UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

School of Physics and Astronomy

# $E_6$ Inspired Supersymmetric Models

by

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# ABSTRACT

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$E_6$ INSPIRED SUPERSYMMETRIC STANDARD MODELS

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This work investigates extensions to the Standard Model that are inspired by supersymmetric models with an $E_6$ gauge group. The models are non-minimal supersymmetric theories which keep the Higgs mass stable against the quantum corrections from higher energy physics, but do not contain the $\mu$-problem or little hierarchy problem of the Minimal Supersymmetric Standard Model (MSSM). Also, unlike conventional Grand Unified Theories, the $E_6$ inspired models do not contain any doublet-triplet splitting and the Minimal $E_6$ Supersymmetric Model (ME$_6$SSM) only contains complete $E_6$ multiplets at low energies. A particularly exciting feature of the ME$_6$SSM is the prediction of gauge coupling unification at the Planck scale rather than the conventional GUT scale, hinting at a potential unification of the Standard Model forces with quantum gravity.

If extended with a discrete non-Abelian family symmetry, the $E_6$ inspired models can explain the masses and mixings of the quarks and leptons that are observed in particle experiments. These are not understood in the Standard Model since they are free parameters, creating a flavour problem for the theory. Extending the Standard Model or MSSM with a family symmetry offers an attractive resolution to the flavour problem, and the recent discovery of neutrino oscillations, which indicate a high-level of symmetry in the lepton mixings, has led to a renewed interest in these models. However, explaining why the Higgs mass is small is essential in these models since it sets the scale for the quark and lepton masses. This motivates the synthesis of a family symmetry with the $E_6$ inspired supersymmetric models, which resolves a number of problems facing the Standard Model including the hierarchy problem and the flavour problem. A particular success of the resulting models is their ability to suppress proton decay and flavour changing neutral currents, from supersymmetry and extended Higgs sectors, using the same family symmetry that is responsible for a tri-bi-maximal mixing of leptons.

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# Declaration of Authorship

I, RICHARD HOWL, declare that this thesis entitled, '$E_6$ Inspired Supersymmetric Models' and the work presented in the thesis are both my own work and have been generated by me as the result of my own original research. I confirm that:

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Date:

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# Abbreviations

| | |
|---|---|
| **CKM** | **C**abibbo **K**obayashi **M**askawa |
| **CSD** | **C**onstrained **S**equential **D**ominance |
| **$E_6$SSM** | **E**xceptional **S**upersymmetric **S**tandard **M**odel |
| **FCNCs** | **F**lavour **C**hanging **N**uetral **C**urrents |
| **GSO** | **G**atto **S**artori **T**onin |
| **GUT** | **G**rand **U**nified **T**heory |
| **IS** | **I**ntermediate **S**ymmetry |
| **M$E_6$SSM** | **M**inimal **E**xceptional **S**upersymmetric **S**tandard **M**odel |
| **MNS** | **M**aki **N**akagawa **S**akata |
| **MSSM** | **M**inimal **S**upersymmetric **S**tandard **M**odel |
| **RG** | **R**enormalization **G**roup |
| **RGEs** | **R**enormalization **G**roup **E**quations |
| **SM** | **S**tandard **M**odel |
| **SUGRA** | **Su**per **Gra**vity |
| **SUSY** | **Su**per**s**ymmetr**y** |
| **VEV** | **V**acuum **E**xpectation **V**alue |

*To my parents*

# Chapter 1

# Introduction

## 1.1 Beyond the Standard Model

For more than thirty years the Standard Model has provided the most accurate description of particle physics and there has been little direct experimental evidence to suggest that the model should be replaced with a new theory. However, the Standard Model cannot explain the recent discovery of neutrino oscillations [1], which suggests that the theory must be modified. Mounting cosmological evidence for dark matter and dark energy also suggests that the model is incomplete [2, 3].

Although it has been experimentally successful, the Standard Model has long been considered to be unsatisfactory in a number of theoretical areas. For example, it is incompatible with General Relativity, our most accurate theory of gravity, and the Higgs mass is unstable with the addition of higher energy physics [4]. There is also a lack of explanation for the observed structure of quark and lepton masses and CKM matrix elements, introducing a flavour problem to the theory. The most popular solution to the instability of the Higgs mass is to treat the Standard Model as a low-energy effective field theory of the Minimal Supersymmetric Standard Model (MSSM) [5], which is the minimal application of supersymmetry to the Standard Model. In the MSSM each Standard Model particle is given a supersymmetric partner so that there is an equal number of boson and fermion degrees of freedom. The Higgs mass is then stable because the quantum corrections from the fermions and bosons cancel [6].

As well as stabilizing the Higgs mass, the MSSM also hints at solutions to a number of other failings of the Standard Model. For example, the MSSM (with R-parity conserved) potentially provides a candidate for dark matter since the lightest supersymmetric particle (LSP) is stable and should be weakly interacting [5]. The MSSM also indicates the existence of a new theory at a very high-energy scale which provides new insights into many theoretical problems of the Standard Model. If the MSSM gauge coupling constants are run to high energies they meet at approximately $3 \times 10^{16}$ GeV, which is called the GUT scale [7]. This suggests that the strong nuclear force and the electroweak force unify at this high-energy scale and that the MSSM is a low-energy approximation to a supersymmetric Grand Unified Theory [8].

Supersymmetric Grand Unified Theories (SUSY GUTs) based on gauge groups such as $SO(10)$ and $E_6$ can explain the mysterious anomaly cancellations of the Standard Model and the quantization of electric charge [9]. They can also predict right-handed neutrinos which, since they do not take part in the gauge interactions of the Standard Model, would be expected to obtain GUT scale masses. A conventional see-saw mechanism then predicts small neutrino masses [10], and the out-of-equilibrium decays of right-handed neutrinos can explain baryon asymmetry through Sphaleron processes [11].

However, despite its obvious attractions, the standard paradigm of SUSY GUTs based on the MSSM faces some serious shortcomings. On the one hand, the failure to discover superpartners or the Higgs boson by the LEP and the Tevatron indicates that the scale of SUSY breaking must be higher than previously thought, leading to fine-tuning at the per cent level [12]. On the other hand experimental limits on proton decay and the requirement of Higgs doublet-triplet splitting provides some theoretical challenges at the high scale. Related to the doublet-triplet splitting problem is the origin of $\mu$, the SUSY Higgs and Higgsino mass parameter, which from phenomenology must be of order the SUSY breaking scale, but which a priori is independent of the SUSY breaking scale [13].

An elegant solution to the $\mu$-problem is to extend the particle content of the MSSM by introducing a new field $S$ that is a singlet of the Standard Model gauge group and couples to the MSSM Higgs doublets such that its dynamically generated vacuum expectation value (VEV) provides an effective TeV scale $\mu$-term that is related to the breaking of supersymmetry [14]. In such theories there is also an advantage to be gained by introducing an additional low-energy Abelian gauge group $U(1)$ since, without a $U(1)$ gauge group, a Goldstone boson would be created by the singlet field's VEV [15]. The $U(1)$ group also explains why there is no explicit $\mu$-term and why $S$ does not get a large Majorana mass.

SUSY GUTs based on an $\mathrm{E}_6$ gauge group naturally contain additional $U(1)$ groups and Standard Model singlets $S$ [16]. This suggests that supersymmetric models based on an $\mathrm{E}_6$ gauge symmetry can be alternatives to the MSSM that do not contain a $\mu$-problem. A low-energy model that is inspired by an $E_6$ SUSY GUT is the $\mathrm{E}_6$ supersymmetric

Standard Model ($E_6$SSM) [17]. This model does not contain the $\mu$-problem or the little hierarchy problem of the MSSM. However, an unsatisfactory aspect of the $E_6$SSM is that, to obtain gauge coupling unification at the GUT scale, the $E_6$SSM contains two electroweak doublets $H'$ and $\overline{H}'$ that do not form complete $E_6$ representations and reintroduce a $\mu'$-problem and a doublet-triplet splitting problem. In this work a new model called the Minimal $E_6$ Supersymmetric Standard Model (M$E_6$SSM) is introduced that only contains complete $E_6$ representations but still predicts unification of the Standard Model gauge coupling constants. This model contains all the benefits of the $E_6$SSM such as a stable Higgs field and no $\mu$-problem, little hierarchy problem or doublet-triplet splitting but does not reintroduce any of these problems. In the M$E_6$SSM the gauge coupling constants are predicted to unify at the Planck scale rather than the GUT scale suggesting a potential unification of the Standard Model forces with quantum gravity.

Another failing of simple SUSY GUTs is their inability to explain the quark and lepton masses and mixing angles that are observed in particle experiments. Since quarks and leptons are unified (or partially unified) into the same representations of the simple gauge group, Grand Unified Theories predict relations between the quark and lepton masses. However they do not explain why there are three generations of quarks and leptons, and why these generations have a strong hierarchical structure. Further, only the unification of the quark and lepton Yukawa couplings for the heaviest generation is successful when renormalized at the electroweak scale [18].

The lack of understanding of quark and lepton masses has seen renewed interest in recent years due to the observation of neutrino masses and lepton mixing angles [19]. An elegant solution to explaining the smallness of neutrino masses is the conventional see-saw mechanism, which naturally occur in Grand Unified Theories such as $SO(10)$ or $E_6$. When combined with a family symmetry this mechanism can also explain the large lepton mixing angles which are, at present, consistent with a tri-bi-maximal symmetry [20]. Family symmetries control the Yukawa couplings of the quarks and leptons to the Higgs field, and discrete non-Abelian family symmetry such as $\Delta_{27}$ are particularly successful at explaining the quark and lepton masses and mixing angles [21]. When applied to supersymmetric theories, non-Abelian family symmetries also provide a solution to the SUSY flavour and CP problems [22]. Extending SUSY GUTs with a family symmetry is thus very successful at resolving the flavour problem of the Standard Model (and MSSM).

In models with a family symmetry the Higgs VEV sets the (upper) scale of the quark and lepton masses and so the Higgs mass must be small (of order the electroweak symmetry) in these models. This strongly suggests extending the $E_6$SSM or M$E_6$SSM with a family symmetry and in this work the ME$_6$SSM and E$_6$SSM are chosen to be extended with a $\Delta_{27}$ family symmetry. The resulting models solve many of the theoretical and experimental problems facing the Standard Model. For example, the Higgs mass is stable, the quark and lepton masses and mixing angles are explained, a dark matter candidate is provided, and, in the ME$_6$SSM models, the gauge coupling constants unify at the Planck scale, which implies unification of the Standard Model forces with quantum gravity.

## 1.2 Structure of Thesis

This thesis is organised as follows: In Chapter 2 the Higgs mechanism of the Standard Model is reviewed and the supersymmetric solution to the instability of the Higgs mass is discussed. Supersymmetric Grand Unified Theories are then motivated and the E$_6$SSM is analysed in the context of the $\mu$-problem of the MSSM. Chapter 3 introduces the $\mu'$-problem of the E$_6$SSM and explains how the Standard Model gauge coupling constants can unify in a SUSY E$_6$ GUT that only contains complete representations of E$_6$, which is equivalent to the particle spectrum of the E$_6$SSM but without the additional electroweak doublets $H'$ and $\overline{H}'$. Chapter 4 uses the results of Chapter 3 to develop an alternative to the E$_6$SSM called the ME$_6$SSM that resolves the $\mu$-problem of the MSSM without reintroducing this problem. Chapter 5 describes the lack of explanation of quark and lepton models in the Standard Model and introduces family symmetries as a potential resolution to this problem. Chapter 6 then extends the E$_6$SSM and ME$_6$SSM with a simple discrete non-Abelian family symmetry to solve the flavour problem of the MSSM and SM. In Chapter 7 a family symmetry is applied to the E$_6$SSM that fully resolves the flavour problem of the model and illustrates how the flavour changing neutral currents from supersymmetric theories with extended Higgs sectors can be suppressed. The overall conclusions to this thesis then follow in Chapter 8.

Appendix A illustrates the two-loop $\beta$-functions that are used in Chapters 3 and 4 for the ME$_6$SSM. Appendix B describes the origin of the $U(1)_X$ group of the ME$_6$SSM in detail, and finally, Appendix C reviews how flavour changing neutral currents (FCNCs) are introduced in models with extended Higgs sectors.

# Chapter 2

# The Higgs Field and Supersymmetry

## 2.1 The Standard Model

The Standard Model is a quantum field theory that is based on the local gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ where $SU(3)_c$ describes the strong nuclear force and $SU(2)_L \times U(1)_Y$ describes the unified electroweak force. The symmetry of the electroweak force $SU(2)_L \times U(1)_Y$ is spontaneously broken in the Standard Model to the weak nuclear force $W^{\pm}, Z^0$ and the electromagnetic force $U(1)_{em}$ [23]. Classically a scalar field called the Higgs field takes on a nonzero global value, which does not respect the $SU(2)_L \times U(1)_Y$ symmetry, at every point in space and causes the symmetry to be broken. This is analogous to a ferromagnet in statistical mechanics that is subjected to an external field with a directional character, which breaks the spatial invariance of the magnet. The material for this Section is based on that in [24].

### 2.1.1 Spontaneous Symmetry Breaking

To illustrate how the electroweak symmetry is broken, consider the Lagrangian of a $U(1)$ gauge field and a charged complex scalar field $h$:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu h|^2 - V(h) \tag{2.1}$$

where $F_{\mu\nu}$ is the field strength of the $U(1)$ gauge field $A_\mu$; $D_\mu$ is the covariant derivative of the scalar field, which describes the interaction between the scalar and gauge fields; and $V(h)$ is the potential of the scalar field. The field strength and covariant derivative are given by Eq.2.2 and Eq.2.3 respectively, and Eq.2.4 represents the most general form for $V(h)$ which provides a renormalizable theory.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.2}$$

$$D_\mu = \partial_\mu + igQ_h A_\mu \tag{2.3}$$

$$V(h) = \mu^2 h^\dagger h + \lambda(h^\dagger h)^2 \tag{2.4}$$

where $g$ is the gauge coupling constant of $U(1)$; $Q_h$ is the charge of the scalar field $h$; and $\mu^2$ and $\lambda$ are coupling constants.

The above scalar potential $V(h)$ for $\mu^2 > 0$ and $\lambda > 0$ is plotted in the left panel of Fig.2.1. In this case the minimum potential energy of the scalar field is at the origin of the potential and respects the $U(1)$ gauge symmetry. However, if we instead assume that $\mu^2 < 0$, then the minimum of the potential is no longer at the origin, as illustrated by the right panel of Fig.2.1. The scalar field will oscillate around its minimum potential energy and it is therefore useful to expand around the minimum $h_0$ by redefining $h$ such that $h(x) = h_0 + H(x)$, where the local $U(1)$ gauge symmetry has been used to make $h(x)$ real-valued at every point $x$.[1] The kinetic energy of the scalar field, given by $|D_\mu h|^2$ in Eq.2.1, now contains a mass term for the $U(1)$ gauge field in the new coordinates:

$$|D_\mu h|^2 = (\partial_\mu H)^2 + g^2 Q_h^2 h_0^2 A_\mu A^\mu + \cdots .$$

Therefore, if the scalar field lives near the minimum of its potential with $\mu^2 < 0$, the $U(1)$ gauge symmetry appears to be spontaneously broken, that is, the gauge boson acquires a mass and there is no $U(1)$ symmetry. The non-zero value of the scalar field's potential energy $h_0$ is called the scalar's vacuum expectation value (VEV), and is given by:

$$v = \sqrt{\frac{-\mu^2}{2\lambda}}. \tag{2.5}$$

By interacting with the complex scalar field $h$ over all space, the $U(1)$ gauge field has thus acquired a mass at every point in space.

### 2.1.2 Electroweak Symmetry Breaking

This argument can be extend to the non-Abelian electroweak theory $SU(2)_L \times U(1)_Y$. In this case the complex scalar field, called the Higgs field $h$, transforms in the spinor representation of $SU(2)_L$ and has $Y = \frac{1}{2}$ hypercharge [25]. The covariant derivative of

[1] Eq.2.1 is invariant under a local $U(1)$ transformation: $\phi(x) \to e^{i\alpha(x)}\phi(x)$ and $A_\mu(x) \to A_\mu(x) - \frac{1}{gQ_h}\partial_\mu\alpha(x)$. We can choose $\alpha(x)$ so that $\phi(x)$ is real-valued at every pint $x$. This is called the unitarity gauge.

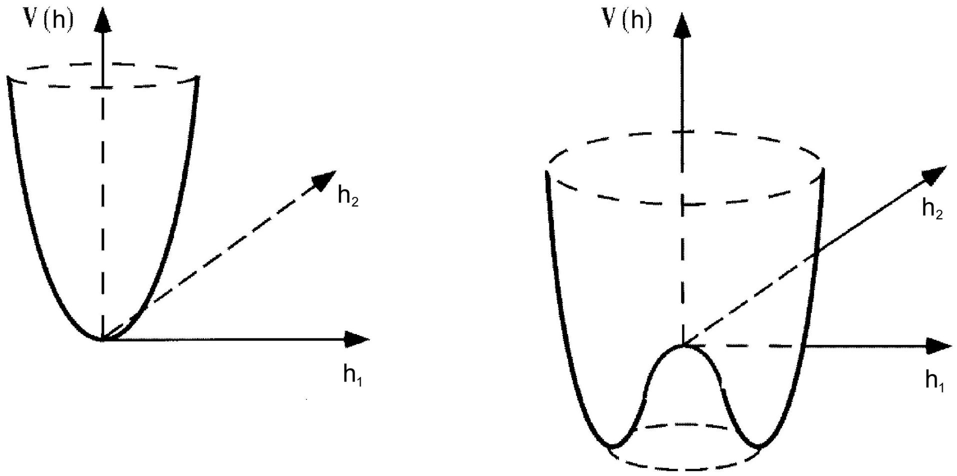



FIGURE 2.1: Shape of the scalar potential $V(h)$ where the left panel is for $\mu^2 > 0$ and the right panel is for $\mu^2 < 0$. $h_1$ and $h_2$ denote the real and imaginary parts of the complex scalar field $h$.

the scalar field is then:

$$D_\mu h = \partial_\mu h + i g_{2L} T_L^a W_{L\mu}^a h + i\frac{1}{2} g_Y B_{Y\mu} h \tag{2.6}$$

where $W_{L\mu}^a$ and $B_{Y\mu}$ are the $SU(2)_L$ and $U(1)_Y$ gauge fields respectively, and the $SU(2)_L$ generators $T_L^a$ are given by $\frac{1}{2}\sigma^a$ where $\sigma^a$ are the Pauli matrices with $a = 1 \ldots 3$.

The Form of the potential $V(h)$ is taken to be the same as in Eq.2.4 and so the scalar field $h$ again obtains a VEV $\langle h \rangle$. We can use the freedom of $SU(2)_L$ rotations to write this VEV in any $SU(2)_L$ component, for example:

$$\langle h \rangle = \begin{pmatrix} 0 \\ \upsilon \end{pmatrix}$$

where $\upsilon = \sqrt{\frac{-\mu^2}{2\lambda}}$ from Eq.2.5.

Expanding around the minimum of the scalar potential, the Kinetic Energy of the Higgs field, given by the mod square of the covariant derivative, then contains the following $SU(2)_L \times U(1)_Y$ gauge field mass terms:

$$\begin{pmatrix} 0 & \upsilon \end{pmatrix} \left( g_{2L} W_{L\mu}^a T_L^a + \frac{1}{2} g_Y B_{Y\mu} \right) \left( g_{2L} W_L^{b\mu\dagger} T_L^{b\dagger} + \frac{1}{2} g_Y B_Y^{\mu\dagger} \right) \begin{pmatrix} 0 \\ \upsilon \end{pmatrix} \tag{2.7}$$

$$= \frac{\upsilon^2}{4} \left[ g_{2L}^2 |W_{L\mu}^1|^2 + g_{2L}^2 |W_{L\mu}^2|^2 + \left| -g_{2L} W_\mu^3 + g_Y B_{Y\mu} \right|^2 \right].$$

The VEV of $h$ therefore generates mass terms for the $SU(2)_L$ fields associated with the Pauli matrices $\tau^1$, $\tau^2$; and mixes the hypercharge field $B_Y$ with the $SU(2)_L$ field associated with $\tau^3$. The mixing of the $A^3_L$ field and $B_Y$ can be written as the matrix product $\frac{1}{4}\upsilon^2\mathbf{B}^T\mathbf{M}\mathbf{B}$ where $\mathbf{B}^T \equiv \begin{pmatrix} B_Y & W^3_L \end{pmatrix}$, and $\mathbf{M}$ is given by:

$$\begin{pmatrix} g_Y^2 & -g_{2L}g_Y \\ -g_{2L}g_Y & g_{2L}^2 \end{pmatrix}.$$

The fields $W^3_L$ and $B_Y$ are the eigenstates of the $SU(2)_L \times U(1)_Y$ interactions but, since they are mixed by the above mass terms, they cannot be the same as the mass eigenstates. These are instead found by diagonalizing the above matrix $\mathbf{M}$. The diagonal matrix $\mathbf{D}$ of $\mathbf{M}$ is defined by $\mathbf{D} \equiv \mathbf{V}^T\mathbf{M}\mathbf{V}$ where $\mathbf{V}$ is the matrix $(\mathbf{v}_1, \mathbf{v}_2)$ of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\mathbf{M}$. The matrix product $\frac{1}{4}\upsilon^2\mathbf{B}^T\mathbf{M}\mathbf{B}$ can therefore be written as $\frac{1}{4}\upsilon^2\mathbf{A}^T\mathbf{D}\mathbf{A}$ where $\mathbf{A} \equiv \mathbf{V}^T\mathbf{B}$ contains the mass eigenstates of the fields and is given by:

$$\begin{pmatrix} A^\gamma \\ Z^0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_Y \\ W^3_{2L} \end{pmatrix}$$

where $\tan\theta = g_Y/g_{2L}$. The eigenstate $A^\gamma$ corresponds to a zero eigenvalue for $\mathbf{M}$ and is therefore a massless field, whereas the $Z^0$ field has acquired a mass $m_Z$ given by:

$$\frac{1}{2}m_Z^2 = \frac{1}{4}\upsilon^2(g_{2L}^2 + g_Y^2). \tag{2.8}$$

Replacing the interaction eigenstates with the above mass eigenstate in the covariant derivative Eq.2.6 then gives:

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\frac{g_{2L}}{\cos\theta}Z_\mu(T^3 - \sin^2\theta Q_{em}) - ieA_\mu Q_{em}$$

where $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu)$ and $T^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$. The coupling constant $e$ and the generator $Q_{em}$ are defined by the following:

$$e = g_{2L}\sin\theta, \tag{2.9}$$

$$Q_{em} = T^3_L + Y. \tag{2.10}$$

The generator $Q_{em}$ leaves the scalar field's vacuum invariant and so it is not affected by the VEV of the scalar field which explains why $A_\mu$ remains massless. The $SU(2)_L \times U(1)_Y$ electroweak symmetry has thus been spontaneously broken to the symmetry of electromagnetism $U(1)_{em}$. The electroweak force is therefore broken in the Standard Model because the vacuum in which all particle interactions takes place is not actually empty but is instead filled with a condensate of particles from the Higgs field. The $W^\pm$, $Z^0$ bosons continuously interact with the Higgs field as they travel through the vacuum, which appears to give them mass. In fact the Higgs field effectively 'slows down' anything that interacts with it, and in the Standard Model all fundamental mass comes from the Higgs field. The way in which the Higgs field gives mass to the quarks and leptons is described in detail in Chapter 5.

Eq.2.10 enables us to determine the hypercharge $Y$ of the various fields in the Standard Model by measuring their electric charge. However we could have just as easily defined hypercharge as:

$$Y = N(Q - T_L^3) \tag{2.11}$$

where $N$ is any real number, as long as we also redefine the gauge coupling constant $g_Y$ as $g_Y/N$ so that the strength of interaction remains the same. In Section 2.4.1 a particular choice of $N$ is introduced that is motivated by higher energy physics.

Eq.2.9 defines the gauge coupling constant of electromagnetism (at the electroweak symmetry breaking scale) in terms of the hypercharge and $SU(2)_L$ gauge coupling constants. This can be re-written as:

$$\frac{1}{\alpha_e} = \frac{1}{\alpha_{2L}} + \frac{1}{\alpha_Y} \tag{2.12}$$

where $\alpha \equiv g^2/4\pi$. This boundary condition applies at the electroweak symmetry breaking scale.

### 2.1.3 The Hierarchy Problem

The previous Section illustrated that if we rewrite the covariant derivative Eq.2.6 in terms of the Higgs field's oscillation around its VEV $h = \langle h \rangle + H$ then mass terms appear for the electroweak gauge fields. Likewise, if we rewrite the whole Lagrangian describing the scalar and the $SU(2)_L \times U(1)_Y$ gauge fields, then we also find a mass term for the scalar field's oscillation $H$ in the scalar potential $V(h)$:

$$\begin{aligned} V(h) &= (\mu^2 + 6\lambda v^2)H^2 + \cdots \\ &\equiv \frac{1}{2}m_H^2 H^2 + \cdots \end{aligned}$$

where $m_H = -2\mu^2 = 4\lambda v^2$ and $\mu^2 < 0$.

The quantum of the field $h(x)$ is called the Higgs boson and has a classical mass $m_H$. Just as with the vector bosons, the mass of this field comes from the product of the VEV of the complex scalar field $h$ and a renormalizable coupling constant. However unlike for the vector bosons the renormalizable coupling constant $\lambda$ is, at the time of writing, undetermined by experiment. This is because the Higgs boson has not yet been observed, although it is hoped to be found at the upcoming Large Hadron Collider in CERN. The present experimental limit on the Higgs boson's mass is set by LEP to be

$m_H > 114.4$ GeV at 95% CL [26].[2] The Tevatron has also given an exclusion region of $160 - 170$ GeV at 95% CL [26]. Theoretical arguments based on the perturbativity of the theory can also be used to place approximate upper and lower bounds upon the Higgs boson's mass [27]. For example, for large Higgs boson masses the coupling $\lambda$ rises with energy and so the theory would eventually become non-perturbative. The requirement that this does not occur below a given energy scale $\Lambda$ defines an upper bound for the Higgs mass. A lower bound is obtained from the study of quantum corrections to the Standard Model and from requiring the effective potential to be positive definite. These theoretical bounds imply that if the Standard Model is to be perturbative up to $M_{GUT} = 10^{16}$ GeV, the Higgs boson mass should be within about 130 and 190 GeV [27].[3]

Since we haven't yet observed the Higgs boson then we cannot say for definite if the Standard Model's explanation of electroweak symmetry breaking is correct. However its successful description of the $W^{\pm}$ and $Z^0$ bosons and the fact that it also provides the quarks and leptons with mass suggests that, if it isn't correct, then the true mechanism of electroweak symmetry breaking must be very similar to that in the Standard Model. There is an awkward element about the Standard Model Higgs mechanism however. This arises when we investigate the quantum corrections to the Higgs boson's mass and find that the square of the Higgs boson's mass $m_H^2$ receives enormous quantum corrections from the virtual effects of every particle that couples to it [4]. This is not a problem so much for the Standard Model itself since the theory is renormalizable, but instead implies a rather disturbing sensitivity of the Higgs potential to new physics in almost any imaginable extension of the Standard Model. This is because quantum corrections to the Higgs boson's mass from new physics would not be eliminated without the physically unjustifiable tuning of counter-terms specifically for that purpose.[4] In fact $m_H^2$ is sensitive to the masses of the heaviest particles that $H$ couples to, so that, if the mass scale of these fields is very large, its effects on the Standard Model do not decouple but instead make it difficult to understand why $m_H^2$ is so small. This problem arises even if there is no direct coupling between the Standard Model Higgs boson and the unknown heavy physics.

This would of course not be a problem if there was no new physics beyond the Standard Model, but this is considered to be very unlikely, particular in light of the expected need for a quantum mechanical description of gravity. We therefore anticipate

[2]For a SUSY theory the limit is $m_H > 92.8$ GeV for the lightest Higgs.

[3]Indirect experimental bounds for the Standard Model Higgs boson mass are obtained from fits to precision measurements of electroweak observables, and to the measured top and $W^{\pm}$ masses. These measurements are sensitive to the logarithm of the Higgs mass, and the latest indirect bounds are: $129^{+70}_{-50}$ GeV [26].

[4]If one introduces a momentum cut-off $\Lambda_{UV}$ rather than using dimensional regularization then the quantum corrections to $m_H^2$ scale as $\Lambda_{UV}^2$.

that, when we include higher energy physics such as quantum gravity, the Higgs mass becomes unstable. Theoretically then we expect that the Higgs mass should be similar to the Planck mass and electroweak symmetry breaking should occur near the Planck scale, which is of course not what we observe experimentally. This is generically called the hierarchy problem of the Standard Model [4].

## 2.2 Supersymmetry

The Higgs field is very important since it sets the scale of everything in the Standard Model, and given that we expect new physics to occur at higher energies, then we must somehow stabilize the Higgs field. Thus, the Standard Model is expected to be embedded in a more fundamental theory which will stabilize the hierarchy between the electroweak scale and the Planck scale in a natural way. The material for this Section is based on that in [13].

The instability of the Higgs mass turns out to be a general property of scalar fields in quantum field theories since, unlike fermions and vector bosons, their mass is not protected from a chiral or gauge symmetry.[5] This suggests that an approach to stabilizing the Higgs mass is to introduce a symmetry for scalar fields. One such symmetry is supersymmetry [28], which transforms a bosonic state into a fermionic state and vice versa: [6]

$$Q^\dagger \text{ or } Q|\text{Boson}>=|\text{Fermion}>, \qquad Q^\dagger \text{ or } Q|\text{Fermion}>=|\text{Boson}>$$

where $Q$ and $Q^\dagger$ are fermionic operators (anti-commuting spinors) since they carry spin angular momentum 1/2. This illustrates that supersymmetry is a spacetime symmetry.

Supersymmetry protects the mass of scalar particles from the virtual effects of heavy particles by cancelling the various contributions to the quantum corrections [28]. For example, at one loop there is a relative minus sign between the fermion and boson contributions to $\Delta m_H^2$ and so, by introducing a boson for every fermion and vice-versa, the contributions to the Higgs mass cancel. This cancellation occurs to all orders of perturbation theory and so the Higgs mass becomes stable.

The single particle states of a supersymmetric theory fall into irreducible representations of the supersymmetry algebra called supermultiplets. Each supermultiplet contains both fermion and boson states, which are commonly known as superpartners

[5] Chiral symmetry requires that the quantum corrections to a fermion's mass are proportional to the mass itself, resulting in much smaller tuning than quantum corrections to scalar masses.

[6] Only the simplest type of supersymmetric algebra, $N = 1$ supersymmetry is considered in this work, where $N$ refers to the number of supersymmetries (the number of distinct copies of $Q, Q^\dagger$).

of each other. Since the generators of supersymmetry commute with the generators of gauge transformations, particles in the same supermultiplet must also be in the same representation of the gauge group.

## 2.3 The Minimal Supersymmetric Standard Model

The Minimal Supersymmetric Standard Model (MSSM) [5] is the result of what is generally considered to be the simplest application of supersymmetry to the Standard Model. In the MSSM every particle of the Standard Model has a supersymmetric partner called a sparticle. For example, the quarks and leptons have scalar partners called squarks and sleptons that together make up chiral supermultiplets, and the Standard Model gauge bosons have fermionic partners, called gauginos, that together form vector supermultiplets.

The Higgs sector of the MSSM however does not just contain the Standard Model Higgs and its fermionic superpartner. Instead it contains two Higgs chiral supermultiplets called the up and down Higgs supermultiplets $h_u$ and $h_d$. Two Higgs fields rather than one are principally required so that the gauge anomalies for the electroweak gauge symmetry cancel. If these didn't cancel then the model would be an inconsistent quantum field theory. The cancellation of gauge anomalies includes the requirement that $\mathrm{Tr}[(T_L^3)^2 Y] = \mathrm{Tr}[Y^3] = 0$, where traces run over all the left-handed Weyl fermionic degrees of freedom in the theory. In the Standard Model, these conditions are already satisfied by the known quarks and leptons, but a fermionic partner of a Higgs field must be a weak isodoublet with weak hypercharge $Y = 1/2$ or $Y = -1/2$. In either case the fermion will make a non-zero contribution to the traces and spoil anomaly cancellation. This can be avoided however if there are two Higgs supermultiplets with opposite hypercharge so that the total contribution to the anomaly traces from the two fermionic members of the Higgs chiral supermultiplets vanishes.

### 2.3.1 The MSSM Superpotential

The superpotential of a supersymmetric model lists all the non-gauge interactions for particles that live in the chiral supermultiplets of the model. The form of the non-gauge couplings, including the mass terms, is highly restricted by the requirement that the action that is invariant under supersymmetry transformations is renormalizable. The superpotential of the MSSM is given below:[7]

$$W_{MSSM} = \lambda_u^{ij} Q_i u_j^c h_u + \lambda_d^{ij} Q_i d_j^c h_d + \lambda_e^{ij} L_i e_j^c h_d + \mu h_u h_d \tag{2.13}$$

where $Q_i$, $L_i$, $e_i^c$, $u_i^c$ and $d_i^c$ are the quark and lepton chiral supermultiplets; $\lambda_u^{ij}, \lambda_d^{ij}, \lambda_e^{ij}, \mu$ are renormalizable parameters; $i, j = 1 \ldots 3$ are flavour indices; and $c$ denotes a charge-conjugate of a left-handed field. $W_{MSSM}$ is the supersymmetric version of the Yukawa interactions of the Standard Model.

Other terms, which are allowed by the gauge symmetry of the MSSM, are not present in the MSSM superpotential because of a discrete $Z_2$ symmetry called R-parity. These terms are $LLh_d$, $QLd^c$, $Lh_u$, which arise because $L$ and $h_d$ are identical under the MSSM gauge group, and $u^c d^c d^c$. These operators would cause phenomenological problems such as rapid proton decay if they aren't forbidden or heavily suppressed.

The first three terms in Eq.2.15 illustrate that two Higgs fields are also required so as to give mass to both the up and down the quarks and charged leptons. If $h_u$ develops a VEV then it will give mass to the up quarks, and if $h_d$ also develops a VEV then it will give mass to the down quarks and charged leptons. Terms such as $Qu^c h_d^*$, $Qd^c h_u^*$ and $Qe^c h_u^*$ are forbidden in the superpotential since it must be analytic in the chiral superfields.

The $\mu h_u h_d$ term in the superpotential, called the $\mu$-term, can be written out as $\mu (h_u)_\alpha (h_d)_\beta \epsilon^{\alpha\beta}$ where $\alpha$, $\beta$ are $SU(2)_L$ indices. Terms such as $h_u^* h_u$ or $h_d^* h_d$ are forbidden in the superpotential since again it must be analytic. The $\mu$-term is therefore the supersymmetric version of the Higgs boson mass in the Standard Model potential Eq.2.4. The full Higgs potential in the MSSM is reviewed in Section 2.3.3.

### 2.3.2 Soft Supersymmetry Breaking

The theory described so far is in strong violation of experimental data since supersymmetry requires that the mass of all superpartners is equal and so we should have observed the various squarks and sleptons in particle accelerators. In the MSSM this problem is avoided by including explicit mass terms for the scalar particles of the chiral supermultiplets and the fermion particles of the vector supermultiplets. These explicit mass terms then break supersymmetry but maintain a hierarchy between the electroweak scale and the Planck (or any other very large) mass scale [29].[8] Excluding the gaugino mass terms,

[7] If we include three right-handed neutrinos $\nu_R$ then there would also be an additional term $\lambda_\nu^{ij} L_i \nu_j^c h_u$.

[8] From a theoretical perspective we expect that supersymmetry should be an exact symmetry that is broken spontaneously. That is, the underlying model should have a Lagrangian density that is invariant under supersymmetry, but a vacuum state that is not, analogous to the electroweak symmetry breaking in the Standard Model.

the supersymmetry-breaking couplings in the MSSM are the following:

$$\begin{aligned}\mathcal{L}_{soft}^{MSSM} = &- (a_u^{ijk}\tilde{Q}\tilde{u^c}h_u - a_d^{ijk}\tilde{Q}\tilde{d^c}h_d - a_e^{ijk}\tilde{Q}\tilde{e^c}h_d + c.c.) \\ &- \tilde{Q}_i^\dagger(m_Q^{ij})^2\tilde{Q}_j - \tilde{L}_i^\dagger(m_L^{ij})^2\tilde{L}_j - \tilde{u^c}_i^\dagger(m_u^{ij})^2\tilde{u^c}_j \\ &- \tilde{d^c}_i^\dagger(m_u^{ij})^2\tilde{d^c}_j - \tilde{e^c}_i^\dagger(m_e^{ij})^2\tilde{e^c}_j \\ &- m_{h_u}^2 h_u^* h_u - m_{h_d}^2 h_d^* h_d - (bh_uh_d + c.c.)\end{aligned} \tag{2.14}$$

where a tilde denotes the scalar component of the chiral superfield.

It has been shown rigorously that a softly broken supersymmetric theory with $\mathcal{L}_{soft}$ as given by Eq.2.14 is free of quadratic divergences for quantum corrections to scalar masses to all orders in perturbation theory [29].

The soft masses in the above equation allow for the Standard Model superpartners (except for the Higgs' superpartners, called the higgsinos) to have a mass which, if large enough, would prevent them from being observable in previous experiments. However, these masses cannot be too large since the Higgs mass is sensitive to the mass difference between the superpartners of a supermultiplet. The fact that we haven't yet observed the superpartners of the Standard Model or Higgs boson introduces a little hierarchy problem to the MSSM [30].

### 2.3.3 The Higgs Potential

The scalar potential $V(\phi, \phi^\dagger)$ of a supersymmetric theory is divided into '$F$-term' and '$D$-term' contributions:

$$V(\phi, \phi^\dagger) = F^{*i}F_i + \frac{1}{2}\sum_a D^a D^a$$

where the sum is over the gauge interactions of the theory; $F$ are complex auxiliary fields; and $D$ are gauge auxiliary fields. The auxiliary fields are just book-keeping devices that are introduced to the supersymmetry algebra to make it consistent off-shell. They therefore do not have a kinetic term and can be eliminated on-shell using their algebraic equation of motion. The $F$-terms are fixed by Yukawa couplings and fermion mass terms, and the $D$-terms are fixed by the gauge interactions.

Ignoring the soft SUSY breaking terms, the Higgs potential of the MSSM would be the following:

$$\begin{aligned} V = &|\mu|^2(|h_u^0|^2 + |h_u^+|^2 + |h_d^0|^2 + |h_d^-|^2) \\ &+ \frac{1}{8}(g_{2L}^2 + g_Y^2)(|h_u^0|^2 + |h_u^+|^2 - |h_d^0|^2 - |h_d^-|^2)^2 \\ &+ \frac{1}{2}g_{2L}^2|h_u^+h_d^{0*} + h_u^0h_d^{-*}|^2. \end{aligned}$$

The terms proportional to $|\mu|^2$ come from the $F$-terms, and the terms proportional to $g^2$ and $g_Y^2$ are the D-term contributions. Since $|\mu|^2 > 0$ this potential takes the form of that in Fig.2.1 for each Higgs field. The minimum of the potential would therefore occur at the origin with $|h_u^0| = |h_d^0| = 0$ and there would be no electroweak symmetry breaking. However, the full Higgs potential of the MSSM also includes the soft SUSY breaking terms for the Higgs fields and is given by:

$$\begin{aligned} V = &(|\mu|^2 + m_{h_u}^2)(|h_u^0|^2 + |h_u^+|^2) + (|\mu|^2 + m_{h_d}^2)(|h_d^0|^2 + |h_d^-|^2) \\ &+ [b(h_u^+h_d^- - h_u^0h_d^0) + c.c.] + \frac{1}{2}g_{2L}^2|h_u^+h_d^{0*} + h_u^0h_d^{-*}|^2 \\ &+ \frac{1}{8}(g_{2L}^2 + g_Y^2)(|h_u^0|^2 + |h_u^+|^2 - |h_d^0|^2 - |h_d^-|^2)^2. \end{aligned} \tag{2.15}$$

With the above soft SUSY terms introduced, the Higgs potential can now have a minimum at which $|h_u^0| = |h_d^0| \neq 0$ and the electroweak symmetry is spontaneously broken. This is effectively because the mass terms for the up (and down) Higgs fields can now be negative since $m_{h_u}^2$ and $m_{h_d}^2$, unlike $|\mu|^2$ can be negative parameters. The form of the potential then becomes a generalization of that in Fig.2.1 which represents the Higgs potential of the Standard Model. Thus the soft SUSY breaking terms are not just required to explain the absence of Standard Model superpartners at previous experiments, but also to break the electroweak symmetry in an analogous way to the Standard Model.

Assuming that the Higgs field obtains a vacuum expectation value and using the freedom of $SU(2)_L \times U(1)_Y$ gauge transformations we can simplify Eq.2.15 to:

$$\begin{aligned} V(h_u^0, h_d^0) = &(|\mu|^2 + m_{h_u}^2)|h_u^0|^2 + (|\mu|^2 + m_{h_u}^2)|h_d^0|^2 - (bh_u^0h_d^0 + c.c) \\ &+ \frac{1}{8}(g_{2L}^2 + g_Y^2)(|h_u^0|^2 - |h_d^0|^2)^2 \end{aligned}$$

where $h_u^0$ and $h_d^0$ are real and positive. CP cannot be spontaneously broken by the Higgs scalar potential, since the VEVs and $b$ can be simultaneously chosen real, as a convention.

For $V$ to really have a minimum the potential must be bounded from below for arbitrarily large values of the scalar fields. In general the scalar quartic interactions in

$V$ will stabilize the potential for almost all arbitrarily large values of $h_u^0$ and $h_d^0$ but, for the special directions in field space $|h_u^0| = |h_d^0|$, the quartic contributions to $V$ are identically zero. Such directions in field space are called D-flat directions, because along them the part of the scalar potential coming from D-terms vanishes. In order for the potential to be bounded from below, the quadratic part of the scalar potential must be positive along the D-flat directions. This requires:

$$2b < 2|\mu|^2 + m_{h_u}^2 + m_{h_d}^2. \tag{2.16}$$

Then, for $V$ to have a stable minimum (or for $h_u^0 = h_d^0 = 0$ to be an unstable minimum) we require that one linear combination of $h_u^0$ and $h_d^0$ has a negative squared mass near $h_u^0 = h_d^0 = 0$. This results in:

$$b^2 > (|\mu|^2 + m_{h_u}^2)(|\mu|^2 + m_{h_d}^2). \tag{2.17}$$

The above inequalities are the necessary conditions for $h_u^0$ and $h_d^0$ to get non-zero VEVs and we can now require that they are compatible with the observed phenomenology of electroweak symmetry breaking. That is, the Higgs' VEVs must satisfy the MSSM version of the of the Standard Model condition given by Eq.2.8:

$$\upsilon^2 \equiv \upsilon_u^2 + \upsilon_d^2 = 2m_Z^2/(g_{2L}^2 + g_Y^2) \approx (174 \text{ GeV})^2 \tag{2.18}$$

where $\upsilon_u \equiv \langle h_u^0 \rangle$ and $\upsilon_d \equiv \langle h_d^0 \rangle$. The ratio of the up and down Higgs VEVs is conventionally denoted by $\tan\beta \equiv \upsilon_u/\upsilon_d$ and is an unknown parameter.

Thus, as long as certain conditions are met, the Higgs potential of the MSSM can break the electroweak symmetry analogous to how it is broken in the Standard Model. This is achieved without the quantum corrections from higher energy physics upsetting the results, that is, the Higgs mass is stable in this theory.

## 2.4 Supersymmetric Grand Unified Theories

In the previous Section the instability of the Standard Model Higgs field to the addition of higher energy physics led us to consider the Standard Model to be an effective low-energy approximation to the MSSM. In this Section we will find that certain aspects of the MSSM then naturally lead us to consider it to be a low-energy approximation to a theory that is, on a logarithmic scale, close to the Planck scale. This new theory solves a number of mysteries about the Standard Model and MSSM such as the quantization of electric charge and gauge anomaly cancellation.

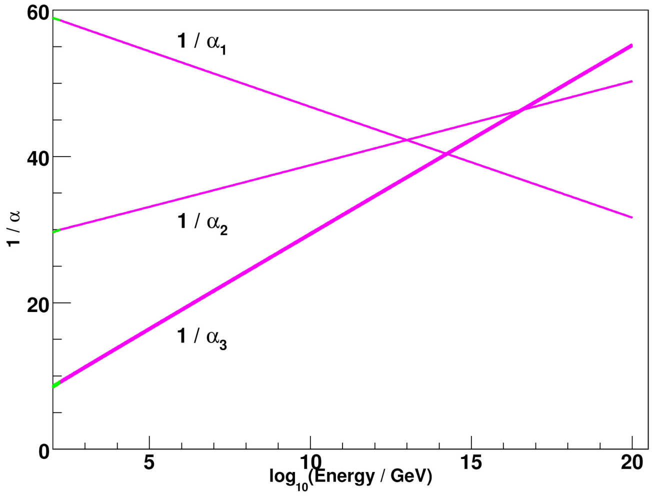



FIGURE 2.2: Two-loop running of the gauge coupling constants in the Standard Model. $\alpha_1 \equiv g_1/4\pi$ is the GUT normalized $U(1)_Y$ gauge coupling constant, $\alpha_2 \equiv g_{2L}/4\pi$ is the $SU(2)_L$ gauge coupling constant, and $\alpha_3 \equiv g_{3c}/4\pi$ is the gauge coupling constant of $SU(3)_c$. The green lines describe the running of the gauge coupling constants in the Standard Model between the mass of the $Z^0$ boson and top quark. The thickness of the lines indicates the experimental uncertainty in the initial values of the gauge coupling constants.

### 2.4.1 Gauge Coupling Unification in the MSSM

If the Standard Model is considered to be an effective approximation to a higher energy theory then the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge coupling constants can usefully be thought of as energy-dependent entities. Using the Standard Model renormalization group equations one can calculate how the gauge coupling constants run with energy to a given order in perturbation theory, and if we run gauge couplings to higher energies then, depending on the normalization chosen for the definition of hypercharge in Eq.2.11, they can meet at a very high-energy scale.[9] The unification of gauge coupling constants would unlikely be a coincidence and would instead imply that something new occurs at the unification scale. A strong possibility is that a theory based on a semi-simple gauge group such as $SU(5)$ spontaneously breaks to the Standard Model gauge group at the unification scale, analogous to how $SU(2)_L \times U(1)_Y$ breaks to $U(1)_{em}$ [18]. Such a theory is called a Grand Unified Theory (GUT) and would of course have just a single gauge coupling constant. However, if $SU(3)_c \times SU(2)_L \times U(1)_Y$ comes from a semi-simple gauge group then the normalization of hypercharge is automatically fixed [9]. This is because $Y$ like $T^a_L$ and $T^a_{3c}$ must come from the generators of the semi-simple group.

For any simple compact Lie group, there is a conventional choice of generators $T_a$ with totally antisymmetric structure constants, which in each reducible or irreducible

[9] If the hypercharge normalization $N$ in Eq.2.11 is taken to be $\sqrt{\frac{13}{10}}$ then the gauge couplings unify at $\approx 10^{17}$ GeV to one-loop, and $4 \times 10^{16}$ GeV to two-loops [31, 32].

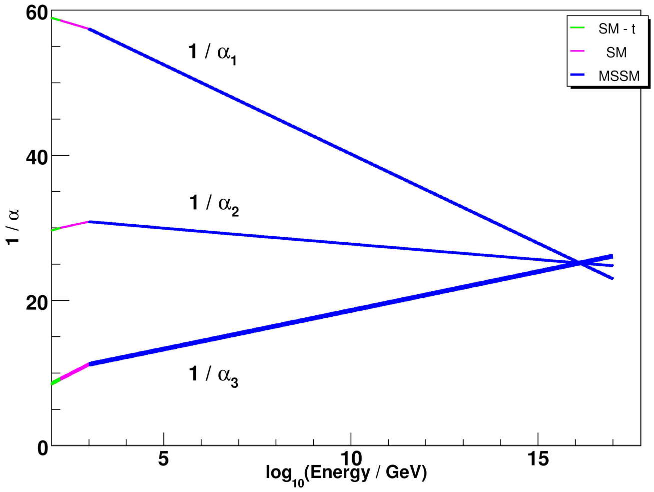



FIGURE 2.3: Running of the $SU(3)_c$, $SU(2)_L$ and GUT normalized $U(1)_Y$ gauge coupling constants for the MSSM using two-loop renormalization group equations. The pink lines are the running of the gauge coupling constants of the Standard Model, below the assumed scale of supersymmetry.

representation $D$ satisfy the following normalization condition:

$$Tr[T_aT_b] = N_D\delta_{ab}.$$

If the Standard Model gauge symmetries come from a single gauge group then we must therefore have $Tr(T_c)^2 = Tr(T_{2L})^2 = Tr(Y)^2$ where $T_c$, $T_{2L}$ and $Y$ are the generators of $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ respectively, and the trace is over all the fermions. These are given by $Tr(T_c)^2 = 6g_3^2$, $Tr(T_{2L})^2 = 6g_{2L}^2$ and $Tr(Y^2) = 10g_Y^2$ which sets $g_3^2 = g_{2L} = (5/3)g_Y^2$ at the GUT scale. Thus the normalization constant in Eq.2.11 is given by $N = \sqrt{\frac{3}{5}}$ so that in this case hypercharge is defined as:

$$Y = \sqrt{\frac{3}{5}}(Q_{em} - T_3). \tag{2.19}$$

If we run the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge coupling constants in the Standard Model with this hypercharge normalization to higher energies, assuming that there are no new particles, then they come close to unifying at a high energy scale, but just miss each other. This is illustrated by Fig.2.2 to two-loops in perturbation theory.

If the MSSM is used instead of the Standard Model however then the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge couplings almost exactly unify at an energy scale of $\approx 3 \times 10^{16}$ GeV, which is illustrated to two-loops by Fig.2.3 [7]. This is under the assumption that there is nothing between the SUSY scale, around 1 TeV, and the so-called GUT scale $\approx 3 \times 10^{16}$ GeV. The huge energy region between these two scales is generically called the Grand Desert.[10]

This suggests that the MSSM is a low-energy approximation to a supersymmetric theory with a semi-simple gauge group that spontaneously breaks at $\approx 3 \times 10^{16}$ GeV. Such a theory is called a SUSY GUT and the next Section provides a brief review for the $SU(5)$ and $SO(10)$ SUSY GUTs.

### 2.4.2 $SU(5)$ and $SO(10)$ SUSY GUTs

The smallest simple Lie group that contains $SU(3) \times SU(2) \times U(1)$ as a subgroup is $SU(5)$. In an $SU(5)$ GUT the Standard Model gauge bosons (and gauginos) are then unified in the adjoint representation, which has dimension 24 [8]. If $SU(5)$ is spontaneously broken to the Standard Model gauge group at $3 \times 10^{16}$ GeV, then the gauge bosons that are not part of the Standard Model would get mass at this high-energy scale, in an analogous way to how $W^{\pm}$ and $Z^0$ get mass from electroweak symmetry breaking.

Just as the MSSM gauge supermultiplets are unified in an $SU(5)$ SUSY GUT, so too are the quark and lepton supermultiplets, although this is only a partial unification. In $SU(5)$ the quarks and leptons fit neatly into the representations $10 + \overline{5}$. The 10 representation contains one generation of the left-handed up and down quarks ($Q$), charged conjugated up quarks ($u^c$) and leptons ($e^c$); whereas the $\overline{5}$ contains one generation of the left-handed leptons ($L$) and charged-conjugated down quarks ($d^c$). In total then three copies of $10 + \overline{5}$ are required to replicate the MSSM matter content. The up and down Higgs doublets must also come from $SU(5)$ representations and the smallest ones available are 5 for $h_u$ and $\overline{5}$ for $h_d$.

Another promising SUSY GUT is that based on the $SO(10)$ gauge group.[11] Unlike in $SU(5)$ SUSY GUTs, one generation of quarks and leptons are unified in a single representation. This is the fundamental spinor representation 16 and three copies of this representation are therefore required. As well as one generations of quarks and leptons, the 16 representation also contains a Standard Model singlet which can be identified as a right-handed neutrino. This particle can be used to explain the resent discovery of neutrino oscillations which is discussed in more detail in Chapter 5.

[10]If complete GUT representations are at a particular scale which lies between these two scales then the unification of gauge coupling constants will still occur at around $10^{16}$ GeV provided that the coupling constants remain in the perturbative regime.

[11]The Lie group involved is not really the special orthogonal group $SO(10)$, but rather its double cover $Spin(10)$.

The MSSM Higgs doublets are also unified into the same $SO(10)$ representation, which is the fundamental representation 10. The MSSM superpotential is then contained in the simple $SO(10)$ tensor product $16 \times 16 \times 10$ for the three copies of the 16 representation. This automatically forbids the R-parity violating operators of the MSSM further illustrating that the MSSM appears to fit neatly inside an $SO(10)$ SUSY GUT.

Another virtue of the $SO(10)$ GUT is its explanation for gauge anomaly cancellation in the Standard Model (and MSSM). This is discussed in the next subsection.

### 2.4.3 Anomaly Cancellation

In theoretical physics a gauge anomaly is a quantum mechanical effect (usually a one-loop diagram) which invalidates the gauge symmetry of a quantum field theory. Therefore all gauge anomalies must cancel out, and this is indeed what happens in the Standard Model. The anomaly in vector gauge anomalies (in gauge symmetries whose gauge boson is a vector) is a chiral anomaly and can be calculated exactly at one-loop level using a Feynman diagram with a chiral fermion running in the loop with $N$ external gauge bosons attached to the loop where $N = 1 + d/2$ and $d$ is the spacetime dimension.[12] The anomaly is proportional to the completely symmetric constant factor $d_{abc}$:

$$d_{abc} \equiv \frac{1}{2} Tr \Big[ \{T_a, T_b\} T_c \Big] \tag{2.20}$$

where $T_a$ is the representation of the gauge algebra on the set of all left-handed fermion and anti-fermion fields, and $Tr$ denotes a sum over these fermion and antifermion species. This condition may be satisfied for any gauge group if the fermion fields furnish a suitable reducible or irreducible representation of the group. In addition, there are some gauge groups for which the above is satisfied for fermions in *any* representation of the group.

The condition is obviously satisfied if the left-handed fermion (and anti-fermion) fields furnish a representation $T_a$ of the gauge algebra that is equivalent to its complex conjugate such that:

$$(iT_a)^* = S(iT_a)S^{-1}$$

or equivalently:

$$T_a^T = -ST_aS^{-1}. \tag{2.21}$$

Inserting this into Eq.2.20 gives $d_{abc} = -d_{abc}$. Such a representation $T_a$ may be either real or pseudoreal, and there is therefore no anomaly for gauge algebras that have

[12]Anomalies occur only in even spacetime dimensions, and since $d = 4$ in the Standard Model, the diagram involved is a triangle diagram with axial and vector currents at one of its vertices.

*only* real or pseudoreal representations, namely $SO(2n+1)$ (including $SU(2) \equiv SO(3)$, $SO(4n)$ for $n \geq 2$, $G_2$, $F_4$, $E_7$ and $E_8$) [9]. A few other algebras also have only representations for which $d_{abc}$ vanishes, even though some representations are neither real nor pseudoreal [33]. These are $SO(4n+2)$ (except for $SO(2) = U(1)$ and $SO(6) = SU(4)$) and $E_6$. Anomalies are thus only possible for gauge algebras that include $SU(n)$ (for $n \geq 3$) or $U(1)$ factors.

Given that the Standard Model is based on the gauge group $SU(3) \times SU(2) \times U(1)$ then the we must rely on the gauge anomalies due to the various quarks and leptons cancelling to make the theory free of anomalies. Fortunately this is exactly what happens. However, from the point of view of the Standard Model, this cancellation amongst the quarks and leptons is mysterious since the apparently arbitrary quantum numbers of the quarks and leptons are just right for the Standard Model to be free of anomalies.

From the point of view of Grand Unified Theories however the cancellation of gauge anomalies in the Standard Model can be neatly understood by noting that $SU(3) \times SU(2) \times U(1)$ may be embedded in $SO(10)$ [34]. All of the representations of $SO(10)$ are anomaly-free, so the same property is inherited by any reducible representation of $SU(3) \times SU(2) \times U(1)$ that furnishes a complete representation of $SO(10)$. As noted in Section 2.4.2, it turns out that the left-handed fields of a single generation of quarks, leptons, antiquarks and antileptons plus one additional ($SU(3) \times SU(2) \times U(1)$)-singlet (a right-handed neutrino) forms a complete 16-dimensional representation of $SO(10)$ (the fundamental spinor representation). The singlet would not contribute to such anomalies, and so there are no anomalies in the gauge symmetries of the Standard Model.

The cancellation of gauge anomalies in the MSSM from the point of view of SUSY GUTs is perhaps less obvious since the left-handed fermions of the MSSM do not come from complete $SO(10)$ representations. This is because the MSSM Higgsinos come from a fundamental 10 representation of $SO(10)$ but, as discussed further in Section 2.4.5.2, their triplet higgsino partners are missing in the MSSM. However, since the triplet higgsinos transform as a $(3,1)_{-\frac{1}{3}}$ and $(\overline{3},1)_{\frac{1}{3}}$ under $SU(3)_c \times SU(2)_L \times U(1)_Y$, they form conjugate representations under the Standard Model gauge group and so their gauge anomalies cancel.

### 2.4.4 Radiative Symmetry Breaking

In Section 2.1.1 we found that the parameter $\mu^2$ must be negative for the Standard Model Higgs field to obtain a VEV in order to break the electroweak symmetry. Similarly in the MSSM the conditions for electroweak symmetry breaking Eq.2.16-Eq.2.17 are helped by $m^2_{h_u}$ (and $m^2_{h_d}$) being negative. However, although there is nothing stopping us choosing these parameters to be negative, it seems a little unnatural, especially when every other parameter in the scalar potential is anticipated to be positive.[13] A solution to this naturalness problem is obtained by using the fact that, just as the gauge coupling constants can run with energy, so also can these Standard Model and MSSM parameters. Assuming the grand desert between the MSSM and GUT scales, it has been shown that $m^2_{h_u}$ can run negative at a low energy scale such as the electroweak energy scales if it starts from a positive value at the GUT scale [35]. This occurs in particular for $m^2_{h_u}$ because the top Yukawa coupling is expected to be $\mathcal{O}(1)$, which reduces the effective value of $m^2_{h_u}$ as the energy scale of interest decreases. In models with $\tan\beta \gg 1$ however, $m^2_{h_d}$ can also run negative since the bottom Yukawa constant is also large in these models.

Generating a negative value for $m^2_{h_u}$ (and $m^2_{h_d}$) in this way is called radiative electroweak symmetry breaking and helps to explain why the electroweak scale is so much smaller than the GUT or Planck scales as it takes a large energy region for $m^2_{h_u}$ (and $m^2_{h_d}$) to run negative from a positive value at the GUT scale (assuming an MSSM spectrum and a Grand Desert).

Radiative electroweak symmetry breaking is particularly well motivated by supergravity theories [35]. These are quantum field theories in which supersymmetry is considered to be a local rather than a global symmetry and offer a candidate for the unification of the Standard Model forces with gravity. In simple supergravity models all the soft SUSY breaking parameters are equal and positive at the GUT scale but run differently with energy to the electroweak scale. Together with the size of the top and bottom Yukawa couplings this then explains why only $m^2_{h_u}$ (and $m^2_{h_d}$) run negative and not other soft MSSM parameters such as the square mass for the selectron $m^2_{\tilde{e}}$.

### 2.4.5 Proton Decay and Doublet-Triplet Splitting

#### 2.4.5.1 Gauge Mediated Proton Decay

Since the quarks and leptons are unified in representations of a GUT's gauge group $G$, interactions with the gauge bosons of $G$ will introduce processes involving violations of

[13] For example we wouldn't want $m^2_{\tilde{t}} < 0$ otherwise it might induce a VEV for the stop and thus break the strong nuclear force.

baryon and lepton number. This can then lead to proton decay, which has not been observed experimentally. The interactions that lead to proton decay are $d = 6$ operators that conserve $B - L$ so that the proton always decays into an antilepton.

The gauge bosons that mediate proton decay are the ones that are not present in the Standard Model and are expected to get GUT scale masses once the symmetry group $G$ is spontaneously broken. The proton decay interactions will therefore be suppressed by the GUT scale masses of the gauge bosons and we can a naive model-independent estimation for the mass of the superheavy gauge bosons using the experimental lower bound on the proton lifetime [36]. Using:

$$\Gamma_p \approx \alpha_{GUT}^2 \frac{m_p^5}{M_V^4}$$

and $\tau(p \to \pi^0 e^+) > 1.6 \times 10^{33}\ yrs$ then a naive lower bound on the superheavy gauge boson masses is $M_V > (2.57 - 3.23) \times 10^{15}$ GeV for $\alpha_{GUT} = 1/40 - 1/25$. This is just below the GUT scale $3 \times 10^{16}$ GeV and therefore general SUSY GUTs are within present experimental limits for proton decay mediated by the gauge bosons.

#### 2.4.5.2 Higgs Triplets

Just as the quarks, leptons and gauge bosons come from GUT representations, so must the MSSM Higgs fields $h_u$ and $h_d$. For example, in SUSY $SU(5)$ $h_u$ fits into a 5 representation, called $5_u$, whereas $h_d$ comes from a $\overline{5}$ representation called $\overline{5}_d$. The MSSM superpotential then comes from the $SU(5)$ superpotential:

$$\lambda_u^{ij} 10_i 10_j 5_u + \lambda_d^{ij} 10_i \overline{5}_j \overline{5}_d + \mu 5_u \overline{5}_d \tag{2.22}$$

where $i, j$ label the three generations, $\lambda_u^{ij}, \lambda_d^{ij}$ are coupling constants, and $\mu$ is an $SU(5)$ generalization of the MSSM $\mu$-parameter. Since these $SU(5)$ Higgs representations are of dimension five they must contain particles other than each MSSM Higgs field. These particles are coloured states called Higgs triplets, which are denoted by $D$ and $\overline{D}$, and transform as $(3,1)_{-\frac{1}{3}}$ and $(\overline{3},1)_{\frac{1}{3}}$ respectively under the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group. This is not just peculiar to the $SU(5)$, all GUTs contain coloured partners to Higgs doublets. This is due to the unification of $SU(3)_c$ with $SU(2)_L \times U(1)_Y$ in GUTs.

Since the MSSM Higgs fields and Higgs triplet fields come from the same GUT multiplet they would be expected to have the same or very similar mass, which should be near to the electroweak scale. However, if we include the Higgs triplet supermultiplets $D$ and $\overline{D}$ at low energies then the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge coupling constants no longer unify. This is a failing of simple GUTs since they don't predict gauge coupling unification, which is the very thing that is required for their existence. Even worse, because of their interactions with the quarks and leptons due to the $SU(5)$ superpotential Eq.2.22, electroweak scale Higgs triplets would cause rapid proton decay in great violation with what we observe in nature [37].

In traditional GUTs this problem is solved by splitting the mass of the MSSM Higgs doublets and their coloured GUT partners. If the Higgs triplets have a GUT scale mass then they obviously won't interfere with the running of the MSSM gauge coupling constants and so won't upset unification. Generally this also pushes the induced proton decay rate just beyond the experimental limits [36]. This however leaves the question of how Higgs triplets have GUT scale mass but MSSM Higgs doublets have electroweak scale mass. This is called the doublet-triplet splitting problem.

In an $SO(10)$ theory, there is potential solution to the doublet-triplet splitting problem known as the 'Dimopoulos-Wilczek' mechanism [38]. In $SO(10)$ the $SU(5)$ representations $5_u$ and $5_d$ are contained in a single fundamental 10 representation. The doublet-triplet splitting can be achieved by coupling this vector to an adjoint Higgs representation $45_H$. The VEV of the $45_H$, when written in the fundamental representation, can take the form $\langle 45_H \rangle \propto diag(a1, a2, a3, a4, a5) \otimes i\tau_2$, where there is no requirement that the trace $\Sigma_i a_i$ vanishes. Thus one can have $\langle 45_H \rangle \propto diag(0, 0, 0, 1, 1) \otimes i\tau_2$ which is just proportional to the $SO(10)$ generator $B - L$. Such a VEV will give mass to the triplets in 10 while leaving the doublets massless. This is not possible in $SU(5)$ since the adjoint field can only have a VEV that is traceless.

However, to arrange for the VEV to align along this direction (and still not mess up the other details of the model) often requires very contrived models. Also, because the adjoint of $SO(10)$ in the fundamental representation is a $10 \times 10$ antisymmetric matrix, two distinct 10 representations must appear in the coupling $10_1 45_H 10_2$. Thus four Higgs doublets, not two, are left massless which would destroy the unification of gauge couplings. The mechanism must then be complicated by the existence of an explicit mass term $M 10_2 10_2$ where $M \gtrsim M_{GUT}$.

Other methods to solving the problems introduced by Higgs triplet fields are motivated by higher energy theories such as string theory. For example, the compactification of extra dimensions via Wilson-line symmetry breaking or orbifolding can be used to split the Higgs triplets from the Higgs doublets.

## 2.5 The $\mu$-problem of the MSSM

Although the MSSM solves the instability of the Higgs mass to higher energy physics, it does not explain why the Higgs mass is so small in the first place (why electroweak symmetry breaking occurs at energies much smaller than the Planck scale). The prediction of gauge coupling unification at $3 \times 10^{16}$ GeV for the MSSM led us to consider the MSSM to be an effective low-energy approximation to a SUSY GUT. Radiative electroweak symmetry breaking in a SUSY GUT can then shed light on why electroweak symmetry breaking occurs at a scale much smaller than the GUT scale, since RGEs can cause $m_{h_u}^2$ to run negative well below the GUT scale. However, this condition is not all that is required for the Higgs field to break the electroweak symmetry. From Section 2.3.3 we found that the Higgs potential also depends on the SUSY respecting parameter $\mu$. Therefore to fully understand why the electroweak symmetry breaking scale is much lower than the GUT or Planck scale we need to understand the origin of this parameter.[14]

We can write the necessary conditions Eq.2.16 and Eq.2.17 for the Higgs potential to have a minimum in terms of $m_Z^2$ and $\tan\beta$ using the phenomenological condition Eq.2.18. These two conditions can then be solved to obtain the following [13]:

$$m_Z^2 = \frac{|m_{h_d}^2 - m_{h_u}^2|}{\sqrt{1-\sin^2(2\beta)}} - m_{h_u}^2 - m_{h_d}^2 - 2|\mu|^2 \tag{2.23}$$

where $\sin(2\beta)$ is given by:

$$\sin(2\beta) = \frac{2b}{m_{h_u}^2 + m_{h_d}^2 + 2|\mu|^2}.$$

Eq.2.23 highlights a slight peculiarity of the MSSM. Without miraculous cancellations, all of the Lagrangian parameters $m_{h_u}^2, m_{h_u}^2, b$ and $|\mu|^2$ ought to be within an order of magnitude or two of $m_{Z^0}^2$. However, in the MSSM, $\mu$ is a supersymmetry-respecting parameter that appears in the superpotential, while $b$, $m_{h_u}^2$ and $m_{h_d}^2$ are supersymmetry-breaking parameters that appear in the soft SUSY potential. Thus there is no a priori reason for the $\mu$ parameter to have a numerical value close to $m_{h_u}^2, m_{h_d}^2$ or $b$ since they are conceptually distinct. Furthermore, given that $\mu$ is a dimensional parameter (the only dimensional parameter) that is supersymmetry-respecting, we might expect it take a value close to the cut-off scale of the MSSM, which is anticipated to be the GUT or Planck scale at $\approx 10^{16}$ GeV or $\approx 10^{19}$ GeV respectively. The fact that the $\mu$-parameter

[14] The origin of the $\mu$-parameter is also related to the doublet-triplet splitting problem since, in grand unified theories, this term is upgraded to a term that also gives mass to the Higgs triplets.

appears to be related to the soft SUSY breaking scale, and not to the Planck or GUT scales, is called the $\mu$-problem of the MSSM.

The rest of this Section reviews non-minimal supersymmetric extensions of the Standard Models. that resolve the $\mu$-problem of the MSSM.

### 2.5.1 The Next-to-Minimal Supersymmetric Standard Model

An elegant solution to the $\mu$-problem is to extend the Higgs sector of the MSSM by introducing a Standard Model singlet field $S$ that couples to the Higgs doublets to generate the term $Sh_uh_d$ in the superpotential. An effective MSSM $\mu$-term would then be generated if $S$ gains a VEV. We will see in the next Section that the VEV of this singlet field can be related to the soft SUSY mass scale and thus an effective MSSM $\mu$-term can be related to the soft SUSY mass scale. With the bare $\mu$-term forbidden, the $\mu$-problem of the MSSM would then be resolved.

However, by forbidding the $\mu$-term of the MSSM and introducing the trilinear term $Sh_uh_d$, one creates a global $U(1)$ symmetry called a Pecci-Quinn symmetry for the superpotential under which the singlet field is charged [39]. A Goldstone boson, which has not been observed in experiments, would therefore be created by the VEV of $S$ [40]. In the Next-to-Minimal Supersymmetric Standard Model (NMSSM) [41] the unwanted Goldstone boson is avoided by explicitly breaking the global $U(1)$ symmetry with a $S^3$ term in the superpotential. However, such an approach is accompanied by additional problems. For example, the $S^3$ term introduces a $Z_3$ discrete symmetry associated with the NMSSM superpotential which should lead to the formation of domain walls in the early universe between regions which were causally disconnected during the period of electroweak symmetry breaking [42]. Such domain structure of vacuum create unacceptably large anisotropies in the cosmic microwave background radiation [43]. In an attempt to break the $Z_3$ symmetry, operators suppressed by powers of the Planck scale could be introduced. But it has been shown that these operators give rise to quadratically divergent tadpole contributions, which destabilise the mass hierarchy once again [44].

### 2.5.2 The USSM

An alternative way to avoid the Goldstone boson is to gauge the global $U(1)$ symmetry [15]. This can be achieved by assuming a local $U(1)$ symmetry, denoted by $U(1)'$, in addition to the Standard model gauge symmetry $SU(3)_c\times SU(2)_L\times U(1)_Y$, for which the field $S$ has a non-zero charge. Supersymmetric models that contain a $U(1)'$ symmetry and a charged Standard Model singlet $S$ field are generically called USSMs. If $S$ gains a VEV in these models, then the $U(1)'$ gauge group eats the Goldstone boson, resulting in an observable massive $Z'$. So far a $Z'$ vector boson has not been detected in particle experiments, which puts a lower limit on its mass of $500 - 600$ GeV [45].

The $U(1)'$ gauge group forbids the MSSM $\mu$-term $\mu h_u h_d$ in the USSM superpotential and replaces it with $\lambda_S S h_u h_d$ where $\lambda_S$ is a dimensionless coupling constant. The $S^3$ of the NMSSM is also forbidden by the $U(1)'$ symmetry. The soft SUSY breaking sector of the USSM contains a soft mass term for the $S$ field $m_S^2|S|^2$ and the b-term of the MSSM is replaced with $a_S b S h_u^0 h_d^0$ where $a_S$ is a dimensional parameter. The singlet field's pure scalar potential is therefore given by [46]:

$$V(S) = m_S^2|S|^2 + \frac{g'^2}{2}(Q_S|S|^2)^2$$

where $g'$ is the gauge coupling constant of the $U(1)'$ gauge field, and $Q_S$ is the $U(1)'$ charge of the singlet field. The quartic terms are from D-term contributions which stabilize the potential and are for obvious reasons not present in the NMSSM.[15] If $m_S^2 < 0$ then minimum of the potential is at:

$$|S|^2 = -\frac{m_S^2}{g'^2 Q_S^2}. \tag{2.24}$$

The VEV of the singlet field $S$ is therefore determined by minimizing a potential that depends on a soft SUSY breaking parameter and so the value of the effective parameter $\mu$ is no longer conceptually distinct from the mechanism of SUSY breaking and should take a value close to the soft SUSY mass scale. The $\mu$-problem of the MSSM is thus resolved in the USSM. Also, there is also no longer an unknown dimensional parameter in the superpotential which would be expected to take a value close to the GUT or Planck scales.

## 2.6 The $E_6$SSM

### 2.6.1 Motivation

As described in Section 2.4.3 the gauge symmetries of a quantum field theory must be anomaly free for the theory to be consistent. We must therefore make sure that, when we add a $U(1)'$ group to the Standard Model, the gauge symmetry does not contain any gauge anomalies. The importance of gauge anomalies in determining models has already been encountered in Section 2.3 where it was shown that two Higgs chiral supermultiplets

[15] The NMSSM uses the $S^3$ term to stabilize the potential instead.

| | $Q$ | $L$ | $u^c$ | $d^c$ | $e^c$ | $\nu^c$ | $h_u$ | $h_d$ | $D$ | $\overline{D}$ | $S$ | $H'$ | $\overline{H}'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Y | $\frac{1}{6}$ | $-\frac{1}{2}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| N | 1 | 2 | 1 | 2 | 1 | 0 | $-2$ | $-3$ | $-2$ | $-3$ | 5 | 2 | $-2$ |

TABLE 2.1: List of the $U(1)_N$ and $U(1)_Y$ charges for the $E_6$SSM chiral supermultiplets.

are required in the MSSM rather than just one. Similarly, for a general $U(1)'$ symmetry, new fields in addition to the $S$ field must be introduced to make it anomaly free. In general however, the required number and type of new fields is not fixed and often requires either the presence of exotic chiral supermultiplets [47] or family-non-universal $U(1)'$ couplings [48]. Any family dependence of the $U(1)'$ charges would result in flavour changing neutral currents (FCNCs) mediated by the $Z'$, which can manifest themselves in rare $B$ decays and $B - \overline{B}$ mixing [49].

### 2.6.2 The $U(1)_N$ Group

If the $U(1)'$ symmetry comes from a GUT group such as $SO(10)$ or $E_6$ however then the gauge anomalies will automatically cancel as long as complete GUT representations survive to the $U(1)'$ symmetry breaking scale. This fixes the number and type of fields required to cancel the anomalies. In particular, SUSY GUTs based on an $E_6$ gauge group turn out to be very promising candidates for USSM models that have no gauge anomalies [16]. $E_6$ is the only exceptional Lie group that has complex representations and therefore the only exceptional group that can be used as a GUT in four dimensions.[16] A supersymmetric model that is inspired by an $E_6$ SUSY GUT is the Exceptional Supersymmetric Standard Model ($E_6$SSM) [17]. The $U(1)'$ symmetry of the $E_6$SSM is called the $U(1)_N$ group and arises from the following symmetry breaking chain [50]:

$$\begin{aligned} E_6 &\to SO(10) \times U(1)_\psi \\ &\to SU(5) \times U(1)_\chi \times U(1)_\psi \\ &\to SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_N \end{aligned}$$

where $SO(10) \times U(1)_\psi$ is a maximal subgroup of $E_6$, $SU(5) \times U(1)_\chi$ is a maximal subgroup of $SO(10)$ [51], and the above symmetry breaking is assumed to take place at the GUT scale. The $U(1)_N$ group is defined as the linear combination of the $U(1)_\psi$ and $U(1)_\chi$ groups for which the right-handed neutrinos are not charged. This combination is defined as:

$$U(1)_\chi \cos\theta + U(1)_\psi \sin\theta$$

[16]Complex representations are required for the theory to be chiral.

where $\theta = \tan^{-1}\sqrt{15}$.

### 2.6.3 Matter Spectrum

The $U(1)_N$ charge assignments are the same for each generation of matter and so the model does not suffer from the FCNC problem of general $U(1)'$ symmetries. In the $E_6$SSM one generation of quarks and leptons is unified into a fundamental $E_6$ representation, which has dimension 27. The fundamental representation of $E_6$ decomposes to the following $SO(10)$ representations:

$$27 \rightarrow 16 + 10 + 1. \tag{2.25}$$

The fundamental $SO(10)$ representation contains Higgs doublet and triplet chiral supermultiplets as in conventional $SO(10)$ SUSY GUTs: $10 = h_u + h_d + D + \overline{D}$. The effective MSSM $\mu$-term is thus forbidden by the $E_6$ symmetry since $27 \times 27$ is not an $E_6$ invariant. For the $U(1)_N$ group to be anomaly free, complete irreducible $E_6$ representations must survive to low energies, and since three generations of quarks and leptons have been observed, three copies of a 27 $E_6$ representation are assumed in the $E_6$SSM. The $U(1)_N$ group of the $E_6$SSM is therefore automatically anomaly free if the particle content forms complete irreducible representations of $E_6$. However, two additional electroweak doublet states $H'$ and $\overline{H}'$ are also included in the $E_6$SSM which form incomplete $E_6$ representations but, since the $H'$ and $\overline{H}'$ states have opposite $U(1)_N$ charges, the gauge anomalies cancel in an analogous way to how the gauge anomalies from $h_u$ and $h_d$ cancel in the MSSM. In total the $E_6$SSM therefore contains the following $SU(3)_c \times SU(2)_L \times U(1)_Y$ representations:

$$3 \times 27 + H', \overline{H}' = 3(Q,\ u^c,\ d^c,\ L,\ e^c,\ \nu^c) + 3(h_u,\ h_d) + 3(D,\ \overline{D}) + 3S + H', \overline{H}'$$

where $S$ denotes the $SO(10)$ singlet in Eq.2.25.

There are thus three generations of quarks and leptons, three copies of (up and down) Higgs doublets and triplets, and three singlet fields $S$. Table 4.5 contains the $U(1)_N$ charges of all the above $E_6$SSM particles. Only the 'third generation' of the up and down Higgs-like fields, denoted by $h_{u3}$ and $h_{d3}$, are defined to obtain electroweak scale VEVs and thus act like the MSSM Higgs fields. The other generations of the up and down Higgs-like fields do not get VEVs and so do not contribute to electroweak symmetry breaking (or the quark and lepton masses). Only the third generation of the singlet fields $S$ is likewise taken to obtain a VEV, which generates the effective $\mu$-term of the MSSM, as discussed in Sections 2.5.1 and 2.5.2.

### 2.6.4 The Effective MSSM Yukawa Interactions

In the $E_6$SSM the effective MSSM Yukawa interactions between the quarks, leptons and Higgs fields come from the $E_6$ tensor product $27 \times 27 \times 27$ which decomposes to the $SO(10)$ tensor products $16 \times 16 \times 10$ and $1 \times 10 \times 10$. For the three generations this $E_6$ product can be written as the following:

$$\lambda^{ijk} 27_i 27_j 27_k = \lambda^{ijk} 16_i 16_j 10_k + \lambda^{ijk} 1_i 10_j 10_k \tag{2.26}$$

where $\lambda^{ijk}$ is a coupling constant and $i, j, k = 1 \ldots 3$ label the three generations.

The $SO(10)$ tensor product $\lambda^{ijk} 16_i 16_j 10_k$ can be written as $\lambda^{ij3} 16_i 16_j 10_3$ $+\lambda^{ij\alpha} 16_i 16_j 10_\alpha$ where $\alpha = 1, 2$. In the $E_6$SSM the effective MSSM Yukawa interactions are contained in $\lambda^{ij3} 16_i 16_j 10_3$ since the third Higgs doublet generations $h_{u3}$ and $h_{d3}$ come from the $10_3$ representation, and are the only Higgs fields that are assumed to get electroweak scale VEVs and thus give mass to the quarks and leptons.

### 2.6.5 Non-Higgs Doublets

The interactions $\lambda^{ij\alpha} 16_i 16_j 10_\alpha$ will create tree-level flavour changing neutral currents due to the exchange of the first and second generation (non-Higgs) doublets $h_{u\alpha}$ and $h_{d\alpha}$. These interactions will violate experimental data unless they are suppressed or forbidden. Appendix C discusses these types of interactions in more detail. The $E_6$SSM includes a discrete $Z_2$ symmetry called $Z_2^H$ that forbids the operators $\lambda^{ij\alpha} 16_i 16_j 10_\alpha$. All $E_6$SSM states are assumed to be odd under this discrete symmetry except the third generation of Higgs doublets and the third generation of MSSM singlets $S_3$. In Section 2.6.7 it is shown that the $Z_2^H$ symmetry is a broken symmetry of the $E_6$SSM however, which can reintroduce these interactions but, as long as the couplings between $h_{u\alpha}$, $h_{d\alpha}$ and the first and second generation of quarks and leptons are sufficiently suppressed, then no experimental observations will be violated. For example, in order to suppress the contribution of new particles and interactions to the $K^0 - \overline{K}^0$ oscillations and to the muon decay channel $\mu \to e^- e^+ e^-$ in accordance with experimental limits, it is necessary to assume that the Yukawa couplings of $h_{u\alpha}$, $h_{d\alpha}$ to the quarks of the first and second generations are less than $10^{-4}$ and their couplings to the leptons of the first two generations are smaller than $10^{-3}$. The couplings to the third generation on the other hand can be as large as $10^{-1}$.

The fact that only the Higgs doublets $h_{u3}$ and $h_{d3}$ couple to the quarks and leptons (in the limit of an exact $Z_2^H$) can be used to explain why only these Higgs fields get electroweak VEVs. This is because only their soft masses can be driven negative by the top (and bottom) Yukawa coupling, generating radiative electroweak symmetry breaking.

### 2.6.6 The Effective MSSM $\mu$-term

The $\lambda^{ijk} 1_i 10_j 10_k$ term in Eq.2.26 can be written as the following:

$$\lambda^{ijk} 1_i 10_j 10_k = \lambda^{333} S_3 10_3 10_3 + \lambda^{3\alpha\beta} S_3 10_\alpha 10_\beta + \lambda^{\alpha\beta 3} S_\alpha 10_\beta 10_3 \qquad (2.27)$$

$$+\lambda^{\alpha\beta\gamma} S_\alpha 10_\beta 10_\gamma + \lambda^{\alpha 33} S_\alpha 10_3 10_3.$$

The $Z_2^H$ symmetry that was introduced in the previous Section forbids the interactions of the second line, leaving only $\lambda^{333} S_3 10_3 10_3$, $\lambda^{3\alpha\beta} S_3 10_\alpha 10_\beta$ and $\lambda^{\alpha\beta 3} S_\alpha 10_\beta 10_3$. Since only the third generation of the singlet fields $S_3$ is assumed to get a VEV, which breaks the $U(1)_N$ symmetry, the operator $S_3 h_{u3} h_{d3}$ in $S_3 10_3 10_3$ generates an effective MSSM $\mu$-term. The VEV of $S_3$ also gives mass to the first and second generation higgsinos $\tilde{h}_{u\alpha}$, $\tilde{h}_{d\alpha}$ because of the operators $S_3 h_{u\alpha} h_{d\beta}$ in $S_3 10_\alpha 10_\beta$. The fermionic partners of the singlet fields $S_\alpha$, singlinos, obtain mass from the operators $S_\alpha h_{u\beta} h_{d3}$ and $S_\alpha h_{d\beta} h_{u3}$ in $S_\alpha 10_\beta 10_3$.

The operator $\lambda^{ijk} 1_i 10_j 10_k$ in Eq.2.27 also includes the term $\lambda^{3ij} S_3 D_i \overline{D}_j$ which gives mass to the Higgs triplets $D_i$ and $\overline{D}_i$ because of the VEV of $S_3$. In the E$_6$SSM the Yukawa coupling constant for the $S_3 D_i \overline{D}_j$ operator can contribute to the renormalization group evolution of the soft singlet mass $m^2_{S_3}$ driving it negative from a positive value at the GUT scale and thus triggering $S$ to gain a VEV [52, 53]. This mechanism for generating a VEV for $S_3$ is analogous to radiative symmetry breaking used in some extensions to the MSSM as discussed in Section .

In addition to solving the $\mu$-problem of the MSSM, the little hierarchy problem of the MSSM should also be resolved by the ME$_6$SSM. This is because there are extra particles below the conventional GUT scale of $10^{16}$ GeV that are not contained in the MSSM. These extra particles are from the three copies of the 27 $E_6$ multiplet. Due to Renormalization Group effects, the extra states increase the value of the Yukawa coupling constant for $S_3 h_{u3} h_{d3}$ at low energies, and hence increase the mass of the lightest CP even Higgs boson [54].

### 2.6.7 Proton Decay and Higgs Triplet Decay

The GUT partners of the Higgs doublets, called the Higgs triplets, $D_i$ and $\overline{D}_i$ do not have equal and opposite $U(1)_N$ charges and so their contributions to a gauge anomaly do not cancel. Therefore these particles cannot have GUT scale masses as in conventional GUTs, and instead must obtain electroweak or TeV scale masses. The operators given by $\lambda^{ijk}16_i16_j10_k$ in Eq.2.26 contain interactions between the Higgs triplets and the quarks and leptons, which are the following:

$$\begin{aligned} W_1 &= \lambda_D^{ijk} D_i Q_j Q_k + \lambda_{\overline{D}}^{ijk} \overline{D}_i d_j^c u_k^c \\ W_2 &= \lambda_d^{ijk} D_i \nu_j^c d_k^c + \lambda_u^{ijk} D_i e_j^c u_k^c + \lambda_Q^{ijk} \overline{D}_i L_j Q_k. \end{aligned} \tag{2.28}$$

If all of these interactions are allowed then baryon number is violated and, if the Higgs triplets have TeV scale masses or lower, then the proton will decay with a lifetime much shorter than that observed. However if all of the above interactions are forbidden, thus avoiding rapid proton decay, then the Higgs triplets $D_i$ and $\overline{D}_i$ can't decay. The Higgs triplets are then stable, strongly interacting particles with small masses. Any heavy stable particle would have been copiously produced during the very early epochs of the Big Bang. The strong (or electromagnetically) interacting fermions and bosons which survive annihilation would subsequently have been confined in heavy hadrons which would annihilate further. The remaining heavy hadrons originating from the Big Bang should be present in terrestrial matter and there are very strong upper limits on the abundances of nuclear isotopes which contain such stable relics in the mass range from 1 GeV to 10 TeV. Different experiments set limits on their relative concentrations from $10^{-15}$ to $10^{-30}$ per nucleon [55]. At the same time various theoretical estimations [56] show that if remnant particles exist in nature today their concentration is expected to be at the level of $10^{-10}$ per nucleon. Therefore $E_6$ inspired models with stable Higgs triplets are ruled out.

However, if either $W_1$ or $W_2$ are forbidden, with the other allowed, then rapid proton decay can be avoided and the Higgs triplets can still decay. A $Z_2$ discrete symmetry is used in the $E_6$SSM to achieve this scenario. This $Z_2$ symmetry can be used in two ways: either the leptons are odd under $Z_2$ (in which case the symmetry is called $Z_2^L$) so that $W_2$ is forbidden but $W_1$ is allowed, or the leptons and Higgs triplets are odd (in which case the symmetry is called $Z_2^B$) so that $W_1$ is forbidden but $W_2$ is allowed. The former case with only $W_1$ allowed is called Model I whereas the latter case with only $W_2$ allowed is called Model II.

Neither $Z_2^L$ nor $Z_2^B$ commute with the $E_6$ symmetry if all the states of the $E_6$SSM (excluding the $H'$ and $\overline{H}'$) come from just three copies of a 27 multiplet. Instead, for either the $Z_2^L$ or $Z_2^B$ symmetry to commute with $E_6$, the quarks and leptons must come from different 27 multiplets to each other. These different multiplets are denoted here by $27_i^Q$ and $27_i^L$ where $i = 1 \ldots 3$ respectively. In the case of $Z_2^B$ the Higgs triplets and quarks must also come from different $E_6$ multiplets. If the $E_6$ symmetry in the $E_6$SSM is a conventional SUSY GUT then a mechanism is required that explains why only the leptons (and Higgs triplets for $Z_2^B$) of $27_2^L$ survive to low-energies but the other states do not, and why the quarks and not the leptons of $27_i^Q$ survive to low-energies. This is similar to the doublet-triplet splitting problem facing conventional GUTs but is expected to be more troublesome since the leptons in $27_2^L$ would also have to be split from the Higgs-doublet states. At present no mechanism has been provided.

An alternative possibility is that the $E_6$ is as a symmetry of a string theory or a quantum field theory that has extra dimensions.[17] Different 27 representations from different $E_6$ multiplets could then potentially arise from the compactification of extra dimensions. For example, by orbifolding the extra dimensions or using Wilson-line symmetry breaking to break the $E_6$ symmetry. Again no particular mechanism has been found to explain the splitting required by the $Z_2^L$ or $Z_2^B$ symmetries in the $E_6$SSM, and this work does not attempt to resolve this problem. Instead a particular unknown mechanism, perhaps string inspired, that solves this problem is assumed.

Another issue arises from the $Z_2^H$ symmetry discussed in Section 2.6.5. Since all the 27 states were assumed to have odd $Z_2^H$ parity except for the third generation of Higgs and singlet fields, both $W_1$ and $W_2$ are forbidden by $Z_2^H$. In the $E_6$SSM it is assumed that $Z_2^H$ is a broken symmetry that allows either $W_1$ or $W_2$ or both. A solution to this problem is proposed in Chapter 7 where an effective $Z_2^H$ symmetry arises from a family symmetry that allows $W_1$ and $W_2$. Other possible solutions include replacing $Z_2^H$ with a $Z_2$ symmetry under which only $h_1$, $h_2$, $S_1$ and $S_2$ are odd, or giving the $D_i$ even $Z_2^H$ parity.

### 2.6.8 $H'$ and $\overline{H}'$ Interactions

As well as the particles from three copies of a 27 $E_6$ representation, the $E_6$SSM also contains two additional electroweak doublet particles $H'$ and $\overline{H}'$ that form incomplete $E_6$ representations. These are required for the unification of gauge coupling constants as discussed in Section 2.6.10. If the $H'$ and $\overline{H}'$ come from a 27 and $\overline{27}$ representation

[17]The $E_6$ symmetry could exist at the Planck scale where it is broken to $SO(10)$ which then breaks to the Standard Model gauge group at the GUT scale.

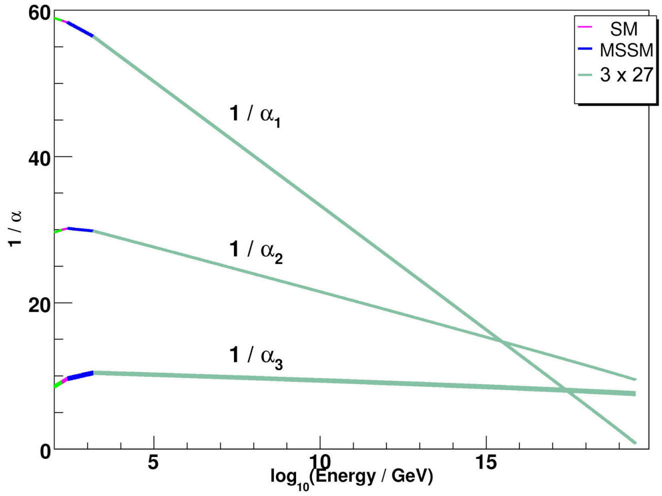



FIGURE 2.4: Two-loop running of the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge coupling constants in a theory with three copies of low-energy complete 27 multiplets of $E_6$.

of $E_6$, denoted by $27'$ and $\overline{27}'$, then the following $E_6$ respecting interactions are allowed:

$$\begin{aligned} W' = \mu' H' \overline{H}' + \mu'^{i} \overline{H}' L_i + \lambda^i \nu_i^c h_{u3} H' + \lambda^i e_i^c h_{d3} H' \\ + \lambda^{i\alpha} \nu_i^c h_{u\alpha} H' + \lambda^{i\alpha} e_i^c h_{d\alpha} H'. \end{aligned} \tag{2.29}$$

If odd under $Z_2^H$ and $Z_2^L$ or $Z_2^B$ then the operators in the second line of Eq.2.29 are forbidden. Adding another $Z_2$ symmetry called $Z_2'$ for which only $H'$ and $\overline{H}'$ then just allows the term $\mu' H' \overline{H}'$.

### 2.6.9 $E_6$-Violating Operators

Since the discrete symmetries $Z_2^H$ and $Z_2^L$ (or $Z_2^B$) do not commute with the $E_6$ symmetry if the $E_6$SSM particles (except for $H'$ and $\overline{H}'$) only come from three complete 27 $E_6$ multiplets, then we must also consider the operators involving these particles that would otherwise violate the $E_6$ symmetry. In both Model I and II (with the $Z_2'$ symmetry) the only operator that disrespects the $E_6$ symmetry is $M_{ij} \nu_i^c \nu_j^c$ which is a Majorana mass term for the right-handed neutrinos.

### 2.6.10 Gauge Coupling Unification in the $E_6$SSM

To cancel gauge anomalies of the $U(1)_N$ group, three copies of 27 $E_6$ representations survive to low-energies which contain three generations of quarks and leptons. If we run the gauge coupling constants with energy using this matter spectrum then they will never meet, as illustrated by Fig.2.4. We have thus lost one of the most important

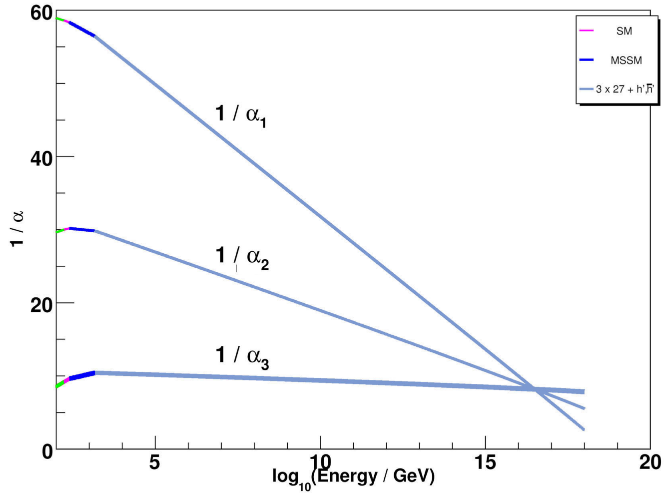



FIGURE 2.5: The two-loop running of the gauge coupling constants for the $E_6$SSM. The particle content of this theory is equivalent to three copies of $E_6$ 27 multiplets and two additional electroweak doublets $H'$ and $\overline{H'}$. The thickness of the lines represents the experimental error in the gauge coupling constants.

predictions of the MSSM and the inspiration for a model based on an $E_6$ unified gauge symmetry.

To rectify this, two additional electroweak doublet states, denoted by $H'$ and $\overline{H}'$, are included at low-energies. With these extra states included the matter spectrum of the $E_6$SSM then looks like the MSSM but with three additional complete $SU(5)$ multiplets $(5+\overline{5}+1)$. The gauge coupling constants will now meet at the conventional GUT scale since we have just added complete GUT states to the MSSM, which is illustrated by Fig.2.5. Note that, although, gauge coupling unification still occurs at the GUT scale (at least at one-loop order), the value of the unified gauge coupling constant is now much larger than it is in the MSSM [57]. However the unified coupling still in the perturbative regime and is similar in size to the QCD coupling at the electroweak symmetry breaking scale.

# Chapter 3

# Intermediate Symmetries and Gauge Coupling Unification

## 3.1 Introduction

In the previous Chapter an $E_6$ inspired supersymmetric model called the $E_6$SSM was proposed as an alternative to the Standard Model. This was motivated by the instability of the Higgs mechanism in the Standard Model to higher energy physics. In the $E_6$SSM (and MSSM) the Higgs mechanism is protected by supersymmetry which cancels all the quantum corrections from fermions and bosons to all orders in perturbation theory. The $E_6$SSM also resolves the $\mu$-problem associated with the Higgs mass in the MSSM. This is achieved without the additional problems of theories such as the NMSSM for example which predicts the formation of domain walls in the early Universe. However a failing of the $E_6$SSM, called the $\mu'$-problem, is highlighted in Section 3.2 which questions the theoretical naturalness of its solution to the $\mu$-problem. The purpose of this Chapter is to resolve the $\mu'$-problem of the $E_6$SSM.

## 3.2 The $\mu'$-Problem of the $E_6$SSM

In Section 2.6 we found that, since the $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_N$ gauge symmetry of the $E_6$SSM is derived from an $E_6$ symmetry, and the quarks and leptons are contained in fundamental 27 representations of $E_6$, three copies of 27 multiplets must survive to low energies for the theory to be free of gauge anomalies. Unfortunately however three copies of low-energy 27 multiplets do not lead to gauge coupling unification, making it difficult to connect the theory to a high energy $E_6$ symmetry. To solve this problem new particles called $H'$ and $\overline{H}'$ that transform as electroweak doublets are included at the TeV scale so that the particle content of $\mathrm{E}_6$SSM resembles the MSSM but with complete $SU(5)$ multiplets that do not upset gauge coupling unification. These new particles must be related to the high-energy $\mathrm{E}_6$ symmetry, and since they have opposite $U(1)_N$ charges so that the additional Abelian group is anomaly free, the simplest possibility is that they come from a 27 and $\overline{27}$ multiplet respectively (called $27'$ and $\overline{27}'$). This then leaves the question of why the rest of the $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_N$ states from these $\mathrm{E}_6$ multiplets do not contribute to the running of the gauge coupling constants.

One possibility is that the rest of the $27'$ and $\overline{27}'$ states gain mass at the GUT scale due to a mechanism that is similar to the doublet-triplet splitting mechanisms used in conventional SUSY GUTs. However, there are more states in the $27'$ and $\overline{27}'$ multiplets than just the coloured partners of $H'$ and $\overline{H}'$. For example, in both $\mathrm{E}_6$ multiplets $27'$ and $\overline{27}'$ there are states that transform in the same way as $H'$ and $\overline{H}'$ respectively under $SU(3)_c \times SU(2)_L \times U(1)_Y$ but just have different $U(1)_N$ charges. Explaining why these particles get GUT masses whereas $H'$ and $\overline{H}'$ get TeV masses would be particularly tricky and at present there is no solution to this problem.[1] This is referred to as the $27'$,$\overline{27}'$ splitting problem.

The $H'$ and $\overline{H}'$ states also introduce a problem analogous to the $\mu$-problem of the MSSM. The mass parameter $\mu'$ in the $\mathrm{E}_6$SSM superpotential Eq.2.29 should not be too large otherwise it spoils gauge coupling, but on the other hand it cannot be too small since $\mu' H' \overline{H}'$ is solely responsible for the mass of the charged and neutral components of the $H'$ and $\overline{H}'$ fermions. In fact we typically require $\mu' \approx \mathcal{O}(1\text{TeV})$ just as $\mu \approx \mathcal{O}(1\text{TeV})$ is required in the MSSM. Unfortunately however we cannot use the $U(1)_N$ gauge group to solve this $\mu'$-problem since the bilinear term $\mu' H' \overline{H}'$ has zero overall $U(1)_N$ charge. If we wish to solve the $\mu'$-problem in a similar way to how the $\mu$-problem of the MSSM is solved in the $\mathrm{E}_6$SSM then we must introduce another $U(1)$ gauge symmetry and a new $\mathrm{E}_6$ singlet field that is charged under the $U(1)$ symmetry. Thus we would have to look for a larger gauge group than $\mathrm{E}_6$.

Within SUGRA models the term $\mu' H' \overline{H}'$ in the superpotential can be induced just after the breakdown of local SUSY if the Kähler potential contains an extra term $(Z(H' \overline{H}') + h.c.)$. This mechanism is analogous to the same one that can used to solve the $\mu$-problem of the MSSM [59]. But in models based on an $E_6$ symmetry, the bilinear terms involving $h_d$ and $h_u$ are forbidden by the $E_6$ symmetry both in the Kähler potential and superpotential since they transform in a 27 representation. As a result

[1] An alternative could be to use a doublet-triplet splitting mechanism that results from the compactification of extra dimensions. For example, orbifolding or Wilson-line symmetry breaking in string inspired theories can split Higgs triplets from Higgs doublets [58]. Explaining why three full 27 representations are present in the low-energy theory but only one electroweak doublet from another 27 representation is also light is likely to be particularly difficult however.

the mechanism mentioned above cannot be applied for the generation of $\mu h_u h_d$ in the $E_6$SSM superpotential. However this mechanism may be used to give mass to the non-Higgs doublets $H'$ and $\overline{H}'$ from additional $27'$ and $\overline{27}'$ since the corresponding bilinear terms are allowed by the $E_6$ symmetry both in the Kähler potential and superpotential. However it is somewhat unappealing that the principal motivation of the $E_6$SSM, to solve the $\mu$-problem of the MSSM, requires the use of a mechanism that can be used as an alternative to solving this problem instead.

On the other hand the only purpose of including the $H'$ and $\overline{H}'$ states however is to achieve gauge coupling unification at $M_{GUT} \approx 10^{16}$ GeV. This allows the possibility of removing these states from the spectrum and thus avoiding the $\mu'$-problem and the $27', \overline{27}'$ mass-splitting problem altogether. Of course we must then search for alternative methods of achieving gauge coupling unification, which is the subject of the rest of this Chapter.

## 3.3 Intermediate Symmetries

An alternative to including the $H'$ and $\overline{H}'$ states at a low-energy scale is to change the gauge symmetry of the $E_6$SSM at a high-energy scale. This would then change the RGEs and the gauge coupling constants of the theory. We can then choose a gauge symmetry such that its gauge coupling constants run with energy until they unify at some high-energy scale, where an $E_6$ would be anticipated to exist. In this case the pattern of symmetry breaking from the $E_6$ unification scale down to the electroweak symmetry breaking scale would be the following:

$$\mathrm{E}_6 \overbrace{\longrightarrow}^{M_{\mathrm{E}_6}} \mathrm{IS} \overbrace{\longrightarrow}^{M_{\mathrm{IS}}} G_{\mathrm{E}_6\mathrm{SSM}} \overbrace{\longrightarrow}^{\mathrm{TeV}} G_{321} \overbrace{\longrightarrow}^{EW} G_{31} \tag{3.1}$$

where $G_{31} \equiv SU(3)_c \times U(1)_{em}$; $G_{321} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y$; $G_{\mathrm{E}_6\mathrm{SSM}} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_N$ is the gauge symmetry of the $E_6$SSM; IS is the new gauge symmetry, which is called an intermediate symmetry; and $M_{\mathrm{E}_6}$ and $M_{IS}$ are respectively the high-energy scales at which the $E_6$ and IS symmetries are broken respectively. Starting from the Standard Model gauge symmetry, the gauge coupling constants would run to the $E_6$SSM scale, where they are joined with a $U(1)_N$ gauge coupling constant, and then continue to run to the scale $M_{\mathrm{IS}}$ where they are replaced with the gauge couplings of IS. The IS gauge couplings then take over which run until they meet at $M_{\mathrm{E}_6}$.

Since the $H'$ and $\overline{H}'$ states would no longer be required, the supersymmetric theory would just contain three copies of a 27 $E_6$ multiplet. We therefore need to search for a symmetry IS that provides unification of the gauge coupling constants in a theory that contains three copies of a 27 multiplet. This symmetry must obviously be large enough to contain the $E_6$SSM gauge symmetry, but small enough to fit inside an $E_6$ symmetry. That is, $G_{E_6SSM}$ must be a subgroup of IS, and IS must be a subgroup of $E_6$.

Unification in supersymmetric models containing three 27 representations of the gauge group $E_6$ was recently considered in [60]. In this paper the authors assumed an intermediate Pati-Salam gauge group $SU(4)\times SU(2)_L\times SU(2)_R$ with a left-right discrete symmetry at the scale $10^{15}$ GeV. At this scale the Standard Model (SM) couplings satisfy $\alpha_1 = \alpha_2$ where $\alpha_1$ and $\alpha_2$ are the $U(1)_Y$ and $SU(2)_L$ gauge coupling constants. The resulting Pati-Salam gauge couplings then subsequently meet at a higher energy scale of about $10^{18}$ GeV.

This suggests that the IS symmetry could be the Pati-Salam symmetry $G_{422} \equiv SU(4) \times SU(2)_L \times SU(2)_R$. However, as will be shown in Section 3.5.2, the condition $\alpha_1 = \alpha_2$ cannot be consistently applied at the Pati-Salam breaking scale and thus the analysis in [60] is incorrect. Instead it will be shown that the Pati-Salam breaking scale must be about an order of magnitude larger than the crossing point $\alpha_1 = \alpha_2$, close to $M_{GUT} \approx 10^{16}$ GeV, with full unification close to $M_p \approx 10^{19}$ GeV. In this case the Standard Model gauge coupling constants will run up to $M_{GUT}$ where they are replaced with the $G_{422}$ gauge coupling constants, which run until they unify at $M_p$. This is illustrated by Fig.3.1 which is discussed in more detail in Section 3.6.

A Pati-Salam gauge symmetry $G_{422}$ is not large enough to contain $G_{E6SSM}$ as a subgroup however and so cannot by itself by the IS symmetry in Eq.3.1. In Chapter 4 it is shown that if the $G_{422}$ gauge group is extended with an extra $U(1)$ group, called $U(1)_\psi$, then it can contain $G_{E6SSM}$ as a subgroup. However in this Chapter the $U(1)_N$ and $U(1)_\psi$ groups are initially ignored to simplify the analysis. This is done because there is no experimental data for a $U(1)_N$ gauge coupling constant and so it will not help to determine the unification scale. $U(1)_\psi$ is thus considered to be broken at the Planck scale in this Chapter.

In the next Section a short introduction to the Pati-Salam Symmetry is provided before the pattern of symmetry breaking and RGEs of the intermediate Pati-Salam symmetry are analysed in Sections and respectively.

## 3.4 Pati-Salam Gauge Symmetry

The Pati-Salam symmetry was first introduced by Jogesh Pati and Abdus Salam in 1974 as a possible extension to the Standard Model gauge group [61]. Under this symmetry the Standard Model leptons are considered to be the '4th colour' of the $SU(4)$ symmetry. Together with the left-handed quarks, the left-handed leptons form the Pati-Salam representation $(4, 2, 1)$, denoted by $F$, whereas the charge-conjugated quark and lepton fields form the $(\overline{4}, 1, 2)$ representation, denoted by $F^c$. This can be represented by the following matrix notation:

$$F^{\alpha a} = (4, 2, 1) = \begin{pmatrix} u_r & u_g & u_b & \nu_e \\ d_r & d_g & d_b & e \end{pmatrix},$$
$$F_{\gamma a} = (\overline{4}, 1, 2) = \begin{pmatrix} d^c_r & d^c_g & d^c_b & e^c \\ u^c_r & u^c_g & u^c_b & \nu^c_e \end{pmatrix}$$

where $u$ fields are left-handed; $u^c$ stands for charge-conjugated left-handed fields; $r, g, b$ stand for colours of $SU(3)_c$; $a = 1 \ldots 4$ is an $SU(4)$ index which labels the columns of the matrices; and $\alpha, \gamma = 1, 2$ are $SU(2)_L$ and $SU(2)_R$ indices respectively that label the rows of the matrix.

The $(\overline{4}, 1, 2)$ representation also contains a state that is not in the Standard Model. It is a singlet of the Standard Model and is an $SU(2)_R$ partner to the charge-conjugated electron $e^c$. This particle is therefore a charge-conjugated neutrino, and since we expect it to have a mass near $10^{12-16}$ GeV in conventional see-saw mechanisms (see Section 5.3.1), we might anticipate that a Pati-Salam symmetry is broken around these high-energy scales.

The $SU(2)_R$ group only couples to right-handed fermions just as the $SU(2)_L$ group only couples to left-handed fermions. The Pati-Salam symmetry, unlike the Standard Model, thus respects parity. A discrete left-right symmetry called $D_{LR}$ can be further applied to the $G_{422}$ gauge group under which the matter multiplets transform as $q_L \rightarrow q^c_L$ where $q$ denotes any matter multiplet, and the gauge groups $SU(2)_L$ and $SU(2)_R$ become interchanged [62].

A supersymmetric Pati-Salam symmetry also looks like a promising extension to the MSSM since the Higgs fields can come from the complete representation $(1, 2, 2)$:

$$h^{\alpha\gamma} = (1, 2, 2) = \begin{pmatrix} h^+_u & h^0_d \\ h^0_u & h^-_d \end{pmatrix}$$

where the $SU(2)_L$ index $\alpha$ labels the rows, and the $SU(2)_R$ index $\gamma$ labels the columns.[2]

## 3.5 Pattern of Symmetry Breaking

The two step pattern of gauge group symmetry breaking analysed in this Section is:

$$E_6 \overset{M_p}{\longrightarrow} G_{422} \times D_{LR} \overset{M_{GUT}}{\longrightarrow} G_{321} \tag{3.2}$$

where $G_{321} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y$. The first stage of symmetry breaking close to $M_p$ will not be considered since it is likely to be a quantum gravity theory. Whatever this quantum gravity theory is, it will involve some high-energy threshold effects, which will depend on the details of the high energy theory, and which is not considered in following analysis.

### 3.5.1 The Pati-Salam Higgs Sector

The second stage of symmetry breaking close to $M_{GUT}$ is within the realm of conventional quantum field theory, and requires a Higgs sector, in addition to the assumed matter content of three 27 representations of the gauge group $E_6$, to break the Pati-Salam symmetry to the Standard Model gauge group. In order to break the Pati-Salam symmetry $G_{422}$ to $G_{321}$ at $M_{GUT}$ the minimal Higgs sector required are the $G_{422}$ representations $H_R = (4,1,2)$ and $\overline{H}_R = (\overline{4},1,\overline{2})$. When these particles obtain VEVs in the right-handed neutrino directions they break the $SU(4) \times SU(2)_R$ symmetry to $SU(3)_c \times U(1)_Y$ with the desired hypercharge assignments, as discussed later.

Although a Higgs sector consisting of $H_R$ and $\overline{H}_R$ is perfectly adequate for breaking Pati-Salam symmetry, it does not satisfy $D_{LR}$. If we wish to satisfy this symmetry we must therefore also consider an extended Higgs sector including their left-right symmetric partners. A minimal left-right symmetric Higgs sector capable of breaking Pati-Salam symmetry consists of the $SO(10)$ Higgs states $16_H$ and $\overline{16}_H$. If complete $E_6$ multiplets are demanded in the entire theory below $M_p$, then the Pati-Salam breaking Higgs sector at $M_{GUT}$ may be assumed to be $27_H$ and $\overline{27}_H$. Therefore in the following analysis two possible Higgs sectors are considered which contribute to the SUSY beta functions in the region between $M_{GUT}$ and $M_p$, namely either the $SO(10)$ states $16_H + \overline{16}_H$ or the $E_6$ states $27_H + \overline{27}_H$, where it is understood that only the Pati-Salam gauge group exists

[2]Note that for the Higgs fields the hypercharge generator $Y$ is equivalent to the $T^3_R$ generator of $SU(2)_R$ (see Section 3.5.2). The matrix can therefore be constructed by considering the $T^3_L$ and $Y$ charges for each Higgs component.

in this region, and these Higgs representations must be decomposed under the Pati-Salam gauge group. No such Higgs sectors were included in the analysis in [60]. For the analysis which involves the $16_H + \overline{16}_H$ states, the rest of the $SO(10)$ representations that together with the $16_H + \overline{16}_H$ states make up complete $E_6$ representations (such as a 27 and $\overline{27}$) are assumed to be at or above the $E_6$ breaking scale and so do not affect the running of the gauge coupling constants below the unification scale.

### 3.5.2 Pati-Salam Symmetry Breaking

When $H_R$ and $\overline{H}_R$ (contained in either the $SO(10)$ states $16_H + \overline{16}_H$ or $E_6$ states $27_H + \overline{27}_H$) develop VEVs in the right-handed neutrino directions they break the $SU(4) \times SU(2)_R$ symmetry to $SU(3)_c \times U(1)_Y$ with the desired hypercharge assignments. Six of the $SU(4)$ and two of the $SU(2)_R$ fields are then given masses related to the VEV of the Higgs bosons and the gauge bosons associated with the $T^{15}$ and $T^3_R$ generators are rotated by the Higgs bosons to create one heavy gauge boson and the gauge boson associated with $U(1)_Y$. In breaking $SU(4) \times SU(2)_R$ to $SU(3)_c \times U(1)_Y$ the SM hypercharge generator is a combination of the diagonal generator $T^{15} = \sqrt{\frac{3}{2}}\ diag(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{2})$ of $SU(4)$ and the diagonal generator of $SU(2)_R$, $T^3_R = \frac{1}{2}\ diag(1, -1)$. $T^{15} = \sqrt{\frac{3}{2}}(B - L)/2$ where $B$ and $L$ are the baryon and lepton number assignments of each Standard Model particle. Comparing these diagonal generators to the hypercharge values we must have $Y = T^3_R + (B - L)/2$. Then, analogous to the electroweak symmetry breaking condition Eq.2.12, one finds the following relation between the hypercharge gauge coupling constant $g_Y$ and the $SU(4)$ and $SU(2)_R$ gauge coupling constants $g_4$ and $g_{2R}$ respectively:

$$\frac{1}{\alpha_Y} = \frac{1}{\alpha_{2R}} + \frac{1}{\frac{3}{2}\alpha_4} \tag{3.3}$$

where $\alpha_Y \equiv \frac{g_Y^2}{4\pi}$, $\alpha_{2R} \equiv \frac{g_{2R}^2}{4\pi}$ and $\alpha_4 \equiv \frac{g_4^2}{4\pi}$.

Because the Pati-Salam symmetry, and hence the standard model, is assumed to come from an $E_6$ group, then all the charges and generators should be correctly normalized.[3] In this case the conventional standard model hypercharge assignments must be modified by a factor of $\sqrt{\frac{5}{3}}$ as discussed in Section 2.4.1. Therefore Eq.3.3 should be rewritten in terms of the 'GUT' normalized hypercharge $g_1 \equiv \sqrt{\frac{5}{3}} g_Y$:

$$\frac{5}{\alpha_1} = \frac{3}{\alpha_{2R}} + \frac{2}{\alpha_4} \tag{3.4}$$

[3]The $E_6$ generators $G^a$ are chosen to be normalized by $Tr(G^aG^b) = 3\delta^{ab}$. It then follows that the Pati-Salam and standard model operators are conventionally normalized by $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$. See Appendix B for more detail.

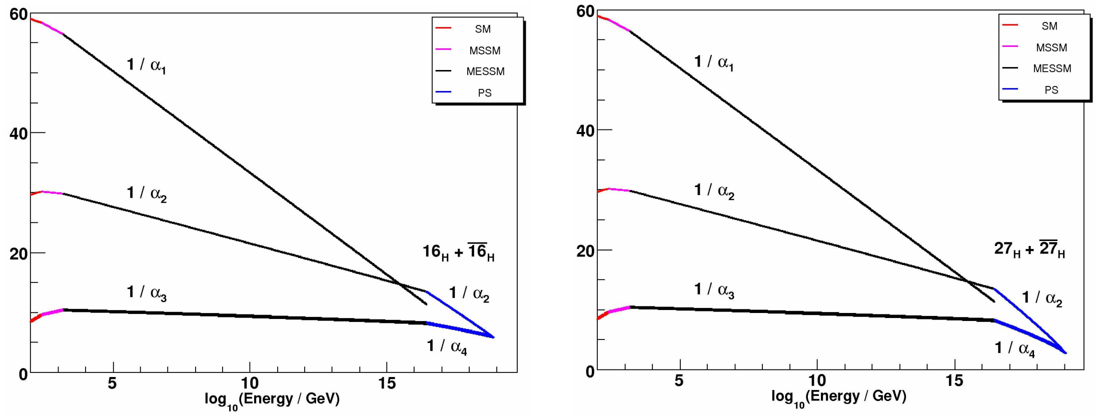



FIGURE 3.1: Two-loop Planck Scale Unification in two supersymmetric models that contain three generations of SUSY 27 particles, and an intermediate left-right symmetric Pati-Salam symmetry. Both models are described in the main body of the text. Near to the conventional GUT scale the $\alpha_3$ gauge coupling constant becomes the $\alpha_4$ gauge coupling constant of $SU(4)$, and the $SU(2)_R$ gauge coupling constant $\alpha_{2R}$ is the combination of the $\alpha_1$ and $\alpha_3$ gauge coupling constants given by Eq.3.4.

where $\alpha_1 \equiv \frac{g_1^2}{4\pi}$. Eq.3.4 is the boundary condition for the gauge couplings at the Pati-Salam symmetry breaking scale, in this case $M_{GUT}$. Due to left-right symmetry, at the Pati-Salam symmetry breaking scale we have the additional boundary condition $\alpha_{2L} = \alpha_{2R}$. In [60] it was assumed that at the Pati-Salam symmetry breaking scale $\alpha_1 = \alpha_{2L} = \alpha_{2R}$ which disagrees with Eq.3.4, since $\alpha_4 \neq \alpha_{2L} = \alpha_{2R}$ at this scale. This is discussed further in the next Section.

## 3.6 Two-loop analysis of gauge coupling unification

In this Section a SUSY two-loop RG analysis of the gauge couplings is performed, corresponding to the pattern of symmetry breaking discussed in the previous Section. Three complete 27 SUSY representations of the group $E_6$ are assumed in the spectrum which survive down to low energies, but, unlike the original E$_6$SSM, there are no additional $H', \overline{H}'$ states so the gauge couplings are not expected to converge at $M_{GUT}$. Instead, the pattern of symmetry breaking shown in Eq.3.2 is envisaged, where above the Pati-Salam symmetry breaking scale $M_{GUT}$ we assume, in addition to the three 27 representations, a Pati-Salam symmetry breaking Higgs sector of either the $SO(10)$ states $16_H + \overline{16}_H$ or $E_6$ states $27_H + \overline{27}_H$ which are assumed to gain masses of order the Pati-Salam symmetry breaking scale $M_{GUT}$, leaving only the three 27 matter representations below this scale.

For the present RG analysis, the couplings are run up from low energies to high energies, using as input the SM couplings measured on the Z-pole at LEP, which are as follows [26]: $\alpha_1(M_Z) = 0.016947(6)$, $\alpha_2(M_Z) = 0.033813(27)$ and $\alpha_3(M_Z) = 0.1187(20)$.

The general two-loop beta functions used to run the gauge couplings are given in Appendix A.

From $M_Z$ up to an assumed MSSM threshold energy of 250 GeV only the non-SUSY SM spectrum is assumed including a top quark threshold at 172 GeV. From 250 GeV to 1.5 TeV all the states of the MSSM are included. From 1.5 TeV up to the Pati-Salam symmetry breaking scale the remaining states which fill out three complete SUSY 27 representations are included. The assumed threshold energies correspond to those in [57], where a full discussion of MSSM and $E_6$SSM threshold effects is given. The only difference is that here the $H', \overline{H}'$ states of the $E_6$SSM are not included, so the gauge couplings do not converge at $M_{GUT}$. Instead $M_{GUT}$ is taken to be the Pati-Salam symmetry breaking scale, which is determined as follows.

In the previous Section the relation in Eq.3.4 between the hypercharge and Pati-Salam coupling constants at the Pati-Salam symmetry breaking scale was discussed. This can be turned into a boundary condition involving purely $G_{321}$ couplings constants at the Pati-Salam breaking scale, since $SU(3)_c$ comes from $SU(4)$ so $\alpha_3 = \alpha_4$ at this scale, and, as remarked, $D_{LR}$ symmetry requires that $\alpha_{2R} = \alpha_{2L}$ at the Pati-Salam symmetry breaking scale. Therefore Eq.3.4 can be re-expressed as:

$$\frac{5}{\alpha_1} = \frac{3}{\alpha_{2L}} + \frac{2}{\alpha_3}. \tag{3.5}$$

Having specified the low-energy matter content, and thresholds, Eq.3.5 allows a unique determination of the Pati-Salam breaking scale, by simply running up the gauge couplings until the condition is satisfied. In practice, $\alpha_3$ runs quite slowly (its one loop beta-function is zero), while the inverse hypercharge coupling decreases most rapidly and the condition is satisfied for a Pati-Salam symmetry breaking scale about an order of magnitude higher than the crossing point of $\alpha_1$ and $\alpha_2$ assumed in [60]. Assuming the above matter content and threshold corrections, the Pati-Salam symmetry is found to be broken at $M_{GUT} = 10^{16.44(4)}$ GeV as illustrated in Fig.3.1. This is close to the conventional GUT energy scale, and justifies the use of the notation $M_{GUT}$ to denote the Pati-Salam breaking scale.

Above the scale $M_{GUT}$ the two Pati-Salam gauge couplings, namely $\alpha_4$ and $\alpha_{2L} = \alpha_{2R}$, are run up including, in addition to the three SUSY 27 matter representations, also a Pati-Salam SUSY Higgs breaking sector consisting of either the $SO(10)$ states $16_H + \overline{16}_H$ or $E_6$ states $27_H + \overline{27}_H$. Fig.3.1 illustrates the running of the gauge coupling constants, where the left panel includes the $16_H + \overline{16}_H$ fields while the right-panel contains the $27_H + \overline{27}_H$ fields. The Pati-Salam couplings are found to converge at either $10^{18.83(7)}$

GeV or $10^{18.97(9)}$ GeV for the left and right-panels respectively.[4] These values are close to the Planck scale $M_p = 1.2 \times 10^{19}$ GeV, and suggests a Planck scale unification of all forces with gravity.

The value of the gauge coupling constant at the unification scales $10^{18.83(7)}$ GeV or $10^{18.97(9)}$ GeV is $\alpha_P = 0.166(7)$ or $\alpha_P = 0.321(46)$ for the $16_H + \overline{16}_H$ or $27_H + \overline{27}_H$ particle spectra, respectively. These values of the unified gauge coupling at the Planck scale are much larger than the conventional values of $\alpha_{GUT}$, and indeed are larger even than $\alpha_3(M_Z)$, however they are still in the perturbative regime.

Of course there are expected to be large threshold corrections coming from Planck scale physics which are not included in this analysis. Indeed, we would expect that QFT breaks down as we approach the Planck scale, so that the RG analysis ceases to be valid as we approach the Planck scale. The precise energy scale $E_{new}$ at which quantum field theories of gravity are expected to break down and new physics takes over is discussed in [63] based on estimates of the scale of violation of (tree-level) unitarity. An upper bound for this new physics energy scale is given by $E_{new}^2 = 20[G(\frac{2}{3}N_s + N_f + 4N_V)]^{-1}$ where $N_s$, $N_f$ and $N_V$ are the number of scalars, fermions and vectors respectively that gravity couples to. Assuming three low-energy 27 multiplets, $E_{new}$ would be equal to $10^{18.6}$ GeV which sets an upper bound for the scale at which the above quantum field theory analysis (and with any corrections from effective quantum gravity theories included) can no longer be trusted. In the above RGEs analysis the gauge coupling constants are predicted to be very close to one another at this scale, and if they are extrapolated, they will unify just below $M_p$. That is, the RGEs have been naively extrapolated up to $M_p$, even though new physics associated with quantum gravity must enter an order of magnitude below this. The fact that the two PS couplings are very close to each other at $E_{new}$, and are on a convergent trajectory must be regarded, at best, as a suggestive hint of a unification of the gauge fields with gravity in this approach. For other discussions of Planck scale unification of gauge coupling constants see for example [64].

## 3.7 Conclusions

This Chapter looked at how gauge coupling unification can be achieved in a supersymmetric model with three copies of 27 $E_6$ multiplets at low energies. It was found that, if the Standard Model gauge group becomes a left-right symmetric Pati-Salam gauge group near the conventional GUT scale, then unification at the planck scale is possible. The

[4]If the $D_{LR}$ symmetry is dropped then, with a minimal Pati-Salam Higgs content consisting of just $H_R$ and $\overline{H}_R$, the equation $\frac{5}{\alpha_1} = \frac{3}{\alpha_{2R}} + \frac{2}{\alpha_3}$ at the Pati-Salam scale would predict that the Pati-Salam symmetry is broken at $10^{14.4(1)}$ GeV and that unification would occur at $10^{19.72(15)}$ GeV.

motivation for considering this pattern of symmetry breaking was to find an alternative to including the additional electroweak doublets $H'$ and $\overline{H}'$ at the TeV scale which is used in the $E_6$SSM to achieve gauge coupling unification. A alternative method was sought because $H'$ and $\overline{H}'$ introduce a number of theoretical problems in the $E_6$SSM, namely the $\mu'$-problem and the $27', \overline{27}'$ splitting problem. No such problems should exist in a theory with just complete $E_6$ representations.

# Chapter 4

# The Minimal Exceptional Supersymmetric Standard Model

## 4.1 Motivation

The previous Chapter looked at how gauge coupling unification could be achieved in a supersymmetric theory with only three complete 27 $E_6$ multiplets surviving to low energies. This then paves a way for a new $E_6$ inspired supersymmetric model that can solve the $\mu$-problem of the MSSM but without the additional complications introduced by the additional $H'$ and $\overline{H}'$ states of the $E_6$SSM. However, for the model to solve the $\mu$-problem we must make sure that a MSSM singlet field $S$ and an additional $U(1)$ gauge group can survive to low energies. This is the topic of the present Chapter in which a Minimal $E_6$ Supersymmetric Standard Model (ME$_6$SSM) is proposed that is based on three low-energy 27 $E_6$ representations. This allows Planck scale unification and provides a solution to the $\mu$-problem and doublet-triplet splitting problem, without re-introducing either of these problems.

Above the conventional GUT scale the ME$_6$SSM is embedded into a left-right symmetric supersymmetric Pati-Salam model with an additional $U(1)$ gauge group, called $U(1)_\psi$, arising from an $E_6$ gauge group broken near the Planck scale. For simplicity the previous analysis in Section 3.6 assumed that the $U(1)_\psi$ gauge group was broken at the Planck scale. Here it is instead assumed that $U(1)_\psi$ remains unbroken down to $M_{GUT}$ and that below $M_{GUT}$ an additional $U(1)_X$ gauge group, consisting of a novel and non-trivial linear combination of $U(1)_\psi$ and two Pati-Salam generators, survives down to low energies. Eventually $U(1)_X$ is broken at the TeV scale by the same singlet that also generates the effective $\mu$ term, resulting in a new low-energy $Z'$ gauge boson.

The $U(1)_X$ group is not in general the same as the $U(1)_N$ group of the E$_6$SSM. However, both groups are low-energy $U(1)'$ groups that allow for a conventional see-saw mechanism since the right-handed neutrinos have zero charge. The $U(1)_X$ group of the ME$_6$SSM thus acts like the $U(1)_N$ group of the E$_6$SSM.

This Chapter is divided up as follows. In Section 4.2 the chain of symmetry breaking used to derive the ME$_6$SSM is described and the origin of the $U(1)_X$ symmetry is discussed. In Section 4.2.2 the two-loop renormalization group running of the gauge coupling constants of the ME$_6$SSM is calculated. In Section 4.3 the superpotential of the ME$_6$SSM is discussed and the suppression of proton decay is illustrated. Section 4.4 then discusses the phenomenology of the $Z'$ of the ME$_6$SSM and compares it to the $Z'$ of the E$_6$SSM to discover how they can be distinguished by their different couplings, which enables the two models to be resolved experimentally. Then finally Section 4.5 concludes the Chapter.

## 4.2 Chain of Symmetry Breaking

The two step pattern of gauge group symmetry breaking that is analysed in this Chapter is the following:

$$E_6 \overset{M_p}{\overbrace{\longrightarrow}} G_{4221} \times D_{LR} \overset{M_{GUT}}{\overbrace{\longrightarrow}} G_{3211} \tag{4.1}$$

where the gauge groups are defined by:

$$\begin{aligned} G_{4221} &\equiv SU(4) \times SU(2)_L \times SU(2)_R \times U(1)_\psi, \\ G_{3211} &\equiv SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_X \end{aligned} \tag{4.2}$$

and it has been assumed that the first stage of symmetry breaking happens close to the Planck scale and that the second stage happens close to the conventional GUT scale. The first stage of symmetry breaking is based on the maximal $E_6$ subgroup $SO(10) \times U(1)_\psi$ and the maximal $SO(10)$ subgroup $G_{422} \times D_{LR}$ corresponding to a Pati-Salam symmetry with $D_{LR}$ being a discrete left-right symmetry. The difference between the pattern of symmetry breaking assumed in this Section to that assumed in Section 3.5 is the inclusion of the $U(1)_\psi$ symmetry, which enables a $U(1)'$ group called $U(1)_X$ to appear after the $G_{4221}$ symmetry is broken.

The first stage of symmetry breaking close to $M_p$ will not be considered explicitly for the same reasoning given in Section 3.5, i.e. quantum field theory is expected to break-down near $M_p$. Under $E_6 \to G_{4221}$ the fundamental $E_6$ representation 27 decomposes as:

$$27 \to F + F^c + h + \mathcal{D} + S \tag{4.3}$$

where $F \equiv (4,2,1)_{\frac{1}{2}}$ contains one family of the left-handed quarks and leptons, $F^c \equiv (\overline{4},1,\overline{2})_{\frac{1}{2}}$ can contain one family of the charge-conjugated quarks and leptons, which includes a charge-conjugated neutrino; $h \equiv (1,2,2)_{-1}$ contains the MSSM Higgs doublets $h_u$ and $h_d$; $\mathcal{D} \equiv (6,1,1)_{-1}$ contains two Higgs triplets; and $S \equiv (1,1,1)_2$ is a MSSM singlet. The subscripts are related to the $U(1)_\psi$ symmetry's charge assignments which are discussed further in Appendix B.

The second stage of symmetry breaking close to $M_{GUT}$ is within the realm of conventional quantum field theory and requires some sort of Higgs sector in addition to the assumed matter content of three 27 representations of the gauge group $E_6$. In order to break the symmetry $G_{4221}$ to $G_{3211}$ at $M_{GUT}$, the minimal Higgs sector required is provided by the $G_{4221}$ representations $\overline{H}_R = (\overline{4},1,\overline{2})_{\frac{1}{2}}$ and $H_R = (4,1,2)_{-\frac{1}{2}}$.[1] These fields are the $G_{4221}$ equivalent to the $G_{422}$ fields $H_R = (4,1,2)$ and $\overline{H}_R = (\overline{4},1,\overline{2})$ described in Section 3.5.1. When these particles obtain VEVs in the right-handed neutrino directions $\langle \overline{H}_R \rangle = \langle \nu^c_H \rangle$ and $\langle H_R \rangle = \langle \nu^H_R \rangle$ they break the $SU(4) \times SU(2)_R \times U(1)_\psi$ symmetry to $SU(3)_c \times U(1)_Y \times U(1)_X$. Six of the off-diagonal $SU(4)$ and two of the off-diagonal $SU(2)_R$ fields receive masses related to the VEV of the Higgs bosons. The gauge bosons associated with the diagonal $SU(4)$ generator $T^{15}_4$, the diagonal $SU(2)_R$ generator $T^3_R$ and the $U(1)_\psi$ generator $T_\psi$, are rotated by the Higgs bosons to create one heavy gauge boson and two massless gauge bosons associated with $U(1)_Y$ and $U(1)_X$. The part of the symmetry breaking $G_{4221}$ to $G_{3211}$ involving the diagonal generators is then:

$$U(1)_{T^{15}_4} \times U(1)_{T^3_R} \times U(1)_\psi \to U(1)_Y \times U(1)_X. \tag{4.4}$$

Note that this is a generalization of the symmetry breaking found in the Standard Model $U(1)_{T^3_L} \times U(1)_Y \to U(1)_{em}$ described in Section 2.6.6 where $U(1)_{T^3_L}$ is the subgroup of $SU(2)_L$ that is associated with the diagonal generator $T^3_L$. The charges of the "right-handed neutrino" component of the Higgs which gets the VEV are:

$$\nu^H_R = \left( -\frac{1}{2}\sqrt{\frac{3}{2}},\ \frac{1}{2},\ -\frac{1}{2}\sqrt{\frac{1}{6}} \right) \tag{4.5}$$

[1] In Appendix B it is shown that the symmetry breaking $G_{4221}$ to $G_{3211}$ also requires an MSSM singlet $S$ from a 27 multiplet of $E_6$ to get a low-energy VEV. The VEV of this MSSM singlet is also used to solve the $\mu$ problem.

under the corresponding correctly $E_6$ normalized diagonal generators:[2]

$$T_4^{15} = \sqrt{\frac{3}{2}}\ diag(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{2}), \quad T_R^3 = \frac{1}{2}\ diag(1, -1), \quad T_\psi/\sqrt{6}. \tag{4.6}$$

Appendix B discusses the symmetry breaking in Eq.4.4 in detail. To simplify the discussion here it is observed that $T^{15} = \sqrt{\frac{3}{2}}\frac{(B-L)}{2}$ where $B$ and $L$ are the baryon and lepton number assignments of each Standard Model particle. The Higgs charges can then be written as

$$\nu_R^H = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \tag{4.7}$$

under the corresponding generators $T_{B-L} = \frac{B-L}{2}$, $T_R^3$ and $T_\psi$. It is then clear to see why the hypercharge generator $Y$ is preserved by the Higgs $H_R$ and $\overline{H}_R$ since

$$Y = T_R^3 + \frac{(B-L)}{2} \tag{4.8}$$

takes a zero value for the right-handed neutrino and anti-neutrino Higgs components which develop VEVs. The generator $Y$ thus leaves the vacuum invariant and its associated gauge field remains massless.

From the analysis in Section 3.5.2, Eq.4.8 provides a relation between the hypercharge gauge coupling constant $g_Y$ and the $SU(4)$ and $SU(2)_R$ gauge coupling constants $g_4$ and $g_{2R}$, which is given by Eq.3.3. The GUT normalized version of this relation is then given by Eq.3.4. This is a boundary condition for the gauge couplings at the Pati-Salam symmetry breaking scale, in this case $M_{GUT}$. Due to the left-right symmetry $D_{LR}$, at the $G_{4221}$ symmetry breaking scale we also have the additional boundary condition $\alpha_{2L} = \alpha_{2R}$, which is used in Eq.3.5.

### 4.2.1 The Additional Abelian Gauge Group

Hypercharge $Y$ is not the only Abelian generator that is preserved by this Higgs sector. The Higgs $H_R$ and $\overline{H}_R$ VEVs also preserve the combinations of generators $T_\psi + T_R^3$ and $T_\psi - T_{B-L}$ which together form the charge $X$ of the $U(1)_X$ group. This is discussed in Appendix B where the charge $X$ is chosen to be defined by:[3]

$$X = (T_\psi + T_R^3) - c_{12}^2 Y \tag{4.9}$$

[2]Note that the $E_6$ generators $G^a$ have been taken to be normalized by $Tr(G^aG^b) = 3\delta^{ab}$. It then follows that the Pati-Salam and standard model operators are conventionally normalized by $Tr(T^aT^b) = \frac{1}{2}\delta^{ab}$. The correctly normalized $E_6$ generator corresponding to $U(1)_\psi$ is $T_\psi/\sqrt{6}$ where $T_\psi$ corresponds to the charges in Eq.4.3. See Appendix B for more detail.

[3]Alternatively we could have defined $X$ to be $\mathbf{g}_{2R}^2(T_\psi + T_R^3) + \mathbf{g}_{B-L}^2(T_\psi - T_{B-L})$.

where $c_{12} = \cos\theta_{12}$ and the mixing angle is given by

$$\tan\theta_{12} = \frac{g_{2R}}{g_{B-L}}, \qquad g_{B-L} = \sqrt{\frac{3}{2}}\, g_4, \tag{4.10}$$

where the $E_6$ normalized Pati-Salam coupling constants $g_{2R}$ and $g_4$ are evaluated at the $G_{4221}$ symmetry breaking scale $M_{GUT}$. Note that this Abelian generator $X$ depends on the values that the Pati-Salam coupling constants take at a particular energy scale. It is easy to prove that it is a general rule that, if three massless gauge fields are mixed, then at least two of the resulting mass eigenstate fields must have a charge that depends on the value of the original gauge coupling constants. See Appendix B for more discussion on this unusual aspect of $X$.

The gauge coupling constant $g_X^0$ of $U(1)_X$ may be expressed in terms of the $SU(4)$, $SU(2)_R$ and $U(1)_\psi$ gauge coupling constants $g_4, g_{2R}$ and $g_\psi$ as:

$$\frac{1}{\alpha_X^0} = \frac{1}{\frac{1}{6}\alpha_\psi} + \frac{1}{\frac{3}{2}\alpha_4 + \alpha_{2R}} \tag{4.11}$$

where $\alpha_X^0 = \frac{(g_X^0)^2}{4\pi}$, $\alpha_{2R} = \frac{g_{2R}^2}{4\pi}$; $\alpha_4 = \frac{g_4^2}{4\pi}$; and $\alpha_\psi = \frac{g_\psi^2}{4\pi}$.

Just as $T_Y \equiv \sqrt{\frac{3}{5}}Y$ is the GUT normalized hypercharge, we can define a GUT (in this case $E_6$) normalized generator for $X$ as:

$$T_X = \frac{1}{N_X}X \tag{4.12}$$

where, from the discussion in Appendix B, the normalization constant $N_X$ is given by:

$$N_X^2 \equiv 7 - 2c_{12}^2 + \frac{5}{3}c_{12}^4. \tag{4.13}$$

In terms of the $E_6$ normalized generator $T_X = X/N_X$, the normalized gauge coupling constant $g_X$ is defined by $g_X \equiv g_X^0 N_X$ so that $\alpha_X = \alpha_X^0 N_X^2$. Thus Eq.4.11 can be written as:

$$\frac{N_X^2}{\alpha_X} = \frac{6}{\alpha_\psi} + \frac{2}{3\alpha_4 + 2\alpha_{2R}}. \tag{4.14}$$

This boundary condition applies at the symmetry breaking scale $M_{GUT}$ and is analogous to the boundary condition for the normalized hypercharge gauge coupling constant $g_1$ given by Eq.3.5. Table 4.1 lists the values that the generators $Y$, $T_{B-L}$, $T_R^3$, $T_\psi$ and $T_\psi + T_R^3$ (and therefore $X$) take for the $G_{3211}$ representations of the 27 multiplet. Note that both $T_\psi + T_R^3$ and $Y$ are zero for $\nu^c$ and therefore neither $B_Y$ or $B_X$ couple to right-handed neutrinos. This is a consequence of Goldstone's theorem [65] since the right-handed neutrino comes from the same $G_{4221}$ representation as the Higgs boson component that gets a VEV to break the symmetry.

| | $Q$ | $L$ | $u^c$ | $d^c$ | $e^c$ | $\nu^c$ | $h_u$ | $h_d$ | $D$ | $\overline{D}$ | $S$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_{B-L}$ | $\frac{1}{6}$ | $-\frac{1}{2}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $0$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | $0$ |
| $T_R^3$ | $0$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $0$ | $0$ |
| $T_\psi$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $-1$ | $-1$ | $-1$ | $2$ |
| $Y$ | $\frac{1}{6}$ | $-\frac{1}{2}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | $1$ | $0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | $0$ |
| $T_\psi + T_R^3$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $1$ | $1$ | $0$ | $-\frac{1}{2}$ | $-\frac{3}{2}$ | $-1$ | $-1$ | $2$ |

TABLE 4.1: List of the $T_{B-L} \equiv \frac{B-L}{2}$, $T_R^3$, $T_\psi$, hypercharge $Y \equiv T_{B-L} + T_R^3$, and $T_\psi + T_R^3$ charge assignments for the $G_{3211}$ representations of the 27 multiplet of $E_6$.

The $U(1)_X$ associated with the preserved generator in Eq.4.9 is an anomaly-free gauge group which plays the same role in solving the $\mu$ problem as the $U(1)_N$ of the E$_6$SSM since it allows the coupling $Sh_uh_d$ that generates an effective $\mu$ term, while forbidding $S^3$ and the $\mu h_u h_d$ (see Section 2.6). $U(1)_X$ is broken by the $S$ singlet VEV near the TeV scale, yielding a physical $Z'$ which may be observed at the LHC. It also allows for a conventional see-saw mechanism since right-handed neutrinos have zero $U(1)_X$ charge.

### 4.2.2 Two-Loop RGEs Analysis for $U(1)_X$

The previous Chapter investigated the SUSY two-loop RG analysis corresponding to the pattern of symmetry breaking discussed in Section 3.5. For simplicity this analysis excluded the running of the gauge coupling constant of $U(1)_X$. The running of this gauge coupling constant is important however to determine its value at the electroweak scale, which is required to understand its phenomenology. It will also affect the running of the other gauge coupling constants at the two-loop order. However it is shown that this effect is negligible and so the results of Section 3.6 are not significantly modified, and it was a good approximation to ignore it.

This Section performs a SUSY two-loop RG analysis of the gauge coupling constants, corresponding to the pattern of symmetry breaking discussed in the Section 4.2. It is assumed that there are three complete 27 SUSY representations of the gauge group $E_6$ which survive down to low energies. Above the $G_{4221}$ symmetry breaking scale $M_{GUT}$, the minimal left-right symmetric Higgs sector capable of breaking the $G_{4221}$ symmetry consists of the $SO(10) \times U(1)_\psi$ Higgs states $(16_H)_{\frac{1}{2}}$ and $(\overline{16}_H)_{-\frac{1}{2}}$, where $(16_H)_{\frac{1}{2}} = (4, 2, 1)_{\frac{1}{2}} + (\overline{4}, 1, \overline{2})_{\frac{1}{2}}$ and $(\overline{16}_H)_{-\frac{1}{2}} = (\overline{4}, \overline{2}, 1)_{-\frac{1}{2}} + (4, 1, 2)_{-\frac{1}{2}}$ is assumed in addition to the three 27 representations. The components which get VEVs are $\overline{H}_R = (\overline{4}, 1, \overline{2})_{\frac{1}{2}}$ and $H_R = (4, 1, 2)_{-\frac{1}{2}}$. If complete $E_6$ multiplets are demanded in the entire theory below $M_p$, then the Pati-Salam breaking Higgs sector at $M_{GUT}$ may be assumed to be $27_H$ and $\overline{27}_H$. For the analysis which involves the $16_H + \overline{16}_H$ states, the rest of the

$SO(10)$ representations that together with the $16_H + \overline{16}_H$ states make up complete $E_6$ representations (such as a 27 and $\overline{27}$) are assumed to be at or above the $E_6$ breaking scale and so do not affect the running of the gauge coupling constants below the unification scale.

The running of the gauge coupling constants at two-loops is therefore investigated for an $E_6$ theory that contains three complete 27 multiplets at low energies and either the $SO(10) \times U(1)_\psi$ states $(16_H)_{\frac{1}{2}} + (\overline{16}_H)_{-\frac{1}{2}}$ or the $E_6$ states $27_H + \overline{27}_H$ above the $G_{4221}$ symmetry breaking scale. The $E_6$ symmetry is assumed to be broken to a left-right symmetric $G_{4221}$ gauge symmetry, which is then broken to the Standard Model gauge group and a $U(1)_X$ group as described in Section 4.2.

As discussed in Section 3.5.2 the relation in Eq.3.4 between the hypercharge and Pati-Salam gauge coupling constants at the $G_{4221}$ symmetry breaking scale can be turned into a boundary condition involving purely Standard Model gauge couplings constants at the $G_{4221}$ breaking scale as given by Eq.3.5. This is because $SU(3)_c$ comes from $SU(4)$ so $\alpha_3 = \alpha_4$ at this scale, and, as remarked, the $D_{LR}$ symmetry requires that $\alpha_{2R} = \alpha_{2L}$ at the $G_{4221}$ symmetry breaking scale. It is also argued that having specified the low energy matter content and thresholds, Eq.3.5 allows a unique determination of the Pati-Salam breaking scale, by running up the gauge couplings until the condition is satisfied.

However the symmetry breaking pattern is slightly different in this Chapter since a $U(1)_X$ symmetry has been included at two-loops the running of the $U(1)_X$ gauge coupling constant will change the running of the Standard Model gauge coupling constants, and the charge of the $U(1)_X$ group $T_X$ depends on the values that the $g_4$ and $g_{2R}$ coupling constants take at the $G_{4221}$ symmetry breaking scale, which is written into the cosine $c_{12}$ of the mixing angle $\tan\theta_{12} \equiv g_{2R}/g_{B-L}$. This upsets the unique determination of the Pati-Salam scale using Eq.3.5.

The running of the gauge coupling constants to two-loops is therefore calculated as follows: First the $U(1)_X$ and $U(1)_\psi$ symmetries are ignored and the two-loop running found in Section 3.6 is used to determine the unification scale. Using one-loop RGEs, the $U(1)_\psi$ gauge coupling is then run down from this unification scale and the $U(1)_X$ gauge coupling is determined at the $G_{4221}$ symmetry breaking scale $M_{GUT}$ from the boundary condition in Eq.4.11. The $U(1)_X$ gauge coupling is then run down to the TeV scale to give a value for the $U(1)_X$ gauge coupling constant at low energies (unlike the Standard Model gauge couplings we do not know the value of the $U(1)_X$ value at low-energies since it has not been observed).

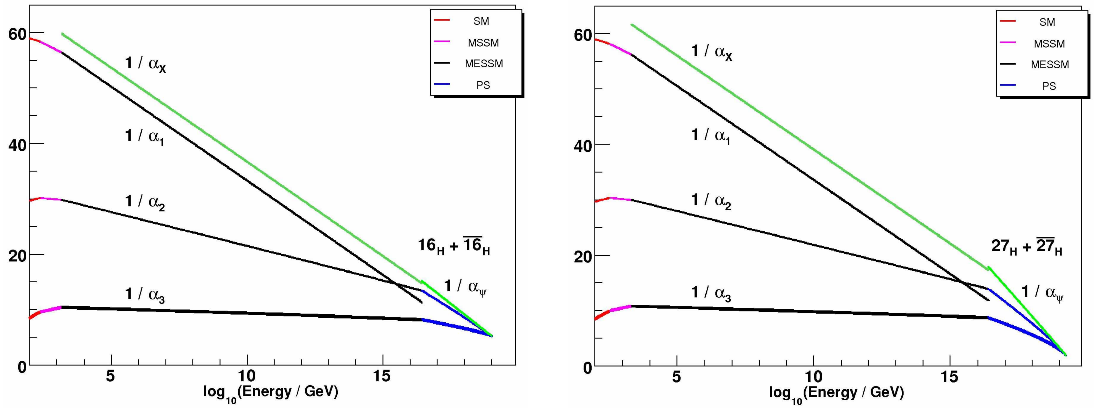



FIGURE 4.1: Two-loop RGEs running of the gauge coupling constants in two ME$_6$SSM toy models that are described in the main body of the text. The thickness of the lines represents the experimental uncertainty in the initial values of the coupling constants. The blue lines represent the Pati-Salam inverse gauge coupling constants $1/\alpha_4$ and $1/\alpha_{2L} = 1/\alpha_{2R}$. Near to the conventional GUT scale the $SU(3)_c$ gauge coupling constant $\alpha_3$ becomes the $SU(4)$ gauge coupling constant $\alpha_4$, and the $SU(2)_R$ and $U(1)_\psi$ gauge coupling constants $\alpha_{2R}$ and $\alpha_\psi$ are the combination of the $SU(3)_c$, $U(1)_Y$ and $U(1)_X$ gauge coupling constants $\alpha_3$, $\alpha_1$ and $\alpha_X$ given by Eq.3.4 and Eq.4.14.

Given this one-loop value for the $U(1)_X$ gauge coupling constant at the TeV scale, all the gauge coupling constants are then run up to the unification scale using two-loop RGEs. Since this is performed at the two-loop order however, the running of the $U(1)_X$ symmetry will affect the running of the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge coupling constants so that the values for $g_4$ and $g_{2R}$ calculated from Eq.3.5 will now differ from those found when we ignored the $U(1)_X$ symmetry, and the unification scale will be slightly modified. Using the new values for $g_4$ and $g_{2R}$ the process is repeated by re-calculating $T_X$ and running the $U(1)_\psi$ gauge coupling using one-loop RGEs down from the new unification scale to determine the value of the $U(1)_X$ gauge coupling constant at low energies. Again this new value is used to re-calculate the running of the gauge couplings to two-loops and determine the unification scale. This process is repeated until the $g_4$ and $g_{2R}$ values and unification scale no longer change to within four significant figures.

After this recursion of the two-loop RGE analysis it is calculated that, with either $(16_H)_{\frac{1}{2}} + (\overline{16}_H)_{-\frac{1}{2}}$ or $27_H + \overline{27}_H$ included above the $G_{4221}$ symmetry breaking scale, $c_{12}^2$ is equal to 0.71 to two significant figures. However, for convenience the physical value of $c_{12}^2$ is taken to be equal to $\frac{5}{7}$ ($\approx 0.71$) so that $T_X$ can be written in terms of fractions. Using this value of $c_{12}^2$ in equation Eq.B.8, $T_X$ is calculated for all the standard model particles of the three low-energy 27 multiplets. The values that $T_X$, $T_Y$ and $T_N$ take for the particles of the 27 multiplets are given in Table 4.2, where $T_N$ is the generator associated with the $U(1)_N$ group in the E$_6$SSM.

The results are shown in Fig.4.1. The left-panel illustrates the running of the gauge coupling constants for the $E_6$ theory that contains the $SO(10) \times U(1)_\psi$ states $(16_H)_{\frac{1}{2}} + (\overline{16}_H)_{-\frac{1}{2}}$ particles. A low-energy effective threshold of 250 GeV for the MSSM states is used in this model and therefore an effective threshold of $(6 \times 250) = 1.5$ TeV is assumed for the rest of states of the three complete 27 multiplets. This was also assumed in Section 3.6, which follows the analysis of effective MSSM thresholds from [57]. The right-panel of Fig.4.1 is for the $E_6$ theory that contains the $E_6$ states $27_H + \overline{27}_H$. The MSSM threshold must be increased to 350 GeV (and hence the 1.5 TeV threshold is increased to 2.1 TeV) in this model to ensure unification for the gauge coupling constants of the $G_{4221}$ symmetry.

The gauge couplings are run up from low energies to high energies, using as input the SM gauge coupling constants measured on the Z-pole at LEP, which are as follows [26]: $\alpha_1(M_Z) = 0.016947(6)$, $\alpha_2(M_Z) = 0.033813(27)$ and $\alpha_3(M_Z) = 0.1187(20)$. The general two-loop beta functions used to run the gauge couplings are described in Appendix A. Using a two-loop renormalization group analysis, it is calculated that the $G_{4221}$ symmetry is broken at $10^{16.45(3)}$ GeV or $10^{16.40(3)}$ GeV and that gauge coupling unification occurs at $10^{18.95(8)}$ GeV or $10^{19.10(10)}$ GeV for the models that contain the $SO(10) \times U(1)_\psi$ states $(16_H)_{\frac{1}{2}} + (\overline{16}_H)_{-\frac{1}{2}}$ or $E_6$ states $27_H + \overline{27}_H$ respectively.

The value of the gauge coupling constant at the unification scales $10^{18.95(8)}$ GeV or $10^{19.10(10)}$ GeV is $\alpha_P = 0.183(10)$ or $\alpha_P = 0.432(121)$ for the $(16_H)_{\frac{1}{2}} + (\overline{16}_H)_{-\frac{1}{2}}$ or $27_H + \overline{27}_H$ particle spectra, respectively. The values of the unified gauge coupling at the Planck scale are much larger than the conventional values of $\alpha_{GUT}$ and indeed are larger even than $\alpha_3(M_Z)$, however they are still in the perturbative regime. Of course there are expected to be large threshold corrections coming from Planck scale physics which are not included in this analysis.

In terms of a logarithmic scale, the Pati-Salam symmetry breaking scale and unification scale have not been significantly changed from the results of Section 3.6 which ignored the $U(1)_X$ and $U(1)_\psi$ symmetries. Planck scale unification and a GUT scale Pati-Salam symmetry breaking are still predicted.

## 4.3 The ME$_6$SSM

In this Section a realistic ME$_6$SSM is formulated, focussing on the model building issues. The ME$_6$SSM has a more 'minimal' particle content than the E$_6$SSM since it only contains three complete 27 multiplets at low energies, whereas the E$_6$SSM contains two additional EW doublets which can be considered as states of incomplete 27 and $\overline{27}$ $E_6$

| | $Q$ | $L$ | $u^c$ | $d^c$ | $e^c$ | $\nu^c$ | $h_u$ | $h_d$ | $D$ | $\overline{D}$ | $S$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y$ | $\frac{1}{6}$ | $-\frac{1}{2}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $X$ | $\frac{8}{21}$ | $\frac{6}{7}$ | $\frac{10}{21}$ | $\frac{16}{21}$ | $\frac{2}{7}$ | 0 | $-\frac{6}{7}$ | $-\frac{8}{7}$ | $-\frac{16}{21}$ | $-\frac{26}{21}$ | 2 |
| $N$ | 1 | 2 | 1 | 2 | 1 | 0 | -2 | -3 | -2 | -3 | 5 |
| $T_Y$ | 0.129 | -0.387 | -0.516 | 0.258 | 0.775 | 0 | 0.387 | -0.387 | -0.258 | 0.258 | 0 |
| $T_X$ | 0.150 | 0.338 | 0.188 | 0.301 | 0.113 | 0 | -0.338 | -0.451 | -0.301 | -0.489 | 0.789 |
| $T_N$ | 0.158 | 0.316 | 0.158 | 0.316 | 0.158 | 0 | -0.316 | -0.474 | -0.316 | -0.474 | 0.791 |

TABLE 4.2: The values that the charges $Y$, $X$ and $N$ take for the all the $G_{3211}$ representations of the $E_6$ 27 multiplet. $Y$ is hypercharge, $X$ is the charge of $U(1)_X$ for the model presented in Section 4.2.2, and $N$ is the charge associated with the $U(1)_N$ group in the E$_6$SSM. The respective GUT normalized charges $T_Y$, $T_X$ and $T_N$ are also given. $N_X$ and $X$ have been calculated for the case when $c_{12}^2 = 5/7$ which, to two significant figures, agrees with the RGEs analysis in Section 4.2.2.

multiplets. From the previous RGEs analysis, unification of the $G_{4221}$ gauge coupling constants occurs near the Planck scale where an $E_6$ symmetry should in principle exist. However, given the expected strength of quantum gravity at this scale, it is likely that any such $E_6$ symmetry is for all practical purposes broken by gravitational effects. Therefore, the model that is proposed in this Section is chosen to not respect an $E_6$ symmetry but instead obey the $G_{4221}$ symmetry that exists between the conventional GUT and Planck scales where quantum gravity effects are anticipated to not be so significant. $G_{4221}$ must be a symmetry of the model since its RGEs were used to determine the scale of gauge coupling unification in Section 4.2.2. The E$_6$ symmetry on the other hand was never used.

Under $E_6 \to SO(10) \times U(1)_\psi \to G_{4221}$, the fundamental $E_6$ representation breaks into the following: $27 \to 16_{\frac{1}{2}} + 10_{-1} + 1_2 \to F + F^c + h + \mathcal{D} + S$. Including three families contained in three $27_i$ reps, then, without further constraints on the theory, the allowed couplings are contained in the tensor products:

$$27_i 27_j 27_k \to F_i F_j^c h_k + F_i F_j \mathcal{D}_k + F_i^c F_j^c \mathcal{D}_k + S_i h_j h_k + S_i \mathcal{D}_j \mathcal{D}_k \tag{4.15}$$

where $i, j, k = 1 \ldots 3$ are family indices. However not all these terms are desirable since the presence of extra Higgs doublets can give rise to flavour changing neutral currents (FCNCs) and the presence of light Higgs triplets can induce proton decay. Therefore extra symmetries are required to control the couplings, a suitable choice being the R-symmetry and the discrete $Z_2^H$ symmetry displayed in Table 4.3, which reduces the allowed couplings to those shown in Table 4.4, where the lowest order non-renormalizable terms are displayed. The physics of the allowed and suppressed terms are now discussed.

| field | $SU(4) \times SU(2)_L \times SU(2)_R \times U(1)_\psi$ | $U(1)_R$ | $Z_2^H$ |
|---|---|---|---|
| $F_i$, $F_i^c$ | $(4,2,1)_{\frac{1}{2}}$, $(\overline{4},1,2)_{\frac{1}{2}}$ | 1 | $-$ |
| $h_3$, $h_\alpha$ | $(1,2,2)_{-1}$ | 0 | $+,-$ |
| $S_3$, $S_\alpha$ | $(1,1,1)_2$ | 2 | $+,-$ |
| $\mathcal{D}_i$ | $(6,1,1)_{-1}$ | 0 | $-$ |
| $H_L$, $\overline{H}_R$ | $(4,2,1)_{\frac{1}{2}}$, $(\overline{4},1,2)_{\frac{1}{2}}$ | 2 | $+$ |
| $\overline{H}_L$, $H_R$ | $(\overline{4},2,1)_{-\frac{1}{2}}$, $(4,1,2)_{-\frac{1}{2}}$ | 0 | $+$ |

TABLE 4.3: All the charge assignments for the $G_{4221}$ representations of the M$E_6SSM$, where $i = 1 \dots 3$ is a family index and $\alpha = 1, 2$. The $U(1)_R$ is an R-symmetry and $Z_2^H$ distinguishes the third family Higgs which get VEVs. The superpotential terms that are allowed by these symmetries are given in Table 4.4. The $h_3$ supermultiplet contains the MSSM Higgs bosons and $S_3$ is the MSSM singlet that generates an effective $\mu$-term.

### 4.3.1 Suppressed Flavour Changing Neutral Currents

The $F_iF_j^ch_3$ superpotential terms are taken to contain the MSSM Yukawa couplings since, as in the E$_6$SSM, the third generation $h_3$ is assumed to contain the MSSM-like Higgs doublets $h_u$ and $h_d$ that gain electroweak VEVs. The other $h_\alpha$ states are taken to not get VEVs and will cause FCNCs unless the superpotential term $F_iF_j^ch_\alpha$, where $\alpha = 1, 2$, is forbidden or highly suppressed by some new symmetry [17]. These terms are forbidden using a $Z_2^H$ discrete symmetry that respects the $G_{4221}$ symmetry but not the Planck-scale $E_6$ symmetry since the latter is assumed to be broken by quantum gravity.

Under this $Z_2^H$ symmetry the 'matter particles' $F_i$ and $F_i^c$ and 'non-Higgs' particles $h_\alpha$ are taken to have $Z_2^H = -1$ and the MSSM Higgs doublets from $h_3$ are assumed to have $Z_2^H = +1$. The FCNC inducing terms $F_iF_j^ch_\alpha$ are therefore forbidden by the $Z_2^H$ symmetry and the effective MSSM superpotential terms $F_iF_j^ch_3$ are allowed. The fact that only the third generation of Higgs doublets $h_3$ couple strongly to the quark and leptons could explain why only these electroweak doublet fields gain VEVs as discussed in Section 2.6.4 for the E$_6$SSM.

The $Z_2^H$ symmetry used here forbids the FCNCs in the same way that the $Z_2^H$ symmetry of the E$_6$SSM forbids the FCNCs from the $h_\alpha$ 'non-Higgs' particles in that model [17]. However, it is shown later that, although the $F_iF_j^ch_\alpha$ terms are forbidden at the renormalizable level by $Z_2^H$, they are still generated from non-renormalizable terms, which are heavily suppressed so that the induced FCNCs are not significant.

Note that the $Z_2^H$ does not commute with an E$_6$ symmetry if all the ME$_6$SSM sates come from only complete representations of E$_6$. It is assumed that the $E_6$ symmetry may not be respected by low-energy symmetries as it is broken by quantum gravity effects. For example if the ME$_6$SSM sates come from four incomplete 27 representations then $Z_2^H$ will commute with E$_6$, which could be explained by compactification of higher dimensions in a quantum gravity theory such as string theory. Such higher energy effects are not considered here however.

### 4.3.2 The $\mu$-Term and Exotic Mass Terms

Following the E$_6$SSM, only the third generation of the $S_i$ states is assumed to get a vacuum expectation value so that the $S_3h_3h_3$ term, from the $G_{4221}$ superpotential term $S_ih_jh_k$, will generate an effective MSSM $\mu$-term. For this term to be allowed by the $Z_2^H$ symmetry, the $S_3$ particles must have $Z_2^H = +1$. No Goldstone boson is created by the VEV of $S_3$ since it is charged under the local $U(1)_X$ group. Instead a $Z'$ boson is created whose phenomenology is discussed in Section 4.4.

In addition to solving the $\mu$-problem of the MSSM, the little hierarchy problem of the MSSM should also be resolved by the ME$_6$SSM. This is because there are extra particles below the conventional GUT scale of $10^{16}$ GeV that are not contained in the MSSM. These extra particles are from the three copies of the 27 $E_6$ multiplet and form two copies of a $5+\overline{5}$ of the $SU(5)$ subgroup of $E_6$, and one Higgs triplet particle. Due to Renormalization Group effects, the extra states increase the value of the Yukawa coupling constant for $S_3h_3h_3$ at low energies, and hence increase the mass of the lightest CP even Higgs boson [17].

The $S_3$ particle is also used to give mass to the 'non-Higgs' particles $h_\alpha$ and Higgs triplet particles $\mathcal{D}_i$ via the terms $S_3h_\alpha h_\beta$ and $S_3\mathcal{D}_i\mathcal{D}_j$ respectively where $\beta = 1, 2$. For general $U(1)'$ models, the $S_3\mathcal{D}_i\mathcal{D}_j$ superpotential term has been shown to induce a VEV for the singlet $S_3$ so that it can generate an effective $\mu$-term [52, 53]. From Table 4.4 the $S_\alpha\mathcal{D}_i\mathcal{D}_j$ and $S_\alpha h_\beta h_\gamma$ (where $\gamma = 1, 2$) superpotential terms are forbidden at tee-level so that the $S_\alpha$ particles should not acquire VEVs. These $S_\alpha$ particles will instead get mass from the $S_\alpha h_\beta h_3$ superpotential terms where $S_\alpha$ has $Z_2^H = -1$. This is exactly the same as in the E$_6$SSM, which was reviewed in Section 2.6.6.

### 4.3.3 Exotic Decay and Suppressed Proton Decay

The remaining $G_{4221}$ superpotential terms to be discussed from Eq.4.15 are $F_iF_j\mathcal{D}_k$ and $F_i^cF_j^c\mathcal{D}_k$. These will cause rapid proton decay in this model unless they are highly suppressed or forbidden by some symmetry [17, 38]. Under $G_{4221} \to G_{3211}$ these terms decompose to the following:

$$\begin{aligned} FF\mathcal{D} &\to QQD + QL\overline{D} \\ F^cF^c\mathcal{D} &\to u^cd^c\overline{D} + u^ce^cD + d^c\nu^cD \end{aligned} \tag{4.16}$$

| **Allowed couplings** | **Physics** |
|---|---|
| $F_iF_j^ch_3$ | MSSM superpotential |
| $S_3h_3h_3$ | Effective MSSM $\mu$-term |
| $S_3h_\alpha h_\beta$ | $h_\alpha$ mass |
| $S_3\mathcal{D}_i\mathcal{D}_j$ | $\mathcal{D}_i$ mass |
| $S_\alpha h_\beta h_3$ | $S_\alpha$ mass |
| $\frac{1}{M_p}\Sigma(F_iF_j\mathcal{D}_k + F_i^cF_j^c\mathcal{D}_k)$ | Allows $\mathcal{D}_i$ and proton decay |
| $\frac{1}{M_P}\Sigma F_iF_j^ch_\alpha$ | Heavily suppressed FCNCs |
| $\frac{1}{M_P}\Sigma S_\alpha\mathcal{D}_i\mathcal{D}_j$ | Harmless |
| $\frac{1}{M_P}\Sigma S_\alpha h_\beta h_\gamma$ | Harmless |
| $\frac{1}{M_P}\Sigma S_\alpha h_3h_3$ | Harmless |
| $\frac{1}{M_p}F_i^cF_j^cH_RH_R$ | $\nu^c$ mass |
| $\frac{1}{M_p}F_iF_j\overline{H}_L\overline{H}_L$ | Harmless |

Table 4.4: The $G_{4221}$ superpotential terms that are obtained from all the renormalizable and first-order non-renormalizable $E_6$ tensor products of $27_i$, the $SO(10)\times U(1)_\psi$ states $(16_H)_{\frac{1}{2}} + (\overline{16}_H)_{-\frac{1}{2}}$ (that are assumed to derive from a $27+\overline{27}$), and $\Sigma$, that are allowed by the $Z_2^H$ and $U(1)_R$ symmetries of the ME$_6$SSM. $i,j,k = 1\ldots3$ and $\alpha,\beta,\gamma = 1,2$ are family indices.

where $D = (3,1)_{-\frac{1}{3}}$, $\overline{D} = (\overline{3},1)_{\frac{1}{3}}$ and the family indices and coupling constants have been dropped for ease of notation. These operators are also found in the E$_6$SSM and are separated into the superpotentials $W_1$ and $W_2$ in Eq.2.28 in Section 2.6.7. In the E$_6$SSM a $Z_2^L$ or $Z_2^B$ discrete symmetry forbids $W_1$ or $W_2$ respectively as discussed in detail in Section 2.6.7. This forbids the otherwise induced proton decay and also allows the $D$ and $\overline{D}$ states to decay. Unfortunately however we cannot use the $Z_2^L$ or $Z_2^B$ symmetry in the ME$_6$SSM since they do not commute with the Pati-Salam gauge symmetry.[4] Therefore a different method to avoid rapid proton decay is required.

The $F_iF_j\mathcal{D}_k$ and $F_i^cF_j^c\mathcal{D}_k$ superpotential terms cannot be forbidden altogether since the $\mathcal{D}_i$ particles would become stable, strongly interacting particles with TeV scale masses. Such particles cannot exist in nature and in fact could potentially cause problems for nucleosynthesis even if they are unstable with a lifetime greater than just 0.1s [56]. This was discussed in more detail in Section 2.6.7 for the E$_6$SSM. Forbidding $F_iF_j\mathcal{D}_k$ over $F_i^cF_j^c\mathcal{D}_k$ or vice versa would not help either since both terms contain parts of $W_1$ *and* $W_2$ as illustrated by Eq.4.16.

The Standard Model representations of $\mathcal{D}_k$ are often found to some degree in other GUTs and the rapid proton decay problems are often solved using some doublet-triplet splitting mechanism that gives large (above the GUT scale) mass to the analogue of the $\mathcal{D}_i$ (triplet) particles, but EW mass to the Higgs doublets. Section 2.4.5 describes

[4] $Z_2^L$ and $Z_2^B$ can commute with $G_{4221}$ if the quarks and leptons are taken to come from separate $F$ and $F^c$ representations. This would be difficult to explain using a conventional field theory however.

such a mechanism in more detail. However, in this model we cannot give a large mass to the $\mathcal{D}_i$ particles because gauge anomalies would then exist, due to the $U(1)_X$ group, and Planck scale unification would be lost. Also, as discussed above, the $\mathcal{D}_i$ particles can be used to help induce a VEV for the $S_3$ particle, around the TeV scale, if they contribute to the low energy theory. We must therefore highly suppress the $F_iF_j\mathcal{D}_k$ and $F_i^cF_j^c\mathcal{D}_k$ superpotential terms using a small Yukawa coupling constant rather than using the general GUT method of creating large $\mathcal{D}_i$ masses. In this case the Yukawa couplings of the quarks and leptons to Higgs doublets and Higgs triplets are 'split' rather than their masses. To achieve this the $F_iF_j\mathcal{D}_k$ and $F_i^cF_j^c\mathcal{D}_k$ superpotential terms are forbidden at the tree-level but generated from the non-renormalizable terms $\Sigma F_iF_j\mathcal{D}_k$ and $\Sigma F_i^cF_j^c\mathcal{D}_k$, where $\Sigma$ is an $E_6$ singlet, by taking both $\Sigma$ and $\mathcal{D}_i$ to have $Z_2^H = -1$. These non-renormalizable superpotential terms are expected to survive from the Planck scale and so will likely be suppressed by a factor of $1/M_p$. We can therefore control the degree of suppression of the $F_iF_j\mathcal{D}_k$ and $F_i^cF_j^c\mathcal{D}_k$ terms by choosing the energy scale at which $\Sigma$ gets a VEV. The level of suppression, and therefore the $\Sigma$ VEV scale, must be such that the induced proton decay has a rate smaller than present experimental limits, but the $\mathcal{D}_i$ states still decay faster than $0.1s$.

In Section 4.3.3.1 the minimum level of suppression required for the proton's lifetime to be within experimental limits is estimated. This is then compared to the maximum level of suppression required for the $\mathcal{D}_i$ particles' lifetime to be greater than $0.1s$ which is estimated in Section 4.3.3.2.

#### 4.3.3.1 Proton Decay

The superpotential terms $\lambda FF\mathcal{D}$ and $\lambda F^cF^c\mathcal{D}$ (with the family indices dropped for simplicity) cause proton decay through $d = 5$ and $d = 6$ operators [36, 66], and the most stringent experimental limits on the proton's lifetime are set by the $d = 5$ $p \to K^+\overline{\nu}$ and $d = 6$ $p \to \pi^0 e^+$ decay channels, which are $1.6 \times 10^{33}$ years and $5.0 \times 10^{33}$ years respectively [26]. The $d = 6$ operators are found in all simple GUTs (including non-SUSY GUTs) and a dimensional analysis estimate for the decay width of the proton is [36]:

$$\Gamma_p \approx |\lambda_{\mathcal{D}u}\lambda_{\mathcal{D}d}|^2 \frac{m_p^5}{m_{\mathcal{D}}^4} \tag{4.17}$$

where $m_{\mathcal{D}}$, $m_p$ are the mass of the $\mathcal{D}_i$ particles and proton respectively; and $\lambda_{\mathcal{D}u}$, $\lambda_{\mathcal{D}d}$ are the strength of couplings between the $\mathcal{D}_i$ mass eigenstate and the up quark (and charged lepton) and down quark mass eigenstates. Taking $m_{\mathcal{D}} = 1.5$ TeV in Eq.4.17 for example requires that $\lambda \lesssim 10^{-13}$ for the proton's lifetime to be greater than $5.0 \times 10^{33}$ years in the approximation that $\lambda_{\mathcal{D}u} = \lambda_{\mathcal{D}d} \equiv \lambda$. In the ME$_6$SSM this suppression $\lambda$ will be approximately given by $\langle\Sigma\rangle/M_p$ so that $\langle\Sigma\rangle \lesssim 10^6$ GeV for the $d=6$ decay $p \to \pi^0 e^+$ to be within experimental limits. Of course this is only a rough order of magnitude estimate and assumes that the operators $\frac{1}{M_p}\Sigma F F \mathcal{D}$ and $\frac{1}{M_p}\Sigma F^c F^c \mathcal{D}$ represent the interactions between the $\mathcal{D}$, quark and lepton mass eigenstates.

The $d=5$ operators are only found in SUSY GUTs since they contain the coloured partners to the higgsinos (the 'triplet higgsinos') and must be dressed with squarks and sleptons to generate proton decay [36]. These operators only exist if the supersymmetric theory contains a mass term that mixes the coloured partners to the up higgsinos with the coloured partners to the down higgsinos. In the ME$_6$SSM this mass term is provided by $S_3\mathcal{D}_i\mathcal{D}_j$ once $S$ gets a VEV. The matrix element for the $d=5$ decay channel $p \to K^+\overline{\nu}$ can be found in [36, 67] and is proportional to $\lambda_{\mathcal{D}u}\lambda_{\mathcal{D}d}/m_{\mathcal{D}}m_{SUSY}$ where $m_{SUSY}$ is the mass scale for the Standard Model's superpartners. For the lifetime of the proton to be within experimental limits, it was found that $m_{\mathcal{D}} \gtrsim 7.6 \times 10^{16}$ GeV was required for a Yukawa suppression of order $\lambda_{hu}\lambda_{hd}$ where $\lambda_{hu}$ and $\lambda_{hd}$ are the Yukawa couplings of the up and down quark with the SUSY Higgs fields [67]. This suppression can be estimated as $m_u m_d/m_t m_b \approx 10^{-10}$, which sets an upper limit for the Yukawa suppression used in [67]. This result can then be scaled to obtain an estimate for the suppression required in the case that $m_{\mathcal{D}} = 1.5$ TeV rather than $m_{\mathcal{D}} \gtrsim 7.6 \times 10^{16}$ GeV:

$$\lambda^2 \approx \frac{m_{\mathcal{D}}}{\overline{m}_{\mathcal{D}}}|\lambda_{hu}\lambda_{hd}|$$

where $\lambda$ is the suppression factor of the superpotential terms $FF\mathcal{D}$ and $F^cF^c\mathcal{D}$ in the ME$_6$SSM; $\overline{m}_{\mathcal{D}} = 7.6 \times 10^{16}$ GeV; $m_{\mathcal{D}} = 1.5$ TeV and the scale $m_{SUSY}$ used in [67] is assumed to be roughly the same as that in the ME$_6$SSM. With $|\lambda_{hu}\lambda_{hd}| \approx 10^{-10}$ then $\lambda \lesssim 10^{-12}$ is required for the $d=5$ decay $p \to K^+\overline{\nu}$ to be within experimental limits.

The $d=5$ decay channel is thus less constraining than the $d=6$ decay channel when the mass of the Higgs triplets is equal to 1.5 TeV. This is the opposite to what is found in conventional SUSY GUTs where the $d=5$ channels set stringent limits on the mass of the triplet higgsinos. For example the $d=6$ channels generically require $m_{\mathcal{D}} \approx 10^{11}$ GeV whereas, as stated above, the $d=5$ channels can require $m_{\mathcal{D}} \approx 7.6 \times 10^{16}$ GeV. The reason that this is not the case in the ME$_6$SSM is because $m_{SUSY} \approx m_{\mathcal{D}}$ so that the matrix elements of the $d=5$ and $d=6$ channels converge.

In summary the proton decay requires that the terms $FF\mathcal{D}$ and $F^cF^c\mathcal{D}$ in the ME$_6$SSM superpotential are suppressed by a factor of $\lambda = 10^{-13}$ or smaller, which is set by the $d=6$ channel $p \to \pi^0 e^+$.

#### 4.3.3.2 Higgs Triplet Decay

The effective superpotential terms $F_i F_j \mathcal{D}_k$ and $F_i^c F_j^c \mathcal{D}_k$, generated from $\frac{1}{M_p}\Sigma F_i F_j \mathcal{D}_k$ and $\frac{1}{M_p}\Sigma F_i^c F_j^c \mathcal{D}_k$, are the only source for the $\mathcal{D}_i$ particles to decay. Assuming that $m_{\tilde{t}} < m_D$, where $m_{\tilde{t}}$ is the mass of the heaviest stop, the $\overline{D}$ Standard Model representation of the $G_{4221}$ $\mathcal{D}$ particle will predominantly decay through the channel $\overline{D} \to \tilde{t} + b$ [17]. Using the standard 2-body decay kinematic formula [26] it is estimated that the decay rate for $\overline{D} \to \tilde{t} + b$, under the assumption that $m_b \ll m_{\tilde{t}}$, is:

$$d\Gamma \approx \frac{1}{32\pi^2}|\mathcal{M}|^2 \frac{m_D^2 - m_{\tilde{t}}^2}{2m_D^3} d\Omega.$$

At tree-level, a rough order of magnitude estimate of the matrix $\mathcal{M}$ for the $\overline{D} \to \tilde{t} + b$ decay channel gives:

$$|\mathcal{M}|^2 \approx 2(m_D^2 - m_{\tilde{t}}^2)\lambda^2.$$

Taking the mass of the stop to be around the TeV scale, it is estimated that the $F_i F_j \mathcal{D}_k$ and $F_i^c F_j^c \mathcal{D}_k$ operators must be multiplied by an effective Yukawa coupling $\lambda$ that is greater than $10^{-13}$ for the $\mathcal{D}_i$ particles to have a lifetime less than 0.1s.[5]

The superpotential terms $\lambda F F \mathcal{D}$ and $\lambda F^c F^c \mathcal{D}$ are effectively generated from the Planck-suppressed operators $\frac{1}{M_p}\Sigma F F \mathcal{D}$ and $\frac{1}{M_p}\Sigma F^c F^c \mathcal{D}$, and so the Yukawa coupling $\lambda$ is effectively given by $\langle\Sigma\rangle/M_p$. To avoid cosmological difficulties from the $\mathcal{D}_i$ particles, the above analysis shows that $\langle\Sigma\rangle \gtrsim 10^6$ GeV, and to avoid experimentally observable proton decay we require that $\langle\Sigma\rangle \lesssim 10^6$ GeV. It is therefore assumed that $\langle\Sigma\rangle \approx 10^6$ GeV and that the generated level of suppression is compatible with both proton decay and Higgs triplet decay.

This small allowed window of couplings warrants a more detailed analysis of both proton decay and triplet decay since it will lead to testable predictions for proton decay and the ME$_6$SSM. The TeV scale Higgs triplet states, which would be quasi-stable at colliders, would also lead to striking signatures at the LHC [68].

### 4.3.4 R-Symmetry and R-Parity

To ensure that the LSP is stable in this model, so that it is a candidate for dark matter, an R-parity is derived from the $U(1)_R$ symmetry [69], which commutes with the $G_{4221}$ symmetry but not the $E_6$ symmetry because the latter may not be respected by low-energy symmetries as it is assumed to be broken by quantum gravity effects. To allow

[5] If the stop has a mass $m_{\tilde{t}} > 1.5$ TeV then a suppression of $10^{-12}$ would be required. The stop must therefore have a mass smaller than 1.5 TeV in the ME$_6$SSM.

the $G_{4221}$ superpotential terms, which respect the $Z_2^H$ discrete symmetry, and to derive a generalization of the MSSM R-parity, the $G_{4221}$ supermultiplets of the three 27 $E_6$ have the following $U(1)_R$ R-charge assignments: $F_i$ and $F_i^c$ have $R = +1$; $h_3$, $h_\alpha$, $\mathcal{D}_i$ and $\Sigma$ have $R = 0$; and $S_3$ and $S_\alpha$ have $R = +2$ (see table 4.3). The $16_H$ state also has $R = +2$ so that, when it gets a VEV, the $U(1)_R$ is broken to a $Z_2$ discrete symmetry called $Z_2^R$. Under this $Z_2^R$ symmetry the scalar components of $F_i$, $F_i^c$ and the fermionic components of $h_3$ (the MSSM sparticles) all have $Z_2^R = -1$ while the fermionic components of $F_i$ and $F_i^c$ and the scalar components of $h_3$ (the MSSM particles) all have $Z_2^R = +1$. The $Z_2^R$ symmetry is therefore equivalent to the R-parity of the MSSM for the $F_i$, $F_i^c$ and $h_3$ supermultiplets.

The $h_\alpha$, $\mathcal{D}_i$, $S_i$ and $\Sigma$ supermultiplets are not in the MSSM. All the scalar components of these 'new' supermultiplets can be shown to have $Z_2^R = +1$ while all the fermionic components have $Z_2^R = -1$. Therefore $F_i$ and $F_i^c$ are the only supermultiplets in the theory which have $Z_2^R = +1$ for their fermionic components and $Z_2^R = -1$ for their scalar components. This $Z_2^R$ symmetry therefore stops the 'non-MSSM' particles from allowing the MSSM LSP to decay as well as operating as the R-parity of the MSSM. The introduction of the $Z_2^R$ symmetry thus ensures a stable dark matter candidate.

Note that the $Z_2^H$ symmetry in Table 4.3 is equivalent to an MSSM matter-parity. Therefore, if it was left unbroken, it would also prevent the MSSM LSP from decaying. However, as discussed in Section 4.3.1, the $Z_2^H$ symmetry is broken by the $E_6$ singlet $\Sigma$ at around $10^6$ GeV generating the effective operators $F_iF_j\mathcal{D}_k$, $F_i^cF_j^c\mathcal{D}_k$ and $F_iF_j^ch_\alpha$ that disrespect $Z_2^H$, and enabling the MSSM LSP to decay. Hence the $Z_2^R$ symmetry must be introduced in addition to the $Z_2^H$ symmetry so that the MSSM LSP is stable.

### 4.3.5 Neutrino Mass

The above R-charge assignments forbid phenomenologically-problematic terms and allow the charge-conjugated neutrinos, from $F_i^c$, to obtain a large Majorana mass $\mathcal{O}(M_{GUT}^2/M_p)$ from a $\frac{1}{M_p}F_i^cF_j^c(\overline{16}_H)_{-\frac{1}{2}}(\overline{16}_H)_{-\frac{1}{2}} \equiv \frac{1}{M_p}F_i^cF_j^cH_RH_R$ superpotential term. This term will create a conventional see-saw mechanism for the left-handed neutrinos together with the superpotential term $F_iF_j^ch_3$.

The operators $\frac{1}{M_p}F_i^cF_j^c(\overline{16}_H)_{-\frac{1}{2}}(\overline{16}_H)_{-\frac{1}{2}} \equiv \frac{1}{M_p}F_i^cF_j^cH_RH_R$ and $\frac{1}{M_p}F_iF_j\overline{H}_L\overline{H}_L$, which is phenomenologically harmless, are the only superpotential terms that contain interactions between the three 27 $E_6$ multiplets and the $(16_H)_{\frac{1}{2}} + (\overline{16}_H)_{-\frac{1}{2}}$ multiplets. In Section 4.2.2 the RGEs analysis was performed for two ME$_6$SSM toy models, one with $16_H + \overline{16}_H$ and the other with $27_H + \overline{27}_H$. If the $27_H + \overline{27}_H$ states are included above the $G_{4221}$ symmetry breaking scale than an additional $Z_2$ symmetry must be added to

Table 4.3 to prevent any phenomenologically problematic terms between these states and the quarks and leptons. Including incomplete $E_6$ states above the $G_{4221}$ symmetry breaking is considered acceptable here because they are split from their $E_6$ particles by $\approx 10^3$ orders of magnitude. This is compared to the splitting between the mass of the top and up quark which is $\approx 10^5$ orders of magnitude.

## 4.4 Phenomenology of the new $Z'$ in the ME$_6$SSM

This Section investigates certain phenomenological implications of the $Z'$ gauge boson in the ME$_6$SSM. The results are compared to those calculated for the $Z'$ in the E$_6$SSM to see if a possible distinction can be made between the two models in future experiments. The covariant derivatives for the E$_6$SSM and ME$_6$SSM symmetries below the GUT scale are first reviewed and then the different $U(1)'$ groups from the two models are compared. In Section 4.4.3 the mixing between the $Z'$ of the ME$_6$SSM and the Standard Model $Z$ gauge boson is then calculated and shown to be negligible as in the E$_6$SSM. In Section 4.4.4 the axial and vector couplings of the $Z'$ to the low energy particle spectrum are calculated and it is shown that the charged lepton vector couplings do differ in the E$_6$SSM and ME$_6$SSM, which could potentially lead to a distinction between the two models in future experiments.

### 4.4.1 The $Z'$ of the E$_6$SSM

In the E$_6$SSM the $E_6$ symmetry is not broken through a Pati-Salam intermediate symmetry but instead breaks to $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_N$ via a $E_6 \rightarrow SO(10) \times U(1)_\psi \rightarrow SU(5) \times U(1)_\chi \times U(1)_\psi$ symmetry breaking chain. The covariant derivative for the $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_N$ symmetry can be written as:

$$D_\mu = \partial_\mu + ig_3 T^n_{3c} A^n_{3c\mu} + ig_{2L} T^s_L A^s_{L\mu} + ig_1 T_Y B_{Y\mu} + ig_N T_N B_{N\mu} \tag{4.18}$$

where $n = 1 \ldots 8$ and $s = 1 \ldots 3$; $A^n_{3c\mu}$, $A^s_{L\mu}$, $B_{Y\mu}$ and $B_{N\mu}$ are the $SU(3)_c$, $SU(2)_L$, $U(1)_Y$ and $U(1)_N$ quantum fields respectively; $g_3$, $g_{2L}$, $g_1$ and $g_N$ denote the universal gauge coupling constants of the respective fields and $T^n_{3c}$, $T^s_L$, $T_Y$ and $T_N$ represent their generators. At low energies the $U(1)_N$ gauge group will be broken giving rise to a massive $Z'$ gauge boson associated with the E$_6$SSM.

The $g_N$ gauge coupling constant is equal to $g_1$ to an excellent approximation [17], independent of the energy scale of interest. This is to be compared to the universal gauge coupling constant $g_X$ of the group $U(1)_X$ in the models presented in this Section, which is always less than $g_1$.

Similar to $T_Y$ and $T_X$, we can split $T_N$ into an $E_6$ normalization constant $N_N$ and a non-normalized charge $N$ so that $T_N \equiv N/N_N$ where the conventional choice is $N_N^2 \equiv 40$ and $N \equiv \frac{1}{4}\chi + \frac{5}{2}T_\psi$ where $\chi \equiv 2\sqrt{10}T_\chi$ [17].

### 4.4.2 The $Z'$ of the ME$_6$SSM

The covariant derivative of the $G_{4221}$ symmetry is discussed in Appendix B and is given by Eq.B.1 as:

$$D_\mu = \partial_\mu + ig_4 T_4^m A_{4\mu}^m + ig_{2L} T_L^s A_{L\mu}^s + ig_{2R} T_R^r A_{R\mu}^r + \frac{1}{\sqrt{6}} ig_\psi T_\psi A_{\psi\mu}$$

where $m = 1 \ldots 15$ and $r, s = 1 \ldots 3$; $A_{4\mu}^m$, $A_{R\mu}^r$ and $A_{\psi\mu}$ are the $SU(4)$, $SU(2)_R$ and $U(1)_\psi$ quantum fields respectively; $g_4$, $g_{2R}$ and $g_\psi$ denote the universal gauge coupling constants of the respective fields; and $T_4^m$, $T_R^r$ and $T_\psi$ represent their generators.

The covariant derivative of the $G_{3211}$ symmetry is also derived in Appendix B and is given by Eq.B.12 as:

$$D_\mu = \partial_\mu + ig_3 T_{3c}^n A_{3c\mu}^n + ig_{2L} T_L^s A_{L\mu}^s + ig_1 T_Y B_{Y\mu} + ig_X T_X B_{X\mu} \tag{4.19}$$

where $n = 1 \ldots 8$ and $s = 1 \ldots 3$; and $B_{X\mu}$ and $T_X$ are the gauge field of the $U(1)_X$ group and its ($E_6$ normalized) charge respectively. At low energies the $U(1)_X$ gauge group will be broken, giving rise to a massive $Z'$ gauge boson associated with the ME$_6$SSM.

As is clear from Table 4.2, for $c_{12}^2 = \frac{5}{7}$, the $T_X$ and $T_N$ charges are different for all of the $G_{3211}$ representations of the 27 multiplets. However, in the limit $c_{12}^2 = \frac{3}{5}$, corresponding to $g_{2R} = g_4$ at the $G_{4221}$ symmetry breaking scale, then $T_X$ and $T_N$ become identical.[6] This can be seen if one sets $g_{2R} = g_4 = \sqrt{\frac{2}{3}} g_{B-L}$ in Eq.4.9 and Eq.4.13, in which case $T_X$ is given by:

$$\begin{aligned} T_X &= \frac{1}{4}\left[\frac{4}{2\sqrt{10}}\Big(T_R^3 - \frac{3}{2}T_{B-L}\Big) + \sqrt{15}\Big(T_\psi/\sqrt{6}\Big)\right] \\ &\equiv \frac{1}{4}\left[T_\chi + \sqrt{15}\Big(T_\psi/\sqrt{6}\Big)\right] \\ &\equiv T_\chi \cos\theta + (T_\psi/\sqrt{6})\sin\theta \\ &\equiv T_N \end{aligned}$$

[6] Although $T_X$ and $T_N$ are identical for $c_{12}^2 = 3/5$, $X$ and $N$ and hence $N_X$ and $N_N$ are not. However, we could have defined $X$ and $N_X$ differently so that they agree with $N$ and $N_N$ when $c_{12}^2 = 3/5$.

where $\theta = \arctan\sqrt{15}$ and $T_\chi$ is the $E_6$ normalized charge for the $U(1)_\chi$ group, which is defined by $SO(10) \to SU(5) \times U(1)_\chi$ [50] [7].

In the E$_6$SSM the $U(1)_N$ group is defined as the linear combination of the two groups $U(1)_\chi$ and $U(1)_\psi$ for which the right-handed neutrino is a singlet of the symmetry [17]. This linear combination is $U(1)_\chi \cos\theta + U(1)_\psi \sin\theta$, where $\theta = \arctan\sqrt{15}$ [17], which is the same linear combination of $U(1)_\chi$ and $U(1)_\psi$ that $U(1)_X$ becomes if $g_R = g_4$ as shown above. Thus if $g_R = g_4$ at the $G_{4221}$ symmetry breaking scale, then the covariant derivative for the E$_6$SSM, Eq.4.18, and the covariant derivative for $G_{3211}$, Eq.4.19, become equivalent because of the reasons stated above. However, in the $E_6$ theories proposed in Section 4.2.2, $c_{12}^2 \approx \frac{5}{7}$ not $\frac{3}{5}$ so that, in general, one expects $g_R \neq g_4$ at the $G_{4221}$ symmetry breaking scale in realistic models. This way of relating the E$_6$SSM and the ME$_6$SSM (i.e. by setting $g_4 = g_{2R}$ at the $G_{4221}$ symmetry breaking scale) is utilized in Chapter 6.

It is clearly of interest to try to distinguish the $Z'$ of the E$_6$SSM from that of the ME$_6$SSM, since the former one is associated with GUT scale unification, while the latter is associated with Planck scale unification. In the remainder of this Section the phenomenology of the new $Z'$ of the ME$_6$SSM is discussed and compared to that of the E$_6$SSM. In principle, different $Z'$ gauge bosons can be distinguished at the LHC by measuring the leptonic forward-backward charge asymmetries as discussed in [70] (providing the mass of the $Z'$ is not much larger than about 2 TeV).

### 4.4.3 Mixing between $Z$ and the $Z'$ of the ME$_6$SSM

This Section investigates the mixing between the $Z$ gauge boson and the $Z'$ gauge boson of $U(1)_X$ which is generated once the Higgs doublets $h_u$ and $h_d$ from $h_3$ get vacuum expectation values and break the electroweak symmetry. When the MSSM singlet particle $S$ from the low-energy 27 multiplets of the ME$_6$SSM gets a VEV, the $U(1)_X$ group will be broken and a heavy $Z'$ gauge boson will be produced. Then, when $h_u$ and $h_d$ get VEVs, the $SU(2)_L \times U(1)_Y$ symmetry will be broken to $U(1)_{em}$ and a heavy neutral $Z$ gauge boson, which is the following mixture of the $SU(2)_L$ and $U(1)_Y$ fields: $Z_\mu = W_\mu^3 \cos\theta_W - A_Y \sin\theta_W$ where $\theta_W$ is the Electroweak (EW) symmetry mixing angle. Since $h_u$ and $h_d$ transform under $U(1)_X$, they couple to $Z'$ and so mix the $Z'$ and $Z$ gauge bosons when they get VEVs. After $S$, $h_u$ and $h_d$ get VEVs the mass squared

[7] When $g_{2L} = g_{2R} = g_4$ the Pati-Salam generators can be thought of as $SO(10)$ generators, on the same footing as the $SU(5)$ and $U(1)_\chi$ generators when their gauge couplings are equal, as in the E$_6$SSM. In this limit the above argument shows that there is no distinction between $U(1)_N$ and $U(1)_X$.

mixing matrix for the $Z$ and $Z'$ gauge bosons is given by [71]:

$$M^2_{ZZ'} = \begin{pmatrix} M_Z^2 & \delta M^2 \\ \delta M^2 & M^2_{Z'} \end{pmatrix}$$

where:

$$\begin{aligned} M_Z^2 &= (g_{2L}^2 + g_Y^2)(Y^h)^2 \upsilon_h^2 \\ M_{Z'}^2 &= g_X^2 \upsilon_h^2 [(T_X^{h_1})^2 \cos^2\beta + (T_X^{h_2})^2 \sin^2\beta] + g_X^2 (T_X^S)^2 s^2 \\ \delta M^2 &= \sqrt{g_{2L}^2 + g_Y^2}\; g_X \; Y^h (T_X^{h_1} \cos^2\beta - T_X^{h_2} \sin^2\beta) \upsilon_h^2 \end{aligned}$$

and $Y^h$ is the magnitude of the $h_u$ and $h_d$ Higgs bosons' hypercharge; $T_X^{h_1}$, $T_X^{h_2}$ and $T_X^S$ are the values that the $E_6$ normalized $U(1)_X$ charge, $T_X$, takes for the $h_1$, $h_2$ and $S$ states respectively; $g_{2L}$ and $g_Y$ are the $SU(2)_L$ and (non-GUT normalized) hypercharge gauge coupling constants evaluated at the EW symmetry breaking scale;[8] $g_X$ is the $U(1)_X$ gauge coupling constant evaluated at the $U(1)_X$ symmetry breaking scale; $s$ is the VEV of the MSSM singlet $S$; $\upsilon_h = \sqrt{\upsilon_u^2 + \upsilon_d^2}$ and $\tan\beta = \frac{\upsilon_d}{\upsilon_u}$ where $\upsilon_u$ and $\upsilon_d$ are the vacuum expectation values for the $h_u$ and $h_d$ MSSM Higgs bosons respectively.

The mass eigenstates generated by this mass mixing matrix are:

$$\begin{aligned} Z_1 &= Z \cos\theta_{ZZ'} + Z' \sin\theta_{ZZ'} \\ Z_2 &= -Z \sin\theta_{ZZ'} + Z' \cos\theta_{ZZ'} \end{aligned}$$

with masses $M^2_{Z_1,Z_2} = \frac{1}{2}\left[M_Z^2 + M_{Z'}^2 \mp \sqrt{(M_Z^2 - M_{Z'}^2) + 4\delta M^4}\,\right]$ respectively, and mixing angle $\tan(2\theta_{ZZ'}) = \frac{2\delta M^2}{M_{Z'}^2 - M_Z^2}$.

In terms of the above mixing angle the covariant derivative for the mass eigenstate gauge bosons $Z_1$ and $Z_2$ is:

$$\begin{aligned} D_\mu = \partial_\mu &+ i\left(\frac{\cos\theta_{ZZ'}}{\sqrt{g_Y^2 + g_{2L}^2}}(g_{2L}^2 T_L^3 - g_Y^2 Y) - g_X T_X \sin\theta_{ZZ'}\right) Z_{1\mu} \\ &+ i\left(g_X T_X \cos\theta_{ZZ'} + \frac{\sin\theta_{ZZ'}}{\sqrt{g_Y^2 + g_{2L}^2}}(g_{2L}^2 - g_Y^2 Y)\right) Z_{2\mu} \end{aligned}$$

where $g_Y$ and $g_{2L}$ are evaluated at the EW symmetry breaking scale and $g_X$ is evaluated at the scale at which $S$ gets a VEV to break the $U(1)_X$ symmetry. Phenomenology constrains the mixing angle $\theta_{ZZ'}$ to be typically less than $2 - 3 \times 10^{-3}$ [72] and the mass of the extra neutral gauge boson to be heavier than $500 - 600$ GeV [45]. It is calculated

[8] The non-GUT normalized hypercharge coupling constant $g_Y$ is identified as $g_Y \equiv \sqrt{\frac{3}{5}} g_1$.

that, if the $S$ particle gets a VEV at 1.5 TeV in the ME$_6$SSM, then $\theta_{ZZ'} = 3 \times 10^{-3}$ and $M_{Z'} = 544$ GeV so that phenomenologically acceptable values are therefore produced for $s > 1.5$ TeV. This vacuum expectation value is consistent with the RGEs analysis in Section 4.2.2 and the scale of electroweak symmetry breaking.

Since the mixing angle $\theta_{ZZ'}$ is very small in the ME$_6$SSM, the two mass eigenstate gauge bosons can be approximated to be just $Z$ and $Z'$. These are the neutral gauge bosons of the broken $SU(2)_L \times U(1)_Y$ and $U(1)_X$ symmetries respectively. The above covariant derivative is then simplified to:

$$D_\mu = \partial_\mu + i\frac{1}{\sqrt{g_Y^2 + g_{2L}^2}} Z_\mu \big(g_{2L}^2 T_L^3 - g_Y^2 Y\big) + i g_X Z'_\mu T_X.$$

### 4.4.4 Axial and Vector Couplings for $Z'$ in the ME$_6$SSM

If the mixing between the $Z$ and $Z'$ gauge bosons is ignored, then the most general Lagrangian for the $U(1)_X$ group is [73]:

$$\mathcal{L}_X = \frac{1}{2} M_{Z'} Z'^\mu Z'_\mu - \frac{g_X}{2} \sum_i \overline{\psi}_i \gamma^\mu (f_V^i - f_A^i \gamma^5) \psi_i Z'_\mu - \frac{1}{4} F'^{\mu\nu} F'_{\mu\nu} - \frac{\sin\chi}{2} F'^{\mu\nu} F_{\mu\nu}$$

where $F'^{\mu\nu}$ and $F^{\mu\nu}$ are the field strength tensors for $U(1)_X$ and $U(1)_Y$ respectively; $\psi_i$ are the chiral fermions; and $f_V^i$ and $f_A^i$ are their vector and axial charges which are given by $f_V^i \equiv \frac{1}{N_X}(X_L^i + X_R^i)$ and $f_A^i \equiv \frac{1}{N_X}(X_L^i - X_R^i)$ where $X_L$ and $X_R$ are the $X$ charges for the left-handed and right-handed particles respectively.

#### 4.4.4.1 Kinetic Term

The $\frac{\sin\chi}{2} F'^{\mu\nu} F_{\mu\nu}$ term in the above Lagrangian represents the kinetic term mixing for the two Abelian symmetries $U(1)_Y$ and $U(1)_X$. In general, the kinetic term mixing for two Abelian gauge groups is non-zero because the field strength tensor is gauge-invariant for an Abelian theory. However, if both Abelian groups come from a simple gauge group, such as $E_6$, then $\sin\chi$ is equal to zero at the tree-level, although non-zero elements could arise at higher orders if the trace of the $U(1)$ charges is not equal to zero for the states lighter than the energy scale of interest [73]. The trace of the $U(1)_Y$ and $U(1)_X$ charges is given by:

$$\mathrm{Tr}\ (T_Y\ T_X) = \sum_{\text{i=chiral fields}} (T_Y^i\ T_X^i).$$

This trace is only non-zero if incomplete GUT multiplets are present in the low-energy particle spectrum. There are no low-energy incomplete $E_6$ multiplets in the ME$_6$SSM

| | $u$ | $d$ | $e$ | $\nu$ | $D$ | $h$ | $S$ |
|---|---|---|---|---|---|---|---|
| $f_V/N_X$ | $\frac{1}{2}-\frac{5}{6}c_{12}^2$ | $-\frac{1}{2}+\frac{1}{6}c_{12}^2$ | $-\frac{1}{2}+\frac{3}{2}c_{12}^2$ | $\frac{1}{2}+\frac{1}{2}c_{12}^2$ | $\frac{2}{3}c_{12}^2$ | $-1+c_{12}^2$ | 2 |
| $f_A/N_X$ | $\frac{1}{2}+\frac{1}{2}c_{12}^2$ | $\frac{3}{2}-\frac{1}{2}c_{12}^2$ | $\frac{3}{2}-\frac{1}{2}c_{12}^2$ | $\frac{1}{2}+\frac{1}{2}c_{12}^2$ | $-2$ | $-2$ | 2 |
| $f_V$ | $-0.0376$ | $-0.1503$ | 0.2255 | 0.3382 | 0.1879 | $-0.1127$ | 0.7892 |
| $f_A$ | 0.3382 | 0.4510 | 0.4510 | 0.3382 | $-0.7892$ | $-0.7892$ | 0.7892 |
| $f_V^0$ | 0.0278 | $-0.1637$ | 0.1081 | 0.2996 | 0.1359 | $-0.1915$ | 0.7906 |
| $f_A^0$ | 0.2996 | 0.4910 | 0.4910 | 0.2996 | $-0.7906$ | $-0.7906$ | 0.7906 |

TABLE 4.5: The axial $f_A$ and vector $f_V$ $U(1)_X$ charge assignments for the $G_{3211}$ representations of the complete 27 $E_6$ multiplet in the ME$_6$SSM. The assignments for a general ME$_6$SSM model and for the model presented in Section 4.2.2, which has $c_{12}^2 = 5/7$, are both given. The $E_6$ normalization factor $N_X$ is given by $N_X^2 = 7 - 2c_{12}^2 + \frac{5}{3}c_{12}^4$. The axial and vector $U(1)_N$ charge assignments $f_V^0$ and $f_A^0$ in the E$_6$SSM are also included.

and so $\sin\chi = 0$ at the tree-level and at higher orders in this particular case. There is therefore no kinetic term mixing for the $U(1)_Y$ and $U(1)_X$ groups in the ME$_6$SSM.

In the E$_6$SSM the two additional EW doublets $H'$ and $\overline{H}'$ from incomplete $E_6$ $27'$ and $\overline{27}'$ multiplets are kept light. In this case, $\sin\chi$ has a non-zero value, which leads to a kinetic term mixing for the $U(1)_N$ and $U(1)_Y$ fields. This can be eliminated by means of a non-unitary transformation of the two $U(1)$ gauge fields [17]. In terms of the new gauge variables, one has the same gauge coupling constant and charge as the hypercharge field, and so can be identified with the hypercharge field $B_{Y\mu}$, whereas the other has a gauge coupling constant that is a particular combination of the $U(1)_N$ and $U(1)_Y$ charge. This results in the charge of the other $U(1)$ field being dependent on the $U(1)_Y$ and $U(1)_N$ gauge coupling constants. This is similar to the fact that the $U(1)_X$ charge depends on the $g_4$ and $g_{2R}$ gauge coupling constants.

#### 4.4.4.2 Interaction with the Fermions

The second term in the $U(1)_X$ Lagrangian $\mathcal{L}_X$ represents the interaction between the $Z'$ gauge boson and the fermions. Table 4.5 lists the vector and axial $U(1)_X$ charges for the $G_{4221}$ representations of the complete 27 low-energy $E_6$ multiplets in a general $E_6$ theory and the ME$_6$SSM, which has $c_{12}^2 = 5/7$. The vector and axial $U(1)_N$ charges of the E$_6$SSM for the low-energy 27 multiplets are also listed for a comparison. The differences between the values of the vector and axial couplings of the two $Z'$ gauge bosons of the $U(1)_X$ and $U(1)_N$ groups are due to the difference in value between the $E_6$ normalized $T_X$ and $T_N$ charges and the fact that the kinetic term mixing between the $U(1)_Y$ and the $U(1)'$ groups is non-zero in the E$_6$SSM but zero in the ME$_6$SSM. The largest difference between the vector and axial couplings of $U(1)_X$ and $U(1)_N$ exists for the charged leptons where the vector coupling for $U(1)_X$ is a factor of two larger than for $U(1)_N$.

As noted above the vector and axial $U(1)_X$ charges depends on the value $c_{12}^2$ and therefore the value of the $g_4$ and $g_{2R}$ gauge coupling constants at the $G_{4221}$ symmetry breaking scale. The presence of additional threshold corrections at the Planck scale will not change the Pati-Salam breaking scale or the values of the Standard Model gauge couplings at this scale to one-loop order. However, since these quantities are determined by running up the couplings from low energies, there will be some sensitivity to TeV scale threshold corrections. Since the vector and axial vector couplings of the $Z'$ are determined from the values of the gauge couplings at the Pati-Salam breaking scale, there will therefore be little sensitivity to Planck scale threshold corrections on the determined vector and axial vector couplings of the $Z'$.

## 4.5 Conclusions

In this Chapter an $\mathrm{E}_6$ inspired supersymmetric model called the Minimal $E_6$ Supersymmetric Standard Model (ME$_6$SSM) was introduced. This model is based on three low-energy 27 $\mathrm{E}_6$ representations and which has many attractive features compared to the MSSM. In particular it provides a solution to the $\mu$ problem and doublet-triplet splitting problem, without re-introducing either of these problems. In addition, the model also resolves the little fine-tuning problem of the MSSM.

Above the conventional GUT scale the ME$_6$SSM is embedded into a left-right symmetric Supersymmetric Pati-Salam model, which allows complete gauge unification at the Planck scale, subject to gravitational uncertainties. At low energies there is an additional $U(1)_X$ gauge group, consisting of a novel and non-trivial linear combination of one Abelian and two non-Abelian Pati-Salam generators. The $U(1)_X$ is broken at the TeV scale by the same singlet that also generates the effective $\mu$-term, resulting in a new low energy $Z'$ gauge boson. The $Z'$ of the ME$_6$SSM (produced via the Pati-Salam breaking chain of $E_6$, where $E_6$ is broken at the Planck scale) was compared to the $Z'$ of the E$_6$SSM (from the $SU(5)$ breaking chain of $E_6$, where $E_6$ is broken at the GUT scale) in Section 4.4.4.2 where it was shown that they could be (in principal) distinguished by their axial and vector different couplings. The possible discovery of such $Z'$ gauge bosons is straightforward at the LHC and the different couplings should enable the two models to be resolved experimentally. In particular, the most significant difference between the vector and axial couplings of the $Z'$ of the E$_6$SSM and ME$_6$SSM is in the vector coupling of the charges leptons, which is twice as large in the ME$_6$SSM as in the E$_6$SSM.

In Section 4.3 an R-symmetry and discrete $Z_2^H$ symmetry were introduced that address the potential major phenomenological problems such as flavour changing neutral currents and proton decay, which would otherwise be introduced to the theory by Higgs triplets and extra non-Higgs doublets from the three copies of the 27 multiplet. In the ME$_6$SSM, right-handed Majorana masses of the correct order of magnitude can naturally arise from the Higgs mechanism that breaks the intermediate Pati-Salam and $U(1)_\psi$ symmetry to the standard model and $U(1)_X$ gauge group, leading to a conventional see-saw mechanism.

In conclusion, the ME$_6$SSM has clear advantages over both the MSSM and NMSSM, and even the E$_6$SSM, which make it a serious candidate SUSY Standard Model. It also has a certain elegance in the way that the low energy theory contains only complete 27 representations that also allow for anomaly cancellation of the gauged $U(1)_X$. It has been shown that the potentially dangerous couplings of the exotic particles can readily be tamed by simple symmetries, leading to interesting predictions at the LHC of exotic colour triplet fermions (triplet higgsinos) and a new $Z'$ with distinctive couplings. The discovery and study of such new particles could potentially provide a glimpse into the physics of unification at the Planck scale.

# Chapter 5

# Family Symmetries and the Flavour Problem

The previous Chapter demonstrated that the ME$_6$SSM can successfully resolve the hierarchy problem of the Standard Model, that is, it can explain why the scale of electroweak symmetry breaking and the Higgs boson's mass are small compared to the GUT scale. However, although this model adequately explains the mass of the $W^{\pm}, Z^0$ bosons and the anticipated mass of the Higgs boson, it does not address the flavour problem in particle physics. That is, it does not provide an adequate explanation for the structure of quark and lepton masses and mixing angles that we observe in particle experiments.

In the Standard Model the quark and lepton masses are created by the VEV of the Higgs field in a similar way to how the $W^{\pm}$ and $Z^0$ bosons obtain mass, and most theories that attempt to explain the structure of the quark and lepton masses retain this Standard Model approach. It is therefore essential that, if these theories are to fully address the flavour problem, then they must also explain why the Higgs boson's mass is small. The ME$_6$SSM thus provides a working extension to the Standard Model to which one can introduce new physics that solves the flavour problem in particle physics.

In the past decade, the flavour problem has been enriched by the discovery of neutrino masses and mixings, leading to an explosion of interest in this area [19]. A common approach is to suppose that the quarks and leptons are described by some family symmetry which is spontaneously broken at a high-energy scale [21]. In particular, the approximately tri-bi-maximal nature of lepton mixing provides a renewed motivation for the notion that the Yukawa couplings are controlled by a spontaneously broken non-Abelian family symmetry which spans all three families. Also, small neutrino masses have long been predicted by conventional see-saw mechanisms and, when combined with family symmetries, can lead naturally to tri-bi-maximal mixing. Grand Unified theories based on $SO(10)$ and $E_6$ naturally contain such see-saw mechanisms suggesting that they should be extended with a family symmetry. The fact that the $ME_6SSM$ is an $E_6$ inspired supersymmetric model that contains a see-saw mechanism, and solves the hierarchy problem, implies that it should be extended with a family symmetry to solve the flavour problem.

In this Chapter the flavour problem in particle physics is reviewed and a brief introduction is given on how the this problem is resolved in SUSY GUTs that have been extended with a family symmetry. In particular, the $\Delta_{27}$ family symmetry model in [74], which is based on an $SO(10)$ SUSY GUT, is reviewed and described. Chapter 6 then extends the $ME_6SSM$ (and $E_6SSM$) with a discrete non-Abelian family symmetry to generate viable models that can resolve the flavour problem of particle physics.

The layout of this Chapter is as follows. Section 5.1 reviews the mechanism used in the Standard Model to generate quark and lepton masses, and highlights its flavour problem. Section 5.3 then shows how the Standard Model can be extended to describe the recent observations of neutrino oscillations. Sections 5.4 and 5.5 illustrate how this mechanism is modified in the MSSM and simple SUSY GUTs such as $SO(10)$. Section 5.6 reviews how family symmetries can explain quark and charged lepton masses and CKM matrix elements. Section 5.6.3 then introduces a discrete non-Abelian family symmetry called $\Delta_{27}$ which will be implemented in Chapters 6 and 7. Section 5.7 demonstrates how this family symmetry predicts tri-bi-maximal mixing using constrained sequential dominance, and finally, in Section 5.8 a short review is given on how non-Abelian family symmetries can solve the SUSY flavour and CP problems.

## 5.1 Quark and Lepton Masses in the Standard Model

In addition to the Higgs field and the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge bosons, the Standard Model also contains fermions called quarks and leptons that transform under its gauge symmetry. The quarks are defined as the fermions that couple to the $SU(3)_c$ gauge bosons and are therefore said to come in three colours, whereas the leptons have no $SU(3)_c$ interactions and are therefore colourless. The way in which the quarks and leptons transform under the Standard Model gauge symmetry is described by their different $SU(3)_c \times SU(2)_L \times U(1)_Y$ representations. Each quark and lepton comes in three copies called generations where each generation transforms in the same way under the gauge symmetries but has a different mass.

The general Lagrangian for a QFT involving fermion and gauge fields contains the covariant derivative term $\overline{\psi} i\gamma^\mu D_\mu \psi$ which describes the interaction between a Dirac fermion $\psi$ and gauge fields $A^a_\mu$, where $D_\mu = \partial_\mu + igT^a A^a_\mu$. This term splits into separate parts for the left-handed and right-handed fermion chiralities $\psi_L$ and $\psi_R$:

$$\overline{\psi} i\gamma^\mu D_\mu \psi = \overline{\psi}_L i\gamma^\mu D_\mu \psi_L + \overline{\psi}_R i D^\mu \partial_\mu \psi_R.$$

We can therefore assign $\psi_L$ and $\psi_R$ to different representations of the gauge group, and this is exactly what we have in the Standard Model where the gauge bosons of $SU(2)_L$ only couple to left-handed chirality states of quarks and leptons. Explicitly, the left-handed quarks $Q_i$ and left-handed leptons $L_i$ form the following $SU(3)_c \times SU(2)_L \times U(1)_Y$ representations:

$$Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix} = (3,2)_{\frac{1}{6}}; \qquad L_i = \begin{pmatrix} \nu_{ei} \\ e_i \end{pmatrix} = (1,2)_{-\frac{1}{2}}$$

where $i = 1\ldots3$ labels the different generation of the quarks and leptons (for example, $u_3$ denotes the left-handed top quark $t$), whereas the right-handed quarks $u_R$, $d_R$ and leptons $e_R$ transform as:[1]

$$u^i_R = (3,1)_{\frac{2}{3}}; \qquad d^i_R = (3,1)_{-\frac{1}{3}}; \qquad e^i_R = (1,1)_{-1}.$$

Unlike the covariant derivative term however, the bare Dirac mass term $m\overline{\psi}\psi$ cannot be split into separate parts for the left-handed and right-handed helicity states. Instead one obtains the following mixed mass terms:

$$m\overline{\psi}\psi = m(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L).$$

This means that bare fermion mass terms cannot be written down for the quarks and leptons in the Standard Model since these would be forbidden by global gauge invariants. For example, $m_e(\overline{e}_L e_R + \overline{e}_R e_L)$ is forbidden since $e_L$ and $e_R$ belong to different $SU(2)_L \times U(1)_Y$ representations. Without mass terms for the different quarks and leptons we would therefore expect that all quarks and leptons should be massless, which is in strong violation with experimental observations.

Fortunately the Higgs field provides a solution to this problem. Just as the Higgs field gives mass to three of the $SU(2)_L \times U(1)_Y$ gauge bosons through its various gauge interactions, the Higgs field can also give mass to the quarks and leptons through its Yukawa interactions. The Yukawa interactions that are allowed by the gauge symmetry of the Standard Model are represented by the following Lagrangian:

$$\mathcal{L}_{Yuk} = \lambda^{ij}_d Q_i \overline{d}_{Rj} h^\dagger + \lambda^{ij}_u Q_i \overline{u}_{Rj} h + \lambda^{ij}_e L_i \overline{e}_{Rj} h^\dagger + h.c. \tag{5.1}$$

[1]By definition there is no right-handed neutrino $\nu_R$ in the Standard Model.

where $i, j = 1 \ldots 3$ label the three different generations of each quark and lepton; and $Qhu^c = \epsilon_{ab} Q^a h^b u^c$ where $a, b = 1 \ldots 2$ are $SU(2)_L$ indices.

When we insert the Higgs field VEV $\upsilon$, the above terms become:

$$\begin{aligned} &\lambda_d^{ij} \upsilon d_{Li} \overline{d}_{Rj} + \lambda_u^{ij} \upsilon u_{Li} \overline{u}_{Rj} + \lambda_e^{ij} \upsilon e_{Li} \overline{e}_{Rj} + h.c. \\ &\equiv m_d^{ij} d_{Li} \overline{d}_{Rj} + m_u^{ij} u_{Li} \overline{u}_{Rj} + m_e^{ij} e_{Li} \overline{e}_{Rj} + h.c. \end{aligned} \tag{5.2}$$

where $m_d^{ij} \equiv \lambda_d^{ij} \upsilon$, $m_u^{ij} \equiv \lambda_u^{ij} \upsilon$ and $m_e^{ij} \equiv \lambda_e^{ij} \upsilon$ are $3 \times 3$ matrices called mass matrices. The terms in Eq.5.2 look like effective dirac mass terms for all the quarks and leptons with each mass given by the product of the particular strength of the interaction with the Higgs field (the Yukawa coupling constant) and the Higgs field's VEV. We are therefore effectively treating the left-handed and right-handed chirality states as different physical states which are mixed to form Dirac fermions by the Higgs field's VEV. In many ways it is a spectacular result that the Higgs field is in just the right representation to break electroweak symmetry and give mass to all the quarks and leptons.

## 5.2 The Flavour Problem

Although the Higgs mechanism in the Standard Model can explain why the quarks and leptons have mass, it does not adequately explain the large differences between the masses that we observe in experiments. For example, the mass of up quark is observed to be $1.5 - 3.3$ MeV [26] whereas the top quark, which has exactly the same $SU(3)_c \times SU(2)_L \times U(1)_Y$ representations, has mass $169.0 - 173.6$ GeV [26]. In the Standard Model the strength of interaction between the top quark and the Higgs field must therefore be $10^5$ orders of magnitude greater than the up quark's interaction. This huge difference is unexplained since the Yukawa coupling constants are renormalizable parameters and so are not predicted by the theory (they are free parameters). Instead we would expect that each copy of a particular quark or lepton has approximately the same mass and that all the masses are of order the scale of electroweak symmetry breaking.

The physical mass of a charged lepton is just the pole of its propagator and can therefore be measured directly. However, since quarks are confined inside hadrons, their masses cannot be measured directly. Instead the only way to determine the quark masses is through the study of their impact on hadron properties. The quark mass parameters in the QCD and electroweak Lagrangians depend both on the renormalization scheme adopted to define the theory and on the scale parameter $\mu$. This dependence reflects the fact that a bare quark is surrounded by a cloud of gluons and quark-antiquark pairs. To get the relative magnitudes of different quark masses in a physically meaningful way, one has to describe all quark masses in the same scheme and at the same scale. It is instructive to consider the light and heavy quark masses at the scale $\mu = M_{Z^0}$, the mass of the $Z^0$ boson, by adopting the $\overline{MS}$ scheme. The advantage of choosing $M_{Z^0}$ as the reference scale is that, for scales above $M_{Z^0}$, extensions of the standard model may naturally appear, and for scales below $M_{Z^0}$, the strong-interaction coupling constant $\alpha_3$ is sizable. The latest experimental values for the quark and charged lepton masses are the following [26]:

$$\begin{array}{lcllcllcl} m_u & = & 0.9\text{-}2.9 \ \text{MeV} & m_d & = & 1.8 - 5.3 \ \text{MeV} & m_e & = & 0.5110 \ \text{MeV} \\ m_c & = & 530 - 680 \ \text{MeV} & m_s & = & 35 - 100 \ \text{MeV} & m_\mu & = & 105.7 \ \text{MeV} \\ m_t & = & 168 - 180 \ \text{GeV} & m_b & = & 2.8 - 3.0 \ \text{GeV} & m_\tau & = & 1.777 \ \text{GeV} \end{array}$$

where the leptons masses are given to four significant figures and the quark masses have been scaled to $\mu = M_{Z^0}$ in the $\overline{MS}$ scheme as discussed above. To get a proper sense of the hierarchy involved with is useful to rewrite the above masses as approximate ratios between the different quark and lepton generations:

$$\begin{aligned} m_t : m_c : m_u &\approx 1 : (0.05)^2 : (0.05)^4 \\ m_b : m_s : m_d &\approx 1 : (0.15)^2 : (0.15)^4 \\ m_\tau : m_\mu : m_e &\approx 1 : 3(0.15)^2 : (0.15)^4/3. \end{aligned}$$

Although only approximate, these illustrate that the three generations obey a strong hierarchical structure, and each hierarchy is slightly different for the different types of $SU(3)_c \times SU(2)_L \times U(1)_Y$ fermions.

### 5.2.1 The CKM Matrix

The interactions between the gauge bosons and the quarks and leptons is highly restricted by the local gauge symmetry since ordinary derivatives are just replaced with covariant derivatives. This does not allow any mixing between the various quark and lepton generations. The coupling of the Higgs field to the quarks and leptons however does not follow from a gauge principle and so does not have any such restrictions. The Higgs couplings will therefore, in general, mix the different generations of quarks and leptons.

We could consider the Lagrangian for the Yukawa operators in Eq.5.1 to be part of the full Standard Model Lagrangian, which includes the quark and lepton gauge interactions. It then seems natural to assume that the full Lagrangian is written in terms of the interaction basis (the basis in which the quarks and leptons are defined to be the eigenstates of the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge symmetry). However, since the couplings of the Higgs field to the quarks and leptons do not follow from a gauge principle, this basis may not may not be the same as the mass basis (the basis in which the quarks and leptons are defined to be the mass eigenstates, which are equivalent to the eigenstates of the mass matrices in Eq.5.1). In this Section it is shown that, although the interaction eigenstates of the strong force are equivalent to the mass eigenstates, the interaction eigenstates of the charged weak force are not.

For ease of notation we can rewrite Eq.5.1 so that the Yukawa couplings $\lambda^{ij}$ are written as $3 \times 3$ matrices and the fermions are column vectors in generation space. Eq.5.1 then becomes the following:

$$\mathcal{L}_q + \mathcal{L}_l = -(\overline{\mathbf{d}}_R \mathbf{M}_d \mathbf{d}_L + \overline{\mathbf{u}}_R \mathbf{M}_u \mathbf{u}_L + \overline{\mathbf{e}}_R \mathbf{M}_e \mathbf{e}_L + h.c.). \tag{5.3}$$

In general the Yukawa matrices are $3 \times 3$ complex matrices and such matrices are diagonalized by two different unitary matrices acting from the left and the right. For example, the up quark Yukawa matrix is diagonalized by:

$$\mathcal{M}_u = \mathbf{V}_{uR} \mathbf{M}_u \mathbf{V}_{uL}^{\dagger} \tag{5.4}$$

where $\mathbf{V}_{uL}$ and $\mathbf{V}_{uR}$ are Unitary matrices and $\mathcal{M}_u = diag(m_u, m_c, m_t)$. If we insert Eq.5.4 into Eq.5.3, we can define the up quark mass eigenstates by:

$$\mathbf{u}_R^m \equiv \mathbf{V}_{uR} \mathbf{u}_R$$
$$\mathbf{u}_L^m \equiv \mathbf{V}_{uL} \mathbf{u}_L$$

and equivalently for $d_L^i$, $d_R^i$ and $e_L^i$, $e_R^i$. Written in terms of the quark and lepton mass eigenstates, the Yukawa interactions are:

$$\mathcal{L}_q + \mathcal{L}_l = -(\overline{\mathbf{d}}_R^m \mathcal{M}_d \mathbf{d}_L^m + \overline{\mathbf{u}}_R^m \mathcal{M}_u \mathbf{u}_L^m + \overline{\mathbf{e}}_R^m \mathcal{M}_e \mathbf{e}_L^m + h.c.).$$

When we also rewrite the fermion gauge interactions (covariant derivatives) of the Standard Model Lagrangian in terms of the mass eigenstates we find that everything is invariant except for the fermion couplings to the $W^{\pm}$ vector bosons, which transforms as the following:

$$\mathcal{L}_{W^{\pm}}^q = \frac{1}{\sqrt{2}} \overline{\mathbf{u}}_L \gamma^{\mu} \mathbf{d}_L \rightarrow \frac{1}{\sqrt{2}} \overline{\mathbf{u}}_L^m \gamma^{\mu} \mathbf{V}_{CKM} \mathbf{d}_L^m$$

where $\mathbf{V}_{CKM} \equiv \mathbf{V}_{uL}^{\dagger} \mathbf{V}_{dL}$ is the Cabibbo-Kobayashi-Maskawa (CKM) Matrix [75]. The fermion couplings to the $W^{\pm}$ vector bosons is not invariant to this change of basis essentially because the $SU(2)_L$ gauge interactions only couple to left-handed fields.

The above CKM matrix can be parameterized by three rotation angles and one complex phase that is CP violating. A popular parametrization is the following:

$$|\mathbf{V}_{CKM}| \approx \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ and $\delta$ is the CP violating phase. The angles $\theta_{12}$, $\theta_{13}$, $\theta_{23}$ are defined as the mixing angles of the various quark fields. The latest experiment values for the CKM matrix elements are given below [26]:

$$|\mathbf{V}_{CKM}| \approx \begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.0010 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.0010 & 0.97334 \pm 0.00023 & 0.0415 \pm 0.0011 \\ 0.00874 \pm 0.00037 & 0.0407 \pm 0.0010 & 0.999133 \pm 0.000044 \end{pmatrix} \tag{5.5}$$

and $J \approx (3.05 \pm 0.20) \times 10^{-5}$. $J$ is the Jarlskog invariant and is related to the CP violating phase by $= c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}\sin\delta$.

Note that the CKM matrix is not diagonal and so the quark eigenstates of the (charged) weak nuclear force are not the same as the quark mass eigenstates. The matrix is almost diagonal however and so the two bases do not differ by very much. There also appears to be a small amount of symmetry in the CKM matrix: it is almost symmetrical and the closer the quark generations are in mass, the larger the CKM entry (and mixing angle). We also find that there are approximate relations between the CKM elements and the quark masses, which will be discussed in Section 5.6.2. These relations and the symmetries of the CKM matrix (as well as the quark and lepton masses) are not explained in the Standard Model.

## 5.3 Neutrino Masses

Unlike the quarks, the leptons are not predicted to have mixing angles in the Standard Model and there is no analogous matrix to the quark CKM matrix. This is because there are no right-handed neutrinos in the Standard Model and so neutrinos are massless particles. Therefore there is no left-handed unitary matrix $\mathbf{V}_{\nu L}$ that transforms between the neutrino mass and interaction eigenstates.

In recent years however there has been growing experimental evidence that neutrinos are not massless and that leptons have large mixing angles [19]. The present experimental data is given below where only the difference between the squares of neutrino masses has been observed, and the lepton mixing angles contain substantial errors (see B. Kayser in [26]):

$$\begin{aligned} &\sin^2(2\theta_{12}) = 0.87 \pm 0.03 & m_{21}^2 &= (7.59 \pm 0.20) \times 10^{-5}\ \mathrm{eV}^2 \\ &\sin^2(2\theta_{23}) > 0.92 & |m_{31}^2| &= (2.43 \pm 0.13) \times 10^{-3}\ \mathrm{eV}^2 \\ &\sin^2(2\theta_{13}) < 0.19\ (90\%\ CL). & & \end{aligned}$$

This suggests that right-handed neutrinos should be included in the Standard Model so that the Yukawa interactions contain operators such as $\lambda_\nu^{ij}\nu_{Li}\overline{\nu}_{Rj}h$ where $i, j$ are generation indices.[2] In that case, when we re-write the modified Standard Model Lagrangian in terms of the mass basis, the charged weak interactions contain a matrix analogous to the quark CKM matrix for the leptons given by $\mathbf{V}_{\nu L}^\dagger \mathbf{V}_{eL}$ which is called the Maki-Nakagawa-Sakata (MNS) matrix [76].

If we assume that there are no large cancellations between the neutrino masses then we expect that the absolute neutrinos masses are of order $10^{-3}\ eV$. This is approximately $10^{14}$ orders of magnitude smaller than the electroweak symmetry breaking scale and is thus inadequately explained by the Standard Model Higgs mechanism with only operators such as $\lambda_\nu^{ij}\nu_{Li}\overline{\nu}_{Rj}h$ included in the Lagrangian.

### 5.3.1 The Conventional See-Saw Mechanism

The neutrinos that are observed (as missing energy) in electroweak processes only act like left-handed neutrinos and not right-handed neutrinos. This can be explained by the fact that right-handed neutrinos would not have any Standard Model gauge interactions since they transform in the trivial singlet representation $(1,1)_0$ of $SU(3)_c \times SU(2)_L \times U(1)_Y$. A Majorana mass term for the right-handed neutrinos $M_{RR}\nu_R\overline{\nu}_R^c$ can thus be included in the Standard Model Lagrangian where $M_{RR}$ is a dimensional parameter, which could take a very large value without upsetting the experimental evidence that supports the Standard Model. If we think of the right-handed and left-handed neutrinos as separate particles (mixed by their Higgs coupling) then this mass term could give a large mass to the right-handed neutrinos, which would explain why we haven't observed their missing energy in experiments, and, in conjunction with the Dirac mass term generated by the Higgs field's VEV, give very small masses to the neutrinos that we observe as missing energy. For example, if we add three right-handed neutrinos to the Standard Model, all with Majorana masses, then, ignoring the gauge interactions, the Lagrangian for the

[2]There are other ways of modifying the Standard Model to generate neutrino masses, such as including $SU(2)_L$-triplet Higgs states. However including right-handed neutrinos is considered to be the most natural explanation for neutrino masses.

neutrinos would be the following:

$$\lambda_\nu^{ij}\nu_{Li}\overline{\nu}_{Rj}h + M_{RR}^{ij}\nu_{Ri}\overline{\nu}_{Rj}^c \ + \ h.c.$$

We can rewrite this as a $2 \times 2$ block matrix:

$$\begin{pmatrix} \nu_{Li} & \nu_{Ri}^c \end{pmatrix} \begin{pmatrix} \mathbf{0} & M_{LR}^{ij} \\ M_{LR}^{ji} & M_{RR}^{ij} \end{pmatrix} \begin{pmatrix} \overline{\nu}_{Lj}^c \\ \overline{\nu}_{Rj} \end{pmatrix}.$$

where $\mathbf{0}$ is a $3 \times 3$ matrix of zeros. Diagonalizing the above matrix in the approximation that $M_{LR}^{ij} \ll M_{RR}^{ij}$ we obtain effective Majorana mass terms for the left-handed and right-handed neutrino states:

$$\begin{pmatrix} \nu_{Li}^m & \nu_{Ri}^{mc} \end{pmatrix} \begin{pmatrix} M_{LL}^{ij} & \mathbf{0} \\ \mathbf{0} & M_{RR}^{ij} \end{pmatrix} \begin{pmatrix} \overline{\nu}_{Lj}^{mc} \\ \overline{\nu}_{Rj}^m \end{pmatrix}$$

where, the superscript $m$ denotes the mass eigenstates, and, in matrix notation, the left-handed Majorana masses are given by:

$$\mathbf{M}_{LL} = \mathbf{M}_{LR}\mathbf{M}_{RR}^{-1}\mathbf{M}_{LR}^T. \tag{5.6}$$

This mechanism for generating effective Majorana masses for the left-handed neutrinos is called the Type I or conventional see-saw mechanism [10]. As an example of the scales involved we can simplify Eq.5.6 by assuming that there is only one generation of neutrinos rather than three, and take the Dirac mass $M_{LR}$ to be of order the weak scale $\approx 80$ GeV. Then to generate an effective left-handed mass $M_{LL}$ of order $10^{-3}$ eV, we would require that $M_{RR} \approx 10^{16}$ GeV, which is of order the GUT scale. This then gives further credence to the idea that the Standard Model is a low-energy approximation to a grand unified theory such as $SO(10)$. This is further discussed in Section 5.5.

By diagonalizing $\mathbf{M}_{LL}$ in Eq.5.6 we finally end up with the mass basis for the left-handed neutrinos. The $SU(2)_L$ gauge interactions are not invariant to this change of basis and we obtain the MNS matrix, which is analogous to the quark CKM matrix. This matrix is given by the product of the (single) unitary matrix $\mathbf{V}_\nu$ that diagonalizes $\mathbf{M}_{LL}$, and the unitary matrix $\mathbf{V}_{eL}$ that diagonalizes the charged lepton mass matrix from the left. Note that the see-saw mechanism has the potential to explain why the lepton mixing angles are so different to the quark mixing angles since the right-handed Majorana neutrino mass terms break the quark-lepton symmetry introduced by the Dirac neutrino mass terms. However, the see-saw mechanism by itself cannot explain why the lepton mixing angles appear to be so different to one another.

## 5.4 Quark and Lepton Masses in the MSSM

In the MSSM the quark and lepton masses come from the Yukawa interactions in the superpotential given by Eq.2.15 where the up quark masses are generated by the VEV of the up Higgs field $h_u$ whereas the down quark and charged lepton masses are generated by the VEV of the down Higgs field $h_d$. The VEV of the additional Higgs field introduces a new parameter for determining the quark and lepton masses that is not present in the Standard Model. This could potentially be used to explain why the mass of the bottom quark $m_b$ is smaller than the mass of the top quark $m_t$. For example, the top and bottom Yukawa coupling constants could both be $\mathcal{O}(1)$ so that the mass of the top quark is of order the up Higgs VEV $\upsilon_u$ and the mass of the bottom quark is of order the down Higgs VEV $\upsilon_d$. A particular scalar potential could then create $\upsilon_u > \upsilon_d$, which would explain why $m_t > m_b$. However, the extra Higgs VEV does not provide any new insight into why the different quarks and leptons have a hierarchical mass structure since this still requires a hierarchical structure for the renormalizable Yukawa coupling constants, which are free parameters of the theory.

In fact with the introduction of TeV scale SUSY the flavour problem increases dramatically due to the undetermined superpartner masses, mixings and phases that must also be explained [77]. Indeed in SUSY extensions of the SM there are typically about a hundred or so additional physical parameters associated with the soft SUSY breaking Lagrangian, depending on the precise nature of the SUSY SM and the origin of neutrino masses and mixings in the SUSY context.

Experimental data seems to imply that the off-diagonal elements of the soft SUSY breaking Lagrangian should be smaller than the diagonal elements, but there is no a prior reason why this should be the case. This is called the SUSY flavour problem. There is also a so-called SUSY CP problem stemming from the fact that, in general, there could be large extra CP phases coming from the soft SUSY breaking sector of the MSSM. However, the Standard Model accounts for the observed CP violating effects to such a level of accuracy that one must impose stringent bounds on such extra contributions to avoid conflict with experiment[77]. This is, however, often at odds with naturalness.

## 5.5 Quark and Lepton Masses in SUSY GUTs

Since Grand Unified Theories unify quarks and leptons into representations of the semi-simple gauge group, the number of renormalizable Yukawa coupling constants is reduced and relations between different quark and lepton Yukawa couplings are introduced. For example, in the simple $SO(10)$ GUT all the Standard Model quarks and leptons come from three copies of the fundamental spinor representation, which has dimension 16. Since the MSSM Higgs fields come from the fundamental 10 representation, all the Standard Model Yukawa interactions are embedded into the $SO(10)$ tensor product $\lambda^{ij}16_i16_j10$ where $i, j$ label the number of generations. This leads to the unification of the Yukawa coupling constants (written in the mass basis) for each generation of up quarks, down quarks and charged leptons. For example, for the third generation we obtain the relation $\lambda_t = \lambda_b = \lambda_\tau$, where $\lambda_t$, $\lambda_b$ and $\lambda_\tau$ are the Yukawa coupling constants for the top quark, bottom quark and tau lepton in the mass basis.[3]. When renormalized at the electroweak scale the relation $\lambda_b/\lambda_\tau = 1$ agrees well with experiment. For large $\tan\beta$ the relation $\lambda_t/\lambda_b = 1$ also works well when renormalized at the electroweak scale.

However, the equivalent relations for the first and second generations are not successful when renormalized at the electroweak scale. A common approach to resolving this problem is to extend the simple $SO(10)$ GUT with a new scalar field, denoted by $H_{45}$, that only couples to the second generation of the quarks and leptons such that, when the field obtains a VEV, the $(2, 2)$ component of the charged lepton Yukawa matrix $\lambda_e^{ij}$ becomes three times larger than the equivalent component of the down quark Yukawa matrix $\lambda_d^{ij}$. If $H_{45}$ is a fundamental scalar field then the smallest dimensional representation it can be is a 45 of $SU(5)$ which comes from a 210 of $SO(10)$ [51]. The factor of three that the $H_{45}$ VEV generates is related to the fact that quarks come in three colours.

When we diagonalize the charged lepton and down quark Yukawa matrix in this case we end up with the relations $\lambda_\mu = 3\lambda_s$ and $\lambda_e = \lambda_d/3$, which work very well when renormalized at the electroweak scale [18]. This leaves the GUT relations $\lambda_u = \lambda_d$ and $\lambda_c = \lambda_s$. If modified to $\lambda_u = \lambda_d/3$ and $\lambda_c = \lambda_s/3$ then these also work well at the electroweak scale but are difficult to generate in $SO(10)$ GUTs. In Section 5.6 a mechanism is introduced that generates these relations just below the GUT scale instead.

Fifteen of the sixteen components of the fundamental spinor representation of $SO(10)$ form one generation of the Standard Model particles. The remaining component is a right-handed neutrino. $SO(10)$ GUTs thus predict that right-handed neutrinos exist and that neutrinos have non-zero masses. The right-handed neutrinos can only obtain a mass once the $SO(10)$ symmetry is broken and GUT scale see-saw mechanisms, which can explain the recently observed neutrino masses as discussed in Section 5.3.1, are thus well motivated in $SO(10)$ GUTs.

[3]Theoretically at the electroweak scale $\frac{\lambda_b}{\lambda_\tau} = \frac{m_b}{m_\tau}$

## 5.6 Family Symmetries

Although Grand Unified Theories, particularly those based on an $SO(10)$ symmetry, improve the explanation provided by the Standard Model for the observed mass structure of the quarks and leptons (by relating the Yukawa couplings of the quarks and leptons that are contained in the same generation), they do not help with understanding why the different generations of the quarks and leptons have hierarchical masses. Inspired by the success of the extra Higgs field in the MSSM for explaining the difference in the top and bottom quark masses, one possibility could be to extend the Higgs sector of an $SO(10)$ SUSY GUT such that there are two Higgs fields for each quark and lepton generation. Each Higgs field could then perhaps couple differently to the various quarks and leptons because of new gauge or global symmetries. The hierarchical structure of the quark and lepton masses might then be explained by a hierarchical structure of the VEVs of the Higgs fields, which would result from a particular Higgs potential and radiative electroweak symmetry breaking.

A number of problems occur if we extend the Higgs sector of the Standard Model and equivalently the MSSM however. In general, extra Higgs fields generate large flavour changing neutral currents for the quarks and leptons which strongly violate experimental data [78], and is the reason for the $Z_2^H$ discrete symmetry in the ME$_6$SSM and E$_6$SSM models, which is expected to prevent the first and second Higgs-doublet generations from obtaining VEVs.[4]

The problems caused by extended Higgs sectors suggest that the quark and lepton masses are the result of a very different mechanism. One possibility is that extra physics is somehow controlling the Yukawa couplings of the quarks and leptons to the Standard Model Higgs field that explains why they take such different values. This can be achieved by extending the Standard Model with a family symmetry [21]. In these models the quarks and leptons are chosen to transform under the family symmetry so that some or all of the Yukawa interactions of the Standard Model are forbidden in the classical Lagrangian. Instead the Yukawa interactions are generated effectively once the family symmetry is spontaneously broken by the VEVs of additional scalar fields. This is then like an extension of the method used by the Standard Model in which the bare Dirac mass terms are generated effectively once the electroweak symmetry is broken by the Higgs VEV. Extending SUSY GUTs with a family symmetry can also help to solve the flavour problem of the MSSM as described in Section 5.8.

[4]Gauge coupling unification in the MSSM with two Higgs fields also suggests that no more Higgs fields exist at the electroweak scale.

| | $U(1)_F$ |
|---|---|
| $16_3$ | 0 |
| $16_2$ | 1 |
| $16_1$ | 3 |
| 10 | 0 |
| $\phi$ | -1 |

TABLE 5.1: This table illustrates a simple $SO(10)$ SUSY GUT with a $U(1)_F$ family symmetry.

### 5.6.1 Abelian Family Symmetries

An example of a simple family symmetry is a gauged $U(1)$ symmetry, called $U(1)_F$, which couples to the different quark and lepton generations with different charges, but doesn't couple to the Higgs field. This symmetry forbids the Standard Model Yukawa interactions such as $Qu^c h$ and we instead assume that the quarks, leptons and Higgs fields couple to very massive particles $H$ through interactions such as $YQhH$ and $Yu^c\phi H$ where $Y$ is some coefficient and $\phi$ is an additional scalar field that carries a $U(1)_F$ charge and is generically called a flavon. Since the $H$ particles, called messenger fields, are much heavier than the electroweak scale we can, to a good approximation, remove them from the theory so that the quarks and leptons interact with the Higgs fields through higher-order operators such as $\frac{Y}{M}Qu^c h\phi$ where $M$ is the mass of the particles $H$ and $Y$ is some coupling constant which we assume to be $\mathcal{O}(1)$. The type of interactions that reduce to $\frac{Y}{M}Qu^c h\phi$ at lower energy scales are illustrated by the Froggart-Nielsen diagrams [79], an example of which is given by Fig.5.1. This is analogous to the Fermi description of the weak nuclear interactions where the $W^{\pm}$ and $Z^0$ vector bosons are removed from the electroweak theory of the Standard Model to leave non-renormalizable interactions between the quarks and leptons. The Fermi theory is an accurate approximation to the electroweak theory at energies much smaller than the mass of $W^{\pm}$ and $Z^0$ since these particles can be integrated out of the theory.

If the flavon field $\phi$ obtains a VEV, spontaneously breaking the $U(1)_F$ symmetry, then the higher-order operators become effective Standard Model Yukawa interactions such as $\frac{Y\langle\phi\rangle}{M}Qu^c h$.[5] In this example the effective Yukawa coefficient is given by $\frac{Y\langle\phi\rangle}{M}$ where we expect $Y$ to be $\mathcal{O}(1)$. The Standard Model Yukawa coefficients are thus determined as the ratio of the scale of the spontaneous breakdown of the $U(1)_F$ family symmetry and the mass of the messenger fields. By assigning the different quark and lepton generations with different $U(1)_F$ charges we can then generate all of the Standard

[5]The scale of the flavon VEV, although smaller than the messenger scale, must be significantly larger than the electroweak scale otherwise the family symmetry would generate rapid transitions between the various quark and lepton generations, which has not been observed.

Model Yukawa interactions in this way. Table 5.1 gives an example of a particular $U(1)_F$ symmetry applied to an $SO(10)$ SUSY GUT. This $U(1)_F$ symmetry allows the following higher-order operators where the messengers have been integrated out to leave behind a mass suppression factor $M$:

$$W_{Yuk} = Y^{33} 16_3 16_3 10 + \frac{1}{M} Y^{23} 16_2 16_3 10 \phi + \frac{1}{M^2} Y^{22} 16_2 16_2 10 \phi^2 \tag{5.7}$$

$$+ \frac{1}{M^3} Y^{13} 16_1 16_3 10 \phi^3 + \frac{1}{M^4} Y^{12} 16_1 16_2 10 \phi^4 + \frac{1}{M^6} Y^{11} 16_1 16_1 10 \phi^6.$$

When we insert the $\phi$ field's VEV the operators in Eq.5.7 become effective Yukawa operators $\lambda^{ij} 16_i 16_j 10$ with coefficients $\lambda^{ij}$ given by different powers of $\epsilon \equiv \langle\phi\rangle / M$. We can write all these Yukawa coefficients in matrix form:

$$\lambda^{ij} = \begin{pmatrix} Y^{11}\epsilon^6 & Y^{12}\epsilon^4 & Y^{13}\epsilon^3 \\ Y^{12}\epsilon^4 & Y^{22}\epsilon^2 & Y^{23}\epsilon \\ Y^{13}\epsilon^3 & Y^{23}\epsilon & Y^{33} \end{pmatrix}. \tag{5.8}$$

To obtain the physical mass eigenstates we must diagonalize this matrix and, if we assume that all coefficients $Y_u^{ij}$ have the same value $Y$, then the diagonal matrix of Eq.5.8 is approximately given by $diag(\epsilon^4, \epsilon^2, 1)Y$. Therefore, with $Y = \mathcal{O}(1)$ and $\epsilon \approx 0.05$ or $\epsilon \approx 0.15$, the $U(1)_F$ symmetry produces approximately the correct mass hierarchy for the up or down quarks respectively. If a $H_{45}$ scalar field attaches itself to the $\frac{1}{M^2} 16_2 16_2 10 \phi^2$ operator then the correct hierarchy for the charged leptons can also be generated if $\epsilon \approx 0.15$ for the reasons given in Section 5.5.

To generate different $\epsilon$ factors for the down quarks and up quarks we could assume that the family symmetry is broken below the GUT scale so that the mass $M_u$ of the messenger fields that couple to right-handed up quarks are different to the mass $M_d$ of the messenger fields that couple to the right-handed down quarks. For this to be allowed the $SU(2)_R$ subgroup of $SO(10)$ must of course be broken before the messenger scale. If we first take the messenger fields that couple to the left-handed quarks to be much heavier than those that couple to the right-handed quarks, then the mass suppression factors in Eq.5.7 will predominantly come from the latter messengers fields. The terms in Eq.5.7 would then be split into sperate terms for the right-handed up and down quark fields. For example, the term $\frac{1}{M^2} 16_2 16_2 10 \phi^2$ will decompose to $\frac{1}{M_u^2} Q_2 u_2^c h_u \phi^2 + \frac{1}{M_d^2} Q_2 d_2^c h_d \phi^2$, which generates different $\epsilon$ factors given by $\epsilon_u = \frac{\langle\phi\rangle}{M_u}$ and $\epsilon_d = \frac{\langle\phi\rangle}{M_d}$ respectively once $\phi$ develops a VEV. If $\langle\phi\rangle = 0.15 M_d$ and $M_u = 3 M_d$ then the correct $\epsilon$ factors are generated.

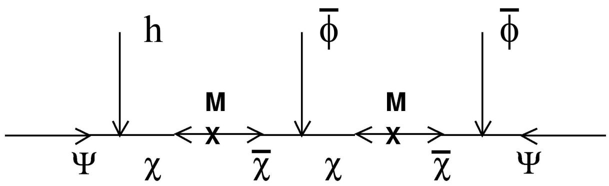



FIGURE 5.1: An example of a Froggatt-Nielsen supergraph that generates quark and lepton masses. $\overline{\phi}$ represents a flavon field that obtains a VEV, $\chi$ the messengers that can be integrated out of the theory at low energies, and $\Psi$ the quarks and leptons.

The operators in Eq.5.7 then should really be written in terms of the Standard Model gauge group rather than the $SO(10)$ (and $SU(2)_R$) gauge group. However it is assumed that the messenger scale is so close to the $SO(10)$ scale that the $SO(10)$ predictions are approximately correct. For instance, a symmetrical Yukawa matrix is still assumed. The $SO(10)$ notation is thus kept for convenience.

### 5.6.2 Yukawa Matrices

The product of the left-handed unitary matrices that diagonalize the up and down quark Yukawa matrices generated by the operators in Eq.5.7 will give an effective CKM matrix. This is because the quark and lepton eigenstates of the $U(1)_F$ family symmetry are the same as the interaction eigenstates of Eq.5.1, since the $U(1)_F$ symmetry is a gauge symmetry. Thus, by diagonalizing the Yukawa matrix Eq.5.8 we are transforming from the interaction basis to the mass basis, and this change of basis generates the CKM matrix as discussed in Section 5.2.1. Unfortunately the effective CKM matrix generated by above $U(1)_F$ symmetry does not agree with experiment, but, if we could determine the form of the up and down quark Yukawa matrices in the interaction basis that reproduces the observed CKM matrix (and mass hierarchies) when transformed to the mass basis, then all we would have to do is search for a new family symmetry that generates this particular form of up and down quark Yukawa matrices. However, while the quark mass matrices and the CKM matrix are intimately related, measurement of the eigenvalues of the mass matrices and the matrix elements of $V_{CKM}$ is not sufficient to determine the structure of the full mass matrix and of the matrix of Yukawa couplings giving rise to them. That is, there is an under-determination in the values of the Yukawa coefficients in the interaction basis when given the CKM matrix elements and quark and lepton masses. This is essentially because the CKM matrix only involves the left-handed unitary transformations and so the full form of the left-handed and right-handed rotation matrices required to diagonalize the quark masses is not known.

Given this under-determination, the phenomenological approach most commonly used is to make some assumption about the structure of the Yukawa matrix and explore the experimental consequences for $V_{CKM}$. For example, the very reasonable assumption that the smallness of the mixing angles is due to the smallness of the mixing angles in both the up and down left-handed bi-unitary matrices $\mathbf{V}_{dL}$ and $\mathbf{V}^{\dagger}_{uL}$, allows one to determine the mass matrix elements on and above the diagonal to good precision for the down quarks and to lesser precision for the up quarks. Another common assumption is that there are zero entries in the up and down quark Yukawa matrices called 'texture zeros'. These lead to relations for the $V_{CKM}$ elements in terms of ratios of quark masses, which do not involve any unknown couplings and hence can be precisely tested.

Experimental data appears to favour a texture zero in the $(1,1)$ position of the up and down Yukawa matrices $\lambda_u^{ij}$, $\lambda_d^{ij}$ (in the interaction basis), and a promising form of such a matrix is given below [80]:

$$\lambda_u^{ij} = \begin{pmatrix} 0 & a_u\epsilon_u^3 & b_u\epsilon_u^3 \\ a_u\epsilon_u^3 & c_u\epsilon_u^2 & d_u\epsilon_u^2 \\ ? & ? & 1 \end{pmatrix} \lambda_t \qquad \lambda_d^{ij} = \begin{pmatrix} 0 & a_d\epsilon_d^3 & b_d\epsilon_d^3 \\ a_d\epsilon_d^3 & c_d\epsilon_d^2 & d_d\epsilon_d^2 \\ ? & ? & 1 \end{pmatrix} \lambda_b \tag{5.9}$$

where $a_q, b_q, c_q, d_q$ with $q = u, d$ are $\mathcal{O}(1)$ coefficients; $\lambda_b$ and $\lambda_t$ are the bottom and top quark Yukawa coefficients; and the question marks indicate that the particular entry is weakly constrained. The above matrices are written in a left-right notation, that is, the left-handed fields $Q$ label rows, and the right-handed fields $d^c$ and $u^c$ label the columns. A fit to the data using this form of matrix was done in the third reference in [80] where a number of different scenarios were found with different $\mathcal{O}(1)$ coefficients. For example, one scenario has $a_u = 1.0,\ b_u = \mathcal{O}(1),\ c_u = 1.0,\ d_u = \mathcal{O}(1);\ a_d = 1.5,\ b_d = 0.4,\ c_d = 1.0,\ d_d = 1.3$ and $\lambda_t = \lambda_b \approx 0.5$ if the matrix is assumed at the GUT scale.

Diagonalizing the above matrices gives the following mass hierarchies $m_{d,u} : m_{s,c} : m_{b,t} = \epsilon_{d,u}^4 : \epsilon_{d,u}^2 : 1$. With $\epsilon_{u,d} \approx 0.05, 0.15$ then a good approximation to the observed mass hierarchies is generated. The product of the unitary transformations that diagonalize the above Yukawa matrices from the left generate an accurate CKM matrix with the following phenomenologically successful relations:

$$\left|\frac{V_{33}}{V_{32}}\right| \approx \sqrt{\frac{m_d}{m_s}} \tag{5.10}$$

$$V_{21} \approx \frac{m_s}{m_b} \tag{5.11}$$

$$V_{12} \approx \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} e^{i\delta} \tag{5.12}$$

where $V_{ij}$ label the $(i,j)$ entries of the CKM matrix $V_{CKM}^{ij}$ and $\delta$ is the CP violating phase entering the Jarlskog invariant.

Note that these approximate relations between the CKM elements and the quark masses are not explained in the Standard Model. However the Gatto-Sartori-Tonin (GSO) relation [81] in Eq.5.12 is motivated by $SO(10)$ GUTs since the $SO(10)$ Yukawa matrix $\lambda^{ij}$ is symmetric. With a symmetric form for the Yukawa matrix for the first two families, and a texture zero in the $(1,1)$ element, this relation gives an excellent fit to $V_{12}$ with $\delta \approx \pm 90°$. An $SO(10)$ Yukawa matrix $\lambda^{ij}$ is symmetric since $16 \times 16 \times 10$ is a symmetric product [51] ($16 \times 16 = 10_s$ where $10_s$ is a symmetric representation).

Although the simple $U(1)_F$ symmetry discussed in Section 5.6.1 produced a symmetric Yukawa matrix with an approximate texture zero in the $(1,1)$ element, it did not generate the full form of the matrices in Eq.5.9 and so didn't generate the correct CKM matrix values. In particular the ratio of the $(2,2)$ and $(2,3)$ matrix elements in Eq.5.9 is not close to 1 which is required to generate the relation Eq.5.12 for the $V_{21}$ entry. In fact, in general, simple Abelian family symmetries are unable to relate the $(2,2)$ and $(2,3)$ entries of the Yukawa matrix, which is seen as a failing of such family symmetries. However, this relation is possible in family symmetries that are based on a non-Abelian gauge group, which is the topic of the next Section.

### 5.6.3 Discrete non-Abelian Family Symmetries

Discrete non-Abelian family symmetries are family symmetries that are based on a discrete non-Abelian symmetry group. In this Section an example of a discrete non-Abelian family symmetry, called $\Delta_{27}$, is described that is taken from [74]. $\Delta_{27}$ is defined as the semi-direct product group $(Z_3 \times Z_3) \rtimes Z_3 \equiv Z_3' \rtimes Z_3$ [82], which is a subgroup of the continuous group $SU(3)$.[6] It only contains triplet and anti-triplet representations (as well as a singlet representation), and Table 5.2 illustrates the way in which these transform under the $Z_3$ and $Z_3'$. The family symmetry is assumed to commute with an $SO(10)$ SUSY GUT and the 16 multiplets that contain the quarks and leptons are taken to transform in the triplet representation of $\Delta_{27}$. The 10 multiplet that contains the up and down Higgs fields on the other hand is taken to be a singlet of the family symmetry.[7]

[6] $\Delta_{27}$ is in fact the smallest subgroup of $SU(3)$ that contains complex representations.

[7] The $\Delta_{27}$ family symmetry is chosen rather than, for example, $A_4$, since it allows complex representations whereas $A_4$ only contains real representations. Complex representations are required in family symmetry models in which the left-handed matter fields $F$ and right-handed matter fields $F^c$ both transform in triplet representations. This is to avoid the trivial combination $FF^ch$.

| Component of Field | $Z_3$ | $Z'_3$ |
|---|---|---|
| $\phi_{i=1}$ | $\phi_1$ | $\phi_2$ |
| $\phi_{i=2}$ | $\alpha\phi_2$ | $\phi_3$ |
| $\phi_{i=3}$ | $\alpha^2\phi_3$ | $\phi_1$ |

TABLE 5.2: Transformation properties of triplet field $\phi_i$ under the non-Abelian discrete group $\Delta_{27} = Z_3 \ltimes Z'_3$ where $\alpha$ is the cube root of unity. This table is taken from [74].

The $\Delta_{27}$ family symmetry then forbids all of the $SO(10)$ Yukawa interactions $\lambda^{ij}16_i16_j10$ since they are not $\Delta_{27}$ invariants. Instead these interactions are generated effectively from higher-order $\Delta_{27}$ invariant operators that come from messenger diagrams which are illustrated by Fig.5.1. This is analogous to the mechanism used to generate the Yukawa interactions in the $U(1)_F$ family symmetry model in Section 5.6.1.

The higher order operators contain flavon fields that transform as triplets and anti-triplets of $\Delta_{27}$. Six different flavon fields are used in this family symmetry: $\phi_3$, $\overline{\phi}_3$, $\phi_{23}$, $\overline{\phi}_1$, $\phi_{123}$ and $\overline{\phi}_{123}$ where the bar indicates that the flavons are anti-triplets of $\Delta_{27}$. The Subscripts indicate the components of $\Delta_{27}$ that develop VEVs, that is:

$$\langle\phi_3\rangle \propto \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \qquad \langle\phi_1\rangle \propto \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad \langle\phi_{123}\rangle \propto \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\langle\overline{\phi}_3\rangle \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \langle\overline{\phi}_{23}\rangle \propto \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \langle\overline{\phi}_{23}\rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Together these flavon fields break the $SU(3)_F$ symmetry to nothing. The way in which the flavon fields and quarks and leptons transform under $\Delta_{27}$ is given by Table 5.3 where an additional $U(1)$ and $Z_2$ symmetry is used to constrain the model. These additional symmetries prevent any phenomenologically disastrous higher-order operators but are flavour independent and therefore not family symmetries. For example, the $U(1)$ symmetry prevents the effective Yukawa operator $\frac{1}{M^2}16_i16_j\phi^i_{123}\phi^j_3$ from appearing in Eq.7.2. The leading order operators that are allowed by the model defined by Table 5.3 are the following [74]:

$$\begin{aligned} W_{Yuk} &= \frac{Y_3}{M^2}16_i16_j10\overline{\phi}^i_3\overline{\phi}^j_3 \\ &+ \frac{Y_2}{M^3}16_i16_j10\overline{\phi}^i_{23}\overline{\phi}^j_{23}H_{45} \\ &+ \frac{Y_1}{M^3}16_i16_j10\overline{\phi}^i_{23}\overline{\phi}^i_{123} \end{aligned} \tag{5.13}$$

where $i, j, k = 1 \ldots 3$ are $\Delta_{27}$ indices.

| Field | $\Delta_{27}$ | $U(1)_R$ | $U(1)$ | $Z_2$ |
|---|---|---|---|---|
| 16 | 3 | 1 | 0 | + |
| 10 | 1 | 0 | 0 | + |
| $\overline{16}$ | $\overline{3}$ | 0 | 0 | + |
| $H_{45}$ | 1 | 0 | 2 | + |
| $\phi_{123}$ | 3 | 0 | -1 | + |
| $\phi_3$ | 3 | 0 | 3 | + |
| $\phi_1$ | 3 | 0 | -4 | - |
| $\overline{\phi}_3$ | $\overline{3}$ | 0 | 0 | - |
| $\overline{\phi}_{23}$ | $\overline{3}$ | 0 | -1 | - |
| $\overline{\phi}_{123}$ | $\overline{3}$ | 0 | 1 | - |

TABLE 5.3: This table illustrates the $\Delta_{27}$ family symmetry model described in Section 5.6.3. The 16 $SO(10)$ multiplet contains the quarks and leptons, and the 10 multiplet contains the up and down Higgs fields. $U(1)_R$ is an R-symmetry, and $U(1) \times Z_2$ are additional symmetries that constrain the model and are family-independent. The table is based on Table 2 in [74].

Once the flavon fields develop VEVs, effective $SO(10)$ Yukawa interactions $\lambda^{ij}16_i16_j10$ are produced with coefficients given by the ratio of the flavon field VEVs and the messenger masses:

$$\lambda^{ij} = \begin{pmatrix} 0 & Y^1\epsilon\delta & Y^1\epsilon\delta \\ Y^1\epsilon\delta & Y^2\epsilon^2 & Y^2\epsilon^2 \\ Y^1\epsilon\delta & Y^2\epsilon^2 & Y^3 \end{pmatrix} \lambda_3$$

where $\langle\overline{\phi}_3\rangle/M \equiv \sqrt{\lambda_3}$, $\langle\overline{\phi}_{23}\rangle/M \equiv \sqrt{\lambda_3}\epsilon$ and $\langle\overline{\phi}_{123}\rangle/M \equiv \sqrt{\lambda_3}\delta$. As for the $U(1)_F$ family symmetry in Section 5.6.1, the $\Delta_{27}$ messenger scale is actually assumed to exist below the $SO(10)$ symmetry breaking scale so that the messengers that couple to the left-handed quarks can be heavier than those that couple to the right-handed quarks. Similarly the mass $M_u$ of the messengers that couple to the $u_i^c$ is assumed to be three times greater than the mass $M_d$ of those that couple to $d_i^c$. If $\langle\overline{\phi}_{23}\rangle = \sqrt{\lambda_3}\epsilon_d M_d$ and $\langle\overline{\phi}_{123}\rangle = \sqrt{\lambda_3}\epsilon_d^2 M_d$ then the operators in Eq.5.13 generate the form of Yukawa matrix given by Eq.5.9 but with the suppression factors $\epsilon$, $\delta$ replaced with $\epsilon_d$, $\epsilon_d^2$ and $\epsilon_u$, $\epsilon_u\epsilon_d$ for the down quark and up quark Yukawa matrix respectively.

The $(3,3)$ entry in the up Yukawa matrix however will be 9 times smaller than the equivalent entry in the down Yukawa matrix because of the VEV of $\overline{\phi}_3$. This would result in the top quark Yukawa coupling constant $\lambda_t$ being much smaller than the bottom Yukawa coupling constant $\lambda_b$ at the $\Delta_{27}$ symmetry breaking scale, which, when renormalized at the electroweak scale, would be in violation with experiment. Instead, if the $\Delta_{27}$ symmetry breaking scale is just below the GUT scale, then we require that $\lambda_t \approx \lambda_b \approx 0.5$. To achieve this the $\overline{\phi}_3$ flavon is chosen to transform as $2 \times 2$ under the $SU(2)_R$ subgroup of $SO(10)$. In this case it may acquire VEVs $a_3^u$, $a_3^d$ in the up and down $SU(2)_R$ directions. Then, with $a_3^u/M_u \approx a_3^d/M_d \approx 0.5$, we have comparable top and bottom Yukawa couplings $\lambda_t \approx \lambda_b \approx 0.5$ as required.[8] The up and down versions of the Yukawa matrix generated by the operators in Eq.5.13 then have the following form [83]:[9]

$$\lambda_u^{ij} \propto \begin{pmatrix} 0 & \epsilon_u^2\epsilon_d & -\epsilon_u^2\epsilon_d \\ \epsilon_u^2\epsilon_d & -2\epsilon_u^2\frac{\epsilon_u}{\epsilon_d} & 2\epsilon_u^2\frac{\epsilon_u}{\epsilon_d} \\ -\epsilon_u^2\epsilon_d & 2\epsilon_u^2\frac{\epsilon_u}{\epsilon_d} & 1 \end{pmatrix}\lambda_t, \qquad \lambda_d^{ij} \propto \begin{pmatrix} 0 & \epsilon_d^3 & -\epsilon_d^3 \\ \epsilon_d^3 & \epsilon_d^2 & -\epsilon_d^2 \\ -\epsilon_d^3 & -\epsilon_d^2 & 1 \end{pmatrix}\lambda_b. \tag{5.14}$$

Higher-order operators than those in Eq.5.13 that are allowed by Table 5.3 then modify the $(1,2)$ and $(1,3)$ up and down quark Yukawa entries so that the full up and down matrices agree with those given by Eq.5.9. These higher-order operators are [74]:

$$\begin{aligned} &\frac{1}{M^5}16_i 16_j 10\overline{\phi}_{123}^i\overline{\phi}_3^j(\overline{\phi}_{123}^m\phi_{1m})H_{45} \\ +&\frac{1}{M^6}16_i 16_j 10\overline{\phi}_{123}^i\overline{\phi}_{123}^j(\overline{\phi}_3^l\phi_{123l})(\overline{\phi}_3^m\phi_{123m}) \end{aligned} \tag{5.15}$$

where the $\mathcal{O}(1)$ coefficients are ignored.

Unlike the $U(1)_F$ family symmetry, the above $\Delta_{27}$ family symmetry model can thus predict an accurate mass hierarchy and CKM matrix for the up and down quarks. The $\Delta_{27}$ model also generates the correct mass hierarchy for the charged leptons due to the $H_{45}$ scalar field in Eq.5.13. The VEV of $H_{45}$ creates a factor of three in $(2,2)$, $(2,3)$ and $(3,2)$ elements of the charged lepton mass matrix compared to the down quark mass matrix. This occurs because the $H_{45}$ is assumed to get a VEV in the hypercharge direction and predominantly couples to the right-handed fields such that:

$$\frac{\langle H_{45}d^c\rangle}{\langle H_{45}e^c\rangle} \approx \frac{Y(d^c)}{Y(e^c)} = 3.$$

The charged lepton Yukawa matrix that is generated by the operators in Eq.5.13 is the following [83]:

$$\lambda_e^{ij} \propto \begin{pmatrix} 0 & \epsilon_d^3 & -\epsilon_d^3 \\ \epsilon_d^3 & 3\epsilon_d^2 & -3\epsilon_d^2 \\ -\epsilon_d^3 & -3\epsilon_d^2 & 1 \end{pmatrix}\lambda_b. \tag{5.16}$$

Another advantage of the $\Delta_{27}$ family symmetry is that it offers a simple explanation for why we have observed exactly three generations of quarks and leptons. This is because they are taken to transform in the triplet representation, which becomes the

[8] This complicated mechanism for creating the third family Yukawa coupling constants is a failing of general non-Abelian family symmetry models.

[9] The factor of three follows from the explanation of electric charge quantization in Grand Unified Theories. That is, the magnitude of charge of the proton is equal to that of the electron because quarks come in three colours.

three generations once the $\Delta_{27}$ symmetry is broken to nothing. For family symmetries based on a continuous non-Abelian symmetry such as $SU(3)$ the quarks and leptons could also be placed in a triplet representation which, once the family symmetry is broken, effectively becomes the three generations. However, in this case we could have just as easily put the quarks and leptons into a different $SU(3)$ representation such as a sextet which would decompose to six generations. In fact since there is an infinite number of representations of $SU(3)$, in principal, we could have put the quarks and leptons into any number of representations. The number of representations present in discrete groups on the other hand is, by definition finite, and some groups such as $A_4$ only have dimension one and (real) dimension three representations, significantly improving the theoretical reasoning for why three generations have been observed.

Discrete symmetries are also well motivated from high-energy theories. For example, discrete non-Abelian symmetries can arise after the compactification of extra spacial dimensions, and this origin of discrete family symmetries has recently been studied in the context of string theory [84].

### 5.6.4 Vacuum Alignment

If the above $\Delta_{27}$ family symmetry is to explain the quark and lepton masses and mixing angles, then we must understand how and why the flavon fields in Eq.5.13 obtain VEVs in certain $\Delta_{27}$ components. For the discrete non-Abelian $\Delta_{27}$ family symmetry a simple mechanism that only involves the D-terms of the flavon fields is used to achieve the desired alignment. This compares to the more complex mechanisms required for continuous non-Abelian family symmetries such as those based on the $SU(3)$ group where additional driving fields [83] are included that arrange the F-terms of the flavons to give a scalar potential whose minimum has the desired vacuum alignment.

Since $\Delta_{27}$ is a discrete subgroup of $SU(3)$, all operators that are invariants of $SU(3)$ are also invariants of $\Delta_{27}$. It is the additional operators that are allowed by $Z_3 \ltimes Z_3'$ and not $SU(3)$ however that determine the vacuum structure of the flavon fields if they appear as higher order terms in the potential. This is because these terms prevent it from being possible to rotate the vacuum expectation value of a triplet field to a single direction, for example the 3 direction, which is conversely always possible for a continuous $SU(3)$ symmetry [74].

To make this more explicit, consider a general $\Delta_{27}$ triplet field $\phi_i$. It will have a SUSY breaking soft mass term in the Lagrangian of the form $m_\phi^2 \phi^{i\dagger}\phi_i$ which is invariant under the approximate $SU(3)$ symmetry. Radiative corrections may drive the mass squared negative at some scale triggering a VEV for the field. At this stage, the VEV of $\phi_i$ can always be rotated to the 3 direction using the approximate $SU(3)$ symmetry. However this does not remain true if higher order terms from messenger field interactions that are allowed by $\Delta_{27}$ but not $SU(3)$ are included. For example suppose that the leading higher order term in the potential is of the form $m^2_{3/2}(\phi^\dagger\phi\phi^\dagger\phi)$. This has two independent quartic $\Delta_{27}$ invariants: $m'^2_\phi\phi^{i\dagger}\phi_i\phi^{j\dagger}\phi_j$ and $m'^2_\phi\phi^{i\dagger}\phi_i\phi^{i\dagger}\phi_i$ where the former is $SU(3)$ invariant but the latter is not. The latter invariant has the potential to remove the vacuum degeneracy in $\phi_i$. For example, if $m'^2_\phi < 0$ then the we must have $\langle\phi\rangle \propto (0,0,1)^T$, which defines the first component, whereas if $m'^2_\phi > 0$ then we instead obtain $\langle\phi\rangle \propto (1,1,1)^T/\sqrt{3}$. The configuration $\langle\phi\rangle \propto (0,-1,1)^T/\sqrt{2}$ can then be generated using a leading higher order term that requires that the VEV be orthogonal to both $(1,0,0)^T$ and $(1,1,1)^T$. All these operators can be used to generate the VEV configurations of the flavons used for the $\Delta_{27}$ family symmetry described in Section 5.6.3, that is, for the flavons $\phi_1$, $\phi_3$, $\phi_{123}$, $\overline{\phi}_3$, $\overline{\phi}_{23}$ and $\overline{\phi}_{123}$ [74]. These VEV configurations where used in Eq.5.13.

## 5.7 Family Symmetries and Tri-Bi-Maximal Mixing

So far we have only been looking at how the quark masses and CKM elements can be explained by family symmetries. In this Section family symmetries are instead used to explain the recently observed neutrino masses and oscillations. The fact that latest experimental data for the neutrino masses and oscillations, given in Section 5.3, contains large errors however makes it difficult to determine what, if any, family symmetry is responsible for the recent observations. To tackle this, the general approach taken is to choose a particular form of MNS matrix $V_{MNS}$ and neutrino hierarchy that is consistent with the present data. A particularly exciting form of the MNS matrix is a tri-bi-maximal matrix in which the $\nu_3$ neutrino mass eigenstate is a 'bi-maximal' mixture of the neutrino flavour eigenstates $\nu_\mu$ and $\nu_\tau$, and the $\nu_2$ neutrino mass eigenstate is a 'tri-maximal' mixture of $\nu_e$, $\nu_\mu$, $\nu_\tau$ [20]. The tri-bi-maximal matrix is defined by:

$$\begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu1}|^2 & |U_{\mu2}|^2 & |U_{\mu3}|^2 \\ |U_{\tau1}|^2 & |U_{\tau2}|^2 & |U_{\tau3}|^2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} \tag{5.17}$$

where $U_{fe}$, with $f = e, \mu, \tau$ and $i = 1 \ldots 3$, are the MNS matrix elements. The lepton mixing angles generated by this matrix are $\theta_{12} = \sin^{-1}(\frac{1}{\sqrt{3}}) = 35.2°$, $\theta_{23} = \sin^{-1}(\frac{1}{\sqrt{2}}) = 45°$, and $\theta_{13} = 0°$.

| | $\Delta_{27}$ | $U(1)$ | $Z_2$ | $Z'_2$ |
|---|---|---|---|---|
| $L_i$ | 3 | 0 | + | + |
| $h_u$ | 1 | 0 | + | + |
| $\nu_{R1}$ | 1 | 2 | - | + |
| $\nu_{R2}$ | 1 | 1 | + | - |
| $\nu_{R3}$ | 1 | 0 | + | + |
| $\phi$ | 1 | 1 | + | + |
| $\overline{\phi}_{23}$ | $\overline{3}$ | -2 | - | + |
| $\overline{\phi}_{123}$ | $\overline{3}$ | -1 | + | - |

TABLE 5.4: A $\Delta_{27} \times U(1) \times Z_2 \times Z'_2$ family symmetry that generates tri-bi-maximal mixing for neutrinos via CSD.

This form of $V_{MNS}$ matrix is very different to the quark CKM matrix $V_{CKM}$ given in Eq.5.5. Therefore, if we are to explain both $V_{MNS}$ and $V_{CKM}$ from a family symmetry, either the family symmetry is acting differently in the quark and lepton sectors [85], or the family symmetry is acting the same in both sectors but something else is distinguishing between them. If the family symmetry commutes with an $SO(10)$ GUT then we can only consider the latter scenario since quark and leptons are unified in the same representation. In Section 5.3 the see-saw mechanism was motivated as being responsible for small neutrino masses and obviously distinguishes the quark and lepton sectors. Indeed, when used in conjunction with certain family symmetries, this mechanism can generate a tri-bi-maximal form for $V_{MNS}$ [85, 86]. The $\Delta_{27}$ family symmetry model in [74] which was described in Section 5.6.3 uses a particular Type I see-saw mechanism called constrained sequential dominance (CSD) [87] to generate a tri-bi-maximal $V_{MNS}$ matrix. In CSD three right-handed neutrinos are assumed with a conventional hierarchical structure and, in the basis in which the Majorana mass matrix for the right-handed neutrinos $\mathbf{M}_{RR}$ is diagonal (see Eq.5.18), the Dirac Yukawa matrix for the neutrinos is of the form given in Eq.5.18 below:

$$\mathbf{M}_{LR} = \begin{pmatrix} 0 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}, \qquad \mathbf{M}_{RR} = \begin{pmatrix} M_A & 0 & 0 \\ 0 & M_B & 0 \\ 0 & 0 & M_C \end{pmatrix} \tag{5.18}$$

where $A_1 = 0$, $|A_2| = |A_3|$, $|B_1| = |B_2| = |B_3|$, $A_2B_2 = A_3B_3$, and $M_A \ll M_B \ll M_C$.

The unitary matrix $V_\nu$ that diagonalized $\mathbf{M}_{LL} = \mathbf{M}_{LR}\mathbf{M}_{RR}^{-1}\mathbf{M}_{LR}^T$ can be shown to be a tri-bi-maximal matrix Eq.5.17. Therefore, if the charged lepton Yukawa matrix is diagonal in this basis, the $V_{MNS}$ matrix is also a tri-bi-maximal matrix. The equivalence of the modulus of the $(1,2)$, $(2,2)$ and $(3,2)$ elements of the above CSD matrix (Eq.5.18) suggests that a $\overline{\phi}_{123}$ flavon is coupling to the left-handed neutrinos [87], where the VEV of $\overline{\phi}_{123}$ is given in Section 5.6.3. The equivalence of the $(2,1)$ and $(3,1)$ elements also suggests a $\overline{\phi}_{23}$ flavon [87]. For example, to generate the matrices in Eq.5.18 required for CSD we could assume a $\Delta_{27} \times U(1)$ family symmetry for which the left-handed neutrinos are triplets of $\Delta_{27}$ but have zero $U(1)$ charge, and the right-handed neutrinos $\nu_{R1}$, $\nu_{R2}$ and $\nu_{R3}$ are singlets of $\Delta_{27}$ with +2, +1, and zero $U(1)$ charges. The symmetries and flavons of this model are illustrated in Table 5.4 and allow the following higher-order operators:

$$W_{LR} = \frac{1}{M} L_i \nu_{R1} h_u \overline{\phi}^i_{23} + \frac{1}{M} L_i \nu_{R2} h_u \overline{\phi}^i_{123}$$

$$W_{RR} = \frac{1}{M^4} \nu_{R1} \nu^c_{R1} \phi^4 + \frac{1}{M^2} \nu_{R2} \nu^c_{R2} \phi^2 + \nu_{R3} \nu^c_{R3}$$

where $\phi$ is a singlet of $\Delta_{27}$ that has a $U(1)$ charge of $-1$ and develops a VEV which is much smaller than $M$.

These operators would generate a diagonal right-handed Majorana with a hierarchical structure and a Dirac mass matrix given by Eq.5.18, which together create a tri-bi-maximal matrix for the $V_{MNS}$ matrix provided that the charged lepton mass matrix is diagonal. For the $\Delta_{27}$ family symmetry described in Section 5.6.3 however the above operators cannot be included in the superpotential. This is because the $\Delta_{27}$ symmetry commutes with an $SO(10)$ symmetry which requires that the charge-conjugated neutrinos $\nu^c$ and left-handed neutrinos $\nu_L$ come from the same 16 representation and thus must come from the same $\Delta_{27}$ representation. Instead the $\Delta_{27}$ family symmetry uses the method of CSD to generate tri-bi-maximal mixing, but in a different basis to the one in which the right-handed Majorana mass matrix is diagonal. This utilizes the fact that the see-saw mechanism, and thus CSD, is invariant to the following non-unitary transformations [88]:

$$\mathbf{M}_{LR} \rightarrow \mathbf{M}_{LR}\mathbf{S}^{-1}, \qquad \mathbf{M}^{-1}_{RR} \rightarrow \mathbf{S}\mathbf{M}^{-1}_{RR}\mathbf{S}^T \tag{5.19}$$

where $\mathbf{S}$ is a non-unitary matrix that is not unique. These transformations leave the effective low-energy neutrino mass matrix $M_{LL}$ given by Eq.5.6 invariant. The $\Delta_{27}$ family symmetry model uses the following $\mathbf{M}_{LR}$ and $\mathbf{M}_{RR}$ matrices [88]:

$$\mathbf{M}_{RR} = \begin{pmatrix} M_A & M_A & 0 \\ M_A & M_A + M_B & 0 \\ 0 & 0 & M_C \end{pmatrix}, \tag{5.20}$$

$$\mathbf{M}_{LR} = \begin{pmatrix} 0 & B & C_1 \\ A & B + A & C_2 \\ -A & B - A & C_3 \end{pmatrix} = \begin{pmatrix} 0 & A & -A \\ A & 2A & 0 \\ -A & 0 & C_3 \end{pmatrix}$$

where $A = B$ is used and symmetric matrices are assumed because of an $SO(10)$ symmetry. The matrices $\mathbf{M}_{LR}$ and $\mathbf{M}_{RR}$ in the original CSD basis are then obtained by the transformations in Eq.5.19 with the $\mathbf{S}$ matrix given by [88]:

$$\mathbf{S}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This illustrates that the $\mathbf{M}_{RR}$ and $\mathbf{M}_{LR}$ matrices used in the $\Delta_{27}$ family symmetry model will generate the same see-saw mechanism as those in Eq.5.18. The $SO(10) \times \Delta_{27}$ operators in Eq.5.13 are responsible for generating the form of $\mathbf{M}_{LR}$ in Eq.5.20. This uses the fact that, contrary to the other messengers, the messengers that couple to the right-handed neutrinos are anticipated to be much heavier than the messengers that couple to the left-handed neutrinos. Then, given that right-handed neutrinos have zero hypercharge, the operators in Eq.5.13 that couple to $H_{45}$ are subdominant since the VEV of $H_{45}$ only picks out the left-handed messengers. The $H_{45}$, in addition to the see-saw mechanism, also distinguishes between the quark and lepton sectors.

The form of matrix $\mathbf{M}_{RR}$ in Eq.5.20 is generated by the following Majorana operators in the $SO(10) \times \Delta_{27}$ model [74]:

$$\begin{aligned} W_{Maj} = & \frac{1}{M_R} 16^i 16^j \overline{16}_i \overline{16}_j \\ & + \frac{1}{M_R^5} 16^i 16^j \overline{16}_k \overline{16}_l \overline{\phi}_{23i} \overline{\phi}_{23j} \phi_{123}^k \phi_3^l \\ & + \frac{1}{M_R^5} 16^i 16^j \overline{16}_k \overline{16}_l \overline{\phi}_{123i} \overline{\phi}_{123j} \phi_{123}^k \phi_{123}^l \end{aligned} \tag{5.21}$$

where $\overline{16}$ is a field of $SO(10)$ that obtains a VEV in the right-handed neutrino direction.

The effective Majorana matrix for the left-handed neutrinos $\mathbf{M}_{LL}$ in the $SO(10) \times \Delta_{27}$ model is then generated by the see-saw mechanism $\mathbf{M}_{LR}\mathbf{M}_{RR}^{-1}\mathbf{M}_{LR}^T$ where $\mathbf{M}_{LR}$ and $\mathbf{M}_{RR}$ are of the form given by Eq.5.20. The unitary matrix $V_\nu$ that diagonalizes this is a tri-bi-maximal matrix due to CSD. From Section 5.3.1 the MNS matrix is given by $V_{MNS} = V_{eL} V_\nu^\dagger$ and therefore we require that $V_{eL} = \mathbf{1}$ for it to be of tri-bi-maximal form. However this is not the case in the $SO(10) \times \Delta_{27}$ model since the charged lepton Yukawa matrix is not diagonal as illustrated by Eq.5.16. Since the off-diagonals of Eq.5.16 are small however, the left-handed unitary matrix that diagonalizes it $V_{eL}$ is close to diagonal and so $V_{MNS}$ does not differ significantly from a tri-bi-maximal form [88]. The predicted lepton mixing angles are found in [74] and are in agreement with experiment.

## 5.8 Family Symmetries and SUSY Flavour Problems

Section 5.4 discussed how the flavour problem of the Standard Model is enlarged in the MSSM because of the introduction of new undetermined free parameters in the soft SUSY breaking Lagrangian. Phenomenology seems to be telling us that the off-diagonal elements in the soft SUSY breaking Lagrangian should be smaller than the diagonal elements in order to suppress SUSY induced flavour changing neutral currents. However, in general, there is no a priori reason for why this should be the case.[10]

Extending the MSSM with a non-Abelian family symmetry can provide a resolution to this SUSY flavour problem [22]. The non-Abelian family symmetries, when combined appropriately with SUSY, can control the structure of the soft mass matrices (as well as the Yukawa couplings), in such a way that SUSY induced flavour changing neutral currents are naturally suppressed. For example, when extended with an $SU(3)$ family symmetry [22] the soft squark and slepton mass squared matrices in the MSSM would have a universal form, proportional to unit matrices, in the limit that the family symmetry is unbroken. However, in this limit the Yukawa and soft trilinear matrices vanish, so the family symmetry must be spontaneously broken, leading simultaneously to flavour in the Yukawa sector, and violations of universality in the soft SUSY breaking sector. The violations of squark and slepton soft mass universality are therefore controlled by the same order parameters $\epsilon$ that are responsible for the origin of Yukawa couplings, resulting in the prediction of suppressed FCNCs. The $SU(3)$ family symmetry thus provides simultaneously a solution to the flavour problem not only in the Standard Model but also in its SUSY extensions such as the MSSM.

Another facet of the SUSY flavour issue is the so called SUSY CP problem stemming from the fact that in general there could be large extra CP phases coming from the soft SUSY breaking sector of the MSSM. However, the Standard Model accounts for the observed CP violating effects to such a level of accuracy that we must impose stringent bounds on such extra contributions to avoid conflict with experiment. This is, however, often at odds with naturalness. In the $SU(3)$ family symmetry models a potential solution to the SUSY CP problem results if the origin of CP violation is due to the spontaneous breaking of the $SU(3)$ family symmetry via flavon vacuum expectation values [22]. Such a scenario leads to suppressed SUSY induced CP violation since CP is preserved in the symmetry limit, and once spontaneously broken, the CP violating effects are in general suppressed in terms of powers of the symmetry breaking flavon VEVs.

[10] Specific frameworks such as minimal supergravity (mSUGRA), under certain assumptions about the hidden sector couplings that break SUSY, can predict universality of soft mass matrices.

# Chapter 6

# Exceptional Supersymmetric Standard Models with Family Symmetry

The purpose of the present Chapter is to extend the ME$_6$SSM (and E$_6$SSM) to include a discrete non-Abelian family symmetry as a step towards solving the flavour problem in these models. In particular, the $\Delta_{27}$ family symmetry [74] that was discussed in Section 5.6.3 is used. This is convenient since the $U(1)_N$ and $U(1)_X$ groups of the E$_6$SSM and ME$_6$SSM are defined to allow a conventional see-saw mechanism, which, together with a $\Delta_{27}$ family symmetry, can generate small neutrino masses and tri-bi-maximal mixing. In a model with a family symmetry the Higgs field's VEV is used to generate the quark and lepton masses, and so the model should ideally also explain the hierarchy problem, that is, it must explain why electroweak symmetry breaking occurs at scales much smaller than the Planck scale. This motivates extending the ME$_6$SSM (and E$_6$SSM) with a family symmetry since in this model the Higgs mass is protected by supersymmetry and there is no $\mu$-problem or little hierarchy problem. Extending the MSSM or a simple $SO(10)$ SUSY GUT with a family symmetry on the other hand generically generates models that suffer from the $\mu$-problem.

The detailed strategy pursued is as follows. The $\Delta_{27}$ family symmetry used in the $SO(10) \times \Delta_{27}$ model described in Section 5.6.3 is introduced to the intermediate Pati-Salam symmetry of the ME$_6$SSM to build a model with a $G_{4221} \times \Delta_{27}$ gauge group where $G_{4221} \equiv SU(4)_{PS} \times SU(2)_L \times SU(2)_R \times U(1)_\psi$. The resulting model can explain the observed mixing angles and mass spectrum of the quarks and leptons, provide a tri-bimaximal mixing for the neutrinos, solve the $\mu$-problem and small fine-tuning problem, and does not involve doublet-triplet splitting. A novel feature of the model is that proton decay is suppressed in a new way by the assumed $\Delta_{27}$ family symmetry and an $E_6$ singlet.

Once a model based on the ME$_6$SSM with $\Delta_{27}$ family symmetry is built we can then relate this model to an E$_6$SSM with $\Delta_{27}$ family symmetry model. This is because, from Section 4.4.2, if $g_4 = g_{2R}$ at the $G_{4221}$ symmetry breaking scale, then the $U(1)_X$ group of the ME$_6$SSM becomes equivalent to the $U(1)_N$ group of the E$_6$SSM. To achieve $g_4 = g_{2R}$ at the $G_{4221}$ symmetry breaking scale we can add the $H'$ and $\overline{H}'$ states of the E$_6$SSM to the ME$_6$SSM so that the gauge coupling constants unify at the conventional GUT scale. Thus by adding $H'$ and $\overline{H}'$ to the ME$_6$SSM with $\Delta_{27}$ family symmetry we will generate a model based on the E$_6$SSM with $\Delta_{27}$ family symmetry. It should be emphasized however that the E$_6$SSM formulated in this way is not exactly the E$_6$SSM described in Section 2.6, which shall be referred to as the 'original' E$_6$SSM. This is because the 'new' E$_6$SSM is built on a Pati-Salam symmetry and so we cannot use the $Z_2^L$ and $Z_2^B$ symmetries of the original E$_6$SSM to forbid the proton decay induced by the Higgs triplet fields. Instead the induced proton decay is suppressed by small Yukawa couplings as in the ME$_6$SSM. Thus, in the original E$_6$SSM the Higgs triplets couple as either diquarks or leptoquarks, whereas the highly suppressed couplings in the new E$_6$SSM imply long-lived TeV mass Higgs triplets with a lifetime typically about 0.1 sec for example. This is the only phenomenological difference between the new and original E$_6$SSM. For convenience the 'new' E$_6$SSM is just refereed to as the E$_6$SSM in the rest of this Chapter.

The resulting models are defined in Table 6.1 where, in addition to the Pati-Salam, $\Delta_{27}$ and $U(1)_\psi$ symmetries, extra discrete and Abelian symmetries are also applied to constrain the models into realistic theories. This is most simply achieved by the combined symmetries $U(1)_R \times U(1) \times Z_2 \times Z_2^H$, where $U(1)_R$ is an R-symmetry that contains the R-parity of the MSSM as a subgroup. The $U(1) \times Z_2$ symmetries are adapted from the $\Delta_{27}$ symmetry in Section 5.6.3, and the $Z_2^H$ is from the E$_6$SSM and ME$_6$SSM.

The next Section reviews the ME$_6$SSM and discusses how it can be extended with the $\Delta_{27}$ family symmetry from Section 5.6.3. Sections 6.2-6.6 then investigate how the different ME$_6$SSM superpotential terms are modified by the inclusion of the $\Delta_{27}$ family symmetry. In particular, Section 6.2 illustrates how the Yukawa couplings are generated in the new model. Section 6.3 looks at how the model predicts approximate tri-bi-maximal mixing for leptons. Section 6.4 then discusses the effective $\mu$-term in the model and how the triplet higgsinos get mass. Section 6.5 describes how the $\Delta_{27}$ family symmetry can be used to tame the proton decay induced by the Higgs triplets, and, Section 6.6 discusses the origin of R-parity in the model. Section 6.7 then adds the $H'$

| Field | $\Delta_{27}$ | $SU(4)_{PS} \times SU(2)_L \times SU(2)_R \times U(1)_\psi$ | $U(1)_R$ | $U(1)$ | $Z_2$ | $Z_2^H$ |
|---|---|---|---|---|---|---|
| $F$ | 3 | $(4,2,1)_{\frac{1}{2}}$ | 1 | 0 | + | - |
| $F^c$ | 3 | $(\overline{4},1,\overline{2})_{\frac{1}{2}}$ | 1 | 0 | + | - |
| $h_3$ ; $h_{1,2}$ | 1 | $(1,2,2)_{-1}$ | 0 | 0 | + | + ; - |
| $\mathcal{D}_{1,2,3}$ | 1 | $(6,1,1)_{-1}$ | 0 | 0 | + | - |
| $S_3$ ; $S_{1,2}$ | 1 | $(1,1,1)_2$ | 2 | 0 | + | + ; - |
| $16_H = \overline{H}_R,\ H_L$ | 3 | $(\overline{4},1,\overline{2})_{\frac{1}{2}},\ (4,2,1)_{\frac{1}{2}}$ | 0 | 0 | + | + |
| $\overline{16}_H = H_R,\ \overline{H}_L$ | $\overline{3}$ | $(4,1,2)_{-\frac{1}{2}},\ (\overline{4},\overline{2},1)_{-\frac{1}{2}}$ | 0 | 0 | + | + |
| $M$ | 1 | $(1,1,1)_0$ | 2 | 0 | + | + |
| $\Sigma$ | 1 | $(1,1,1)_0$ | 0 | 5 | - | - |
| $H_{45}$ | 1 | $(15,1,3)_0$ | 0 | 2 | + | + |
| $\phi_{123}$ | 3 | $(1,1,1)_0$ | 0 | -1 | + | + |
| $\phi_3$ | 3 | $(1,1,1)_0$ | 0 | 3 | + | + |
| $\phi_1$ | 3 | $(1,1,1)_0$ | 0 | -4 | - | + |
| $\overline{\phi}_3$ | $\overline{3}$ | $(1,1,2\times 2)_0$ | 0 | 0 | - | + |
| $\overline{\phi}_{23}$ | $\overline{3}$ | $(1,1,1)_0$ | 0 | -1 | - | + |
| $\overline{\phi}_{123}$ | $\overline{3}$ | $(1,1,1)_0$ | 0 | 1 | - | + |

Table 6.1: All the particles (excluding the messengers) contained in the ME$_6$SSM with a $\Delta_{27}$ family symmetry model. $U(1)_R \times U(1) \times Z_2 \times Z_2^H$ are additional constraining symmetries that are family-independent. The addition of the $H'$ and $\overline{H}'$ fields from split $G_{4221}$ representations generates a model based on the E$_6$SSM with a $\Delta_{27}$ family symmetry, where the $E_6$ symmetry is broken via the Pati-Salam chain.

and $\overline{H}'$ states to create a model based on the E$_6$SSM with $\Delta_{27}$ family symmetry, and Sections 6.8 and 6.9 explore how the running of the ME$_6$SSM and E$_6$SSM gauge coupling constants are modified by the inclusion of a $\Delta_{27}$ family symmetry. Finally, Section 6.10 concludes the Chapter.

## 6.1 The ME$_6$SSM with a $\Delta_{27}$ Family Symmetry

In Section 5.6.3 a $\Delta_{27}$ family symmetry was applied to an $SO(10)$ GUT to solve the flavour problem of the Standard Model (and the MSSM). That is, the formulated $SO(10) \times \Delta_{27}$ model explained the different masses and mixings of quarks and leptons that we observe in particle experiments, but which are unexplained in the Standard Model. This Section describes how this $\Delta_{27}$ family symmetry can be applied to the ME$_6$SSM, which was constructed in Chapter 4.

The ME$_6$SSM is an E$_6$ inspired supersymmetric theory where the E$_6$ symmetry is assumed to come from a non-QFT theory that is broken near the Planck scale. Between the Planck scale and the conventional GUT scale however a conventional QFT is assumed that has a $G_{4221}$ gauge symmetry. This gauge symmetry must be a symmetry of a QFT since its RGEs were used to predict gauge coupling unification at the Planck scale. The $G_{4221}$ symmetry is thus a symmetry of the ME$_6$SSM whereas the E$_6$ symmetry is not. That is, the low-energy physics must obey a $G_{4221}$ symmetry but not necessarily an E$_6$

symmetry. However, although $E_6$ is not a symmetry of the ME$_6$SSM, it is assumed to contain the $G_{4221}$ states that make up three copies of a fundamental 27 representation of $E_6$ at low energies.[1]

Since $G_{4221}$ is a maximal subgroup of $E_6$, the $G_{4221}$ superpotential for three 27 representations is the same as the $G_{4221}$ superpotential derived from the $E_6$ superpotential $\lambda^{ijk}27_i 27_j 27_k$, where $i, j, k = 1 \ldots 3$ label the three copies and $\lambda^{ijk}$ are coupling constants:

$$27_i 27_j 27_k = F_i F_j^c h_k + F_i F_j \mathcal{D}_k + F_i^c F_j^c \mathcal{D}_k + S_i h_j h_k + S_i \mathcal{D}_j \mathcal{D}_k \tag{6.1}$$

where the coupling constants $\lambda^{ijk}$ have been omitted for clarity.

If the ME$_6$SSM is to be extended with a family symmetry then it must commute with the $G_{4221}$ symmetry but does not necessarily commute with an $E_6$ symmetry. We must therefore formulate a theory based on $G_{4221} \times \Delta_{27}$. Sections 6.2-6.5.1 investigate how the above ME$_6$SSM superpotential terms are modified by the addition of a $\Delta_{27}$ family symmetry.

## 6.2 Yukawa Interactions

In the ME$_6$SSM the quarks and leptons come from the Pati-Salam representations $F_i$ and $F_i^c$, and the Higgs fields that break the electroweak symmetry and give mass to the quarks and lepton are defined as the third generation of the $h_i$ representations, where $i = 1 \ldots 3$. In the $SO(10) \times \Delta_{27}$ model described in Section 5.6.3 the quarks and leptons come from the fundamental spinor representation 16 of $SO(10)$ and are taken to transform as triplets of $\Delta_{27}$. The Higgs fields on the other hand come from the fundamental representation 10 of $SO(10)$ and are singlets of $\Delta_{27}$. The Pati-Salam states contained in these $SO(10)$ representations are the following: $16 = F + F^c$ and $10 = h + \mathcal{D}$. Therefore, following the $\Delta_{27}$ family symmetry, the $F_i$ and $F_i^c$ of the ME$_6$SSM are taken to transform as $\Delta_{27}$ triplets, and $h_3$ as a singlet. This forbids the superpotential term $\lambda^{ij} F_i F_j^c h_3$ in Eq.6.1, where $\lambda^{ij}$ are theoretically undetermined Yukawa coefficients. Instead higher-order operators are allowed that contain $\Delta_{27}$ flavon fields. The VEVs of these flavon fields then break the $\Delta_{27}$ family symmetry and generate effective Yukawa interactions.

[1]Hypothetically, the $E_6$ symmetry could, for example, be a symmetry of a string theory which is broken via Wilson lines to the $G_{4221}$ symmetry at the Planck scale. The $G_{4221}$ states could then come from different 27 $E_6$ multiplets [58], which, if taken to come from the same $E_6$ multiplets, do not commute with an $E_6$ symmetry.

The same type of flavon fields that were used in Section 5.6.4 are assumed to couple to the quarks and leptons. These flavon fields are $\overline{\phi}_3$, $\overline{\phi}_{23}$, $\overline{\phi}_{123}$, $\phi_1$ and $\phi_{123}$ where the subscripts denote the components of $\Delta_{27}$ that obtain VEVs. The leading higher-order operators allowed by the symmetries are then:

$$\frac{Y^{33}}{M_R^2} F_i F_j^c h_3 \overline{\phi}_3^i \overline{\phi}_3^j \tag{6.2}$$

$$\frac{Y^{22}}{M_R^3} F_i F_j^c h_3 \overline{\phi}_{23}^i \overline{\phi}_{23}^j H_{45} \tag{6.3}$$

$$\frac{1}{M_R^2} F_i F_j^c h_3 (Y^{13} \overline{\phi}_{123}^i \overline{\phi}_{23}^j + Y^{31} \overline{\phi}_{123}^j \overline{\phi}_{23}^i) \tag{6.4}$$

$$\frac{1}{M_R^5} F_i F_j^c h_3 (Y^{12} \overline{\phi}_3^i \overline{\phi}_{123}^j + Y^{21} \overline{\phi}_3^j \overline{\phi}_{123}^i)(\overline{\phi}_{123}^k \phi_{1k}) H_{45} \tag{6.5}$$

$$\frac{Y^{11}}{M_R^6} F_i F_j^c h_3 \overline{\phi}_{123}^i \overline{\phi}_{123}^j (\overline{\phi}_3^k \phi_{123k})(\overline{\phi}_3^l \phi_{123l}) \tag{6.6}$$

where the Latin indices refer to the $\Delta_{27}$ symmetry, $Y^{ij}$ are order one coupling constants, and $M_R$ is the mass of right-handed messengers, which is explained below. The $H_{45}$ in Eq.6.3 and Eq.6.5 is a $\Delta_{27}$ singlet that transforms as $(15, 1, 3)_0$ under the $G_{4221}$ symmetry. $(15, 1, 3)$ is the Pati-Salam component of the $H_{45}$ field used in Section 5.6.3, which is a 210 multiplet of $SO(10)$. This field gets a VEV in the hypercharge direction generating the Georgi-Jarlskog factor for Eq.6.3.

The high-order superpotential terms given by Eq.6.2-Eq.6.6 are assumed to come from renormalizable, high-energy interactions involving heavy vector-like messengers that transform in the same way as the quark and lepton fields under the $G_{4221}$ symmetry. These messengers are integrated out of the high energy theory to generate the above suppressed superpotential terms. To distinguish the Yukawa matrices for the up and down quarks we require that the $SU(2)_R$ messengers dominate over the $SU(2)_L$ messengers and, for the correct up and down Yukawa matrices, we require that the up and down right-handed messengers have mass $M_u$ and $M_d$ related by $M_u \approx \frac{1}{3} M_d$. $M_R$ is used to denote the right-handed messenger scale, which could be $M_u$ or $M_d$ depending on the interactions involved.

The above higher-order operators are essentially the Pati-Salam versions of the higher-order Yukawa operators in Eq.5.13 and Eq.5.15 for the $SO(10) \times \Delta_{27}$ model. The $SO(10)$ product $16 \times 16 \times 10$ contains the following Pati-Salam products $FF^c h + FF\mathcal{D} + F^c F^c \mathcal{D}$ and the operators that contain the products $FF\mathcal{D}$ and $F^c F^c \mathcal{D}$ were not considered to be not important in the $SO(10) \times \Delta_{27}$ model since the $\mathcal{D}$ states can get GUT scale masses from a doublet-triplet splitting mechanism. Section 5.6.3 showed that the higher-order operators in Eq.5.13 and Eq.5.15 can create the experimentally observed values of quark and (charged) lepton masses and CKM matrix elements. However, the operators in Eq.5.13 are invariant to an $SO(10)$ GUT symmetry which causes the Yukawa matrices Eq.5.14 to be (approximately) symmetrical. $SO(10)$ is not a symmetry of the ME$_6$SSM and so there is no a prior reason why the Yukawa matrices generated by the operators in Eq.6.2-Eq.6.6 are symmetrical. Instead the first three operators in Eq.6.2-Eq.6.4 generate the following up and down quark Yukawa matrices [74]:

$$\lambda_d^{ij} \propto \begin{pmatrix} 0 & Y^{13}\epsilon_d^3 & -Y^{13}\epsilon_d^3 \\ Y^{31}\epsilon_d^3 & \epsilon_d^2 & -\epsilon_d^2 \\ -Y^{31}\epsilon_d^3 & -\epsilon_d^2 & \epsilon_3^2 \end{pmatrix} \lambda_b, \tag{6.7}$$

$$\lambda_u^{ij} \propto \begin{pmatrix} 0 & Y^{13}\epsilon_u^2\epsilon_d & -Y^{13}\epsilon_u^2\epsilon_d \\ Y^{31}\epsilon_u^2\epsilon_d & -2\epsilon_u^2\frac{\epsilon_u}{\epsilon_d} & 2\epsilon_u^2\frac{\epsilon_u}{\epsilon_d} \\ -Y^{31}\epsilon_u^2\epsilon_d & 2\epsilon_u^2\frac{\epsilon_u}{\epsilon_d} & \epsilon_3^2 \end{pmatrix} \lambda_t$$

where all higher-order coupling constants $Y^{ij}$ have been suppressed except for $Y^{13}$ and $Y^{31}$ which are taken to be approximately the same for both the up and down quark interactions. These matrices were obtained by assuming the same flavon VEV scales as in Section 5.6.3. That is, the flavons $\overline{\phi}_3+\phi_3$, $\overline{\phi}_{23}+\phi_1$ and $\overline{\phi}_{123}+\phi_{123}$ get VEVs of order $\sqrt{\lambda_b}M_d$, $\sqrt{\lambda_b}\epsilon_d M_d$ and $\sqrt{\lambda_b}\epsilon_d^2 M_d$ respectively.

Symmetrical up and down Yukawa matrices are required for the first two generations to generate the phenomenologically successful Gatto-Sartori-Tonin relation given by Eq.5.12. This requires that $Y^{13}=Y^{31}$ in Eq.6.3. One way to achieve this is to assume that $F_i$ and $F_i^c$ come from the same E$_6$ representation at the Planck scale so that $Y^{13}=Y^{31}$, and that the RGEs from the Planck scale to the $\Delta_{27}$ symmetry breaking scale do not upset this relation. With this assumption and $\lambda_t=\lambda_b\approx 0.7$, $\epsilon_u\approx 0.05$, $\epsilon_d\approx 0.15$, then, the above matrices agree with those in Section 5.6.3, which, after radiative corrections from a high energy scale, are able to generate quark masses and CKM values that are in good agreement with the observed values once the corrections from the higher order operators Eq.6.5 and Eq.6.6 are included.

It should be noted that in the ME$_6$SSM the RGEs are very different from those in the MSSM since there are three copies of a supersymmetric $E_6$ 27 multiplet below the conventional GUT scale (and two additional electroweak doublets in the E$_6$SSM model) rather than just the MSSM particle spectrum. The Yukawa terms in the $\Delta_{27}$ model [74] were assumed to be formulated at the GUT scale and, after running the assumed MSSM from the GUT scale to the electroweak scale, the results agree with the observed quark and lepton mixing angles and masses. In the ME$_6$SSM with $\Delta_{27}$ family symmetry model the running effects will clearly be different, but the main features of the low-energy spectrum are not expected to be qualitatively very different. Section 6.8

investigates how the running of the gauge coupling constants in the ME$_6$SSM is likely to be modified by the $\Delta_{27}$ family symmetry.

## 6.3 Majorana Interactions

The $U(1)_X$ group of the ME$_6$SSM is defined such that a conventional see-saw mechanism can be used to generate small neutrino masses. The $(4,1,2)_{-\frac{1}{2}}$ particle, denoted by $H_R$, that breaks the $G_{4221}$ symmetry once it develops a GUT-scale VEV, gives mass to the right-handed neutrinos using the Planck suppressed operators $\frac{1}{M_p}\lambda^{ij}F_i^cF_j^cH_RH_R$ (see Section 4.3.5). This non-renormalizable term, together with the Yukawa interaction involving the neutrinos, can explain the small masses of the neutrinos but not the observed hierarchical structure of neutrino masses and large mixing angles without setting the couplings $\lambda^{ij}$ by hand.

In the $SO(10)\times\Delta_{27}$ model of Section 5.6.3, the particles that give mass to the right-handed neutrinos transform as $\overline{16}$ of $SO(10)$ and anti-triplets of $\Delta_{27}$. It is the $(4,1,2)$ Pati-Salam representation of these particles that obtains a VEV. With an anti-triplet $\Delta_{27}$ assignment, these particles dynamically generate the observed hierarchical structure of neutrino masses and a tri-bi-maximal mixing using the CSD mechanism discussed in Section 5.7. Following the $\Delta_{27}$ family symmetry model, the $H_R$ particle of the ME$_6$SSM is thus taken to transform as an anti-triplet of $\Delta_{27}$. The Majorana interactions allowed by the symmetries are then:

$$
\begin{aligned}
W_{Maj} = & \frac{1}{M_R}F_i^cF_j^cH_R^iH_R^j \\
& + \frac{1}{M_R^5}F_i^cF_j^c\overline{\phi}_{23}^i\overline{\phi}_{23}^jH_R^kH_R^l\phi_{123k}\phi_{3l} \\
& + \frac{1}{M_R^5}F_i^cF_j^c\overline{\phi}_{123}^i\overline{\phi}_{123}^jH_R^kH_R^l\phi_{123k}\phi_{123l}.
\end{aligned}
$$

The above operators are exactly the relevant Pati-Salam versions of those in Eq.5.13 from Section 5.6.3 but with $H_R$ transforming in a $(\overline{4},1,\overline{2})_{\frac{1}{2}}$ representation of the $G_{4221}$ symmetry rather than a $(\overline{4},1,\overline{2})$ representation of the Pati-Salam symmetry. Together with the neutrino and charged lepton Yukawa matrix generated by Eq.6.2-Eq.6.6, the above interactions produce a $V_{MNS}$ matrix with approximate tri-bimaximal mixing and a hierarchical structure of neutrino masses in agreement with the observed values [89] exactly as discussed in Section 5.6.3. This is however reliant on the assumption made in the previous Section that $F$ and $F^c$ come from the same E$_6$ representation so that the Yukawa matrices are symmetrical.

## 6.4 The $\mu$-Term and Higgs Triplet Mass

In the ME$_6$SSM and E$_6$SSM the superpotential term $S_3h_3h_3$ solves the $\mu$-problem of the MSSM if $S_3$ obtains a vacuum expectation value at the TeV scale as discussed in Sections 2.6 and 4.3.2. This term is not present in the $SO(10) \times \Delta_{27}$ model described in Section 5.6.3 and so we are free to take $S_3$ to transform in any $\Delta_{27}$ representation. Taking $S_3$ to transform as a singlet under $\Delta_{27}$ allows the superpotential term $S_3h_3h_3$ and thus keeps the simple solution to the $\mu$-problem.

The $S_3\mathcal{D}_{1,2,3}\mathcal{D}_{1,2,3}$ terms in the ME$_6$SSM superpotential give mass to the $\mathcal{D}_{1,2,3}$ states once $S_3$ develops a TeV scale VEV. This suggests that the $\mathcal{D}_{1,2,3}$ particles should also transform as $\Delta_{27}$ singlets, so that they may all acquire TeV scale masses. If we had instead assumed them to be $\Delta_{27}$ triplets then at least one of their masses would be expected to be lower than the electroweak symmetry breaking scale, in violation of the direct experimental limits. This is because we would expect the effective couplings $S_3\mathcal{D}_{1,2,3}\mathcal{D}_{1,2,3}$, with $S_3$ obtaining a VEV at the TeV scale, to have a strongly hierarchical mass structure, as in the case of ordinary quarks, with at least the first generation, $D_1$, possibly having a mass lower than the electroweak symmetry breaking scale. Instead, with $\mathcal{D}_{1,2,3}$ as $\Delta_{27}$ singlets, they will all obtain TeV scale masses from the (unsuppressed) superpotential terms $S_3\mathcal{D}_{1,2,3}\mathcal{D}_{1,2,3}$. Similarly, the first two generations of $h$ from the fundamental 27 multiplets, denoted by $h_{1,2}$, are taken to transform as $\Delta_{27}$ singlets so that they obtain TeV scale masses from the $S_3h_{1,2}h_{1,2}$ superpotential terms.[2]

## 6.5 Proton Decay and Higgs Triplet Decay

In the ME$_6$SSM the superpotential terms $\lambda^{ijk}F_iF_j\mathcal{D}_k$ and $\lambda^{ijk}F_i^cF_j^c\mathcal{D}_k$ in Eq.6.1 are forbidden by the $Z_2^H$ symmetry, under which $F_i$, $F_i^c$ and $\mathcal{D}_i$ are all odd. The terms are instead generated effectively, but highly suppressed, from higher-order operators that involve a new particle $\Sigma$ that is odd under $Z_2^H$ and is a $G_{4221}$ singlet. In Section 4.3.3 we found that if the level of suppression is of order $10^{-13}$ then the proton's lifetime is consistent with present experimental data, and the $\mathcal{D}$ states decay fast enough to avoid any cosmological problems.

With $F_i$ and $F_i^c$ as $\Delta_{27}$ triplets and $\mathcal{D}_{1,2,3}$ as singlets, the terms $\lambda^{ijk}F_iF_j\mathcal{D}_k$ and $\lambda^{ijk}F_i^cF_j^c\mathcal{D}_k$ are automatically forbidden by the $\Delta_{27}$ symmetry. Once the $\Delta_{27}$ family symmetry is broken however, proton decay operators will reappear suppressed by flavon

[2]Note that the first two generations of $h$ and $\mathcal{D}$ can fit inside a $10_{-1}$ multiplet of $SO(10)\times U(1)_\psi$, but the third generations cannot due to opposite $Z_2^H$ parity assignments. Also note that the required TeV scale VEV of $S_3$ implies an effective $\mu$-term of similar magnitude, leading to a slight tuning required for electroweak symmetry breaking.

and other VEVs, and it becomes a quantitative question whether these operators are sufficiently suppressed. This suggets that a combination of the discrete $Z_2^H$ symmetry and the $\Delta_{27}$ family symmetry can be used to create a $10^{-13}$ level of suppression.

With the $Z_2^H$ and $\Delta_{27}$ symmetries chosen as in Table 6.1, the only way to generate the proton-decay inducing terms is from higher-order terms involving flavons (to repair the $\Delta_{27}$ symmetry), and the $E_6$ singlet $\Sigma$ (to repair the $Z_2^H$ symmetry). Taking $\Sigma$ to have $U(1) = +5$ and $Z_2 = -1$, the smallest suppressed proton decay terms are:

$$\begin{aligned} W_{trip} = &\frac{1}{M_S M_d^6}\Sigma\mathcal{D}_{1,2,3}F_iF_j\overline{\phi}^i_{123}\overline{\phi}^j_{23}(\phi_{123k}\overline{\phi}^k_3)(\phi_{1l}\overline{\phi}^l_3) \ + \ (F_{i,j}\to F^c_{i,j}) \\ &+ \frac{1}{M_S M_d^6}\Sigma\mathcal{D}_{1,2,3}(\epsilon^{ijk}F^c_i\phi_{123j}\phi_{3k})(\epsilon^{lmn}F^c_l\phi_{1m}\phi^{3n})(\phi_{1l}\overline{\phi}^l_{123}) \ + \ (F_{i,j}\to F^c_{i,j}). \end{aligned} \tag{6.8}$$

These operators are suppressed by the square of a string scale $M_S$, which is taken to be of order $10^{17.5}$ GeV. This assumes that the messengers that couple the $\Sigma$ particle to the $F^cF^cD_{1,2,3}$ superpotential term are different to the messengers that couple the flavons and $H_R$ to the quarks and leptons in the Yukawa and Majorana interactions of Sections 6.2 and 6.3. The former messengers are assumed to reside at the unification scale which is taken to be $M_S \approx 10^{17.5}$ GeV. This is further discussed in Section 6.8.

The effective terms $F^cF^cD_{1,2,3}$ are then suppressed by a factor of about $\epsilon_d^6\lambda_t\frac{\langle\Sigma\rangle}{M_S}$ where $\sqrt{\lambda_t} \equiv \langle\overline{\phi}_3\rangle/M_R$. For $\epsilon_d \approx 0.15$, $\sqrt{\lambda_t} \approx 0.7$, $\langle\Sigma\rangle \approx 10^{10}$ GeV, and $M_S \approx 10^{17.5}$ GeV, this suppression factor is around $10^{-13}$. From the discussion in Section 4.3.3 this level of suppression should be just sufficient to prevent proton decay from being observable in present experiments if the Higgs triplets have mass greater than about 1.5 TeV. At the same time it should also be sufficient to permit the Higgs triplets to decay with a lifetime smaller than 0.1s.

### 6.5.1 FCNCs From Additional Higgs-Doublet Fields

The other 'Higgs generations' $h_{1,2}$ are taken to transform in the same way as the $\mathcal{D}$ particles in Table 6.1. This forbids the interactions $FF^ch_{1,2}$ at tree-level but allows higher-order operators that are equivalent to Eq.6.8 but with $FF\mathcal{D}$ and $F^cF^c\mathcal{D}$ replaced with $FF^ch_{1,2}$. These higher-order operators become effective $FF^ch_{1,2}$ interactions at low energies but with a suppression factor of order $10^{-13}$. Such operators will cause FCNCs as discussed in Section 4.3.1 for the ME$_6$SSM. However the level by which they are suppressed puts them well within the present experimental limits.

## 6.6 R-parity and $H_R + \overline{H}_R$ Mass

Not all the components of $H_R$ and $\overline{H}_R$ obtain mass by absorbing the broken Pati-Salam gauge bosons when they acquire vacuum expectation values in the right-handed neutrino direction. To give the rest of $H_R$ and $\overline{H}_R$ (and $H_L$ and $\overline{H}_L$ from the $SO(10)$ multiplets $16_H$ and $\overline{16}_H$) mass, a a singlet $M$ has been included in Table 6.1. This singlet is assumed to get a GUT scale VEV, giving mass to $16_H + \overline{16}_H$ from the superpotential term $M16_H\overline{16}_H$. Since $M$ carries a $U(1)_R$ charge of $+2$, its VEV also breaks $U(1)_R$ to an R-parity. This R-parity is a generalization of R-parity in the MSSM and keeps the LSP stable, providing a dark matter candidate.

Note that the $U(1)_R$ symmetry of $16_H$ used in Table 6.1 is different to that used in the ME$_6$SSM defined by Table 4.3. This R-symmetry prevents the bilinear term $16_H\overline{16}_H$ in the superpotential.

## 6.7 The E$_6$SSM with a $\Delta_{27}$ Family Symmetry

As discussed at the start of this Chapter, if we introduce two additional electroweak doublets $H'$ and $\overline{H}'$ with TeV scale masses to the above ME$_6$SSM with $\Delta_{27}$ family symmetry then we can generate an E$_6$SSM with $\Delta_{27}$ family symmetry model. All the above operators of the ME$_6$SSM with $\Delta_{27}$ family symmetry are also present in this E$_6$SSM model. However, gauge coupling unification now occurs at the GUT scale (rather than the Planck or String Scale) where an E$_6$ symmetry is assumed to exist. In this case it is easier to understand how the Yukawa matrices Eq.6.7 can be symmetrical since we don't have to neglect any RGE effects from the Planck scale to the GUT scale. Each type of Yukawa matrix will be symmetrical as long as the right-handed up quarks, down quarks, charged leptons, and neutrinos come from the same E$_6$ multiplet as their left-handed counterparts, which is perfectly acceptable in the E$_6$SSM.

To prevent the two additional electroweak doublets $H'$ and $\overline{H}'$ from introducing gauge anomalies for the $U(1)_N$ gauge group, they must have opposite $U(1)_N$ charges. One possibility would be that $H'$ and $\overline{H}'$ transform as $(1,2,1)_x$ and $(1,2,1)_{-x}$ under $G_{4221}$, but such multiplets cannot be derived from $E_6$ multiplets making it difficult to relate the E$_6$SSM to any E$_6$ symmetry. This requires that $H'$ and $\overline{H}'$ must come from split Pati-Salam representations. For example, $H'$ could come from $(1,2,2)_x$ and $\overline{H}'$ from $(1,2,2)_{-x}$, or alternatively $H'$ could come from $(4,2,1)_x$ and $\overline{H}'$ from $(\overline{4},1,\overline{2})_{-x}$ where $x$ is some $U(1)_\psi$ charge. If these split Pati-Salam multiplets come from 27 representations of E$_6$ then in the former case $x = 1$ whereas in the latter case $x = 1/2$. A mechanism is not provided to explains why the $H'$ and $\overline{H}'$ are split from their Pati-Salam (and $E_6$) partners.

The symmetries of the model couple the $E_6$ singlet $\Sigma$ to the $H'$ and $\overline{H}'$ through the non-renormalizable term $(1/M_S)\Sigma\Sigma H'\overline{H}'$. If $\Sigma$ obtains a vacuum expectation value at $\approx 10^{10}$ GeV then this would give $H'$ and $\overline{H}'$ approximately TeV scale masses so that the Standard Model gauge couplings unify at the GUT scale and $g_4 = g_{2R}$ at the $G_{4221}$ symmetry breaking scale. It is emphasized that this is not a solution to the $\mu'$-problem however since the VEV of $\Sigma$ has not been related to the (soft SUSY) TeV scale.

## 6.8 Unification and Symmetry Breaking in the ME$_6$SSM

This Section describes how the pattern of symmetry breaking for the ME$_6$SSM is modified when we apply a $\Delta_{27}$ family symmetry. In the ME$_6$SSM the $E_6$ symmetry is assumed to be broken at the Planck scale to a left-right symmetric Pati-Salam gauge group $SU(4)\times SU(2)_L\times SU(2)_R\times D_{LR}$ (a maximal subgroup of $SO(10)$) and an Abelian gauge group $U(1)_\psi$. The left-right symmetric gauge group is then broken to the Standard Model gauge group with an additional Abelian gauge group $U(1)_X$, which is a combination of the charge of the $U(1)_\psi$ group, the diagonal generator $\tau_R^3$ of the $SU(2)_R$ group, and the diagonal generator associated with the $U(1)_{B-L}$ subgroup of $SU(4)$ defined by $SU(4) \to SU(3)_c \times U(1)_{B-L}$. This breaking is achieved by the ME$_6$SSM equivalent to the $H_R + \overline{H}_R$ particles from Section 6.3 gaining VEVs in the right-handed neutrino directions. At the scale of this symmetry breaking the gauge couplings of the Abelian groups $U(1)_{B-L}$, $U(1)_{\tau_R^3}$ and $U(1)_Y$ must satisfy Eq.3.4 with $\alpha_{B-L} = \alpha_4$ and $\alpha_{\tau_R^3} = \alpha_{2R}$.

When the $\Delta_{27}$ family symmetry is introduced to the ME$_6$SSM however, the pattern of symmetry breaking is likely to change from the above discussion. This is due to the inclusion of the higher-order messengers introduced by the $\Delta_{27}$ family symmetry. From Section 5.6.3 we require that the messengers that couple to the right-handed up quarks are heavier than the messengers that couple to the right-handed down quarks. Since these messenger fields must come from the same $G_{4221}$ then this difference in mass can only occur once the $SU(2)_R$ symmetry is assumed to be broken. However, if these messenger fields have mass equal to or lighter than the $G_{4221}$ breaking scale then they will cause the gauge coupling constants to blow up before they unify. To prevent this from happening the $SU(2)_R$ group is assumed to be broken to its $U(1)_{\tau_R^3}$ subgroup at some higher energy scale. The right-handed messenger fields would then gain mass at this higher energy scale and would not significantly alter the running of the gauge coupling constants of the ME$_6$SSM.

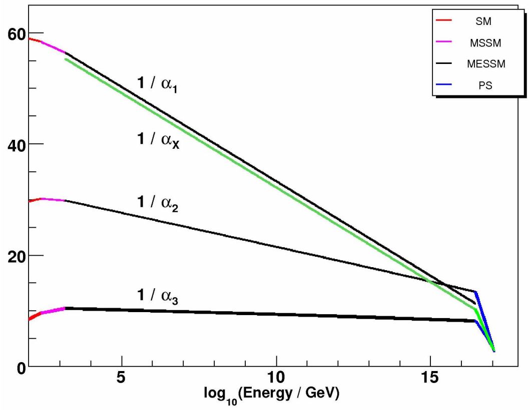
The messengers that couple to the left-handed quarks must be heavier than their right-handed counterparts. To prevent these messenger fields from upsetting the running of the gauge coupling constants in the $\mathrm{ME_6SSM}$ they are assumed to they gain mass at the unification scale. This means that the $SU(2)_R$ breaking scale must be slightly below the unification scale. Note that the difference in the mass of the left-handed and right-handed messengers violates the left-right discrete symmetry of the $\mathrm{ME_6SSM}$ and will change the running of the gauge coupling constants.

The $G_{4211} \equiv SU(4) \times SU(2)_L \times U(1)_{\tau_R} \times U(1)_\psi$ symmetry is broken by the VEV of the $H_R + \overline{H}_R$ multiplets. This mixes the $U(1)_{B-L} \times U(1)_{\tau_R} \times U(1)_\psi$ groups to generate the $U(1)_X$ and $U(1)_Y$ symmetries, as well as breaking $SU(4)$ to the $SU(3)_c$ symmetry of the Standard Model.[3] The $H_R + \overline{H}_R$ particles also transform under the $\Delta_{27}$ family symmetry and get VEVs in the third component so that they break the $\Delta_{27}$ symmetry at the same scale as the $G_{4211}$ symmetry. The remaining part of the family symmetry, which is a subgroup of $\Delta_{27}$, will be broken by the VEV of the $\overline{\phi}_{23}$ flavon at the scale $\epsilon_d M_d$ where the right-handed messengers mass $M_d$ should be above the $\Delta_{27}$ symmetry breaking scale, otherwise wavefunction insertions of the invariant operator $\overline{\phi}_3 \overline{\phi}_3^\dagger / M_R^2$ on a third family propagator can spoil the perturbative expansion if $\langle \overline{\phi}_3 \rangle \ > \ M_R$ [90].

The scale of the $E_6$ symmetry breaking in the $\mathrm{ME_6SSM}$ is also expected to be modified when the $\Delta_{27}$ symmetry is included. Instead of Planck scale $E_6$ symmetry breaking, the $E_6$ symmetry is expected to be broken at a string scale. This is mainly due to the number of additional particles (messengers) to the $\mathrm{ME_6SSM}$ states at and above the $G_{4211}$ symmetry breaking scale, which are required for the $\Delta_{27}$ family symmetry to accurately describe the observed quark and fermion masses and mixing angles. These extra states cause the gauge coupling constants to increase rapidly above the $G_{4211}$ symmetry breaking scale, bringing forward the unification scale. Other modifications to the $E_6$ symmetry breaking scale in the $\mathrm{ME_6SSM}$ will come from the running of the gauge coupling constant for the Abelian $U(1)_{\tau_R^3}$ group, and the breaking of the left-right discrete symmetry at the compactification scale.

The pattern of symmetry breaking in this case is thus expected to proceed as follows: the $SU(2)_R$ group is broken to $U(1)_{\tau_R^3}$ at a compactification scale $M_C$, which, along with the $SU(4) \times SU(2)_L \times U(1)_\psi$ symmetry, is broken at a lower scale to $G_{3211} \equiv SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_X$ by the $H_R + \overline{H}_R$ particles. The left-right discrete symmetry $D_{LR}$ is also expected to be broken since the left-handed messengers are heavier than and right-handed messengers. The pattern of symmetry breaking for the $\mathrm{E_6}$ group

[3] One could alternatively consider the VEV of $H_{45}$ to break $SU(4)$ to $SU(3)_c \times U(1)_{B-L}$. This depends on whether the VEV of $H_{45}$ is chosen to be at a greater or smaller energy scale than the $H_R + \overline{H}_R$ VEV. In [83] and (the second reference in) [90], for example, the $H_{45}$ VEV is taken to be of order $3M_d$ and $3\epsilon_d M_d$ respectively.



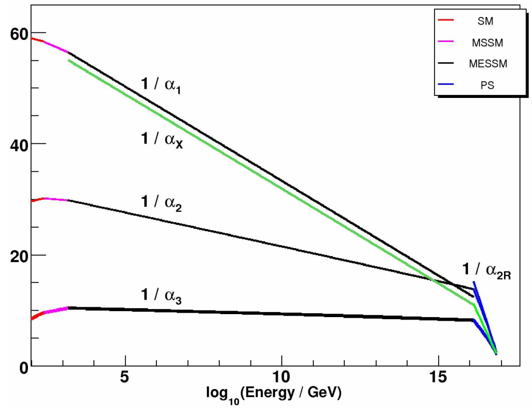




FIGURE 6.1: The two-loop RGEs running of the gauge coupling constants for two models based on the ME$_6$SSM with $\Delta_{27}$ family symmetry. Both models are described in detail in the main body of the text. The thickness of the lines indicates the error in the coupling constants due to the experimental uncertainty in their initial values.

is summarized as:

$$E_6 \overset{M_S}{\longrightarrow} G_{4221} \overset{M_C}{\longrightarrow} G_{4211} \overset{M_{GUT}}{\longrightarrow} G_{3211} \overset{\text{TeV}}{\longrightarrow} G_{321}.$$

where the $\Delta_{27}$ family symmetry is also broken at $M_{GUT}$.

### 6.8.1 Two-Loop RGEs Analysis

Unification of the gauge coupling constants may in fact no longer be possible when all of these changes from the ME$_6$SSM are calculated, but Fig.6.1 demonstrates that gauge coupling unification still occurs for two simple models of the ME$_6$SSM with $\Delta_{27}$ symmetry.

For both models the $SU(2)_R$ symmetry breaking scale is taken to be approximately equal to the $G_{4211}$ symmetry breaking scale. Both models therefore have an intermediate $G_{4221}$ symmetry as in the ME$_6$SSM. However, for the model in the right panel of Fig.6.1, the left-right discrete symmetry is assumed to be broken at the unification scale due to the different masses for the left-handed and right-handed messengers. In both panels of Fig.6.1 three copies of an $E_6$ 27 multiplet, which contain all the MSSM states as well as new (non-MSSM) states, have mass at low energies are used and, following the ME$_6$SSM, effective MSSM and non-MSSM thresholds of 250 GeV and 1.5 TeV respectively are assumed.

At the $G_{4221} \times \Delta_{27}$ symmetry breaking scale, additional particles that break the symmetry and play a part in the $\Delta_{27}$ family symmetry's description of quark and lepton masses are also assumed. In the left panel these extra particles are taken to consist of all the $G_{4221}$ states from five copies of $27+\overline{27}$ multiplets, except for the $(6,1,1)_{\frac{1}{2}}+(6,1,1)_{-\frac{1}{2}}$ states which we assume have mass at the unification scale, as well as all the flavons given in Table 6.1 and a left-handed partner for $\overline{\phi}_3$. The additional $27+\overline{27}$ states contain the $16_H + \overline{16}_H$ particles that break the $G_{4221} \times \Delta_{27}$ symmetry and provide the Majorana interactions, the $16+\overline{16}$ particles that give the $H_{45}$ as a composite, and messengers that also transform as a $16+\overline{16}$ of $SO(10)$. The $H_{45}$ is taken to be a composite of a $16+\overline{16}$ state since a fundamental $H_{45}$ particle (and its left-handed partner) would affect the running of the $SU(4)$ gauge couplings by an amount that causes it to blow up before any unification of gauge couplings is possible. We would also need to explain why the rest of the 650 $E_6$ multiplet, that contains the $H_{45}$, have larger mass. On top of the five copies of the $27+\overline{27}$ multiplets, additional Higgs messengers that transform as a triplet and an anti-triplet of the $\Delta_{27}$ family symmetry are assumed. These are required for unification of the gauge coupling constants.

The right panel assumes the same states as the left panel but without the left-handed messengers as these are expected to get much larger masses than their right-handed components. The scales of unification and $G_{4221}$ symmetry breaking are at $10^{17.1}$, $10^{16.9}$ GeV and $10^{16.4}$, $10^{16.1}$ GeV for the left, right panel respectively. Since the $G_{4221}$ symmetry breaking scales are close to the Grand Unification scale in conventional GUTs they are denoted by $M_{GUT}$.

It should be emphasized that the above models do not represent accurate predictions for the running of the gauge coupling constants of the ME$_6$SSM with $\Delta_{27}$ family symmetry and are only used to demonstrate that, with the inclusion of the $\Delta_{27}$ messenger states to the ME$_6$SSM, gauge coupling unification is still possible but at a scale that is closer to the string scale than the Planck scale.

## 6.9 Unification and Symmetry Breaking in the E$_6$SSM

Including the extra electroweak states $H'$ and $\overline{H}'$ at the TeV scale, in addition to the Pati-Salam representations of three copies of an E$_6$ 27 multiplet, causes the Standard Model gauge coupling constants to unify at the conventional GUT scale (but with a higher value than the MSSM prediction for the unification gauge coupling constant). At the unification scale an $E_6$ symmetry is assumed to exist. However, from Section 5.6.3 we require that the messengers that couple to the right-handed quarks are lighter than the messengers that couple to the left-handed quarks. Since these messenger fields must come from the same $G_{4221}$ multiplet, then the difference in their mass would violate the E$_6$ symmetry.

To overcome this problem the left-handed messengers are assumed to have a mass equal to the GUT scale, and the up and down right-handed messenger fields are taken to gain mass just below the conventional GUT scale. To compensate for the effect on the running of the Standard Model gauge coupling constants caused by the right-handed up and down messengers (which would upset unification), additional fields are included that, together with the messenger fields, form complete $SU(5)$ representations, which in this case would be a complete $10+\overline{5}$. The extra fields below the GUT scale will increase the MSSM prediction for the value of the unification gauge coupling constant but keep the unification scale as the conventional GUT scale. Of course too many messengers, and too small messenger masses, would cause the Standard Model gauge coupling constants to blow up before they unify. Here it is simply assumed that the minimal number of messengers required to generate the correct quark and lepton masses and mixing angles does not prevent the unification of the Standard Model gauge coupling constants at the GUT scale.

## 6.10 Conclusions

In this Chapter the ME$_6$SSM and E$_6$SSM have been extended to include a $\Delta_{27}$ family symmetry which is broken just below the conventional GUT scale. To provide realistic models additional $U(1)\times Z_2\times Z_2^H\times U(1)_R$ symmetries are also applied where $U(1)_R$ is an R-symmetry which results in a conserved R-parity. The resulting supersymmetric models solve a number of problems facing the MSSM, including the little fine-tuning problem, the $\mu$-problem and the SUSY flavour problem. The $\Delta_{27}$ family symmetry accounts for the quark and lepton masses and mixing angles, with tri-bimaximal neutrino mixing resulting from vacuum alignment and constrained sequential dominance. A particularly attractive feature of the models is that the proton decay induced by the Higgs triplets is naturally suppressed by the $\Delta_{27}$ family symmetry.

# Chapter 7

# Solving the Flavour Problem of Supersymmetric Standard Models with Three Higgs Families

In the previous Chapter the $E_6$SSM and M$E_6$SSM were extended with a $\Delta_{27}$ family symmetry to solve the flavour problem. These models are more powerful than the $SO(10) \times \Delta_{27}$ model in Section 5.6.3 for example because they have the potential to explain why the Higgs mass, which is indirectly related to the quark and lepton masses, is much smaller than the Planck mass. This is because the $E_6$ models do not contain the $\mu$-problem of the MSSM, whereas in the $SO(10) \times \Delta_{27}$ model for example, there is nothing preventing the bilinear term $\mu 10.10$, which is the GUT version of the $\mu$-term, from being included in the superpotential.

However, although the family symmetry solves the flavour problem of the effective MSSM states present in the $E_6$SSM and M$E_6$SSM, it does not explain the flavour of the non-MSSM states present in these models. For example, the $\Delta_{27}$ family symmetry accounts for the three generations of (up and down) quarks and (charged and neutral) leptons in the $E_6$SSM and M$E_6$SSM, but does not explain why there are also three copies of (up and down) Higgs fields, three copies of ($D$ and $\overline{D}$) Higgs triplet fields, and three copies of MSSM singlet fields $S$.

In this Chapter a more ambitious application of a family symmetry is introduced that solves the full flavour problem of the $E_6$SSM. The approach taken is to assume that all the $E_6$SSM states that fill three complete 27 representations of $E_6$ transform in triplet representations of a $\Delta_{27}$ family symmetry. This then explains why there are exactly three copies of all these fields in the $E_6$SSM. Table 7.1 describes how all the states from a 27 representation, as well as the flavons, transform under the family symmetry and the additional symmetries that constrain the model. Only the $E_6$SSM is concentrated on in this Chapter, since, unlike the previous synthesis, the family symmetry cannot suppress the proton decay induced by the Higgs triplets. The $Z_2^B$ or $Z_2^L$ symmetry of the $E_6$SSM is therefore used to avoid the induced proton decay, which violates a Pati-Salam gauge symmetry.

Once the family symmetry is broken the full mass structure of the $E_6$SSM is determined, including the masses and mixings of the quarks and leptons. In particular the model predicts tri-bi-maximal mixing for leptons, two almost degenerate LSPs and two almost degenerate families of triplet higgsinos. The broken $\Delta_{27}$ family symmetry also explains why only the third generation of the Higgs fields interacts with the quarks and leptons, thus forbidding FCNCs that would be caused by the additional Higgs-like families. The broken $\Delta_{27}$ family symmetry therefore provides a high-energy theoretical understanding of the $Z_2^H$ symmetry which is somewhat ad hoc in the $E_6$SSM and adds an additional complication to the flavour problem of the model that is not present in the MSSM or NMSSM. This method of avoiding FCNCs should, in theory, be applicable to any general supersymmetric model with three families of Higgs fields, and this Chapter only uses the $E_6$SSM as an example of such a theory.

The outline of this Chapter is as follows. In Section 7.1 the renormalizable $E_6$SSM superpotential in the absence of any family symmetry is reviewed. The rest of the Chapter is then divided into different sections which investigate how each term in this superpotential is generated from the $\Delta_{27}$ family symmetry: Section 7.2 introduces the non-renormalizable operators allowed by $\Delta_{27}$ that lead to the quark and lepton Yukawa interactions with the Higgs fields, Section 7.3 illustrates how the $Z_2^H$ symmetry of the $E_6$SSM effectively emerges from the high-energy theory, Section 7.4 then discusses how tri-bi-maximal mixing is generated from the $\Delta_{27}$ family symmetry and constrained sequential dominance, Section 7.5 describes how the effective $\mu$-term of the MSSM and the mass structure of the LSPs that are formed from the inert higgsinos and singlinos are generated, Sections 7.6 and 7.7 explain the mass structure of the triplet higgsinos and discusses their decay channels, and Section 7.8 introduces the vacuum alignment required for the various $\Delta_{27}$ flavon fields. Finally, in Section 7.9, the Chapter is concluded.

| | $\Delta_{27}$ | $U(1)$ | $Z_2$ | $Z_2^h$ | $Z_2^S$ | $U(1)_R$ |
|---|---|---|---|---|---|---|
| $F$ | 3 | 0 | + | + | + | 1 |
| $F^c$ | 3 | 0 | + | + | + | 1 |
| $h$ | 3 | 0 | + | - | + | 0 |
| $\mathcal{D}$ | 3 | 0 | + | - | + | 0 |
| $S$ | 3 | 0 | + | + | - | 0 |
| $\phi_3$ | $\overline{3}$ | 0 | - | + | + | 0 |
| $\phi_{23}$ | $\overline{3}$ | -1 | - | + | + | 0 |
| $\phi_{123}$ | $\overline{3}$ | 1 | - | + | + | 0 |
| $\overline{\phi}_1$ | 3 | -4 | - | + | + | 0 |
| $\overline{\phi}_3$ | 3 | 3 | + | + | + | 0 |
| $\overline{\phi}_{123}$ | 3 | -1 | + | + | + | 0 |
| $\phi_3^h$ | $\overline{3}$ | 0 | + | - | + | 0 |
| $\phi_3^S$ | $\overline{3}$ | 0 | + | + | - | 0 |
| $\overline{\phi}_3^h$ | 3 | 0 | + | + | + | 0 |
| $\overline{\phi}_3^S$ | 3 | 0 | + | + | - | 0 |
| $H_{45}$ | 1 | 2 | + | + | + | 0 |
| $H_R$ | $\overline{3}$ | 0 | + | + | + | 0 |
| $H'$ | 1 | 0 | + | + | + | 1 |
| $\overline{H}'$ | 1 | 0 | + | + | + | 1 |

TABLE 7.1: This table illustrates how all the flavon fields and Pati-Salam states transform under the $\Delta_{27}$ family symmetry and the additional constraining $U(1)\times Z_2\times Z_2^h\times Z_2^S$ symmetry. An R-symmetry is also applied to the model which breaks to an R-parity once $S_3$ obtains a vacuum expectation value.

## 7.1 Review of E$_6$SSM Superpotential

In terms of a Pati-Salam notation, and dropping all couplings and indices for clarity, the E$_6$SSM superpotential terms from the E$_6$ tensor product $27\times 27\times 27$ are the following:

$$27\times 27\times 27 \to FF^ch + Shh + S\mathcal{D}\mathcal{D} + FF\mathcal{D} + F^cF^c\mathcal{D}. \tag{7.1}$$

The interactions between the quarks and leptons and the Higgs fields, $\lambda^{ijk}F_iF^c_jh_k$, is the subject of the next Section. Section 7.5 discusses the superpotential term $\lambda_{ijk}S^ih^jh^k$ from which the MSSM effective $\mu$-term is generated. Section 7.6 describes the term $\lambda^{ijk}S_i\mathcal{D}_j\mathcal{D}_k$ from which the Higgs triplet states get mass, and Section 7.7 looks at the operators $\lambda^{ijk}F_iF_j\mathcal{D}_k + \lambda^{ijk}F^c_iF^c_j\mathcal{D}_k$ which provide their decay channels.

The above operators are only written in a Pati-Salam notation for ease of notation. The actual gauge symmetry of the model presented in this Chapter is the E$_6$SSM gauge symmetry $SU(3)_c\times SU(2)_L\times U(1)_Y\times U(1)_N$ rather than a Pati-Salam gauge symmetry. For the rest of this Chapter a Pati-Salam notation is used unless stated otherwise.

## 7.2 The Effective Yukawa Operators

In the previous synthesis of the $\Delta_{27}$ family symmetry with the ME$_6$SSM, the quarks and leptons were taken to transform as triplets but the Higgs states were singlets. Here instead the Higgs states are also taken to transform in a triplet representation to explain why three Higgs doublet type fields are present in the E$_6$SSM. This then allows the E$_6$SSM $\times\Delta_{27}$ superpotential term $\epsilon^{ijk}F_iF^c_jh_k$ where $i,j,k = 1\ldots3$ are $\Delta_{27}$ indices and $\epsilon^{ijk}$ is the totally anti-symmetric tensor. This however contains operators such as $F_1F^c_2h_3 - F_2F^c_1h_3$ which must be forbidden since they would give too large a mixing between the first and second generation quarks. To forbid these terms the Higgs states $h_i$ are taken to be odd under a new discrete symmetry called $Z^h_2$, which forbids the entire $\epsilon^{ijk}F_iF^c_jh_k$ superpotential. To 'repair' the $Z^h_2$ symmetry, a $\Delta_{27}$ flavon denoted by $\overline{\phi}^h_3$ is included that transforms as an anti-triplet and is odd under $Z^h_2$. Two flavons that are also anti-triplets must then couple to the quarks and leptons to form a $\Delta_{27}$ invariant. Table 7.1 describes how the quarks, lepton, Higgs and all other the Pati-Salam states from a 27 representation transform under the family symmetry. It also contains the additional symmetries that constrain the model such as $Z^h_2$ symmetry which distinguishes the Higgs fields (but unlike $Z^H_2$ treats all three Higgs families identically) as well as the $\Delta_{27}$ flavon fields.

The lowest order Yukawa superpotential consistent with the symmetries of Table 7.1 is:

$$\begin{aligned}W_{Yuk} \sim\ & \frac{1}{M^3}F_iF^c_jh_k\overline{\phi}^i_3\overline{\phi}^j_3(\overline{\phi}^h_3)^k \\ &+ \frac{1}{M^4}F_iF^c_jh_k\overline{\phi}^i_{23}\overline{\phi}^j_{23}(\overline{\phi}^h_3)^kH_{45} \\ &+ \frac{1}{M^3}F_iF^c_jh_k(\overline{\phi}^i_{123}\overline{\phi}^j_{23}+\overline{\phi}^j_{123}\overline{\phi}^i_{23})(\overline{\phi}^h_3)^k \\ &+ \frac{1}{M^6}F_iF^c_jh_k(\overline{\phi}^i_{123}\overline{\phi}^j_3+\overline{\phi}^j_{123}\overline{\phi}^i_3)(\overline{\phi}^m_{123}\phi_{1m})(\overline{\phi}^h_3)^kH_{45} \\ &+ \frac{1}{M^7}F_iF^c_jh_k\overline{\phi}^i_{123}\overline{\phi}^j_{123}(\overline{\phi}^l_3\phi_{123l})(\overline{\phi}^m_3\phi_{123m})(\overline{\phi}^h_3)^k\end{aligned} \tag{7.2}$$

where all $\mathcal{O}(1)$ coupling constants are suppressed.

Note that the above superpotential is exactly that found in Section 5.6.3 but with $h_3$ replaced with $h_i(\overline{\phi}^h_3)^i$. This flavon field and $\overline{\phi}_3$ are assumed to get a VEV in the third component of $\Delta_{27}$, and the other flavon fields are equivalent to those used in the $\Delta_{27}$ family symmetry described in Section 5.6.3.

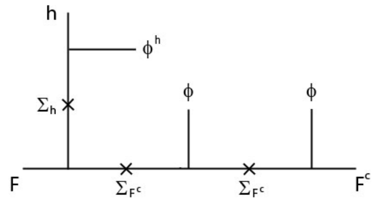



FIGURE 7.1: This Figure contains the type of messenger diagram that provides the dominant contribution to the Yukawa operators in Eq.7.2.

### 7.2.1 The Messenger Fields

The messenger fields $\Sigma$ that are responsible for the suppression factors in Eq.7.2 include fields that transform in the same way as quarks and leptons under the Standard Model gauge group and as singlets, triplets and anti-triplets of $\Delta_{27}$. For convenience these type of messenger fields are referred to as quark and lepton-like messengers $\Sigma_{F,F^c}$. In addition there are also messengers that are singlets of $\Delta_{27}$ and transform in the same way as Higgs fields under the Standard Model gauge group. These messenger fields are called Higgs-like messengers $\Sigma_h$. All these messenger fields are taken to carry positive $Z_2^h$ parity, and the Higgs-like messengers $\Sigma_h$ are assumed to be heavier than the quark and lepton-like messengers $\Sigma_{F,F^c}$ so that the latter dominate the messenger diagrams. Also, as in Chapters 5 and 6, the right-handed quark and lepton messengers $\Sigma_{F^c}$ are assumed to be heavier than their left-handed counterparts $\Sigma_F$ (except for the neutrino messengers) so that the former dominate over the latter. The messenger diagrams are illustrated by Fig.7.1.

To create a smaller hierarchy in the down quark sector compared to the up quark sector, the mass of the 3 and $\overline{3}$ up and down Higgs messengers $M_3^h$ are assumed to be equal, but the up right-handed quark messengers $\Sigma_{u^c}$ that are 3 and $\overline{3}$ and singlets of $SU(3)$ are taken to have a mass $M^u$ that is greater than the mass of the right-handed down quark messengers $\Sigma_{d^c}$ by approximately a factor of three. This then creates $\epsilon_d = 3\epsilon_u$ as in Chapters 6 and 7.

For the top Yukawa coupling constant to be greater than the bottom Yukawa coupling constant the $\overline{\phi}_3$ flavon is again taken to transform as a $3 \oplus 1$ of the $SU(2)_R$ subgroup of $E_6$, and its VEV is chosen so that $\langle \overline{\phi}_3 \rangle / M_d = \langle \overline{\phi}_3 \rangle / M_u$. In terms of these messenger masses, the VEV scales for the various flavon fields are then taken to be the following:

$$\frac{\langle \overline{\phi}_3^h \rangle}{M_3^h} \approx \frac{\langle \overline{\phi}_3 \rangle}{M^u} \approx 0.8, \quad \frac{\langle \overline{\phi}_{23} \rangle}{M^u} \approx \epsilon_d, \quad \frac{\langle \overline{\phi}_{123} \rangle}{M^u} \approx \epsilon_d^2 \tag{7.3}$$

where $\epsilon_d \approx 0.15$. At the GUT scale the Yukawa coupling for the top and bottom quark is expected to be about 0.5 in third family Yukawa unification models based on the

MSSM with large $\tan\beta$ [80]. It is therefore assumed that $\langle\overline{\phi}_3^h\rangle/M_3^h \approx \langle\overline{\phi}_3\rangle/M^u \approx 0.8$. By comparison, in the model formulated in the previous Chapter, in which the Higgs is a singlet, $\langle\overline{\phi}_3\rangle/M^u$ is assumed to be about 0.7 [74].

Inputting the above flavon VEVs into the superpotential given by Eq.7.2 generates the effective Yukawa matrices given by 5.9 for the quarks and leptons, which were shown to produce a realistic CKM matrix and realistic mass hierarchies for the up and down quarks in Section 5.6.3. This is essentially because the superpotential in 5.9 is exactly that found in Section 6.2 but with $h_3$ replaced with $h_i(\overline{\phi}_3^h)^i$, which becomes $h_3$ once $\overline{\phi}_3^h$ gets a VEV.

## 7.3 Preventing Flavour Changing Neutral Currents

Note that since $\overline{\phi}_3^h$ transforms under $Z_2^h$ it will only couple to the Higgs fields and not to the quarks and leptons. This can be understood by considering the messenger diagrams of the above higher-order operators where $\overline{\phi}_3^h$ will only be allowed to attach itself to the Higgs fields (and the Higgs-like messenger fields) if all the messenger fields are even under $Z_2^h$. This is illustrated by Fig.7.1. Once $\overline{\phi}_3^h$ gets a VEV, only the third 'generation' of the up and down Higgs fields $h_3$ will couple to the quarks and leptons. It is these up and down Higgs fields which we therefore take to obtain electroweak scale VEVs, and thus act like the up and down Higgs fields of the $E_6$SSM.

The $Z_2^h$ and $\Delta_{27}$ symmetries prevent the first and second generation of Higgs fields from interacting with the quarks and leptons at tree-level and so there can be no tree-level FCNC processes involving the neutral scalar components of these fields. In the $E_6$SSM the $Z_2^H$ symmetry is applied to all the 27 fields except for the third generation of Higgs fields and singlet fields to prevent the first and second generation of Higgs fields from interacting with the quarks and leptons at tree-level. The $Z_2^h$ in this model is therefore acting as the $Z_2^H$ symmetry of the $E_6$SSM even though it does not distinguish between the different Higgs fields.

This then illustrates how the flavour problem in general supersymmetric models with three (up and down) Higgs fields can be solved: the model should be extended with a family symmetry for which the Higgs are in a triplet representation. This then explains why there are three Higgs doublets, and with the addition of a simple flavour-independent $Z_2$ symmetry, also explains why there are no FCNCs from the additional Higgs fields. This can be achieved using the same family symmetry that generates the masses and mixings of the leptons and quarks.

## 7.4 Tri-Bi-Maximal Mixing

Tri-bi-maximal mixing for the leptons is created using the $\Delta_{27}$ family symmetry and constrained sequential dominance exactly as in the previous $E_6$SSM model with $\Delta_{27}$ family symmetry since the right-handed neutrino Majorana operators that are allowed by the symmetries are again given by Eq.5.21. Table 7.1 illustrates how the $H_R$ fields that give mass to the right-handed neutrinos transform under the symmetries of the model.

## 7.5 The Effective $\mu$-Term and Inert Higgsino and Singlino Masses

The Pati-Salam superpotential term $\lambda^{ijk}S_i h_j h_k$ from Eq.7.1 is used in the $E_6$SSM to generate higgsino and singlino masses as well as an effective MSSM $\mu$-term. In terms of the Standard Model gauge group this superpotential term reduces to $\lambda^{ijk}S_i h_{uj} h_{dk}$ where $h_i$ and $h_d$ denote up and down Higgs fields. To explain why three copies of the singlet fields $S$ are in the $E_6$SSM they are taken to form a triplet representation of $\Delta_{27}$. This then allows the $E_6$SSM $\times\Delta_{27}$ superpotential term $\epsilon^{ijk}S_i h_{uj} h_{dk}$ and forbids terms such as $S_3 h_{u3} h_{d3}$, which is used by the $E_6$SSM to generate an effective $\mu$-term, because of the $\Delta_{27}$ symmetry. To avoid this the singlet fields are taken to be odd under a new $Z_2^S$ discrete symmetry which forbids all the $\epsilon^{ijk}S_i h_{uj} h_{dk}$ operators. To repair this symmetry new flavon fields $\overline{\phi}_3^S$ and $\phi_3^S$ are introduced that are odd under the $Z_2^S$ discrete symmetry and form anti-triplet and triplet representations respectively of $\Delta_{27}$. The following higher-order operators are then allowed:

$$\begin{aligned} W_\mu \sim\ & \frac{1}{M^3} S_i h_{uj} h_{dk} (\overline{\phi}_3^S)^i (\overline{\phi}_3^h)^j (\overline{\phi}_3^h)^k \\ & + \frac{1}{M^2}\epsilon^{jkl} S_i h_{uj} h_{dk} (\overline{\phi}_3^S)^i (\phi_3^h)_l \\ & + \frac{1}{M^2}\epsilon^{ijl} S_i h_{uj} h_{dk} (\phi_3^S)_l (\overline{\phi}_3^h)^k + \frac{1}{M^2}\epsilon^{ijl} S_i h_{uk} h_{dj} (\phi_3^S)_l (\overline{\phi}_3^h)^k \end{aligned} \tag{7.4}$$

where $\phi_3^h$ is a 3 flavon that has even $Z_2^S$ parity but odd $Z_2^h$ parity. The scale of the flavon VEVs are taken to be $\langle\phi_3^S\rangle/M_S = \epsilon_S$, $\langle\overline{\phi}_3^S\rangle/M_h = \overline{\epsilon}_S$ and $\langle\overline{\phi}_3^h\rangle/M_S = \overline{\epsilon}_h$ where $M_S$ is the mass scale of the singlet-like messengers, $M_h$ is the mass scale of the Higgs-like messengers and it is assumed that $\epsilon_S \ll 1$. The messenger diagrams responsible for generating the above higher-order operators are represented by Fig.7.2. All Higgs-like and singlet-like messengers are assumed to carry even $Z_2^h$ parity but the Higgs-messengers, unlike the singlet-messengers, can carry both odd and even $Z_2^S$ parity.

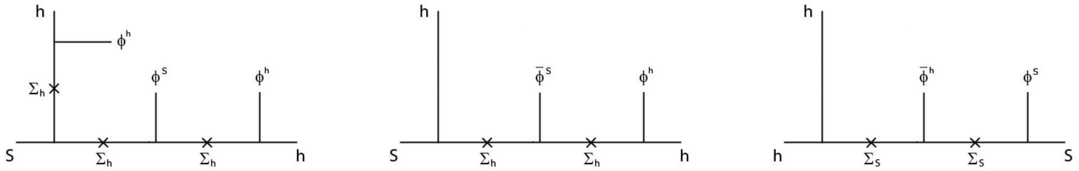



FIGURE 7.2: This Figure illustrates the type of messenger diagrams that provide the dominant contribution to the effective $\mu$-term and higgsino mass operators in Eq.7.4.

The first operator in Eq.7.4 is responsible for generating an effective $\mu$-term for the third family of Higgs fields once the flavons fields and the third family singlet field $S_3$ obtain VEVs. Since only the third family of Higgs obtains a VEV, this effective $\mu$-term acts like the $\mu$-term of the MSSM Higgs fields. The effective $\mu$-term will have a value $(0.8)^3\langle S_3\rangle$, which will be approximately 1 TeV if $\langle S_3\rangle = 2$ TeV, which is consistent with the experimental bound for the mass of a $Z'$ (see Section 4.4.3).

The second and third operators in Eq.7.4 are responsible for providing mass to the first and second families of higgsinos and singlinos once the third family of Higgs fields and singlet field obtain VEVs. This results in a mixing between all of these states which is represented by the following matrix:

$$M^{inert} = \begin{pmatrix} A_{22} & A_{21} \\ A_{21}^T & A_{11} \end{pmatrix}$$

This matrix is written in the basis $(\tilde{h}^0_{d2}, \tilde{h}^0_{u2}, \tilde{S}_2|\tilde{h}^0_{d1}, \tilde{h}^0_{u1}, \tilde{S}_1)$ so that $A_{\alpha\beta}$ are $3\times3$ matrices where $\alpha, \beta = 1, 2$. Because of the anti-symmetric tensor in the Eq.7.4 we find that $A_{11} = A_{22} = \mathbf{0}$, whereas $A_{21}$ is given by the following:

$$A_{21} = \begin{pmatrix} 0 & \epsilon_S\bar{\epsilon}_h\langle S^3\rangle & \bar{\epsilon}_S\langle h^3_u\rangle \\ \epsilon_S\bar{\epsilon}_h\langle S^3\rangle & 0 & \bar{\epsilon}_S\langle h^3_d\rangle \\ \bar{\epsilon}_S\langle h^3_u\rangle & \bar{\epsilon}_S\langle h^3_d\rangle & 0 \end{pmatrix}, \tag{7.5}$$

where this matrix couples the states $(\tilde{h}^0_{d2}, \tilde{h}^0_{u2}, \tilde{S}_2)$ to the states $(\tilde{h}^0_{d1}, \tilde{h}^0_{u1}, \tilde{S}_1)$. In the limit of exact $Z_2^h$ and $Z_2^S$ symmetry these higgsino and singlino states will decouple from the usual inert USSM states such as the third family of Higgsinos, singlinos, wino and hypercharge bino fields. A full discussion on the mixing between the usual USSM states and the additional $\mathrm{E_6SSM}$ states can be found in [91] where it is also shown that the mixing between the $U(1)_N$ bino and Higgsino and singlino fields is expected to be small.

The above Higgsino and singlino neutral states combine to form two degenerate LSP states, approximately consisting of a Dirac state formed from (dropping the tildes) $S_1$ and $S_2$, together with two generally heavier approximately degenerate Dirac states formed from $h^0_{d1}$ and $h^0_{u2}$ on the one hand and $h^0_{d2}$ and $h^0_{u1}$ on the other hand. With exact R-parity the Dirac LSP state formed from $S_1$ and $S_2$ becomes a dark matter candidate. However the masses of the degenerate LSPs $S_1$ and $S_2$ can be split if the first and second generation of Higgs and singlet fields are distinguished from one another. One way of achieving this is to assume that the flavon field $\phi^h_3$ gets small vacuum expectation values in its first and second components of $\Delta_{27}$ such that $\langle\phi^h_3\rangle^T \propto (\delta_1, \delta_2, 1)$ where $\delta_1,\ \delta_2 \ll 1$. This might be expected to occur from higher-order operators that affect the vacuum alignment of the fields. Two WIMPs that are almost degenerate in mass have been recently used to explain the DAMA data [92].[1] More work is required to determine whether the model considered here can be used to explain this data.

Note that although the $Z^h_2$ and $Z^S_2$ symmetries of this model have combined to operate in a similar manner to the original $Z^H_2$ symmetry of the E$_6$SSM, they allow fewer operators than the latter. The operators allowed by the original $Z^H_2$ symmetry but which are not present in this model are $S_3 h_{u\alpha} h_{d\alpha}$, $S_\alpha h_{u\alpha} h_{d3}$ and $S_\alpha h_{u3} h_{d\alpha}$. Such operators are responsible for the $A_{22}$ and $A_{11}$ matrices being non-zero in the E$_6$SSM.

## 7.6 Higgs Triplet Mass Terms

The Pati-Salam superpotential $\lambda^{ijk} S_i \mathcal{D}_j \mathcal{D}_k$, which is derived from the E$_6$ superpotential of the E$_6$SSM given by Eq.7.1, is used in the E$_6$SSM to give mass to the Higgs triplets $\mathcal{D}_i$. In terms of a Standard Model gauge symmetry this operator becomes $\lambda^{ijk} S_i D_j \overline{D}_k$ where $D$ is a triplet of the strong force gauge group $SU(3)_c$ but $\overline{D}$ is an anti-triplet.

To explain the three copies of the $\mathcal{D}$ states in the E$_6$SSM they are assumed to transform in a triplet representation of $\Delta_{27}$. As for the Higgs doublet-like states, the $\mathcal{D}$ are also taken to have odd $Z^h_2$ parity but even $Z^S_2$ parity. The allowed higher-order operator thus mirrors the allowed operators that provide effective $\mu$-terms for the Higgs fields:

$$\begin{aligned} W_D \sim\ & \frac{1}{M^3} S_i D_j \overline{D}_k (\overline{\phi}^S_3)^i (\overline{\phi}^h_3)^j (\overline{\phi}^h_3)^k \\ &+ \frac{1}{M^2} \epsilon^{jkl} S_i D_j \overline{D}_k (\overline{\phi}^S_3)^i (\phi^h_3)_l \\ &+ \frac{1}{M^2} \epsilon^{ijl} S_i D_j \overline{D}_k (\phi^S_3)_l (\overline{\phi}^h_3)^k \\ &+ \frac{1}{M^2} \epsilon^{ijl} S_i D_k \overline{D}_j (\phi^S_3)_l (\overline{\phi}^h_3)^k . \end{aligned} \tag{7.6}$$

[1] If $\langle\phi^h_3\rangle^T \propto (\delta_1, \delta_2, 1)$ then the first operator in Eq.7.4 will mix the Higgs doublet-like flavour eigenstates $h_1$, $h_2$ and $h_3$ so that the mass eigenstate $h^m_3$ is a mixture of all these Higgs doublet-like states. When inserted into the operators in Eq.7.2, FCNCs will be generated by the additional Higgs doublet-like fields. However, with $\delta_1,\ \delta_2 \ll 1$ then these FCNCs will be heavily suppressed and will be well within experimental limits.

The mass scale for the exotic-like messengers $\Sigma_{D,\overline{D}}$ responsible for the operators in Eq.7.6 however need not be the same as the Higgs messengers. The messenger scales are defined such that $M_D = M_{\overline{D}}$, $\langle \overline{\phi}_3^h \rangle / M_D \equiv \epsilon_D$, $\langle \phi_3^h \rangle / M_S \equiv \overline{\epsilon}_D$ and $\langle \phi_3^S \rangle / M_D \equiv \overline{\epsilon}'_S$. The exotic-like messengers, like the Higgs-like messengers are also assumed to only have even $Z_2^h$ parity and carry either even or odd $Z_2^S$ parity. The messenger diagrams that are responsible for generating the higher-order operators in Eq.7.6 are analogous to those in Fig.7.2 but with the Higgs fields and Higgs-like messenger fields replaced with exotic fields and exotic-like messenger fields respectively.

The fermion components of the Higgs triplet fields (triplet higgsinos) thus obtain mass once the flavons and $S_3$ obtain an expectation value. The masses are written in matrix form $M^{Dij} D_i \overline{D}_j$ where $M^{Dij}$ is the following:[2]

$$M^{Dij} = \begin{pmatrix} 0 & \epsilon_S \overline{\epsilon}_D & 0 \\ \epsilon_S \overline{\epsilon}_D & 0 & 0 \\ 0 & 0 & \epsilon_S \epsilon_D^2 + \epsilon_D^3 \end{pmatrix} \langle S^3 \rangle .$$

The parameters $\epsilon_S$, $\epsilon_D$ and $\overline{\epsilon}_D$ can then be chosen for the masses to be larger than the experimental bound of 300 GeV. Two of the triplet higgsinos are predicted to be degenerate in mass with the third also being degenerate in the approximation that $\epsilon_D^2 = \overline{\epsilon}_D$ and $\epsilon_D \ll \epsilon_S$. This mass structure is in stark contrast to the hierarchical structure of the quarks and leptons despite all the states being triplets of the family symmetry.

## 7.7 Higgs Triplet Decay and Proton Decay Suppression

If the triplet Higgs particles $\mathcal{D}_i$ are taken to have the same $\Delta_{27}$, $Z_2^h$ and $Z_2^S$ quantum numbers as the Higgs fields $h_i$, then they can decay via the following non-renormalizable operators:

$$\begin{aligned} W_{Exotic} \sim & \frac{1}{M^3} F_i^c F_j^c \mathcal{D}_k \overline{\phi}_3^i \overline{\phi}_3^j (\overline{\phi}_3^h)^k \\ & + \frac{1}{M^4} (F_i F_j + F_i^c F_j^c) \mathcal{D}_k \overline{\phi}_{23}^i \overline{\phi}_{23}^j (\overline{\phi}_3^h)^k H_{45} \\ & + \frac{1}{M^3} (F_i F_j + F_i^c F_j^c) \mathcal{D}_k \overline{\phi}_{123}^i \overline{\phi}_{23}^j (\overline{\phi}_3^h)^k \\ & + \frac{1}{M^3} (F_i F_j + F_i^c F_j^c) \mathcal{D}_k (\overline{\phi}_{123}^i \overline{\phi}_3^j + \overline{\phi}_3^i \overline{\phi}_{123}^j) (\overline{\phi}_{123}^m \phi_{1m}) (\overline{\phi}_3^h)^k H_{45}. \end{aligned} \tag{7.7}$$

[2]The scalar components of the Higgs triplets will also obtain mass from soft terms in the SUSY-breaking Lagrangian.

However not all these operators can be allowed otherwise this would lead to very rapid proton decay. Thus, either the $Z_2^B$ or $Z_2^L$ discrete symmetries that is used in the E$_6$SSM [17] are assumed. From Section 2.6.7, under the $Z_2^B$ symmetry the leptons and $D$ states are odd whereas, under the $Z_2^L$ symmetry, only the leptons are odd and all other particles are even. Thus these symmetries differentiate between different fermion $F, F^c$ components and therefore break the Pati-Salam gauge symmetry (but not the E$_6$SSM gauge symmetry).

In the limit that $\langle\phi_3^h\rangle^T \propto (0,0,1)$ exactly, the decay channels of the Higgs triplets states will be different to those of the E$_6$SSM since only the third generation of the Higgs triplets couples directly to quarks and leptons, whereas all three generations of the Higgs triplets in the E$_6$SSM interact directly with the quarks and leptons. The difference between the two models occurs because the Higgs triplets transform under $Z_2^h$, which results in an effective $Z_2^H$ symmetry for only the first and second generation of Higgs triplets. In the E$_6$SSM however all three generations transform under $Z_2^H$. This application of the $Z_2^h$ symmetry results in the decay products of the first and second generation of Higgs triplets always involving a singlet field $S_i$.

If instead $\langle\phi_3^h\rangle^T \propto (\delta_1, \delta_2, 1)$ as discussed in Section 7.5, then all the $\Delta_{27}$ components of the Higgs triplets will mix via the mass terms presented in Section 7.6. This results in the same Higgs triplets channels as used in the E$_6$SSM but with some being more suppressed since $\delta_1,\ \delta_2 \ll 1$.

## 7.8 Vacuum Alignment

The vacuum alignment assumed for the flavon fields $\overline{\phi}_3$, $\overline{\phi}_{23}$, and $\overline{\phi}_{123}$ is assumed to be the same as that discussed in Section 5.6.3. However this did not include the vacuum alignment for the new flavon fields $\phi_3^h$, $\overline{\phi}_3^h$, $\phi_3^S$ and $\overline{\phi}_3^S$. For these additional flavon fields to get the required direction of vacuum expectation values, the following D-terms are used: $m_{3/2}^2((\overline{\phi}_3^h)^i\phi_{3i}\phi_{3i}^\dagger(\overline{\phi}_3^{h\dagger})^i)$ and $m_{3/2}^2(\overline{\phi}_3^i\phi_{3i}^h\phi_{3i}^{h\dagger}\overline{\phi}_3^{\dagger i})$ both with negative coefficients, and similarly for the $\phi_3^S$ and $\overline{\phi}_3^S$ flavons. These terms cause $\overline{\phi}_3^h$ and $\phi_3^h$ to get VEVs in the same direction as the pre-aligned fields $\overline{\phi}_{3i}$ and $\phi_3^i$ respectively, which obtain their vacuum structure from the operators analogous to those discussed in Section 5.6.4 and are discussed in detail in [74].

## 7.9 Conclusions

In the previous Chapter a $\Delta_{27}$ was applied to the ME$_6$SSM and E$_6$SSM with the purpose of creating a model that can solve the flavour problems of the MSSM and SM. This was motivated by the ability of the ME$_6$SSM and E$_6$SSM to explain the small Higgs mass in comparison to the Planck mass. Together with the $\Delta_{27}$ family symmetry these models can then explain the masses of the quarks and leptons and in particular why they are small (compared to the Planck scale). However this come at the expense of additional flavour introduced by the ME$_6$SSM and E$_6$SSM which were not fully explained by the $\Delta_{27}$ family symmetry. This extra flavour comes from the non-MSSM states such as the additional Higgs doublet-like fields and Higgs triplet fields that are included in the ME$_6$SSM and E$_6$SSM.

The purpose of this Chapter was to find an alternative application of a $\Delta_{27}$ which can fully solve the flavour problem of the E$_6$SSM and thus present a truly viable alternative to the MSSM. This was achieved by taking all the E$_6$SSM states that fill 27 multiplets, which includes the Higgs fields, to transform as triplets under the $\Delta_{27}$ family symmetry. The breaking of the $\Delta_{27}$ family symmetry then resolves the fermion mass and mixing puzzle present in the SM, the SUSY FCNC problems introduced by the MSSM, and predicts the mass structure of the non-E$_6$SSM states. The main phenomenological predictions of the model are tri-bi-maximal mixing for leptons, two almost degenerate LSPs and two almost degenerate families of triplet higgsinos.

A particular success of the model illustrated in this Chapter is that it demonstrates how FCNC's in models with three families of Higgs fields may be tamed by the same family symmetry that predicts tri-bi-maximal lepton mixing and provides a solution to the SUSY FCNC and CP problems. This is because the $\Delta_{27}$ family symmetry, together with a vertical $Z_2^h$ symmetry, gives rise effectively to the $Z_2^H$ symmetry of the E$_6$SSM, which solves the flavour changing neutral current problem of the three families of Higgs fields.

A disadvantage of the application of the $\Delta_{27}$ family symmetry presented in this Chapter however is that it can only be used for the E$_6$SSM and not the ME$_6$SSM. This is because this application, unlike that in the previous Chapter, does not suppress the proton decay induced by the Higgs triplet fields $\mathcal{D}$. Instead the induced proton decay can only be suppressed by the method adopted by the E$_6$SSM where discrete symmetries are used to prevent the decay. This leaves the theoretically undesirable fields $H'$ and $\overline{H}'$ in the low-energy particle spectrum which introduce a $\mu'$-problem as discussed in Section 3.2.

# Chapter 8

# Conclusions and Outlook

The hierarchy problem remains a principal incentive for new physics beyond the Standard model and embedding the model in the MSSM is the most studied solution to this problem. However, although this solves the instability of the Higgs mass against higher energy physics, it does not adequately explain why its mass is small in the first place. This is related to the $\mu$-problem of the MSSM. Non-minimal supersymmetric models inspired by an $E_6$ symmetry on the other hand can naturally stabilize the Higgs mass without introducing the $\mu$-problem or little hierarchy problem of the MSSM. An example is the $E_6$SSM which contains three copies of a 27 representation of $E_6$ and two additional electroweak doublets whose sole purpose is to generate gauge coupling unification in the model. However, because these electroweak doublets come from incomplete $E_6$ representations, they introduce a number of theoretical problems to the model.

In this work a new model called the $ME_6$SSM has been proposed as an alternative to the $E_6$SSM that only contains complete $E_6$ representations at low energies and so does not contain any of the theoretical problems that come from incomplete representations. As well as solving the hierarchy problem of the Standard Model, the $ME_6$SSM also predicts gauge coupling unification at the Planck scale, suggesting a potential unification with quantum gravity.

Another motivation for physics beyond the Standard model (and the MSSM) comes from the flavour problem, which has seen a renewed interest in recent years due to the discovery of neutrino oscillations. In this work the $E_6$SSM and $ME_6$SSM have been extended with a non-Abelian discrete family symmetry as a step towards solving the flavour problem. The quantitatively new feature of the resulting models is that the same family symmetry that explains the observed masses and mixings of the quarks and leptons, including tri-bi-maximal mixing for leptons, also naturally tames the proton decay induced by Higgs triplet fields or the FCNCs mediated by the extended Higgs sectors.

A failing of the ME$_6$SSM (and E$_6$SSM) however is that a low-energy $Z_2^H$ discrete symmetry does not commute with an E$_6$ symmetry if the chiral superfields comes from the same high-energy E$_6$ multiplets. If on the other hand the superfields come from different E$_6$ multiplets then a complicated mass splitting mechanism is required to allow only three copies of a 27 representation to survive to low-energies. More work is therefore required to relate the ME$_6$SSM (and E$_6$SSM) to the high-energy E$_6$ symmetry.

One possibility may be to embed the models in a string inspired theory in which the E$_6$ symmetry is broken by compactification. Split E$_6$ multiplets that together look like complete E$_6$ representations at low energies could then arise from Wilson-line symmetry breaking for example [58]. This could also potentially explain the origin of the non-Abelian discrete family symmetry [84].

# Appendix A

# $\beta$-Functions for the ME$_6$SSM

All of the parameters of a renormalizable field theory can usefully be thought of as scale-dependent entities. The scale dependence is described by simple differential equations called renormalization group equations (RGEs). The rate of flow of a coupling constant as a function of momentum is defined by the $\beta$-function:

$$\beta(\overline{g}) = \frac{d\overline{g}(p;g)}{d\ln(p/M)} \qquad \overline{g}(M;g) = g$$

where $M$ is a renormalization scale, and $\overline{g}(p)$ is called the running coupling constant which is the coupling constant $g$ obtained by integrating out degrees of freedom down to the scale $p$. We can calculate the $\beta$-function of a gauge coupling constant for a general quantum field theory to a given order of perturbation. Ignoring the small contributions from any Yukawa couplings of the theory, the $\beta$-function of a gauge coupling constant $g_i$ to two-loops is given by [93]:[1]

$$\begin{aligned} \frac{d\overline{\alpha}_i}{dt} &= -b_i\overline{\alpha}_i^2 - \overline{\alpha}_i^2\Big(\sum_j b_{ij}\overline{\alpha}_j\Big) \\ \Rightarrow \frac{d(1/\overline{\alpha}_i)}{dt} &= b_i + \sum_j b_{ij}\overline{\alpha}_j \end{aligned} \tag{A.1}$$

where $\overline{\alpha}_i \equiv g_i^2/(4\pi)^2$; $t \equiv \ln(p/M)$; $b_i$ and $b_{ij}$ are group factors from the group representations of the various particles of the quantum field theory; and the indices $i$ and $j$ run over all the gauge coupling constants of the quantum field theory. The first term in Eq.A.1 is the one-loop contribution and the second term is from two-loops.

[1] At one-loop the graph of $1/\alpha_i$ versus $t$ is a straight line. At two-loops the graph is a curve that is generally close to a straight line since the two-loop effects are respectively small by definition of perturbation theory.

For a non-supersymmetric quantum field theory the group factors $b_i$ and $b_{ij}$ are the following [93]:

$$b_i = -\frac{11}{3}C_i(G) + \frac{2}{3}\sum_f n_f C_i(r_f) + \frac{1}{6}\sum_s n_s C_i(r_s) \tag{A.2}$$

$$\begin{aligned} b_{ij} = &\frac{34}{3}C_i(G)C_{j=i}(G) - \frac{10}{3}\sum_f n_f C_i(G)C_{j=i}(r_f) - \frac{1}{3}\sum_s n_s C_i(G)C_{j=i}(r_s) \\ &- 2\sum_f n_f C_i(r_f)C^2_{j\neq i}(r_f) - 2\sum_s n_s C_i(r_s)C^2_{j\neq i}(r_s). \end{aligned}$$

The first term in Eq.A.2 comes from the gauge bosons, and the second and third terms come from all the chiral fermions and (real) scalars, respectively, that live in the different irreducible representations $r_f$ and $r_s$ of the gauge group that has the gauge coupling constant $g_i$. $n_f$, $n_s$ are the number of scalars and fermions that live in the representations $r_f$ and $r_s$ of the gauge group. $C(G)_i$ is the Casmir operator for the adjoint representation of the group, $C(r)_i$ is the Casmir operator for the irreducible representation $r$ of the group, and $C_i^2(r)$ is the quadratic Casmir operator for the irreducible representation $r$ of group that has the gauge coupling constant $g_i$.

For an $SU(N)$ group $C(G) = N$, and, for the defining fundamental representation $N$ and its conjugate $\overline{N}$, $C(N) = C(\overline{N}) = 1/2$. For an irreducible representation $r$ of $SU(N)$ the quadratic Casmir operator is given by:

$$C_i^2(r) = \frac{C(r)_i d(G)_i}{d(r)_i}$$

where $d(G)_i$ is the dimension of the adjoint representation of $SU(N)$ which is $N^2 - 1$, and $d(r)_i$ is the dimension of the irreducible representation $r$. For an Abelian group $U(1)$, $C(G) = 0$, and $C(r)$ and $C^2(r)$ are replaced with $Q_i^2$, the square of the charge of the particle that couples to $U(1)$.

In a supersymmetric quantum field theory $b_i$ and $b_{ij}$ are given by the following [94]:

$$b_i = -3C_i(G_i) + \sum_c n_c C_i(r_c) \tag{A.3}$$

$$\begin{aligned} b_{ij} = &-6C_i(G)C_{j=i}(G) + 2C_i(G)\sum_c n_c C_{j=i}(r_c) \\ &+ 4\sum_c n_c C_i(r_c)C^2_{j\neq i}(r_c). \end{aligned}$$

In Eq.A.3 the first term is a vector supermultiplet contribution and the second term comes from chiral supermultiplets where $c$ labels the different chiral supermultiplets in the supersymmetric quantum field theory.

Below the RGEs of the ME$_6$SSM are calculated but with the $U(1)_X$ and $U(1)_\psi$ groups ignored for simplicity. At energies below the $TeV$ scale the ME$_6$SSM is equivalent to the Standard Model, above $TeV$ it is a supersymmetric theory based on the gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X$, and above the GUT scale it is a supersymmetric theory based on the gauge group $SU(4)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_\psi$. Each energy regime is looked at in turn.

## A.1 The Standard Model

The Standard Model contains the gauge coupling constants $g_1$, $g_2$ and $g_3$ for the gauge groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_c$, where the hypercharge gauge coupling constant $g_1$ is GUT normalized. The $b_i$ and $b_{ij}$ group factors for the Standard Model are given below:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 22/3 \\ -11 \end{pmatrix} + n_g \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} + n_h \begin{pmatrix} 1/10 \\ 1/6 \\ 0 \end{pmatrix}$$

$$\begin{aligned} b_{ij} = & \begin{pmatrix} 0 & 0 & 0 \\ 0 & -136/3 & 0 \\ 0 & 0 & -102 \end{pmatrix} + n_g \begin{pmatrix} 19/5 & 3/5 & 44/15 \\ 1/5 & 49/3 & 4 \\ 11/30 & 3/2 & 76/3 \end{pmatrix} \\ & + n_h \begin{pmatrix} 9/50 & 9/10 & 0 \\ 3/10 & 13/6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

where $n_g$ are the number of generations of quarks and leptons, and $n_h$ are the number of Higgs fields. In the Standard Model $n_g = 3$ and $n_h = 1$.

The gauge coupling constants in Section 3.6 are run from their initial values measured at the $Z^0$ pole. Since the top quark is more massive than the $Z^0$ vector boson the initial RGEs do not depend on the top quark (its degrees of freedom have been integrated out of the theory). The top quark contributes the following to the fermionic part of the group factors $b_i$ and $b_{ij}$:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1/30 \\ 1 \\ 1/3 \end{pmatrix}$$

$$b_{ij} = \begin{pmatrix} 1/600 & 3/40 & 1/3 \\ 1/20 & 49/4 & 4 \\ 1/60 & 3/4 & 19/3 \end{pmatrix}. \tag{A.4}$$

## A.2 The ME$_6$SSM below the GUT Scale

The group factors $b_i$ and $b_{ij}$ for a supersymmetric quantum field theory with the gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ that contains the particles in the ME$_6$SSM are the following:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ -9 \end{pmatrix} + n_g \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + n_h \begin{pmatrix} 3/10 \\ 1/2 \\ 0 \end{pmatrix} + n_D \begin{pmatrix} 1/5 \\ 0 \\ 1/2 \end{pmatrix}$$

$$b_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -54 \end{pmatrix} + n_g \begin{pmatrix} 38/15 & 6/5 & 88/15 \\ 2/5 & 14 & 8 \\ 11/15 & 3 & 68/3 \end{pmatrix}$$
$$+ n_h \begin{pmatrix} 9/50 & 9/10 & 0 \\ 3/10 & 7/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + n_D \begin{pmatrix} 4/75 & 0 & 16/15 \\ 0 & 0 & 0 \\ 2/15 & 0 & 17/3 \end{pmatrix}.$$

where $n_g$ is the number of generations of the quark and lepton supermultiplets, $n_h$ is the number of Higgs-doublet supermultiplets, and $n_D$ is the number of Higgs triplet superfields. In the MSSM $n_g = 3$, $n_h = 2$ and $n_D = n_{H'} = 0$, whereas in the E$_6$SSM, $n_g = 3$, $n_h = 8$ (including the $H'$ and $\overline{H}'$ states) and $n_D = 6$. The ME$_6$SSM below the GUT scale contains $n_g = 3$, and $n_h = n_D = 6$. Note that in the E$_6$SSM and ME$_6$SSM the $\beta$-function for $g_3$ is zero at one-loop order and receives a positive contribution at two-loops. QCD therefore looses asymptotic freedom in these models because of the additional states that are not in the MSSM.

## A.3 The ME$_6$SSM above the GUT Scale

Above the GUT scale the ME$_6$SSM is a supersymmetric field theory based on the gauge group $SU(4) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_\psi$. $g_4$, $g_{2R}$, $g_{2L}$ are defined to be the gauge coupling constants of $SU(2)_R$, $SU(2)_L$ and $SU(4)$ respectively and the group factors are the following:

$$\begin{pmatrix} b_{2R} \\ b_{2L} \\ b_4 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \\ -12 \end{pmatrix} + n_g \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + n_h \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + n_D \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} b_{ij} = & \begin{pmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -96 \end{pmatrix} + n_g \begin{pmatrix} 14 & 0 & 15 \\ 0 & 14 & 15 \\ 3 & 3 & 31 \end{pmatrix} \\ & + n_h \begin{pmatrix} 7 & 3 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix} + n_D \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 18 \end{pmatrix} \\ & + n_{H_R} \begin{pmatrix} 14 & 0 & 15 \\ 0 & 0 & 0 \\ 3 & 0 & 31/2 \end{pmatrix} + n_{H_L} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 14 & 15 \\ 0 & 3 & 31/2 \end{pmatrix}. \end{aligned}$$

where now $n_g$ are the number of generations of the $F = (4, 2, 1)$ and $F^c = (\overline{4}, 1, \overline{2})$ multiplets that contain the quarks and leptons; $n_h$ is the number of $h = (1, 2, 2)$ multiplets that contain Higgs fields; $n_D$ is the number of $\mathcal{D} = (6, 1, 1)$ triplet Higgs fields; $n_{H_L}$ is the number of $H_L = (4, 2, 1)$ and $\overline{H}_L = (\overline{4}, \overline{2}, 1)$ states; and $n_{H_R}$ is the number of $H_R = (4, 1, 2)$ and $\overline{H}_R = (\overline{4}, 1, \overline{2})$ states. In the ME$_6$SSM, $n_g = n_h = n_D = 3$ and $n_{H_L} = n_{H_R} = 2$.

All the above group factors $b_i$ and $b_{ij}$ are used in Eq.A.1 to determine the two-loop running of the gauge coupling constants in the ME$_6$SSM. The results are plotted in Fig.3.1 and Fig.4.1 where Fig.4.1 also uses the group factors for the $U(1)_X$ and $U(1)_\psi$ groups which have not been included in this Appendix.

# Appendix B

# The $U(1)_X$ Symmetry

Since the $U(1)_X$ group does not appear to have been considered in the literature, this Appendix illustrates in detail how it is generated from a $G_{4221} = SU(4) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_\psi$ symmetry once $H_R = (4, 1, 2)_{-\frac{1}{2}}$ and $\overline{H}_R = (\overline{4}, 1, \overline{2})_{\frac{1}{2}}$ obtain VEVs. The $G_{4221}$ symmetry is then broken to the $G_{3211} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X$ symmetry.

The covariant derivative of the $G_{4221}$ symmetry is given by:

$$D_\mu = \partial_\mu + ig_4 T_4^m A_{4\mu}^m + ig_{2L} T_L^s A_{L\mu}^s + ig_{2R} T_R^r A_{R\mu}^r + \frac{1}{\sqrt{6}} ig_\psi T_\psi A_{\psi\mu} \tag{B.1}$$

where $m = 1 \ldots 15$ and $r, s = 1 \ldots 3$; $A_{4\mu}^m$, $A_{L\mu}^s$, $A_{R\mu}^r$ and $A_{\psi\mu}$ are the $SU(4)$, $SU(2)_L$, $SU(2)_R$ and $U(1)_\psi$ quantum fields respectively; $g_4$, $g_{2L}$, $g_{2R}$ and $g_\psi$ denote the universal gauge coupling constants of the respective fields and $T_4^m$, $T_L^s$, $T_R^r$ and $T_\psi$ represent their generators. All of the $T_4^m$, $T_R^r$, $T_L^s$ and $T_\psi$ generators are derived from components of the $E_6$ generators $G^a$, which are chosen to be $E_6$ normalized, for the fundamental representation 27, by:

$$\mathrm{Tr}(G^a \, G^b) = 3\delta^{ab} \tag{B.2}$$

where $a, b = 1 \ldots 78$.

Then, with this normalization, the Pati-Salam generators $T_4^m$, $T_R^r$ and $T_L^s$ are normalized for the fundamental representations of $SU(4)$, $SU(2)_R$ and $SU(2)_L$ respectively, by:[1]

$$Tr(T_4^m\ T_4^n) = \frac{1}{2}\delta^{mn},$$
$$Tr(T_R^r\ T_R^s) = Tr(T_L^r\ T_L^s) = \frac{1}{2}\delta^{rs}$$

where $m, n = 1 \ldots 15$.

The $U(1)_\psi$ charge $\frac{1}{\sqrt{6}}T_\psi$ is a diagonal $E_6$ generator, which is chosen to be the 78th generator $G^{78} = \frac{1}{\sqrt{6}}T_\psi$, and is therefore normalized by Eq.B.2 to give:

$$\frac{1}{6}\sum_{27} T_\psi^2 = 3 \tag{B.3}$$

where the sum is over all the $G_{4221}$ representations that make up the fundamental 27 multiplet of $E_6$.

The scalar fields $H_R$ and $\overline{H}_R$ are used to break $G_{4221}$ to $G_{3211}$. These are the smallest $G_{4221}$ multiplets that can be used to break the Pati-Salam symmetry directly to the standard model gauge group. When $H_R$ and $\overline{H}_R$ develop VEVs in the $\nu_R$ and $\nu^c$ components respectively, they will break $SU(4) \to SU(3)_c$ [61] and mix the field associated with the remaining $SU(4)$ diagonal generator, $A_4^{15}$, with the field associated with the diagonal generator of $SU(2)_R$, $A_R^3$, and the $U(1)_\psi$ field $A_\psi$. The rest of the $SU(4)$ and $SU(2)_R$ fields are given square mass proportional to $\upsilon^2$, the sum of the square of the $H_R$ and $\overline{H}_R$ VEVs.

The diagonal generators for the $A_4^{15}$ and $A_R^3$ fields are $T_4^{15}$ and $T_R^3$. For the fundamental representations of $SU(4)$ and $SU(2)_R$ respectively [51] :

$$T_4^{15} = \sqrt{\frac{3}{2}}\ diag(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{2}), \qquad T_R^3 = diag(\frac{1}{2}, -\frac{1}{2}).$$

The part of the symmetry breaking $G_{4221}$ to $G_{3211}$ involving the diagonal generators $T_4^{15}$, $T_R^3$ and $T_\psi$ is then equivalent to:

$$U(1)_{T_4^{15}} \otimes U(1)_{T_R^3} \otimes U(1)_\psi \to U(1)_Y \otimes U(1)_X.$$

In the rest of this Appendix this particular symmetry breaking is explained in detail. Using the $G_{4221}$ covariant derivative, Eq.B.1, the covariant derivative for the $U(1)_{T_4^{15}} \otimes$

[1]These normalizations are necessary for the Standard Model generators $T_{SM}$ of $SU(3)_c$ and $SU(2)_L$ to be normalized in the conventional way: $Tr(T_c^d T_c^e) = \frac{1}{2}\delta^{de}$ and $Tr(T_L^r T_L^s) = \frac{1}{2}\delta^{rs}$ for the fundamental representations, where $T_c$ and $T_L$ are the generators for the $SU(3)_c$ and $SU(2)_L$ groups respectively and $d, e = 1 \ldots 8$.

$U(1)_{T_R^3} \otimes U(1)_\psi$ symmetry is:

$$\begin{aligned} D_\mu &= \partial_\mu + ig_4 T_4^{15} A_{4\mu}^{15} + ig_{2R} T_R^3 A_{R\mu}^3 + \frac{1}{\sqrt{6}} ig_\psi T_\psi A_{\psi\mu} \\ &\equiv \partial_\mu + ig_{B-L} T_{B-L} A_{4\mu}^{15} + ig_{2R} T_R^3 A_{R\mu}^3 + ig_{N\psi} T_\psi A_{\psi\mu} \end{aligned} \tag{B.4}$$

where $g_{B-L} \equiv \sqrt{\frac{3}{2}} g_4$, $g_{N\psi} \equiv \frac{1}{\sqrt{6}} g_\psi$, $T_{B-L} \equiv \sqrt{\frac{2}{3}} T_4^{15} = \frac{(B-L)}{2}$ and $B$ and $L$ are baryon and lepton number respectively.

In terms of the diagonal generators $T_{B-L}$, $T_R^3$ and $T_\psi$, the $\nu_R$ component of $H_R$ and the $\nu^c$ component of $\overline{H}_R$ transform under $U(1)_{T_4^{15}} \otimes U(1)_{T_R^3} \otimes U(1)_\psi$ as:

$$\nu_R^H = \left(-\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right), \qquad \nu_{\overline{H}}^c = \left(\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right). \tag{B.5}$$

Therefore, once $H_R$ and $\overline{H}_R$ get their VEVs, the square of the covariant derivative for the $A_4^{15}$, $A_R^3$ and $A_\psi$ fields becomes:

$$\left|D_\mu \nu_R^H\right|^2 = \frac{1}{4} \upsilon^2 \Big( - \mathbf{g}_{B-L} A_{4\mu}^{15} + \mathbf{g}_{2R} A_{R\mu}^3 - \mathbf{g}_{N\psi} A_{\psi\mu} \Big)^2$$

where $\mathbf{g}_{B-L}$, $\mathbf{g}_{2R}$ and $\mathbf{g}_{N\psi}$ are the $g_{B-L}$, $g_{2R}$ and $g_{N\psi}$ gauge coupling constants evaluated at the $G_{4221}$ symmetry breaking scale. The above squared covariant derivative can be written in matrix form as:

$$\frac{1}{4}\upsilon^2 \begin{pmatrix} A_R^3 & A_4^{15} & A_\psi \end{pmatrix} \begin{pmatrix} \mathbf{g}_{2R}^2 & -\mathbf{g}_{2R}\,\mathbf{g}_{B-L} & -\mathbf{g}_{2R}\,\mathbf{g}_{N\psi} \\ -\mathbf{g}_{2R}\,\mathbf{g}_{B-L} & \mathbf{g}_{B-L}^2 & \mathbf{g}_{B-L}\,\mathbf{g}_{N\psi} \\ -\mathbf{g}_{2R}\,\mathbf{g}_{N\psi} & \mathbf{g}_{B-L}\,\mathbf{g}_{N\psi} & \mathbf{g}_{N\psi}^2 \end{pmatrix} \begin{pmatrix} A_R^3 \\ A_4^{15} \\ A_\psi \end{pmatrix}. \tag{B.6}$$

Diagonalizing this matrix equation determines the mass eigenstate fields generated by the mixing of the $G_{4221}$ fields $A_R^3$, $A_4^{15}$ and $A_\psi$. The $3\times 3$ square mass mixing matrix has two zero eigenvalues and one non-zero eigenvalue so that two massless gauge bosons and one massive gauge boson appear to have been created by the mixing. The massive gauge boson $B_H$ is the following mixture of $G_{4221}$ fields:

$$B_H = \frac{1}{b}\Big( - \mathbf{g}_{2R} A_R^3 + \mathbf{g}_{B-L} A_4^{15} + \mathbf{g}_{N\psi} A_\psi \Big)$$

where $b^2 \equiv \mathbf{g}_{2R}^2 + \mathbf{g}_{B-L}^2 + \mathbf{g}_{N\psi}^2$.

This massive field is an unique mass eigenstate field. However, the degeneracy in the zero-eigenvalue eigenvectors of the square mass mixing matrix implies that all orthogonal combinations of any chosen two massless eigenstate fields also describe two massless eigenstate fields. All the orthogonal combinations of two massless eigenstate fields are physically distinct and so the symmetry breaking mechanism does not generate two unique massless eigenstate fields.[2] We therefore require something in addition to this symmetry breaking mechanism that lifts the degeneracy of the zero-eigenvalue eigenvectors and selects two unique massless gauge fields.

It is shown below that when we include the low-energy VEV of the $S$ particle from the third generation of the 27 multiplets, the degeneracy in the zero-eigenvalue eigenvectors is lifted and the two massless gauge fields are uniquely chosen to be the gauge field $B_Y$ of the Standard Model hypercharge group and an (effectively massless) gauge field that we call $B_X$. The $B_Y$ and $B_X$ gauge fields are generated from orthogonal zero-eigenvalued eigenvectors of the above $3 \times 3$ square mass mixing matrix and are the following mixture of $G_{4221}$ fields:

$$\begin{aligned} B_Y &= \frac{1}{a}\Big(\mathbf{g}_{B-L} A_R^3 + \mathbf{g}_{2R} A_4^{15}\Big), \\ B_X &= \frac{1}{ab}\Big(\mathbf{g}_{2R}\mathbf{g}_{N\psi} A_R^3 - \mathbf{g}_{B-L}\mathbf{g}_{N\psi} A_4^{15} + (\mathbf{g}_{2R}^2 + \mathbf{g}_{B-L}^2) A_\psi\Big) \end{aligned}$$

where $a^2 \equiv \mathbf{g}_{2R}^2 + \mathbf{g}_{B-L}^2$.

In terms of the diagonal generators $T_{B-L}$, $T_R^3$ and $T_\psi$, the $S$ particle transforms under the $U(1)_{T_4^{15}} \otimes U(1)_{T_R^3} \otimes U(1)_\psi$ symmetry as:

$$S = \Big(0,\ 0,\ 2\Big).$$

The $S$ particle only couples to $A_\psi$ and so its VEV, $s$, therefore introduces a perturbation proportional to $s^2/v^2$ to the $(3,3)$ component of the $3 \times 3$ square mass mixing matrix in Eq.B.6. From Section 4.2.2, $v$ is determined to be of the order $10^{16}$ GeV and we require that $s \approx 10^3$ GeV for EW symmetry breaking.

Diagonalizing the $3 \times 3$ square mass mixing matrix with this extremely small perturbation in the $(3,3)$ component determines the mass eigenstate fields to be the massless hypercharge gauge field $B_Y$, and an extremely small mass gauge field and large mass gauge field that can be taken to be the $B_X$ and $B_H$ gauge fields, respectively, in the excellent approximation that $s^2/v^2 = 0$.[3]

It is easy to see why the hypercharge gauge field of the Standard Model is the exact massless gauge field of this symmetry breaking. The hypercharge field is the only massless gauge field generated by the $H_R$ and $\overline{H}_R$ VEVs that does not contain the $A_\psi$ field and therefore the only massless gauge field that $S$ does not couple to. If the $A_\psi$ field is removed from the $G_{4221}$ symmetry then the mixing of the remaining $G_{4221}$ diagonal

[2] All the orthogonal combinations are physical since the kinetic term part of the Lagrangian is invariant to orthogonal transformations of the fields.

[3] The VEV of the Standard Model Higgs field is ignored in this symmetry breaking.

generators becomes equivalent to $U(1)_{T_4^{15}} \otimes U(1)_{T_R^3} \rightarrow U(1)_Y$ when $H_R$ and $\overline{H}_R$ get VEVs [61].

The mass eigenstate fields $B_Y$, $B_X$ and $B_H$ can be written in terms of the $G_{4221}$ fields $A_R^3$, $A_4^{15}$ and $A_\psi$ in the following matrix form:

$$\begin{pmatrix} B_Y \\ B_X \\ B_H \end{pmatrix} = \begin{pmatrix} \mathbf{g}_{B-L}/a & \mathbf{g}_{2R}/a & 0 \\ \mathbf{g}_{2R}\mathbf{g}_{N\psi}/ab & -\mathbf{g}_{B-L}\mathbf{g}_{N\psi}/ab & (\mathbf{g}_{2R}^2 + \mathbf{g}_{B-L}^2)/ab \\ -\mathbf{g}_{2R}/b & \mathbf{g}_{B-L}/b & \mathbf{g}_{N\psi}/b \end{pmatrix} \begin{pmatrix} A_R^3 \\ A_4^{15} \\ A_\psi \end{pmatrix}. \tag{B.7}$$

This orthogonal $3 \times 3$ matrix can be parameterized in terms of rotation and reflection matrices in the following way:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ c_{23}s_{12} & -c_{23}c_{12} & s_{23} \\ -s_{23}s_{12} & s_{23}c_{12} & c_{23} \end{pmatrix}$$

where $c_{12} = \mathbf{g}_{B-L}/a$, $s_{12} = \mathbf{g}_{2R}/a$, $c_{23} = \mathbf{g}_{N\psi}/b$ and $s_{23} = a/b$. The mixing angles $\theta_{12}$ and $\theta_{23}$ are therefore given by $\tan\theta_{12} = \mathbf{g}_{2R}/\mathbf{g}_{B-L}$ and $\tan\theta_{23} = a/\mathbf{g}_{N\psi}$.

Taking the transpose of Eq.B.7, the $G_{4221}$ fields $A_R^3$, $A_4^{15}$ and $A_\psi$ can be written in terms of the mass eigenstate fields $B_Y$, $B_X$ and $B_H$ as:

$$\begin{pmatrix} A_R^3 \\ A_4^{15} \\ A_\psi \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12}c_{23} & -s_{12}s_{23} \\ s_{12} & -c_{12}c_{23} & c_{12}s_{23} \\ 0 & s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} B_Y \\ B_X \\ B_H \end{pmatrix}.$$

Putting this matrix equation into the covariant derivative for the $U(1)_{T_4^{15}} \otimes U(1)_{T_R^3} \otimes U(1)_\psi$ symmetry, Eq.B.4, determines the covariant derivative for the massless gauge fields $B_Y$ and $B_X$ to be:

$$D_\mu = \partial_\mu + ig_Y Y B_{Y\mu} + ig_X^0 X B_{X\mu}$$

where:

$$Y = T_R^3 + T_{B-L} = T_R^3 + (B-L)/2$$

is the Standard Model hypercharge [61], and:[4]

$$X = (T_\psi + T_R^3) - c_{12}^2 Y \tag{B.8}$$

[4]Alternatively we could have defined $X$ to be $\mathbf{g}_{2R}^2(T_\psi + T_R^3) + \mathbf{g}_{B-L}^2(T_\psi - T_{B-L})$ and redefined $g_X^0$ equivalently.

is the non-normalized charge of the $B_X$ gauge field. $g_Y$ and $g_X^0$ are the non-normalized universal gauge coupling constants of the $B_Y$ and $B_X$ fields respectively and, at the $G_{4221}$ symmetry breaking scale, are given by Eq.B.9 and Eq.B.10:[5]

$$g_Y = \frac{\mathbf{g}_{2R}\ \mathbf{g}_{B-L}}{a} \tag{B.9}$$

$$g_X^0 = \frac{a}{b}\mathbf{g}_{N\psi}. \tag{B.10}$$

Eq.B.9 and Eq.B.10 can be written is terms of $\alpha_Y = \frac{g_Y^2}{4\pi}$ and $\alpha_X^0 = \frac{(g_X^0)^2}{4\pi}$, see Eq.3.3 and Eq.4.11 in Section 4.2.1. The charges $X$ and $Y$ are not $E_6$ normalized and the respective charges are defined as $T_X$ and $T_Y$ where:

$$T_X = X/N_X, \qquad T_Y = Y/N_Y$$

and the normalization constants $N_X$ and $N_Y$ are given by:

$$N_X^2 = 7 - 2c_{12}^2 + \frac{5}{3}c_{12}^4, \qquad N_Y^2 = \frac{3}{5}$$

Note that the Abelian generator $T_Y$ is just the conventional GUT normalized hypercharge. $T_X$ and $T_Y$ have been $E_6$ normalized using Eq.B.2 which is equivalent to:

$$\sum_{27} T_Y^2 = \sum_{27} T_X^2 = 3$$

where the sum is over all the $G_{3211} \equiv SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X$ representations of the fundamental 27 $E_6$ multiplet and $U(1)_X$ is the unitary group of the $B_X$ field.[6]

In terms of the $E_6$ normalized charges $T_X$ and $T_Y$, the covariant derivative for the $B_X$ and $B_Y$ gauge fields becomes:

$$D_\mu = \partial_\mu + ig_1 T_Y B_{Y\mu} + ig_X T_X B_{X\mu} \tag{B.11}$$

where $g_1$ and $g_X$ are the *normalized* universal gauge coupling constants of the $B_Y$ and $B_X$ fields respectively. At the $G_{4221}$ symmetry breaking scale, the normalized gauge coupling constants $g_1$ and $g_X$ are the following combinations of $G_{4221}$ gauge coupling constants:

$$g_1 = N_Y \frac{\mathbf{g}_{2R}\ \mathbf{g}_{B-L}}{a}, \qquad g_X = N_X\ \frac{a}{b}\mathbf{g}_{N\psi}.$$

[5]Note that Eq.B.9 is the relation that $g_Y$ must satisfy if the Pati-Salam symmetry, without the $U(1)_\psi$, was broken to the Standard Model gauge group using a Higgs boson that transforms as $(4, 1, 2)$ and gets a VEV in the $\nu_R$ direction.

[6]$X$ and $N_X$ could have been defined differently as long as $T_X$ is the same. Here $X$ has been defined so that it can be written in terms of hypercharge $Y$.

From Eq.B.8, the charge $T_X$ of the $U(1)_X$ group depends on the Pati-Salam gauge coupling constants $g_{2R}$ and $g_{B-L}$ evaluated at the $G_{4221}$ symmetry breaking scale. Therefore, under the excellent approximation that $s^2/v^2 = 0$, a massless gauge boson exists that couples to particles with a charge that depends on the values that certain coupling constants take at some high energy scale. Although this may be unusual, it does not appear to pose any problems. Indeed, like any other quantum charge, $T_X$ is a dimensionless constant that is independent of the energy scale at which the interaction between the particle and the $A_X$ field occurs and, although the numbers that $X$ takes may not be able to be arranged into fractions like $Y$, they are still discrete and sum to zero for a complete $E_6$ representation. However, unlike conventional $U(1)$ charges, $T_X$ is obviously very model dependent since different $E_6$ models with an intermediate Pati-Salam symmetry will, in general, contain different values of the gauge coupling constants $g_{2R}$ and $g_4$ evaluated at the $G_{4221}$ symmetry breaking scale. It is easy to prove that it is a general rule that, if three massless gauge fields are mixed, then at least two of the resulting mass eigenstate fields must have a charge that depends on the value of the original gauge coupling constants. Therefore this gauge coupling dependence is not peculiar to the Higgs symmetry breaking mechanism discussed in this Appendix, but to any symmetry breaking mechanism involving three fields.

In conclusion, this Appendix has illustrated how the $G_{4221} \equiv SU(4) \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_\psi$ symmetry can be broken to the symmetry $G_{3211} \equiv SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X$ when the $G_{4221}$ multiplets $H_R$, $\overline{H}_R$ and $S$ obtain vacuum expectation values. Using the covariant derivatives for the $G_{4221}$ symmetry, Eq.B.1, and the $U(1)_Y \otimes U(1)_X$ symmetry, Eq.B.11, the covariant derivative for the $G_{3211}$ symmetry is given by:

$$D_\mu = \partial_\mu + ig_3 T^n_{3c} A^n_{3c\mu} + ig_{2L} T^s_L A^s_{L\mu} + ig_1 T_Y B_{Y\mu} + ig_X T_X B_{X\mu} \tag{B.12}$$

where $A^n_{3c}$ and $T^n_{3c}$ are the $SU(3)_c$ fields and generators derived from the $SU(4)$ symmetry respectively (with $n = 1 \ldots 8$) and $g_{3c}$ is the universal gauge coupling constant of $A^n_{3c}$.

This $G_{3211}$ symmetry can be considered to be an effective high energy symmetry under the assumption that the low-energy VEVs of the MSSM singlet $S$ and MSSM Higgs bosons can be neglected at higher energy scales.

# Appendix C

# FCNC Processes from Extended Higgs Sectors

Models with extended Higgs sectors can potential contain tree-level FCNCs that are mediated by the exchange of the neutral Higgs states [78]. In the Standard Model and the MSSM, these effects are absent at the tree-level, since the coupling of the quark-quark-Higgs mass eigenstates is flavour conserving. This arises from having the Yukawa couplings proportional to the quark mass matrices, so that diagonalizing the mass matrices also diagonalizes the Yukawas. This is illustrated below.

For a supersymmetric theory with three generations of up and down Higgs doublet-like fields $h_u^i$ and $h_d^i$, where $i = 1 \ldots 3$, the general superpotential involving the quarks and Higgs fields is the following [95]:

$$W = \sum_{i=1}^{i=3} h_u^i \overline{\mathbf{u}}_R \mathbf{Y}_u^i \mathbf{u}_L + \sum_{i=1}^{i=3} h_d^i \overline{\mathbf{d}}_R \mathbf{Y}_d^i \mathbf{d}_L$$

where the quark fields are column vectors in generation space, and the Yukawa couplings $Y_q^i$ are $3 \times 3$ matrices in generation space. The index $i$ labels the different generations of Higgs doublet fields, not the different quark and lepton generations. Assuming that all the Higgs fields have VEVs $\langle h_u^i \rangle \equiv \upsilon_u$, $\langle h_d^i \rangle \equiv \upsilon_d$ then the above superpotential becomes:

$$W = \overline{\mathbf{u}}_R \mathbf{M}_u \mathbf{u}_L + \overline{\mathbf{d}}_R \mathbf{M}_d \mathbf{d}_L$$

where:

$$\mathbf{M}_q \equiv \sum_{i=1}^{i=3} \mathbf{Y}_q^i \upsilon_q^i$$

with $q = u, d$.

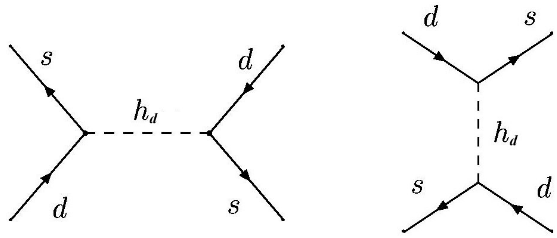



FIGURE C.1: This figure illustrates tree-level Feynman diagrams that contribute to $K^0 - \overline{K}^0$ mixing mediated by an extended Higgs sector.

The above superpotential is written in terms of the quark and lepton interaction eigenstates (the eigenstates of the gauge symmetries of the model). To obtain the mass eigenstates we must diagonalize the matrices $\mathbf{M}_u$ and $\mathbf{M}_d$. These are diagonalized as the following:

$$\begin{aligned}\mathcal{M}_u &= \mathbf{V}_{uR}\mathbf{M}_u\mathbf{V}^\dagger_{uL} = diag(m_u,\ m_c,\ m_t),\\ \mathcal{M}_d &= \mathbf{V}_{dR}\mathbf{M}_d\mathbf{V}^\dagger_{dL} = diag(m_d,\ m_s,\ m_b)\end{aligned}$$

where $\mathbf{V}_{uL}$, $\mathbf{V}_{dL}$, $\mathbf{V}_{uR}$ and $\mathbf{V}_{dR}$ are unitary matrices. The quark mass eigenstates $\mathbf{u}^m$ and $\mathbf{d}^m$ are then given in terms of the interaction eigenstates by the following transformations:

$$\begin{aligned}\mathbf{u}^m_L &= \mathbf{V}_{uL}\mathbf{u}_L & \mathbf{u}^m_R &= \mathbf{V}_{uR}\mathbf{u}_R,\\ \mathbf{d}^m_L &= \mathbf{V}_{dL}\mathbf{d}_L & \mathbf{d}^m_R &= \mathbf{V}_{dR}\mathbf{d}_R.\end{aligned}$$

The observable CKM matrix is then given by:

$$\mathbf{V}_{CKM} = \mathbf{V}^\dagger_{uL}\mathbf{V}_{dL}.$$

We can now re-write the interaction superpotential in terms of the quark mass eigenstates:

$$\begin{aligned}W &= \sum_{i=1}^{i=3} h^i_u\overline{\mathbf{u}}^m_R(\mathbf{V}^\dagger_{uR}\mathbf{Y}^i_u\mathbf{V}_{uL})\mathbf{u}^m_L + \sum_{i=1}^{i=3} h^i_d\overline{\mathbf{d}}^m_R(\mathbf{V}^\dagger_{dR}\mathbf{Y}^i_d\mathbf{V}_{dL})\mathbf{d}^m_L\\ &\equiv \sum_{i=1}^{i=3} h^i_u\overline{\mathbf{u}}^m_R\mathbf{W}^i_u\mathbf{u}^m_L + \sum_{i=1}^{i=3} h^i_d\overline{\mathbf{d}}^m_R\mathbf{W}^i_d\mathbf{d}^m_L \qquad (C.1)\end{aligned}$$

where $\mathbf{W}^i_q \equiv \mathbf{V}^\dagger_{qR}\mathbf{Y}^i_q\mathbf{V}_{qL}$.

In the MSSM and the Standard Model the Yukawa couplings are proportional to the mass matrices and so the $\mathbf{W}^i_q$ matrices become the identity matrix. This illustrates that the Higgs fields do not mediate tree-level FCNCs. In models with extended Higgs sectors however the Higgs fields will, in general, interact with the quarks to generate tree-level FCNCs with the interactions described by the matrices $\mathbf{W}_q^i$ in Eq.C.1. Fig.C.1 illustrates this tree-level contribution for $K^0 - \overline{K}^0$ mixing. Of course we must also write the Higgs fields in Eq.C.1 in terms of their mass eigenstates to find the physical interactions between the Higgs fields and quarks that generate the observable tree-level FCNCs. Since experimental data is in good agreement with the Standard Model predictions, the potentially large contributions arising from the tree-level interactions must be suppressed in order to have a model which is experimentally viable.

In the $\mathrm{E_6}$SSM and $\mathrm{ME_6}$SSm a $Z_2^H$ symmetry is applied to the first two generations of Higgs fields $h_{u\alpha}$, $h_{d\alpha}$ where $\alpha = 1, 2$. In the exact symmetry limit, $Z_2^H$ forbids these Higgs doublets from interacting with the quarks and leptons, and so the quark mass matrices are given by the product of the VEVs and Yukawa couplings of the $h_{u3}$ and $h_{d3}$ Higgs fields. The mass matrices are thus proportional to the Yukawa matrices and so there are no tree-level FCNCs that are mediated by neutral scalar Higgs fields.

# Bibliography


[1] J. N. Bahcall, arXiv:physics/0406040; B. Pontecorvo, Sov. Phys. JETP **7** (1958) 172 [Zh. Eksp. Teor. Fiz. **34** (1957) 247]; B. Pontecorvo, Sov. Phys. JETP **26** (1968) 984 [Zh. Eksp. Teor. Fiz. **53** (1967) 1717]; L. Wolfenstein, Phys. Rev. **D17** (1978) 2369; S. Mikheyev and A. Yu. Smirnov, Sov. J. Nucl. Phys. **42** (1985) 913; Y. Fukuda *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **81**, 1562 (1998); S. N. Ahmed *et al.* [SNO Collaboration], arXiv:nucl-ex/0309004; K. Eguchi *et al.* [KamLAND Collaboration], Phys. Rev. Lett. **90** (2003) 021802 [arXiv:hep-ex/0212021]; J. Maricic and J. G. Learned, Contemp. Phys. **46** (2005) 1; M. H. Ahn et al. [K2K Collaboration], Phys. Rev. D **74** (2006) 072003 [arXiv:hep-ex/0606032]; M. Apollonio et al. [CHOOZ Collaboration], Phys. Lett. B 466 (1999) 415 [arXiv:hep-ex/9907037].

[2] Rubin, V. & Ford, W. 1970, *ApJ*, **159**, 379; Briel, U.G. and Henry, J.P. 1997, arXiv:astro-ph/9711237; Bennett, C. *et al.* 2003, Astrophys. J. Suppl. **148**, 97; Adelman-McCarthy, J.K. *et al* 2005, astro-ph/0507711; Komatsu, E. *et al.* 2008, arXiv:0803.0547.

[3] Perlmutter, S. *et al.* 1999, ApJ, 517, 565; Riess, A.G. *et al.* 1999, AJ, 118, 2675; Riess, A.G. et al 2004, Astrophys. J. **607**, 665.

[4] E. Gildener, S. Weinberg, Phys. Rev. D **13** (1976) 3333; E. Gildener, Phys. Rev. D 14 (1976) 1667; L. Susskind, Phys. Rev. D 20, 2619 (1979); G. t Hooft, in *Recent developments in gauge theories*, Proceedings of the NATO Advanced Summer Institute, Cargese 1979, (Plenum, 1980).

[5] J. Ellis, S. Kelley, D.V. Nanopoulos, Phys. Lett. B **260** (1991) 131; P. Langacker, M. Luo, Phys. Rev. D **44** (1991) 817; U. Amaldi, W. de Boer, H. Furstenau, Phys. Lett. B **260** (1991) 447; F. Anselmo, L. Cifarelli, A. Peterman, A. Zichichi, Nuovo Cimento **104**A (1991) 1817, **105**A (1992) 581.

[6] S. Dimopoulos and S. Raby, Nucl. Phys. B **192**, 353 (1981); E. Witten, Nucl. Phys. B **188**, 513 (1981); M. Dine, W. Fischler and M. Srednicki, Nucl. Phys. B **189**, 575 (1981); S. Dimopoulos and H. Georgi, Nucl. Phys. B **193**, 150 (1981); N. Sakai, Z. Phys. C **11**, 153 (1981); R.K. Kaul and P. Majumdar, Nucl. Phys. B **199**, 36 (1982).

[7] P. Langacker, N. Polonsky, Phys. Rev. D **52** (1995) 3081; P.H. Chankowski, Z. Pluciennik, S. Pokorski, C.E. Vayonakis, Phys. Lett. B **358** (1995) 264; J. Bagger, K. Matchev, D. Pierce, Phys. Lett. B **348** (1995) 443; S. Dimopoulos, S. Raby and F. Wilczek, Phys. Rev. D **24** (1981) 1681.

[8] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32** (1974) 438.

[9] For example see Weinberg, *The quantum theory of fields. Vol. 2: Modern applications, Cambridge, UK: Univ. Pr. (1996) 489 p*

[10] P. Minkowski, Phys Lett 67B 421 (1977); M. Gell-Mann, P. Ramond and R. Slansky in Sanibel Talk, CALT-68-709, Feb 1979, and in *Supergravity* (North Holland, Amsterdam 1979); T. Yanagida in *Proc. of the Workshop on Unified Theory and Baryon Number of the Universe*, KEK, Japan, 1979; S.L.Glashow, Cargese Lectures (1979); R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44** (1980) 912; J. Schechter and J. W. Valle, Phys. Rev. D **25** (1982) 774

[11] M. Fukugita and T. Yanagida, Phys. Lett. B 174 (1986) 45; V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **155** (1985) 36; V. A. Rubakov and M. E. Shaposhnikov, Usp. Fiz. Nauk, **166** (1996) 493.

[12] For example see D. J. H. Chung, L. L. Everett, G. L. Kane, S. F. King, J. D. Lykken and L. T. Wang, Phys. Rept. **407** (2005) 1 [arXiv:hep-ph/0312378].

[13] For a recent review see S. P. Martin, arXiv:hep-ph/9709356.

[14] J. R. Ellis, J. F. Gunion, H. E. Haber, L. Roszkowski and F. Zwirner, Phys. Rev. D **39** (1989) 844; D. Suematsu and Y. Yamagishi, Int. J. Mod. Phys. A **10** (1995) 4521 [arXiv:hep-ph/9411239]; M. Cvetic and P. Langacker, Mod. Phys. Lett. A **11** (1996) 1247 [arXiv:hep-ph/9602424]; D. Suematsu, Phys. Rev. D **59** (1999) 055017 [arXiv:hep-ph/9808409]; P. Langacker, arXiv:0801.1345 [hep-ph].

[15] J. E. Kim and H. P. Nilles, Phys. Lett. B **138** (1984) 150; E. J. Chun, J. E. Kim and H. P. Nilles, Nucl. Phys. B **370** (1992) 105.

[16] P. Binetruy, S. Dawson, I. Hinchliffe, M. Sher, Nucl. Phys. B **273** (1986) 501; J. R. Ellis, K. Enqvist, D. V. Nanopoulos, F. Zwirner, Nucl. Phys. B **276** (1986) 14; L. E. Ibanez, J. Mas, Nucl. Phys. B **286** (1987) 107; J. F. Gunion, L. Roszkowski, H. E. Haber, Phys. Lett. B **189** (1987) 409; H. E. Haber, M. Sher, Phys. Rev. D **35** (1987) 2206; J. R. Ellis, D. V. Nanopoulos, S. T. Petcov, F. Zwirner, Nucl. Phys. B **283** (1987) 93; M. Drees, Phys. Rev. D **35** (1987) 2910; H. Baer, D. Dicus, M. Drees, X. Tata, Phys. Rev. D **36** (1987) 1363; J. F. Gunion, L. Roszkowski, H. E. Haber, Phys. Rev. D **38** (1988) 105.

[17] S. F. King, S. Moretti and R. Nevzorov, Phys. Rev. D **73** (2006) 035009 [arXiv:hep-ph/0510419]; S. F. King, S. Moretti and R. Nevzorov, Phys. Lett. B **634** (2006) 278 [arXiv:hep-ph/0511256]; P. Athron, S. F. King, D. J. . Miller, S. Moretti, R. Nevzorov and R. Nevzorov, arXiv:0901.1192 [hep-ph]; P. Athron, S. F. King, D. J. Miller, S. Moretti and R. Nevzorov, arXiv:0904.2169 [hep-ph].

[18] H. Georgi and C. Jarlskog, Phys. Lett. B **86** (1979) 297.

[19] For recent reviews see e.g.: S. F. King, arXiv:0712.1750 [physics.pop-ph]; R. N. Mohapatra and A. Y. Smirnov, Ann. Rev. Nucl. Part. Sci. **56** (2006) 569 [arXiv:hep-ph/0603118]; R. N. Mohapatra *et al.*, arXiv:hep-ph/0510213; S. F. King, Rept. Prog. Phys. **67** (2004) 107 [arXiv:hep-ph/0310204]; G. Altarelli and F. Feruglio, New J. Phys. **6** (2004) 106 [arXiv:hep-ph/0405048]; R. N. Mohapatra, "Understanding neutrino masses and mixings within the seesaw framework," arXiv:hep-ph/0306016.

[20] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B **458**, 79 (1999) [arXiv:hep-ph/9904297]; P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B **530**, 167 (2002) [arXiv:hep-ph/0202074].

[21] See for example: R. Barbieri, L. J. Hall, S. Raby and A. Romanino, Nucl. Phys. B **493**, 3 (1997) [arXiv:hep-ph/9610449]. M. C. Chen and K. T. Mahanthappa, Phys. Rev. D **62**, 113007 (2000) [arXiv:hep-ph/0005292]. G. Altarelli, F. Feruglio and I. Masina, JHEP **0011**, 040 (2000) [arXiv:hep-ph/0007254]. J. L. Chkareuli, C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B **626**, 307 (2002) [arXiv:hep-ph/0109156]. J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, Prog. Theor. Phys. **109**, 795 (2003) [Erratum-ibid. **114**, 287 (2005)] [arXiv:hep-ph/0302196]. G. L. Kane, S. F. King, I. N. R. Peddie and L. Velasco-Sevilla, JHEP **0508**, 083 (2005) [arXiv:hep-ph/0504038]. R. Dermisek and S. Raby, Phys. Lett. B **622**, 327 (2005) [arXiv:hep-ph/0507045]. R. Dermisek, M. Harada and S. Raby, Phys. Rev. D **74**, 035011 (2006) [arXiv:hep-ph/0606055].

[22] G. G. Ross and O. Vives, Phys. Rev. D **67** (2003) 095013 [arXiv:hep-ph/0211279]; G. G. Ross, L. Velasco-Sevilla and O. Vives, Nucl. Phys. B **692** (2004) 50 [arXiv:hep-ph/0401064]; S. Antusch, S. F. King and M. Malinsky, arXiv:0708.1282 [hep-ph].

[23] S.L. Glashow, Nucl. Phys. **20**, 579 (1961); S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967); A. Salam, *Elementary Particle Theory*, eds.: Svartholm, Almquist, and Wiksells, Stockholm, 1968; S. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. **D2**, 1285 (1970).

[24] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory, Reading, USA: Addison-Wesley (1995) 842 p*

[25] P.W. Higgs, Phys. Rev. Lett. **12**, 132 (1964); idem., Phys. Rev. **145**, 1156 (1966); F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964); G.S. Guralnik, C.R. Hagen, and T.W. Kibble, Phys. Rev. Lett. **13**, 585 (1964).

[26] C. Amsler *et al.* [Particle Data Group], Phys. Lett. B **667** (2008) 1.

[27] N. Cabibbo et al., Nucl. Phys. B158, 295 (1979); See, e.g., G. Altarelli and G. Isidori, Phys. Lett. **B337**, 141 (1994); J.A. Casas, J.R. Espinosa, and M. Quiros, Phys. Lett. **B342**, 171 (1995) idem., Phys. Lett. **B382**, 374 (1996); T. Hambye and K. Riesselmann, Phys. Rev. **D55**, 7255 (1997).

[28] P. Ramond, Phys. Rev. D **3**, 2415 (1971); A. Neveu and J.H. Schwarz, Nucl. Phys. B **31**, 86 (1971); J.L. Gervais and B. Sakita, Nucl. Phys. B **34**, 632 (1971); Yu. A. Golfand and E. P. Likhtman, JETP Lett. **13**, 323 (1971); J. Wess and B. Zumino, Nucl. Phys. B **70** (1974) 39; D.V. Volkov and V.P. Akulov, Phys. Lett. B **46**, 109 (1973).

[29] E. Witten, Nucl. Phys. B **188** (1981) 513; N. Sakai, Z. Phys. C **11** (1981) 153; S. Dimopoulos, H. Georgi, Nucl. Phys. B **193** (1981) 150; R. K. Kaul, P. Majumdar, Nucl. Phys. B **199** (1982) 36.

[30] P. Chankowski, J. Ellis and S. Pokorski, Phys. Lett. B **423** (1998) 327; R. Barbieri and A. Strummia, Phys. Lett. B **433** (1998) 63; P. Chankowski, J. Ellis, M. Olechowski and S. Pokorski, hep-ph/9808275; Kwok Lung Chan, Utpal Chattopadhyay, and Pran Nath, hep-ph/9710473; G.L. Kane and S.F. King, Naturalness implications of LEP results, Phys. Lett. B **451** (1999) 113 [hep-ph/9810374].

[31] For a complete review, see K. R. Dienes, Phys. Rept. **287** (1997) 447 [hep-th/9602045].

[32] G. C. Cho and K. Hagiwara, Phys. Lett. B **419** (1998) 199 [hep-ph/9709279].

[33] H. Georgi, and S. L. Glashow Phys. Rev. D **6** (1972) 429.

[34] H. Georgi, in *Particles and Fields* 1974, ed. C. Carlson (Amer. Inst. of Physics, New York 1975).

[35] L.E. Ibáñez and G.G. Ross, Phys. Lett. **B** 110, 215 (1982).

[36] P. Nath and P. F. Perez, arXiv:hep-ph/0601023.

[37] N. Sakai and T. Yanagida, Nucl. Phys. B **197**, 533 (1982); S. Weinberg, Phys. Rev. D **26**, 287 (1982).

[38] S. Dimopoulos, S. Raby and F. Wilczek, Phys. Lett. B **112** (1982) 133.

[39] R.D. Peccei and H.R. Quinn, Phys. Rev. D **16** (1977) 1791.

[40] P. Fayet, Nucl. Phys. B **78** (1974) 14;

[41] H. P. Nilles, M. Srednicki, D. Wyler, Phys. Lett. B **120** (1983) 346; J. M. Frere, D. R. T. Jones, S. Raby, Nucl. Phys. B **222** (1983) 11; J. P. Derendinger, C. A. Savoy, Nucl. Phys. B **237** (1984) 307; M. I. Vysotsky, K. A. Ter-Martirosian, Sov. Phys. JETP **63** (1986) 489; J. Ellis, J. F. Gunion, H. Haber, L. Roszkowski, F. Zwirner, Phys. Rev. D **39** (1989) 844; L. Durand, J. L. Lopez, Phys. Lett. B **217** (1989) 463; L. Drees, Int. J. Mod. Phys. A **4** (1989) 3635.

[42] Ya. B. Zeldovich, I. Yu. Kobzarev, L. B. Okun, Sov. Phys. JETP **40** (1975) 1.

[43] A. Vilenkin, Phys. Rept. **121** (1985) 263; A. Vilenkin, E. P. S. Shellard, Cosmic Strings and Other Topological Defects (Cambridge University Press, 1994).

[44] S. A. Abel, S. Sarkar, P. L. White, Nucl. Phys. B **454** (1995) 663.

[45] F. Abe *et al.* [CDF Collaboration], Phys. Rev. Lett. **79** (1997) 2192.

[46] P. Langacker, N. Polonsky and J. Wang, Phys. Rev. D **60** (1999) 115005 [arXiv:hep-ph/9905252].

[47] M. Cvetič, D. Demir, J. R. Espinosa, L. L. Everett, P. Langacker, Phys. Rev. D **56** (1997) 2861; J. Erler, Nucl. Phys. B **586** (2000) 73.

[48] D. A. Demir, G. L. Kane, T. T. Wang, hep-ph/0503290.

[49] P. Langacker, M. Plumacher, Phys. Rev. D **62** (2000) 013006; V. Barger, C. W. Chiang, P. Langacker, H. S. Lee, Phys. Lett. B **580** (2004) 186; V. Barger, C. W. Chiang, J. Jiang, P. Langacker, Phys. Lett. B **596** (2004) 229.

[50] E. Keith and E. Ma, Phys. Rev. D **56** (1997) 7155 [arXiv:hep-ph/9704441].

[51] R. Slansky, Phys. Rept. **79** (1981) 1.

[52] For a recent review see e.g. N. Polonsky, arXiv:hep-ph/9911329.

[53] M. Dine and A. E. Nelson, Phys. Rev. D **48** (1993) 1277 [arXiv:hep-ph/9303230]. K. Agashe and M. Graesser, Nucl. Phys. B **507** (1997) 3 [arXiv:hep-ph/9704206].

[54] M. Bastero-Gil, C. Hugonie, S.F. King, D.P. Roy and S. VemPati, Phys. Lett. B **489** (2000) 359 [hep-ph/0006198]; E. Keith and E. Ma, Phys. Rev. D **54** (1996) 3587 [hep-ph/9603353]; E. Ma, Phys. Rev. Lett. **60** (1988) 1363.

[55] J. Rich, D. Lloyd Owen and M. Spiro, Phys. Rept. **151** (1987) 239; P. F. Smith, Contemp. Phys. **29** (1988) 159. T. K. Hemmick et al. , Phys. Rev. D **41** (1990) 2074.

[56] S. Wolfram, Phys. Lett. B **82** (1979) 65; C. B. Dover, T. K. Gaisser, G. Steigman, Phys. Rev. Lett. **42** (1979) 1117. T. K. Hemmick *et al.*, Phys. Rev. D **41** (1990) 2074.

[57] S. F. King, S. Moretti and R. Nevzorov, arXiv:hep-ph/0701064.
(1991) 131; U. Amaldi, W. de Boer, H.

[58] For example see P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. **B258** (1985) 46; E. Witten, Nucl. Phys. B **258**, 75 (1985); L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261** (651) 1985; Nucl. Phys. **B274** (1986) 285; L. Ibanez, H. P. Nilles, and F. Quevedo, Phys. Lett. **B187** (1987) 25; L. Ibanez, J. E. Kim, H. P. Nilles, and F. Quevedo, Phys. Lett. **B191** (1987) 282; M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," *On Mathematical Physics)*; L. J. Hall and Y. Nomura, Phys. Rev. D **64** (2001) 055003 [arXiv:hep-ph/0103125]; J. E. Kim, hep-th/0301177.

[59] G. F. Giudice, A. Masiero, Phys. Lett. B **206** (1988) 480; J. A. Casas, C. Muñoz, Phys. Lett. B **306** (1993) 288.

[60] W. Kilian and J. Reuter, Phys. Lett. B **642** (2006) 81 [arXiv:hep-ph/0606277].

[61] J. C. Pati and A. Salam, Phys. Rev. D **10** (1974) 275.

[62] D. Chang, R. N. Mohapatra and M. K. Parida, Phys. Rev. D **30** (1984) 1052.

[63] T. Han and S. Willenbrock, Phys. Lett. B **616** (2005) 215 [arXiv:hep-ph/0404182].

[64] J. D. Lykken and S. Willenbrock, Phys. Rev. D **49** (1994) 4902 [arXiv:hep-ph/9309258].

[65] J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. **127** (1962) 965.

[66] S. Wiesenfeldt, Mod. Phys. Lett. A **19** (2004) 2155 [arXiv:hep-ph/0407173].

[67] H. Murayama and A. Pierce, Phys. Rev. D **65** (2002) 055009 [arXiv:hep-ph/0108104].

[68] A. Nisati, S. Petrarca and G. Salvini, Mod. Phys. Lett. A **12** (1997) 2213 [arXiv:hep-ph/9707376].

[69] For a recent review see e.g. R. Barbier *et al.*, Phys. Rept. **420** (2005) 1 [arXiv:hep-ph/0406039].

[70] M. Dittmar, A. S. Nicollerat and A. Djouadi, Phys. Lett. B **583** (2004) 111 [arXiv:hep-ph/0307020].

[71] J. Kang and P. Langacker, Phys. Rev. D **71** (2005) 035014 [arXiv:hep-ph/0412190].

[72] P. Abreu *et al.* [DELPHI Collaboration], Phys. Lett. B **485** (2000) 45 [arXiv:hep-ex/0103025]; R. Barate *et al.* [ALEPH Collaboration], Eur. Phys. J. C **12** (2000) 183 [arXiv:hep-ex/9904011].

[73] K. S. Babu, C. F. Kolda and J. March-Russell, Phys. Rev. D **54** (1996) 4635 [arXiv:hep-ph/9603212].

[74] I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B **648** (2007) 201 [arXiv:hep-ph/0607045].

[75] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531; M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973) 652.

[76] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28** (1962) 870.

[77] D. J. H. Chung et al., Phys. Rept. **407** (2005) 1203, [hep-ph/0312378].

[78] S. L. Glashow and S. Weinberg, Phys. Rev. D **15**, 1958 (1977).

[79] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B **147** (1979) 277.

[80] R. G. Roberts, A. Romanino, G. G. Ross and L. Velasco-Sevilla, Nucl. Phys. B **615** (2001) 358 [arXiv:hep-ph/0104088]; G. G. Ross and L. Velasco-Sevilla, Nucl. Phys. B **653** (2003) 3 [arXiv:hep-ph/0208218]; G. Ross and M. Serna, Phys. Lett. B **664** (2008) 97 [arXiv:0704.1248 [hep-ph]].

[81] R. Gatto, G. Sartori and M. Tonin, Phys. Lett. B **28** (1968) 128.

[82] A. Bovier, M. Lling, and D. Wyler, J. Math. Phys. **22**:1543, 1981. C. Luhn, S. Nasri and P. Ramond, J. Math. Phys. **48** (2007) 073501 [arXiv:hep-th/0701188].

[83] I. de Medeiros Varzielas and G. G. Ross, Nucl. Phys. B **733** (2006) 31 [arXiv:hep-ph/0507176].

[84] T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby and M. Ratz, Nucl. Phys. B **768** (2007) 135 [arXiv:hep-ph/0611020]; P. Ko, T. Kobayashi, J. h. Park and S. Raby, Phys. Rev. D **76** (2007) 035005 [Erratum-ibid. D **76** (2007) 059901] [arXiv:0704.2807 [hep-ph]].

[85] E. Ma, New J. Phys. **6**, 104 (2004) [arXiv:hep-ph/0405152]; G. Altarelli and F. Feruglio, arXiv:hep-ph/0504165; [check if now in a journal] G. Altarelli and F. Feruglio, Nucl. Phys. B **741** (2006) 215 [arXiv:hep-ph/0512103]; G. Altarelli, F. Feruglio and C. Hagedorn, JHEP **0803** (2008) 052 [arXiv:0802.0090 [hep-ph]].

[86] I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B **644** (2007) 153 [arXiv:hep-ph/0512313]; S. F. King and M. Malinsky, Phys. Lett. B **645** (2007) 351 [arXiv:hep-ph/0610250]; G. Altarelli, F. Feruglio and C. Hagedorn, arXiv:0802.0090 [hep-ph].

[87] S. F. King, Phys. Lett. B **439** (1998) 350 [arXiv:hep-ph/9806440]; Nucl. Phys. B **562** (1999) 57 [arXiv:hep-ph/9904210]; Nucl. Phys. B **576** (2000) 85 [arXiv:hep-ph/9912492];JHEP **0209**, 011 (2002) [arXiv:hep-ph/0204360]. S. F. King, Phys. Lett. B **439** (1998) 350 [arXiv:hep-ph/9806440]; S. F. King, Nucl. Phys. B **576** (2000) 85 [arXiv:hep-ph/9912492]; S. F. King, JHEP **0209** (2002) 011 [arXiv:hep-ph/0204360]; S. Antusch and S. F. King, New J. Phys. **6** (2004) 110 [arXiv:hep-ph/0405272]; S. Antusch and S. F. King, Phys. Lett. B **591** (2004) 104 [arXiv:hep-ph/0403053]; S. F. King, JHEP **0508** (2005) 105 [arXiv:hep-ph/0506297];

[88] S. F. King, Nucl. Phys. B **786** (2007) 52 [arXiv:hep-ph/0610239].
tri-bi exp

[89] F. Plentinger and W. Rodejohann, Phys. Lett. B **625** (2005) 264 [arXiv:hep-ph/0507143].
G. Altarelli, F. Feruglio and I. Masina, Phys. Lett. B **472** (2000) 382 [arXiv:hep-ph/9907532]; A. Y. Smirnov, Phys. Rev. D **48** (1993) 3264 [arXiv:hep-ph/9304205].

[90] S. F. King and G. G. Ross, Phys. Lett. B **520**, 243 (2001) [arXiv:hep-ph/0108112]; S. F. King and G. G. Ross, Phys. Lett. B **574**, 239 (2003) [arXiv:hep-ph/0307190].

[91] J. P. Hall and S. F. King, arXiv:0905.2696 [hep-ph].

[92] S. Chang, G. D. Kribs, D. Tucker-Smith and N. Weiner, Phys. Rev. D **79** (2009) 043513 [arXiv:0807.2250 [hep-ph]].

[93] W. E. Caswell, Phys. Rev. Lett. **33** (1974) 244; D. R. T. Jones, Nucl. Phys. B **75** (1974) 531. M. E. Machacek and M. T. Vaughn, Nucl. Phys. B **222** (1983) 83.

[94] D. R. T. Jones, Nucl. Phys. B **87** (1975) 127; D. R. T. Jones and L. Mezincescu, Phys. Lett. B **136** (1984) 242; S. P. Martin and M. T. Vaughn, Phys. Rev. D **50** (1994) 2282 [Erratum-ibid. D **78** (2008) 039903] [arXiv:hep-ph/9311340].

[95] N. Escudero, C. Munoz and A. M. Teixeira, Phys. Rev. D **73** (2006) 055015 [arXiv:hep-ph/0512046].