak{P}}_1^{(a)}(\mathcal{T})\Big) \;+ \\ +\; \mathcal{U}^0\Big(\mathfrak{P}_3(\mathcal{T}) - S\,\widehat{\mathfrak{P}}_3(\mathcal{T})\Big) \;+\; \mathcal{U}^b\Big(\mathfrak{P}_2^{(b)}(\mathcal{T}) - S\,\widehat{\mathfrak{P}}_2^{(b)}(\mathcal{T})\Big) \end{array} \right) \qquad (7.1.3)$$

The sequence of fluxes induced by T duality each define a cubic polynomial in the complex structure moduli, coupling differently to the Kähler moduli. We now have polynomials which are cubic in the Kähler moduli and which couple differently to the complex structure moduli. This reformulation of the superpotential in (7.1.3), due to the symmetry in the moduli of the two dimensional sub-tori, and the symmetry in their kinetic terms of (7.1.1) suggests that on $\mathcal{M}_{\mathbb{Z}_2^2}$ it is possible to write a IIB construction in the form of a Type IIA construction and vice versa.

This reformulation would be guided by the nature of the holomorphic forms given in Table 7.2.

$$\mathcal{M}^{Q}(\mathsf{T},S)\otimes\mathcal{M}^{K}(\mathsf{U})\to\mathcal{M}_{\mathsf{T}}\otimes\mathcal{M}_{S}\otimes\mathcal{M}_{\mathsf{U}}\to\mathcal{M}^{\widetilde{Q}}(\mathsf{U},S)\otimes\mathcal{M}^{\widetilde{K}}(\mathsf{T})$$

The factorisation of the superpotential into expressions dependent on those new holomorphic forms is non-trivial due to the dilaton couplings, a point we shall see in the next section. Given a factorisation of the superpotential such that it is dependent on these modified holomorphic forms we would expect the resultant constraints to be inequivalent to the original ones, given a comparision between (7.1.2) and (7.1.3). If T duality invariance on (7.1.2) induces $\mathrm{SL}(2,\mathbb{Z})$ modular invariance on the complex structure moduli then we would argue there is a duality related to (7.1.3) which induces $\mathrm{SL}(2,\mathbb{Z})$ invariance on the Kähler moduli. We will refer to as T′ duality and whose precise nature we will construct shortly.

This reformulation is motivated by the symmetry between the moduli types in the $\mathcal{M}_{\mathbb{Z}_2^2}$ superpotential, which is a manifestation of the fact a two dimensional torus is self mirror dual and thus we might associate the additional constraints due to T′ invariance to this enhanced symmetry. As such, in our more general discussion we shall eventually restrict our considerations to those spaces which satisfy $\mathcal{M}=\mathcal{W}$, even though some of the methodology does not require this. To examine this more quantitatively, give some justification for our speculation and for spaces other than $\mathcal{M}_{\mathbb{Z}_2^2}$ we shall consider the many different ways we have of constructing superpotential-like expressions from objects thus far examined.

7.2 Alternate Superpotentials

Since we will be discussing how various derivatives and their matrix representations relate to one another we will dispense with the different D's used for different derivatives in previous sections. Instead we will simply label them with an index,

IIA		
$\mho \in \mathcal{M}^K(\mathfrak{T})$	$\to$	$\Omega \in \mathcal{M}^{\widetilde{K}}(\mathfrak{U})$
$\Omega_c^{(\prime)} \in \mathcal{M}^Q(\mathfrak{U}, S)$	$\to$	$\mho_c^{(\prime)} \in \mathcal{M}^{\widetilde{Q}}(\mathfrak{T}, S)$
IIB		
$\Omega \in \mathcal{M}^K(\mathsf{U})$	$\to$	$\mho \in \mathcal{M}^{\widetilde{K}}(\mathsf{T})$
$\mho_c^{(\prime)} \in \mathcal{M}^Q(\mathsf{T}, S)$	$\to$	$\Omega_c^{(\prime)} \in \mathcal{M}^{\widetilde{Q}}(\mathsf{U}, S)$

Table 7.2: Holomorphic forms of reformulated Type II moduli spaces.

D_i, and likewise with their associated matrix representations. In the case of Type IIB we have M_i representing the action on the $\Delta^+(\mathsf{E}^*)$ basis and N_j on the $\Delta^3(\mathsf{E}^*)$ basis. In Type IIB we can construct objects which have a superpotential-like form in two different ways; one of which is the Type IIB superpotential and the second is obtained from the first by exchanging the roles of the holomorphic forms in line with Table 7.2.

$$\begin{aligned} \mathsf{W}_1 &= \int_{\mathcal{W}} \big\langle\, \Omega, \Big(\mathsf{D}_1(\mho_c) + \mathsf{D}'_1(\mho'_c)\Big) \,\big\rangle_\pm \\ &\qquad = \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \Big(\mathsf{C} \cdot \mathsf{M}_1 + \mathsf{C}' \cdot \mathsf{M}'_1\Big) \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathsf{U}} \qquad (7.2.1) \\ \mathsf{W}_2 &= \int_{\mathcal{W}} \big\langle\, \mho, \Big(\mathsf{D}_2(\Omega_c) + \mathsf{D}'_2(\Omega'_c)\Big) \,\big\rangle_\pm \\ &\qquad = \underline{\mathsf{U}}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \Big(\widetilde{\mathsf{C}} \cdot \mathsf{N}_2 + \widetilde{\mathsf{C}}' \cdot \mathsf{N}'_2\Big) \cdot \mathsf{g}_\nu \cdot \underline{\mathsf{T}} \qquad (7.2.2) \end{aligned}$$

In general, namely $\mathcal{M} \neq \mathcal{W}$, these are the only[1] two expressions which can be formed of integrals and from pairs of elements of either $\Delta^3(\mathsf{E}^*)$ or $\Delta^+(\mathsf{E}^*)$. It is still possible to construct Type IIB scalar products which are of the same general

[1] We do not consider $\Omega \wedge \mathsf{D}(\mho_c)$ and $\mathbb{D}(\Omega) \wedge \mho_c$ as different due to the same manner in which the dilaton couples to the moduli.

factorisation, but from the bilinear forms g and h defined in Type IIA.

$$\mathsf{W}_3 \;=\; \underline{\mathsf{T}}^\top \cdot h_{\mathfrak{a}} \cdot \Big(\mathsf{C} \cdot M_3 + \mathsf{C}' \cdot M_3' \Big) \cdot g_\nu \cdot \underline{\mathsf{U}} \tag{7.2.3}$$

$$\mathsf{W}_4 \;=\; \underline{\mathsf{U}}^\top \cdot h_{\nu} \cdot \Big(\widetilde{\mathsf{C}} \cdot N_4 + \widetilde{\mathsf{C}}' \cdot N_4' \Big) \cdot g_{\mathfrak{a}} \cdot \underline{\mathsf{T}} \tag{7.2.4}$$

These two expressions are constructable using matrices because the dimensions of such pairs as h_ν and $h_{\mathfrak{a}}$ are equal by $h^{1,1}(\mathcal{W}) = h^{2,1}(\mathcal{M})$. This allows us to build forms such as $\underline{\mathsf{T}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\nu)}$, hybrids of terms defined in different spaces and in different Type II theories. However, this fact means that generally such constructs are ill defined. The expression $\underline{\mathsf{T}}^\top \cdot \mathsf{h}_{\mathfrak{a}} \cdot \mathbf{e}_{(\nu)}$ can be built in $\mathcal{W}$ if $h^{1,1} = h^{2,1}$ and provided[2] it is also possible to choose $\Delta^3(\mathsf{E}^*)$ bases in $\mathcal{W}$ and $\mathcal{M}$ such that $h_{\mathfrak{a}} = \mathsf{h}_{\mathfrak{a}}$. This is a reflection of the link between the Kähler moduli space of $\mathcal{W}$ and the complex structure moduli space of $\mathcal{M}$, $\mathsf{T}_I \leftrightarrow \mathfrak{U}_I$. If the link is to be between the two moduli spaces of $\mathcal{W}$ itself then we instead wish to consider the equivalence $\mathsf{T}_A \leftrightarrow \mathsf{U}_I$. Such an equivalence is only possible if $h^{1,1} = h^{2,1}$ and also $\mathsf{U}_A \leftrightarrow \mathsf{U}_I$. As such we have the motivation fo the narrowing of our considerations to those spaces which satisfy $\mathcal{W} = \mathcal{M}$, the self mirror dual spaces. Such a restriction automatically allows us to make the equivalence $g_{\mathfrak{a}} = \mathsf{g}_{\mathfrak{a}}$ and likewise with the other bilinear forms because of the equality of the Hodge numbers[3]. As a result it is possible to construct the Type IIB form $\widetilde{\Omega} \equiv \Omega\Big|_{\mathsf{U}\to\mathsf{T}} = \underline{\mathsf{T}}^\top \cdot \mathsf{h}_{\mathfrak{a}} \cdot \mathbf{f}_{(\mathfrak{a})}$ on $\mathcal{W}$. With this equality between the tilded and untilded bilinear forms on $\mathcal{W} = \mathcal{M}$ both (7.2.3) and (7.2.4) therefore obtain an integral representation, in terms of $\widetilde{\Omega}$ and $\widetilde{\mho} \equiv \mho\Big|_{\mathsf{T}\to\mathsf{U}}$.

$$\mathsf{W}_3 = \int_{\mathcal{W}} \big\langle\, \widetilde{\mho}, \Big(\mathsf{D}_3(\widetilde{\Omega}_c) + \mathsf{D}_3'(\widetilde{\Omega}_c') \Big) \,\big\rangle_{\pm} \tag{7.2.5}$$

$$\mathsf{W}_4 = \int_{\mathcal{W}} \big\langle\, \widetilde{\Omega}, \Big(\mathsf{D}_4(\widetilde{\mho}_c) + \mathsf{D}_4'(\widetilde{\mho}_c') \Big) \,\big\rangle_{\pm} \tag{7.2.6}$$

[2]Without this particular requirement there is no reason to expect a bijective equivalence between the two constructions.

[3]It should be noted that although the complex structure indices $I, J, \dots$ and the Kähler indices $A, B, \dots$ range over the same values we retain their distinction for the purposes of clarity.

To illustrate this more explicitly we consider an integral similar to that of (7.2.1), namely using non-complexified holomorphic forms and use the properties of the symplectic basis and the equality of the Hodge number to convert it into something similar to (7.2.5). Thus illustrating a rearrangement of the superpotential akin to that between (7.1.2) and (7.1.3).

$$\begin{aligned}
\int_{\mathcal{W}} \langle\, \Omega, \mathsf{D}_1(\mho)\,\rangle_{\pm} &= \int_{\mathcal{W}} \Big(\mathsf{U}_A \mathfrak{a}_A - \mathsf{U}^B \mathfrak{b}^B\Big) \wedge \left[\begin{array}{r} \mathsf{T}_I\Big(\mathsf{F}_{(I)A}\mathfrak{a}_A - \mathsf{F}_{(I)}{}^B \mathfrak{b}^B\Big) \\ \pm\mathsf{T}^J\Big(\mathsf{F}^{(I)}{}_A \mathfrak{a}_A - \mathsf{F}^{(J)B}\mathfrak{b}^B\Big) \end{array}\right] \\
&= \int_{\mathcal{W}} \Big(\mathsf{T}_I \mathfrak{a}_I - \mathsf{T}^J \mathfrak{b}^J\Big) \wedge \left[\begin{array}{r} \mathsf{U}_A\Big((\mp\mathsf{F}^{(I)A})\mathfrak{a}_I - \mathsf{F}_{(J)}{}^B \mathfrak{b}^J\Big) \\ -\mathsf{U}^B\Big((\mp\mathsf{F}^{(I)}{}_A)\mathfrak{a}_I - \mathsf{F}_{(J)B}\mathfrak{b}^J\Big) \end{array}\right] \\
&= \int_{\mathcal{W}} \langle\, \widetilde{\Omega}, \mathsf{D}_4(\widetilde{\mho})\,\rangle_{\pm} \qquad (7.2.7)
\end{aligned}$$

We have had to make the assumption that I, J and A, B range over the same indices and that the sympletic structure of $\mathcal{W}$ is equivalent to that of $\mathcal{M}$, as such expressions as $\mathsf{T}_A \mathfrak{a}_A - \mathsf{T}^B \mathfrak{b}^B$ are the Type IIA holomorphic 3-form Ω but with the moduli labelled in the Type IIB manner. The general fact that these expressions bear a striking resemblence to the Type IIA superpotential integrals prompts us to now turn our attention to those superpotential-like integrals defined in Type IIA on a generic $\mathcal{M}$. As with Type IIB, there are two expressions which can be written as integrals and two which, in general, cannot. We label the Type IIA derivatives with an index, $\mathcal{D}_i$, and likewise with their associated matrix representations, which in the case of Type IIA has M_i representing the action on the $\Delta^3(\mathsf{E}^*)$ basis and N_j on the $\Delta^+(\mathsf{E}^*)$ basis.

$$\begin{aligned}
W_1 &= \int_{\mathcal{M}} \langle\, \mho, \Big(\mathcal{D}_1(\Omega_c) + \mathcal{D}_1'(\Omega_c')\Big)\,\rangle_{\pm} \\
&\qquad = \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \Big(\mathbb{C} \cdot M_1 + \mathbb{C}' \cdot M_1'\Big) \cdot g_\nu \cdot \underline{\mathfrak{T}} \qquad (7.2.8) \\
W_2 &= \int_{\mathcal{M}} \langle\, \Omega, \Big(\mathcal{D}_2(\mho_c) + \mathcal{D}_2'(\mho_c')\Big)\,\rangle_{\pm} \\
&\qquad = \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot \Big(\widetilde{\mathbb{C}} \cdot N_2 + \widetilde{\mathbb{C}}' \cdot N_2'\Big) \cdot g_{\mathfrak{a}} \cdot \underline{\mathfrak{U}} \qquad (7.2.9)
\end{aligned}$$

W_1 is the Type IIA superpotential on $\mathcal{M}$ as obtained by taking the moduli dual of the Type IIB superpotential defined on $\mathcal{W}$, the exchange of the holomorphic forms and the alteration $\mathsf{D} \to \mathcal{D}$. Comparing (7.2.3) with (7.2.8) we have $M_1 = M_3$ and $M_1' = M_3'$, once we account for the different ways of labelling the moduli degrees of freedom and similarly for the pair (7.2.4) and (7.2.9). Given this relationship between non-integral expression of Type IIB with integral expressions of Type IIA we would expect the reverse to also be true, the integrals of Type IIB are equal to expression in Type IIA which do not in general have an integral expression.

$$W_3 \;=\; \underline{\mathfrak{U}}^\top \cdot \mathsf{h}_\nu \cdot \Big(\mathbb{C} \cdot \mathsf{M}_3 + \mathbb{C}' \cdot \mathsf{M}_3'\Big) \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathfrak{T}} \tag{7.2.10}$$

$$W_4 \;=\; \underline{\mathfrak{T}}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \Big(\widetilde{\mathbb{C}} \cdot \mathsf{N}_4 + \widetilde{\mathbb{C}}' \cdot \mathsf{N}_4'\Big) \cdot \mathsf{g}_\nu \cdot \underline{\mathfrak{U}} \tag{7.2.11}$$

As expected, the Type IIB expressions each have a Type IIA partner which is not always expressible as an integral over $\mathcal{M}$, (7.2.1) with (7.2.10) and (7.2.2) with (7.2.11). For the case of $\mathcal{W} = \mathcal{M}$ it is possible to construct integral representations in the same manner as the Type IIB case and we again take $\widetilde{\mho}$ and $\widetilde{\Omega}$ to represent the holomorphic forms which have had their moduli dependencies exchanged.

$$W_3 = \int_{\mathcal{M}} \widetilde{\Omega} \wedge \Big(\mathcal{D}_3(\widetilde{\mho}_c) + \widetilde{\mathsf{D}_3'}(\widetilde{\mho}_c')\Big) \;,\; W_4 = \int_{\mathcal{M}} \widetilde{\mho} \wedge \Big(\mathcal{D}_4(\widetilde{\Omega}_c) + \widetilde{\mathsf{D}_4'}(\widetilde{\Omega}_c')\Big) \tag{7.2.12}$$

The set of expressions $\mathsf{W}_- = \{\mathsf{W}_1, \mathsf{W}_3, W_1, W_3\}$ are linked by moduli relabelling and mirror symmetry and as such the constraints arising from the nilpotency of the related derivatives should all be equivalent. This is clearly seen when considering the pairing of W_1 with W_3 and W_3 with W_1, related by relabellings of moduli, and we have previously seen it for the mirror map related pairing of W_1 and W_1 but repeat here.

$$\underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \Big(\mathsf{C} \cdot \mathsf{M}_1 + \mathsf{C}' \cdot \mathsf{M}_1'\Big) \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathsf{U}} \quad = \quad \underline{\mathfrak{U}}^\top \cdot h_\mathfrak{a} \cdot \Big(\mathbb{C} \cdot M_1 + \mathbb{C}' \cdot M_1'\Big) \cdot g_\nu \cdot \underline{\mathfrak{T}}$$

Upon accounting for the moduli relabelling we can equate the matrices defining the expressions without having to transpose one of them and because of $\mathsf{C} = \mathbb{C}$

the complexification matrices can be factorised out and thus neglected.

$$\mathsf{h}_\nu \cdot \mathsf{M}_1 \cdot \mathsf{g}_{\mathfrak{a}} = h_{\mathfrak{a}} \cdot M_1 \cdot g_\nu$$

The interdependence of the flux matrices is determined by the choice of $\langle\ \rangle_\pm$.

$$\begin{array}{lcllll} \langle\ \rangle_\pm \to \langle\ \rangle_+ & \Rightarrow & M_1 = h_{\mathfrak{a}} \cdot \mathsf{M}_1 \cdot \mathsf{h}_{\mathfrak{a}} & , & N_1 = -h_\nu \cdot \mathsf{N}_1 \cdot \mathsf{h}_\nu \\ \langle\ \rangle_\pm \to \langle\ \rangle_- & \Rightarrow & M_1 = \mathsf{M}_1 & , & N_1 = \mathsf{N}_1 \end{array}$$

The primed cases are exactly the same because both flux sectors have their associated derivatives acting on the complexified holomorphic forms and from this it is straightforward to see that $\mathsf{D}_1^2 = 0$ is equivalent to $\mathcal{D}_1^2 = 0$. The algebra required to show this is considerably simplified by the fact the complexification matrices do not play a part in how the IIA and IIB fluxes relate to one another. This is a result of the fact that each element of W_- has the dilaton coupling the same degrees of freedom[4].

$$\begin{array}{lclcl} \text{Type IIA} & : & \Omega_c & = & - S\,\mathsf{U}_0\mathfrak{a}_0 + \mathsf{U}_i\mathfrak{a}_i - \mathsf{U}^j\mathfrak{b}^j + S\,\mathsf{U}^0\mathfrak{b}^0 \in \mathcal{M}^Q(\mathfrak{U}, S) \\ \text{Type IIB} & : & \mho_c & = & - S\,\mathfrak{T}_0\nu_0 + \mathfrak{T}_i\nu_i + \mathfrak{T}^j\widetilde{\nu}^j - S\,\mathfrak{T}^0\widetilde{\nu}^0 \in \mathcal{M}^Q(\mathsf{T}, S) \end{array}$$

As seen in our consideration of the Type IIA R-R sector it is immaterial which holomorphic form the derivatives act on, the important point is which moduli combine with the dilaton to make $\mathcal{M}^Q$. Ultimately we aim to construct a Type II theory with equivalent moduli spaces and this is most easily done by considering the $\mathrm{SL}(2,\mathbb{Z})_S$ symmetric Type IIB superpotentials, where each derivative acts on a holomorphic form of the same moduli space. As a result of this and for the sake of following on from previous results we will consider only superpotential-like expressions which have the derivatives acting on the $\mathcal{M}^Q$ holomorphic forms.

By the same reasoning the set of expressions $\mathsf{W}_+ = \{\mathsf{W}_2, \mathsf{W}_4, W_2, W_4\}$ are linked by moduli relabelling and mirror symmetry. Each one has the dilaton

[4]We temporarily drop the distinction between the Kähler and complex structure index ranges since $\mathcal{W} = \mathcal{M}$ to illustrate equivalent degrees of freedom.

coupling to the same degrees of freedom and so the complexification matrices can again be factorised out when comparing the expressions. From this it is straightforward to obtain the relationship between the different flux matrices and to show the nilpotency conditions to be equal. As an example we consider W_2 and W_2 where the complexification matrices combine with either T or U, as is the case for any other pairwise comparision of W_+ elements.

$$\underline{\mathsf{U}}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \left(\widetilde{\mathsf{C}} \cdot \mathsf{N}_2 + \widetilde{\mathsf{C}}' \cdot \mathsf{N}_2' \right) \cdot \mathsf{g}_\nu \cdot \underline{\mathsf{T}} = \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot \left(\widetilde{\mathbb{C}} \cdot N_2 + \widetilde{\mathbb{C}}' \cdot N_2' \right) \cdot g_\mathfrak{a} \cdot \underline{\mathfrak{U}}$$

Accounting for the different moduli labelling and removing the complexification matrices we have the relationship between the $\mathsf{N}_2^{(\prime)}$ and $N_2^{(\prime)}$.

$$h_\mathfrak{a} \cdot \mathsf{N}_2 \cdot \mathsf{g}_\nu = h_\nu \cdot N_2 \cdot g_\mathfrak{a}$$

As in the previous case the specific relationship between the flux matrices depends on $\langle\ \rangle_\pm$ and in each case it follows quickly that their nilpotency conditions are equivalent.

$$\begin{array}{lllll} \langle\ \rangle_\pm \to \langle\ \rangle_+ & \Rightarrow & N_2 \ = \ \mathsf{h}_\mathfrak{a} \cdot \mathsf{N}_2 \cdot h_\mathfrak{a} & , & M_2 \ = \ h_\nu \cdot \mathsf{M}_2 \cdot \mathsf{h}_\nu \\ \langle\ \rangle_\pm \to \langle\ \rangle_- & \Rightarrow & N_2 \ = \ \mathsf{N}_2 & , & M_2 \ = \ \mathsf{M}_2 \end{array}$$

In both W_- and W_+ this same factorisation has occured and led to very similar results. However, if we are to compare an element of W_- with an element of W_+ this simplification is no longer applicable. Rather than linking two superpotential-like expressions which have the same $\mathcal{M}^K \times \mathcal{M}^Q$ moduli space construction we are now comparing different moduli space constructions. This alteration of dilaton coupling presents the further complication that the two flux sectors mix. Though we have constructed the expressions in $\mathsf{W}_\pm$ such that their polynomial forms are equal the fluxes, their related covariant derivatives, components and flux matrices are not and contributions from both flux sectors, as defined in W_-, will appear in each derivative for a superpotential in W_+. We shall denote the map which converts the standard Type IIB fluxes and derivatives of W_1 into those of W_2 by

π, whose general behaviour is to map (7.1.2) to (7.1.3), which we wish to express in terms of derivatives and holomorphic forms.

7.3 Alternate Fluxes

The Kähler moduli in (7.2.1) arise due to the Kähler forms $\mathscr{J}^{(n)}$ with $\mho = \sum \mathscr{J}^{(n)}$ and we defined the complex structure equivalent of them $\mathfrak{J}^{(n)}$ in (4.3.3). Using the expressions in (4.1.42) as a guide we shall choose the non-standard way of writing the superpotential in (7.2.2) to be the form of the superpotential we examine. This expression can be broken down into simpler expressions by expressing Ω_c using the decomposition of (4.3.3).

$$\int_{\mathcal{W}} \left\langle \mho, \mathsf{D}_2(\Omega_c) \right\rangle_{\pm} \;=\; \int_{\mathcal{W}} \left\langle \mho, \mathsf{D}_2 \left(-S\,\mathfrak{J}^{(0)} + \mathfrak{J}^{(1)} + \mathfrak{J}^{(2)} - S\,\mathfrak{J}^{(3)} \right) \right\rangle_{\pm}$$

The fluxes which couple to the $\mathfrak{J}^{(n)}$ define a set of flux multiplets, $\mathfrak{F}_n$ and $\widehat{\mathfrak{F}}_m$. In the Type IIB superpotential the non-standard coupling of the Kähler moduli required us to define the flux multiplets of D in the form $\star\mathsf{F}_n$, given in (4.2.13). We also saw in the tadpole expressions of Table 5.3 that this induced in the Type IIA mirror an altering of the way in which the fluxes couple to the complex structure moduli. We also demonstrated that for both holomorphic forms in both Type II constructions if the matrice associated to this alteration, such as L and K in Type IIB , were symplectic then we could work on the level of the derivatives, such as D in Type IIB, rather than the fluxes, $\mathbf{G}$ in Type IIB. As a result precisely how we denote the flux multiplets is reduced to a matter of convention, we are not attempting to derive their string theoretic or compactification origins. We choose to use the same notation as (4.2.13).

$$\begin{aligned}
\mathsf{D}_2(\Omega_c) &= -\; S\,(\star\widehat{\mathfrak{F}}_0)\cdot\mathfrak{J}^{(0)} + (\star\mathfrak{F}_1)\cdot\mathfrak{J}^{(1)} + (\star\mathfrak{F}_2)\cdot\mathfrak{J}^{(2)} - S\,(\star\widehat{\mathfrak{F}}_3)\cdot\mathfrak{J}^{(3)} \\
\mathsf{D}_2'(\Omega_c') &= (\star\mathfrak{F}_0)\cdot\mathfrak{J}^{(0)} - S\,(\star\widehat{\mathfrak{F}}_1)\cdot\mathfrak{J}^{(1)} - S\,(\star\widehat{\mathfrak{F}}_2)\cdot\mathfrak{J}^{(2)} + (\star\mathfrak{F}_3)\cdot\mathfrak{J}^{(3)}
\end{aligned}$$

Given these definitions for the flux multiplets of $\mathsf{D}_2^{(\prime)}$ and the expansion of the $\mathfrak{J}^{(n)}$ in terms of the $\Delta^3(\mathsf{E}^*)$ basis we can define the components of the fluxes in $\Delta^*(\mathfrak{E})$ in the same way as (4.2.14) and we relate them to the actual flux multiplets which define the D_2 version of $\mathbf{G}$.

$$\begin{array}{rcl} \star\widehat{\mathfrak{F}}_0 \sim \widehat{\mathfrak{F}}_0 & : & \Big(\mathfrak{F}^{(0)}{}_I \nu_I \pm \mathfrak{F}^{(0)J}\widetilde{\nu}^J \Big)\, \iota_{\mathfrak{b}^0} \\ \star\widehat{\mathfrak{F}}_2 \sim \widehat{\mathfrak{F}}_1 & : & \Big(\mathfrak{F}^{(b)}{}_I \nu_I \pm \mathfrak{F}^{(b)J}\widetilde{\nu}^J \Big)\, \iota_{\mathfrak{b}^b} \\ \star\widehat{\mathfrak{F}}_3 \sim \widehat{\mathfrak{F}}_3 & : & \Big(\mathfrak{F}_{(0)I} \nu_I \pm \mathfrak{F}_{(0)}{}^J\widetilde{\nu}^J \Big)\, \iota_{\mathfrak{b}^0} \\ \star\widehat{\mathfrak{F}}_1 \sim \widehat{\mathfrak{F}}_2 & : & \Big(\mathfrak{F}_{(a)I} \nu_I \pm \mathfrak{F}_{(a)}{}^J\widetilde{\nu}^J \Big)\, \iota_{\mathfrak{a}_a} \end{array} \tag{7.3.1}$$

The superpotential is then straightforward to express in terms of these fluxes, in the same manner as the standard Type IIB case and we including the sign choice due to $\langle\ \rangle_\pm$.

$$\int_{\mathcal{W}} \big\langle\, \mho, \mathsf{D}_2(\Omega_c)\,\big\rangle_\pm = \left(\begin{array}{l} -S\,\mathsf{U}_0\Big(\mathfrak{F}_{(0)I}\mathsf{T}^I \pm \mathfrak{F}_{(0)}{}^J\mathsf{T}_J\Big) + \mathsf{U}_a\Big(\mathfrak{F}_{(a)I}\mathsf{T}^I \pm \mathfrak{F}_{(a)}{}^J\mathsf{T}_J\Big) \\ +S\,\mathsf{U}^0\Big(\mathfrak{F}^{(0)}{}_I\mathsf{T}^I \pm \mathfrak{F}^{(0)J}\mathsf{T}_J\Big) - \mathsf{U}^b\Big(\mathfrak{F}^{(b)}{}_I\mathsf{T}^I \pm \mathfrak{F}^{(b)J}\mathsf{T}_J\Big) \end{array} \right) \tag{7.3.2}$$

$$\int_{\mathcal{W}} \big\langle\, \mho, \mathsf{D}'_2(\Omega'_c)\,\big\rangle_\pm = \left(\begin{array}{l} \mathsf{U}_0\Big(\widehat{\mathfrak{F}}_{(0)I}\mathsf{T}^I \pm \widehat{\mathfrak{F}}_{(0)}{}^J\mathsf{T}_J\Big) - S\,\mathsf{U}_a\Big(\widehat{\mathfrak{F}}_{(a)I}\mathsf{T}^I \pm \widehat{\mathfrak{F}}_{(a)}{}^J\mathsf{T}_J\Big) \\ -\mathsf{U}^0\Big(\widehat{\mathfrak{F}}^{(0)}{}_I\mathsf{T}^I \pm \widehat{\mathfrak{F}}^{(0)J}\mathsf{T}_J\Big) + S\,\mathsf{U}^b\Big(\widehat{\mathfrak{F}}^{(b)}{}_I\mathsf{T}^I \pm \widehat{\mathfrak{F}}^{(b)J}\mathsf{T}_J\Big) \end{array} \right)$$

By comparing these two ways of writing the superpotential we can obtain the components of the $\mathfrak{F}$ in terms of the usual fluxes F and $\widehat{\mathsf{F}}$, which are given in Table 7.3. The $\widehat{\mathfrak{F}}$ cases follow in the same manner. The global factor of ∓ 1 follows from the choice of $\langle\ \rangle_\pm$, with the $\langle\ \rangle_+$ case allowing an overall factor of -1 to arise since it is symmetric on $\Delta^+(\mathsf{E}^*)$ and antisymmetric on $\Delta^-(\mathsf{E}^*)$. In terms of the $\mathfrak{F}$ components we can express the action of the derivative on $\Delta^*(\mathsf{E}^*)$ in the same way as was done for the F.

$$\begin{array}{rcl} \mathsf{D}_2(\mathfrak{a}_A) &=& \mathfrak{F}_{(A)I}\nu_I \pm \mathfrak{F}_{(A)}{}^J\widetilde{\nu}^J \\ \mathsf{D}_2(\mathfrak{b}^B) &=& \mathfrak{F}^{(B)}{}_I\nu_I \pm \mathfrak{F}^{(B)J}\widetilde{\nu}^J \end{array} \quad \Leftrightarrow \quad \begin{array}{rcl} \mathsf{D}_2(\nu_I) &=& \mp\mathfrak{F}^{(A)I}\mathfrak{a}_A \pm \mathfrak{F}_{(B)}{}^I\mathfrak{b}^B \\ \mathsf{D}_2(\widetilde{\nu}^J) &=& \mathfrak{F}^{(A)}{}_J\mathfrak{a}_A - \mathfrak{F}_{(B)J}\mathfrak{b}^B \end{array} \tag{7.3.3}$$

The action of π on the various objects of Type IIB theory can now be written in a more explicit manner, one which bears close resemblence to the action of mirror

$\mathfrak{F}$	:	$\mathfrak{F}_{(0)0}$	$\mathfrak{F}_{(0)i}$	${\mathfrak{F}_{(0)}}^{0}$	${\mathfrak{F}_{(0)}}^{j}$
$\in W$	:	$-S\,\mathsf{U}_0\mathsf{T}^0$	$-S\,\mathsf{U}_0\mathsf{T}^i$	$-S\,\mathsf{U}_0\mathsf{T}_0$	$-S\,\mathsf{U}_0\mathsf{T}_j$
F	:	$\mp\mathsf{F}^{(0)0}$	$\mp\widehat{\mathsf{F}}^{(i)0}$	$\mp{\mathsf{F}_{(0)}}^{0}$	$\mp{\widehat{\mathsf{F}}_{(j)}}^{0}$
$\mathfrak{F}$	:	$\mathfrak{F}_{(a)0}$	$\mathfrak{F}_{(a)i}$	${\mathfrak{F}_{(a)}}^{0}$	${\mathfrak{F}_{(a)}}^{j}$
$\in W$	:	$\mathsf{U}_a\mathsf{T}^0$	$\mathsf{U}_a\mathsf{T}^i$	$\mathsf{U}_a\mathsf{T}_0$	$\mathsf{U}_a\mathsf{T}_j$
F	:	$\mp\widehat{\mathsf{F}}^{(0)a}$	$\mp\mathsf{F}^{(i)a}$	$\mp{\widehat{\mathsf{F}}_{(0)}}^{a}$	$\mp{\mathsf{F}_{(j)}}^{a}$
$\mathfrak{F}$	:	${\mathfrak{F}^{(0)}}_{0}$	${\mathfrak{F}^{(0)}}_{i}$	$\mathfrak{F}^{(0)0}$	$\mathfrak{F}^{(0)j}$
$\in W$	:	$S\,\mathsf{U}^0\mathsf{T}^0$	$S\,\mathsf{U}^0\mathsf{T}^i$	$S\,\mathsf{U}^0\mathsf{T}_0$	$S\,\mathsf{U}^0\mathsf{T}_j$
F	:	$\mp{\mathsf{F}^{(0)}}_{0}$	$\mp{\widehat{\mathsf{F}}^{(i)}}_{0}$	$\mp\mathsf{F}_{(0)0}$	$\mp\widehat{\mathsf{F}}_{(j)0}$
$\mathfrak{F}$	:	${\mathfrak{F}^{(b)}}_{0}$	${\mathfrak{F}^{(b)}}_{i}$	$\mathfrak{F}^{(b)0}$	$\mathfrak{F}^{(b)j}$
$\in W$	:	$-\mathsf{U}^b\mathsf{T}^0$	$-\mathsf{U}^b\mathsf{T}^i$	$-\mathsf{U}^b\mathsf{T}_0$	$-\mathsf{U}^b\mathsf{T}_j$
F	:	$\mp{\widehat{\mathsf{F}}^{(0)}}_{b}$	$\mp{\mathsf{F}^{(i)}}_{b}$	$\mp\widehat{\mathsf{F}}_{(0)b}$	$\mp\mathsf{F}_{(j)b}$

Table 7.3: π defined components of $\mathfrak{F}$ in terms of the components of F and $\widehat{\mathsf{F}}$ and associated superpotential coefficients.

symmetry on the R-R sector but without $\mathcal{M} \leftrightarrow \mathcal{W}$ or Type IIA $\leftrightarrow$ Type IIB.

$$\pi \quad : \quad \begin{array}{ccccccc} \mathsf{D}_1^{(\prime)} & \leftrightarrow & \mathsf{D}_2^{(\prime)} & , & (\mathsf{F},\widehat{\mathsf{F}}) & \leftrightarrow & (\mathfrak{F},\widehat{\mathfrak{F}}) \\ \mathsf{M}_1^{(\prime)} & \leftrightarrow & \mathsf{M}_2^{(\prime)} & , & \Omega & \leftrightarrow & \mho \\ \mathsf{N}_1^{(\prime)} & \leftrightarrow & \mathsf{N}_2^{(\prime)} & & & & \end{array} \tag{7.3.4}$$

These actions are such that the superpotential is left invariant by π but the fluxes and derivatives are redefined. It is noteworthy also that π satisfies $\pi^2 = \mathrm{Id}$, where Id is the identity map which leaves all objects in (7.3.4) unchanged.

7.4 Alternate Flux Matrices

Before considering the constraints induced on the fluxes of $\mathsf{D}_2^{(\prime)}$ we shall derive the dependence of those fluxes on the usual $\mathsf{D}_1^{(\prime)}$ fluxes by equating the two ways of writing the superpotential in terms of flux matrices in (7.2.1) and (7.2.2).

$$\underline{\mathsf{T}}^{\top} \cdot \mathsf{h}_{\nu} \cdot \left(\mathsf{C} \cdot \mathsf{M}_1 + \mathsf{C}' \cdot \mathsf{M}'_1 \right) \cdot \mathsf{g}_{\mathfrak{a}} \cdot \underline{\mathsf{U}} = \underline{\mathsf{U}}^{\top} \cdot \mathsf{h}_{\mathfrak{a}} \cdot \left(\widetilde{\mathsf{C}} \cdot \mathsf{N}_2 + \widetilde{\mathsf{C}}' \cdot \mathsf{N}'_2 \right) \cdot \mathsf{g}_{\nu} \cdot \underline{\mathsf{T}} \tag{7.4.1}$$

The right hand expression is akin to the Type IIA superpotential but since it is defined within Type IIB the moduli are the same as the left hand expression. This is in contrast to comparing the mirror dual superpotentials, as done in Section 4.4, where the inclusion of quantum corrections alters how we related the moduli on each side of the mirror transformation. Since the Type IIB superpotential is naturally written in terms of M_1 and M'_1 we wish to express N_2 and N'_2 in terms of them. Hence, because of the non-trivial dilaton coupling caused by the inability to neglect the complexification matrices we must consider the NS-NS and R-R sector simultaneously.

$$\mathsf{C} \cdot \mathsf{M}_1 + \mathsf{C}' \cdot \mathsf{M}'_1 = \mathsf{h}_{\nu} \cdot \mathsf{g}_{\nu}^{\top} \cdot \left(\mathsf{N}_2^{\top} \cdot \widetilde{\mathsf{C}} + (\mathsf{N}'_2)^{\top} \cdot \widetilde{\mathsf{C}}' \right) \cdot \mathsf{h}_{\mathfrak{a}} \cdot \mathsf{g}_{\mathfrak{a}}^{\top}$$

The complexification matrices, tilded and not, are all diagonal and commute with the bilinear forms in both Type IIA and Type IIB and though we are assuming $h^{1,1} = h^{2,1}$ we retain the distinction between $\mathsf{C}^{(\prime)}$ and $\widetilde{\mathsf{C}}^{(\prime)}$ and their definition in terms of other matrices.

$$\mathsf{C} \;=\; \mathsf{A} - S\,\mathsf{B} \;=\; \mathcal{A}_{h^{1,1}} - S\,\mathcal{B}_{h^{1,1}} \qquad \mathsf{C}' \;=\; \mathsf{B} - S\,\mathsf{A} \;=\; \mathcal{B}_{h^{1,1}} - S\,\mathcal{A}_{h^{1,1}}$$
$$\widetilde{\mathsf{C}} \;=\; \mathcal{A} - S\,\mathcal{B} \;=\; \mathcal{A}_{h^{2,1}} - S\,\mathcal{B}_{h^{2,1}} \qquad \widetilde{\mathsf{C}}' \;=\; \mathcal{B} - S\,\mathcal{A} \;=\; \mathcal{B}_{h^{2,1}} - S\,\mathcal{A}_{h^{2,1}}$$

Commuting the $\mathsf{C}^{(\prime)}$ through the bilinear forms we can reexpress the $\mathsf{N}_2^{(\prime)}$ in terms of $\mathsf{M}_2^{(\prime)}$ through the use of (4.1.35) so that all the transposed matrices are removed. The result is $\langle\;\rangle_{\pm}$ dependent as it involves bilinear forms.

$$\mathsf{C} \cdot \mathsf{M}_1 + \mathsf{C}' \cdot \mathsf{M}'_1 \quad = \quad \mathsf{g}_{\nu} \cdot g_{\mathfrak{a}}^{\top} \cdot \left(\mathsf{M}_2 \cdot \widetilde{\mathsf{C}} + \mathsf{M}'_2 \cdot \widetilde{\mathsf{C}}' \right)$$

Inserting the specific definitions of the bilinear forms for each $\langle\;\rangle_{\pm}$ we again see that the choice $\langle\;\rangle_{-}$ leads to an expression which is particularly simple.

$$\mathsf{C} \cdot \mathsf{M}_1 + \mathsf{C}' \cdot \mathsf{M}'_1 \quad = \quad \begin{cases} \left(\mathsf{M}_2 \cdot \widetilde{\mathsf{C}} + \mathsf{M}'_2 \cdot \widetilde{\mathsf{C}}' \right) & : \; \langle\;\rangle_{\pm} \to \langle\;\rangle_{-} \\ -h_{\mathfrak{a}} \cdot \left(\mathsf{M}_2 \cdot \widetilde{\mathsf{C}} + \mathsf{M}'_2 \cdot \widetilde{\mathsf{C}}' \right) & : \; \langle\;\rangle_{\pm} \to \langle\;\rangle_{+} \end{cases} \tag{7.4.2}$$

This reformulation is that obtained by altering which holomorphic form the derivative acts upon, as previously constructed for $\langle\ \rangle_{\pm}$.

$$
\begin{aligned}
\int_{\mathcal{W}} \langle\, \Omega_c, \mathsf{D}_2(\mho)\,\rangle_- &= \int_{\mathcal{W}} \langle\, \mho, \mathsf{D}_2(\Omega_c)\,\rangle_- \\
\underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{M}_2 \cdot \mathsf{g}_\mathfrak{a} \cdot \widetilde{\mathsf{C}} \cdot \underline{\mathsf{U}} &= \underline{\mathsf{U}}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \widetilde{\mathsf{C}} \cdot \mathsf{N}_2 \cdot \mathsf{g}_\nu \cdot \underline{\mathsf{T}}
\end{aligned}
$$

By considering dilaton couplings this decomposes into a pair of equations, each involving all of the flux matrices, which can be written in terms of the $\underset{\sim\sim}{\mathsf{F}_n}$ and $\underset{\sim\sim}{\widehat{\mathsf{F}}_m}$ matrices defined in (5.3.14). To formalise this we restrict our attention to the $\langle\ \rangle_-$ case as the $\langle\ \rangle_+$ case is different only by an overall factor of $-h_\mathfrak{a}$ as given in (7.4.2) and elaborate on the expressions of (5.3.14).

$$
\begin{aligned}
\langle\, \Omega, \mathsf{D}_1(\mho_c) + \mathsf{D}_1'(\mho_c')\,\rangle_- &\equiv \langle\, \Omega, \mathsf{F}_1(\mho) - S\,\widehat{\mathsf{F}}_1(\mho)\,\rangle_- \\
&\equiv \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \Big(\underset{\sim\sim}{\mathsf{F}_1} - S\,\underset{\sim\sim}{\widehat{\mathsf{F}}_1}\Big) \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathsf{U}}
\end{aligned}
$$

Repeating this for D_2 and D_2' we make use of the anti-self adjoint properties of the derivatives on $\langle\ \rangle_-$ to change the argument of the derivatives and as a result the definition of $\underset{\sim\sim}{\mathsf{F}_2}$ and $\underset{\sim\sim}{\widehat{\mathsf{F}}_2}$ differ from the $\underset{\sim\sim}{\mathsf{F}_1}$ and $\underset{\sim\sim}{\widehat{\mathsf{F}}_1}$ cases in line with (7.4.2).

$$
\begin{aligned}
\langle\, \mho, \mathsf{D}_2(\Omega_c) + \mathsf{D}_2'(\Omega_c')\,\rangle_- &\equiv \langle\, \mho, \mathsf{F}_2(\Omega) - S\,\widehat{\mathsf{F}}_2(\Omega)\,\rangle_- \\
&\equiv \langle\, \mho, \mathsf{F}_2(\Omega)\,\rangle_- - S\,\langle\, \mho, \widehat{\mathsf{F}}_2(\Omega)\,\rangle_- \\
&\equiv \langle\, \Omega, \mathsf{F}_2(\mho)\,\rangle_- - S\,\langle\, \Omega, \widehat{\mathsf{F}}_2(\mho)\,\rangle_- \qquad (7.4.3) \\
&\equiv \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \Big(\underset{\sim\sim}{\mathsf{F}_2} - S\,\underset{\sim\sim}{\widehat{\mathsf{F}}_2}\Big) \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathsf{U}} \\
&\equiv \underline{\mathsf{T}}^\top \cdot \mathsf{h}_\nu \cdot \Big(\mathsf{M}_2 \cdot \widetilde{\mathsf{C}} + \mathsf{M}_2' \cdot \widetilde{\mathsf{C}}'\Big) \cdot \mathsf{g}_\mathfrak{a} \cdot \underline{\mathsf{U}}
\end{aligned}
$$

Comparing the definitions of $\underset{\sim\sim}{\mathsf{F}_1}$ and $\underset{\sim\sim}{\widehat{\mathsf{F}}_1}$ with those of $\underset{\sim\sim}{\mathsf{F}_2}$ and $\underset{\sim\sim}{\widehat{\mathsf{F}}_2}$ it follows that for $\langle\ \rangle_-$ they are equal, while for $\langle\ \rangle_+$ there is an overall factor, on the left, of $h_\mathfrak{a}$.

$$
\begin{aligned}
\underset{\sim\sim}{\mathsf{F}_2} &= \mathsf{M}_2 \cdot \mathcal{A} \;+\; \mathsf{M}_2' \cdot \mathcal{B} = \mathsf{A} \cdot \mathsf{M}_1 \;+\; \mathsf{B} \cdot \mathsf{M}_1' = \underset{\sim\sim}{\mathsf{F}_1} \\
\underset{\sim\sim}{\widehat{\mathsf{F}}_2} &= \mathsf{M}_2 \cdot \mathcal{B} \;+\; \mathsf{M}_2' \cdot \mathcal{A} = \mathsf{B} \cdot \mathsf{M}_1 \;+\; \mathsf{A} \cdot \mathsf{M}_1' = \underset{\sim\sim}{\widehat{\mathsf{F}}_1}
\end{aligned}
$$

Using the properties of $\mathcal{A}$ and $\mathcal{B}$ these simultaneous equations allow us to express $\mathsf{M}_2^{(\prime)}$ entirely in terms of $\mathsf{M}_1^{(\prime)}$.

$$\begin{aligned}
\mathsf{M}_2 &= \Big(\mathsf{A}\cdot\mathsf{M}_1 + \mathsf{B}\cdot\mathsf{M}_1'\Big)\cdot\mathcal{A} + \Big(\mathsf{B}\cdot\mathsf{M}_1 + \mathsf{A}\cdot\mathsf{M}_1'\Big)\cdot\mathcal{B} \\
&= \underset{\sim}{\mathsf{F}_1}\cdot\mathcal{A} + \underset{\sim}{\widehat{\mathsf{F}}_1}\cdot\mathcal{B} \\
\mathsf{M}_2' &= \Big(\mathsf{A}\cdot\mathsf{M}_1 + \mathsf{B}\cdot\mathsf{M}_1'\Big)\cdot\mathcal{B} + \Big(\mathsf{B}\cdot\mathsf{M}_1 + \mathsf{A}\cdot\mathsf{M}_1'\Big)\cdot\mathcal{A} \\
&= \underset{\sim}{\mathsf{F}_1}\cdot\mathcal{B} + \underset{\sim}{\widehat{\mathsf{F}}_1}\cdot\mathcal{A}
\end{aligned}$$

Previously, when discussing S duality transformations in Type IIB, it was convenient to view the two flux matrices as doublet partners due to their relationship with the $\mathrm{SL}(2,\mathbb{Z})_S$ doublets and the same is true here; we can express the relationship between the $\mathsf{M}_2^{(\prime)}$ and the $\mathsf{M}_1^{(\prime)}$ in terms of transformations on a two component vector using the same transformation matrices that relate the S duality flux doublet with the flux matrix doublet, as in (5.3.15).

$$\begin{pmatrix}\mathsf{M}_2\\ \mathsf{M}_2'\end{pmatrix} = \begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{B} & \mathsf{A}\end{pmatrix}_L \begin{pmatrix}\mathsf{M}_1\\ \mathsf{M}_1'\end{pmatrix}\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B} & \mathcal{A}\end{pmatrix}_R = \begin{pmatrix}\underset{\sim}{\mathsf{F}_1}\\ \underset{\sim}{\widehat{\mathsf{F}}_1}\end{pmatrix}\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B} & \mathcal{A}\end{pmatrix}_R \tag{7.4.4}$$

The L and R subscripts define the direction of multiplication.

$$\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{B} & \mathsf{A}\end{pmatrix}_L \begin{pmatrix}X\\ Y\end{pmatrix} \equiv \begin{pmatrix}\mathcal{A}\cdot X + \mathcal{B}\cdot Y\\ \mathcal{A}\cdot X + \mathcal{B}\cdot Y\end{pmatrix} \quad , \quad \begin{pmatrix}X\\ Y\end{pmatrix}\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B} & \mathcal{A}\end{pmatrix}_R \equiv \begin{pmatrix}X\cdot\mathcal{A} + Y\cdot\mathcal{B}\\ X\cdot\mathcal{A} + Y\cdot\mathcal{B}\end{pmatrix}$$

The $\mathsf{N}_i^{(\prime)}$ forms of these expressions are straightforward to construct from (7.4.4).

$$\begin{aligned}
\mathsf{N}_2 &= \mathcal{A}\cdot\Big(\mathsf{N}_1\cdot\mathsf{A} + \mathsf{N}_1'\cdot\mathsf{B}\Big) + \mathcal{B}\cdot\Big(\mathsf{N}_1\cdot\mathsf{B} + \mathsf{N}_1'\cdot\mathsf{A}\Big) \\
&= \Big(\mathcal{A}\cdot\mathsf{N}_1 + \mathcal{B}\cdot\mathsf{N}_1'\Big)\cdot\mathsf{A} + \Big(\mathcal{A}\cdot\mathsf{N}_1' + \mathcal{B}\cdot\mathsf{N}_1\Big)\cdot\mathsf{B} \\
\mathsf{N}_2' &= \mathcal{B}\cdot\Big(\mathsf{N}_1\cdot\mathsf{A} + \mathsf{N}_1'\cdot\mathsf{B}\Big) + \mathcal{A}\cdot\Big(\mathsf{N}_1\cdot\mathsf{B} + \mathsf{N}_1'\cdot\mathsf{A}\Big) \\
&= \Big(\mathcal{B}\cdot\mathsf{N}_1 + \mathcal{A}\cdot\mathsf{N}_1'\Big)\cdot\mathsf{A} + \Big(\mathcal{B}\cdot\mathsf{N}_1' + \mathcal{A}\cdot\mathsf{N}_1\Big)\cdot\mathsf{B}
\end{aligned} \tag{7.4.5}$$

These form the same kind of tranformed doublet structure as in (7.4.4)

$$\begin{pmatrix}\mathsf{N}_2\\ \mathsf{N}_2'\end{pmatrix} = \begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B} & \mathcal{A}\end{pmatrix}_L \begin{pmatrix}\mathsf{N}_1\\ \mathsf{N}_1'\end{pmatrix}\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{B} & \mathsf{A}\end{pmatrix}_R \tag{7.4.6}$$

7.5 Alternate Bianchi Constraints

7.5.1 T′ duality constraints

Given the two actions of D_2 on the $\Delta^3(\mathsf{E}^*)$ and $\Delta^+(\mathsf{E}^*)$ light forms of (7.3.3) we can construct the D_2^2 expressions.

$$\begin{aligned}
\mathsf{D}_2^2(\mathfrak{a}_A) &= \pm\Big(\mathfrak{F}_{(A)}{}^{J}\mathfrak{F}^{(B)}{}_{J} - \mathfrak{F}_{(A)I}\mathfrak{F}^{(B)I} \Big)\mathfrak{a}_B \pm \Big(\mathfrak{F}_{(A)I}\mathfrak{F}_{(B)}{}^{I} - \mathfrak{F}_{(A)}{}^{J}\mathfrak{F}_{(B)J} \Big)\mathfrak{b}^B \\
\mathsf{D}_2^2(\mathfrak{b}^B) &= \pm\Big(\mathfrak{F}^{(B)J}\mathfrak{F}^{(A)}{}_{J} - \mathfrak{F}^{(B)}{}_{I}\mathfrak{F}^{(A)I} \Big)\mathfrak{a}_A \pm \Big(\mathfrak{F}^{(B)}{}_{I}\mathfrak{F}_{(A)}{}^{I} - \mathfrak{F}^{(B)J}\mathfrak{F}_{(A)J} \Big)\mathfrak{b}^A \\
\mathsf{D}_2^2(\nu_I) &= \pm\Big(\mathfrak{F}_{(B)}{}^{I}\mathfrak{F}^{(B)}{}_{J} - \mathfrak{F}^{(A)I}\mathfrak{F}_{(A)J} \Big)\nu_J + \Big(\mathfrak{F}_{(B)}{}^{I}\mathfrak{F}^{(B)J} - \mathfrak{F}^{(B)I}\mathfrak{F}_{(B)}{}^{J} \Big)\widetilde{\nu}^J \\
\mathsf{D}_2^2(\widetilde{\nu}^J) &= +\Big(\mathfrak{F}^{(A)}{}_{J}\mathfrak{F}_{(A)I} - \mathfrak{F}_{(B)J}\mathfrak{F}^{(B)}{}_{I} \Big)\nu_I \pm \Big(\mathfrak{F}^{(A)}{}_{J}\mathfrak{F}_{(A)}{}^{I} - \mathfrak{F}_{(B)J}\mathfrak{F}^{(B)I} \Big)\widetilde{\nu}^I
\end{aligned} \tag{7.5.1}$$

Although we can use Table 7.3 to convert these expressions into the F and $\widehat{\mathsf{F}}$ components, it is more convenient to work with flux matrices, as the generalisation to the S duality case is more forthcoming in that formulation. In terms of flux matrices the constraints on the fluxes as a result of the nilpotency of D_2 are not equivalent to the D_1 nilpotency constraints, due to the existence and placement of the projection-like matrices $\mathcal{A}$ and $\mathcal{B}$. To examine this we redefine our notation for each of the flux matrices such that the expressions relating to $\mathsf{M}_i \cdot \mathsf{h}_\mathfrak{a} \cdot \mathsf{N}_i = 0$ simplify and we again use $\cdot \mathsf{h}_\mathfrak{a} \cdot = \diamond$.[5]

$$\begin{pmatrix} \mathsf{m}_2 \\ \mathsf{m}_2' \end{pmatrix} = \begin{pmatrix} \mathsf{M}_2 \\ \mathsf{M}_2' \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}_R \quad , \quad \begin{pmatrix} \mathsf{n}_2 \\ \mathsf{n}_2' \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}_L \begin{pmatrix} \mathsf{N}_2 \\ \mathsf{N}_2' \end{pmatrix}$$

Due to the orthogonality of $\mathcal{A}$ and $\mathcal{B}$ half of the terms in the expansion of $\mathsf{M}_2^{(\prime)} \diamond \mathsf{N}_2^{(\prime)}$ as linear combinations of $\mathsf{M}_1^{(\prime)} \diamond \mathsf{N}_1^{(\prime)}$ are identically zero, as was seen when considering S duality constraints. With each of the four cases being of the same format, only differing by location and number of primed flux matrices, without

[5] The case of $\mathsf{N}_i \cdot \mathsf{h}_\nu \cdot \mathsf{M}_i = 0$ follows in the same manner if we did a different redefinition in which we factorised out the matrices $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}$.

much loss of generality we explicitly consider the first case.

$$\mathsf{M}_2 \diamond \mathsf{N}_2 = \mathsf{m}_2 \cdot \mathcal{A} \diamond \mathcal{A} \cdot \mathsf{n}_2 + \mathsf{m}'_2 \cdot \mathcal{B} \diamond \mathcal{B} \cdot \mathsf{n}'_2$$
$$= \begin{pmatrix} \mathsf{A} \cdot \Big(\mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 + \mathsf{M}'_1 \cdot \mathcal{B} \diamond \mathsf{N}'_1 \Big) \cdot \mathsf{A} \\ + \mathsf{B} \cdot \Big(\mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 + \mathsf{M}_1 \cdot \mathcal{B} \diamond \mathsf{N}'_1 \Big) \cdot \mathsf{A} \\ + \mathsf{A} \cdot \Big(\mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 + \mathsf{M}'_1 \cdot \mathcal{B} \diamond \mathsf{N}_1 \Big) \cdot \mathsf{B} \\ + \mathsf{B} \cdot \Big(\mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 + \mathsf{M}_1 \cdot \mathcal{B} \diamond \mathsf{N}_1 \Big) \cdot \mathsf{B} \end{pmatrix} \tag{7.5.2}$$

This bears a strong resemblence to (5.3.20), except that there are $\mathcal{A}$ and $\mathcal{B}$ factors between the two flux matrices as well as being external to each term. By using the projection properties of the external $\mathcal{A}$ and $\mathcal{B}$ we can compare the components of $\mathsf{M}_2 \diamond \mathsf{N}_2$ with those of $\mathsf{M}_1 \diamond \mathsf{N}_1$ and $\mathsf{M}'_1 \diamond \mathsf{N}'_1$, as well as $\mathsf{M}'_2 \diamond \mathsf{N}'_2$. In order to drop the non-degenerate external factors dependent upon any bilinear forms we consider the ideals generated by the components of the flux matrices.

$$\begin{aligned} \Big\langle \mathsf{A} \cdot \mathsf{M}_2 \diamond \mathsf{N}_2 \cdot \mathsf{A} \Big\rangle &= \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 + \mathsf{M}'_1 \cdot \mathcal{B} \diamond \mathsf{N}'_1 \Big) \cdot \mathsf{A} \Big\rangle \\ \Big\langle \mathsf{A} \cdot \mathsf{M}'_2 \diamond \mathsf{N}'_2 \cdot \mathsf{A} \Big\rangle &= \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}_1 \cdot \mathcal{B} \diamond \mathsf{N}_1 + \mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 \Big) \cdot \mathsf{A} \Big\rangle \\ \Big\langle \mathsf{A} \cdot \mathsf{M}_1 \diamond \mathsf{N}_1 \cdot \mathsf{A} \Big\rangle &= \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}_1 \cdot \mathbb{I} \diamond \mathsf{N}_1 + \mathsf{M}'_1 \cdot 0 \diamond \mathsf{N}'_1 \Big) \cdot \mathsf{A} \Big\rangle \\ \Big\langle \mathsf{A} \cdot \mathsf{M}'_1 \diamond \mathsf{N}'_1 \cdot \mathsf{A} \Big\rangle &= \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}_1 \cdot 0 \diamond \mathsf{N}_1 + \mathsf{M}'_1 \cdot \mathbb{I} \diamond \mathsf{N}'_1 \Big) \cdot \mathsf{A} \Big\rangle \end{aligned} \tag{7.5.3}$$

It is clear from the fact $\mathcal{A}$ and $\mathcal{B}$ are internal to the flux matrix pairings of $\mathsf{M}_2^{(\prime)} \diamond \mathsf{N}_2^{(\prime)}$ that they cannot be written as some linear combination of the $\mathsf{M}_1^{(\prime)} \diamond \mathsf{N}_1^{(\prime)}$ and so the T' constraints associated with the derivatives defining W_2 in (7.2.2) provide different constraints to those of W_1 in (7.2.1). However, it is clear from (7.5.3) that the constraints are equivalent on a slightly weaker level, in that the sum of the two terms associated with W_1 is equal to the sum of the terms associated with W_2.

$$\begin{aligned} \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}_2 \diamond \mathsf{N}_2 + \mathsf{M}'_2 \diamond \mathsf{N}'_2 \Big) \cdot \mathsf{A} \Big\rangle &= \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}_1 \diamond \mathsf{N}_1 + \mathsf{M}'_1 \diamond \mathsf{N}'_1 \Big) \cdot \mathsf{A} \Big\rangle \\ \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}'_2 \diamond \mathsf{N}_2 + \mathsf{M}'_2 \diamond \mathsf{N}_2 \Big) \cdot \mathsf{A} \Big\rangle &= \Big\langle \mathsf{A} \cdot \Big(\mathsf{M}_1 \diamond \mathsf{N}'_1 + \mathsf{M}'_1 \diamond \mathsf{N}_1 \Big) \cdot \mathsf{A} \Big\rangle \end{aligned}$$

These kinds of flux combinations have been previously seen in our analysis of S duality, forming terms in $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets. Since we have explicitly assumed both NS-NS and R-R fluxes are all potentially non-zero we have to consider what kind of flux structures are induced by S duality.

7.5.2 S duality constraints

In order to examine this further we repeat the method used to examine the S duality of the Type IIB W_1 superpotential but now we look at W_2, by expressing $\mathsf{M}_2 \cdot \widetilde{\mathsf{C}} + \mathsf{M}'_2 \cdot \widetilde{\mathsf{C}}'$ as an inner product.

$$\mathsf{M}_2 \cdot \widetilde{\mathsf{C}} + \mathsf{M}'_2 \cdot \widetilde{\mathsf{C}}' = \begin{pmatrix} \mathsf{M}_2 & \mathsf{M}'_2 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\mathsf{C}} \\ \widetilde{\mathsf{C}}' \end{pmatrix}$$

Using previous results for how the complexification matrices transform under $\mathrm{SL}(2,\mathbb{Z})_S$ we have the transformation properties of the doublet formed of the two flux matrices and the transformation on the $\mathsf{N}_2^{(\prime)}$ follow or can be obtained directly from the definition of W_2.

$$\begin{aligned} \begin{pmatrix} \widetilde{\mathsf{C}} \\ \widetilde{\mathsf{C}}' \end{pmatrix} &\to && \Big(\big(\Gamma_S^{\top}\big)^{-1} \otimes \mathcal{A} + \big(\sigma \cdot \Gamma_S^{\top} \cdot \sigma\big)^{-1} \otimes \mathcal{B} \Big) \begin{pmatrix} \widetilde{\mathsf{C}} \\ \widetilde{\mathsf{C}}' \end{pmatrix} \\ \begin{pmatrix} \mathsf{M}_2 & \mathsf{M}'_2 \end{pmatrix} &\to & \begin{pmatrix} \mathsf{M}_2 & \mathsf{M}'_2 \end{pmatrix} & \big(\Gamma_S^{\top} \otimes \mathcal{A} + \big(\sigma \cdot \Gamma_S^{\top} \cdot \sigma\big) \otimes \mathcal{B}\big) \\ \begin{pmatrix} \mathsf{N}_2 \\ \mathsf{N}'_2 \end{pmatrix} &\to && \big(\Gamma_S \otimes \mathcal{A} + \big(\sigma \cdot \Gamma_S \cdot \sigma\big) \otimes \mathcal{B}\big) \begin{pmatrix} \mathsf{N}_2 \\ \mathsf{N}'_2 \end{pmatrix} \end{aligned} \tag{7.5.4}$$

These are precisely those transformations seen in our previous analysis S duality in (5.3.16) and (5.3.18) but with certain relabellings.

$$\mathsf{M}_{1,2} \leftrightarrow \mathsf{N}_{2,1} \ , \ \mathsf{M}'_{1,2} \leftrightarrow \mathsf{N}'_{2,1} \ , \ \mathcal{A} \leftrightarrow \mathsf{A} \ , \ \mathcal{B} \leftrightarrow \mathsf{B}$$

The immediate implication of this fact is that we can deduce all the $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets associated to W_2 from the known $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets associated to W_1. Applying these relabellings to $\mathbf{3}_{\mathsf{A}/\mathsf{B}}$ of (5.3.22) we obtain $\mathbf{3}_{\mathcal{A}/\mathcal{B}}$ and the pair of singlets $\mathbf{1}_{\mathcal{A}/\mathcal{B}}$ follow in the same manner from $\mathbf{1}_{\mathsf{A}/\mathsf{B}}$ in (5.3.24).

$$\begin{aligned}
\mathbf{3}_{\mathcal{A}} &\equiv \langle\ \mathsf{M}_2 \cdot \mathcal{A} \diamond \mathsf{N}_2 \ \ ,\ \ \mathsf{M}'_2 \cdot \mathcal{A} \diamond \mathsf{N}'_2 \ \ ,\ \ \mathsf{M}'_2 \cdot \mathcal{A} \diamond \mathsf{N}_2 + \mathsf{M}_2 \cdot \mathcal{A} \diamond \mathsf{N}'_2\ \rangle \\
\mathbf{3}_{\mathcal{B}} &\equiv \langle\ \mathsf{M}_2 \cdot \mathcal{B} \diamond \mathsf{N}_2 \ \ ,\ \ \mathsf{M}'_2 \cdot \mathcal{B} \diamond \mathsf{N}'_2 \ \ ,\ \ \mathsf{M}'_2 \cdot \mathcal{B} \diamond \mathsf{N}_2 + \mathsf{M}_2 \cdot \mathcal{B} \diamond \mathsf{N}'_2\ \rangle \\
\mathbf{1}_{\mathcal{A}} &\equiv \langle\ \mathsf{M}'_2 \cdot \mathcal{A} \diamond \mathsf{N}_2 \ -\ \mathsf{M}_2 \cdot \mathcal{A} \diamond \mathsf{N}'_2\ \rangle \\
\mathbf{1}_{\mathcal{B}} &\equiv \langle\ \mathsf{M}'_2 \cdot \mathcal{B} \diamond \mathsf{N}_2 \ -\ \mathsf{M}_2 \cdot \mathcal{B} \diamond \mathsf{N}'_2\ \rangle
\end{aligned} \tag{7.5.5}$$

The introduction of these $\mathcal{A}$ and $\mathcal{B}$ terms inside the flux matrix pairings allows us to make use of (7.4.4) and (7.4.6) to compare these W_2 multiplets with the W_1 multiplets. Due to the linearly independent decomposition (5.3.19) $\mathbf{3}_{\mathcal{A}}$ can be written as a union of ideals defined by this decomposition.

$$\begin{aligned}
\mathbf{3}_{\mathcal{A}} = &\langle\, \mathsf{A} \cdot \mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 \cdot \mathsf{A}\ ,\ \mathsf{A} \cdot \mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 \cdot \mathsf{A}\ ,\ \mathsf{A} \cdot (\mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 + \mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1) \cdot \mathsf{A}\,\rangle \\
&\cup \langle\, \mathsf{B} \cdot \mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 \cdot \mathsf{A}\ ,\ \mathsf{B} \cdot \mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 \cdot \mathsf{A}\ ,\ \mathsf{B} \cdot (\mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 + \mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1) \cdot \mathsf{A}\,\rangle \\
&\cup \langle\, \mathsf{A} \cdot \mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 \cdot \mathsf{B}\ ,\ \mathsf{A} \cdot \mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 \cdot \mathsf{B}\ ,\ \mathsf{A} \cdot (\mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 + \mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}_1) \cdot \mathsf{B}\,\rangle \\
&\cup \langle\, \mathsf{B} \cdot \mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 \cdot \mathsf{B}\ ,\ \mathsf{B} \cdot \mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}_1 \cdot \mathsf{B}\ ,\ \mathsf{B} \cdot (\mathsf{M}_1 \cdot \mathcal{A} \diamond \mathsf{N}'_1 + \mathsf{M}'_1 \cdot \mathcal{A} \diamond \mathsf{N}_1) \cdot \mathsf{B}\,\rangle
\end{aligned}$$

By considering the splittings and decompositions due to $\mathcal{A}$, $\mathcal{B}$, A and B it can be seen that the union of all the $\mathrm{SL}(2,\mathbb{Z})_S$ ideals of W_1 is equal to the union of all the $\mathrm{SL}(2,\mathbb{Z})_S$ ideals of W_2 but individually the ideals are not equal to one another.

$$\mathbf{3}_{\mathcal{A}} \cup \mathbf{3}_{\mathcal{B}} \cup \mathbf{1}_{\mathcal{A}} \cup \mathbf{1}_{\mathcal{B}} = \mathbf{3}_{\mathsf{A}} \cup \mathbf{3}_{\mathsf{B}} \cup \mathbf{1}_{\mathsf{A}} \cup \mathbf{1}_{\mathsf{B}}$$

The second set of $\mathrm{SL}(2,\mathbb{Z})_S$ triplets on W_2 follow (5.3.22) and (5.3.23) by the same relabelling, including the bilinear forms.

$$\begin{aligned}
\mathbf{3}_{\mathcal{A}\mathcal{A}} &\equiv \langle\ \mathcal{A} \cdot \mathsf{N}_2 \triangleleft \mathsf{M}_2 \cdot \mathcal{A}\ \ ,\ \ \mathcal{A} \cdot \mathsf{N}'_2 \triangleleft \mathsf{M}'_2 \cdot \mathcal{A}\ \ ,\ \ \mathcal{A} \cdot (\ \mathsf{N}'_2 \triangleleft \mathsf{M}_2 + \mathsf{N}_2 \triangleleft \mathsf{M}'_2\ ) \cdot \mathcal{A}\ \rangle \\
\mathbf{3}_{\mathcal{A}\mathcal{B}} &\equiv \langle\ \mathcal{A} \cdot \mathsf{N}'_2 \triangleleft \mathsf{M}_2 \cdot \mathcal{B}\ \ ,\ \ \mathcal{A} \cdot \mathsf{N}_2 \triangleleft \mathsf{M}'_2 \cdot \mathcal{B}\ \ ,\ \ \mathcal{A} \cdot (\ \mathsf{N}_2 \triangleleft \mathsf{M}_2 + \mathsf{N}'_2 \triangleleft \mathsf{M}'_2\ ) \cdot \mathcal{B}\ \rangle \\
\mathbf{3}_{\mathcal{B}\mathcal{A}} &\equiv \langle\ \mathcal{B} \cdot \mathsf{N}_2 \triangleleft \mathsf{M}'_2 \cdot \mathcal{A}\ \ ,\ \ \mathcal{B} \cdot \mathsf{N}'_2 \triangleleft \mathsf{M}_2 \cdot \mathcal{A}\ \ ,\ \ \mathcal{B} \cdot (\ \mathsf{N}'_2 \triangleleft \mathsf{M}'_2 + \mathsf{N}_2 \triangleleft \mathsf{M}_2\ ) \cdot \mathcal{A}\ \rangle \\
\mathbf{3}_{\mathcal{B}\mathcal{B}} &\equiv \langle\ \mathcal{B} \cdot \mathsf{N}'_2 \triangleleft \mathsf{M}'_2 \cdot \mathcal{B}\ \ ,\ \ \mathcal{B} \cdot \mathsf{N}_2 \triangleleft \mathsf{M}_2 \cdot \mathcal{B}\ \ ,\ \ \mathcal{B} \cdot (\ \mathsf{N}_2 \triangleleft \mathsf{M}'_2 + \mathsf{N}'_2 \triangleleft \mathsf{M}_2\ ) \cdot \mathcal{B}\ \rangle
\end{aligned}$$

These can then be written in terms of the W_1 flux matrices using (7.4.4) and (7.4.5), though we only do so explicitly for $\mathbf{3}_{\mathcal{A}\mathcal{A}}$ due to the length of the expressions. The remaining multiplets follow the same general structure but with appropriate (un)priming of the flux matrices and bilinear forms.

$$\begin{aligned}\mathbf{3}_{\mathcal{AA}} = \quad & \langle \mathcal{A}\cdot(\ \mathsf{N}_1\cdot\mathsf{A}\lhd\mathsf{M}_1 \ + \ \mathsf{N}'_1\cdot\mathsf{A}\lhd\mathsf{M}'_1\ )\cdot\mathcal{A}\rangle \ \cup \\ \cup\ & \langle \mathcal{A}\cdot(\ \mathsf{N}_1\cdot\mathsf{B}\lhd\mathsf{M}_1 \ + \ \mathsf{N}'_1\cdot\mathsf{A}\lhd\mathsf{M}'_1\ )\cdot\mathcal{A}\rangle \ \cup \\ \cup\ & \langle \mathcal{A}\cdot(\ \mathsf{N}_1\lhd\mathsf{M}'_1 \ + \ \mathsf{N}'_1\lhd\mathsf{M}_1\ )\cdot\mathcal{A}\rangle\end{aligned}$$

As before the singlets are the third term of each triplet with a sign change.

$$\begin{aligned}\mathbf{1}_{\mathcal{AA}} &\equiv \langle\ \mathcal{A}\cdot(\ \mathsf{N}'_2\lhd\mathsf{M}_2 \ - \ \mathsf{N}_2\lhd\mathsf{M}'_2\ )\cdot\mathcal{A}\ \rangle \\ \mathbf{1}_{\mathcal{AB}} &\equiv \langle\ \mathcal{A}\cdot(\ \mathsf{N}_2\lhd\mathsf{M}_2 \ - \ \mathsf{N}'_2\lhd\mathsf{M}'_2\ )\cdot\mathcal{B}\ \rangle \\ \mathbf{1}_{\mathcal{BA}} &\equiv \langle\ \mathcal{B}\cdot(\ \mathsf{N}'_2\lhd\mathsf{M}'_2 \ - \ \mathsf{N}_2\lhd\mathsf{M}_2\ )\cdot\mathcal{A}\ \rangle \\ \mathbf{1}_{\mathcal{BB}} &\equiv \langle\ \mathcal{B}\cdot(\ \mathsf{N}_2\lhd\mathsf{M}'_2 \ - \ \mathsf{N}'_2\lhd\mathsf{M}_2\ )\cdot\mathcal{B}\ \rangle\end{aligned}$$

However, when expressed in terms of the W_1 flux matrices this simple relation between the triplet and singlet generator functions is lost.

$$\mathbf{1}_{\mathcal{AA}} = \langle\mathcal{A}\cdot(\ \mathsf{N}'_1\cdot(\mathsf{A}-\mathsf{B})\lhd\mathsf{M}_1 \ - \ \mathsf{N}_1\cdot(\mathsf{A}-\mathsf{B})\lhd\mathsf{M}'_1\ )\cdot\mathcal{A}\rangle$$

In our examination of the usual formulation of the fluxes and superpotential we noted that not all of these $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets are Bianchi constraints, some of them are non-zero and measure tadpole contributions due to branes and their S duality images. Which type of constraint a particular multiplet fell into was given in Table 5.4 and we would expect a similar behaviour in these multiplets. The simplest tadpole considered was the C_4 potential which coupled to the external space filling D3 branes whose flux contribution $H_3\wedge F_3 \sim \widehat{\mathsf{F}}_0\wedge\mathsf{F}_0$ could be written in terms of derivatives as being proportional to $\iota_{\nu_0}(\mathsf{D}_1\mathsf{D}'_1 - \mathsf{D}'_1\mathsf{D}_1)(\widetilde{\nu}^0) \in \mathbf{1}_{\mathsf{BB}}$. The π image of this is obtained by replacing the W_1 derivatives with those of W_2 and the flux polynomials associated to that appear in the $\mathbf{1}_{\mathcal{BB}}$ singlet and can be written in terms of derivatives as $\iota_{\nu_0}(\mathsf{D}_2\mathsf{D}'_2 - \mathsf{D}'_2\mathsf{D}_2)(\widetilde{\nu}^0)$. How the tadpole contributions are to be viewed in terms of the action of π on the branes of the Type IIB theory is a question we shall not address other than to comment that $\iota_{\nu_0}(\mathsf{D}_2\mathsf{D}'_2 - \mathsf{D}'_2\mathsf{D}_2)(\widetilde{\nu}^0)$ contains the fluxes found on branes other than the D3 branes in the formulation of the W_1 superpotential, including extended objects which are the NS-NS counterparts of the D branes.

7.5.3 Reduced superpotential expression

If we assume that the formulation of W_2 is as valid as that of W_1 then we can express the superpotential in a way which is symmetric in its treatment of the moduli spaces. By using (4.1.42) as a guide we have obtained the relationship between the flux matrices of W_1 in (7.2.1) and those of W_2 in (7.2.2). To motivate this further we consider a superpotential-like expression $\mathsf{W}_{\mathfrak{D}}$ which is defined as a scalar product involving the moduli vector $\underline{\Phi}$ and a matrix $\underline{\underline{\mathfrak{D}}}$. We do not treat $\underline{\underline{\mathfrak{D}}}$ as the matrix associated to a derivative, only a linear operator on the cohomology bases so that the associated flux matrices $\mathsf{M}_{\mathfrak{D}}$ and $\mathsf{N}_{\mathfrak{D}}$ are independent. However, we use notation which follows previous superpotential-like scalar products.

$$\mathsf{W}_{\mathfrak{D}} \equiv \underline{\Phi}^{\top} \cdot h \cdot \mathfrak{C} \cdot \underline{\underline{\mathfrak{D}}} \cdot g \cdot \underline{\Phi} \qquad \underline{\underline{\mathfrak{D}}} = \begin{pmatrix} 0 & \mathsf{M}_{\mathfrak{D}} \\ \mathsf{N}_{\mathfrak{D}} & 0 \end{pmatrix} \qquad \mathfrak{C} = \begin{pmatrix} \mathsf{C} & 0 \\ 0 & \widetilde{\mathsf{C}} \end{pmatrix}$$

With $h^{1,1} = h^{2,1}$ the complexification matrices are equal, $\mathsf{C} = \widetilde{\mathsf{C}}$, and so for convenience we use $\mathfrak{C} = \mathbb{I}_2 \otimes \mathsf{C}$. Expanding $\mathsf{W}_{\mathfrak{D}}$ out in terms of the individual moduli sectors results in a pair of terms, one of the form seen in W_1 and the other of the form seen in W_2.

$$\mathsf{W}_{\mathfrak{D}} \; = \; \underline{\mathsf{T}}^{\top} \cdot \mathsf{h}_{\nu} \cdot \mathsf{C} \cdot \mathsf{M}_{\mathfrak{D}} \cdot \mathsf{g}_{\mathfrak{a}} \cdot \underline{\mathsf{U}} \; + \; \underline{\mathsf{U}}^{\top} \cdot \mathsf{h}_{\mathfrak{a}} \cdot \mathsf{C} \cdot \mathsf{N}_{\mathfrak{D}} \cdot \mathsf{g}_{\nu} \cdot \underline{\mathsf{T}} \tag{7.5.6}$$

This is in contrast to previous superpotential expressions considered, where the matrices $\underline{\underline{\Omega}}$ and $\underline{\underline{\mho}}$ are defined with a projection matrix $\mathcal{P}^{\pm}$ so that one of the two terms is projected out. In general there are two contributions to the superpotential due to the different flux sectors so if the two moduli spaces are equivalent we would expect it to be possible to express the full superpotential in the same manner as (7.5.6). On the assumption that $\mathsf{W}_2 \equiv \pi(\mathsf{W}_1) = \mathsf{W}_1$ the superpotential W, which is normally written as having the form of W_1, is proportional to $\mathsf{W}_1 + \mathsf{W}_2$ and the proportionality constant can be gauged to 1.

$$W = \mathsf{W}_1 + \mathsf{W}_2 = \begin{pmatrix} \mathsf{T}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C} \cdot \mathsf{M}_1 \cdot \mathsf{g}_\mathfrak{a} \cdot \mathsf{U} \; + \; \mathsf{U}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \mathsf{C} \cdot \mathsf{N}_2 \cdot \mathsf{g}_\nu \cdot \mathsf{T} \\ + \; \mathsf{T}^\top \cdot \mathsf{h}_\nu \cdot \mathsf{C}' \cdot \mathsf{M}'_1 \cdot \mathsf{g}_\mathfrak{a} \cdot \mathsf{U} \; + \; \mathsf{U}^\top \cdot \mathsf{h}_\mathfrak{a} \cdot \mathsf{C}' \cdot \mathsf{N}'_2 \cdot \mathsf{g}_\nu \cdot \mathsf{T} \end{pmatrix}$$

Given the fact $\pi^2 = \mathrm{Id}$ by construction we have that $W = \mathsf{W}_1 + \pi(\mathsf{W}_1)$ is π invariant and therefore the two moduli spaces are treated in the same manner. Comparing the scalar product expression for W with (7.5.6) we can see that the two pairs of terms, relating to primed and non-primed flux matrices, suggest we consider a pair of matrices where there is no mixing between the primed and non-primed flux matrices.

$$\begin{aligned} \underline{\underline{\mathfrak{D}}} &= \begin{pmatrix} 0 & \mathsf{M}_1 \\ \mathsf{N}_2 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathsf{M}_1 \\ \mathsf{N}_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} 0 & \mathsf{M}_2 \\ \mathsf{N}_2 & 0 \end{pmatrix} \\ &\equiv P^+ \cdot \underline{\underline{\mathsf{D}_1}} + P^- \cdot \underline{\underline{\mathsf{D}_2}} \\ \underline{\underline{\mathfrak{D}'}} &= \begin{pmatrix} 0 & \mathsf{M}'_1 \\ \mathsf{N}'_2 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathsf{M}'_1 \\ \mathsf{N}'_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} 0 & \mathsf{M}'_2 \\ \mathsf{N}'_2 & 0 \end{pmatrix} \\ &\equiv P^+ \cdot \underline{\underline{\mathsf{D}'_1}} + P^- \cdot \underline{\underline{\mathsf{D}'_2}} \end{aligned}$$

Due to the non-trivial mixing between the NS-NS and R-R sectors in (7.4.4) and (7.4.5) the distinction between the two flux sectors is no longer a simple one but with $\Phi = \underline{\Phi}^\top \cdot h \cdot \mathbf{e} = \mho + \Omega$ we are able to express the superpotential in a way which treats the two moduli spaces in the same manner, using $\mathfrak{D}_\circ\mathfrak{C}(\Phi) \equiv \mathfrak{D}\big(\mathfrak{C}(\Phi)\big) = \underline{\Phi}^\top \cdot \mathfrak{C} \cdot \underline{\underline{\mathfrak{D}}} \cdot \mathbf{e}$. This is not equivalent to $\mathfrak{D}(\mathfrak{C}(\Phi))$ as the actions of $\mathfrak{D}$ on $\Delta^3(\mathsf{E}^*)$ and $\Delta^+(\mathsf{E}^*)$ are not equivalent and thus $\mathfrak{D}$ as a derivative is ill defined.

$$\begin{aligned} W &= \underline{\Phi}^\top \cdot \mathsf{h} \cdot \left(\mathfrak{C} \cdot \underline{\underline{\mathfrak{D}}} + \mathfrak{C}' \cdot \underline{\underline{\mathfrak{D}'}} \right) \cdot \mathsf{g} \cdot \underline{\Phi} \\ &= \mathsf{g}\Big(\Phi \, , \, (\mathfrak{D}_\circ\mathfrak{C} + \mathfrak{D}'_\circ\mathfrak{C}')(\Phi) \Big) \\ &= \int_{\mathcal{W}} \Big(\mho + \Omega \Big) \wedge \Big(\mathfrak{D}_\circ\mathfrak{C} + \mathfrak{D}'_\circ\mathfrak{C}' \Big) \Big(\mho + \Omega \Big) \end{aligned}$$

In our examination of S duality we found it convenient to consider the invariance of $\mathsf{C} \cdot \mathsf{M} + \mathsf{C}' \cdot \mathsf{M}' = \underset{\sim}{\mathsf{F}} - S \widehat{\underset{\sim}{\mathsf{F}}}$, from which we could deduce the S duality transformation properties of the fluxes. Now that we have combined the two moduli spaces we can extend this further.

$$\mathfrak{C} \cdot \underline{\underline{\mathfrak{D}}} + \mathfrak{C}' \cdot \underline{\underline{\mathfrak{D}'}} \equiv \underline{\underline{\mathsf{F}}} - S\,\underline{\underline{\widehat{\mathsf{F}}}} \quad \Rightarrow \quad \begin{pmatrix} \mathfrak{C} \\ \mathfrak{C}' \end{pmatrix} \cdot \begin{pmatrix} \underline{\underline{\mathfrak{D}}} & \underline{\underline{\mathfrak{D}'}} \end{pmatrix} = \begin{pmatrix} 1 \\ -S \end{pmatrix} \cdot \begin{pmatrix} \underline{\underline{\mathsf{F}}} & \underline{\underline{\widehat{\mathsf{F}}}} \end{pmatrix}$$

Given the same schematic structure the flux dependent matrices can be related to one another in the same manner as (5.3.15), except that the dimensions of the matrices have increased so we denote $\mathbb{I}_2 \otimes \mathcal{A}$ by $\underline{\underline{A}}$ and likewise for $\underline{\underline{B}}$.

$$\begin{pmatrix} \underline{\underline{\mathsf{F}}} \\ \underline{\underline{\widehat{\mathsf{F}}}} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}} & \underline{\underline{A}} \end{pmatrix} \begin{pmatrix} \underline{\underline{\mathfrak{D}}} \\ \underline{\underline{\mathfrak{D}'}} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \underline{\underline{\mathfrak{D}}} \\ \underline{\underline{\mathfrak{D}'}} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}} & \underline{\underline{A}} \end{pmatrix} \begin{pmatrix} \underline{\underline{\mathsf{F}}} \\ \underline{\underline{\widehat{\mathsf{F}}}} \end{pmatrix}$$

This allows us to much more succinctly state the S duality transformation properties of the fluxes in this moduli symmetric formulation under $S \to \frac{aS+b}{cS+d}$.

$$\begin{pmatrix} \underline{\underline{\mathfrak{D}}} \\ \underline{\underline{\mathfrak{D}'}} \end{pmatrix} \to \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}} & \underline{\underline{A}} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}} & \underline{\underline{A}} \end{pmatrix} \begin{pmatrix} \underline{\underline{\mathfrak{D}}} \\ \underline{\underline{\mathfrak{D}'}} \end{pmatrix}$$

This result combined with the flux matrix definitions of $\underline{\underline{\mathfrak{D}}}$ and $\underline{\underline{\mathfrak{D}'}}$ and the $\mathbb{I}_2$ term in $\underline{\underline{A}}$ and $\underline{\underline{B}}$ again illustrates that the $\mathsf{M}_1^{(\prime)}$ and $\mathsf{N}_2^{(\prime)}$ have equivalent $\mathrm{SL}(2,\mathbb{Z})_S$ transformations, as noted in (7.5.4) and required by definition (7.4.1). With this definition of $\underline{\underline{\mathsf{F}}}$ and $\underline{\underline{\widehat{\mathsf{F}}}}$ we can reduce the superpotential down to a simple form.

$$W = \underline{\Phi}^\top \cdot \mathsf{h} \cdot \left(\mathfrak{C} \cdot \underline{\underline{\mathfrak{D}}} + \mathfrak{C}' \cdot \underline{\underline{\mathfrak{D}'}} \right) \cdot \mathsf{g} \cdot \underline{\Phi} = \underline{\Phi}^\top \cdot \mathsf{h} \cdot \left(\underline{\underline{\mathsf{F}}} - S\,\underline{\underline{\widehat{\mathsf{F}}}} \right) \cdot \mathsf{g} \cdot \underline{\Phi}$$

This formulation makes S duality transformation properties and the symmetry in moduli treatment manifest. Previously we had seen that the complex structure moduli sector possesses $\mathrm{Sp}(h^{1,1}+1)$ invariance due to its $\mathbf{f}_{(\mathfrak{a})}$ definition and the Kähler moduli sector's $\mathbf{f}_{(\nu)}$ has a symmetry group isomorphic to $\mathrm{O}(h^{2,1}+1, h^{2,1}+1)$ if $\langle\ \rangle_\pm \to \langle\ \rangle_+$ and $\mathrm{Sp}(h^{2,1}+1)$ if $\langle\ \rangle_\pm \to \langle\ \rangle_-$. In this combined moduli space formulation it is possible an enhancement to these symmetries occurs, namely those transformations which leave $\mathsf{g} = \mathsf{g}_{\mathfrak{a}} \oplus \mathsf{g}_\nu$ invariant, of which these $\mathrm{Sp}(n)$ or $\mathrm{O}(m,m)$ groups are obvious subgroups.

7.6 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orientifold

We have already discussed some of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold's symmetries between its Type IIA and Type IIB formulations and we again use it as an explicit example for the results just derived.

7.6.1 Alternative Fluxes

We can combine the Type IIB F fluxes given in Table 6.5 with the relationship between the $\mathsf{F} \in \mathsf{W}_1$ and $\mathfrak{F} \in \mathsf{W}_2$ fluxes given in Table 7.3 to obtain the Type IIB $\mathfrak{F}$ cohomology components in the Type IIB $\Lambda^p(\mathsf{E}^*)$ components. These are stated in Table 7.4.

7.6.2 Alternative Bianchi Constraints

We shall restrict ourselves to a particular case, rather than consider the full set of constraints, and to fall inline with our previous work on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold we consider the fluxes which survive the orientifold projection in Type IIB/O3. We shall also not consider S duality induced constraints but rather compare T duality constraints to those of T$'$ duality, though for simplicity we consider all fluxes induced by S duality in Type IIB/O3. To begin with we must determine which $\mathfrak{F}$ components survive the orientifold projection.

$$\left.\begin{array}{l} \widehat{\mathsf{F}}_{(i)A} \ , \ \widehat{\mathsf{F}}_{(i)}{}^{B} \ , \ \mathsf{F}^{(0)}{}_{A} \ , \ \mathsf{F}^{(0)B} \\ \mathsf{F}_{(i)A} \ , \ \mathsf{F}_{(i)}{}^{B} \ , \ \widehat{\mathsf{F}}^{(0)}{}_{A} \ , \ \widehat{\mathsf{F}}^{(0)B} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \mathfrak{F}^{(A)i} \ , \ \mathfrak{F}_{(B)}{}^{i} \ , \ \mathfrak{F}^{(A)}{}_{0} \ , \ \mathfrak{F}_{(B)0} \\ \widehat{\mathfrak{F}}^{(A)i} \ , \ \widehat{\mathfrak{F}}_{(B)}{}^{i} \ , \ \widehat{\mathfrak{F}}^{(A)}{}_{0} \ , \ \widehat{\mathfrak{F}}_{(B)0} \end{array}\right.$$

Recalling the T duality Bianchi constraints of D in (5.1.12) we can apply the orientifold projection to obtain the T duality constraints for D_1 in Type IIB/O3. The cases of $\mathsf{D}_1^2(\mathfrak{a}_A)$ and $\mathsf{D}_1^2(\mathfrak{b}^B)$ are trivial and we can restrict the summed indices in the cases of D_1 on the $\Delta^+(\mathsf{E}^*)$ basis.

$$\begin{aligned} \mathsf{D}_1^2(\nu_i) &= \Big(\mathsf{F}_{(i)}{}^{B}\mathsf{F}^{(0)}{}_{B} - \mathsf{F}_{(i)A}\mathsf{F}^{(0)A} \Big)\nu_0 + \Big(\mathsf{F}_{(i)A}\mathsf{F}_{(j)}{}^{A} - \mathsf{F}_{(i)}{}^{B}\mathsf{F}_{(j)B} \Big)\widetilde{\nu}^j \\ \mathsf{D}_1^2(\widetilde{\nu}^0) &= \Big(\mathsf{F}^{(0)B}\mathsf{F}^{(0)}{}_{B} - \mathsf{F}^{(0)}{}_{A}\mathsf{F}^{(0)A} \Big)\nu_0 + \Big(\mathsf{F}^{(0)}{}_{A}\mathsf{F}_{(i)}{}^{A} - \mathsf{F}^{(0)B}\mathsf{F}_{(i)B} \Big)\widetilde{\nu}^i \end{aligned}$$

	$\mathfrak{F}_{(0)0}$	$\mathfrak{F}_{(0)1}$	$\mathfrak{F}_{(0)2}$	$\mathfrak{F}_{(0)3}$	$\mathfrak{F}_{(0)}{}^{0}$	$\mathfrak{F}_{(0)}{}^{1}$	$\mathfrak{F}_{(0)}{}^{2}$	$\mathfrak{F}_{(0)}{}^{3}$	
$\mathfrak{F}_{(0)I}$	$-\widehat{\mathsf{F}}_{135}$	$+\widehat{\mathsf{F}}^{2}_{35}$	$+\widehat{\mathsf{F}}^{4}_{51}$	$+\widehat{\mathsf{F}}^{6}_{13}$	$-\widehat{\mathsf{F}}^{246}$	$-\widehat{\mathsf{F}}^{46}_{1}$	$-\widehat{\mathsf{F}}^{62}_{3}$	$-\widehat{\mathsf{F}}^{24}_{5}$	$\mathfrak{F}_{(0)}{}^{J}$

	$\mathfrak{F}_{(a)0}$	$\mathfrak{F}_{(a)1}$	$\mathfrak{F}_{(a)2}$	$\mathfrak{F}_{(a)3}$	$\mathfrak{F}_{(a)}{}^{0}$	$\mathfrak{F}_{(a)}{}^{1}$	$\mathfrak{F}_{(a)}{}^{2}$	$\mathfrak{F}_{(a)}{}^{3}$	
$\mathfrak{F}_{(1)I}$	$-\mathsf{F}_{146}$	$+\mathsf{F}^{2}_{46}$	$-\mathsf{F}^{3}_{61}$	$-\mathsf{F}^{5}_{14}$	$-\mathsf{F}^{235}$	$-\mathsf{F}^{35}_{1}$	$+\mathsf{F}^{52}_{4}$	$+\mathsf{F}^{23}_{6}$	$\mathfrak{F}_{(1)}{}^{J}$
$\mathfrak{F}_{(2)I}$	$-\mathsf{F}_{236}$	$-\mathsf{F}^{1}_{36}$	$+\mathsf{F}^{4}_{62}$	$-\mathsf{F}^{5}_{23}$	$-\mathsf{F}^{145}$	$+\mathsf{F}^{45}_{2}$	$-\mathsf{F}^{51}_{3}$	$+\mathsf{F}^{14}_{6}$	$\mathfrak{F}_{(2)}{}^{J}$
$\mathfrak{F}_{(3)I}$	$-\mathsf{F}_{245}$	$-\mathsf{F}^{1}_{45}$	$-\mathsf{F}^{3}_{52}$	$+\mathsf{F}^{6}_{24}$	$-\mathsf{F}^{136}$	$+\mathsf{F}^{36}_{2}$	$+\mathsf{F}^{61}_{4}$	$-\mathsf{F}^{13}_{5}$	$\mathfrak{F}_{(3)}{}^{J}$

	$\mathfrak{F}^{(0)}{}_{0}$	$\mathfrak{F}^{(0)}{}_{1}$	$\mathfrak{F}^{(0)}{}_{2}$	$\mathfrak{F}^{(0)}{}_{3}$	$\mathfrak{F}^{(0)0}$	$\mathfrak{F}^{(0)1}$	$\mathfrak{F}^{(0)2}$	$\mathfrak{F}^{(0)3}$	
$\mathfrak{F}^{(0)}{}_{I}$	$+\widehat{\mathsf{F}}_{246}$	$+\widehat{\mathsf{F}}^{1}_{46}$	$+\widehat{\mathsf{F}}^{3}_{62}$	$+\widehat{\mathsf{F}}^{5}_{24}$	$-\widehat{\mathsf{F}}^{135}$	$+\widehat{\mathsf{F}}^{35}_{2}$	$+\widehat{\mathsf{F}}^{51}_{4}$	$+\widehat{\mathsf{F}}^{13}_{6}$	$\mathfrak{F}^{(0)J}$

	$\mathfrak{F}^{(b)}{}_{0}$	$\mathfrak{F}^{(b)}{}_{1}$	$\mathfrak{F}^{(b)}{}_{2}$	$\mathfrak{F}^{(b)}{}_{3}$	$\mathfrak{F}^{(b)0}$	$\mathfrak{F}^{(b)1}$	$\mathfrak{F}^{(b)2}$	$\mathfrak{F}^{(b)3}$	
$\mathfrak{F}^{(1)}{}_{I}$	$+\mathsf{F}_{235}$	$+\mathsf{F}^{1}_{35}$	$-\mathsf{F}^{4}_{52}$	$-\mathsf{F}^{6}_{23}$	$-\mathsf{F}^{146}$	$+\mathsf{F}^{46}_{2}$	$-\mathsf{F}^{61}_{3}$	$-\mathsf{F}^{14}_{5}$	$\mathfrak{F}^{(1)J}$
$\mathfrak{F}^{(2)}{}_{I}$	$+\mathsf{F}_{145}$	$-\mathsf{F}^{2}_{45}$	$+\mathsf{F}^{3}_{51}$	$-\mathsf{F}^{6}_{14}$	$-\mathsf{F}^{236}$	$-\mathsf{F}^{36}_{1}$	$+\mathsf{F}^{62}_{4}$	$-\mathsf{F}^{23}_{5}$	$\mathfrak{F}^{(2)J}$
$\mathfrak{F}^{(3)}{}_{I}$	$+\mathsf{F}_{136}$	$-\mathsf{F}^{2}_{36}$	$-\mathsf{F}^{4}_{61}$	$+\mathsf{F}^{5}_{13}$	$-\mathsf{F}^{245}$	$-\mathsf{F}^{45}_{1}$	$-\mathsf{F}^{52}_{3}$	$+\mathsf{F}^{24}_{6}$	$\mathfrak{F}^{(3)J}$

Table 7.4: Explicit π defined components for fluxes $\mathfrak{F}_n$ for $\langle\ \rangle_-$. For $\langle\ \rangle_+$ there is a global factor of -1. The $\widehat{\mathfrak{F}}_m$ follow by the (un)hatting of all components.

The flux polynomial $\iota_{\nu_0}\mathsf{D}_1^2(\widetilde{\nu}^0)$ is non-trivial but due to the antisymmetric nature in I and J of $\mathsf{F}^{(I)B}\mathsf{F}^{(J)}{}_B - \mathsf{F}^{(I)}{}_A\mathsf{F}^{(J)A}$ it vanishes under the orientifold projection. The previously noted symmetry of $\iota_{\widetilde{\nu}^j}\mathsf{D}_1^2(\widetilde{\nu}^0) = \iota_{\nu_0}\mathsf{D}_1^2(\nu_j)$ reduces the number of independent flux polynomials which define the nilpotency conditions and we define a vector and a matrix from them.

$$\begin{aligned}
\iota_{\widetilde{\nu}^j}\mathsf{D}_1^2(\widetilde{\nu}^0) &= \chi_j \;\Rightarrow\; \chi_j = \mathsf{F}^{(0)}{}_A\mathsf{F}_{(j)}{}^A - \mathsf{F}^{(0)B}\mathsf{F}_{(j)B} \\
\iota_{\widetilde{\nu}^j}\mathsf{D}_1^2(\nu_i) &= \chi_{ij} \;\Rightarrow\; \chi_{ij} = \mathsf{F}_{(i)A}\mathsf{F}_{(j)}{}^A - \mathsf{F}_{(i)}{}^B\mathsf{F}_{(j)B}
\end{aligned}$$

We now repeat this for the D_2 derivative by applying the orientifold projection to the Bianchi constraints of (7.5.1). The cases of $\mathsf{D}_2^2(\mathfrak{a}_A)$ and $\mathsf{D}_2^2(\mathfrak{b}^B)$ are once again trivial due to the projection and we can restrict the summation range of the indices for the $\Delta^+(\mathsf{E}^*)$ nilpotency expressions.

$$\begin{aligned}
\mathsf{D}_2^2(\nu_i) &= \pm\Big(\mathfrak{F}_{(B)}{}^i\mathfrak{F}^{(B)}{}_0 - \mathfrak{F}^{(A)i}\mathfrak{F}_{(A)0}\Big)\nu_0 + \Big(\mathfrak{F}_{(B)}{}^i\mathfrak{F}^{(B)j} - \mathfrak{F}^{(B)i}\mathfrak{F}_{(B)}{}^j\Big)\widetilde{\nu}^j \\
\mathsf{D}_2^2(\widetilde{\nu}^0) &= +\Big(\mathfrak{F}^{(A)}{}_0\mathfrak{F}_{(A)0} - \mathfrak{F}_{(B)0}\mathfrak{F}^{(B)}{}_0\Big)\nu_0 \pm \Big(\mathfrak{F}^{(A)}{}_0\mathfrak{F}_{(A)}{}^i - \mathfrak{F}_{(B)0}\mathfrak{F}^{(B)i}\Big)\widetilde{\nu}^i
\end{aligned} \tag{7.6.1}$$

The flux polynomial $\iota_{\nu_0}\mathsf{D}_2^2(\widetilde{\nu}^0)$ is non-trivial but it too vanishes under the orientifold projection. The symmetry of $\iota_{\widetilde{\nu}^j}\mathsf{D}_2^2(\widetilde{\nu}^0) = \iota_{\nu_0}\mathsf{D}_2^2(\nu_j)$ reduces the number of independent flux polynomials which define the nilpotency conditions and we define a second pair of a vector and a matrix from them.

$$\begin{aligned}
\iota_{\widetilde{\nu}^j}\mathsf{D}_2^2(\widetilde{\nu}^0) &= \xi_j \;\Rightarrow\; \xi_j = \pm\mathfrak{F}^{(A)}{}_0\mathfrak{F}_{(A)}{}^i \mp \mathfrak{F}_{(B)0}\mathfrak{F}^{(B)i} \\
\iota_{\widetilde{\nu}^j}\mathsf{D}_2^2(\nu_i) &= \xi_{ij} \;\Rightarrow\; \xi_{ij} = \mathfrak{F}_{(B)}{}^i\mathfrak{F}^{(B)j} - \mathfrak{F}^{(B)i}\mathfrak{F}_{(B)}{}^j
\end{aligned}$$

Since we have included the S duality induced fluxes of Type IIB/O3 the form of D_2's nilpotency constraints are of the same schematic form as those of D_1. To examine this further we use Table 7.3 to convert the fluxes of D_1 into those of D_2, and vice versa, for the χ and ξ terms.

$$\begin{aligned}
\chi_j &= \mathsf{F}^{(0)}{}_0\mathsf{F}_{(j)}{}^0 + \mathsf{F}^{(0)}{}_a\mathsf{F}_{(j)}{}^a - \mathsf{F}^{(0)0}\mathsf{F}_{(j)0} - \mathsf{F}^{(0)b}\mathsf{F}_{(j)b} \\
&= \mathfrak{F}^{(0)}{}_0\widehat{\mathfrak{F}}_{(0)}{}^j + \widehat{\mathfrak{F}}^{(a)}{}_0\mathfrak{F}_{(a)}{}^j - \mathfrak{F}_{(0)0}\widehat{\mathfrak{F}}^{(0)j} - \widehat{\mathfrak{F}}_{(b)0}\mathfrak{F}^{(b)j} \\
\xi_j &= \pm\mathfrak{F}^{(0)}{}_0\mathfrak{F}_{(0)}{}^i \pm \mathfrak{F}^{(a)}{}_0\mathfrak{F}_{(a)}{}^i \mp \mathfrak{F}_{(0)0}\mathfrak{F}^{(0)i} \mp \mathfrak{F}_{(b)0}\mathfrak{F}^{(b)i} \\
&= \pm\mathsf{F}^{(0)}{}_0\widehat{\mathsf{F}}_{(i)}{}^0 \pm \widehat{\mathsf{F}}^{(0)}{}_a\mathsf{F}_{(i)}{}^a \mp \mathsf{F}^{(0)0}\widehat{\mathsf{F}}_{(i)0} \mp \widehat{\mathsf{F}}^{(0)b}\mathsf{F}_{(i)b}
\end{aligned}$$

With these expansions we can see the mixing of the NS-NS and R-R fluxes in the constraints of D_2, despite us not considering $\mathrm{SL}(2,\mathbb{Z})_S$ transformations. To see this further we make use of Table 7.4 and consider χ_1 and ξ_1.

$$\begin{aligned}\chi_1 &= -\widehat{\mathsf{F}}_{246}\mathsf{F}^{46}_1 - \widehat{\mathsf{F}}_{235}\mathsf{F}^{35}_1 + \widehat{\mathsf{F}}_{145}\mathsf{F}^{45}_2 + \widehat{\mathsf{F}}_{136}\mathsf{F}^{36}_2 \\ &\quad +\widehat{\mathsf{F}}_{135}\mathsf{F}^{35}_2 + \widehat{\mathsf{F}}_{146}\mathsf{F}^{46}_2 - \widehat{\mathsf{F}}_{236}\mathsf{F}^{36}_2 - \widehat{\mathsf{F}}_{245}\mathsf{F}^{45}_2 \propto \widehat{\mathsf{F}}_{pq[1}\mathsf{F}^{pq}_{2]} \\ \xi_1 &= -\widehat{\mathsf{F}}_{246}\widehat{\mathsf{F}}^{46}_1 - \widehat{\mathsf{F}}_{235}\widehat{\mathsf{F}}^{35}_1 + \widehat{\mathsf{F}}_{145}\widehat{\mathsf{F}}^{45}_2 + \widehat{\mathsf{F}}_{136}\widehat{\mathsf{F}}^{36}_2 \\ &\quad +\widehat{\mathsf{F}}_{135}\widehat{\mathsf{F}}^{35}_2 + \widehat{\mathsf{F}}_{146}\widehat{\mathsf{F}}^{46}_2 - \widehat{\mathsf{F}}_{236}\widehat{\mathsf{F}}^{36}_2 - \widehat{\mathsf{F}}_{245}\widehat{\mathsf{F}}^{45}_2 \propto \widehat{\mathsf{F}}_{pq[1}\widehat{\mathsf{F}}^{pq}_{2]}\end{aligned}$$

Repeating this with χ_{ij} and ξ_{ij} we use Table 7.3 to convert the fluxes of D_1 into those of D_2, and vice versa.

$$\begin{aligned}\chi_{ij} &= \mathsf{F}_{(i)0}\mathsf{F}_{(j)}{}^{0} + \mathsf{F}_{(i)a}\mathsf{F}_{(j)}{}^{a} - \mathsf{F}_{(i)}{}^{0}\mathsf{F}_{(j)0} - \mathsf{F}_{(i)}{}^{b}\mathsf{F}_{(j)b} \\ &= \widehat{\mathfrak{F}}^{(0)i}\widehat{\mathfrak{F}}_{(0)}{}^{j} + \mathfrak{F}^{(a)i}\mathfrak{F}_{(a)}{}^{j} - \widehat{\mathfrak{F}}_{(0)}{}^{i}\widehat{\mathfrak{F}}^{(0)j} - \mathfrak{F}_{(i)}{}^{b}\mathfrak{F}_{(j)b} \\ \xi_{ij} &= \mathfrak{F}_{(0)}{}^{i}\mathfrak{F}^{(0)j} + \mathfrak{F}_{(b)}{}^{i}\mathfrak{F}^{(b)j} - \mathfrak{F}^{(0)i}\mathfrak{F}_{(0)}{}^{j} - \mathfrak{F}^{(b)i}\mathfrak{F}_{(b)}{}^{j} \\ &= \widehat{\mathsf{F}}_{(i)}{}^{0}\mathsf{F}_{(j)0} + \mathsf{F}_{(i)}{}^{b}\mathsf{F}_{(j)b} - \mathsf{F}_{(i)0}\mathsf{F}_{(j)}{}^{0} - \mathsf{F}_{(i)b}\mathsf{F}_{(j)}{}^{b}\end{aligned}$$

To examine this further we make use of Table 7.4 and consider χ_{12} and ξ_{12}.

$$\begin{aligned}\chi_{12} &= -\mathsf{F}^{35}_2\mathsf{F}^{62}_3 + \mathsf{F}^{46}_2\mathsf{F}^{52}_4 + \mathsf{F}^{36}_1\mathsf{F}^{51}_3 - \mathsf{F}^{45}_1\mathsf{F}^{61}_4 \\ &\quad +\mathsf{F}^{46}_1\mathsf{F}^{51}_4 - \mathsf{F}^{35}_1\mathsf{F}^{61}_3 - \mathsf{F}^{45}_2\mathsf{F}^{62}_4 + \mathsf{F}^{36}_2\mathsf{F}^{52}_3 \propto \mathsf{F}^{p[5}_q\mathsf{F}^{6]q}_p \\ \xi_{12} &= -\widehat{\mathsf{F}}^{46}_1\widehat{\mathsf{F}}^{51}_4 + \mathsf{F}^{35}_1\mathsf{F}^{61}_3 + \mathsf{F}^{45}_2\mathsf{F}^{62}_4 - \mathsf{F}^{36}_2\mathsf{F}^{52}_3 \\ &\quad +\widehat{\mathsf{F}}^{35}_2\widehat{\mathsf{F}}^{62}_3 - \mathsf{F}^{46}_2\mathsf{F}^{52}_4 - \mathsf{F}^{36}_1\mathsf{F}^{51}_3 + \mathsf{F}^{45}_1\mathsf{F}^{61}_4 \not\propto \widehat{\mathsf{F}}^{p[5}_q\widehat{\mathsf{F}}^{6]q}_p\end{aligned}$$

Unlike ξ_j it is not possible to express ξ_{ji} in terms of the D_1 fluxes in a straight forward manner. The orientifold projection's effect on the fluxes of D_1 is to remove two of the four fluxes induced by T duality but in terms of the fluxes of D_2 half of the components of each of the four fluxes are projected out. This is analogous to the way in which the orientifold projection affects the fluxes of Type IIA compared to those of Type IIB. In this case the complication is not due to the NS-NS fluxes mixing in a non-trivial manner but the two flux sectors being mixed by the action of π.

Summary

In this chapter we have considered the way in which the polynomial form of a general U duality invariant superpotential has a symmetry in the two moduli types, particularly on those spaces which are their own mirror duals. Naturally the moduli of the compact space and the stringy dilaton modulus arrange themselves into Kähler manifolds in a way which depends on which Type II construction is used for the effective theory and in previous chapters we constructed the relevant U duality invariant superpotentials in terms of the holomorphic sections of these Kähler manifolds. Under T dualities or mirror transformations Type II theories are exchanged and the type of moduli the dilaton combines into the $\mathcal{M}^Q$ Kähler manifold changes. We have argued that for self-mirror spaces this implies that a polynomial form of the superpotential can be reformulated into being in terms of either the holomorphic forms associated to a Type IIB construction or the holomorphic forms of a Type IIA construction.

$$\int_{\mathcal{M}} \left\langle \Omega, \mathsf{D}_1(\mho_c) + \mathsf{D}'_1(\mho'_c) \right\rangle_{\pm} = \int_{\mathcal{M}} \left\langle \mho, \mathsf{D}_2(\Omega_c) + \mathsf{D}'_2(\Omega'_c) \right\rangle_{\pm}$$

In each case the fluxes are associated to a set of derivatives and due to the non-trivial way in which the dilaton coupling enters into the definition of the fluxes we found that these derivatives have Bianchi constraints which are inequivalent to those of the standard formulation. Despite this inequivalence of the constraints the two alternative formulations share many of the same structures under such symmetries as $\mathrm{SL}(2,\mathbb{Z})_S$ modular invariance. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold has again provided us with a convenient explicit example to illustrate the results and due to its parallelisability it also demonstrates how the formulation can be written in terms of the $\Lambda^p(\mathsf{E}^*)$ defined fluxes. In such cases we obtained expressions which were independent of our choice of $\langle\ \rangle_{\pm}$, as would be expected in such a construction.

Chapter 8

Summary and Conclusions

This thesis has considered string dualities in the effective theories of both Type IIA and Type IIB string theories and illustrated the construction of maximally invariant $\mathcal{N} = 2$ Type II flux configurations. We have seen how fluxes obtained by compactification determine the structure of the full duality extended superpotential, which possesses a natural construction in terms of generalised geometry and SU(3) structure defined cohomologies. Constructing the superpotentials in terms of holomorphic forms and their flux dependent derivatives suggests the existence of additional fluxes in the same manner as the Lie algebra structure constant formulation does. These two approaches have different advantages and disadvantages. The derivative representation allows the use of powerful generalise geometry methods and provides a natural basis for the moduli definitions, in terms of $\Delta^{p}(\mathsf{E}^{*})$. However, with moduli obtaining flux induced masses the distinction between heavy and light fields for the effective theory can be lost. The Lie algebra formulation does not neglect any modes but moduli dependence and algebraic methods are less forthcoming when working with $\Lambda^{p}(\mathsf{E}^{*})$ defined components and requires that the compact space be parallelisable. Different symmetries are manifest in the different formulations; the truncated basis of $\Delta^{*}(\mathsf{E}^{*})$ allows the Kähler structure of the moduli spaces to be clearly seen, the complex or symplectic, depending on the inner product used, nature of the Kähler moduli and the symplectic nature of the complex structure moduli, while for parallelisable spaces the flux components defined in the $\Lambda^p(\mathsf{E}^*)$ construction have $\mathrm{GL}(6,\mathbb{Z})$ invariance.

The vast majority of our analysis centred on the generalised geometry construction, within which we could demonstrate the equivalence of T or mirror dual Type II constraints, $\mathcal{N}=1$ field content via the orientifold projection and entirely classify $\mathrm{SL}(2,\mathbb{Z})_S$ multiplets for both Bianchi and tadpole constraints. In terms of generalised geometry the structure and construction of the Bianchi and tadpole constraints can be unified into a single description. However, we observed that unlike the T duality or S duality only cases the physical construction of U duality induced tadpoles is not clear in terms of charged extended objects coupling to fields. For the T and S invariant Type IIB/O3 construction of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold such expressions were seen in terms of D3, D7 branes and their S duality images but this was not the case for the more general $\mathcal{N}=2$ U duality constructions on other compact spaces. For parallelisable cases the $\mathcal{N}=2$ constraints in the $\Lambda^p(\mathsf{E}^*)$ construction follow from non-trivial twelve dimensional Lie algebras but upon the application of a particular orientifold projection in Type IIB sets of six dimensional subalgebras arise. The immediate implication of this and the $\mathrm{GL}(6,\mathbb{Z})$ invariance was that we could make use of the full classification of six dimensional Lie algebras to construct $\mathrm{GL}(6,\mathbb{Z})$ isomorphisms between the non-geometric fluxes and Lie algebra canonical structure constants.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold has served as a simple example of the methods and results, in both the construction of $\mathcal{N}=2$ U duality invariant superpotentials and the methods of solving $\mathcal{N}=1$ U duality invariant Bianchi constraints. Due to the actions of our choosen orbifold group the generalised geometry $\Delta^p(\mathsf{E}^*)$ fluxes of the derivatives are expressible in terms of a single $\Lambda^p(\mathsf{E}^*)$ component and allows for the isotropy restriction. The orbifold group also reduces the $\mathrm{GL}(6,\mathbb{Z})$ invariance of the orientifolded Lie algebra to the subgroup $G \subset \mathrm{GL}(6,\mathbb{Z})$ invariant under the group generators. This reduction, as well as in the number of independent fluxes, allowed us to explicitly construct the G isomorphisms between the non-geometric fluxes and five non-trivial isotropic Lie algebras. Unlike a generic compact space the generating functions of the ideal defined by the Bianchi constraints on the isotropic orientifold are sufficiently simple to be prime decomposed on a computer. The structure of elements of G resulted in a modular invariance of the complex structure moduli, viewed as being induced by T duality and in line with the known properties of moduli for a two dimensional torus. This combined with the modular invariance of the dilaton induced by S duality such that the resultant non-geometric contributions to the superpotential depends purely on the choice of Lie algebra the non-geometric fluxes are isomorphic to. This modular invariance considerably reduced the complexities in constructing example vacua with $\Lambda \leq 0$ cosmological constant and partly broken supersymmetry.

Finally, the constructed T duality induced modular symmetry in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold, the known $\mathrm{SL}(2,\mathbb{Z})^7$ symmetry of the space and the symmetries in the $\mathcal{N} = 2$ Type IIB superpotential motivated us to reexamine the full U duality invariant $\mathcal{N} = 2$ superpotential. We had already constructed, through the use of T and S dualities, the origin of the $\mathrm{SL}(2,\mathbb{Z})^4 \subset \mathrm{SL}(2,\mathbb{Z})^7$ and by noting how symmetric the orientifold is in its treatment of the moduli spaces we hypothesised the origin of the remaining $\mathrm{SL}(2,\mathbb{Z})^3$ symmetries associated to the Kähler moduli. This was done by noting that the role of the holomorphic forms in the superpotential is not naturally symmetric due to the dilaton and yet the moduli space $\mathcal{M}_{\mathcal{M}}$ is locally a product of the two geometric moduli spaces and the dilaton's moduli space, $\mathcal{M}_{\mathcal{M}} = \mathcal{M}_{\mathsf{T}} \times \mathcal{M}_{\mathsf{U}} \times \mathcal{M}_S$. The manner in which these recombine into Kähler manifolds $\mathcal{M}^K$ and $\mathcal{M}^Q$ is theory dependent but for self mirror spaces we argued either formulation should be applicable. As a result of this we constructed a superpotential in Type IIB where the roles of the complex structure and Kähler moduli were exchanged; in a Type IIB construction the complex structure moduli and dilaton were combined into a quaternionic manifold, rather than the Kähler moduli and dilaton.

Due to the manner in which the fluxes were defined in terms of the image of the $\mathcal{M}^Q$ holomorphic form under a generalised derivative this reformulation $\mathcal{M}^Q(\mathsf{T},S) \to \mathcal{M}^{Q'}(\mathsf{U},S)$ resulted in entirely different flux structures with inequivalent constraints. The new fluxes retained the same kind of Bianchi and tadpole-like structures, with analogous transformation properties, and allowed for the superpotential to be written in an extremely symmetric way which treated both moduli spaces of the internal space in the same manner. Since a space which is self mirror dual has an enhanced symmetry over a generic compact space it is tempting to associate the inequivalent Bianchi and tadpole constraints with the extra constraints such a symmetry would impose. Such a reformulation and the construction of new flux constraints was motivated entirely on the grounds of symmetry in the effective theory but the fact $\mathrm{SL}(2,\mathbb{Z})_S$ Bianchi and tadpole multiplets arise with the same schematic structure in each formalism lends further weight to this notion. However, without a more direct string based construction it is unclear if such reformulations and their structures are coincidences or a sign of something deeper.

Aside from further investigation into such reformulations there are a number of other possible continuations of this thesis. In our general geometry we neglected

$\Delta^1(\mathsf{E}^*)$ and $\Delta^5(\mathsf{E}^*)$ contributions. Through the use of Hitchin functionals this simplification is not required but the initial examination of Calabi-Yau manifolds is not possible as by definition their $\Delta^1(\mathsf{E}^*)$ and $\Delta^5(\mathsf{E}^*)$ are empty. The physical nature of certain U duality induced structures is unclear, a problem which is worsened by the issue of even some T duality structures having little or no interpretation in terms of objects which appear in the non-compactified theory. We have concentrated on Type II constructions but due to our use of dualities we could extend this analysis to heterotic or M theory models too. Type I is related to Type IIB via the use of O9-plane defining orientifolding and while Type IIB is self S dual the S dual of Type I is the SO(32) heterotic string theory which is itself T dual to $E_8 \times E_8$ heterotic string theory. Both $E_8 \times E_8$ and Type IIA are related to compactified M theory by dilaton transformations. Using this web of dualities we could construct superpotentials for any of these theories given the Type II results we have seen. However, the symmetries of the Type II superpotentials are particularly manifest in the $\mathcal{N} = 2$ case but Type I and its dual heterotic constructions are $\mathcal{N} = 1$. As a result it would not be possible to map $\mathcal{N} = 2$ results we have seen to the heterotic constructions, we have to apply the orientifold projection first, but analogous constructions may none-the-less be possible in the heterotic effective theories.

The use of non-geometric fluxes has required us to leave behind the familiar notions of metrics and geometric interpretations of the space within which we construct our physical models. However, they are essential and unavoidable in any full model which possesses the symmetries inherent to string theory. As a result we have gone beyond the statement made by Poincaré.

"Geometry is not true, it is advantagous."

We have seen that in string theory geometry can be neither true nor advantagous.

Appendix A

$\mathcal{N} = 2$ Geometry

In order to motivate the structures of the superpotential and fluxes used in this thesis we shall briefly review the basics of Kähler geometry and its application to the moduli spaces found in the literature. These results are discussed in much deeper detail in Refs. [74, 75, 18, 64, 69, 78, 80] but our discussion differs from them all in the case of Kähler moduli since our notation is such that the symplectic Kähler structure of the moduli space is not manifest.

A.1 Spinors and Differential Forms

A.1.1 SU(3) × SU(3) Structures and Generalised Geometry

The ten dimensional gravitini of the Type II string theory descend to the effective theory by the same splitting as seen in metric. The two Type IIA gravitini descend to two spinors whose six dimensional parts differ in six dimensional chirality while the Type IIB six dimensional are equal.

$$\begin{aligned} \eta^1_{IIA} &\to \chi^1_+ \otimes \xi^1_+ + \chi^1_- \otimes \xi^1_- \quad , \quad \eta^1_{IIB} \to \chi^1_+ \otimes \xi^1_- + \chi^1_- \otimes \xi^1_+ \\ \eta^2_{IIA} &\to \chi^2_+ \otimes \xi^2_- + \chi^2_- \otimes \xi^2_+ \quad , \quad \eta^2_{IIB} \to \chi^2_+ \otimes \xi^2_- + \chi^2_- \otimes \xi^2_+ \end{aligned} \tag{A.1.1}$$

Using two of these six dimensional spinors we can define a matrix whose decomposition in terms of the basis Clifford algebra of $\mathcal{M}$ defines a set of rank p coefficients, where $0 \leq p \leq 6$, for a spin bundle element.

$$\xi_+ \otimes \overline{\xi}'_\pm = \frac{1}{4}\sum_{p=0}^{6}\frac{1}{p!}\left(\overline{\xi}'_\pm \gamma_{m_1\ldots m_p}\xi_+\right)\gamma^{m_1\ldots m_p} \in S^\pm \tag{A.1.2}$$

Since the two spinors $\xi \neq \xi'$ transform seperately under Spin(6) we obtain an SU(3)$\times$SU(3) structure. The set of coefficients suggest that there is an equivalent formulation in terms of p-forms. In order to obtain a manifestly SU(3) $\times$ SU(3) structured geometry it is convenient to define the generalised frame bundle $\mathfrak{E} = \mathsf{E} \oplus \mathsf{E}^*$ [66, 73, 76], where E is the frame bundle of $\mathcal{M}$ and E^* its dual. This space carries with it a natural O(6, 6) metric which is independent of the space-time metric.

$$X \in \mathsf{E} \quad , \quad \xi \in \mathsf{E}^* \quad \Rightarrow \quad (X+\xi, X+\xi) = \xi(X) \equiv \xi_m X^m$$

A generic element of $\mathfrak{E}$ acts on an element of $\Omega^*(\mathsf{E}^*)$ by the natural action of the subspaces and thus provides a representation of a Clifford algebra $\mathsf{CL}(\mathfrak{E}, \delta^n_m)$ whose generators are γ_m and γ^n.

$$(X+\xi)\cdot\chi = \iota_X\chi + \xi\wedge\chi = X^m\gamma_m\cdot\chi + \xi_n\gamma^n\cdot\chi \equiv \mathcal{X}_M\Gamma^M\cdot\chi$$

The spinorial construction of (A.1.2) can be related to a set of forms and the bundle splitting of such generic Spin(6, 6) spin bundles $S \to S^\pm$ corresponds to the splitting of the forms $\Omega^*(\mathsf{E}^*) \to \Omega^\pm(\mathsf{E}^*)$. The precise isomorphism is obtained by using $\sqrt{\epsilon}$, where ϵ is the volume form on $\Omega^6(\mathsf{E}^*)$, to give the required spinor transformations.

$$\xi_+ \otimes \overline{\xi}'_\pm\sqrt{\epsilon} = \frac{1}{4}\bigoplus_{p=0}^{6}\frac{1}{p!}\left(\overline{\xi}'_\pm \gamma_{m_1\ldots m_p}\xi_+\right)\eta^{m_k}\wedge\ldots\wedge\eta^{m_1} \in \Omega^\pm(\mathsf{E}^*)$$

This morphism allows the description of the SU(3) structures of $\mathcal{M}$ to be worded either in terms of spinors or differential forms. In each Type II theory we have four six dimensional spinors with which to construct p-forms but it is convenient to make the simplification $\xi^1_+ = \xi^2_+ = \xi$ and use $\xi_- = \xi^c_+$. As a result the

SU(3) × SU(3) structure is reduced to a single SU(3). Two spinors can then be constructed such that they are in different $\Omega^{\pm}(\mathsf{E}^*)$ subspaces of $\Omega^*(\mathsf{E}^*)$.

$$\begin{aligned} \Phi^+ &= \xi_+ \otimes \overline{\xi}_+ \sqrt{\epsilon} \sim \tfrac{1}{8} e^J \in \Omega^+(\mathsf{E}^*) \\ \Phi^- &= \xi_+ \otimes \overline{\xi}_- \sqrt{\epsilon} \sim -\tfrac{i}{8}\Omega \in \Omega^-(\mathsf{E}^*) \end{aligned} \quad \text{(A.1.3)}$$

A.1.2 Non-Degenerate Inner Products

The superpotential is defined by an integral over $\mathcal{M}$ and thus only integrand terms in $\Omega^6(\mathsf{E}^*)$ may contribute and the integrand factorises into either a pair of $\Omega^+(\mathsf{E}^*)$ elements or a pair of $\Omega^-(\mathsf{E}^*)$ elements. This can be modified by considering the integrand as formed by a non-degenerate inner product $\langle \; , \; \rangle$ between pairs of elements in $\Omega^{\pm}(\mathsf{E}^*)$.

$$\langle \phi, \psi \rangle_s \equiv \begin{cases} s\phi_0 \wedge \psi_6 + \phi_2 \wedge \psi_4 + s\phi_4 \wedge \psi_2 + \phi_6 \wedge \psi_0 & : \Omega^+(\mathsf{E}^*) \\ s\phi_1 \wedge \psi_5 + \phi_3 \wedge \psi_3 + s\phi_5 \wedge \psi_1 & : \Omega^-(\mathsf{E}^*) \end{cases} \quad \text{(A.1.4)}$$

The parameter s defines the parity structure of the inner product. For $s = 1$ the inner product is symmetric on $\Omega^+(\mathsf{E}^*)$ and antisymmetric on $\Omega^-(\mathsf{E}^*)$ while $s = -1$ makes the inner product antisymmetry on both $\Omega^{\pm}(\mathsf{E}^*)$. $\langle \; \rangle_+$ is equivalent to the standard wedge product as those elements $\phi \wedge \psi \notin \Omega^6(\mathsf{E}^*)$ do not contribute to the superpotential. The $\langle \; \rangle_-$ is the Mukai inner product[1] and is regularly used in the literature as it makes the Kähler structure of the moduli spaces manifest. In each case we are motivated to choose explicit bases for the $\Omega^{\pm}(\mathsf{E}^*)$ so as to simplify these expressions. We reduce our considerations to the SU(3) case so $\Omega^1(\mathsf{E}^*)$ and $\Omega^5(\mathsf{E}^*)$ are neglected and therefore $\Omega^-(\mathsf{E}^*) = \Omega^3(\mathsf{E}^*)$. Though these general structures can be discussed in the infinite dimensional $\Omega^*(\mathsf{E}^*)$ space we select a finite dimensional subspace $\Delta^* \subset \Omega^*(\mathsf{E}^*)$ which decomposes into the even and odd form subspaces $\Delta^{\pm}$. The physical motation for this is given in Appendix

[1] In order to provide the simplest sign structure to our analysis we have actually defined the negative of the Mukai inner product. Never the less, the schematic structures are unchanged compared to the literature.

B.1.1. Until then it is sufficient to consider the generic basis $(\varpi_M, \widetilde{\varpi}^N)$. If this is the basis of Δ^- then it is defined to be symplectic and if it is the basis of Δ^+ we set $\varpi_M \in \Omega^2(\mathsf{E}^*) \oplus \Omega^6(\mathsf{E}^*)$ while $\widetilde{\varpi}^M \in \Omega^0(\mathsf{E}^*) \oplus \Omega^4(\mathsf{E}^*)$ so that the s dependence on the components is simplified.

$$\begin{aligned} s = +1 \ &: \ \Phi^\pm \ = \ \Phi_M \varpi_M \pm \Phi^N \widetilde{\varpi}^N \\ s = -1 \ &: \ \Phi^\pm \ = \ \Phi_M \varpi_M - \Phi^N \widetilde{\varpi}^N \end{aligned} \tag{A.1.5}$$

The inner product intersection numbers are such that this kind of sign structure is preserved in any superpotential integrand. To illustrate this we consider Φ and Ψ with components defined in the same manner as (A.1.5).

$$\begin{aligned} \big\langle\, \Phi, \Psi \,\big\rangle \ &= \ \big\langle\, \Phi_M \varpi_M \pm \Phi^N \widetilde{\varpi}^N, \Psi_M \varpi_M \pm \Psi^N \widetilde{\varpi}^N \,\big\rangle \\ &= \ \pm \Phi_M \Psi^N \big\langle\, \varpi_M, \widetilde{\varpi}^N \,\big\rangle \pm \Phi^N \Psi_M \big\langle\, \widetilde{\varpi}^N, \varpi_M \,\big\rangle \end{aligned} \tag{A.1.6}$$

We have made use of the fact all forms are self-orthogonal in terms of this inner product, thus dropping two terms from the expansion. For $(\varpi_M, \widetilde{\varpi}^N)$ being the basis of $\Delta^3(\mathsf{E}^*)$ for $\langle\ \rangle_\pm$ or the $\Delta^+(\mathsf{E}^*)$ for $\langle\ \rangle_-$ we have $\big\langle\, \widetilde{\varpi}^N, \varpi_M \,\big\rangle = -\big\langle\, \varpi_M, \widetilde{\varpi}^N \,\big\rangle$ and in the above expression we set $\pm \to -$.

$$\int \big\langle\, \Phi, \Psi \,\big\rangle \ = \ \Phi^N \Psi_N - \Phi_M \Psi^M$$

For $(\varpi_M, \widetilde{\varpi}^N)$ being the basis of $\Delta^+(\mathsf{E}^*)$ for $\langle\ \rangle_+$ we have $\big\langle\, \widetilde{\varpi}^N, \varpi_M \,\big\rangle = \big\langle\, \varpi_M, \widetilde{\varpi}^N \,\big\rangle$ and in the above expression we set $\pm \to +$.

$$\int \big\langle\, \Phi, \Psi \,\big\rangle \ = \ \Phi^N \Psi_N + \Phi_M \Psi^M$$

A.1.3 Hitchin Functions

Hitchin functions [73, 75, 77] provide a natural way to write the Kähler potential of the special Kähler moduli spaces such that the full $SU(3) \times SU(3)$ structures are obtained [76]. Not all $\mathrm{Spin}(6,6)$ spinors construct viable $SU(3)$ structures, only those which are 'stable' [74] can be used and they form an open set in the space of Spin$(6,6)$ spinors. When the morphism from spinors to forms is used we can construct a generic Hitchin function for both of the two spinors of (A.1.3) using the inner product $\langle\ \rangle_{\pm}$.

$$H(\Phi^{\pm})_s \equiv i^{\rho} \int_{\mathcal{M}} \langle\, \Phi^{\pm}, \overline{\Phi^{\pm}}\, \rangle_s \tag{A.1.7}$$

Since we are taking $\Phi^{\pm}$ to be a pure form the inner product is of definite sign and thus $\langle\, \Phi^{\pm}, \overline{\Phi^{\pm}}\, \rangle$ is either purely real or purely imaginary. The exponent ρ is such that $H(\Phi^{\pm})$ is real.

$$\begin{aligned} s = +1 \ &:\ H(\Phi^{\pm})_+ \ =\ i^{(1\pm1)/2}\ (\Phi^N\overline{\Phi}_N \pm \Phi_M\overline{\Phi}^M) \\ s = -1 \ &:\ H(\Phi^{\pm})_- \ =\ i\ (\Phi^N\overline{\Phi}_N - \Phi_M\overline{\Phi}^M) \end{aligned} \tag{A.1.8}$$

Furthermore, the factors of i and the sign choice in Φ expansions are such that the Hitchin function is either the real or imaginary part of $\Phi_M\overline{\Phi}^M$, depending on whether the inner product is symmetric or antisymmetric on the appropriate $\Omega^{\pm}(\mathsf{E}^*)$.

$$\begin{aligned} \text{Symmetric} \ &:\ H(\Phi) \ =\ 2\mathrm{Re}(\Phi_M\overline{\Phi}^M) \\ \text{Antisymmetric} \ &:\ H(\Phi) \ =\ 2\mathrm{Im}(\Phi_M\overline{\Phi}^M) \end{aligned} \tag{A.1.9}$$

Since the choice of inner product only arises in $\Omega^{+}(\mathsf{E}^*)$ we consider the generic form of the associated holomorphic form, e^{ψ}, with the exponent splitting into real and imaginary parts $\psi = \phi + i\chi$. We surpress the $\wedge$ for convenience.

$$\begin{aligned} H(e^{\psi})_+ \ &=\ \frac{1}{3!}\left(\overline{\psi}^3 + 3\psi\overline{\psi}^2 + 3\psi^2\overline{\psi} + \psi^3\right) \ =\ \frac{2^3}{3!}\phi^3 \\ H(e^{\psi})_- \ &=\ \frac{1}{3!}\left(\overline{\psi}^3 - 3\psi\overline{\psi}^2 + 3\psi^2\overline{\psi} - \psi^3\right) = i\frac{2^3}{3!}\chi^3 \end{aligned}$$

Since $\langle\ \rangle_{\pm}$ is dependent on only one of the real or imaginary parts of ψ we can use either inner product to construct the same function. Of special note is the case $\psi = \mathcal{J} = J + iB$.

$$H(e^{\mathcal{J}})_+ = -iH(e^{i\mathcal{J}})_- \tag{A.1.10}$$

A.2 Moduli Spaces

The moduli spaces associated to the Kähler and complex structure deformations in $(\mathcal{M}, G, \mathsf{J})$ are special Kähler manifolds [74]. More specifically one moduli space is special Kähler naturally while the second is embedded in a quaternionic manifold of twice the dimension but which is which depends on the Type II construction being considered. In Type IIA the moduli space embedded in a quaternionic manifold is associated to the complex structure while in Type IIB it is the moduli space associated to the Kähler moduli.

A.2.1 Special Kähler $\mathcal{M}^K$

We follow the definitions from Ref. [74]. Given a set of q complex scalar fields φ_m, belonging to $\mathcal{N} = 2$ supermultiplets, they form a local special Kähler manifold of dimension $2q$ holomorphically embedded into a holomorphic vector bundle $\mathcal{V}$ with $\mathrm{Sp}(2q+2)$ structure $\omega(\ ,\ )$ provided the Kähler potential is written in particular manner.

$$K = -\ln i\,\omega(\Phi, \overline{\Phi}) \quad , \quad \omega(\Phi, \partial\Phi) = 0$$

The coordinates Φ_M of $\mathcal{V}$ can be choosen such that the Kähler potential is written in a straightforward manner in terms of holomorphic functions Ψ^M.

$$\omega(\Phi, \overline{\Phi}) = \overline{\Phi}_M \Phi^M - \Phi_N \overline{\Phi}^N$$

The $2q$ dimensional manifold is then embedded by defining $\varphi_m = \Phi_m/\Phi_0$. The condition $\omega(\Phi, \partial\Phi) = 0$ implies that the holomorphic functions are derivatives of a single holomorphic function $\mathcal{P}_\Phi$, the prepotential. The symplectic structure $\omega(\ ,\ )$ can be written on a basis $(\varpi_M, \widetilde{\varpi}^N)$ which is symplectic under $\langle\ \rangle_-$.

$$\Phi = \varphi_M \varpi_M - \frac{\partial \mathcal{P}_\Phi}{\partial \varphi_N} \widetilde{\varpi}^N \quad , \quad K = -\ln H(\Phi)_-$$

This prepotential $\mathcal{P}_\Phi(\Phi_M)$ is homogeneous of degree two in the Φ_M and given the relationship between the φ_m and Φ_m it takes a specific form.

$$\mathcal{P}_\Phi(\Phi_M) = -\frac{1}{3!\Phi_0}(\mathcal{P}_\Phi)_{mnp}\Phi_m\Phi_n\Phi_p = -\Phi_0^2\frac{1}{3!}(\mathcal{P}_\Phi)_{mnp}\varphi_m\varphi_n\varphi_p$$

This construction makes explicit use of a symplectic inner product. As such if we tried to repeat this with $\langle\ \rangle_+$ it would fail due to a discrepency in signs between Φ_0 and Φ_m terms and so using the $\langle\ \rangle_+$ inner product does not allow us to construct a manifestly local special Kähler manifold. However the inner products can define the same Hitchin functions due to (A.1.10) and therefore the same superpotentials can be built so our choice of inner product does not affect the effective theory.

$$K = -\ln H(\Phi)_\pm \quad , \quad \Phi = \Phi_M\varpi_M \pm \Phi^N\widetilde{\varpi}^N \tag{A.2.1}$$

A.2.2 Special Kähler in Quaternionic $\mathcal{M}^Q$

We follow the definitions from Ref. [75]. A quaternionic manifold is not automatically Kähler, special or otherwise, but it is possible to embedd a manifold of lower dimension into it such that the submanifold is special Kähler and after the application of the orientifold projection the resultant $\mathcal{N} = 1$ multiplets are chiral. In Type IIA and Type IIB the multiplets are such that they possess dilaton dependence. We shall not reproduce the entire argument given in Ref. [75] and instead simply quote results and discuss their relevance. Though not generally being equal in dimension to $\mathcal{M}^K$ we again take M,N to vary over the $2h+2$ un-orientifolded complex coordinates (Φ_M, Φ^N) of $\mathcal{M}^Q$. Furthermore in either Type II theory the holomorphic section includes a dilaton S dependence.

$$\Phi_c \;=\; -S\,\Phi_0\varpi_0 \;+\; \Phi_m\varpi_m \;\pm\; \Phi^m\widetilde{\varpi}^m \;\mp\; S\,\Phi^0\widetilde{\varpi}^0 \tag{A.2.2}$$

How this dilaton relates to the $\mathcal{N} = 2$ fields is dependent upon how the $\mathcal{N} = 2$ is projected down to $\mathcal{N} = 1$ [74] and is not something we will consider in depth. None the less, the Kähler potential of the chiral multiplets contributes both the geometric moduli and the dilaton modulus terms, though the full expression is quite complex [75]. Instead we shall construct a function analogous to the Hitchin function such that the chiral multiplets with constant dilaton result in the known dilaton and vector multiplet Kähler potentials of toroidal orientifolds. With this in mind we must consider the dilatonic complement of Φ_c, Φ'_c.

$$\Phi'_c \;=\; \Phi_0\varpi_0 \;-\; S\,\Phi_m\varpi_m \;\mp\; S\,\Phi^m\widetilde{\varpi}^m \;\pm\; \Phi^0\widetilde{\varpi}^0 \tag{A.2.3}$$

In Type IIB this holomorphic form, $\Phi'_c = \mho'_c$, couples to the R-R flux sector and so its origin can be taken as being the S dual of $\Phi_c = \mho_c$. In Type IIA the lack of a simple way to examine modular transformations on the dilaton makes the origin of $\Phi'_c = \Omega'_c$ less obvious but we can motivate its existence via a mirror duality on the Type IIB sector dependent upon $\mho'_c$. As such we have made the choice in signs so as to fall inline with the known Type IIB S duality transformations of the fluxes. As a result these two expressions are not $\mathrm{SL}(2,\mathbb{Z})_S$ inversions of one another but once they are coupled to fluxes within the superpotential the two sectors are S dual with the fluxes transforming in the appropriate manner. Hence we define $\Gamma_S\varphi(S) = S\varphi(-1/S)$ and apply this operator to $\Phi_c^{(\prime)}$.

$$\begin{aligned} \Gamma_S(\Phi_c) \;&=\; \Phi_0\varpi_0 \;+\; S\,\Phi_m\varpi_m \;\pm\; S\,\Phi^m\widetilde{\varpi}^m \;\pm\; \Phi^0\widetilde{\varpi}^0 \\ \Gamma_S(\Phi'_c) \;&=\; S\,\Phi_0\varpi_0 \;+\; \Phi_m\varpi_m \;\pm\; \Phi^m\widetilde{\varpi}^m \;\pm\; S\,\Phi^0\widetilde{\varpi}^0 \end{aligned} \tag{A.2.4}$$

We choose particular combinations of these four objects.

$$\begin{aligned} \big\langle\, \Phi_c, \overline{\Gamma_S(\Phi_c)}\,\big\rangle_\pm \;&=\; \Big(\;\overline{S}\;\Big(\;\overline{\Phi}_n\Phi^n \pm \Phi_m\overline{\Phi}^m \;\Big) \;-\; S\Big(\;\overline{\Phi}_0\Phi^0 \pm \Phi_0\overline{\Phi}^0 \;\Big) \Big)\mathrm{vol}_6 \\ \big\langle\, \Phi'_c, \overline{\Gamma_S(\Phi'_c)}\,\big\rangle_\pm \;&=\; \Big(\;\overline{S}\;\Big(\;\overline{\Phi}_0\Phi^0 \pm \Phi_0\overline{\Phi}^0 \;\Big) \;-\; S\Big(\;\overline{\Phi}_n\Phi^n \pm \Phi_m\overline{\Phi}^m \;\Big) \Big)\mathrm{vol}_6 \end{aligned}$$

These combinations are such that their sum factorises into terms seperately dependent on $\mathrm{Im}(S)$ and the Φ_M coordinates.

$$\big\langle\, \Phi_c, \overline{\Gamma_S(\Phi_c)}\,\big\rangle_\pm + \big\langle\, \Phi'_c, \overline{\Gamma_S(\Phi'_c)}\,\big\rangle_\pm = (\overline{S} - S)(\overline{\Phi}_N\Phi^N \pm \Phi_M\overline{\Phi}^M) \tag{A.2.5}$$

The inner product expressions take the same general form as the Hitchin function of (A.1.7) but the second argument has had one of its moduli transformed. To that end we define a new function is a modified form of (A.1.7).

$$H(\Phi,\phi)_s \;\equiv\; i^\rho \int_{\mathcal{M}} \big\langle\, \Phi, \overline{\Gamma_\phi(\Phi)}\,\big\rangle_s \tag{A.2.6}$$

The logarithm of (A.2.5) is close to the expected Kähler potential of a moduli space which is locally the product of local special Kähler manifold with coordinates (Φ_N, Φ^M) and the dilaton moduli space.

$$\begin{aligned} K_S + K_\Phi \;&\sim\; -\ln\big(\overline{S} - S\big) - \ln\Big(\overline{\Phi}_N \Phi^N \pm \Phi_M \overline{\Phi}^M\Big) \\ &\sim\; -\ln\big(\overline{S} - S\big) - \ln H(\Phi)_\pm \\ &\sim\; -\ln\big(H(\Phi_c, S)_\pm + H(\Phi'_c, S)_\pm\big) \end{aligned}$$

This form of the Kähler potential for the dilaton dependent moduli space is seen explicitly in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold [53, 60, 61, 92, 93] and other toroidal compactifications [56] as they have kinetic terms of the form given in (7.1.1).

A.2.3 Reduction to $\mathcal{N} = 1$

The effect on the moduli associated to $\mathcal{M}^Q$ of the orientifold projection is to split it into two disjoint parts, as defined by its ± 1 eigenvalues and given in Table 2.1. Since the $\Delta^\pm(\mathsf{E}^*)$ are affected in the same general manner we denote the generic space spanned by the $(\varpi_M, \widetilde{\varpi}^N)$ basis as Δ and the action of the projection as $\Delta \to \mathcal{O}_+(\Delta) \oplus \mathcal{O}_-(\Delta)$. The structure of the projection is made more explicit by defining the orientifold action on the basis; The orientifold projection removes h_- of the Φ_M and we take λ, κ to vary from 1 to h_+ and Σ, Ξ to vary from $1 + h_+$ to h.

$$\begin{aligned} \Phi_c \;&=\; -S\,\Phi_0\varpi_0 \;+\; \Phi_m\varpi_m \;\pm\; \Phi^m\widetilde{\varpi}^m \;\mp\; S\,\Phi^0\widetilde{\varpi}^0 \;\in\; \Delta \\ \sigma_+(\Phi_c) \;&=\; -S\,\Phi_0\varpi_0 \;+\; \Phi_\Sigma\varpi_\Sigma \;\pm\; \Phi^\lambda\widetilde{\varpi}^\lambda \;\in\; \mathcal{O}_+(\Delta) \\ \sigma_-(\Phi_c) \;&=\; \Phi_\lambda\varpi_\lambda \;\pm\; \Phi^\Sigma\widetilde{\varpi}^\Sigma \;\mp\; S\,\Phi^0\widetilde{\varpi}^0 \;\in\; \mathcal{O}_-(\Delta) \end{aligned} \tag{A.2.7}$$

The $\mathcal{O}_\pm(\Delta)$ are such that the projection removes one of the two p-forms which combine pairwise to construct each of the terms in (A.1.4) and therefore it follows

[75] that $\mathcal{O}_{\pm}(\Delta)$ are Lagrangian manifolds [13] of Δ in terms of the inner products $\left\langle\ \right\rangle_s$.

$$\left\langle\, \mathcal{O}_{\pm}(\Delta), \mathcal{O}_{\pm}(\Delta)\,\right\rangle_s = 0$$

Appendix B

Bases and Fluxes

A number of different notations are used throughout this thesis, relating to the $\Omega^*(\mathsf{E}^*)$ p-form basis, the $\Delta^*(\mathsf{E}^*)$ light mode basis and the scalar product defined superpotentials. In this Appendix we define the scalar product notation and demonstrate several identities on the $\Omega^p(\mathsf{E}^*)$ bases as well as motivate our use of the light mode basis.

B.1 The Effective Theory Degrees of Freedom

B.1.1 The Truncation

We review the discussion of truncating to a finite dimensional space the basis over which the fluxes and pure forms have support given in Ref. [74]. In the previous Appendix section we assumed that the different forms Ω and $\mho$ associated to the pure spinors $\Phi^{\pm}$ have support in a finite dimensional space $\Delta^*(\mathsf{E}^*) \subset \Omega^*(\mathsf{E}^*)$.

$$\phi \in \Delta^0(\mathsf{E}^*) \qquad J\,,\, B \in \Delta^2(\mathsf{E}^*) \qquad \Omega \in \Delta^3(\mathsf{E}^*) \qquad C_p \in \Delta^p(\mathsf{E}^*)$$

We have taken Ω as a form of pure degree such that the $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure reduces to $\mathrm{SU}(3)$. The finite dimensional requirement follows from the physical interpretation of the modes, only the lightest modes descend into the four dimensional effective theory and each mode is associated to the supermultiplet. If the moduli associated to these light modes are to form the special Kähler manifolds required for supersymmetry then the inner product $\langle\ \rangle_{\pm}$ must not be degenerate on $\Delta^{*}(\mathsf{E}^{*})$ and it therefore follows that $\Delta^{n}(\mathsf{E}^{*})$ and $\Delta^{6-n}(\mathsf{E}^{*})$ are of equal dimension. Since the fluxes can be viewed as terms in a derivative the light modes must also be closed under exterior differentiation.

$$\mathrm{d} : \Delta^{p}(\mathsf{E}^{*}) \to \Delta^{p+1}(\mathsf{E}^{*}) \tag{B.1.1}$$

The requirement that the spinors defining the pure $\Phi^{\pm}$ are singlets in SU(3) is equivalent to projecting out any SU(3) triplets and therefore the light modes in $\Delta^{2}(\mathsf{E}^{*})$ and $\Delta^{3}(\mathsf{E}^{*})$ are orthogonal.

$$\begin{array}{rcl} \chi_2 & \in & \Delta^{2}(\mathsf{E}^{*}) \\ \varphi_3 & \in & \Delta^{3}(\mathsf{E}^{*}) \end{array} \quad \Rightarrow \quad \chi_2 \wedge \varphi_3 = 0 \in \Delta^{5}(\mathsf{E}^{*}) \tag{B.1.2}$$

This is related to the reduction of SU(3) × SU(3) structure to just SU(3) since this restricts Φ^{-} to having no support in $\Delta^{1}(\mathsf{E}^{*})$ or $\Delta^{5}(\mathsf{E}^{*})$. In the case of $\Delta^{1}(\mathsf{E}^{*})$ this condition restricts $\Delta^{0}(\mathsf{E}^{*})$ to containing only constant functions and so is one dimensional. Finally the identity $\star\Omega = -i\Omega$ requires $\Delta^{-}(\mathsf{E}^{*})$ to be closed under the Hodge star and this is taken to extend to $\Delta^{+}(\mathsf{E}^{*})$. As a result $\Delta^{p}(\mathsf{E}^{*})$ and $\Delta^{6-p}(\mathsf{E}^{*})$ have the same number of dimensions, as found by the non-degeneracy of the inner product $\langle\ \rangle_{\pm}$.

$$\chi \in \Delta^{p}(\mathsf{E}^{*}) \quad \Rightarrow \quad \star\chi \in \Delta^{6-p}(\mathsf{E}^{*}) \tag{B.1.3}$$

Since the fields on $\mathcal{M}$ are written in terms of this basis it must be able to allow fields which satisfy their equations of motion and Bianchi constraints. This is done in the simplest way on Calabi-Yaus as the moduli are defined by expanding the pure spinors in the harmonic p-forms. Since harmonic forms are both closed and co-closed the Calabi-Yau provides a background which automatically satisfies the equations of motion for the fields.

$$\chi \in \mathfrak{H}^{p}(\mathsf{E}^{*}) \quad \to \quad \Delta\chi = 0 \quad \to \quad \mathrm{d}\chi = \mathrm{d}\star\chi = 0$$

With the inclusion of non-zero fluxes the $\Delta^*(\mathsf{E}^*)$ are no longer harmonic. As a result the Kaluza-Klein tower of masses is no longer a set of clearly seperated levels, with the moduli possibly gaining masses of order the first excited states or the modes in the first excited state reducing in mass. However, the use of the harmonic forms as a basis is a sufficiently good approximation in cases where the fluxes are small or the volume of $\mathcal{M}$ is large. As such we use the Calabi-Yau harmonic forms as an initial basis for $\Delta^*(\mathsf{E}^*)$ when the switching on of the fluxes deforms the space and induces masses in some or all of the holomorphic forms.

B.1.2 Fluxes in Parallelisable Generalised Frame Bundles

The constuction of the truncated basis above considers the light $\Delta^p(\mathsf{E}^*)$ as a subspace of $\Omega^p(\mathsf{E}^*)$ and thus the fluxes[1] can have their action written in a simple schematic manner.

$$\mathcal{F}_n : \Omega^{2n}(\mathsf{E}^*) \to \Omega^3(\mathsf{E}^*) \tag{B.1.4}$$

The Type IIA NS-NS fluxes of (4.1.10) induced by sequences of T duality transformations on toroidal orientifolds have $\Lambda^3(\mathfrak{E})$ defined components with the decomposition into the spaces $\mathcal{F}_n$ have support on given in (4.1.4). This $\mathfrak{E}$ component definition is essential if the flux components are to be put into a Lie algebra context and the $\mathrm{GL}(6,\mathbb{Z})$ invariance used to find flux parameterisations. Furthermore it allows us to define the components of the fluxes without acting them on p-forms elements and thus the action of $\mathcal{F}_n$ on $\Omega^{2n}(\mathsf{E}^*)$ in (B.1.4) uniquely determines the action of $\mathcal{F}_n$ on $\Omega^3(\mathsf{E}^*)$. However the resultant requirement that the components are constant puts a limit on the number of independent components each flux can have. Denoting the number of independent components of $\mathcal{F}_n$ as $|\mathcal{F}_n|$ the index symmetries give an upper bound on these for $\dim(\mathcal{M}) = 6$, independent of any

[1]Since the methodology is much the same in each case we consider the Type IIA NS-NS fluxes as our explicit example.

underlying symmetries in $\mathcal{M}$.

$$\mathcal{F}_n \in \Lambda^n(\mathsf{E}) \wedge \Lambda^{3-n}(\mathsf{E}^*) \quad \Rightarrow \quad |\mathcal{F}_n| \leq C^6_n \cdot C^6_{3-n} \quad \Rightarrow \quad \begin{array}{c} |\mathcal{F}_0| \leq 20 \;\;,\;\; |\mathcal{F}_1| \leq 90 \\ |\mathcal{F}_2| \leq 90 \;\;,\;\; |\mathcal{F}_3| \leq 20 \end{array}$$

These upper bounds can be seen to violated by giving the basis elements of $\Delta^*(\mathsf{E}^*)$ their $\Omega^*(\mathsf{E}^*)$ representations. We consider the explicit case of the $\omega_a \in \Delta^2(\mathsf{E}^*)$ and suppose that $\mathcal{F}_1(\omega_a) \in \Delta^3(\mathsf{E}^*)$.

$$\begin{aligned} \mathcal{F}_1 \cdot \omega_a &= \frac{1}{2!}\mathcal{F}^r_{pq}\eta^{pq}\iota_r\left(\frac{1}{2!}(\omega_a)_{mn}\eta^{mn}\right) \\ &= \frac{1}{2!}\mathcal{F}^r_{pq}(\omega_a)_{rn}\eta^{pqn} \\ &= \frac{1}{3!}\left(f_{(a)I}(\alpha_I)_{pqn} - f_{(a)}{}^J(\beta^J)_{pqn}\right)\eta^{pqn} \end{aligned}$$

Comparing the degrees of freedom we have that $\mathcal{F}_1$ must map $h^{1,1}$ basis elements of $\Delta^2(\mathsf{E}^*)$ to the $2(h^{2,1}+1)$ basis elements of $\Delta^3(\mathsf{E}^*)$ and thus in general has $2h^{1,1}(h^{2,1}+1)$ independent components in its $\Delta^*(\mathsf{E}^*)$ definition. Since the bound $2h^{1,1}(h^{2,1}+1) \leq 90$ is violated in specific[2] Calabi-Yaus the representation of $\mathcal{F}_n$ in terms of $\Lambda^3(\mathfrak{E})$ components is not applicable to all spaces. However, for parallelisable spaces it is applicable and this representation has a number of important or useful properties. Contributions which do not otherwise appear in the effective theory light $\Delta^*(\mathsf{E}^*)$ space are included. These additional non-light terms are such that the Lie algebra interpretation of the fluxes is possible and so the $\mathrm{GL}(6,\mathbb{Z})$ invariance can be used to construct parameterisations of the non-geometric fluxes. The explicit construction of the $\Delta^p(\mathsf{E}^*) \subset \Omega^p(\mathsf{E}^*)$ for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold also allows a consistency check on results constructed in terms of the finite truncated basis of $\Delta^p(\mathsf{E}^*)$ as either representation should be valid on the orientifold.

[2] The example given in the discussion of Calabi-Yaus in Ref. [8] is a Calabi-Yau defined by a homogeneous degree 5 polynomials in $\mathbb{CP}^4$ with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 101)$.

B.1.3 Fluxes in Generalised Light Bundles

Our guiding principle in this section is that we wish construct a formalism which extends the results for parallelisable $\mathcal{M}$, where the fluxes can act on either $\Omega^{\pm}(\mathsf{E}^*)$, in such a way as to reduce to the parallelisable result on such spaces. In parallelisable $\mathcal{M}$ the flux components can be obtained via the factorisation of $\mathcal{D}(\mho) \in \Delta^{-}(\mathsf{E}^*)$ into the holomorphic form $\mho \in \Delta^{+}(\mathsf{E}^*)$ and the derivative $\mathcal{D} \in \Delta^{*}(\mathfrak{E})$ and then the splitting $\mathcal{D}$ into its individual fluxes. The fluxes in parallelisable $\mathcal{M}$ can be regarded as elements of $\Lambda^3(\mathfrak{E})$, as given in (4.1.3) and (4.1.4). For those non-parallelisable $\mathcal{M}$ the degrees of freedom within the fluxes are defined by the expansion of the flux image of elements of $\Delta^{+}(\mathsf{E}^*)$ in the $\Delta^{-}(\mathsf{E}^*)$ basis, $\mathcal{F}_n(\Delta^{+}(\mathsf{E}^*)) \in \Delta^{-}(\mathsf{E}^*)$ and we view the fluxes purely as a linear map belonging to $\mathrm{End}(\Delta^{*}(\mathsf{E}^*))$. To that end we denote the operator which maps the coefficient of $\xi \in \Delta^{*}(\mathsf{E}^*)$ to becoming the coefficient of ζ by $\mathfrak{f}(\zeta, \xi)$. We consider $\mathcal{F}_1$ as an explicit example.

$$\mathcal{F}_1 = \mathcal{F}_{(a)I}\,\mathfrak{f}(\mathfrak{a}_I, \widetilde{\nu}^a) - \mathcal{F}_{(a)}{}^{J}\,\mathfrak{f}(\mathfrak{b}^J, \widetilde{\nu}^a)$$

If we assume the same factorisation $\mathcal{D}(\mho)$ to $\mathcal{D}$ and $\mho$ for non-parallelisable $\mathcal{M}$ we generalise (4.1.4) to the light forms and find $\mathcal{F}_n$ belong to specific subspaces of $\mathrm{End}(\Delta^{*}(\mathsf{E}^*))$.

$$\mathcal{F}_n : \Delta^{n,n}(\mathsf{E}^*) \to \Delta^{3}(\mathsf{E}^*) \quad \Rightarrow \quad \mathcal{F}_n \in \Delta^{3}(\mathsf{E}^*) \wedge \Delta^{n,n}(\mathsf{E}) \in \mathrm{End}(\Delta^{*}(\mathsf{E}^*)) \tag{B.1.5}$$

As in the $\Lambda^p(\mathfrak{E})$ case, despite the more general action $\mathcal{F}_n : \Omega^p(\mathsf{E}^*) \to \Omega^{p+3-2n}(\mathsf{E}^*)$ the fact $\mathcal{F}_n(\mathcal{J}^{(n)})$ couples to $\Omega \in \Delta^{3}(\mathsf{E}^*)$ causes only the $p = 2n$ case to be relevant. The basis of $\Delta^{p,q}(\mathsf{E})$ we shall define to be dual in some sense to the basis of $\Delta^{p,q}(\mathsf{E}^*)$. This is motivated by the wish to have an operator which has the factorisation of $\mathrm{End}(\Delta^{*}(\mathsf{E}^*))$ of (B.1.5) manifest and we define the operator ι_ξ by this factorisation.

$$\mathfrak{f}(\zeta, \xi) \equiv \zeta \wedge \iota_\xi = \zeta\,\iota_\xi \tag{B.1.6}$$

For the components of $\mathcal{F}_n$ the $\mathfrak{f}(\zeta,\xi)$ are such that $\xi \in \Delta^{\pm}(\mathsf{E}^*)$ and $\zeta \in \Delta^{\mp}(\mathsf{E}^*)$. The group structure of the $\mathfrak{f}$ follow in a straightforward manner. With this factorisation we can consider the ι_ξ individually. We initially examine the $(\omega_A, \widetilde{\omega}^B)$ and (α_I, β^J) basis so that motivation for the change of basis can be demostrated more clearly.

$$\iota_{\omega_A}(\omega_B) = \delta^A_B = \iota_{\widetilde{\omega}^B}(\widetilde{\omega}^A) \quad , \quad \iota_{\alpha_I}(\alpha_J) = \delta^I_J = \iota_{\beta^J}(\beta^I)$$

Not all possible combinations have been explicitly stated. In the cases of $\iota_{\omega_A}(\widetilde{\omega}^B)$ and $\iota_{\widetilde{\omega}^B}(\omega_A)$ the ξ in (B.1.6) is such that the expression evaluates to zero identically or provide no contribution to the superpotential. For the $\iota_{\alpha_I}(\beta^J)$ and $\iota_{\beta^I}(\alpha_J)$ cases it is not immediate that they vanish but this is proven in the next section for parallelisable spaces and we assume the extension to non-parallelisable ones. These identities are also demonstrated for the explicit basis of $\Delta^*(\mathsf{E}^*)$ for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold.

B.1.4 $\Delta^*(\mathfrak{E})$ Basis Identities

To examine the action of the interior forms in $\Delta^-(\mathsf{E})$ on elements of $\Delta^-(\mathsf{E}^*)$ we restrict ourselves to parallelisable $\mathcal{M}$ such that we have the embedding $\Delta^-(\mathsf{E}) \subset \Lambda^-(\mathsf{E})$ and $\Delta^-(\mathsf{E}^*) \subset \Lambda^-(\mathsf{E}^*)$.

$$\begin{aligned} \alpha_I &= \tfrac{1}{3!}(\alpha_I)_{abc}\eta^{abc} \quad , \quad \beta^J = \tfrac{1}{3!}(\beta^J)_{ijk}\eta^{ijk} \\ \iota_{\alpha_I} &= \tfrac{1}{3!}(A^I)^{abc}\iota_{cba} \quad , \quad \iota_{\beta^J} = \tfrac{1}{3!}(B_J)^{ijk}\iota_{kji} \end{aligned}$$

The symplectic definition of the $\Delta^-(\mathsf{E}^*)$ basis puts constraints on their $\Lambda^3(\mathsf{E}^*)$ components.

$$\delta^J_I = \int_{\mathcal{M}} \alpha_I \wedge \beta^J = \frac{1}{3!}\frac{1}{3!}(\alpha_I)_{abc}\epsilon^{abcijk}(\beta^J)_{ijk} \tag{B.1.7}$$

By the same methodology the dual conditions between the elements of $\Lambda^3(\mathsf{E})$ and $\Lambda^3(\mathsf{E}^*)$ constraint the components of both sets of basis elements.

$$\begin{aligned} \delta^J_I &= \iota_{\alpha_J}(\alpha_I) = \tfrac{1}{3!}\tfrac{1}{3!}(A^J)^{abc}(\alpha_I)_{ijk}\iota_{cba}(\eta^{ijk}) = \tfrac{1}{3!}(A^J)^{abc}(\alpha_I)_{abc} \\ \delta^J_I &= \iota_{\beta^I}(\beta^J) = \tfrac{1}{3!}\tfrac{1}{3!}(B_I)^{abc}(\beta^J)_{ijk}\iota_{cba}(\eta^{ijk}) = \tfrac{1}{3!}(B_I)^{abc}(\beta^J)_{abc} \end{aligned} \tag{B.1.8}$$

Comparing the three coefficient expansions for δ_I^J we obtain the $(A^J)^{abc}$ and $(B_I)^{abc}$ in terms of the $(\alpha_I)_{abc}$, $(\beta^J)_{abc}$ and the antisymmetric ϵ.

$$(A^J)^{abc} = \frac{1}{3!}\epsilon^{abcijk}(\beta^J)_{ijk} \quad , \quad (B_I)^{abc} = \frac{1}{3!}\epsilon^{abcijk}(\alpha_I)_{ijk} \tag{B.1.9}$$

With these explicit expressions for the coefficients we can construct $\iota_{\alpha_J}(\beta^I)$ and $\iota_{\beta^I}(\alpha_J)$ in terms of the α_I and β^J components.

$$\begin{aligned} \iota_{\alpha_I}(\beta^J) &= \tfrac{1}{3!}\epsilon^{abcijk}(\beta^I)_{ijk}\iota_{cba}(\beta^J)_{pqr}\eta^{pqr} &= (\beta^I)_{ijk}\epsilon^{abcijk}(\beta^J)_{abc} \\ \iota_{\beta^J}(\alpha_I) &= \tfrac{1}{3!}\epsilon^{abcijk}(\alpha_J)_{ijk}\iota_{cba}(\alpha_I)_{pqr}\eta^{pqr} &= (\alpha_J)_{ijk}\epsilon^{abcijk}(\alpha_I)_{abc} \end{aligned} \tag{B.1.10}$$

Making note of the identity $\epsilon^{abcdef}\mathrm{vol}_6 = \eta^{abcdef}$ these expressions can be written entirely in terms of an integral defined on the symplectic basis.

$$\iota_{\alpha_I}(\beta^J) = \int_{\mathcal{M}} (\beta^J)_{abc}\eta^{abc}(\beta^I)_{ijk}\eta^{ijk} = \int_{\mathcal{M}} \beta^J \wedge \beta^I = 0 \tag{B.1.11}$$

By the same method we obtain the second expression.

$$\iota_{\beta^J}(\alpha_I) = \int_{\mathcal{M}} (\alpha_I)_{abc}\eta^{abc}(\alpha_J)_{ijk}\eta^{ijk} = \int_{\mathcal{M}} \alpha_I \wedge \alpha_J = 0 \tag{B.1.12}$$

B.1.5 Non-Geometric Flux Operator Action

The schematic action of an adjoint derivative $\mathfrak{d}^\dagger : \Omega^p(\mathsf{E}^*) \to \Omega^{p-1}(\mathsf{E}^*)$ is the same schematic action as $\mathcal{F}_2$ and so we can rephrase $\mathcal{F}_2$ in terms of the adjoint action of an exterior derivative. To that end we define $\mathfrak{d}$ and its $\langle\!\langle\, , \,\rangle\!\rangle$ adjoint $\mathfrak{d}^\dagger$ by the action of $\mathcal{F}_2$ on $\Delta^{2,2}(\mathsf{E}^*)$.

$$\mathfrak{d}^\dagger(\widetilde{\omega}^b) \equiv \mathcal{F}_2(\widetilde{\omega}^b) \equiv (\mathcal{F}_2)^{(a)}{}_I\alpha_I - (\mathcal{F}_2)^{(a)J}\beta^J$$

As in the $\mathcal{F}_1$ case we can project out particular coefficients of $\mathcal{F}_2(\widetilde{\omega}^b)$ by taking its inner product with particular $\Delta^3(\mathsf{E}^*)$ basis elements, allowing us to then make use of the adjoint properties of the inner product.

$$(\mathcal{F}_2)^{(a)}{}_I = \langle\!\langle\, \mathcal{F}_2(\widetilde{\omega}^a), \alpha_I \,\rangle\!\rangle = \langle\!\langle\, \mathfrak{d}^\dagger\widetilde{\omega}^a, \alpha_I \,\rangle\!\rangle \equiv \langle\!\langle\, \widetilde{\omega}^a, \mathfrak{d}\alpha_I \,\rangle\!\rangle$$

Making use of Stokes theorem again we can change which form the derivative $\mathfrak{d}$ acts upon, which is not possible to do when working with $\mathfrak{d}^\dagger$.

$$\begin{aligned} 0 = \int_{\mathcal{M}} \mathfrak{d}(\alpha_I \wedge \omega_a) &= \int_{\mathcal{M}} \mathfrak{d}\alpha_I \wedge \omega_a - \int_{\mathcal{M}} \alpha_I \wedge \mathfrak{d}\omega_a \\ &= \langle\!\langle\, \mathfrak{d}\alpha_I, \widetilde{\omega}^a \,\rangle\!\rangle - \int_{\mathcal{M}} \alpha_I \wedge \star(\star^{-1}\mathfrak{d}\star)\widetilde{\omega}^b \end{aligned}$$

Having obtained an expression for $\mathfrak{d}$ acting on an element of $\Delta^3(\mathsf{E}^*)$ we need to revert back to expressing derivatives as $\mathfrak{d}^\dagger$. This is done by using the definition of adjoint derivatives in terms of Hodge stars and derivatives, taking note that the action of the derivatives on the symplectic basis elements acquires an additional factor of -1.

$$\mathfrak{d}^\dagger \quad = \quad \begin{cases} \star^{-1}\mathfrak{d}\star & \mathfrak{d} : \Delta^+(\mathsf{E}^*) \to \Delta^3(\mathsf{E}^*) \\ -\star^{-1}\mathfrak{d}\star & \mathfrak{d} : \Delta^3(\mathsf{E}^*) \to \Delta^+(\mathsf{E}^*) \end{cases}$$

Inverting this relationship, to express $\mathfrak{d}$ in terms of $\mathfrak{d}^\dagger$, we obtain the alternative action of $\mathfrak{d}^\dagger$ on $\Delta^3(\mathsf{E}^*)$ by noting that $\star = \star^{-1}$ on $\Delta^+(\mathsf{E}^*)$ due to the intersection numbers of the basis elements being the Kronecker delta.

$$(\mathcal{F}_2)^{(a)}{}_I = \langle\!\langle\, \widetilde{\omega}^a, \mathfrak{d}\alpha_I \,\rangle\!\rangle = \langle\!\langle\, \widetilde{\omega}^a, -\star\, \mathfrak{d}^\dagger \star^{-1} \alpha_I \,\rangle\!\rangle = \langle\!\langle\, \omega_a, \mathfrak{d}^\dagger \beta^I \,\rangle\!\rangle$$

Repeating this method but projecting out the remaining fluxes in $\mathcal{F}_2$ gives terms related to the $\mathcal{F}_2$ image of the α_I.

$$(\mathcal{F}_2)^{(a)J} = \langle\!\langle\, \widetilde{\omega}^a, \mathfrak{d}\beta^J \,\rangle\!\rangle = \langle\!\langle\, \widetilde{\omega}^a, -\star\, \mathfrak{d}^\dagger \star^{-1} \beta^J \,\rangle\!\rangle = \langle\!\langle\, \omega_a, \mathfrak{d}^\dagger(-\alpha_J) \,\rangle\!\rangle$$

B.1.6 Alternate Hodge Star

Since much of our analysis is done in terms of the inner product $\langle\ \rangle_\pm$ rather than the usual $\langle\!\langle\ \rangle\!\rangle$ inner product, which is associated to the Hodge star $\star$, we define the $\langle\ \rangle_\pm$ equivalent $*$ in terms of the $\Delta^*(\mathsf{E}^*)$ basis elements. As in the previous Appendix we use the representative basis $(\varpi_M, \widetilde{\varpi}^N)$ and state the non-trivial defining expressions.

$$\int \langle\, \varpi_M, *\varpi_N \,\rangle = \delta_{MN} \quad , \quad \int \langle\, \widetilde{\varpi}^M, *\widetilde{\varpi}^N \,\rangle = \delta^{MN}$$

The explict forms of $*\varpi_M$ and $*\widetilde{\varpi}^N$ are easily deduced and we convert back to the $\Delta^*(\mathsf{E}^*)$ bases.

$$*\mathfrak{a}_I = \mathfrak{b}^I \quad , \quad *\mathfrak{b}^J = -\mathfrak{a}_J \quad , \quad *\nu_A = \widetilde{\nu}^A \quad , \quad *\widetilde{\nu}^B = \pm\nu_A$$

Unless otherwise stated we use the $*$ associated to $\langle\ \rangle_-$, giving both $\Delta^\pm(\mathsf{E}^*)$ bases a symplectic structure. From this it follows that (4.1.24) can be rewritten as (4.1.25) and the $\mathfrak{f}$ of (B.1.6) obeys those identities.

$$\mathfrak{f}(\zeta,\xi) = \zeta\,\iota_\xi = -*\xi\,\iota_{*\zeta} = -\mathfrak{f}(*\xi,*\zeta)$$

The overall factor of -1 follows from definition of the $(\nu_A,\widetilde{\nu}^B)$ basis in terms of the $(\omega_A,\widetilde{\omega}^B)$ basis and the definition of $\langle\ \rangle_-$ in (A.1.4).

B.2 Scalar Product Representations

Having defined a finite basis for the $\Delta^*(\mathsf{E}^*)$ we can define the associated structures to this space of forms, such as the non-degenerate bilinear form or forms associated to inner products. As elsewhere, we take all Hodge number dependent statements to be defined in terms of the topology of $\mathcal{M}$, rather than its mirror dual $\mathcal{W}$, so I,J range over $\{0,\ldots,h^{2,1}\}$ and A,B over $\{0,\ldots,h^{1,1}\}$.

B.2.1 Inner Product $\langle\ \rangle_+$

We begin with the inner product $\langle\ \rangle_+$ which is equivalent to simple exterior multiplication and so is symmetric on $\Delta^+(\mathsf{E}^*)$ and antisymmetric on $\Delta^-(\mathsf{E}^*)$.

$$\int_{\mathcal{M}} \langle\,\phi,\varphi\,\rangle_+ = \int_{\mathcal{M}} \phi\wedge\varphi \equiv g(\phi,\varphi) \qquad \phi,\varphi\in\Delta^\pm(\mathsf{E}^*)$$

The bilinear form associated to the inner product is g. If $\underline{\mathsf{e}}$ and $\underline{\mathsf{e}}'$ are vectors of forms then the entries of g are defined by $g(\mathsf{e}_m,\mathsf{e}'_n)$. The basis vector for the complex structure moduli space we take to be $\mathbf{e}_{(\mathfrak{a})}$ and for the Kähler moduli space $\mathbf{e}_{(\nu)}$.

$$g_{\mathfrak{a}} \equiv g(\mathbf{e}_{(\mathfrak{a})},\mathbf{e}_{(\mathfrak{a})}) = \begin{pmatrix} g(\mathfrak{a}_I,\mathfrak{a}_J) & g(\mathfrak{a}_I,\mathfrak{b}^J) \\ g(\mathfrak{b}^I,\mathfrak{a}_J) & g(\mathfrak{b}^I,\mathfrak{b}^J) \end{pmatrix} = \begin{pmatrix} 0 & \delta_I^J \\ -\delta_J^I & 0 \end{pmatrix}$$

$$g_{\nu} \equiv g(\mathbf{e}_{(\nu)},\mathbf{e}_{(\nu)}) = \begin{pmatrix} g(\nu_A,\nu_B) & g(\nu_A,\widetilde{\nu}^B) \\ g(\widetilde{\nu}^A,\nu_B) & g(\widetilde{\nu}^A,\widetilde{\nu}^B) \end{pmatrix} = \begin{pmatrix} 0 & \delta_A^B \\ \delta_B^A & 0 \end{pmatrix}$$

By construction $g_{\mathfrak{a}}$ has $\mathrm{Sp}(n)$ symmetry for $n = h^{2,1}+1$ and g_ν has $\mathrm{O}(m,m)$ symmetry for $m = h^{1,1}+1$. In each case $g^\top = g^{-1}$. The second bilinear form is $h(\underline{\mathsf{e}},\underline{\mathsf{e}}') \equiv g(\underline{\mathsf{e}},\Sigma\cdot\underline{\mathsf{e}}')$ where Σ is the canonical $\mathrm{O}(n,n)$ inner product bilinear form and $2n$ is the dimension of the $\underline{\mathsf{e}}$.

$$h_{\mathfrak{a}} \equiv g(\mathbf{e}_{(\mathfrak{a})},\Sigma_{\mathfrak{a}}\cdot\mathbf{e}_{(\mathfrak{a})}) = \begin{pmatrix} \delta_I^J & 0 \\ 0 & -\delta_I^J \end{pmatrix} \quad , \quad h_\nu \equiv h(\mathbf{e}_{(\nu)},\Sigma_\nu\cdot\mathbf{e}_{(\nu)}) = \begin{pmatrix} \delta_A^B & 0 \\ 0 & \delta_A^B \end{pmatrix}$$

In cases where the dimensionality of the canonical $\mathrm{O}(n,n)$ bilinear form is unambigious we shall drop the subscript and simply use Σ. In terms of the matrices of the bilinear forms we have the general relationship $h = g\cdot\Sigma$. This, along with $h = h^\top$ and $\Sigma = \Sigma^\top$, can be used to construct a number of identities relating to how any two of these three matrices will combine, which we will make considerable use of.

$$\begin{array}{ccccc} h\cdot g = \Sigma & , & h\cdot\Sigma = g & , & g\cdot\Sigma = h \\ g^\top\cdot h = \Sigma & , & \Sigma\cdot h = g^\top & , & \Sigma\cdot g^\top = h \end{array} \tag{B.2.1}$$

Depending on the basis, $\mathbf{e}_{(\mathfrak{a})}$ or $\mathbf{e}_{(\nu)}$, being considered we have that $g^\top = \pm g$ and so to have expressions true in either basis we must retain the transpositions. However, if we put this identity into the above expressions we obtain (anti)commutation relations.

$$\begin{array}{lllll} g^\top = +g & \Rightarrow & \big[h,g\big] = 0 \ , & \big[h,\Sigma\big] = 0 \ , & \big[g,\Sigma\big] = 0 \\ g^\top = -g & \Rightarrow & \big\{h,g\big\} = 0 \ , & \big\{h,\Sigma\big\} = 0 \ , & \big\{g,\Sigma\big\} = 0 \end{array} \tag{B.2.2}$$

Given these bilinear forms on $\Delta^\pm(\mathsf{E}^*)$ we can express elements of $\Delta^*(\mathsf{E}^*)$ as scalar products which in term provide a scalar product representation of superpotential like expressions. To that end we associated elements of $\Delta^{\pm}(\mathsf{E}^*)$ with sets of vectors.

$$
\begin{aligned}
\xi &= \xi_I \mathfrak{a}_I - \xi^J \mathfrak{b}^J = \begin{pmatrix} \xi_I & \xi^J \end{pmatrix} \begin{pmatrix} \delta_I^J & 0 \\ 0 & -\delta_I^J \end{pmatrix} \begin{pmatrix} \mathfrak{a}_I \\ \mathfrak{b}^J \end{pmatrix} \equiv \underline{\xi}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \\
\Pi &= \Pi_A \nu_A - \Pi^B \widetilde{\nu}^B = \begin{pmatrix} \Pi_A & \Pi^B \end{pmatrix} \begin{pmatrix} \delta_A^B & 0 \\ 0 & \delta_A^B \end{pmatrix} \begin{pmatrix} \nu_A \\ \widetilde{\nu}^B \end{pmatrix} \equiv \underline{\Pi}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)}
\end{aligned}
$$

The vector associated to the holomorphic forms are the moduli vector, $\Omega = \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})}$ and $\mho = \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)}$. Using these vector expressions we can represent the inner product between two 3-forms ξ and ζ or two even forms Π and Θ in a particularly straight forward manner.

$$
\begin{aligned}
g(\xi,\zeta) &= \int_{\mathcal{M}} \xi \wedge \zeta = \zeta_I \xi^I - \zeta^J \xi_J = \underline{\xi}^\top \cdot h_{\mathfrak{a}} \cdot g_{\mathfrak{a}} \cdot h_{\mathfrak{a}}^\top \cdot \underline{\zeta} = \underline{\zeta}^\top \cdot g_{\mathfrak{a}} \cdot \underline{\xi} \\
g(\Pi,\Theta) &= \int_{\mathcal{M}} \Pi \wedge \Theta = \Theta_A \Pi^A + \Theta^B \Pi_B = \underline{\Pi}^\top \cdot h_\nu \cdot g_\nu \cdot h_\nu^\top \cdot \underline{\Theta} = \underline{\Theta}^\top \cdot g_\nu \cdot \underline{\Pi}
\end{aligned}
$$

With these definitions we can obtain an expression for a generic Type II superpotential contribution due to some $\varphi \in \Delta^-(\mathsf{E}^*)$ coupling to Ω or $\Pi \in \Delta^+(\mathsf{E}^*)$ coupling to $\mho$.

$$
\begin{aligned}
\int_{\mathcal{M}} \big\langle\, \Omega, \xi \,\big\rangle_+ &= g\Big(\Omega,\xi\Big) = g_{\mathfrak{a}}\Big(\Omega,\xi\Big) = \underline{\xi}^\top \cdot g_{\mathfrak{a}} \cdot \underline{\mathfrak{U}} \\
\int_{\mathcal{M}} \big\langle\, \mho, \Pi \,\big\rangle_+ &= g\Big(\mho,\Pi\Big) = g_\nu\Big(\mho,\Pi\Big) = \underline{\Pi}^\top \cdot g_\nu \cdot \underline{\mathfrak{T}}
\end{aligned}
$$

These two constructions in $\Delta^{\pm}(\mathsf{E}^*)$ combine to define vectors for all elements of $\Delta^*(\mathsf{E}^*)$, $\chi = \underline{\chi} \cdot h \cdot \mathbf{e}$.

$$
g = g_{\mathfrak{a}} \oplus g_\nu \quad \Rightarrow \quad g(\xi+\Pi, \zeta+\Theta) = g_{\mathfrak{a}}(\xi,\zeta) + g_\nu(\Pi,\Theta)
$$

These intersection numbers obey the identities (B.2.1) except that now Σ is associated to $\mathrm{O}(h^{2,1}+h^{1,1}+2, h^{2,1}+h^{1,1}+2)$. The Hodge star can be defined by the construction of an inner product which combines two elements of $\Delta^p(\mathsf{E}^*)$, rather than elements of $\Delta^p(\mathsf{E}^*)$ and $\Delta^{6-p}(\mathsf{E}^*)$. In terms of vectors the natural inner product on the $\Delta^p(\mathsf{E}^*)$ takes the form $\big\langle\!\big\langle\, \phi, \chi \,\big\rangle\!\big\rangle = \underline{\chi}^\top \cdot \underline{\phi}$ and we define the

Hodge star's matrix representation $\underline{\underline{\star}}$ by reexpressing this in terms of $\langle\ \rangle_+$ via $\star(\chi) = \underline{\chi}^\top \cdot h \cdot \underline{\underline{\star}} \cdot \mathbf{e}$.

$$
\begin{aligned}
\langle\!\langle\, \phi, \chi \,\rangle\!\rangle \equiv g\Big(\phi, \star(\chi)\Big) &= \int_{\mathcal{M}} \langle\, \underline{\phi}^\top \cdot h \cdot \mathbf{e} \,,\, \underline{\chi}^\top \cdot h \cdot \underline{\underline{\star}} \cdot \mathbf{e} \,\rangle_+ \\
&= \int_{\mathcal{M}} \langle\, \Big(\underline{\phi}^\top \cdot h \cdot \mathbf{e}\Big) \,,\, \Big(\underline{\chi}^\top \cdot \mathrm{Ad}(\underline{\underline{\star}}) \cdot h \cdot \mathbf{e}\Big) \,\rangle_+ \\
&= \underline{\chi}^\top \cdot \mathrm{Ad}_h(\underline{\underline{\star}}) \cdot g \cdot \underline{\phi}
\end{aligned}
$$

We have used the $\mathrm{GL}(n,\mathbb{Z})$, for the appropriate n, group adjoint operator $\mathrm{Ad}_x(y)$ to move h to the right of $\underline{\underline{\star}}$.

$$
\mathrm{Ad}_x(y) \equiv x \cdot y \cdot x^{-1} \qquad x, y \in \mathrm{GL}(n,\mathbb{Z}) \tag{B.2.3}
$$

If the inner product is to evaluate to $\underline{\chi}^\top \cdot \underline{\phi}$ we require $\mathrm{Ad}_h(\underline{\underline{\star}}) = g^{-1}$ and it follows that $\underline{\underline{\star}} = g$.

B.2.2 Inner Product $\langle\ \rangle_-$

The $\langle\ \rangle_-$ of (A.1.4) is antisymmetric on both $\Delta^\pm(\mathsf{E}^*)$. In Chapter 4 the change of basis $(\omega_A, \widetilde{\omega}^B) \to (\nu_A, \widetilde{\nu}^B)$ is motivated by algebraic simplification when considering flux operators. This is precisely that used in (A.1.5) to simplify the sign structure of the holomorphic section of a special Kähler manifold. We again use g and h to represent the associated bilinear forms.

$$
g(\phi, \varphi) \equiv \int_{\mathcal{M}} \langle\, \phi, \varphi \,\rangle_- \quad , \quad h(\phi, \varphi) \equiv g(\phi, \Sigma \cdot \varphi)
$$

With both $\Delta^\pm(\mathsf{E}^*)$ possessing manifest special Kähler structure the bilinear form g is symplectic in each case and applying Σ to one of the arguments of the g gives the associated h.

$$
g_{\mathfrak{a}} = \begin{pmatrix} 0 & \delta_I^J \\ -\delta_J^I & 0 \end{pmatrix} \quad g_\nu = \begin{pmatrix} 0 & \delta_A^B \\ -\delta_B^A & 0 \end{pmatrix} \quad h_{\mathfrak{a}} = \begin{pmatrix} \delta_I^J & 0 \\ 0 & -\delta_I^J \end{pmatrix} \quad h_\nu = \begin{pmatrix} \delta_A^B & 0 \\ 0 & -\delta_A^B \end{pmatrix}
$$

With both $\Delta^\pm(\mathsf{E}^*)$ having symplectic structure we have $g^\top = g^{-1} = -g$ and the identities of (B.2.1) can be simplified and we obtain the anticommutation relations of (B.2.2). The vectors associated to the elements of $\Delta^*(\mathsf{E}^*)$ are defined in the same manner and the moduli can be defined by this.

$$\begin{aligned} \xi &= \underline{\xi}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \quad , \quad \Pi = \underline{\Pi}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)} \\ \Omega &\equiv \underline{\mathfrak{U}}^\top \cdot h_{\mathfrak{a}} \cdot \mathbf{e}_{(\mathfrak{a})} \quad , \quad \mho \equiv \underline{\mathfrak{T}}^\top \cdot h_\nu \cdot \mathbf{e}_{(\nu)} \end{aligned} \tag{B.2.4}$$

Since h_ν is different from the $\langle\ \rangle_+$ case the Kähler moduli will be different, as required to make the special Kähler structure manifest, but the scalar product expressions are of the same schematic form. With these expansion of the holomorphic forms we can construct superpotential-like expressions.

$$\begin{aligned} \int_{\mathcal{M}} \langle\, \Omega, \varphi \,\rangle_- &= g\Big(\Omega, \varphi\Big) = g_{\mathfrak{a}}\Big(\Omega, \varphi\Big) = \underline{\varphi}^\top \cdot g_{\mathfrak{a}} \cdot \underline{\mathfrak{U}} \\ \int_{\mathcal{M}} \langle\, \mho, \chi \,\rangle_- &= g\Big(\mho, \chi\Big) = g_\nu\Big(\mho, \chi\Big) = \underline{\chi}^\top \cdot g_\nu \cdot \underline{\mathfrak{T}} \end{aligned}$$

The Hodge star on $\langle\ \rangle_-$ takes the same form as the $\langle\ \rangle_+$ case but the $\underline{\underline{*}} = g$ is altered to being symplectic on both $\Delta^\pm(\mathsf{E}^*)$.

B.2.3 Mirror Bilinear Forms

In the previous sections we defined the bilinear forms on $\mathcal{M}$. The construction of these bilinear forms on the mirror of $\mathcal{M}$, $\mathcal{W}$, follows the same method but with the Hodge number exchange $h^{p,q}(\mathcal{M}) = h^{q,p}(\mathcal{W})$. We denote those bilinear forms defined in Type IIB on $\mathcal{W}$ as g and h. In each Type II theory the bilinear forms are inner product dependent and we have different algebraic relations for each inner product $\langle\ \rangle_\pm$. For $\langle\ \rangle_+$ the Kähler moduli bilinear form is equivalent to Σ.

$$\langle\ \rangle_\pm \to \langle\ \rangle_+ \quad \Rightarrow \quad g_\nu = \Sigma \quad , \quad \mathsf{g}_\nu = \Sigma$$

We have not explicitly distinguished between the Σ in $\mathcal{M}$ and the Σ in $\mathcal{W}$ as their dimensionality is clear from the context. For $\langle\ \rangle_-$ there are several useful relationships.

$$\langle\ \rangle_\pm \to \langle\ \rangle_- \quad \Rightarrow \quad g_{\mathfrak{a}} = \mathsf{g}_\nu \quad , \quad h_{\mathfrak{a}} = \mathsf{h}_\nu \quad , \quad \mathsf{h}_{\mathfrak{a}} = h_\nu \quad , \quad \mathsf{h}_{\mathfrak{a}} = h_\nu$$

Appendix C

Algebraic Geometry

C.1 Mathematical Background

We review basic algebraic geometry definitions and results relevant to the main text, following the notation of [97].

C.1.1 Ideals and Varieties

Given a set of functions $f = (f_1, \ldots, f_n)$ where $f_i \in k[x_1, \ldots, x_m]$, polynomials in m variables over field k, the set of points which are roots of all polynomials define a variety $V(f_1, \ldots, f_n)$.

$$V(f_1, \ldots, f_n) = \big\{(a_1, \ldots, a_m) \in k^m \,:\, f_i(a_1, \ldots, a_m) = 0 \quad \forall f_i\big\}$$

Varieties can be combined via normal set operators, intersection and union.

$$\begin{array}{rcl} V &=& V(f_1, \ldots, f_s) \\ W &=& W(g_1, \ldots, g_t) \end{array} \quad \Rightarrow \quad \begin{array}{rcl} V \cap W &=& V(f_1, \ldots, f_s, g_1, \ldots, g_t) \\ V \cup W &=& V\big(f_i g_j : i \in \{1, \ldots, s\}\ ,\ j \in \{1, \ldots, t\}\big) \end{array}$$

A set $I \subset k[x_1, \ldots, x_n]$ is an <u>ideal</u> if it satisfies a number of conditions.

- $0 \in I$
- If $f, g \in I$ then $f + g \in I$

- If $f \in I$ and $h \in k[x_1, \ldots, x_n]$ then $fh \in I$

From this we can define an ideal I generated by a set of polynomials $\{f_i\}$.

$$\langle I \rangle = \langle f_1, \ldots, f_n \rangle = \left\{ \sum_{i=1}^{n} h_i f_i \,:\, h_j \in k[x_1, \ldots, x_m] \right\} \tag{C.1.1}$$

This ideal can be linked to a variety $\mathbf{V}(I)$ by identifying the ideal $\langle f_1, \ldots, f_n \rangle$ with the affine variety defined by $f_i = 0\ \forall i$. The choice of generating polynomials for a finitely generated ideal does not alter the variety the ideal defines, in the same way the choice of basis does not alter the vector space it spans. If $\langle I \rangle = \langle f_1, \ldots, f_n \rangle = \langle g_1, \ldots, g_m \rangle$, then $\mathbf{V}(f_1, \ldots, f_n) = \mathbf{V}(g_1, \ldots, g_m)$. For a given ideal I there is a particularly convenient choice of generating basis, known as the Groebner basis, whose definition is given in [97]. This basis plays an important role in the question of whether or not a given polynomial f is in I or not. If I is generated by $\langle f_1, \ldots, f_n \rangle$, then $f \in I$ iff $1 \in \tilde{I} \equiv \langle f_1, \ldots, f_n, 1 - yf \rangle \subset k[x_1, \ldots, x_m, y]$. That is, if the Groebner basis of $\tilde{I}$ is $\{1\}$, then $f \in I$.

Just as ideals define varieties, varieties can define ideal. An ideal can be defined from V by considering the points in the variety as zeros of the generating functions of the ideal.

$$\mathbf{I}(V) = \big\{ f \in k[x_1, \ldots, x_n] \,:\, f(a_1, \ldots, a_n) = 0 \quad \forall (a_1, \ldots, a_n) \in V \big\} \tag{C.1.2}$$

$\mathbf{V}$ and $\mathbf{I}$ can be thought of as maps between the geometric and algebraic formalisms. $\mathbf{I}$ take varieties to ideals and $\mathbf{V}$ does vice versa, though they are not inverses of one another. These maps are inclusion reversing, in that if $I_1 \subset I_2$, $\mathbf{V}(I_1) \supset \mathbf{V}(I_2)$ and if $V_1 \subset V_2$ then $\mathbf{I}(V_1) \supset \mathbf{I}(V_2)$. The ideals and varieties can be combined a number of straight forward ways.

- Summation : $I = \langle f_1, \ldots, f_n \rangle$ and $J = \langle g_1, \ldots, g_m \rangle$.

$$I + J = \langle f_1, \ldots, f_n, g_1, \ldots, g_m \rangle \quad , \quad \mathbf{V}(I + J) = \mathbf{V}(I) \cap \mathbf{V}(J)$$

- Product : $I = \langle f_1, \ldots, f_n \rangle$ and $J = \langle g_1, \ldots, g_m \rangle$.

$$I.J = \langle f_i g_j \,:\, i \in [1,n] \quad j \in [1,m] \rangle \quad , \quad \mathbf{V}(I.J) = \mathbf{V}(I) \cup \mathbf{V}(J)$$

- Intersection : $I, J \subset k[x_1, \ldots, x_r]$.

$$I \cap J = \big(tI + (1-t)J \big) \cap k[x_1, \ldots, x_r] \quad , \quad \mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J)$$

These basis operators on ideals and varieties follow from the standard properties of either the ring of polynomials defining the ideal or the set construction defining the variety. From these a number of less trivial operations can be defined.

C.1.2 Quotient and Prime Ideals

A quotient ideal can be constructed from $I \subset k[x_1, \ldots, x_r]$ for $J = \langle g \rangle$.

$$I : J = \big\{ f \in k[x_1, \ldots, x_r] \text{ s.t. } fg \in I \quad \forall g \in J \big\} \tag{C.1.3}$$

More generally, for $J = \langle f_1, \ldots, f_n \rangle$, this extends to the intersection of individual quotient ideals.

$$I : J \;=\; \bigcap_{i=1}^{r} (I : f_i) \tag{C.1.4}$$

The saturation of I by f is the quotienting by all powers of f.

$$(I : f^\infty) \;\equiv\; \big\{ g \in k[x_1, \ldots, x_n] \text{ s.t. } f^m g \in I \quad \text{for some } m > 0 \big\}$$

A variety V is irreducible if whenever $V = V_1 \cup V_2$, where V_i are affine varieties, then either $V_1 = V$ or $V_2 = V$. The equivalent description of ideals is that $I \subset k[x_1, \ldots, x_n]$ is prime if whenever $f, g \in k[x_1, \ldots, x_n]$ and $fg \in I$ then either $f \in I$ or $g \in I$. V is irreducible iff $\mathbf{I}(V)$ is prime. An irreducible variety cannot be split into simpler varieties, so a line through $\mathbb{C}^n$ is irreducible, but a plane unioned with a line is not. A maximal ideal is one in which the only ideal which strictly contains it is $k[x_1, \ldots, x_n]$ and for any field k a maximal ideal in $k[x_1, \ldots, x_n]$ is prime. If k is algebraicly closed, then every maximal ideal in $k[x_1, \ldots, x_n]$ has the generating set of the form $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $a_i \in k$. Any variety V can be decomposed into finitely many irreducible varieties, $V = V_1 \cup \ldots \cup V_n$, where V_i is irreducible. There is a unique decomposition of this form if $V_i \not\subset V_j$ for $i \neq j$. If a variety can be expressed as a unique union of irreducible varieties, then the ideal $I \equiv \mathbf{I}(V)$ can be expressed as the unique interaction of prime ideals (since $\cap \leftrightarrow \cup$ for $I \leftrightarrow V$), $I = P_1 \cap \ldots P_m$, where $P_i \not\subset P_j$, provided k is algebraicly closed.

$$I = \bigcap_{i=1}^{n} P_i \tag{C.1.5}$$

This is I's minimal representation as an intersection of prime ideals.

C.1.3 Groeber Basis

Though we shall not give its formal definition there is a particularly convenient basis to use for a given ideal, that of the Groebner basis. Its utility comes in determining if an ideal already contains a particular function, because of the duality between ideals and varieties. Given an ideal $I = \langle f_1, \ldots, f_n \rangle$ in $\mathbb{C}[x_1, \ldots, x_N]$ it has variety $\mathbf{V}(I)$, whose points are the zeros of the generating functions of I, f_i. A function g has associated variety $\mathbf{V}(\langle g \rangle)$, which are the zeros of g. If $\mathbf{V}(I) \cap \mathbf{V}(\langle g \rangle) = \emptyset$ then by the inclusion-exclusion reversal of map between ideals and varieties $I \cup \langle g \rangle = \langle f_1, \ldots, f_n, g \rangle = \langle 1 \rangle = \mathbb{C}[x_1, \ldots, x_N]$. The polynomial 1 has no zeros, so its associated variety is empty and generates all of $\mathbb{C}[x_1, \ldots, x_N]$. It is not always clear whether a given ideal $I = \langle f_1, \ldots, f_n \rangle$ is trivial but if its Groebner generating basis if 1 then it is and it follows that there are no values of x_n which satisfy $f_m = 0$ simultaneously for all m.

C.1.4 Variables and Coefficients

Thus far we have discussed only elements of $\mathbb{K}[x_1, \ldots, x_m]$, those polynomials formed from the variables x_i and with coefficients in the field $\mathbb{K}$, typically taken to be $\mathbb{C}$. However, we may wish to define polynomials with unknown coefficients that are not variables. If a is a coefficient and x a variable then $\langle \mathsf{a} \rangle = \langle 1 \rangle \neq \langle x \rangle$, it is possible to divide through by a but not x. To distinguish between unknowns which are variables to be solved and unknowns which are parameters to be specified we define a new type of ring $\mathbb{K}[x_1, \ldots, x_m\,;\, \mathsf{a}_1, \ldots, \mathsf{a}_n]$.

C.2 Computational Methods

While the methods and algorithms for algebraic geometry can be explained and proved through short amounts of algebra and logic, practical applications of such methods require large quantities of repetitive processes, ideally suited for computers. Mathematica has several functions built into it which impliment algorithms to do such things are prime decompositions, saturations, taking the radical of an ideal and computing dimensions. For small ideals this would normally be sufficent but the complexity and size of the ideals produced in practical systems of SUGRA are sufficently big to require a more specialised program.

One such program is Singular [98]. While not the most flexible, it has all the required routines programmed into it and is considerably faster (between 10 and 1000 times) than Mathematica in the application of such algorithms. However, it is devoid of much of the functionality of Mathematica. For instance, it cannot do the algebraic manipulation to go from K and W to V and ∂V, plot graphs or do numerical solutions once the systems are broken down to sufficently simple components. Fortunately, there are 'bridging' routines such as [99] which convert Mathematica outputs into Singular inputs and vice versa, allowing the seemless use of functionality both programs provide. Alternatively there is StringVACUA [100], which is a Mathematica front end for Singular specifically written for the purposes of constructing supergravity vacua in terms of moduli and fluxes and does not require more than a minimal amount of algebraic geometry knowledge.

C.2.1 Singular Algorithms

The Singular algorithms relevant to the construction of vacua or flux compactifications implement the algebraic geometry methods previously outlined.

`simplify`	:	$\{g_1,\ldots,g_m\} \to \{g'_1,\ldots,g'_n\}$	:	$\mathrm{LC}(g'_i)=1\,,\ g'_i \neq k g'_j\,,\ g'_k \neq 0$
`intersect`	:	$\{I_1,I_2\} \to J$	:	$J = I_1 \cap I_2$
`radical`	:	$I \to J$	:	$J = \sqrt{I}$
`facstd`	:	$I \to \{I_1,.,I_n\}$	:	$\sqrt{I} = \sqrt{I_1} \cap \ldots \cap \sqrt{I_n}$
`MinAssGTZ`	:	$I \to \{P_1,.,P_n\}$	:	$I = P_1 \cap \ldots \cap P_n$

In all cases the ring of polynomials over which Singular applies the algebraic geometry algorithms must be stipulated.

C.3 Supergravity Applications

Having provided a short dictionary for algebraic geometry methods and results we now consider how these can be applied to the construction of string vacua. There are two general ways in which algebraic geometry can be used, solving the equations defined in terms of the scalar potential V to obtain local minima for the moduli and solving the integrability and cohomology condictions on non-geometric fluxes which arise in the context of Type IIB Bianchi constraints. The former methods are not used explicitly in this thesis, vacua are found analytically for restricted cases, but we cover them for the purposes of completeness. The application of algebraic geometry to string vacua is suggested and discussed in [26, 27], including non-perturbative contributions such as gaugino condensates, and are then implimented in StringVACUA [100]. However, StringVACUA does not include native algorithms for the parameterisations of the non-geometric fluxes to be explained shortly.

C.3.1 Vacua Construction

Given the scalar potential V of (2.4.1) the vacua are the local minima of the potential and thus are the solutions to $\partial_M V = 0$ such that the Hessian of V is positive definite. With the Kähler potential being a logarithm of polynomials and the superpotential being a polynomial the scalar potential is itself a polynomial and thus so are all its n'th derivatives. Translating this into algebraic geometry we have an ideal I generated by the numerator polynomials of $\partial_M V$ and the associated variety $\mathbf{V}(I)$ are all moduli values which are solutions to said polynomials.

$$\partial_M V = \frac{V n_M}{V d_M} \quad \Rightarrow \quad I \equiv \langle V n_M \rangle \tag{C.3.1}$$

Not all solutions to $\partial_M V = 0$ are minima, all turning points satisfy such an expression, including local maxima. It is also possible that V has regions which are flat in particular directions, resulting in regions which satisfy $\partial_M V = 0$, not just isolated points. However, this faciliates a decomposition of $\mathbf{V}(I)$. Each turning point, point of inflextion or partially flat region in V defines an irreducible subvariety within $\mathbf{V}(I)$. Furthermore, any stable local minimum will have no flat directions and thus it defines a zero dimensional variety in moduli space. A prime decomposition of I splits the $\partial_M V = 0$ solution space into its component varieties whose dimensions can be computed and only those which are of zero dimension can be possible stable vacuum states. Only a finite amount of such varieties can exist, a large reduction on the possibly infinite amount of solutions to $\partial_M V = 0$ induced by flat directions. The Hessian of V at each of the finitely many points can be computed, a reduction in computing the Hessian everywhere in moduli space.

A further restriction arises from supersymmetry, where the vacuum expectation value of the F-term F_M defines the supersymmetry breaking scale of the M'th modulus. Enforcing complete supersymmetry preservation on the vacuum reduces the scalar potential to $V = -3e^K|W|^2 \leq 0$, which does not allow for de Sitter solutions without the inclusion of non-perturbative effects, but makes analytic analysis easier. It is possible to check for the possible existence of entirely supersymmetry vacua by noting such vacua would be in the intersection of the varieties associated to $\langle \partial_M V \rangle$ and $\langle F_N \rangle$.

$$\langle \partial_M V \, , \, F_N \rangle = \langle 1 \rangle \quad \Rightarrow \quad \text{No completely supersymmetic vacua} \tag{C.3.2}$$

This can be further refined by selecting a subset of the F_M, F_m, to be definitely supersymmetric and the supersymmetric status of the remaining moduli undetermined by computing the Groebner basis of $\langle \partial_M V \, , \, F_m \rangle$. Although a trivial Groebner basis immediately excludes the possibility of such a vacuum a non-trivial basis is not automatically a set of polynomials which are any easier to solve than $\partial_M V$ and F_m. $\mathbf{V}(\langle \partial_M V \rangle)$ is the space of all extrema while $\mathbf{V}(\langle \partial_M V, F_1 \rangle)$ is the space of all extrema which preserve supersymmetry in the first modulus and is a subset of the first. This can be stated more generally for sets A and B.

$$A = A \cap (B \cup \neg B) = (A \cap B) \cup (A \cap \neg B) \tag{C.3.3}$$

Either a modulus is supersymmetry or not and so we can split $\mathbf{V}(\langle \partial_M V \rangle)$ into two parts, supersymmetric or not, for each modulus and this is done algebraicly by saturation.

$$\mathbf{V}(\langle \partial_M V \rangle) = \mathbf{V}(\langle \partial_M V, F_1 \rangle) + \mathbf{V}(\langle \partial_M V : F_1^\infty \rangle) \tag{C.3.4}$$

The variety $\mathbf{V}(\langle \partial_M V \, : \, F_1^\infty \rangle)$ represents the space of moduli values which are extrema of the potential but for which the first modulus is not supersymmetric.

This ability to pick and choose which moduli are and are not supersymmetric provides a way to split $\langle \partial_M V \rangle$ into $2^{h^{1,1}+h^{2,1}+1}$ seperate ideals, within which only those with zero dimensional varieties would be possible candidates for stable vacuum states. It is noteworthy that it is computationally preferable to stipulate supersymmetry breakings, or any other dichotomy inducing condition, before attempting a prime decomposition.

In all of these cases it is the moduli which are the variables to be solved for and if there are still fluxes which are not stipulated then they are parameters, not variables. Denoting the set of moduli as Φ_M and the fluxes as f_n the ring over which Singular, or any other algebraic geometry program, applies its algorithms is $\mathbb{C}[\Phi_M\,;\,\mathsf{f}_n]$.

C.3.2 Non-Geometric Flux Parameterisations

Two algebraic geometry methods are used in constructing the solutions to the Type IIB/O3 Bianchi constraints on the non-geometric flux $\mathsf{F}_2 = Q$ and its S duality partner, $\widehat{\mathsf{F}}_2 = P$. For clarity we recall the constraint equations on Q in (6.4.1) and its S duality partners.

$$Q^{[ab}_{e}Q^{c]e}_{d} = 0 \quad , \quad P^{[ab}_{e}P^{c]e}_{d} = 0 \quad , \quad Q^{[ab}_{e}P^{c]e}_{d} + P^{[ab}_{e}Q^{c]e}_{d} = 0$$

We consider the first of these three expressions, where Q is playing the role of a structure constant for a six dimensional Lie algebra. Such Lie algebras have been completely classified [101]. However, Q is further restrained by the properties of the orbifold, any Lie algebra must be invariant under the orbifold group and this is particularly constrained by isotropy. Under the cyclic permutation of the three subtori the only Lie algebras which can possibly match such a symmetry are those whose structure constants are isotropic. The six dimensional Lie algebra obtains two generators from each torus and these form two triplets, $X_a \to (T_\rho, \widetilde{T}_\sigma)$ and we can write down the most general isotropic commutation relations for these two triplets, though it is most conveniently done in the dual $(\xi^{\rho},\widetilde{\xi}^{\sigma})$ basis.

$$\begin{aligned} \mathrm{d}\xi^{\rho} &= \lambda_1\,\xi^{\sigma}\wedge\xi^{\tau} \;+\; \lambda_2\,\widetilde{\xi}^{\sigma}\wedge\xi^{\tau} \;+\; \lambda_3\,\widetilde{\xi}^{\sigma}\wedge\widetilde{\xi}^{\tau} \\ \mathrm{d}\widetilde{\xi}^{\rho} &= \lambda_4\,\xi^{\sigma}\wedge\xi^{\tau} \;+\; \lambda_5\,\widetilde{\xi}^{\sigma}\wedge\xi^{\tau} \;+\; \lambda_6\,\widetilde{\xi}^{\sigma}\wedge\widetilde{\xi}^{\tau} \end{aligned} \qquad \text{(C.3.5)}$$

Not all values of λ_i satisfy the Bianchi/Jacobi constraints of the algebra. Only those values which give the derivatives of Table 6.13 automatically satisfy the Jacobi identity. Since this is a complete classification of all six dimensional Lie algebras and the constraints on Q are $\mathrm{GL}(6,\mathbb{Z})$ invariant then the only consistent flux configurations for Q are those obtained by a $\mathrm{GL}(6,\mathbb{Z})$ transformation on one of the five Lie algebras of Table 6.13. Since, by construction, the canonical structure constants of the five Lie algebras match the orbifold symmetries, as do the flux components of Q, only those elements of $\mathrm{GL}(6,\mathbb{Z})$ satisfying the orbifold symmetries can be used. We denote such a generic member as M which is responsible for transforming Q^{pq}_{r} to the canonical form g^{xy}_{z} and is, due to orbifold symmetries, dependent on the four modular parameters of (6.4.4).

$$M^{x}_{p}\,M^{y}_{q}\,Q^{pq}_{r}\,(M^{-1})^{r}_{z} = g^{xy}_{z} \qquad \text{(C.3.6)}$$

This expression represents an array of equations and by rearranging them we can define an ideal whose generating functions are said equations.

$$I(Q) \equiv \langle\, M^{x}_{p}\,M^{y}_{q}\,Q^{pq}_{r}\,(M^{-1})^{r}_{z} - g^{xy}_{z}\,\rangle \qquad \text{(C.3.7)}$$

It is at this point important to note the difference between the entries of Q, the c_i, and the modular parameters in M, m_j. The ring over which Singular, or any other algebraic geometry program, applies its algorithms is $\mathbb{C}[\mathsf{c}_i\,;\,\mathsf{m}_j]$. The moduli do not factor into the parameterisations of the non-geometric fluxes as the Bianchi constraints are moduli independent. Due to the symmetry properties of the structure constants and the orbifold, the number of unique generating functions is very small. This is because expressions which differ by an overall factor only contribute one generating function to the ideal and trivial cases of generating functions being identically zero are ignored. In order to be sure that (6.4.3) is a well defined equation we saturate the initial ideal with respect to the condition $|\Gamma_M| \neq 0$ so as to remove possible solutions for which the transformation matrix is degenerate. Under degenerate transformations it is possible to transform different Lie algebras into one another, which is inconsistent with the notion of Q being isomorphic to a specific algebra.

The third member of the S duality triplet is not of the same format, mixing the two non-geometric fluxes. As outlined in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold chapter the two non-geometric fluxes are transformed by different sets of modular parameters, $M = M(\mathsf{m}_i)$ and $M' = M'(\mathsf{m}'_j)$, solving $QQ = 0 = PP$ seperately. The mixing then reduces to constraints on m_i and m'_j, as $\langle QP + PQ \rangle$ belongs to the ring $\mathbb{C}[\mathsf{m}_i, \mathsf{m}'_j]$ and it is over this ring that Singular applies its prime decomposition algorithm.

$$\langle QP + PQ \rangle = P_1(\mathsf{m}_i, \mathsf{m}'_j) \cap \ldots \cap P_n(\mathsf{m}_i, \mathsf{m}'_j) \tag{C.3.8}$$

Given a solution to the constraints $QP + PQ = 0$ it must belong to one of the varieties $\mathbf{V}(P_i)$. Splitting the ideal down into its prime components generally simplifies the generating functions, which are the constraints on the fluxes. Solving the constraints of P_i individually provides inequivalent solutions to the constraints of $QP + PQ = 0$ but considering all P_i covers all possible solutions.

Algebra		Geometry
radical ideals		varieties
I	$\rightarrow$	$\mathbf{V}(I)$
$\mathbf{I}(V)$	$\leftarrow$	V
addition of ideals		intersection of varieties
$I + J$	$\rightarrow$	$\mathbf{V}(I) \cap \mathbf{V}(J)$
$\sqrt{\mathbf{I}(V) + \mathbf{I}(W)}$	$\leftarrow$	$V \cap W$
product of ideals		union of varieties
IJ	$\leftarrow$	$\mathbf{V}(I) \cup \mathbf{V}(J)$
$\sqrt{\mathbf{I}(V)\mathbf{I}(W)}$	$\rightarrow$	$V \cup W$
intersection of ideals		union of varieties
$I \cap J$	$\leftarrow$	$\mathbf{V}(I) \cup \mathbf{V}(J)$
$\mathbf{I}(V) \cap \mathbf{I}(W)$	$\rightarrow$	$V \cup W$
quotients of ideals		difference of varieties
$I : J$	$\rightarrow$	$\overline{\mathbf{V}(I) - \mathbf{V}(J)}$
$\mathbf{I}(V) : \mathbf{I}(W)$	$\leftarrow$	$\overline{V - W}$
prime ideal	$\leftarrow$	irreducible variety
maximal ideal	$\leftarrow$	point of affine space

Table C.1: Equivalent algebraic and geometric structures.

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