

TOPOLOGICAL SUPERRIGIDITY

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ABSTRACT. The geometric superrigidity theorem states broadly, that for Γ in a wide class of co-compact lattices, a non-constant, equivariant, harmonic map, with target in a suitable non-positively curved manifold is a totally geodesic embedding up to renormalisation. In this paper we propose a topological analogue: for a wide class of manifolds every codimension-1, π_1 -injective map is a finite cover of an embedding up to homotopy.

1. INTRODUCTION

We use superscripts to denote real dimension e.g. a manifold denoted N^k has dimension k . Once the dimension is established we omit the superscript so that the manifold N^k is also denoted N .

The presence of a group action on a space often allows one to promote an existing structure to another of a more strictly controlled kind. Examples of this phenomenon in topology include Papakyriokopoulos's sphere theorem, Waldhausen's torus theorem and the geometric superrigidity theorem of Mok et al. In this paper we propose the following topological result which has features in common with them all.

Theorem 1. *Let N^n be a closed, orientable, aspherical topological manifold with n even and $n \geq 6$, satisfying the following properties:*

- (1) *every cellular action of $\pi_1(N)$ on a $CAT(0)$ cubical complex has a global fixed point,*
- (2) *the projective class group $\widetilde{K}_0(C)$ vanishes for any finite extension C of $\pi_1(N)$.*

Given any closed, orientable, aspherical topological manifold M^{n+1} and any π_1 -injective, continuous function $j : N \rightarrow M$, there is a diagram as follows which commutes up to homotopy

$$\begin{array}{ccc}
 N_0 & \xleftarrow{h'} & N \\
 \downarrow p & & \downarrow j \\
 N' & & \\
 \downarrow i & & \\
 M' & \xleftarrow{h} & M
 \end{array}$$

where:

- (1) M', N_0 are closed, orientable, aspherical, topological manifolds and h, h' are homotopy equivalences,

- (2) The map $p : N_0 \rightarrow N'$ is a finite degree cover, and
- (3) $i : N' \rightarrow M'$ is a two sided topological embedding.

If $\pi_1(M)$ and $\pi_1(N)$ both satisfy the Borel conjecture, or, alternatively, if $\pi_1(N)$ satisfies the Borel conjecture and is square root closed in $\pi_1(M)$, then the homotopy equivalences h, h' in Theorem 1 are homotopic to homeomorphisms and the commutative diagram simplifies as below. In this case the theorem asserts that every π_1 -injective codimension-1 map $j : N \rightarrow M$ factors, up to homotopy, as a finite cover of a 2-sided embedding.

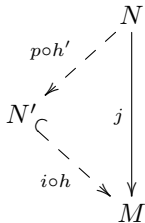


FIGURE 1. When $\pi_1(M)$ and $\pi_1(N)$ satisfy the Borel conjecture the map j factors up to homotopy as a finite cover, $p \circ h'$, of an embedding $i \circ h$.

WE SHOULD EXPAND THIS SECTION A BIT TO EXPLAIN WHY THE STATEMENT ABOUT HYPERBOLIC AND CAT(0) GROUPS IS TRUE (FINITE EXTENSIONS OF FBOX GROUPS ARE FBOX, FINITE EXTENSIONS OF HYPERBOLIC GROUPS ARE HYPERBOLIC, FINITE EXTENSIONS SIT INSIDE WREATH PRODUCTS. The hypotheses on N are rigidity constraints which, for example, are satisfied when N is a closed, orientable, Riemannian manifold such that the universal cover $\tilde{N}$ is quaternionic hyperbolic space or the Cayley hyperbolic plane. More generally, by [16], condition 1 is satisfied whenever $\pi_1(N)$ satisfies Kazhdan's property (T) and by the results of Bartels and Lück [1] condition 2 is satisfied when $\pi_1(N)$ is a word hyperbolic group or a CAT(0) group. (We say that a group is CAT(0) if it admits a co-compact isometric proper action on a finite dimensional CAT(0) space.) Hence we obtain as a corollary:

The topological superrigidity theorem. *Let M^{n+1} and N^n be closed, aspherical, orientable topological manifolds with n even and $n \geq 6$ such that $\pi_1(N)$ is either a word hyperbolic or a CAT(0) group and satisfies Kazhdan's property (T). If M satisfies the Borel conjecture then every π_1 -injective map $j : N \rightarrow M$ is, a finite cover of an embedding (up to homotopy).*

We regard this as a topological counterpart to the celebrated geometric super-rigidity theorem:

The Geometric Superrigidity Theorem (Ngaiming Mok, Yum-Tong Siu, Sai-kee Yeung, [14]). *Let $\tilde{N}$ be a globally symmetric irreducible Riemann manifold of non-compact type. Assume that either $\tilde{N}$ is of rank at least 2, or is the quaternionic hyperbolic space of dimension at least 8 or the hyperbolic Cayley plane. Let H be a cocompact discrete subgroup of the group of isometries of $\tilde{N}$ acting freely. Let $\tilde{M}$ be a Riemann manifold. Let f be a non-constant H -equivariant harmonic map from*

$\tilde{N}$ to $\tilde{M}$. When the rank of $\tilde{N}$ is at least 2, the Riemann sectional curvature is assumed to be non-positive. When the rank of $\tilde{N}$ is one, the complexified sectional curvature is assumed to be nonpositive. Then the covariant derivative of the differential of f is identically zero. As a consequence, f is a totally geodesic isometric embedding (up to a renormalization constant).

To compare the two results note that the map $j : N \rightarrow M$ induces a $\pi_1(N)$ -equivariant map between the universal covers. The constraint on the curvature of the target $\tilde{M}$ in the geometric super rigidity theorem is dropped (together with the requirement that the target is smooth) in favour of a statement that $\tilde{M}$ is contractible and $\pi_1(M)$ satisfies the Borel conjecture. Instead of a harmonic map, we start with a continuous function which is codimension-1. The conclusion that the map is totally geodesic up to renormalisation is replaced by the conclusion that it is a finite cover of an embedding up to homotopy. Deforming the map j to a harmonic map in the same homotopy class allows us to combine the conclusions of the two results to see that the embedding provided by Theorem 1 is homotopic to a totally geodesic surface in M .

The reader may find it helpful in visualising the results to consider the analogous statements in lower dimensions. First we consider the case of π_1 -injective loops on a 2-torus T . The fundamental group $\pi_1(T)$ is free abelian of rank 2 and so any homotopy class of curves is represented by a pair of integers (m, n) . It is an elementary fact that a non-trivial curve is homotopic to a simple closed curve if and only if the pair (m, n) is coprime, and it follows that in general a curve representing a pair (a, b) is homotopic to a degree d cover of the embedded curve representing the pair $(a/d, b/d)$ where $d = \gcd(a, b)$. It follows that every closed π_1 -injective curve is homotopic to a finite cover of an embedded loop. This is a direct analogue of the topological super rigidity theorem.

Now, in contrast, consider the case of loops on a *hyperbolic* surface. Every orientable hyperbolic surface Σ admits π_1 -injective maps $\gamma : S^1 \rightarrow \Sigma$ which do not factorise up to homotopy as a finite cover of an embedding. Recall that the free homotopy class of a closed loop contains a unique geodesic, and that this minimises the self intersection number for curves in that class; on the other hand, intersection numbers are multiplicative on powers, and it follows that any self intersecting closed geodesic which has intersection number 1 with some simple closed curve provides a loop which does not finitely cover an embedded loop so the analogous statement fails in this case, though we note that such surfaces do carry many splitting curves, and these can often be obtained from immersed totally geodesic curves by the somewhat different methods of cut and paste.

We note in passing that in dimension 3 the situation worsens: Kahn and Markovic have recently proved, [9], that every hyperbolic 3-manifold contains an immersed π_1 -injective surface, however, there are examples which do not contain any embedded π_1 -injective surfaces. For example, in [17], Reid constructs a non-Haken manifold M^3 which admits a finite cover homeomorphic to a hyperbolic surface bundle over S^1 . The fibre in the finite cover yields an immersed surface in M but there are no embedded surfaces in M which it could cover so there is in general no available analogue of topological super rigidity in low dimensions. This leaves open the question of what happens in the case $n = 4$.

The strategy in the proof of Theorem 1 is to first establish the existence of a splitting of the fundamental group $\pi_1(M)$ over a suitable subgroup using methods from geometric group theory. This result applies in the context of Poincaré duality groups and is of independent interest.

Theorem 2. *Let G be an orientable PD^n group and H be an orientable PD^{n-1} subgroup of G . If every action of H on a $CAT(0)$ cube complex has a global fixed point (in particular, by [16], if H satisfies Kazhdan's property (T)), then G splits over a subgroup C containing H as a finite index subgroup.*

In a companion paper we give a group theoretic application of Theorem 2, obtaining a canonical decomposition of Poincaré duality groups over codimension-1 property (T) Poincaré duality subgroups. This may be viewed as an analogue of the JSJ decomposition studied by Kropholler, Dunwoody, Scott and others.

While Theorem 1 is stated in the topological category the surgery technology applied in this paper also works in the smooth category and if the map j is smooth then the map i it constructs is smooth also. We will use this fact to deduce the following obstruction result from Theorem 1.

Corollary 3. *Let M^{4d+1} be a closed, orientable, aspherical, smooth manifold such that $d \geq 2$ and the first Betti number $b_1(M)$ is zero. Let N^{4d} be a closed, orientable, aspherical, smooth manifold with at least one non-zero Pontryagin number such that $\pi_1(N)$ is either word hyperbolic or a $CAT(0)$ group and satisfies Kazhdan's property (T). Then there are no π_1 -injective continuous maps $f: N \rightarrow M$.*

The paper is organised as follows. In order to extract the topological splitting of Theorem 1 from the algebraic splitting furnished by Theorem 2 we will apply Borel rigidity together with tools from surgery theory and Poincaré duality. In Section 2 we set up and prove Theorem 2 while in section 3 we outline the topological ingredients required for the proof of Theorem 1. In Section 4 we present the proof of Theorem 1, and in section 5 we give the proof of Corollary 3.

2. SPLITTING POINCARÉ DUALITY GROUPS

The proof of the main Theorem relies on ideas from geometric group theory, surgery theory, homological algebra and rigidity theory. The first ingredient is a strengthening of a result from geometric group theory which we proved in [10]. To set up notation we will define Poincaré duality groups. For further details, we refer the reader to [12].

Definition 4. *Let R be a commutative ring with unity. A group G is said to be a duality group of dimension n over R if there is an RG module D_G such that one has natural isomorphisms*

$$H^k(G; A) \cong H_{n-k}(G; D_G \otimes_R L)$$

for all $k \in \mathbb{Z}$ and for all RG modules L .

The module D_G is called the dualising module and if in addition, D_G is one-dimensional then we say that G is a Poincaré duality group of dimension n over R , or a $PD^n(R)$ group, for short. When G is the fundamental group of a closed aspherical manifold M , the dualising module over $\mathbb{Z}$ is a trivial G module if and only if M is orientable. We extend this to the definition of an orientable Poincaré duality group over $\mathbb{Z}$.

Suppose G is a $PD^n(R)$ group. For any R -module L , consider the RG -module $L \otimes_R RG$. By Section 9.2 of [2], we have

$$H^k(G; L \otimes_R RG) = 0, \quad k \neq n \text{ and } H^n(G; L \otimes_R RG) = D_G \otimes_R L.$$

In particular, when $L = R$, we have $H^k(G; RG) = 0$, $k \neq n$ and $H^n(G; RG) \cong D_G$, and in fact this is equivalent to the PD^n condition, see [3, Section VIII, 10.1]. Hence the cohomological dimension of G over R , written $\text{cd}_R G$, is precisely n . Moreover, setting $R = \mathbb{Z}$ and $L = \mathbb{F}_2$, we see that every $PD^n(\mathbb{Z})$ group is a $PD^n(\mathbb{F}_2)$ group. We will follow the convention of writing PD^n for $PD^n(\mathbb{Z})$.

The geometric and algebraic end invariants which play a crucial role in this paper may be defined in terms of the cohomology groups of a group with exotic coefficients. In order to define them we will need the following modules.

Let $\mathcal{P}G$ denote the collection of all subsets of G . Then, $\mathcal{P}G$ is an $\mathbb{F}_2$ -vector space with respect to the operation of symmetric difference. One checks that $\mathcal{P}G$ is also a G module. Moreover, $\mathcal{P}G \cong \text{Coind}_1^G \mathbb{F}_2$. On the other hand

$$\mathcal{F}_H(G) = \{A \subseteq G : A \subseteq HF \text{ for some finite set } F\}$$

is the $\mathbb{F}_2 G$ -module $\text{Ind}_H^G \mathcal{P}H$. Similarly the power set $\mathcal{P}(H \backslash G)$ of $H \backslash G$ and the collection of finite subsets of $H \backslash G$, written $\mathcal{F}(H \backslash G)$ are $\mathbb{F}_2[G]$ modules. In fact, $\mathcal{P}(H \backslash G) \cong \text{Coind}_H^G \mathbb{F}_2$ and $\mathcal{F}(H \backslash G) \cong \text{Ind}_H^G \mathbb{F}_2$.

We recall the following definitions:

Definition 5. For a subgroup $H < G$ the algebraic end invariant $\tilde{e}(G, H)$ is defined to be $\dim_{\mathbb{F}_2} (\mathcal{F}_H(G) \backslash \mathcal{P}G)^G$, and for an infinite index subgroup we have

$$\tilde{e}(G, H) = 1 + \dim_{\mathbb{F}_2} H^1(G, \mathcal{F}_H(G))$$

Similarly the geometric end invariant is defined as

$$e(G, H) = \dim_{\mathbb{F}_2} (\mathcal{F}(H \backslash G) \backslash \mathcal{P}(H \backslash G))^G.$$

Convention A group G is said to split over a subgroup H if G has one of the following descriptions:

Case I (Free product with amalgamation) $G = G_1 *_H G_2$ with $G_1 \neq H \neq G_2$,

Case II (HNN extension) $G = J *_H$.

Theorem 2. Let G be an orientable PD^n group and H be an orientable PD^{n-1} subgroup of G . If every action of H on a $CAT(0)$ cube complex has a global fixed point (in particular, by [16], if H satisfies Kazhdan's property (T)), then G splits over a Poincaré duality subgroup C containing H as a finite index subgroup.

Proof. Let G and H be as in the statement of the theorem. As PD^k groups are also $PD^k(\mathbb{F}_2)$ groups, we may work over $\mathbb{F}_2$. Since the dualizing module $H^n(G, \mathbb{F}_2 G) \cong \mathbb{F}_2$ is trivial the duality isomorphism gives $H^k(G; \mathcal{F}_H(G)) \cong H_{n-k}(G; \mathcal{F}_H(G))$ for all $k \in \mathbb{Z}$ so we have

$$H^1(G, \mathcal{F}_H(G)) \cong H_{n-1}(G, \mathcal{F}_H(G)) \cong H_{n-1}(G, \text{Ind}_H^G(\mathcal{P}H)) \cong H_{n-1}(H, \mathcal{P}H),$$

where the last isomorphism is given by Shapiro's Lemma.

Since H is an orientable PD^{n-1} group, $H_{n-1}(H, \mathcal{P}H) \cong H^0(H, \mathcal{P}H) \cong \mathbb{F}_2$, hence, $\tilde{e}(G, H) = 2$. We now invoke Lemma 2.5 of [12] to get a subgroup H' of at most index 2 in H such that $e(G, H') = \tilde{e}(G, H) = 2$.

Applying Sageev's construction [18, Theorem 2.3] we obtain a CAT(0) cube complex X such that G acts essentially on X and H' is the stabiliser of an oriented codimension-1 hyperplane J . As H' has finite index in the group H the action of H' on the CAT(0) cube complex J has a global fixed point. One now extracts from the fixed point of the action, a proper H' almost invariant subset B of G such that $H'BH' = B$ [18, Lemma 2.5].

The singularity obstruction $S_B(G, H')$, introduced in [15], is defined as the collection $\{g \in G : gB \cap B \neq \emptyset, gB^c \cap B \neq \emptyset, gB \cap B^c \neq \emptyset \text{ and } gB^c \cap B^c \neq \emptyset\}$, where $B^c = G \setminus B$. We now apply [11, Lemma 4.17] as follows to deduce that the subgroup $K_g = H' \cap gH'g^{-1}$ satisfies $\tilde{e}(G, K_g) \geq 2$.

- When $g \in (B^*)^{-1} \cap B^*$ then $gB \cap B$ is a K_g -almost invariant subset.
- When $g \in (B^*)^{-1} \cap B$ then $gB \cap B^*$ is a K_g -almost invariant subset.
- When $g \in B^{-1} \cap B^*$ then $gB^* \cap B$ is a K_g -almost invariant subset.
- When $g \in B^{-1} \cap B^*$ then $gB^* \cap B^*$ is a K_g -almost invariant subset.

We will use this to show that the elements of $S_B(G, H')$ all lie in the commensurator of H' , allowing us to apply the generalised Stallings' theorem from [15].

Claim For any $PD^n(\mathbb{F}_2)$ group Γ with subgroup Γ' , if $\tilde{e}(\Gamma, \Gamma') \geq 2$ then $\text{cd}_{\mathbb{F}_2} \Gamma' = n - 1$.

Let Γ, Γ' be as in Claim. If $\tilde{e}(\Gamma, \Gamma') \geq 2$, then Γ' has infinite index in Γ and so by Strebel's theorem $\text{cd}_{\mathbb{F}_2} \Gamma' \leq n - 1$. We will show that in fact we have equality. Suppose not and $\text{cd}_{\mathbb{F}_2} \Gamma' \leq n - 2$.

As before $\tilde{e}(\Gamma, \Gamma') \geq 2$ implies that $H^1(\Gamma, \text{Ind}_{\Gamma'}^{\Gamma} \mathcal{P}\Gamma')$ is non-zero. However, by duality for Γ and Shapiro's Lemma, we have $H^1(\Gamma, \text{Ind}_{\Gamma'}^{\Gamma} \mathcal{P}\Gamma') \cong H_{n-1}(\Gamma', \mathcal{P}\Gamma')$.

As $\text{cd}_{\mathbb{F}_2} \Gamma' \leq n - 2$, there is a projective resolution P of $\mathbb{F}_2$ by $\mathbb{F}_2\Gamma'$ -modules of length $n - 2$. By definition, $H_{n-1}(\Gamma', \mathcal{P}\Gamma')$ is the $(n - 1)$ -th homology of the complex $P \otimes_{\mathbb{F}_2\Gamma'} \mathcal{P}\Gamma'$. Clearly, the latter vanishes, contradicting the fact that $H^1(\Gamma, \text{Ind}_{\Gamma'}^{\Gamma} \mathcal{P}\Gamma')$ is non-zero. This proves the Claim.

Returning to the proof, we deduce from the Claim that if $g \in S_B(G, H')$ then K_g has the same cohomological dimension as H' and as H'^g . It follows, again by Strebel's theorem that K_g has finite index in both. Hence, g lies in the commensurator $\text{Comm}_G(H')$ of H' as required. Therefore by Theorem B of [15], G splits over a subgroup C commensurable with H' and hence with H . More precisely, G is either a non-trivial amalgamated free product $G \cong A *_C B$ or an HNN extension $A *_C$ such that C and H are commensurable.

Now consider the action of G on the Bass-Serre tree for the splitting. Since H is commensurable with an edge stabiliser, and G acts with no edge inversions, the subgroup H fixes a vertex v so up to conjugation (and switching the roles of A, B if necessary in the amalgamated free product case) we may assume that H lies in the vertex stabiliser A . Now H is commensurable with an edge stabiliser C^g for some g , but, *a priori*, having already taken a specific conjugate of A to ensure that A contains H , it is not clear that H is commensurable with C . Suppose that C^g fixes an edge e so that $H \cap C^g$ fixes both e and v . It then fixes the first edge e' on the geodesic from v to e . So the stabiliser of this edge is commensurable with H . Up to conjugation within A this stabiliser is C , so we may express G as an amalgamated free product $G \cong A *_C B$ or as an HNN extension $A *_C$ with $H < A$ and so that H and C are commensurable.

By [4, Theorem V.8.2] we obtain a PD^n pair (A, Ω) , where Ω is the set of cosets of C in A . There are two types of PD^n pairs, the I -bundle type where Ω has two elements and the general type where Ω is infinite.

As remarked in [12, Section 2.1], if (A, Ω) is of general type, then the subgroup C is self-commensurating in A . If (A, Ω) is the I -bundle type and A acts trivially on Ω then $A = C$. So C is self-commensurating in A in both these cases. As commensurable subgroups have the same commensurators and $H < A$ we conclude in both these cases that H is a finite index subgroup of C as required.

It remains to consider the situation when (A, Ω) is of twisted I -bundle type i.e. A acts non-trivially on Ω . Here it is no longer evident that C is self-commensurating in A and the proof that H is a subgroup of C is slightly different. Recall that Ω is a 2-element set and A acts nontrivially on Ω so that the stabiliser of any point in Ω is isomorphic to C . This implies that C has index 2 in A and by [3, Proposition VIII.10.2], A is a PD^{n-1} group. We argue that in this case A is non-orientable, using the following observation.

Observation If (Γ, Ω) is an orientable PD^n pair of I -bundle type in which Γ is an orientable PD^{n-1} group then Γ acts trivially on Ω .

The proof given here of the observation above is due to Jonathan Cornick and Peter Kropholler, and we are grateful to them for allowing us to include it. Consider the long exact sequence of cohomology for the pair with coefficients in the integral group ring $\mathbb{Z}\Gamma$. Cohomology groups with these coefficients inherit an action of Γ . We obtain the following short exact sequence

$$0 \rightarrow H^{n-1}(\Gamma, \mathbb{Z}\Gamma) \rightarrow \text{Ext}_{\mathbb{Z}\Gamma}^{n-1}(\mathbb{Z}\Omega, \mathbb{Z}\Gamma) \rightarrow H^n((\Gamma, \Omega), \mathbb{Z}\Gamma) \rightarrow 0.$$

The orientability of the pair (Γ, Ω) implies $H^n((\Gamma, \Omega), \mathbb{Z}\Gamma)$ is isomorphic to the trivial module $\mathbb{Z}$. The orientability of Γ implies that $H^{n-1}(\Gamma, \mathbb{Z}\Gamma)$ is isomorphic to the trivial module as well. From this it follows that the action of Γ on the middle group $\text{Ext}_{\mathbb{Z}\Gamma}^{n-1}(\mathbb{Z}\Omega, \mathbb{Z}\Gamma)$ is either trivial or of infinite order.

Let H be the stabiliser of one of the points of Ω . Let K be the unique maximal orientable PD^{n-1} subgroup in H . The action of K on $\text{Ext}_{\mathbb{Z}\Gamma}^{n-1}(\mathbb{Z}\Omega, \mathbb{Z}\Gamma)$ is trivial and K has index at most 4 in Γ . From this it follows that Γ acts trivially on $\text{Ext}_{\mathbb{Z}\Gamma}^{n-1}(\mathbb{Z}\Omega, \mathbb{Z}\Gamma)$. In particular, Γ acts trivially on Ω . This proves the observation.

Returning to the proof of Theorem 2, we apply the observation with $\Gamma = A$. As the pair (A, Ω) is of twisted I -bundle type, A must be non-orientable, so it has a non-trivial dualising module D . Proposition VIII.10.2 of [3] says that any PD^{n-1} subgroup of A inherits its dualising module by restriction from the dualising module of A , so A has a unique maximal orientable PD^{n-1} subgroup, A^+ . Since A is not orientable A^+ is a proper subgroup and since C is orientable $C < A^+$. Finally since the index $[A : C] = 2$ we conclude that $A^+ = C$. Since H is also orientable, $H < C$ as required.

□

3. PREREQUISITES FROM SURGERY THEORY

In this section (and in particular in Lemmas 6 and 7 below), for a smooth (respectively, PL or topological) manifold, submanifold means a smooth (respectively, PL locally flat or topological locally flat) submanifold.

Let M, N be manifolds as in the statement of Theorem 1 and $j : N \rightarrow M$ a π_1 -injective map. Then $\pi_1(N), \pi_1(M)$ satisfy the hypotheses of Theorem 1 providing

a splitting of $\pi_1(M)$ as either an amalgamated free product $\pi_1(M) = A *_C B$ or as an HNN extension $A *_C$ where $\pi_1(N) < C$ is a subgroup of finite index. In [6] Capell provided tools to geometrize such a splitting. In particular he proved that if C is square root closed in $\pi_1(M)$ then there is a closed aspherical embedded submanifold $i : N' \hookrightarrow M$ such that i induces the inclusion of C in $\pi_1(M)$ and, by Van Kampen's theorem, induces the splitting of $\pi_1(M)$. Asphericity then yields the homotopy commutative Figure 1 of the introduction.

The idea of Cappell's proof is to use surgery techniques to build a cobordism from M to a manifold M' which does split in the required way. This provides the homotopy equivalences h, h' required by Theorem 1 and Cappell's square root closed hypothesis on C can be used to promote the homotopy equivalences to a homeomorphism $M \cong M'$, bypassing the need for $\pi_1(M)$ to satisfy the Borel conjecture. Since we prefer not to invoke either the Borel conjecture or the square root closed hypothesis in our statement of Theorem 1 it is necessary to unpack the proof of Cappell's splitting theorem.

As usual, surgery below the middle dimension requires no additional hypotheses, as is captured by the following lemma. We have amended the notation to fit our situation:

Lemma 6. [6, Lemma I.1] *Let Y be an $(n+1)$ dimensional Poincaré complex and X a codimension-1 sub-Poincaré complex with trivial normal bundle in Y and with $\pi_1(X) \rightarrow \pi_1(Y)$ injective. Let M be an $(n+1)$ dimensional closed manifold with $f : M \rightarrow Y$ a homotopy equivalence, $n \geq 5$. Assume we are given $m < (n-1)/2$; then f is homotopic to a map, which we continue to call f , which is transverse regular to X (whence $N' = f^{-1}(X)$ is a codimension-1 submanifold of M) and such that the restriction $f|_{N'} : N' \rightarrow X$ induces isomorphisms $\pi_i(N') \rightarrow \pi_i(X)$, $i \leq m$.*

Lemma 7 describes the obstruction to carrying out surgery in the middle dimension which by Lemma 8 is all that is then required. It is carried by the surgery kernels $K_i(N')$ defined in [6, Section I.4], and the projective class group $\widetilde{K}_0(C)$ appearing in Lemma 7 below. As recorded in the proof of [6, Lemma I.2], if $\pi_i(f^{-1}(X)) \rightarrow \pi_i(X)$ is an isomorphism then $K_i(N') = 0$.

Lemma 7 (Lemma II.1 from [6]). *Let $n = 2k$, M be a closed manifold and Y a Poincaré complex of dimension $(n+1)$. Assume we are given X a sub-Poincaré complex of dimension n of Y with trivial normal bundle and with $C = \pi_1(X) < \pi_1(Y)$. Assume further that $f : M \rightarrow Y$ is a homotopy equivalence transverse regular to X with, writing $N' = f^{-1}(X)$, N' connected and $\pi_1(N') \rightarrow \pi_1(X)$ an isomorphism and $K_i(N') = 0$, $i < k$. Then letting $K_k(N') = P \oplus Q$ denote the decomposition of $\mathbb{Z}C$ modules defined in [6] I.4,*

- (1) $K_k(N')$ is a stably free $\mathbb{Z}C$ module and $[P] = -[Q]$. Moreover, in Case I, $[P] \in \ker(\widetilde{K}_0(C) \rightarrow \widetilde{K}_0(G_1) \oplus \widetilde{K}_0(G_2))$, and in Case II, $[P] \in \ker(\widetilde{K}_0(C) \rightarrow \widetilde{K}_0(J))$.
- (2) Any finite set of elements of P (respectively, Q) can be represented by embedded disjoint framed spheres in N' for $k > 2$. The intersection pairing of $K_k(N')$ is trivial when restricted to P (resp; Q) and $Q \cong P^*$. Thus, $[P] = -[P^*]$.

- (3) If $[P] = 0$, f is homotopic to a map f' with $N' = f'^{-1}(X) \rightarrow X$ k -connected and so that, abusing notation by writing $K_k(N') = P \oplus Q$ for the decomposition of [6] I.4, P and Q are free $\mathbb{Z}C$ -modules.

Applying Lemma 6 up to dimension $n/2 - 1$ ensures the vanishing of the surgery kernels $K_i(N')$ while hypothesis 2 of Theorem 1 ensures the vanishing of $\widetilde{K}_0(C)$ so that $[P] = 0$ as required by condition 3 of Lemma 7.

Cappell's lemmas along with his Nilpotent Normal Cobordism Construction allow us to realise the splitting obtained in Theorem 2 by an embedded submanifold N' in a homotopic manifold M' which is aspherical up to the middle dimension. Given that we are working with Poincaré duality groups the following elementary lemma from homotopy theory guarantees that this is sufficient to ensure that N' is aspherical.

Lemma 8. *Let X^n be a closed, orientable manifold and let k be the largest integer less than or equal to $\frac{n}{2}$. Suppose the universal cover $\tilde{X}$ of X is k -connected and that $G = \pi_1(X)$ is a duality group of dimension n . Then $\pi_i(X) = \{0\}$ for all $i \geq 2$.*

Proof. Let X be a manifold as in the statement of the Lemma. Suppose that $\pi_i(X) \neq \{0\}$ for some $i \geq 2$ and choose the smallest such p . Evidently, $\tilde{X}$ is $(p-1)$ -connected. By the Hurewicz Theorem, $H_i(\tilde{X}) = 0$ for all $i = 1, \dots, (p-1)$ and $\pi_p(\tilde{X}) \cong H_p(\tilde{X})$. Observe that $k+1 \leq p \leq n$. Duality for orientable non-compact manifolds implies that $H_p(\tilde{X}) \cong H_c^{n-p}(\tilde{X})$. Here, $H_c^*(\tilde{X})$ refers to the cohomology with compact supports for $\tilde{X}$. We claim that $H_c^{n-p}(\tilde{X}) \cong H^{n-p}(G, \mathbb{Z}G)$.

Recall that $H_c^i(\tilde{X}) = \varinjlim_K H^i(\tilde{X}, \tilde{X} - K)$, as K varies over compact subsets of $\tilde{X}$ and hence is the i -th homology of the complex $\varinjlim_K \text{Hom}(C(\tilde{X}, \tilde{X} - K), \mathbb{Z})$. Note that every element of $\varinjlim_K \text{Hom}(C(\tilde{X}, \tilde{X} - K), \mathbb{Z})$ is represented by a module homomorphism from $C(\tilde{X}, \tilde{X} - K)$ to $\mathbb{Z}$ for some compact K .

Recalling that G acts freely and properly on the universal cover $\tilde{X}$ with quotient X , consider the partial augmented singular chain complex $C_p(\tilde{X}) \rightarrow C_{p-1}(\tilde{X}) \rightarrow \dots \rightarrow C_0(\tilde{X}) \rightarrow \mathbb{Z} \rightarrow 0$. Since $\tilde{X}$ is $(p-1)$ -connected this is a partial free G -resolution of $\mathbb{Z}$ and may be completed to a projective resolution P . As the action of G is proper, for $i < p$, the cohomology of the cocomplex $\text{Hom}_G(C(\tilde{X}), \mathbb{Z}G)$ is the cohomology of X with local coefficients in $\mathbb{Z}G$ and hence is the i -th cohomology of G with $\mathbb{Z}G$ coefficients.

We now show that for $i < p$, $\text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G) \cong \varinjlim_K \text{Hom}(C_i(\tilde{X}, \tilde{X} - K), \mathbb{Z})$, where K varies over compact subsets of $\tilde{X}$. Let $F \in \text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G)$. As a $\mathbb{Z}$ -module map, $F : C_i(\tilde{X}) \rightarrow \mathbb{Z}G$ is induced by the assignment $\sigma \mapsto \sum_g f_g(\sigma) \cdot g$, where the sum is finite and f_g is a module homomorphism from $C_i(\tilde{X})$ to $\mathbb{Z}$. A straightforward computation shows that F is a G -module homomorphism if and only if $f_g(\sigma) = f_1(g^{-1}\sigma)$ for all $g \in G$. Note that f_1 is zero outside of a compact subset K of $\tilde{X}$ and hence we have a map $\text{Hom}_G(C_i(\tilde{X}), \mathbb{Z}G) \rightarrow \varinjlim_K \text{Hom}(C_i(\tilde{X}, \tilde{X} - K), \mathbb{Z})$ given by $F \mapsto f_1$. One checks that this is an isomorphism with inverse coming from the prescription $f \mapsto (F : \sigma \mapsto \sum_g f(g^{-1}\sigma) \cdot g)$ and hence $H_c^{n-p}(\tilde{X}) \cong H^{n-p}(G, \mathbb{Z}G)$.

Finally, putting all the isomorphisms together, $\pi_p(X) \cong H^{n-p}(G, \mathbb{Z}G)$. As $0 \leq n - p \leq k - 1$ and G is a duality group of dimension n group, $H^i(G, \mathbb{Z}G) = 0$ except possibly when $i = n$, so in particular $H^{n-p}(G, \mathbb{Z}G) = 0$ which contradicts our hypothesis on $\pi_p(X)$. $\square$

4. PROOF OF THEOREM 1

Proof. Let M and N be as in the statement of Theorem 1. Set $G = \pi_1(M)$ and $H = \pi_1(N)$. Then G is an orientable PD^{n+1} group and H is an orientable PD^n subgroup of G and by Theorem 2, G splits as either a non-trivial amalgamated free product $G \cong A *_C B$ or as an HNN extension $A *_C$ such that C is a Poincaré duality subgroup containing H as a subgroup of finite index.

In the case when the splitting is a non-trivial amalgamated free product, both subgroups A, B have infinite index in G and Strebel's theorem [19, Theorem 3.2] shows that they both have cohomological dimension at most n . Since A, B both contain C all three have cohomological dimension equal to n . In the case of the HNN decomposition, again the subgroups A, C have cohomological dimension n . By the Eilenberg-Ganea theorem, [7], since $n > 2$, the cohomological dimensions of A, B and C are equal to the geometric dimensions of A, B and C respectively.

We now carry out the standard mapping cylinder construction of an Eilenberg MacLane space for G . When the group G splits as an amalgamated free product $G = A *_C B$ let X_A, X_B and X_C denote the n -dimensional $K(\pi, 1)$ complexes for A, B and C . Note that these complexes may be taken to be simplicial complexes. Since C is of type FP, we can apply the Mayer Vietoris sequence, to see that the groups A and B are also of type FP so we may choose the complexes to be of finite type.

Let $\phi_A : X_C \rightarrow X_A$ and $\phi_B : X_C \rightarrow X_B$ be maps inducing the inclusions $C \hookrightarrow A$ and $C \hookrightarrow B$. We construct the mapping cylinders M_A, M_B of these maps and making the standard identifications yields a $K(\pi, 1)$ for the group G which we will denote by Y . In the case when G splits as an HNN extension we carry out a similar process building a $K(\pi, 1)$ from X_A and $X_C \times [-1, 1]$ by glueing the two ends $X_C \times \{\pm 1\}$ to X_A using maps which induce the two inclusions of C into A defining the HNN extension. In either case we denote the $K(\pi, 1)$ by Y .

Note that Y is a Poincaré complex, and moreover, the Poincaré subcomplex $X_C \times \{0\} \subset Y$ cuts the neighbourhood $X_C \times (-1/2, 1/2) \subset Y$ into two components so that the normal bundle of X_C in Y is trivial (see Introduction of [6]). By construction, the composition $X_C \rightarrow X_C \times \{0\} \hookrightarrow Y$ is π_1 -injective. Applying Lemma 6, we may replace f (up to homotopy) by a map, which we continue to call f , which is transverse regular to $X_C \times \{0\}$ and such that the restriction of f to $N' = f^{-1}(X_C \times \{0\}) \rightarrow X_C$ induces isomorphisms $\pi_i(N') \rightarrow \pi_i(X_C)$, for all $i \leq (n/2 - 1)$. In particular, $\pi_i(N') = 0$ for all $i = 2, \dots, (n/2 - 1)$. Note that the transverse regularity of f ensures that the pre-image N' is a codimension-1 submanifold of M . Our aim is to further modify the map f so that it remains transverse regular and so that its restriction to N' induces an isomorphism on all homotopy groups, thus making N' aspherical.

Recall that n is even so that $n = 2k$ with $k \geq 3$. As recorded in the proof of Lemma I.2 of [6], the isomorphisms $\pi_i(N') \rightarrow \pi_i(X_C)$, for all $i \leq (k - 1)$ ensure that each of the modules $K_i(N')$ for $i \leq k - 1$ vanishes.

Since C is a torsion free finite extension of H , condition 2 of Theorem 1 ensures the vanishing of $\widetilde{K}_0(C)$. Invoking Lemma 7, we replace f by a map f' homotopic to f which is transverse regular and such that $\pi_1(N') \cong C$ where $N' = f'^{-1}(X)$ and the restriction of f' to N' is k -connected. In addition, both P and Q in $K_k(N') = P \oplus Q$ are free C -modules. This means that one can perform Cappell's Nilpotent Normal Cobordism Construction on M , a process we will now describe.

Note that cutting M along N' we get a decomposition $M = M_A \cup_{N'} M_B$ in Case I and $M = M_A / \{N'_A \cong N' \cong N'_B\}$ in Case II. The covering of M corresponding to the image of $\pi_1(X) \rightarrow \pi_1(Y) \cong \pi_1(M)$ is labelled $\hat{M}$ and the universal covering of M is labelled $\tilde{M}$. The group $C = \pi_1(N')$ acts by covering transformations on $\tilde{M}$, M_L and M_R (where $\tilde{M} = M_L \cup_{\tilde{N}'} M_R$) with quotients being $\hat{M}$, M_l and M_r . Hence, $\hat{M} = M_l \cup_{N'} M_r$. Let $I = [0, 1]$ and $I' = [-2, 2]$. We select a tubular neighbourhood $N' \times I'$ of N' in M such that when the lift of N' to $\hat{M}$ is extended to a lift of $N' \times I'$, we have $N' \times \{-2\} \subset M_l$ and $N' \times \{2\} \subset M_r$.

Let $\{x_i\}_{i=1}^s$ denote a $\pi_1(N')$ -free basis for P and let $\{y_i\}_{i=1}^s$ denote the dual basis for Q under the intersection pairing of $K_k(N')$. We choose disjoint framed spheres $\{X_i\}_{i=1}^s$ and $\{Y_i\}_{i=1}^s$ representing the bases for P and Q respectively. We can assume given the choice of bases, that for $i \neq j$, $X_i \cap Y_j = \emptyset$ and for any i , X_i intersects Y_i in a point. We kill the spheres X_i by k -surgery on N' to obtain a new manifold N'_P yielding a cobordism T_P of N' with N'_P ; similarly killing the spheres Y_i by k -surgery we get a cobordism T_Q of N' with a new manifold N'_Q . Clearly, $\pi_1(N'_P) \cong C \cong \pi_1(N'_Q)$ and by Lemma 8 both N'_P, N'_Q are aspherical and thus homotopic to X_C .

Now consider the trivial cobordism $M \times I$. We will extend it to a cobordism from M to M' by applying the cobordism extension lemma, gluing $T_P \times [-2, -1]$ to $M \times \{1\}$ along $N' \times [-2, -1] \times \{1\}$ and glueing $T_Q \times [1, 2]$ to $M \times \{1\}$ along $M \times [1, 2]$.

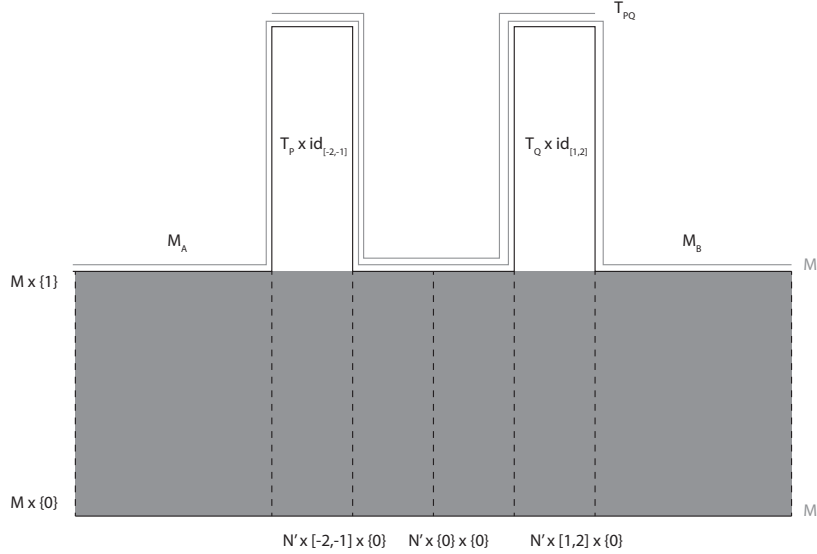


FIGURE 2. Extending the trivial cobordism on M via surgery along the spheres $\{X_i\}$ and $\{Y_j\}$.

The boundary of the resulting manifold T has two components, one being M and the other, a new manifold $M' = (M_A \cup_{N'} T_P) \cup_{N'_P} T_{PQ} \cup_{N'_Q} (T_Q \cup_{N'} M_B)$ where,

$$T_{PQ} = (N'_P \times [-2, -1]) \cup_{N'_P} T_P \cup_{N'} (N' \times [-1, 1]) \cup_{N'} (T_Q \cup_{N'_Q} N'_Q \times [1, 2]).$$

Observe that the manifold T_{PQ} sits naturally inside M' and gives a cobordism between N'_P and N'_Q . Moreover each of N'_P , N'_Q and T_{PQ} is homotopic to X_C . The resulting cobordism T is homotopy equivalent to Y (say, via $F : T \rightarrow Y$) and F restricts to a split homotopy equivalence h of M' with Y such that $T_{PQ} = h^{-1}(X_C)$.

Hence we obtain a manifold M' which is homotopic to M and split along $X_C \subset Y$ via the embedded submanifold we continue to call N' . Recall also that H is a finite index subgroup of C . Hence, there is a finite cover N_0 of N' with fundamental group H . As N_0 and N are aspherical manifolds with isomorphic fundamental groups there exists a homotopy equivalence h' between them. This completes the proof of Theorem 1. $\square$

5. AN APPLICATION: COROLLARY 3

Recall the following classical fact: if M^{4d+1} is a smooth manifold such that the first Betti number $b_1(M)$ of M is 0 and N^{4d} has non-zero signature then there are no immersions of N into M . This follows using Hirzebruch's signature theorem: since f is a codimension-1 immersion, $f^* : H^{4d}(M, \mathbb{Q}) \rightarrow H^{4d}(N, \mathbb{Q})$ maps the Hirzebruch L-class $L_d(M)$ onto $L_d(N)$. It follows from Poincaré duality that $H^{4d}(M, \mathbb{Q})$ is isomorphic to the torsion free part of $H_1(M)$. Therefore, the vanishing of $b_1(M)$ forces $L_d(N)$ to be zero. However Hirzebruch's signature theorem says that $L_d(N)$ is equal to the signature of N . This applies, for example, to show that there are no codimension-1 immersions of an orientable quaternionic hyperbolic or Cayley hyperbolic manifold N^{4d} into a smooth orientable aspherical manifold M^{4d+1} with $b_1(M) = 0$.

Appealing to the non-vanishing of Pontryagin numbers instead of the non-vanishing of the signature, we obtain the following generalisation as a corollary to Theorem 1, which obstructs the existence of π_1 -injective continuous maps rather than immersions. Note that while our hypotheses on N imply that it satisfies the Borel conjecture so that we can take the homotopy equivalence h' to be a homeomorphism, we do not assume in the statement that the target manifold satisfies the Borel conjecture, nor do we assume Cappell's square root closed hypothesis, since in the proof we are applying Theorem 1 in its most general form.

Corollary 3. *Let M^{4d+1} be a closed, orientable, aspherical, smooth manifold such that $d \geq 2$ and the first Betti number $b_1(M)$ is zero. Let N^{4d} be a closed, orientable, aspherical, smooth manifold with at least one non-zero Pontryagin number such that $\pi_1(N)$ is either word hyperbolic or a $CAT(0)$ group and satisfies Kazhdan's property (T) . Then there are no π_1 -injective continuous maps $f : N \rightarrow M$.*

Proof. Carrying out the Cappell surgery arguments in the smooth category, the proof of Theorem 1 shows that there is a 2-sided embedded smooth submanifold N' in a smooth manifold M' homotopy equivalent to M that realises the splitting. Furthermore, since $\pi_1(N)$ satisfies the Borel conjecture, the map $p \circ h' : N \rightarrow N'$ is a covering map. Now if N' is separating, write M_1 for one of the connected components of the manifold obtained from cutting M' along N' . Then M_1 is a

smooth orientable manifold with boundary N' and hence N' bounds orientably. This means that all Pontryagin numbers for N' must vanish. On the other hand we are assuming that N has at least one non-zero Pontryagin number (for the case when N is quaternionic or Cayley hyperbolic, see [13]). Since Pontryagin numbers vary multiplicatively with degree on covering maps the Pontryagin numbers of N' are also non-zero and so N' cannot bound orientably. This means that N' cannot be separating and so intersection with N' yields an element of infinite order in the first cohomology of M' . By duality and the universal coefficients theorem the first Betti number $b_1(M')$ and hence $b_1(M)$ is non-zero. $\square$

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