Autonomous Underwater Vehicle Minimum-Time Navigation in a Current Field

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***Abstract*—The problem of navigation in a spatially variable current is reviewed, and for a certain class of mathematically-describable functions, solved for minimum time in closed form.**

Nomenclature

$\dot{\theta}$	rate of change of heading (angular velocity)	
$\dot{x}$	x-directed component of vehicle velocity	
$\dot{y}$	y-directed component of vehicle velocity	
F	n-dimensional constraint vector	
λ	n Lagrange multipliers	
θ	vehicle course or heading (control input)	$\theta \in U$
U	m-dimensional control input, decision vector	
$\vec{\imath}$	unit vector	
$\vec{\jmath}$	unit vector	
$\vec{c}$	velocity of current	
X	n-dimensional state vector	
$g(x,y)$	x-directed component of current	
$h(x,y)$	y-directed component of current	
J	a scalar functional	
L	performance index	
m	number of control inputs	
n	number of components of state vector	
r	representative length	
V	vehicle velocity relative to the water	
x	Cartesian component (abscissa)	$x \in X$
y	Cartesian component (ordinate)	$y \in X$

I. Introduction

To optimise navigation with current in two-dimensional space is known as Zermelo's problem, from [5]. The ability to compensate for current is a vital tool for Autonomous Underwater Vehicles (AUVs). Waypoints, docks and other quasi-stationary targets need to be reached by the AUV in minimum time, if its range is to be maximised under conditions of constant velocity. The solution to this problem is available under certain conditions in closed form, which allows the validation of algorithms designed to govern the navigation of these vehicles. One alternate means to specify the problem that will not be investigated here is that the energy consumed by the AUV be minimised over the trajectory from A to B. Another unexplored option in the same current field is to control the velocity in order to ensure a straight-line course. This last option is not recommended for practical applications because under some circumstances the current may exceed the maximum velocity of the vehicle.

Zermelo's problem is solved optimally here by means of the Pontryagin maximum principle [2] with the methodology of Bryson and Ho [1, §2.7]. Extensions to this class of problem were published by Zlobec [6]. The paper is subdivided into an abstraction, an analysis, and the case of linear current distribution is solved.

Smith *et al.* [3] have recently investigated the use of the three-dimensional current predictions of the JPL OurOcean portal and found that an unscented Kalman filter (UKF) algorithm seems reliable to predict vehicle paths over a 2km range. The 2km range is here taken as a representative measure; the AUV is tasked with minimum time rendez-vous with the waypoint.

II. Abstraction

The functional that is minimised here is just

$$J = \int_{t_A}^{t_B} L(X(t), U(t))dt \tag{1}$$

which must be a stationary point in the two hyperplanes, X and U, of the Hamiltonian, $\varkappa$:

$$\delta J = \frac{\partial \varkappa}{\partial U}\delta U + \frac{\partial \varkappa}{\partial X}\delta X \tag{2}$$

The vector X represents the state (position) vector, while U represents the control input vector. The function $L(X,U) \equiv 1$, which implies

$$J = t_B - t_A \tag{3}$$

and allows the statement of a minimum-time navigation problem, subject to constraints

$$\dot{X} \equiv \frac{dX}{dt} = F(X,U) \tag{4}$$

where the overdot notation is employed to indicate the time derivative, and optimality conditions

$$0 = F_u^T \lambda \tag{5}$$

obtain. Subscripts, in general, denote differentiation. Here, the Hamiltonian

$$\varkappa = L + F_u^T \lambda \tag{6}$$

is comprised of the performance index, L, and the Lagrange multipliers, λ, compounded by the constraint functions, F, transposed. This abstraction maps onto the navigation problem in what follows with $[x, y] \in X$ and $[\theta] \in U$. That is, the state vector is identical to the position of the vehicle, while the heading angle is the only control variable.

III. Analysis

The current field is of known magnitude and direction,

$$\vec{c} = g(x,y)\vec{\imath} + h(x,y)\vec{\jmath} \tag{7}$$

in the Cartesian plane. Vehicle velocity relative to the water is V, and constant in magnitude, and as a result the equations of motion can be written

$$\dot{x} = V\cos\theta + g(x,y) \tag{8}$$
$$\dot{y} = V\sin\theta + h(x,y) \tag{9}$$

where overdot notation is written for the time derivative and θ represents the vehicle course with respect to an earth-fixed orthogonal coordinate frame. The Hamiltonian of the system is

$$\varkappa = \lambda_x\left(V\cos\theta + g\right) + \lambda_y\left(V\sin\theta + h\right) + 1 \tag{10}$$

so

$$\dot{\lambda_x} = -\frac{\partial\varkappa}{\partial x} = -\lambda_x\frac{g}{x} - \lambda_y\frac{h}{x} \tag{11}$$
$$\dot{\lambda_y} = -\frac{\partial\varkappa}{\partial y} = -\lambda_x\frac{g}{y} - \lambda_y\frac{h}{y} \tag{12}$$
$$0 = \frac{\partial\varkappa}{\partial\theta} = V\left(-\lambda_x\sin\theta + \lambda_y\cos\theta\right) \tag{13}$$

where Eq. 13 implies that

$$\tan\theta = \frac{\lambda_y}{\lambda_x} \tag{14}$$

The Hamiltonian is time invariant. Moreover, it equates to zero because an extremal (minimum-time) solution is desired. The system of equations 11 and 12 is solved[1] for λ_x and λ_y:

$$\lambda_x\left(V\cos\theta + g\right) + \lambda_y\left(V\sin\theta + h\right) = -1 \tag{15}$$
$$\lambda_x\left(-V\sin\theta\right) + \lambda_y\left(V\cos\theta\right) = 0 \tag{16}$$

to obtain

$$\lambda_x = \frac{-\cos\theta}{V + g\cos\theta + h\sin\theta} \tag{17}$$
$$\lambda_y = \frac{-\sin\theta}{V + g\cos\theta + h\sin\theta} \tag{18}$$

and an equation for the rate of change of heading angle is the result:

$$\dot{\theta} = \sin^2\theta\frac{\partial h}{\partial x} + \sin\theta\cos\theta\left(\frac{\partial u}{\partial x} - \frac{\partial h}{\partial y}\right) - \cos^2\theta\frac{\partial g}{\partial y} \tag{19}$$

The three rate equations, Eqs. 8, 9 and 19, will determine the minimum-time paths through a terminal point B when the initial coordinates A and course, θ_A, are set. What follows will place the destination at the origin $(0,0)$ of the coordinate axes, and the initial point A somewhere in the domain.

[1]see Appendix

IV. Linear Current Distribution

The case of a linear current distribution, $g = -V/r y$, $h = 0$ is addressed here. This current is irrotational and meant to model a shear flow in y. The terminal heading angle θ_B is assumed to be known and collinear with the terminal velocity.

$$\cos\theta = \frac{\cos\theta_B}{1 + y/r\cos\theta_B} \tag{20}$$
$$\cos\theta + \frac{y}{r}\cos\theta_B\cos\theta = \cos\theta_B \tag{21}$$
$$\frac{y}{r}\cos\theta_B\cos\theta = \cos\theta_B - \cos\theta \tag{22}$$
$$\frac{y}{r} = \frac{1}{\cos\theta} - \frac{1}{\cos\theta_B} \tag{23}$$
$$\frac{y}{r} = \sec\theta - \sec\theta_B \tag{24}$$

The rate equation for θ is solved next:

$$\dot{\theta} = \sin^2\theta\frac{\partial h}{\partial x} - \cos^2\theta\frac{\partial g}{\partial y} + \tag{25}$$
$$\sin\theta\cos\theta\left(\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y}\right) \tag{26}$$
$$\dot{\theta} = -\cos^2\theta\frac{\partial g}{\partial y} \tag{27}$$
$$\frac{d\theta}{dt} = \cos^2\theta\frac{V}{r} \tag{28}$$
$$\frac{dt}{d\theta} = \frac{r}{V}\sec^2\theta \tag{29}$$
$$\int_A^B \frac{V}{r}dt = \int_{\theta_A}^{\theta_B}\sec^2\theta d\theta \tag{30}$$
$$\frac{V}{r}\left(t_B - t_A\right) = \tan\theta - \tan\theta_B \tag{31}$$

Equation 31, which encodes the functional J from Eq. 3, is also known as the performance index.

The rate equation for x is solved last:

$$\frac{dx}{dt} = V\cos\theta + g \tag{32}$$
$$= V\cos\theta - V\frac{y}{r} \tag{33}$$
$$\frac{dx}{d\theta}\frac{d\theta}{dt} = V\cos\theta - V\sec\theta + V\sec\theta_B \tag{34}$$
$$\frac{dx}{d\theta} = \left[V\cos\theta - V\sec\theta + V\sec\theta_B\right]\frac{dt}{d\theta} \tag{35}$$
$$\frac{dx}{d\theta} = \left[V\cos\theta - V\sec\theta + V\sec\theta_B\right]\frac{r}{V}\sec^2\theta \tag{36}$$
$$\frac{dx}{d\theta} = \frac{V\cos\theta - V\sec\theta + V\sec\theta_B}{V/r\cos^2\theta} \tag{37}$$
$$dx = \left[r\sec\theta - r\sec^3\theta + r\sec\theta_B\sec^2\theta\right]d\theta \tag{38}$$
$$\frac{x}{r} = \int_{\theta_B}^{\theta_A}\left[\sec\theta - \sec^3\theta + \sec\theta_B\sec^2\theta\right]d\theta \tag{39}$$
$$= -\frac{1}{2}\left[\sec\theta_B(\tan\theta_B - \tan\theta_A) - \log\frac{\tan\theta_B + \sec\theta_B}{\tan\theta_A + \sec\theta_A} - \tan\theta_A(\sec\theta_B - \sec\theta_A)\right] \tag{40}$$

V. Results

Equations 24 and 40 constitute an implicit system for θ_A and θ_B, and are solved by numerical methods. Suppose an AUV desires to travel from $(x_A/r, y_A/r) = (3.66, -1.86)$ to the origin, then

$$\frac{y_A}{r} = -1.86 = \sec\theta_A - \sec\theta_B \quad (41)$$

$$\frac{x_A}{r} = 3.66 = -\frac{1}{2}\left[\sec\theta_B(\tan\theta_B - \tan\theta_A) - \log\frac{\tan\theta_B + \sec\theta_B}{\tan\theta_A + \sec\theta_A} - \tan\theta_A(\sec\theta_B - \sec\theta_A)\right] \quad (42)$$

This system of equations is intractable because it is composed of the trigonometric tangent and secant functions, both of which are in places unbounded. Equations 41 and 42 are modified to equate to zero, and the map of the L_2-norm is derived over a Cartesian solution plane composed of θ_A and θ_B. The map is generated on every odd-numbered degree in order to avoid the infinities, and then scrutinised in regions which contain small values. The algorithm, although robust, can be said to be in want of refinement. The solution, which requires at most 32,400 iterations of two cosines, two tangents, one logarithm and one square root function, is found to be for this example,

$$\theta_A = 105^\circ \qquad \theta_B = 240^\circ \quad (43)$$

VI. Conclusions and Future Work

Although one is unlikely to meet a well-ordered linear shear current in nature, the present result can be employed off-line to validate algorithms for navigation in a current. The abstraction needs to be automated in order to introduce the ability to navigate in regions with arbitrary current distributions.

Acknowledgements

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Appendix

A. Isolation of λ_x, λ_y

The system of equations is solved as follows. Isolate from Eq. 16 λ_x

$$\lambda_x = \tfrac{-V\cos\theta}{-V\sin\theta}\lambda_y \quad (44)$$

$$= \tfrac{\cos\theta}{\sin\theta}\lambda_y \quad (45)$$

and substitute the result into Eq. 15:

$$-1 = \lambda_y\left(\frac{\cos\theta}{\sin\theta}\right)[V\cos\theta + g] + \lambda_y[V\sin\theta + h] \quad (46)$$

$$-\sin\theta = \lambda_y\cos\theta[V\cos\theta + g] + \lambda_y\left[V\sin^2\theta + h\sin\theta\right] \quad (47)$$

$$-\sin\theta = \lambda_y\left[V\cos^2\theta + g\cos\theta + V\sin^2\theta + h\sin\theta\right] \quad (48)$$

$$-\sin\theta = \lambda_y[V + g\cos\theta + h\sin\theta] \quad (49)$$

$$\lambda_y = \frac{-\sin\theta}{V + g\cos\theta + h\sin\theta} \quad (50)$$

Isolate from Eq. 16 λ_y

$$\lambda_y = \lambda_x\frac{\sin\theta}{\cos\theta} \quad (51)$$

and substitute the result into Eq. 15:

$$0 = \lambda_x[V\cos\theta + g] + \lambda_x\frac{\sin\theta}{\cos\theta}[V\sin\theta + h] + 1 \quad (52)$$

$$-1 = \lambda_x\left[\frac{V\cos\theta^2 + g\cos\theta + V\sin\theta^2 + h\sin\theta}{\cos\theta}\right] \quad (53)$$

$$-1 = \lambda_x\left[\frac{V + g\sin\theta + h\sin\theta}{\cos\theta}\right] \quad (54)$$

$$\lambda_x = \frac{-\cos\theta}{V + g\cos\theta + h\sin\theta} \quad (55)$$

B. Integration of Equation 39

The integral is split into its additive components. The observation was made by Stewart [4] on page 122 that

$$\frac{d}{dx}\tan x = \sec^2 x \quad (56)$$

On the inside back cover of Stewart [4] it is observed that Eq. 71 of his Table of Integrals has

$$\int\sec^3 u du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\log|\sec u + \tan u| + C \quad (57)$$

Finally it is known that

$$\int\sec x dx = \log|\sec x + \tan x| \quad (58)$$

The logarithmic term in Eq. 57, the integral of the cube of the secant, is subtracted from the last-mentioned integral, and this results in Eq. 40.

References

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[2] Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishschenko, *The mathematical theory of optimal processes*, Interscience Publishers, London, 1962. Authorized translation from the Russian, translator K. N. Trirogoff.

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[4] Stewart, J., *Calculus*, Brooks/Cole Publishing Company, Pacific Grove, CA, 1987.

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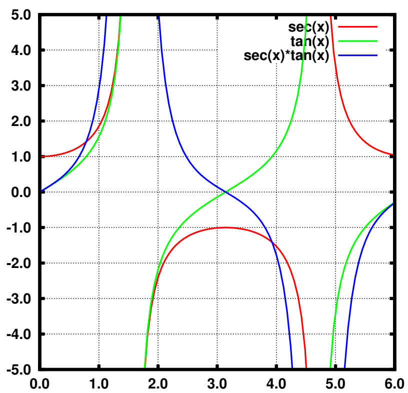
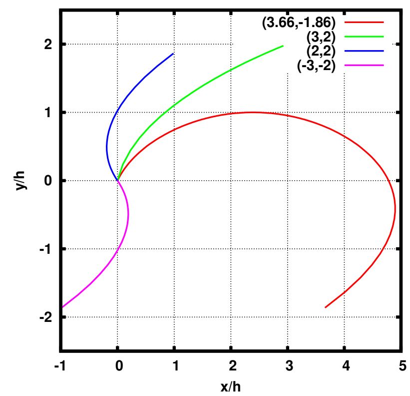
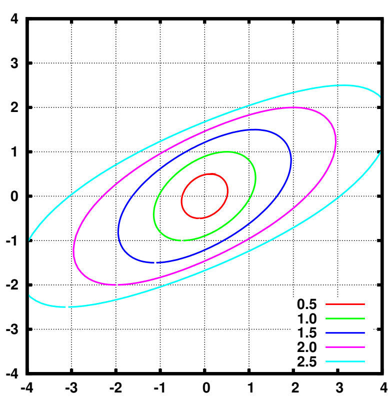
[6] Zlobec, S., *Partly convex programming and Zermelo's navigation problems*, Journal of Global Optimization, 7 (1995), pp. 229–259.



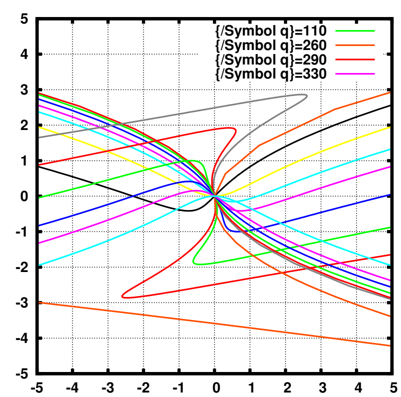
(a) Some useful trigonometric graphs.



(b) Various paths of minimum time navigation.



(c) Constant time contours.



(d) Various paths of constant-course angle.

Fig. 1. Figures for 'Autonomous Underwater Vehicle Minimum-Time Navigation in a Current Field'.