Journal of Mathematical Analysis and Applications **244,** 425–441 (2000)
doi:10.1006/jmaa.2000.6711, available online at http://www.idealibrary.com on IDEAL®

On Energy Inequalities

Vesna Perišić

Fachbereich Mathematik, Universität des Saarlandes, Postfach 15 11 50, D-66041 Saarbrücken, Germany

Submitted by George Hagedorn

Received January 4, 1999

We consider the Schrödinger operator $H = -\frac{\Delta}{2} + q$ in a bounded domain D in R^d, $d \geq 3$, with q in the Kato class $\mathscr{K}^d(D)$ and finite gauge $g(x) = E_x[\exp \int_0^\tau q(X_s)\, ds]$, where (X_t) is a Brownian motion and τ is the first exit time of the Brownian motion (X_t) from the domain D. Let K and G denote the Green operators in D for H and $\frac{\Delta}{2}$, respectively. We prove that there is a positive constant α such that

$$\frac{1}{\alpha}(Gf, f) \leq (Kf, f) \leq \alpha(Gf, f),$$

where f is a real, measurable function for which $(G|f|, |f|)$ is finite. As a direct consequence of this double inequality, we have that the potential Gf is in the Sobolev space $H_0^1(D)$ if and only if $Kf \in H_0^1(D)$.

1. NOTATION AND PRELIMINARIES

Let $d \geq 3$ and q be a measurable function on R^d. We say that q belongs to the Kato class $\mathscr{K}^d$ if

$$\lim_{\alpha \downarrow 0} \sup_x \int_{\{\|x-y\| \leq \alpha\}} \frac{|q|(y)}{\|x-y\|^{d-2}}\, dy = 0. \tag{1}$$

For a bounded domain D in R^d, $d \geq 3$, we say q belongs to $\mathscr{K}^d(D)$ if $q 1_D \in \mathscr{K}^d$, with 1_D the indicator function of the set D.

We consider the Schrödinger operator $H = -\frac{\Delta}{2} + q$, where the Laplacian operator $\Delta = \sum_{i=1}^d (\partial^2/\partial x_i^2)$ is understood in the weak or distributional sense and q is in the Kato class.

0022-247X/00 $35.00

Let $X = \{X_t,\ t \geq 0\}$ be the standard Brownian motion process in R^d, $d \geq 3$, with continuous paths; P_x and E_x are the probability and expectation for the paths starting at x. The first exit time of the Brownian motion X from the domain D is defined to be

$$\tau = \inf\{t > 0;\ X_t \notin D\}. \tag{2}$$

For the Feynman–Kac multiplicative functional we use the notation

$$e_q(t) = \exp \int_0^t q(X_s)\, ds, \qquad t > 0. \tag{3}$$

The class of real, Borel measurable functions on a set D is denoted by $\mathscr{B}(D)$; the subscript b denotes the subclass of bounded functions.

We will need a theorem called by Chung the Gauge Theorem. It was first proved by Chung and Rao in [2] for an arbitrary bounded domain D and bounded q.

THEOREM 1.1. *Let D be a bounded domain in R^d, $d \geq 3$, and $q \in \mathscr{K}^d(D)$. The following conditions on (D, q) are equivalent to each other*:

(i) $E_x e_q(\tau) \not\equiv \infty$ *in* D.

(ii) $\sup_x E_x e_q(\tau) < \infty$.

(iii) $E_x e_q(\tau)$ *is continuous on* $\overline{D}$.

(iv) $\sup_x E_x \int_0^\tau e_q(t)\, dt < \infty$.

(v) *There exists a continuous function u on D such that*

$$\left(\frac{\Delta}{2} + q\right) u(x) = 0, \qquad x \in D,$$

and $0 < \inf_{x \in D} u(x) < \sup_{x \in D} u(x) < \infty$.

Proof. Ref. [15].

REMARK 1.2. *If (D, q) satisfies one of the conditions* (i)–(v) *above*, *then the Dirichlet boundary value problem for Schrödinger equation*

$$\begin{aligned} \left(\frac{\Delta}{2} + q\right) u(x) &= 0, \qquad x \in D, \\ u(x) &= g(x), \qquad x \in \partial D, \quad g \in C(\partial D), \end{aligned} \tag{4}$$

has the unique solution which can be expressed by

$$u_g(x) = E_x\left[e_q(\tau) g(X_\tau)\right], \qquad x \in D. \tag{5}$$

Proof. Ref. [15].

Throughout this paper we assume that D is a bounded domain in R^d, $d \geq 3$, $q \in \mathscr{K}^d(D)$, and the gauge $g(x) := E_x e_q(\tau)$ is bounded in D.

For each $t \geq 0$, we introduce the operators S_t and T_t as follows: for $f \in \mathscr{B}_b(D)$,

$$S_t f(x) = E_x\left[e_q(t) f(X_t); t < \tau\right], \qquad x \in D, \tag{6}$$

and

$$T_t f(x) = E_x\left[f(X_t); t < \tau\right], \qquad x \in D. \tag{7}$$

The family of operators $\{S_t,\ t \geq 0\}$ forms a semigroup on $\mathscr{B}_b(D)$, called the Feynman–Kac semigroup, and for $q \equiv 0$ the Feynman–Kac semigroup reduces to the semigroup of the killed Brownian motion $\{T_t,\ t \geq 0\}$.

We define the potential operator K of the semigroup $\{S_t;\ t \geq 0\}$ by

$$K = \int_0^\infty S_t\, dt. \tag{8}$$

The potential operator K is also called the Green operator for Schrödinger operator H in D. Zhao has shown in [15] under some smoothness assumptions on the boundary ∂D that the operator K is an integral operator with nonnegative, symmetric kernel $K(\cdot, \cdot)$ which is extended continuous and finite off the diagonal. By Kf we denote the K-potential of f. Since the gauge g is bounded, for $f \in \mathscr{B}_b(D)$ the integral

$$E_x \int_0^\tau e_q(t) f(X_t)\, dt \tag{9}$$

is finite according to Theorem 1.1, and we have

$$\begin{aligned} Kf(x) &= \int_0^\infty S_t f(x)\, dt = E_x \int_0^\tau e_q(t) f(X_t)\, dt \\ &= \int_D K(x, y) f(y)\, dy, \qquad x \in D. \end{aligned}$$

In the case when $q \equiv 0$, the Feynman–Kac semigroup is reduced to the semigroup of killed Brownian motion

$$T_t f(x) = E_x\left[f(X_t), t < \tau\right], \qquad t > 0, \quad x \in D,$$

and the potential operator K is reduced to the potential operator G of this semigroup, also called the Green operator for $\frac{\Delta}{2}$ in D.

It is well known that the operator G is an integral operator with a symmetric kernel $G(\cdot, \cdot)$ extended continuous and finite off the diagonal. We denote by Gf the G-potential of f. For $f \in \mathscr{B}_b(D)$, the integral

$$E_x \int_0^\tau f(X_t)\, dt \tag{10}$$

is finite and so we have

$$Gf(x) = \int_0^\infty T_t f(x)\, dt = E_x \int_0^\tau f(X_t)\, dt = \int_D G(x, y) f(y)\, dy.$$

With the kernels $G(\cdot, \cdot)$ and $K(\cdot, \cdot)$, or corresponding potentials, we connect the notion of energy. More generally, we say that a signed measure μ, or its potential $G\mu$, has finite G-energy if

$$\|\mu\|_e^2 := \|G\mu\|_e^2 = (G\mu, \mu) = \int_D G\mu(x)\, d\mu(x) < \infty, \tag{11}$$

and call $\|\mu\|_e$ the G-energy of μ or of its potential $G\mu$. For an absolutely continuous measure μ with density f, we use the notation as above, namely Gf for G-potential and (Gf, f) for G-energy. It can be shown using integration by parts that

$$\|G\mu\|_e^2 = \int_D G\mu(x)\, d\mu(x) = \int_D |\nabla G\mu|^2(x)\, dx, \tag{12}$$

with ∇ denoting the gradient operator.

Analogously, we say that a signed measure μ, or its potential $K\mu$, has finite K-energy if

$$\int K|\mu|(x)\, d|\mu|(x) < \infty. \tag{13}$$

It is well known that the completion of the set of all G-potentials $G\mu$ of finite G-energy with respect to the energy norm $\| \ \|_e$ is the Sobolev space $H_0^1(D)$ (Rao [9] or Landkof [8]).

We will show that on the space of K-potentials $K\mu$ with finite K-energy, we can define a norm $\| \ \|_*$ by

$$\|K\mu\|_*^2 := \int K\mu(x)\, d\mu(x) \tag{14}$$

and the completion of this space of functions with respect to $\| \ \|_*$ is again the Sobolev space $H_0^1(D)$.

2. MAIN RESULT

It is well known that the operator G satisfies the *strong maximum principle*. We will show that the operator K (or its kernel $K(\cdot,\cdot)$) satisfies the *weak maximum principle* as defined in Landkof [8]; that is, if for a nonnegative, measurable function f on D

$$Kf \le 1 \qquad \text{on } \operatorname{supp} f \subseteq D,$$

then

$$Kf \le h \qquad \text{everywhere on } D,$$

where h is a certain positive constant ($h \ge 1$). In order to prove the weak maximum principle for the operator K, we will need a simple lemma.

LEMMA 2.1. *Let $D \subset R^d$, $d \ge 3$, be a bounded domain, $q \in \mathscr{K}^d(D)$, and the gauge g is bounded. Then for a stopping time T, the function*

$$h(x) = E_x\left[e_q(T); T < \tau\right] \tag{15}$$

is a bounded function on D.

Proof. Using the strong Markov property and Jensen's inequality, for $x \in D$ we have

$$\begin{aligned} E_x[e^{\int_0^\tau q(X_s)\,ds}; T < \tau] &= E_x\left[e^{\int_0^T q(X_s)\,ds} E_{X_T} e^{\int_0^\tau q(X_s)\,ds}; T < \tau\right] \\ &\ge E_x\left[e^{\int_0^T q(X_s)\,ds} e^{E_{X_T}\int_0^\tau q(X_s)\,ds}; T < \tau\right] \\ &\ge E_x\left[e^{\int_0^T q(X_s)\,ds} e^{-E_{X_T}\int_0^\tau |q|(X_s)\,ds}; T < \tau\right]. \end{aligned}$$

Since $E_x[\int_0^\tau |q|(X_s)\,ds]$ is a bounded function of x on D ([4, Theorem 3.2]), h is bounded. ∎

Now we are ready to prove the weak maximum principle.

PROPOSITION 2.2. *The potential operator K satisfies the weak maximum principle.*

Proof. Let $f\colon D \to R$ be a nonnegative function with $\operatorname{supp} f \subseteq D$ such that

$$Kf \le 1 \qquad \text{on } \{f > 0\}.$$

Let F_n be compact subsets of $\{f > 0\}$ such that $f_n = f 1_{F_n}$ increase to f a.e. Thus $f_n > 0$ on F_n and $f_n = 0$ outside; i.e., f_n has support in F_n. Then $Kf_n \uparrow Kf$ everywhere.

Let T_n be the stopping time defined by

$$T_n = \inf\{t > 0;\ X_t \in F_n,\ t < \tau\}.$$

For $x \in D$, simple calculations and the strong Markov property give

$$\begin{aligned} Kf_n(x) &= E_x\left[\int_{T_n}^{\tau} e_q(t) f_n(X_t)\, dt;\ T_n < \tau\right] \\ &\le E_x\left[e_q(T_n) Kf(X_{T_n});\ T_n < \tau\right] \\ &\le E_x\left[e_q(T_n);\ T_n < \tau\right] \le M. \end{aligned}$$

The last inequality follows from Lemma 2.1. Now taking the limit, we have the claim. ∎

REMARK 2.3. *By the same proof, replacing $f(X_t)\,dt$ by dA_t, where A is the additive functional of a measure μ on D, we get the weak maximum principle for K (or its kernel $K(\cdot,\cdot)$); i.e., if for a positive measure μ, $K\mu \le 1$ on the support of μ, then $K\mu \le M$ everywhere, where M is some constant ($M \ge 1$).*

REMARK 2.4. *For a nonnegative measure μ, the potential $G\mu$ is lower semicontinuous relative to the measure μ. That means if*

$$\lim_{n\to\infty} \int_D f(x)\, d\mu_n(x) = \int_D f(x)\, d\mu(x), \qquad f \in C_{cp}^{\infty}(D),$$

then

$$G\mu(x) \le \liminf_{n\to\infty} G\mu_n(x), \qquad x \in D. \tag{16}$$

Additionally, if $\{\mu_n\}$ is monotonically increasing, then

$$G\mu(x) = \lim_{n\to\infty} G\mu_n(x), \qquad x \in D. \tag{17}$$

Proof. Ref. [8].

Note that the potential $K\mu$ is lower semicontinuous as well since K has a lower semicontinuous kernel.

PROPOSITION 2.5. *The operator K has a positive definite character; that is, the inequality*

$$(Kf, f) \ge 0 \tag{18}$$

holds for any real, measurable function f on D such that $(K|f|, |f|) < \infty$.

Proof. Using symmetry of S, we have

$$\begin{aligned}(Kf, f) &= \int_0^\infty (S_t f, f)\, dt \\ &= \int_0^\infty (S_{t/2} f, S_{t/2} f)\, dt \\ &= \int_0^\infty \|S_{t/2} f\|_2^2\, dt \geq 0.\end{aligned}$$

∎

From the positive character of the operator K, we have that the *energy inequality* for K holds, i.e.,

$$|(Kf, q)| \leq (Kf, f)^{1/2} (Kq, q)^{1/2}. \tag{19}$$

REMARK 2.6. *Let D be a bounded domain in R^d, $d \geq 3$, and $q \in \mathscr{K}^d(D)$. Then the energies $(G|q|, |q|)$ and $(K|q|, |q|)$ are finite.*

Proof. The claims of the remark follow directly since the potentials $G|q|$ and $K|q|$ are bounded functions [4], and $q \in L^1(D)$ for a bounded domain D. ∎

In order to prove our main result, we will need an important relation between K- and G-potential which is given in the next proposition.

PROPOSITION 2.7. *Let D be a bounded domain in R^d, $d \geq 3$, and $q \in \mathscr{K}^d(D)$ with finite gauge.*

(i) *Let f: $D \to R$ be a measurable function such that $(G|f|, |f|) < \infty$. Then for almost all x, $K(|q|G|f|)(x) < \infty$, and for each such x,*

$$Kf(x) = Gf(x) + K(qGf)(x). \tag{20}$$

(ii) *Let f: $D \to R$ be a measurable function such that $(K|f|, |f|) < \infty$. Then for almost all x, $G(|q|K|f|)(x) < \infty$, and for each such x,*

$$Kf(x) = Gf(x) + G(qKf)(x). \tag{21}$$

Proof. (i) If $(G|f|, |f|) < \infty$, then $(|q|, G|f|) < \infty$. Indeed, by the energy inequality for the operator G and Remark 2.6, we have

$$(|q|, G|f|) \leq (|q|, G|q|)^{1/2} (|f|, G|f|)^{1/2} < \infty.$$

Further,

$$\begin{aligned}(K(|q|G|f|), 1) &= (|q|G|f|, K1) \\ &\leq \|K1\|_\infty (|q|G|f|, 1) < \infty,\end{aligned}$$

where the first equality follows by the symmetry of K, and $\|K1\|_\infty$ is finite because of finite gauge (Theorem 1.1). So for each x such that $K(|q|G|f|)(x) < \infty$, we have by the Markov property

$$K(|q|G|f|)(x) = E_x\left[\int_0^\tau e_q(t)|q|(X_t)G|f|(X_t)\,dt\right]$$
$$= E_x\left[\int_0^\tau e_q(t)|q|(X_t)\int_t^\tau |f|(X_u)\,du\,dt\right] < \infty.$$

It follows that Fubini's theorem applies, and for such x we have

$$K(qGf)(x) = E_x\left[\int_0^\tau e_q(t)q(X_t)Gf(X_t)\,dt\right]$$
$$= E_x\left[\int_0^\tau e_q(t)q(X_t)\int_t^\tau f(X_u)\,du\,dt\right]$$
$$= E_x\left[\int_0^\tau f(X_u)\int_0^u e_q(t)q(X_t)\,dt\,du\right]$$
$$= E_x\left[\int_0^\tau f(X_u)\big(e_q(u) - 1\big)\,du\right]$$
$$= Kf(x) - Gf(x).$$

Thus, for each x such that $K(|q|G|f|)(x) < \infty$, we have

$$Kf(x) = Gf(x) + K(qGf)(x).$$

(ii) Similarly repeating the arguments as in (i), we have the claim. ∎

REMARK 2.8. *As the proof shows, the relation* (20) *holds almost everywhere if*

$$\int_D |q|(x)G|f|(x)\,dx < \infty \tag{22}$$

and the relation (21) *holds almost everywhere if*

$$\int_D |q|(x)K|f|(x)\,dx < \infty. \tag{23}$$

Recall that $K(\cdot,\cdot)$ is a kernel satisfying a weak maximum principle for K. So from [11] we have for each measure μ,

$$(K\mu)^2 \le cK[\mu(K\mu)], \tag{24}$$

where c is a positive constant.

THEOREM 2.9. *Let* $\{\mu_n\}$ *be a sequence of measures on* D *such that the corresponding sequence of potentials* $\{K\mu_n\}$ *is bounded in energy. Then there exist a measure* μ *and a subsequence* $\{\mu_{n_i}\} \subseteq \{\mu_n\}$ *such that* $\mu_{n_i} \to \mu$ *vaguely.*

Proof. According to Theorem 0.6 in [8], it is enough to show that $\{\mu_n\}$ is bounded on compacts; that is, for any compact set $C \subseteq D$, there exists a constant M_C such that

$$\mu_n(C) \leq M_C, \qquad \text{for every } n \in N. \tag{25}$$

Let $\varphi\colon D \to R$ be a positive, bounded function. Then, since $K\varphi$ is lower semicontinuous and strictly positive, we have

$$\inf\{K\varphi(x);\, x \in C\} > 0 \tag{26}$$

for any compact set $C \subseteq D$. The energy inequality

$$(\varphi, K\mu_n) \leq (K\varphi, \varphi)^{1/2}(K\mu_n, \mu_n)^{1/2}$$

and symmetry of K give that $(K\varphi, \mu_n)$ is a bounded sequence. Using simple calculation, we obtain

$$\begin{aligned} 0 &< \inf\{K\varphi(x);\, x \in C\}\,\mu_n(C) \\ &\leq \int_C K\varphi(x)\, d\mu_n(x) \leq \int_D K\varphi(x)\, d\mu_n(x), \end{aligned}$$

and the assertion follows. ∎

THEOREM 2.10. *Let* $\{\mu_n\}$ *be a sequence of measures supported in* D *such that* $\{K\mu_n\}$ *is bounded in energy and* $\mu_n \to \mu$ *vaguely. Then* $K\mu_n \to K\mu$ *weakly in energy.*

Proof. Ref. [8].

THEOREM 2.11. *Let* ν *be a signed measure of finite K-energy. There exists a measure* μ *of finite K-energy such that*

(i) $|K\nu| \leq K\mu$ *a.e.*

(ii) $(K\mu, \mu) \leq 16(K\nu, \nu)$.

Proof. Let

$$\alpha := \inf\{(K(\nu - \lambda), \nu - \lambda);\ \lambda \text{ a measure of finite energy}\}. \tag{27}$$

Let $\mu_n \geq 0$ such that

$$\big(K(\nu - \mu_n), \nu - \mu_n\big) \to \alpha. \tag{28}$$

Then $(K\mu_n, \mu_n)$ is bounded.

We use the previous two theorems to extract a measure μ' such that $\mu_n \to \mu'$ vaguely and $K\mu_n \to K\mu'$ weakly in energy as $n \to \infty$. Then, since K has a lower semicontinuous kernel, we have

$$\liminf_{n\to\infty} (K\mu_n, \mu_n) \geq (K\mu', \mu') \tag{29}$$

and

$$\begin{aligned}(K(\nu - \mu'), \nu - \mu') &= (K\nu, \nu) + (K\mu', \mu') - 2(K\nu, \mu') \\ &\leq (K\nu, \nu) + \liminf_{n\to\infty}\big((K\mu_n, \mu_n) - 2(K\nu, \mu_n)\big) \\ &= \liminf_{n\to\infty}\big(K(\nu - \mu_n), \nu - \mu_n\big).\end{aligned}$$

Thus $\alpha = (K(\nu - \mu'), \nu - \mu')$. This implies that, for any positive measure ϵ and $t > 0$,

$$(K(\nu - \mu' - t\epsilon), \nu - \mu' - t\epsilon) \geq (K(\nu - \mu'), \nu - \mu'). \tag{30}$$

Expanding, dividing by t, and letting $t \to 0$, we obtain

$$(K(\nu - \mu'), \epsilon) \leq 0. \tag{31}$$

Hence, clearly

$$K\nu \leq K\mu' \qquad \text{a.e.} \tag{32}$$

Again from the definition of the measure μ', we have

$$(K(\nu - \mu'), \nu - \mu') \leq (K(\nu - \epsilon), \nu - \epsilon) \tag{33}$$

for any measure ϵ of finite K-energy. Especially for $\epsilon \equiv 0$ we obtain

$$(K(\nu - \mu'), \nu - \mu') \leq (K\nu, \nu). \tag{34}$$

Using symmetry of K and the Cauchy–Schwarz inequality, Eq. (34) leads to

$$(K\mu', \mu') \leq 4(K\nu, \nu). \tag{35}$$

Similarly, replacing ν by $-\nu$, there is a measure μ'' such that

$$-K\nu = K(-\nu) \leq K\mu'' \quad \text{a.e.} \qquad \text{and} \qquad (K\mu'', \mu'') \leq 4(K\nu, \nu). \tag{36}$$

Now, it is easy to see that the measure $\mu = \mu' + \mu''$ satisfies the properties of the theorem. ∎

COROLLARY 2.12. *Let ν be a signed measure of finite G-energy. There exists a measure μ such that*

(i) $|G\nu| \le G\mu$ *a.e.*

(ii) $(G\mu, \mu) \le 16(G\nu, \nu)$.

Proof. The claim follows directly from Theorem 2.11 with $q \equiv 0$. ∎

In the next theorem are given important inequalities for a signed measure of finite energies.

THEOREM 2.13. *For $q \in \mathscr{K}^d(D)$ and a signed measure μ of finite G- and K-energy, there exists a constant c such that*

$$\int_D |q|(x)(K\mu)^2(x)\,dx \le c(K\mu, \mu) \tag{37}$$

and

$$\int_D |q|(x)(G\mu)^2(x)\,dx \le c(G\mu, \mu). \tag{38}$$

Proof. According to Theorem 2.11, we can find a positive measure ν such that

$$|K\mu| \le K\nu \quad \text{a.e.} \qquad \text{and} \qquad (K\nu, \nu) \le c'(K\mu, \mu),$$

with c' a positive constant. Then, using in addition the inequality (24), symmetry of K, and the fact that $K|q|$ is bounded, we have

$$\begin{aligned} \int_D |q|(x)(K\mu)^2(x)\,dx &\le \int_D |q|(x)(K\nu)^2(x)\,dx \\ &\le c_0 \int_D |q|(x) K(\nu K\nu)(x)\,dx \\ &= c_0 \int_D K\nu(x) K|q|(x)\,d\nu(x) \\ &\le c_1 \int_D K\nu(x)\,d\nu(x) \\ &\le c_2(K\mu, \mu), \end{aligned}$$

where c_0, c_1, and c_2 are positive constants. Similarly, we prove the other inequality. ∎

The K- and G-energy are equivalent. This equivalence is the main result of this paper and it is given in the next theorem.

THEOREM 2.14. *Let D be a bounded domain in R^d, $d \geq 3$, $q \in \mathscr{K}^d(D)$, and let the gauge g be finite for some $x \in D$. Then there exists a positive constant α such that, for any real measurable function f on D with finite energy $(G|f|, |f|)$, we have*

$$\frac{1}{\alpha}(Gf, f) \leq (Kf, f) \leq \alpha(Gf, f). \tag{39}$$

Proof. Assume both $(K|f|, |f|)$ and $(G|f|, |f|)$ are finite. Using the relation (21) of Proposition 2.7 between G- and K-potential, symmetry of G, the Cauchy–Schwarz inequality, and the last theorem, we have

$$\begin{aligned}(Kf, f) &= (Gf, f) + (G(qKf), f)\\ &= (Gf, f) + (qKf, Gf)\\ &\leq (Gf, f) + (|q|Kf, Kf)^{1/2}(|q|Gf, Gf)^{1/2}\\ &\leq (Gf, f) + c(Kf, f)^{1/2}(Gf, f)^{1/2},\end{aligned}$$

with some positive constant c. From this inequality it follows easily that $(Kf, f) \leq (1 + 2c^2)(Gf, f)$. Using similar manipulations as above, we prove the second inequality. ∎

REMARK 2.15. *Let $\mathscr{E} = \{Kf;\ (K|f|, |f|) < \infty\}$ and define $\|Kf\|_*^2 = (Kf, f)$. Then $\|\ \|_*$ defines a norm on $\mathscr{E}$.*

COROLLARY 2.16. *Let f be a real, measurable function on D and assume that Gf is in the Sobolev space $H_0^1(D)$. Then Kf is in the Sobolev space $H_0^1(D)$. Conversely, if $(K|f|, |f|) < \infty$, then Gf is in the Sobolev space $H_0^1(D)$.*

Proof. We have to prove that

$$\frac{\partial Kf}{\partial x_j} \in L^2(D), \qquad j = 1, 2, \ldots, d, \tag{40}$$

where the partial derivatives are taken in the sense of distributions. Using (ii) of Proposition 2.7, that is,

$$Kf = Gf + G(qKf) \qquad \text{a.e.} \quad \text{on } D, \tag{41}$$

and the Minkowski–Riesz inequality [6], ∇Kf is in $L^2(D)$ if ∇Gf and $\nabla G(qKf)$ are in $L^2(D)$. Equation (12) gives that for $Gf \in H_0^1(D)$, the energy (Gf, f) is finite. Since $\nabla Gf \in L^2(D)$ according to the assumption, it remains to prove that $\nabla G(qKf) \in L^2(D)$. First, we define

$$L^2(D, |q|) = \left\{ f\colon D \to R;\ \int_D |q|(x) f^2(x)\, dx < \infty \right\}. \tag{42}$$

From Theorem 2.13 and Theorem 2.14, it follows that for a function f such that (Gf, f) is finite, the potential Kf is in $L^2(D, |q|)$. Consequently, it is enough to show that for $\psi \in L^2(D, |q|)$, the integral

$$\int_D |q|(x) |\psi|(x) G(|q|\,|\psi|)(x)\, dx \tag{43}$$

is finite.

For $\psi \in L^2(D, |q|)$, let

$$\psi_n(x) := (|\psi| \wedge n)(x), \qquad x \in D. \tag{44}$$

By The Cauchy–Schwarz inequality, we have

$$\begin{aligned} &\int_D |q|(x) \psi_n(x) G(|q|\psi_n)(x)\, dx \\ &\quad \le \left(\int_D |q|(x) \psi_n^2(x)\, dx \right)^{1/2} \left(\int_D |q|(x) \big(G(|q|\psi_n)(x)\big)^2\, dx \right)^{1/2}, \end{aligned} \tag{45}$$

and by the inequality (38) of Theorem 2.13, we have

$$\int_D |q|(x) \big(G(|q|\psi_n)(x)\big)^2\, dx \le c \int_D |q|(x) \psi_n(x) G(|q|\psi_n)(x)\, dx \tag{46}$$

for some positive constant c.

Thus from Eqs. (45) and (46), we obtain

$$\begin{aligned} &\int_D |q|(x) \psi_n(x) G(|q|\psi_n)(x)\, dx \\ &\quad \le c \left(\int_D |q|(x) \psi_n^2(x)\, dx \right)^{1/2} \left(\int_D |q|(x) \psi_n(x) G(|q|\psi_n)(x)\, dx \right)^{1/2}. \end{aligned} \tag{47}$$

Further, we have

$$0 < \int_D |q|(x) \psi_n(x) G(|q|\psi_n)(x)\, dx \le n^2 \int_D |q|(x) G(|q|)(x)\, dx. \tag{48}$$

The expression on the right-hand side of the inequality (48) is finite since $q \in \mathscr{K}^d(D)$ has finite G-energy, and so dividing both sides of the inequality in (47) by

$$\left(\int_D |q|(x)\,\psi_n(x) G(|q|\psi_n)(x)\,dx\right)^{1/2},$$

we obtain

$$\left(\int_D |q|(x)\,\psi_n(x) G(|q|\psi_n)(x)\,dx\right)^{1/2} \le c\left(\int_D |q|(x)\,\psi_n^2(x)\,dx\right)^{1/2}. \quad (49)$$

From the definition of ψ_n, the integral on the right-hand side is dominated by

$$\int_D |q|(x)\,\psi^2(x)\,dx,$$

which is finite since ψ is in $L^2(D, |q|)$. So, we have

$$\int_D |q|(x)\,\psi_n(x) G(|q|\psi_n)(x)\,dx \le c' \int_D |q|(x)\,\psi^2(x)\,dx < \infty. \quad (50)$$

Now, applying Lebesgue's dominated convergence theorem and Remark 2.4, we have shown that the integral in Eq. (43) is finite and so we have shown the first part of the corollary. The second part follows directly using the first part and Theorem 2.14. ∎

Further interesting result considering K-potential is given in the next theorem.

THEOREM 2.17. *For $f \in L^1(D)$ and $q \in K^d(D)$, the operator T defined by*

$$Tf = f + qKf \quad (51)$$

is one-to-one and onto $L^1(D)$.

Proof. The theorem will be proved in a few steps.

Step 1. First we note that $T\colon L^1(D) \to L^1(D)$. This follows from symmetry of K, Fubini's theorem, and the fact that $K|q|$ is bounded.

Step 2. T is one-to-one. Indeed, suppose

$$Tf = f + qKf = 0. \quad (52)$$

Then $G[f + qKf] = 0$. If $f \in L^1(D)$, then $\int_D |q|(x) K|f|(x)\,dx < \infty$ and so, as Remark 2.8 shows,

$$Kf = G[f + qKf],$$

and hence $Kf = 0$. So from Eqs. (52), $f \equiv 0$. Thus T is one-to-one.

Step 3. The range of T is dense in $L^1(D)$. Otherwise there exists $0 \not\equiv g \in L^\infty(D)$ such that

$$(f + qKf, g) = 0 \qquad \text{for every } f \in L^1(D). \tag{53}$$

From Eq. (53) it follows

$$(f, g + Kqg) = 0 \qquad \text{for every } f \in L^1(D), \tag{54}$$

which leads to

$$g + Kqg = 0, \tag{55}$$

and so for $q \in \mathscr{K}^d(D)$ we obtain

$$qg + qKqg = 0. \tag{56}$$

But since $qg \in L^1(D)$, we get as before $qg = 0$, which implies $Kqg = 0$ and so $g \equiv 0$.

Step 4. The set $\{|q|K|f|;\ \|f\|_1 \le 1\}$ is uniformly integrable. To see this, first note that

$$\int_D |q|(x)|Kf|(x)\,dx \le \int_D |f|(x) K|q|(x)\,dx \le \|K|q|\,\|_\infty. \tag{57}$$

Second, if A has small measure, then (because $|q|$ is in the Kato class) $K(|q|1_A)$ is uniformly small. So

$$\int_A |q|(x)|Kf|(x)\,dx \le \int_D |f|(x) K(|q|1_A)(x)\,dx \le \|K(|q|1_A)\|_\infty \tag{58}$$

is small. This proves uniform integrability.

Step 5. Let $f_n \to f$ weakly in $L^1(D)$, i.e., $\int_D f_n(x) g(x)\,dx \to \int_D f(x) g(x)\,dx$ for every $g \in L^\infty(D)$. Then $qKf_n \to qKf$ strongly in $L^1(D)$.

To see this, suppose first that $f_n \ge 0$. Since K has a lower semicontinuous kernel,

$$\liminf_{n\to\infty} Kf_n \ge Kf. \tag{59}$$

Also since $f_n \to f$ weakly in $L^1(D)$, we have

$$\lim_{n\to\infty}\int_D Kf_n(x)\,dx = \lim_{n\to\infty}\int_D f_n(x)K1(x)\,dx$$
$$= \int_D f(x)K1(x)\,dx = \int_D Kf(x)\,dx.$$

We conclude

$$\lim_{n\to\infty} Kf_n = Kf \qquad \text{a.e.} \tag{60}$$

It follows that qKf_n converges to qKf a.e. Since qKf_n is uniformly integrable, we have

$$qKf_n \to qKf \qquad \text{strongly in } L^1(D). \tag{61}$$

Splitting f_n into f_n^+ and f_n^-, we get the general case.

Step 6. $\inf\{\|Tf\|_1;\ \|f\|_1 = 1\} > 0$. If not, we can find f_n such that $\|f_n\|_1 = 1$ and

$$\|Tf_n\|_1 \le 2^{-n}. \tag{62}$$

But, as Step 4 shows, qKf_n is uniformly integrable, and since $\|f_n + qKf_n\|_1 \to 0$, we see that f_n is uniformly integrable. By choosing a subsequence if necessary, we may assume $f_n \to f \in L^1(D)$ weakly.

From Step 5 we get that $qKf_n \to qKf$ strongly. Since $\|f_n + qKf_n\|_1 \to 0$, we get $f_n \to f$ strongly and

$$f + qKf = 0. \tag{63}$$

Now using Step 2, we get $f \equiv 0$. This contradicts $\|f_n\|_1 = 1$ and $f_n \to 0$ strongly.

Step 7. Step 6 says that the range of T is closed, and from Step 3 the range of T is dense in $L^1(D)$. Thus T is onto $L^1(D)$, proving the theorem. ∎

ACKNOWLEDGMENT

I thank Professor Murali Rao for his immeasurable help in the preparation of this paper.

REFERENCES

1. M. Aizenman and B. Simon, Brownian motion and Harnack's inequality for Schrödinger operator, *Comm. Pure Appl. Math.* **35** (1982), 209–271.
2. K. L. Chung and M. Rao, Feynman–Kac functional and the Schrödinger equation, in "Seminar on Stochastic Processes," 1–29, Birkhäuser, Boston, 1981.
3. K. L. Chung and M. Rao, General gauge theorem for multiplicative functionals, *Trans. Amer. Math. Soc.* **306** (1988), 819–836.
4. K. L. Chung and Z. Zhao, "From Brownian Motion to Schrödinger's Equation," Springer-Verlag, Berlin, New York, 1995.
5. M. Cranston, E. Fabes, and Z. Zhao, Conditional gauge and potential theory for the Schrödinger operator, *Trans. Amer. Math. Soc.* **307** (1988), 174–194.
6. R. M. Dudley, "Real Analysis and Probability," Wadsworth and Brooks/Cole Advanced Books and Software, 1989.
7. R. Khas'minskii, On positive solution of the equation $Uu + Vu = 0$, *Theory Probab. Appl.* **4** (1959), 309–318.
8. N. S. Landkof, "Foundation of Modern Potential Theory," Springer-Verlag, Berlin, New York, 1972.
9. M. Rao, Brownian Motion and Classical Potential Theory, Aarhus University Lecture Notes **47**, 1977.
10. M. Rao, Capacitary inequality for energy, *Israel J. Math.* **61** (1988), 179–191.
11. M. Rao and H. Šikić, Potential inequality, *Israel J. Math.* **83** (1993), 97–127.
12. B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982), 447–526.
13. Z. Zhao, Local Feynman–Kac semigroup, *J. Syst. Sci. Math. Sci.* **2** (1982), 314–326.
14. Z. Zhao, Conditional gauge with unbounded potential, *Z. Wahrsch. Verw. Gebiete* **65** (1983), 13–18.
15. Z. Zhao, Green function for Schrödinger operator and conditional Feynman–Kac gauge, *J. Math. Anal. Appl.* **116** (1986), 309–334.