A HOMOLOGICAL CHARACTERIZATION OF TOPOLOGICAL AMENABILITY

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ABSTRACT. Generalizing Block and Weinberger's characterization of amenability we introduce the notion of uniformly finite homology for a group action on a compact space and use it to give a homological characterization of topological amenability for actions. By considering the case of the natural action of G on its Stone-Čech compactification we obtain a homological characterization of exactness of the group, answering a question of Nigel Higson.

There are two well known homological characterizations of amenability for a countable discrete group G. One, given by Johnson [8], states that a group is amenable if and only if a certain cohomology class in the first bounded cohomology $H^1_b(G,\ell^1_0(G)^{**})$ vanishes, where $\ell^1_0(G)$ is the augmentation ideal. By contrast Block and Weinberger [2] described amenability in terms of the non-vanishing of a homology class in the 0-dimensional uniformly finite homology of G, $H^{uf}_0(G,\mathbb{R})$. The relationship between these characterizations is explored in [3].

Amenable actions on a compact space were extensively studied by Anantharaman-Delaroche and Renault in [1] as a generalization of amenability which is sufficiently strong for applications and yet is exhibited by almost all known groups. A group is amenable if and only if the action on a point is amenable and it is exact if and only if it acts amenably on its Stone-Čech compactification, βG, [7, 6, 10]. It is natural to consider the question of whether or not the Johnson and Block-Weinberger characterizations of amenability can be generalized to this much broader context. In particular Higson asked for such a characterization of exactness.

In [4] we showed how to generalize Johnson's result in terms of bounded cohomology with coefficients in a specific module $N_0(G,X)^{**}$ associated to the action. In this paper we turn our attention to the Block-Weinberger theorem, studying a related module $W_0(G,X)$ (the *standard module of the action*), and define the *uniformly finite homology of the action*, $H^{uf}_*(G \curvearrowright X)$ as the group homology with coefficients in $W_0(G,X)^*$. The modules $N_0(G,X)^{**}$ and $W_0(G,X)^*$ should be thought of as analogues of the modules $(\ell^\infty(G)/\mathbb{R})^*$ and $\ell^\infty(G)$ respectively. The two characterizations are intimately related, and we consider this relationship in section 3.

In the case of Block and Weinberger's uniformly finite homology the vanishing of the 0-dimensional homology group is equivalent to vanishing of a fundamental

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class $[\sum_{g\in G} g] \in H_0^{uf}(G,\mathbb{R})$, however the homology group $H_0^{uf}(G \curvearrowright X)$ is rarely trivial even when the action is topologically non-amenable. Indeed if X is a compactification of G then the homology group is always non-zero, see Theorem 6 below. A similar phenomenon can be observed for controlled coarse homology [9], which is another generalization of uniformly finite homology: only the vanishing of the fundamental class has geometric applications. Here we show that topological amenability is detected by a fundamental class $[G \curvearrowright X] \in H_0^{uf}(G \curvearrowright X)$ for the action, and we obtain a homological characterization of topological amenability generalizing the Block-Weinberger theorem, Theorem 7, which may be summarized as follows:

Theorem. *Let G be a finitely generated group acting by homeomorphisms on a compact Hausdorff topological space X. The action of G on X is topologically amenable if and only if the fundamental class $[G \curvearrowright X]$ is non-zero in $H_0^{uf}(G \curvearrowright X)$.*

To recover the Block-Weinberger result, consider the case when X is a point so that $[G \curvearrowright X] = [\sum_{g\in G} g]$, and

$$H_0^{uf}(G,\mathbb{R}) \simeq H_0(G,\ell^\infty(G)) \simeq H_0^{uf}(G, W_0(G,\mathrm{pt})^*) = H_0^{uf}(G \curvearrowright X).$$

1. The uniformly finite homology of an action

Let G be a group generated by a *finite* set $S = S^{-1}$, acting by homeomorphisms on a compact Hausdorff space X. Let 1_X denote the constant function 1 on X. We denote by $W_{00}(G,X)$ the linear space

$$\left\{\xi : G \to C(X) \colon \operatorname{supp}\xi \text{ is finite and } \exists c \in \mathbb{R} \text{ s.t.} \sum_{g\in G} \xi_g = c1_X\right\},$$

and equip it with the norm

$$\|\xi\| = \left\| \sum_{g\in G} |\xi_g| \right\|_\infty .$$

Definition 1 ([4])**.** *Denote by $W_0(G,X)$ the closure of $W_{00}(G,X)$. We call $W_0(G,X)$ the* standard module of the action *of G on X.*

We define a functional $\sigma \in W_0(G,X)^*$ given by the continuous extension of the map

$$\sigma(\xi) = \sum_g \xi_g,$$

for $\xi \in W_{00}(G,X)$. We denote $N_0(G,X) = \ker\sigma$.

$W_0(G,X)$ is equipped with a natural action of G,

$$(g\cdot\xi)_h = g * \xi_{g^{-1}h},$$

for each $g,h \in G$, where $*$ denotes the translation action of G on $C(X)$: $g * f(x) = f(g^{-1}x)$ for $f \in C(X)$. It is easy to see that we have the following short exact sequence of G-modules:

$$0 \longrightarrow N_0(G,X) \xrightarrow{\ i\ } W_0(G,X) \xrightarrow{\ \sigma\ } \mathbb{R} \longrightarrow 0.$$

It is also worth pointing out that when X is a point we have $W_0(G,X) = \ell^1(G)$ and $N_0(G,X) = \ell^1_0(G)$. The above modules and decompositions were introduced in [4] for a compact X and in [5] in the case when $X = \beta G$, the Stone-Čech compactification of G.

Definition 2. *Let G be a group acting by homeomorphisms on a compact Hausdorff space. The action of G on X is said to be* amenable *if there exists a sequence of elements* $\xi^n \in W_{00}(G,X)$ *such that*

(1) $\xi^n_g \geq 0$ *in* $C(X)$ *for every* $n \in \mathbb{N}$ *and* $g \in G$*,*
(2) $\sigma(\xi^n) = 1$ *for every n,*
(3) $\sup_{s\in S} \|\xi^n - s\cdot\xi^n\| \to 0.$

Universality of the Stone-Čech compactification leads to the observation that a group acts amenably on some compact space if and only if it acts amenably on βG, which is equivalent to exactness. Amenable actions on compact spaces (lying between the point and βG) form a spectrum of generalized amenability properties interpolating between amenability and exactness.

Now consider the coboundary map

$$W_0(G,X) \xrightarrow{\ \delta\ } \left(\bigoplus_{s\in S} W_0(G,X)\right)_\infty,$$

where

$$(\delta\xi)_s = \xi - s\cdot\xi,$$

for $\xi \in W_0(G,X)$, where the (finite) direct sum is equipped with a supremum norm. The operator δ is clearly bounded. Since S is finite the dual of δ is

$$W_0(G,X)^* \xleftarrow{\ \delta^*\ } \left(\bigoplus_{s\in S} W_0(G,X)^*\right)_1,$$

where the direct sum is equipped with an ℓ^1-norm and the adjoint map is given by

$$\delta^*\psi = \sum_{s\in S} \psi_s - s^{-1}\cdot\psi_s.$$

The functional σ can be used to detect amenability of the action.

Theorem 3. *Let G be a finitely generated group acting on a compact space X by homeomorphisms. The following conditions are equivalent:*

(1) *the action of G on X is topologically amenable,*
(2) $\sigma \notin \overline{\mathrm{Image}(\delta^*)}^{\|\cdot\|}$,
(3) $\sigma \notin \mathrm{Image}(\delta^*)$,

Proof. (1) $\Longrightarrow$ (2). Assume first that the action is amenable. Take μ to be the weak-* limit of a convergent subnet of ξ_β as in the definition of amenable actions. Then

$$\mu(\sigma) = \lim_\beta \sigma(\xi_\beta) = 1,$$

and in particular σ is not in the kernel of μ. On the other hand

$$|\mu(\delta^*\psi)| = \lim_\beta |\delta^*\psi(\xi_\beta)| = \lim_\beta |\psi(\delta\xi_\beta)| \le \lim_\beta \left(\|\psi\| \sup_{s\in S} \|\xi_\beta - s\cdot\xi_\beta\| \right) = 0,$$

for every $\psi \in \bigoplus_{s\in S} W_0(G,X)^*$. Thus

$$\mathrm{Image}(\delta^*) \subseteq \ker\mu.$$

Since $\ker\mu$ is norm-closed, we conclude

$$\overline{\mathrm{Image}(\delta^*)}^{\|\cdot\|} \subseteq \ker\mu.$$

Thus $\sigma \notin \overline{\mathrm{Image}(\delta*)}^{\|\cdot\|}$ and (2) follows.

$(2) \implies (3)$ is obvious.

To prove $(3) \implies (1)$ we suppose there exists a constant $D > 0$ such that

$$(\dagger) \qquad \|\delta\xi\| \ge D|\sigma(\xi)|$$

for all ξ, and seek a contradiction. Consider a functional $\psi : \delta(W_0(G,X)) \to \mathbb{R}$, defined by

$$\psi(\delta\xi) = \sigma(\xi).$$

This is well defined, since $\delta : W_0(G,X) \to \bigoplus_{s\in S} W_0(G,X)$ is injective. By inequality (†), ψ is continuous on $\delta(W_0(G,X))$ and, by the Hahn-Banach theorem, we can extend it to a continuous functional Ψ on $\bigoplus_{s\in S} W_0(G,X)$. By definition, for $\xi \in W_0(G,X)$ we have

$$[\delta^*(\Psi)](\xi) = \Psi(\delta\xi) = \psi(\delta\xi) = \sigma(\xi),$$

hence σ is in the image of δ^*, contradicting (3).

It follows that there is no $D > 0$ such that inequality (†) holds for all $\xi \in W_0(G,X)$, hence there exists a sequence $\xi^n \in W_0(G,X)$ such that $\sigma(\xi^n) = 1$ for all n, and $\|\delta\xi^n\| \to 0$. Since $W_{00}(G,X)$ is dense in $W_0(G,X)$, we may assume without loss of generality that $\xi^n \in W_{00}(G,X)$, and applying the standard normalization argument we deduce that the action is amenable. □

We now introduce the notion of uniformly finite homology for a group action.

Definition 4. *Let G be a finitely generated group acting by homeomorphisms on a compact space X. We define the* uniformly finite homology of the action *by setting*

$$H_n^{uf}(G \curvearrowright X) = H_n(G, W_0(G,X)^*),$$

for every $n \ge 0$, where H_n denotes group homology.

As mentioned earlier, when X is a point we have $W_0(G,X)^* = \ell^\infty(G)$ and the uniformly finite homology of the action $H_n^{uf}(G \curvearrowright X)$ reduces to $H_n^{uf}(G,\mathbb{R})$, the uniformly finite homology of G with real coefficients [2].

2. Non-vanishing elements in $H_0^{uf}(G \curvearrowright X)$ and characterizing amenability

A certain homology class in the uniformly finite homology of the action will be of particular importance to us.

Definition 5. *Let G act by homeomorphisms on a compact space X. The* fundamental class of the action*, denoted $[G \curvearrowright X]$, is the homology class in $H_0^{uf}(G \curvearrowright X)$ represented by σ.*

Unlike the Block-Weinberger case, vanishing of this fundamental class does not in general imply the vanishing of $H_0^{uf}(G \curvearrowright X)$.

Theorem 6. *Let X be a compact G space, containing an open G-invariant subspace U on which G acts properly. Then $H_0^{uf}(G \curvearrowright X)$ is non-zero. In particular $H_0^{uf}(G \curvearrowright \overline{G})$ is non-zero for any compactification $\overline{G}$ of G.*

Proof. If G is finite, and the action of G on X is trivial, then $H_0^{uf}(G \curvearrowright X) = W_0(G,X)^*$ which is non-zero.

Otherwise we may assume that the action of G on U is non-trivial, replacing U with X if G is finite. Thus we may pick a point x_0 in U, and $x_1 = g_1x_0$ in Gx_0 with $x_0 \neq x_1$. Let $f \in C(X)$ be a positive function of norm 1, with $f(x_0) = 1$ and with the support K of f contained in $U \setminus \{x_1\}$. By construction $x_0 \notin g_1^{-1}K$.

Define $\xi \in W_0(G,X)$ by $\xi_e = f, \xi_{g_1} = -f$, and $\xi_g = 0$ for $g \neq e, g_1$. We note that ξ is in $W_0(G,X)$ as required, indeed it is in $N_0(G,X)$, since $\sum_{g\in G}\xi_g$ is identically zero. We now form the sequence

$$\xi^n = \sum_{k\in G}\phi_n(k)k\cdot\xi, \text{ where } \phi_n(k) = \max\left\{\frac{n-d(e,k)}{n}, 0\right\}.$$

If $\xi^n_g(x)$ is non-zero then x is in gK or $gg_1^{-1}K$. By properness of the action there are only finitely many $h \in G$ such that hK meets K. Let N be the number of such h. If $x \in hK$, then $x \in gK \cup gg_1^{-1}K$ for at most $2N$ values of g, hence for each $x \in X$, the set of g with $\xi^n_g(x) \neq 0$ has cardinality at most $2N$.

For $s \in S$ consider

$$\xi^n - s\cdot\xi^n = \sum_{g\in G}\phi_n(g)(g\cdot\xi - sg\cdot\xi) = \sum_{g\in G}(\phi_n(g) - \phi_n(s^{-1}g))g\cdot\xi.$$

Since $|(g\cdot\xi)_h(x)| \leq 1$ for all x and $|\phi_n(g) - \phi_n(s^{-1}g)| \leq \frac{1}{n}$ it follows that $|(\xi^n - s\cdot\xi^n)_h(x)| \leq \frac{1}{n}$ for all h, x. On the other hand, for a given x, $(\xi^n - s\cdot\xi^n)_h(x)$ is non-zero for at most $4N$ values of h, hence $\|\xi^n - s\cdot\xi^n\| \leq \frac{4N}{n}$. We thus have a sequence ξ^n in $W_0(G,X)$ with $\|\delta\xi^n\| \to 0$. It follows that if ζ is a weak-* limit point of ξ^n in $W_0(G,X)^{**}$ then $\delta^{**}\zeta = 0$, so ζ is a cocycle defining a class $[\zeta]$ in $H^0(G, W_0(G,X)^{**})$.

Let $\mathrm{ev}_{e,x_0} \in W_0(G,X)^*$ be the evaluation functional $\eta \mapsto \eta_e(x_0)$, and consider the homology class $[\mathrm{ev}_{e,x_0}] \in H_0^{uf}(G \curvearrowright X)$. We have

$$\mathrm{ev}_{e,x_0}(\xi^n) = \xi^n_e(x_0) = \phi_n(e)(e\cdot\xi)_e(x_0) + \phi_n(g_1^{-1})(g_1^{-1}\cdot\xi)_e(x_0)$$

since the other terms in the sum vanish. The first term is $\phi_n(e)f(x_0) = 1$, while $(g_1^{-1}\cdot\xi)_e(x_0) = (g_1^{-1} * \xi_{g_1})(x_0) = 0$ since x_0 is not in $g_1^{-1}K$. Thus $\mathrm{ev}_{e,x_0}(\xi^n) = 1$ for all n. It follows that the pairing of $[\mathrm{ev}_{e,x_0}]$ with $[\zeta]$ is 1, hence $[\mathrm{ev}_{e,x_0}]$ is a non-trivial element of $H_0^{uf}(G \curvearrowright X)$. □

We remark that there is a surjection from $H_0^{uf}(G \curvearrowright X)$ onto $H_0(G, N_0(G,X)^*)$, and the non-trivial elements constructed in the proposition remain non-trivial after applying this map.

We are now in the position to prove the main theorem, which is stated here in a more general form. The *reduced homology* $\overline{H}_n^{uf}(G \curvearrowright X) = \overline{H}_n(G, W_0(G,X)^*)$ in the statement is defined, as in the context of L^2-(co)homology, by taking the closure of the images in the chain complex.

Theorem 7. *Let G be a finitely generated group acting by homeomorphisms on a compact space X. The following conditions are equivalent*

(1) *the action of G on X is topologically amenable,*
(2) $[G \curvearrowright X] \neq 0$ *in* $\overline{H}_0^{uf}(G \curvearrowright X)$,
(3) $[G \curvearrowright X] \neq 0$ *in* $H_0^{uf}(G \curvearrowright X)$,
(4) *the map* $(i^*)_* : H_0^{uf}(G \curvearrowright X) \to H_0(G, N_0(G,X)^*)$ *is not injective,*
(5) *the map* $(i^*)_* : H_1^{uf}(G \curvearrowright X) \to H_1(G, N_0(G,X)^*)$ *is surjective.*

Proof. The equivalence (1)⟺(2)⟺(3) follows from Theorem 3. Indeed, we have $H_0(G,M) = M_G$, where M_G is the coinvariant module, namely the quotient of M by the module generated by elements of the form $g\cdot m - m$. Since G is finitely generated it is enough to consider only sums of elements of the form $s \cdot m - m$, where s are the generators. Indeed, if $g = s_1 s_2 \dots s_n$ for $s_i \in S$, we can write

$$g\cdot m - m = \left(\sum_{i=1}^{n-1} s_i \cdot m_i - m_i\right) + s_n \cdot m - m,$$

where $m_i = (s_{i+1}\dots s_n)\cdot m$ for $i \leq n$. Hence $W_0(G,X)^*_G$ is exactly the quotient $W_0(G,X)^*$ by the image of δ^*.

Additionally, we have the following short exact sequence of modules:

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ \sigma^*\ } W_0(G,X)^* \xrightarrow{\ i^*\ } N_0(G,X)^* \longrightarrow 0.$$

which induces the long exact sequence in homology:

$$\dots \longrightarrow H_1^{uf}(G \curvearrowright X) \xrightarrow{(i^*)_*} H_1(G, N_0(G,X)^*) \longrightarrow$$

$$H_0(G,\mathbb{R}) \xrightarrow{(\sigma^*)_*} H_0^{uf}(G \curvearrowright X) \xrightarrow{(i^*)_*} H_0(G, N_0(G,X)^*) \longrightarrow 0.$$

Since the action on $\mathbb{R}$ is trivial, we have $H_0(G,\mathbb{R}) = \mathbb{R}$. Denote by $[1]$ the generator of that group. It is easy to check that

$$[G \curvearrowright X] = (\sigma^*)_*[1].$$

Thus $[G \curvearrowright X] \neq 0$ if and only if the map $(\sigma^*)_*$ is non-zero, or equivalently the kernel of $(i^*)_*$ is non-zero. Thus it follows that (1) is equivalent to (4).

Also by exactness of the sequence $[G \curvearrowright X] \neq 0$ if and only if $[1]$ is not in the image of the connecting map, or equivalently the connecting map is zero, and we obtain condition the equivalence of (1) and (5). □

3. The interaction between uniformly finite homology and bounded cohomology

We conclude with some remarks concerning the interaction of the uniformly finite homology of an action and the bounded cohomology with coefficients introduced in [4]. These illuminate the special role played by the Johnson class in $H^1_b(G, N_0(G,X)^{**}$ and the fundamental class in $H^{uf}_0(G \curvearrowright X)$ and extend the results in [3] which considered the special case of the action of G on a point.

In [4] we showed that topological amenability of the action is encoded by triviality of an element $[J]$ in $H^1_b(G, N_0(G,X)^{**})$, which we call the Johnson class for the action. This class is the image of the class $[1] \in H^0_b(G, \mathbb{R})$ under the connecting map arising from the short exact sequence of coefficients

$$0 \to N_0(G,X)^{**} \to W_0(G,X)^{**} \to \mathbb{R} \to 0$$

which is dual to the short exact sequence appearing in the proof of Theorem 7.

By applying the forgetful functor from bounded to ordinary cohomology, we obtain a pairing of $H^1_b(G, N_0(G,X)^{**})$ with $H_1(G, N_0(G,X)^*)$, and clearly if the Johnson class $[J]$ is trivial then its pairing with any $[c] \in H_1(G, N_0^*)$ is zero.

Now suppose that every $[c] \in H_1(G, N_0(G,X)^*)$ pairs trivially with the Johnson class. Since the Johnson class $[J]$ is obtained by applying the connecting map to the generator $[1]$ of $H^0_b(G, \mathbb{R}) = \mathbb{R}$, pairing $[J]$ with $[c] \in H_1(G, N_0(G,X)^*)$ is the same as pairing $[1]$ with the image of $[c]$ under the connecting map in homology. As this pairing (between $H^0(G, \mathbb{R}) = H^0_b(G, \mathbb{R})$ and $H_0(G, \mathbb{R})$) is faithful, it follows that the image of $[c]$ under the connecting map is trivial for all $[c]$, so the connecting map is zero, which we have already noted is equivalent to amenability of the action. Thus in the case when the group is non-amenable, the non-triviality of the Johnson element must be detected by the pairing.

On the other hand, we can run a similar argument in the opposite direction: if pairing $[G \curvearrowright X]$ with every element $[\phi] \in H^0_b(G, W_0(G,X)^{**})$ we get zero, then since $[G \curvearrowright X] = (\sigma^*)_*[1]$, we have that the pairing of $(\sigma^{**})_*[\phi] \in H^0_b(G, \mathbb{R})$ with $[1] \in H_0(G, \mathbb{R})$ is trivial, whence $(\sigma^{**})_*[\phi] = 0$ (again by faithfulness of the pairing). Thus, by exactness, the connecting map on cohomology is injective and the Johnson class is non-trivial. So when the action is amenable, (and hence the Johnson class is trivial), non-triviality of $[G \curvearrowright X]$ must be detected by the pairing.

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