UNIVERSITY OF SOUTHAMPTON

FACULTY OF LAW, ARTS AND SOCIAL SCIENCES

School of Education

The Use of Real-World Contextual Framing in UK University Entrance Level Mathematics Examinations

by

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ABSTRACT

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THE USE OF REAL-WORLD CONTEXTUAL FRAMING IN UK UNIVERSITY ENTRANCE LEVEL MATHEMATICS EXAMINATIONS

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Although there has been considerable research into real-world contexts in elementary mathematics, little work has been done at a more advanced, post-16 level. This thesis explores the origin, function and effect of real-world contextual framing (RWCF) in GCE A/AS mathematics examinations. The study develops an evaluation framework (ARTA) based on the notions of accessibility, realism and task authenticity, derived from assessment theory, and considers ‘context’ in relation to theoretical ideas such as Realistic Mathematics Education, construct validity and construct-irrelevant variance.

The function and effect of RWCF are investigated using the ARTA framework on samples of A/AS questions. Its effect is explored using sequence questions with the same solutions with and without real-world context, set to a sample of nearly 600 students, together with a questionnaire that surveys students’ attitudes to RWCF.

Quantitative differences in the use of RWCF are established and traced to early project syllabuses such as SMP and MEI. The study finds that RWCF in general adds to the difficulty of questions, unless they can be solved by ‘thinking within the context’. The *accessibility* of questions with RWCF is a function of comprehensibility of language, and the explicitness of the match between context and mathematical model. The study distinguishes between *natural* and *synthetic* contexts, according to the extent to which the context matches reality, or reality is configured to match the mathematics. Natural contexts are more *realistic*; but synthetic contexts can serve the purpose of reifying abstract mathematical ideas. At best, RWCF in examination questions require solvers to engage in *pseudo-modelling*: they cannot test aspects of the modelling cycle such as discussing assumptions, refining, and critical reading of longer arguments. There is, moreover, a *gender* difference in students’ attitudes to RWCF, with boys in general expressing more favourable views about its use in pure mathematics questions.

These findings have the following implications for A/AS assessment. Current examination questions are not able to satisfy current QCDA (Qualifications and Curriculum Authority, 2002) assessment objectives on mathematical modelling. Questions with RWCF need to be authentic, and require careful construction to ensure that language is precise and unambiguous. Longer questions, which present and invite comparison of more than one model, are desirable, in order that students appreciate the relationship between reality and mathematical models.

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DECLARATION OF AUTHORSHIP

I, Christopher Thomas Little, declare that the thesis entitled

The use of real-world contextual framing in UK university entrance level mathematics examinations

and the work presented in this thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- the work was done wholly while in candidature for a research degree at this university;
- no part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- parts of this work have been published as:

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LIST OF ABBREVIATIONS

a	'algebraic' (AP/GP study)
A/AS	General Certificate of Education Advanced /Advanced Subsidiary
AO	Assessment Objective
AP	Arithmetic progression
AQA	Assessment and Qualifications Alliance
ARTA	Accessibility – Realism – Task Authenticity evaluation tool
BP	British Petroleum
BSRLM	British Society for Research in Learning Mathematics
CCEA	Council for the Curriculum Examinations and Assessment (Northern Ireland)
e	'explicit' (AP/GP study)
FSMQ	Free-Standing Mathematics Qualification
G	'Growth'
GP	Geometric progression
ICMI	International Committee on Mathematical Instruction
ICTMA	International Conferences on the Teaching of Mathematical Modelling and its Applications (ICTMA)
KS	Key Stage
MEI	Mathematics for Education and Industry Project
MME	Midland Mathematics Experiment
OCR	Oxford, Cambridge and RSA Examinations
OECD	Organisation for Economic Co-operation and Development
p	'pattern' (AP/GP study)
PISA	Programme for International Student Assessment
QCA	Qualifications and Curriculum Authority
QCDA	Qualifications and Curriculum Development Authority (formerly QCA)
RQ	Research Question
RME	Realistic Mathematics Education
RWCF	Real-world contextual framing
SMP	School Mathematics Project
UCLES	University of Cambridge Local Examinations Syndicate
w	'word' (AP/GP study)
WJEC	Welsh Joint Education Committee

CHAPTER 1

INTRODUCTION

1.1 Real-world Contextual Framing

Consider the mathematics questions in Figures 1.1.1 and 1.1.2.

An arithmetic progression has first term 7 and common difference 3.

(i) Which term of the progression equals 73?

(ii) Find the sum of the first 30 terms of the progression.

Fig. 1.1.1 Arithmetic progression question without real-world contextual framing

Chris saves money regularly each week. In the first week, he saves £7. Each week after that, he saves £3 more than the previous week.

(i) In which week does he save £73?

(ii) Find his total savings after 30 weeks.

Fig. 1.1.2 Arithmetic progression question with real-world contextual framing

Figures 1.1.1 and 1.1.2 show two mathematics questions which on first glance seem to be quite different, but which have identical solutions. In the first question, the language used is purely mathematical; in the second, the mathematics is framed in the everyday context of savings. This is an example of what, in this thesis, I shall call *real-world contextual framing* (RWCF), in which pure mathematics questions are presented through reference to a narrative taken from outside the world of abstract mathematics.

Both these types of questions are typical examples of questions from the General Certificate of Education Advanced and Advanced Subsidiary level (GCE A/AS level) Mathematics examination, which is a national university entrance examination for students in England, Wales and Northern Ireland. This examination is offered at two levels: the Advanced

Subsidiary (AS-) level is normally completed after the first year of post-16 study; the Advanced level (A-level) is completed after a further year of post-16 study.

1.2 Real-world contextual framing and mathematics examinations

The influence of high-stakes public examinations such as, in the UK, A/AS level, on the curriculum has been widely researched, and readily acknowledged by teachers. For example, many of the papers in the 1992 International Committee on Mathematical Instruction (ICMI) study on assessment discuss the potentially negative effects of summative assessment tasks on classroom practice. Given this influence, the nature of the assessment tasks presented in examinations is an important area of research.

Currently, in England, Wales and Northern Ireland, there are six different General Certificate of Education A/AS level Mathematics specifications, developed and administered by government approved examining groups - Edexcel, Assessment and Qualifications Alliance (AQA), Council for the Curriculum Examinations and Assessment (CCEA, Northern Ireland), Welsh Joint Education Committee (WJEC) and Oxford, Cambridge and RSA Examinations (OCR), the latter offering two (Syllabus A and Syllabus B (MEI)). Although these specifications are required to satisfy national criteria laid down by the Qualifications and Curriculum Development Authority, the examining groups have the freedom to interpret these criteria in different ways, and the specifications therefore vary in both the mathematical content of units, and the nature and style of the assessments.

One of the differences between specifications and their assessment methods is the degree to which pure mathematical questions are framed within real-world contexts. At the time of writing, the subject criteria for A/AS Mathematics specifies the following assessment objectives (AOs) which relate mathematics to real-life contexts and modelling, and the approximate weighting to be assigned to them in the scheme of assessment:

> AO3 Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinements of such models. (10%)
>
> AO4 Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications. (5%)

Thus, both translating realistic situations into mathematics, and using mathematical models to make predictions, are embedded in the A/AS Mathematics construct. It is to be expected that these assessment objectives would be prevalent in the applied mathematics units, which comprise statistics, mechanics, and decision mathematics. However, knowledge and classroom experience of these specifications suggests that the extent to which pure mathematical questions are formulated in, or used to develop mathematical models of, real-world contexts varies. For example, the OCR MEI specification (Oxford Cambridge and RSA Examinations, 2004b) has a unit in pure mathematics (C4) entitled 'Applications of Pure Mathematics', in which 50% of the marks of the timed written paper (4754A) are allocated to a section B which involves two pure mathematics questions in real-world contexts, together with a Comprehension Paper (4754B) in which candidates study an article in which mathematics is used to model a real-world context, and then answer questions to test their understanding. In contrast, although the OCR 'A' Specification (Oxford Cambridge and RSA Examinations, 2004a) does contain some questions from the pure mathematics papers which do contain real-world context, my experience suggests that there is less emphasis on real-world modelling, and more on pure mathematical skills such as algebraic technique and manipulation (see section 6.3 for a detailed analysis of this claim).

These assessment practices could be taken as reflecting different approaches to the teaching of subject. A predominantly techniques-based approach may be said to emphasise formal notation and mathematical techniques. The contextualised approach, on the other hand, emphasises mathematical modelling skills, such as formulation, problem solving and interpretation of solutions.

These differences of approach are reflected in the schemes of assessment used by the examination syllabuses. Some use coursework assessment more freely, as this allows greater freedom to develop applications of the subject. Examination papers vary in the length of questions asked: longer questions provide greater opportunity to develop a context and test students' comprehension and modelling skills. On the other hand, the demands on pure mathematical technique required may be less than in examinations with technique-based questions.

There is a substantial body of research on the effects of real-world contextualisation in elementary mathematics questions. Cooper and Dunne (2000), for example, have questioned the validity of using 'realistic' items in UK national tests of attainment, finding that many students, faced with mathematics questions in context, totally misunderstand the intention of the question setter. They have also questioned the fairness of such questions, finding that social class appears to be a factor in the quality of response.

Little research, however, appears to have been carried out on the effect of contextualising the assessment of pure mathematics at a more advanced level. The aim of this research is to review the research on context in pre-16 mathematics questions, and extend this to consider the approaches to context and modelling in post-16 A/AS mathematics assessment, and what the effects of different approaches might be for the A/AS mathematics in general.

My experience as a Principal Examiner in setting A-level questions on pure mathematical concepts in context suggests that it is only certain mathematical ideas and concepts which prove to be amenable to framing in real-world terms. For example, it is difficult to develop a real-world context for embedding the binomial theorem, or binomial series expansions, except in apparently artificial terms – such as using it to approximate a square root such as $\sqrt{4.01}$ with a fraction. Yet it would be hard to envisage *not* relating a question on the binomial probability distribution to a real-world context which, on the one hand, serves as tangible model for it, and on the other, establishes its utility in describing certain extra-mathematical situations.

The role of real-world context appears to be different in applications of mathematics, such as statistics, mechanics and discrete mathematics. While it is may be possible to strip away these of real-world content, the fact that they fall under the title of ‘applied mathematics’ implies an element of utility which one would expect to be reflected in A/AS-level questions designed to test the knowledge of students (although the extent to which traditional mechanics questions, set in the pseudo-real world of light inextensible strings, point masses, frictionless pulleys and vacuums without air resistance etc. constitutes a form of reality as we experience it is open to question).

There appears to be a spectrum of ‘contextualisability’ – the extent to which real-world contexts can be found which embed the pure mathematics in an accessible, authentic way. At the ‘pure’ end of pure mathematics lie concepts such as function or mapping, algebraic techniques such as the remainder theorem or the afore-mentioned binomial theorem. At the more applied end of ‘pure’ lies topics such as calculus, differential equations, and specific functions such as trigonometric, exponential and logarithmic, which serve not just as models of, but models for, real-world contexts that are familiar and accessible to most students.

In the middle of this ‘contextualisability’ spectrum lie topics which would seem to be assessable with or without real-world contexts. An example of such a topic is arithmetic and geometric series. These can be embedded in contexts such as finance, geometrical patterns, or kinematics – the range and types of such contexts are explored in chapter 8.

The exponential function is another such example, capable of development in the classroom either as pure mathematics or as a model which serves many real-world contexts, for example population growth, radioactivity, and Newton's Law of Cooling. Reduction to linear form using logarithmic transformations of data serves to provide the pure mathematical theory of logarithms with a purpose and utility which would seem to be motivating to students of science. Differential equations can be treated as exercises in technique – separation of variables, integrating factors, etc – or as models both of and for real-world contexts.

It is topics like these which offer a choice of questions: on the one hand, questions could be restricted to 'pure mathematics', reinforcing the view that mathematics is a subject in its own right which requires the development of techniques such as algebraic manipulation, axiomatic development, careful use of mathematical notation, and proof. On the other hand, real-world contexts could be actively sought out in which to embed the problems, thus presenting the mathematics as a subject which can be used to enhance our knowledge of the 'real world'.

It is this fundamental choice that this study explores. In terms of current A/AS level specifications, these can be seen to differ in the extent to which they embrace context in their pure mathematics papers, although, as has been previously pointed out, current QCDA subject criteria (Qualifications and Curriculum Authority, 2002) require all specifications to test candidates abilities to translate real-world situations into mathematical models, and use mathematical models to solve real-world problems.

Is this difference of approach simply a matter of taste? To some extent this might be the case – different specifications offering different styles of question could be seen as offering choice to the 'consumer', be they teacher or student. However, the consequences of such a choice for the 'A-level mathematics' construct would seem to be worth exploring.

My interest in researching context in mathematics stems from my personal experience of teaching, curriculum development and examining. I started teaching in the 1970s, and my early classroom experiences of A-level pure mathematics were of a 'traditional' syllabus which utilised little real-world context in A-level pure mathematics questions. At the same time, the influence of the 'modern mathematics' movement was becoming apparent in the pre-16 mathematics curriculum. In 1985, I joined the School Mathematics Project, and became involved in curriculum development work for an 11-16 course. This leant heavily on a real-world contextualised approach, and was highly influential in the mathematics curriculum, being used at the time by over 40% of all English secondary schools. At the same time, I became involved in A-level curriculum development, leading a working group of teachers, and subsequently editing, a revised edition of the SMP A-level course (School Mathematics Project, 1988), and contributing to a new SMP 16-19 A-level course.

My examining work began in 1991, when I became a Principal Examiner for OCR, responsible for setting and marking A-level questions, many of which were required to use real-world contexts. I have in recent years performed a similar role for the OCR MEI Mathematics A/AS specification.

My experiences of mathematics learning, teaching, curriculum development and assessment, over a period of over 30 years, have led me to question the role of real-world context in the teaching and learning of mathematics. On the one hand, as a student of mathematics, I have always been fascinated by the subject per se, and my natural aptitudes lay in pure mathematical topics such as abstract algebra and topology. On the other hand, as a teacher, my experience of students suggests that few share this fascination with mathematics, and many are put off by its abstract nature, and are learning mathematics in order to pursue other subjects, such as physics, economics or psychology which require a solid base of mathematical skills and knowledge. It would seem appropriate with such students to emphasise mathematical modelling at the expense of the development of highly sophisticated pure mathematical skills, such as algebraic manipulation.

My natural inclination as a teacher, and curriculum developer, has been to embed pure mathematics in contexts which are meaningful to students, in order to motivate the subject and to teach them necessary mathematical modelling skills. However, my experience of setting examination questions which deploy real-world contexts has led me to doubt the effectiveness of such questions in developing modelling skills. While some questions seem to satisfactorily marry the requirement to test the mathematics with a realistic and worthwhile context, in others the real-world context seems to be an artificial, synthetic distraction from the mathematical task. I have also been concerned that setting wordy, contextualised questions might be unfair to students whose knowledge of English is less secure, thus jeopardising the validity of the questions in a mathematics examination.

Referring to the QCDA assessment objectives quoted earlier, my experiences as a teacher would support these as important aspects of the A-level mathematics construct, which are required to be tested through schemes of assessment. However, whether schemes which rely entirely on timed written examinations, composed of closed, short, questions, effectively test these assessment objectives is open to question.

The last forty years have seen a national debate on the merits of 'coursework' assessment, which in the UK is taken to mean the inclusion of tasks, some of which might be extended in nature, conducted by students outside the examination hall. At the time of writing the position is that a large majority of public examinations in mathematics use timed written examinations exclusively.

My experience suggests that, without the inclusion of extended pieces of coursework, many A/AS syllabuses in mathematics are failing to meet their assessment objectives. It is germane, therefore, to research the extent to which real-world contextualisation deployed in examination questions effectively encourages the teaching and assessment of mathematical modelling skills.

1.3 Research Questions

The introductory discussion suggests a number of broad research questions to be addressed by the study. These are as follows.

Research question 1	What has led to the introduction of real-world context and mathematical modelling in A-level mathematics?
Research question 2	To what degree are 'pure' mathematics questions in A/AS level examinations capable of being framed within real-world contexts, and what is the nature of these contexts?
Research question 3	What functions are served by real-world contextual framing (RWCF) of pure A-level mathematics questions, and what are its effects?

While the methodology to address these questions is developed in chapter 4, and is informed by earlier chapters, it is appropriate to expand upon these questions and sketch here the general approach adopted in this thesis.

The first question seeks to explain the historical roots of real-world contextual framing. When did it develop? What were the reasons for its development? The intention is to consider the way in which A-level pure mathematics questions have changed over time, and look for pointers towards the historical reasons for these changes, considering sources on syllabus development in A-level mathematics.

The second question investigates the extent to which pure mathematics questions at A/AS level are amenable to real-world contextual framing. The aim is to analyse, in detail, past papers from two alternatives specifications in order to confirm and quantify the differences in the use of real-world contextual framing utilised in the pure mathematics questions.

Analysis of past paper questions may also enhance understanding of the types of contexts which are used. What can we learn about the nature of the relationship between mathematics and the real world from this analysis?

The third question relates the intended *functions* of real-world contextual framing. The research literature on real-world context in mathematics proposes a number of functions, such as motivation, utility, and providing mental 'scaffolding' to help students to develop mathematical concepts (see chapter 2). Are these functions applicable to real-world contexts as they are used in A-level pure mathematics questions?

The second issue addressed in this question is the *effects* of RWCF. Testing mathematics through real-world contexts changes the nature of the questions. Usually, questions are longer, as it takes more words to explain a context than to pose a pure mathematics question. Does this added length make questions more difficult for students, or do questions become more understandable? In addition to testing mathematical knowledge, questions usually assume some knowledge of the context. Does this affect their validity as assessment tools?

In practice, it is difficult to consider functions and effects separately, as they tend to be interlinked. For example, if one accepts that testing mathematical modelling is a function of RWCF, then this is likely to affect judgement of the validity of test items. If this function is accepted, then how effective are examination questions which utilise RWCF in testing mathematical modelling skills?

The aim is that by investigating the roots, degree, function and effect of RWCF, this study can help to understand more about how mathematics can be assessed. Given a better understanding of the functions and effects of setting questions in real-world contexts, can questions with RWCF be evaluated to find out which use context more effectively than others? Is there a tool which enables questions to be analysed with a view to improving quality? This study aims to propose such a framework.

1.4 Thesis structure

The structure employed by this thesis is as follows. Chapters 1 – 4 establish the background, the research questions, the research context, theoretical underpinning and methodology of the study. After this introductory chapter, Chapter 2 surveys the current research literature that is relevant to the research questions. This includes studies of real-world context in elementary mathematics, such as research into ‘word problems’, a review of Realistic Mathematics Education (RME), research on mathematical modelling, authentic assessment, and research into UK mathematics examinations. The relevance of this to more advanced, post-16, mathematics is then discussed.

Chapter 3 considers the theoretical constructs relevant to real-world context and assessment, in particular discussing the key concept of validity. It then uses these ideas, together with the issues developed in Chapter 2, to propose a theoretical framework for assessing the function and effect of RWCF in examination questions. Chapter 3 also considers the nature of ‘context’ in more detail and provides a definition that is used for the purposes of this study. Chapter 4 then discusses the research methods used in the study.

Chapters 5 – 9 present the main findings of the study, subdivided into three parts. In the first part, Chapter 5 traces the origins of RWCF to curriculum projects in the 1960s, 70s and 80s, and provides examples of how examination questions have evolved in time to include RWCF. Part II (Chapters 6 and 7) then analyses the use of RWCF using questions drawn from recent A/AS examination specifications, both quantitatively and qualitatively, using a theoretical framework drawn from the ideas presented in chapter 3. Chapter 7 then applies this to analyse and classify questions on a particular topic – that of arithmetic and geometric sequences (APs and GPs).

In part III, this classification is then used as a basis for a large-scale study of the effect of RWCF, using a student questionnaire and a versioned topic test on APs and GPs. The results of this study are reported and discussed in chapter 8. Finally, chapter 9 concludes the study by discussing and summarising the findings of the research, reviewing the methodology, and proposing ideas for future research.

CHAPTER 2

LITERATURE REVIEW

Overview

In this chapter, the existing research literature relating to the use of real-world context in mathematics education is reviewed. The aim is to see what can be learned from this about the origins, degree, functions and effects of setting mathematics questions in context. In what follows a wide interpretation of ‘context’ is taken. Further consideration is given to the nature of ‘context’ in chapter 3.

The first research question of this thesis asks about the historical development of real-world contextual framing (RWCF) in A-level Mathematics. The literature relating to this is considered in Chapter 5.

My second and third questions relate to the degree, function and effect of RWCF. The research literature suggests a number of themes relating to these questions. First, there is substantial research on what is often termed ‘word problems’ that explores younger children’s responses to questions framed in real-world everyday contexts. This research suggests that children can struggle to understand and interpret word problems. This issue is reviewed in section 2.1. A second theme is that real-world context may assist the solver of mathematical problems by providing a ‘mental scaffolding’ which enables them to think about the mathematics. Further, some theories of learning mathematics propose that real-world context plays a mediating role in developing mathematical concepts. Research relevant to these ideas is discussed in section 2.2. The relationship between RWCF and the large and diverse range of research on mathematical modelling is considered in Section 2.3. Another theme which emerges relates to the degree of realism or artificiality of real-world context as it is deployed in mathematics questions, as posed in both the classroom and examinations. This is considered in section 2.4. Section 2.5 relates the relevance of the research discussed to post-16 mathematics, and the final section summarises the chapter.

2.1 Research on real-world contexts in mathematics word problems

The use of real-world context in what may be termed ‘word problems’ has been the subject of extensive research (see, for example, Verschaffel et al., 2000, Verschaffel et al., 2009). Such research has found that young children often fail to apply common-sense considerations to real-world contexts found in mathematics problems. An example is the now well-known ‘bus’ item, in which children, when asked to work out how many 36-seater buses would be required to transport 1128 soldiers, included fractions of a bus in their answers (Silver, 1993)

Similarly, Verschaffel, De Corte & Lasure (1994) found that children can fail to apply realistic considerations to their solutions of word problems. A follow-up study (Verschaffel et al., 1997) found that pre-service teachers tended to exclude real-world knowledge from their own spontaneous solutions of school mathematics word problems as well as from their appreciation of the pupils’ answers from their solutions of such word problems.

A number of reasons have been proposed for this phenomenon. Greer (1997) accounts for the apparent blindness to real-world considerations not through some cognitive defect of the children, but in terms of the culture of the classroom, wherein word problems are presented in stereotyped fashion, with an implicit assumption that a solution involving the application of one or more basic arithmetic operations to the numbers mentioned in the text is appropriate and un-problematical. He proposes an alternative conceptualisation of word problems, as situations calling for mathematical modelling, taking account of real-world knowledge where appropriate.

Gravemeijer (1997), a researcher with a background in the Realistic Mathematics Education (RME) movement, following the ideas of Freudenthal (1991) and Treffers (1987), likens inappropriate application of arithmetic operations to stereotypical word problems to automated behaviour, as when finding your way round a city is internalised.

Others have criticised the assumption of these earlier studies that the misinterpretation of contextualised questions by children is primarily an issue of classroom culture. Cooper and Dunne (2000) studied a number of National Curriculum test items for mathematics at Key Stage 2 (age 11) and Key Stage 3 (age 14). In analysing children’s responses, they found a range of similar sorts of ‘misinterpretations’ as Silver *et al* and Verschaffel *et al*. However, they suggest that children’s knowledge and experiences outside the classroom are equally, if not more, significant in determining their reactions to contextualised problems:

> ‘There has been a neglect, especially within research on mathematics education, on the ways in which cultural differences between children from differing social classes might influence their success and failure in mathematics. This is partly a result of a reaction against what were seen as ‘deficit’ theories and partly because of a relative falling away of concern with social class differences in educational achievement in comparison with the post-war period. Much more energy has been expended on the more important areas of gender and ethnicity. But a little reflection on the existing literature on social class differences in attitudes to formal knowledge and problem-solving suggests that this area deserves further attention.’ (p, 6)

They therefore carry their analysis a stage further by relating the children's responses to family social class. Through this analysis, they argue that the way children apply set mathematical procedures, or 'play the assessment game' by the correct rules, is influenced by social class, and hence that these National Curriculum test items were unreliable and did not measure student attainment fairly.

An alternative explanation, it could be argued, is that the items themselves were flawed in their design, in the sense that the answers were such that a degree of realism brought to the task by those taking the tests invited a range of responses, perhaps equally valid, which were not taken sufficiently into account by the assessment mark schemes. This raises the issue of the nature and degree of 'realism' presented in assessment items and what influence this might have on the range of responses obtained.

Real-world contextualisation is an issue which affects subjects other than mathematics. Ahmed and Pollitt (2007) investigated the effects of context in science questions with year 9 students. They propose that when students read contextualised questions, the cognitive processes provoked by the context can interfere with their understanding of the science in the question. Validity is then compromised in the sense that a question is only valid if the students' minds are doing 'what we want them to show us they can do'. They define the 'focus' of a context for a question as the extent to which the most salient aspects of the context correspond to the main issues addressed in the question, and propose that the validity of contextualized questions may be enhanced by setting them in more focused contexts. In their study, questions in a Key Stage 3 science test were manipulated to alter the focus of their context, and the effects of these changes on the difficulty and validity of the questions. The relevance of this study to mathematics questions is open to question; however, the methodology of adapting questions is of interest, and is indeed adopted later in this study (see chapter 8).

2.2 Real-world contexts as 'mental scaffolding' and Realistic Mathematics Education

As Vappula and Clausen-May (2006) say, "defining what constitutes a context in a maths test question is more difficult than may at first appear" in that "contexts may serve at least two different functions". One function, according to Vappula and Clausen-May, can be thought of as relating to the match the selected context might have with the 'reality' of those tackling the examination question, while the second, and quite different function, might relate to what Clausen-May (2005) calls a "model to think with". In this latter function, the context within which the examination question is set could be seen to act as mental scaffolding for the student, such that the mathematical concept is embedded in a real-life situation which exemplifies it.

Nickson and Green (1996) investigated the effects of context in the assessment of mathematical learning of 10 and 11 year-olds. They studied the cognitive levels at which context aids or obstructs pupil performance, and identified which elements of context contributed to these effects, and how. They identified 'operatives' in the form of one or two key ideas which pupils identify and use to solve the problem. The results supported the 'mental scaffolding' view that the context can provide the pupil with something with which to reason as well as a goal towards which to work.

The idea that real-world contexts can serve as 'mental scaffolding' for using mathematical concepts to solve problems is developed further in the Realistic Mathematics Education (RME) theory of learning mathematics, which proposes that mathematical ideas and concepts develop through the mathematisation of real-world situations (see Freudenthal, 1991, Treffers, 1987). The RME movement grew out of the ideas of Hans Freudenthal. As a professional mathematician, Freudenthal (1961) criticised the process whereby mathematical discoveries are 'sanitised' through a process of *anti-didactical inversion* :

> 'No mathematical idea has ever been published in the way it was discovered. Techniques have been developed and are used, if a problem has to be solved, to turn the solution procedure upside down, or if it is a larger complex of statements and theories, to turn definitions into propositions, and propositions into definitions, the hot invention into icy beauty. This then, if it has affected teaching matter, is the *didactical* inversion, which as it happens may be *anti-didactical*. Rather than behaving anti-didactically, one should recognise the learner is entitled to recapitulate in a fashion the learning process of mankind. Not in the trivial manner of an abridged version, but equally we cannot require the new generation to start just at the point where their predecessors left off.' (p. ix)

This recapitulation of the historical process of mathematical discovery relies on the notion of *guided reinvention*, which is an elaboration of a Socratic process using 'thought experiments' (Gravemeijer and Terwel, 2000 p. 786). Freudenthal sees the roots of abstract mathematical ideas as tools to organise the real world:

> 'Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world. *Phenomenology* of a mathematical concept, structure, or idea means describing it in relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind, and, as far as

> this description is concerned with the learning process of the young generation, it is *didactical phenomenology*, a way to show the teacher the places where the learner might step into the learning process of mankind.' (ibid)

Thus, as an example, Freudenthal analyses the phenomenology of 'length' by providing mathematical definition to the additive and multiplicative structure of the length concept, followed by a didactical phenomenology, which develops the concept through semantic analysis, invariance properties, congruence and similarity, rigidity and flexibility, distance, conservation and reversibility, and so on. It is possible to trace the provenance of these ideas in the Piagetian theory of development of logico-mathematical relationships; but Freudenthal develops these links between our physical experience and mathematical organisers in considerable detail. In order to give a flavour of this analysis, I quote a section on the didactical phenomenology of planes:

> 'I start with planes, or rather what one imagines to be infinitely extended planes. There are reasons why I do not bestow priority on lines – planes come earlier. First of all, in the topographical context, horizontal and vertical planes, floors, ceilings, walls, bottoms, covers. Among the oblique planes, the most striking are roofs, covers of chests and slides. Objects with faces can be bounded by oblique planes, depending on their position. Water in a vessel does not behave as a rigid body; its surface remains horizontal even if the vessels are inclined. (A glass with powder, beads, or peas behaves as though it were halfway between liquid and solid matter. Contrary to what Piaget claims it has nothing to do with logic but all to do with physics whether such a surface is horizontal or inclined and how much it is inclined).' (p.297)

Freudenthal emphasised the notion of mathematics as a human activity (Gravemeijer and Terwel, 2000). Rather than teaching mathematics as a process of abstraction, he believed in teaching *mathematising*, which he saw as a process of organising reality, for generality, certainty (proof), for exactness and for brevity (p. 781). Treffers (1987) makes the distinction between *horizontal* and *vertical* mathematisation, the former involving a process of converting a contextual problem into a mathematical problem, the latter taking mathematics onto a higher plane. Freudenthal characterises the difference thus (quoted in Gravemeijer and Terwel, 2000):

> 'Horizontal mathematisation leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one, symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly; this is vertical mathematisation. The world of life is what is experienced as reality, as is a symbol world with regard to abstraction. To be sure the frontiers of these worlds are vaguely marked. The worlds can expand and shrink – also at one another's expense.' (p. 782)

One might draw an analogy here between the RME concept of horizontal mathematisation and real-world contextual modelling. For example, different real-world embodiments of arithmetic and geometric series might be construed as horizontal mathematisations of the algebraic theory of sequences and series. However, one should be cautious here in equating 'real world' with 'reality': Freudenthal's notion of 'reality' is a relative one, which can encompass mathematics itself:

> 'I prefer to apply the term 'reality' to what common sense experiences as real at a certain stage.' (p. 783)

Gravemeijer (1997) gives a useful illustration of the difference between the type of realistic modelling employed in solving traditional word problems, and the organising activity implied by mathematising. The former is simply a process of transfer, where the context of the word problem is translated into mathematics, for example a division problem. The latter involves an active process of engagement with the context by the learner. For example, in a problem involving the division of 36 sweets between three girls, one student achieved a solution to this problem by allocating the sweets one by one to each child, whilst crossing them from an array of 36 sweets, until all the sweets were exhausted (p.395). This process of acting with the model – a *referential model* – is gradually reified and, at the general level, the student no longer needs to think of the model. The 'reified model' can then function as a model *for* mathematical reasoning:

> 'The distinction between a 'model of' and a 'model for' can be characterised as a distinction between a 'referential level' and a 'general level'. At the referential level, the model refers to the situation sketched in the problem statement. The model is meaningful to the student, because of this reference to a concrete situation. When the student gains more experience of acting with this model, the attention shifts from the original situation to the mathematical relations involved. The process of acting with the model is gradually reified, and, at the general level, the student no longer needs to think of the problem situation to give meaning to the model. This 'reified model' then can function as a *model for* mathematical reasoning.' (p 394)

The focus is therefore on modelling as an *organising* activity.

RME is not just a theory of learning. Freudenthal was instrumental in setting up the Institute for Development of Mathematics Education (IOWO) in Holland, now known as the Freudenthal Institute, to develop a curriculum along the principles of his theories of learning. This has exerted considerable influence over curriculum development projects in other countries, including the UK and US.

De Lange (1991) relates Realistic Mathematics Education research to constructivist theories of knowledge:

> ‘The theory of realistic mathematics education evolved after 20 years of developmental research which seems to be related to a constructivist approach. There are, however, some differences. The social constructivist theory is in the first place a theory of learning in general, while the realistic mathematics theory is a theory of learning and instruction, and in mathematics only. One of the key components of realistic mathematics education is that students re-construct or re-invent mathematical ideas and concepts by exposing them to a large and varied number of ‘real-world’ problems and situations which have a real-world character or model character.
>
> This process takes place by means of progressive schematisation, and horizontal and vertical mathematisation. Here, the students are given the opportunity to choose their own pace and route in the concept building process. At some moment abstraction, formalisation and generalisation takes place – although not necessarily for all students.
>
> After the process of conceptual mathematisation the newly developed concepts are applied and used in ‘real-world’ situations. This leads to reinforcement of the concepts and adjustment of the student’s real world. It goes without saying that (mental) construction and production play an essential role in realistic mathematics education, and it will come as no surprise that learning strands are intertwined and that student interaction is essential.’

De Lange then develops a theory of assessment which reflects the theory of instruction. He characterises three levels in assessment:

- a lower level, which concerns ‘objects’, ‘definitions’, ‘technical skills’ and ‘standard algorithms’;

- a middle level, characterised by ‘making connections’, ‘integration’ and ‘problem solving’;

- a higher level, encompassing tasks which involve mathematical thinking and reasoning, communication, critical attitude, interpretation, reflection, creativity, generalising and mathematising.

He goes on to define the roles of context, following Treffers and Goffree (1985), as concept forming, model forming, applicability (uncovering reality as a source and domain of applications), and exercise of specific abilities in applied situations. He discusses various ‘degrees of reality’ of a context, from ‘no context’, to ‘camouflage’ (zero order), in which context is used to ‘dress up’ the mathematical problem, and ‘relevant and essential’ (first order), tasks which use context in a real, or authentic, as opposed to artificial, way.

These ideas are developed into an assessment framework (De Lange, 1999), and an 'assessment pyramid' which embraces the dimensions levels of thinking, difficulty of questions, and domains of mathematics. De Lange sees contexts as playing a major role as a vehicle for assessing insight, understanding and concepts. He quotes Meyer's five roles for context as motivation, application, as a source of mathematics, as a source of solutions strategies, and as an anchor for student understanding. He defines the 'distance to students' of contexts, the closest being in private life, then school life, work and sports, then scientific contexts.

A useful distinction which De Lange makes here is between 'real' and 'virtual' contexts:

> 'It seems clear that when we emphasise mathematics education that will prepare our citizens to be intelligent and informed citizens, we have to deal with all kinds of real contexts. We have to deal with pollution problems, with traffic safety, with population growth. But does this not mean that we have to exclude artificial and virtual contexts? The answer is no, but we need to be aware of the differences for students.
>
> A virtual context contains elements that are not drawn from any existing physical, social, practical or scientific reality. They are of idealised, stylised or generalised nature. For example, if a stylised street layout of a city C is considered for an idealised traffic problem, it is only the labels 'street', 'city', 'traffic' that are real – the city, streets, and traffic are not real or authentic.
>
> An artificial context deals, for instance, with fairy tales – non-existent objects or constructs. This class of context is easier to separate from the real context and should be used with care. Students will not always be able to co-fantasize within this artificial setting or engage in a world that is clearly not real.'

The curriculum development work of the Freudenthal Institute has extended to post-16 courses in mathematics. Gravemeijer and Doorman (1999) describe a contextualised approach to calculus, which draws upon the historical development of calculus by guiding students to 'reinvent' concepts of instantaneous velocity and area under graph as displacement, using a problem posed by Galileo of calculating the distance travelled by a body travelling with constant acceleration. They refer to Sfard's (1991) characterisation of the history of mathematics as an ongoing process of reification in which processes are re-interpreted as objects.

How does RME relate to the question of RWCF in teaching and learning? As with constructivist theories of learning (Ernest, 1996), the RME model proposes that mathematical knowledge cannot simply be transmitted, through a process of definition, example and practice, to learners. Meaning needs to be constructed through a process, or negotiation, individual or social.

Some mathematics educators argue that the short, closed, timed written questions found in summative examinations tend to limit and proscribe this process. To quote from Goldin and Kaput (1996):

> 'Most often, the goals of instruction in mathematics are defined in terms of the type of problems we want students to be able to solve, or the particular skills and concepts we wish them to have. But these formulations of learning goals tend to limit the vision we bring to mathematics education. The reason for this is that such goals do not embody capabilities for spontaneous new constructions, for extension to unfamiliar situations, for synthesis of new strategies when necessary, or for creative mathematical acts.' (p. 425)

It could be argued that teaching approaches, such as RME, have no direct relevance to the issue of real-world contextual framing in assessment: teachers select teaching methods irrespective of the styles of summative assessment in external examinations. However, the short time span of the A/AS mathematics course (effectively 20 months), together with the high-stakes nature of the assessment, suggest that A/AS level courses are indeed, as Goldin and Kaput imply, defined in terms of the style of question posed in examinations (see, for example, Niss, 1993, Cockcroft, 1982 page 161). Emphasising mathematics as a human activity which can be used to organise real-world situations perhaps implies a real-world contextualising approach to assessment. However, the extent to which A/AS questions which use RWCF require the solver to engage in genuine modelling in RME terms is a question which is considered in more detail later in this study (see, for example, sections 6.4 and 9.1).

2.3 Real-world context and mathematical modelling

A growing area of research in mathematics education throughout the last 30 years has concerned the process and application of mathematical modelling (see, for example, Burkhardt, 1981, Lesh and Lamon, 1992, Niss et al., 1991, Lesh et al., 2010).

An early example of a book which advocates such an approach to the mathematics curriculum is Burkhardt (1981). This outlines a range of 'real-world' problems which can be tackled using a modelling approach, and discusses pedagogical approaches to introducing real-world modelling in the classroom. However, this book has only a little to say about assessment:

> 'The direct testing of particular skills in model formulation, interpretation and validation is possible, but research is needed to ensure that these are indeed crucial components in being a good, realistic applied mathematician. The ability to generate ideas, variables of relations when faced with a problem can be tested separately – the number and range of ideas are clearly separate factors. The ability to choose helpful and sensible lines of attack, to carry them through, to interpret answers, and to devise searching tests of a model's validity are all under study. As practical possibilities for assessment, these are for the future – for the moment it seems sensible to de-emphasise assessment and confine mainly to overall success with the problem.' (p. 104)

Burkhardt refers to a contemporary 'Mathematics Applicable' course developed at the time as an examination course under the direction of Ormell, a consistent advocate in the UK for a modelling approach to mathematics in post-16 classroom (see, for example, Ormell, 1972, Ormell, 1975, Ormell, 1991). In a volume devoted to teaching mathematical modelling (Niss et al., 1991), Ormell makes the case for a modelling approach (Ormell, 1991):

> 'The idea of looking for a 'new view' of mathematics which takes modelling capabilities fully into account is that we might establish a perspective which simultaneously increases the motivation of students and clarifies the substantial, long-term social purposes of the subject.' (p. 63)

Since the 1980s, a considerable literature has grown up on the design of suitable tasks and projects at all levels which are amenable to mathematical modelling,. Niss et al (1991) includes UK secondary school examples such as the Shell Centre's *Numeracy through Problem Solving Project* (Swan, 1991), the *Enterprising Mathematics Project* (Francis and Hobbs, 1991) and the Northern Ireland *Further Mathematics Project* (Houston, 1991), as well as a host of other projects from Portugal, Denmark, Holland, Italy, Austria, Germany and Hungary. The Shell Centre in Nottingham, UK have been a leading advocate of a modelling approach to mathematics, and examples of 'Balanced Assessment Tasks' are currently available through the Mathematics Assessment Resource Service (MARS, 2010).

In the US, Lesh and Lamon (1992) describe a number of *model-eliciting tasks* which require students to respond to open-ended task situations, and formulate mathematical models as solutions to these. This work in the US mirrors work in Holland (De Lange, 1999) and, following the publication of the influential Cockcroft Report (1982) in the UK, the development of extended coursework assessment in GCSE and A/AS level (Little, 1993, Brown, 1993).

This burgeoning of interest in modelling approaches to mathematics has, at the time of writing, spawned thirteen International Conferences on the Teaching of Mathematical Modelling and its Applications (ICTMA) (Lesh et al., 2010).

How does this body of research relate to real-world contextual framing? The term mathematical modelling is a term whose application is extremely wide, being used to characterise mathematical activity from elementary word problems to university-level modelling of complex real-world problems with sophisticated mathematical techniques. Ormell (1991) refers to the 'baffling complexity' of these different kinds of applications of mathematics, and outlines no less than twelve levels, ranging from everyday check-ups to meta-mathematics.

The use of real-world context in A/AS mathematics questions may be regarded as a form of mathematical modelling, albeit a relatively simple form. Kaiser and Sriraman (2006) classify five perspective on modelling in order to understand the inter-relations between different researchers and practitioners:

> A. Realistic or applied modelling (using authentic examples and concerned with understanding of the real world and modelling competencies);
>
> B. contextual modelling (with subject-related goals such as solving word problems or psychological goals such as fostering learners' motivation;
>
> C. educational modelling with a didactical or conceptual focus (the most popular approach looking at the structure of the learning processes and introducing new mathematical concepts, methods and principles;
>
> D. socio-critical modelling (promoting critical thinking about the role of mathematics in society);
>
> E. epistemological or theoretical modelling (promoting theory development).

Within this framework, it would appear that RWCF, as defined in this study, falls well within the compass of B above, as well as satisfying some elements of A. The A/AS level assessment objectives quoted in section 1.1 appear to require A/AS schemes of assessment to incorporate aspects of the modelling cycle, as described in Fig. 2.3.1, which is reproduced from the OCR MEI A/AS Specification. Flow diagrams, such as Fig. 2.3.1, describing the modelling cycle are a common feature of the mathematical modelling literature – see Burkhardt (1981) for early examples. Their evolution is discussed in detail in Haines and Crouch (2010).

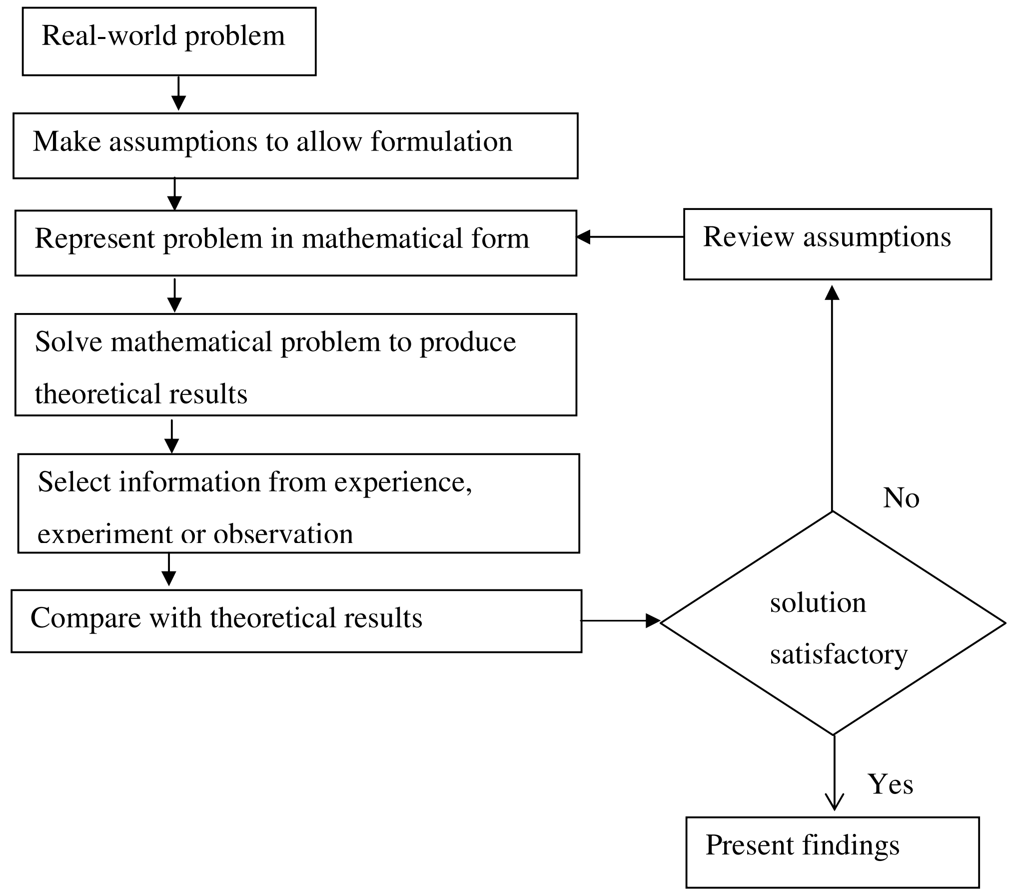


Fig. 2.3.1 Diagram of the mathematical modelling cycle

The time restrictions of examinations would seem to limit the extent to which candidates can engage in genuine mathematical modelling as outlined by this modelling cycle. In particular, the initial stages of the cycle – making assumptions and formulating the problem in mathematical language – would seem to be difficult to test using the conventional timed written paper items which are the subject of this research. However, it is pertinent to discuss the extent to which the utilisation of real-world contexts in such items encourages modelling skills. The issue of when, and indeed why, aspects of mathematical modelling became incorporated in A/AS Mathematics specifications will be taken up later in this study (see chapter 6).

2.4 Problems of artificiality and transfer of real-world contexts

Not all authors see the role of real-world context in problems as necessarily benign. In the UK, Wiliam (1997) has criticised what he sees as inappropriate use of context in mathematical tasks. He utilises Alfred Hitchcock's metaphor of a 'McGuffin'—a plot device primarily intended to motivate the action in a film, and to which relatively little attention is paid—to illustrate the use of 'realistic' contexts in mathematics education. He classifies contexts used for mathematics teaching into three kinds:

- contexts which bear little or no relation to the mathematics being taught, and which serve primarily to legitimate the subject matter ('maths looking for somewhere to happen');
- contexts having an inherent structure with elements that can be mapped onto the mathematical structures being taught ('realistic mathematics'); and
- contexts in which the primary aim is the resolution of a problem in which no particular (or even any!) mathematics need necessarily be used ('real problems').

He then cites examples of the first two types (1997, p.3):

Example A: Alan drank 5/8 of his pint of beer. What fraction was left?

I quote Wiliam's analysis of this item in full:

> 'Beer is not measured in eighths, and even when it is so measured, it is unlikely Alan would be thinking about the measurement when drinking. We have a particular task, which is 1 − 5/8, set in a particular context, but there is nothing about this particular context that might help the student identify an appropriate strategy. Again, the context here is, essentially, a McGuffin – the mathematics is 'looking for somewhere to happen'. The choice of this particular context, included as it is in a book entitled 'Mathematics at Work', seems to be used solely to convince low attaining fifteen- and sixteen-year-olds that this is 'real' mathematics, done by 'real' people. In a very real sense, the situation is a 'con'-text – a deception that the activity is worthwhile.'

Example B: Addition and scalar multiplication of matrices exemplified using a context (size and make of jeans, house number and variety of milk bottles).

Here, the context is used as a metaphor for establishing the mathematics: the real-world context is, to a certain degree, isomorphic with the mathematical structure. Wiliam points out, however, that this metaphor can lose plausibility when its range is extended to, for example, multiplication of a matrix and a column vector. Wiliam describes this isomorphism between context and mathematics thus:

> 'The primary use of the context here is not motivational, but structural. The *relevance* of the context is almost completely incidental to the choice of context. The most important thing about the context is that aspects of the structure of the real situation can be represented by the mathematical structure of matrices. In other words they are *metaphors* for the mathematical

> structure of matrices. Contexts are employed in curricular materials to capitalise on *scripts*, *schemas* or *frames* that learners already have in order to produce responses from the learner that converge on the desired mathematical responses.' (1997, p.7)

Wiliam's paper then goes on to discuss three attributes of this type of context or model:

- the *commonality*: the extent to which the script required to access the task metaphor is shared – universally, commonly or not commonly;
- the *match* between the structure of the task metaphor and the mathematics: certain attributes of the former will not match the latter;
- the *range*: how far the task metaphor gets you with the mathematics.

Wiliam concludes by pointing to an inherent and unavoidable tension in using contexts for assessing and teaching mathematics:

> 'If the teacher claims that a problem is 'real', then a student may come to 'own' the problem, but may well produce resolutions of the problem that are perfectly acceptable to the student, but involve no mathematics. Of course, this rarely happens, because, by virtue of their lengthy enculturation into the practices of mathematics teachers, most students know that the activity in which they are involved is a form of 'glass bead game', in which knowledge of the world outside the mathematics classroom is never needed, is hardly ever even useful and, as often as not, will lead one in the wrong direction.' (1997, p.9)

Boaler (1994), focusing on classroom practice, cites the following reasons for learning in context:

- to provide students with a familiar metaphor from concrete, familiar experiences to make learning more accessible;
- to motivate students, providing students with examples which enliven and enrich the curriculum;
- to enhance transfer through demonstrating links between school maths and real-world problems.

She claims that research, however, suggests that transfer does not happen, and that students do not perceive the links between the mathematics learned in school and problems in the 'real world'. She identifies the problem of context: students are required to engage partly as though the context in a task were real whilst simultaneously ignoring factors pertinent to the 'real-life' version. Students are required to enter what Wiliam (1992) has christened 'mathsland':

> 'Over the last eight years, I have visited a lot of maths classrooms, and it seemed to me that in most of them, it was as if there were a kind of check-in desk just outside the classroom door labelled 'common sense', and as the pupils filed into the classroom, they left their common sense at the check-in desk saying 'Well, we won't be needing *this* in here.' (p 3)

Boaler provides the following critique of the contexts which are used:

> 'Contexts are often used in order to provide meaningful situations which students can learn and generalise from when often they are not perceived by students as anything with which they can identify. The learning which results from such situations is completely tied to the specificity of the situation and often forgotten when students go through the classroom door.'
>
> 'Contexts can aid transfer if students examine and reflect upon the underlying structures and processes which connect experiences. Good or model contexts can encourage this type of thinking (Treffers, 1987); unreal, textbook contexts in short atomistic questions are likely to suppress and devalue it.' (p 5)

Boaler's study involved offering questions to students from two schools, one from classrooms which encouraged an investigative, process-based approach, and another from classrooms which used more traditional 'content' based methods. She found that the former students were more successful in overcoming problems of transfer (for more on 'situated' learning, see Lave, 1988).

How 'real' contexts appear to students may also be gender-related. Boaler (1994) reported on a small-scale study of contextualised assessment items in the classroom, the results of which suggested that some girls did better on questions set in a football context compared to items in a fashion context. She reasoned that this may be because girls' knowledge of fashion was, in fact, distracting them from the intended responses. She goes further, suggesting that the use of artificially-contrived contexts to develop mathematical content may contribute to a lack of transfer of mathematical skills to solving real-world problems. She compares this to a process-rich classroom methodology using contexts in a more natural, open-ended way, suggesting that a more active engagement with practical real-world tasks may lead to a more effective ability to transfer mathematical knowledge and techniques to real-world contexts (Boaler, 1993a).

Boaler (1993b) points to the inconsistency of performance in tasks using different real-world contexts:

> 'The degree to which the context of a task may affect students' performance has, for many years, been widely underestimated. When context *is* recognised as a powerful determinant, misconceptions still prevail such as the belief that mathematics in an 'everyday' context is

> easier than its abstract equivalent and that learning mathematics in an everyday context can ensure transfer to the 'everyday' lives of students. Lave (1988) has suggested that the specific context within a mathematical task is capable of determining not only general performance but choice of mathematical procedure.' (p.1)

In the same paper, Boaler distinguishes three meanings to the phrase 'learning in context', one as the use of context in mathematical examples in described situations, one in which students formulate a real-world problem and pursue this as a more extended task, and thirdly to describe the general environment in which students learn mathematics, including the room, the people in it, the mathematical examples and the student's overall goal structures. She cites socio-mathematical and ethno-mathematics research as being influential in recognising the importance of context, ownership and subjectivity in the development of 'mathematical meaning':

> 'Two concerns are raised by the discussions of folk and ethno mathematics. One reflects the need to acknowledge that the mathematics classroom is itself a place of values with its own cultural and value perspectives. The second is the need to acknowledge the the 'cultural' solutions offered by students in the real world are also mathematical. This mathematics is a part of students' social and cultural lives and the social and cultural is a part of their experience of the mathematics classroom.' (p.11)

In recognising the wider social context of the classroom, Boaler emphasises the diversity of individuals' responses to real-world contexts:

> 'Consideration of the context of a task, activity or example suggests that students do not perceive school mathematics tasks as 'real' merely by the coating of a real world 'veneer', yet their mathematical procedure and performance can be largely determined by the context used. This suggests that students interact with the context of a task in many different and unexpected ways and that this interaction is, by its nature, individual. Students are constructing their own meaning in different situations and it is inappropriate the assume a generality of familiarity or understanding in presenting students with a 'context'.' (p.11)

The authenticity of real-world contexts in mathematics is the subject of critique by Cumming and Maxwell (1999). They characterise *authentic achievement* as

- production of knowledge instead of reproduction;
- disciplined enquiry, dependent on an a priori knowledge base, in-depth understanding, integration;
- value beyond assessment - aesthetic, utilitarian, personal.

Authentic assessment should focus on achievement of these authentic learning outcomes. They too warn against the danger of using real-world contextualisation to 'camouflage' more traditional forms of assessment, referring to its usage in mathematics:

> 'Camouflage occurs when a traditional form of assessment is 'dressed up' to appear authentic, often by the introduction of 'real-world' elements or tokenism. The most flimsy are usually found in maths and problem solving.'

A number of US state assessment programmes have attempted to implement the ideas of authentic assessment, using a variety of assessment tools, (for more on this, see Pandey, 1990).

Palm (2009), in proposing a theory of authentic task situations, comments that terms such as 'authentic' are used differently by different authors:

> 'The meaning of terms like "really real", "realistic", and "authentic" tasks differ between authors, are sometimes vaguely defined, and are sometimes not clarified in a publication at all. There is also a lack of frameworks to guide research and synthesize research results, which may be one of the reasons for the lack of synthesized research results in this area.' (p. 5)

Palm analyses real-life situations by considering aspects which he considers important in their simulation using mathematical tasks:

- *Event*.

  Is the event likely to have taken place? For example, picking coloured marbles from urns occurs commonly in mathematical tasks, but rarely in the real world;

- *Question*

  Is there concordance between the assignment in the school task and the corresponding out-of-school situation?

- *Information / data*

  This explores the existence, realism and specificity of the information and data in the task.

- *Presentation*

  This refers to the way the task is presented to students, e.g. orally or in written form;

- *Language use*

  This considers the language used in the task, and compares the linguistic demands made with the language used in the out-of-school situation.

- *Availability*

What is the match between solution strategies available for solving the school task and those available in the practical situation?

- *Circumstances*

The social context in which the problem is solved, including such aspects as availability of external tools (calculator, map, ruler, etc.), guidance, consultation and cooperation, discussion opportunities, time and consequences of success or failure.

- *Solution requirements*

Discussion of solution methods and final answers.

- *Purpose in the figurative context*

The appropriateness of the answer when compared to the purpose of the task.

Palm hypothesizes that the representativeness of mathematical simulations of real-life situations to students is correlated with the similarity between their behaviour when dealing with the in- and out-of-school task situations: the higher the representativeness of the simulation is, the larger will be the proportion of students that make proper use of their real-world knowledge when working with a word problem.

2.5 Research on UK public examinations in mathematics

Considerable research has been undertaken in the UK on examination comparability and standards (see, for example, Bramley, 2005, Fitzgibbon and Vincent, 1994, Pollitt et al., 2000, Pollitt and Ahmed, 2001, Pollitt et al., 2007, Ahmed and Pollitt, 2007, Quinlan, 1995, Fisher-Hoch et al., 1997, McLone and Patrick, 1990). Some of this work has a direct bearing on the use of real-world context, and is referenced elsewhere in this study.

The UK Qualifications and Curriculum Development Authority has published a comprehensive study (Newton et al., 2007) of comparability techniques, which summarises work on examination standards conducted on public examinations. The use of real-world contexts in public examinations raises the question of how such contextual framing affects the difficulty of examination questions.

Pollitt, Ahmed and Crisp (2007), in their paper from this publication, make the distinction between examination *demand* and *difficulty*. 'Demand' is essentially a concern pre-test, and attempts to judge the difficulty of questions before they are operational; difficulty, on the other hand, is defined and analysed post-test, after the examination has been conducted, and based on data on students' responses. They illustrate the difference using a question from the Third International Mathematics and Science study, which was answered correctly by 75% of Scottish children, but only 59% of English children. In this case, the demand of the question was the same for each group, but the difficulty was different, due to differences in classroom experiences prior to the test being taken. In practice, it is impossible to establish a theoretical basis for assessing 'demand', and most studies rely upon the judgment and experience of examiners to establish this.

Notwithstanding this body of research, there appears to be little research into quantitative evidence of the effect of using real-world contexts in mathematics questions. To what extent is the 'demand' of questions affected by the introduction of real-world context? This question is explored later using a comparative study of questions framed with and without context (see chapter 8).

2.6 Relevance to the post-16 context

Most of the research reviewed so far in this chapter has related to pre-16 school mathematics, and, for this reason, its relevance to post-16 mathematics needs consideration. Clearly, simple word problems in primary school involving the selection and application of four-rules arithmetic are quite different in character and purpose to real-life modelling in a post-16 curriculum.

Cooper and Dunne's sociological analysis, and their findings that students from different social backgrounds respond to contextualised items in different ways, may have relevance to the post-16 curriculum and assessment regime, though it is likely that older students will be more successful, experienced and therefore better versed in the 'assessment game'.

How relevant is the idea of real-world context as 'mental scaffolding' in questions at post-16 level? One might speculate that this might have greater relevance to younger children tackling mathematical problems, and that the sort of formal mathematical thinking required of students engaged in a post-16 mathematics course might require them to detach the mathematical modelling from the context.

Similarly, it could be argued that an RME approach to teaching mathematics might be more relevant to the development of fundamental concepts in mathematics such as number, ratio and scale, and might require modification at later stages in mathematical development. Nevertheless, as pointed out earlier, some work has been done in applying RME principles to post-16 mathematics (Gravemeijer and Doorman, 1999). It is also possible to distinguish the two distinct deployments of context suggested by Gravemeijer in the sixth-form classroom where, for example, the exponential function can be taught as a *model of*, for example, population growth, whilst other differential equations might be proposed as *models for* a population's growth or decay in time.

De Lange's (1991) 'distance to students' of contexts would also appear to depend on their educational attainments, for example in science. The range of real-world contexts which become amenable to mathematical modelling may presumably become greater at later stages in schooling. Similarly, the range of mathematics which students are familiar with, such as trigonometric, exponential and logarithmic functions, and calculus, may have a bearing on the level of mathematical modelling which are able to bring to bear on real-world problems.

De Lange's (1999) 'higher-level' contexts seem to approach a more genuine form of mathematical modelling in the post-16 curriculum, as alluded to in the growing body of research literature on mathematical modelling reviewed in section 2.3. However, most of this research refers to longer, extended, open-ended projects which require students to formulate models, mathematise the problem by choosing from a variety of mathematical approaches, and evaluate the mathematical solutions. It is debatable whether timed, written examination questions deploy real-world context in ways which may be classified as higher-level in De Lange's sense.

The criticism by Wiliam, that some contexts are artificially selected in order to motivate mathematics through real life, needs to be considered in a post-16 context, where students have positively selected to study mathematics, not as part of a statutory curriculum for all, but because they enjoy the subject in its own right, or have been convinced of its utility through exposure to applied mathematics such as statistics and probability or mechanics. Classroom experience suggests that many students welcome real-world application, and are often put off by pure mathematics, and fail to understand its relevance.

How far does Wiliam's classification of usage of context apply to post-16 mathematics? Is it possible to identify similar degrees of 'McGuffinism' in the deployment of context in more advanced questions? Is the metaphorical usage of context he identifies in elementary mathematics identifiable in more advanced mathematics? These questions are explored in this thesis in relation to advanced level questions.

Wiliam's third category is 'real problems' in which the primary aim of the context is to resolve a problem in which no particular mathematics need necessarily be used. In relation to assessment items for A-level mathematics, one would expect contexts to be used which precipitate the use of mathematical methods which are to be assessed and, to this extent, the aim of the context is not a utilitarian, extra-mathematical one. However, is it possible to set problems in contexts in which mathematics is deployed in a quasi-authentic fashion, which uses mathematics in a way which approaches or simulates its applications in the real-world?

2.7 Summary

The research literature described above indicates a number of possible functions and effects of real-world context in both the learning and assessment of mathematics. On the one hand, as proposed by RME, real-world context might play a crucial mediating role in learning abstract mathematical concepts. Assessing mathematics through real-world contexts may serve the function of scaffolding mathematical concepts by placing them in meaningful everyday schema. Real-world context may emphasise the utility and applicability of mathematics for solving real-world problems. They may introduce students to some elements of mathematical modelling.

On the other hand, poorly adapted use of real-world context might undermine the validity of the assessment, by introducing uncertainty and potential sources of misunderstanding in the solution of the questions. It may be unfair to certain classes of student, for example those whose knowledge of English might hinder their understanding of the context. Far from motivating students by showing genuine applications of mathematics, RWCF might have the opposite effect if the context appears artificial, or there simply to camouflage the mathematics. How does the RWCF reflect the aims of authentic assessment, or achieving authentic learning outcomes?

How can one reconcile these contradictory positions on the function and effect of real-world contextual framing? How do they relate to A-level mathematics questions? These are fundamental questions to be addressed in this study.

When used in public examinations, the issue of comparability between attainment on questions with and without RWCF becomes relevant. What is the effect of introducing RWCF into questions? Are they more difficult than ‘pure’ mathematics questions?

Finally, there is the issue of the extent to which RWCF encourages mathematical modelling skills, as mandated by the current assessment objectives of A/AS mathematics. The extent to which these objectives can be satisfied by the short, written A/AS examination questions is the issue considered in this study.

CHAPTER 3

REAL-WORLD CONTEXTUAL FRAMING: THEORETICAL CONSIDERATIONS

Overview

The literature review conducted in chapter 2 suggests a dilemma concerning the use of real-world contextual framing (RWCF) in mathematics questions. Real-world context, on the one hand, might play a mediating role in the teaching and learning mathematics, and as such would be expected, perhaps, to be part of its assessment. On the other hand, it might be said to compromise the validity of assessment items, through inappropriately 'camouflaging' the mathematics, failing to address practical modelling considerations, or leading the solver into misconceptions. This chapter sets out the theoretical notions which are used to analyse this dilemma in more detail.

At the outset, it is important to clarify, as far as possible, what is meant by real-world contextual framing, and this is tackled in section 3.1. In section 3.2, the theoretical basis of the study is laid down in terms of concepts from assessment theory, in particular Messick's unified theory of construct validity. Section 3.3 uses this theoretical basis, together with the ideas developed from the literature review conducted in chapter 2, to propose an evaluative framework for considering RWCF in examination questions.

3.1 What is real-world contextual framing?

Before considering the theoretical ideas and concepts relevant to the rationale, degree, function and effect of using real-world contexts in A/AS mathematics questions, it is important to clarify, as far as possible, what is meant by *real-world contextual framing* (RWCF).

Figures 3.1.1 and 3.1.2 repeat the initial example, quoted at the start of this thesis, of two questions, the solutions of which are identical, which are expressed with and without real-world context.

An arithmetic progression has first term 7 and common difference 3.

(i) Which term of the progression equals 73?

(ii) Find the sum of the first 30 terms of the progression.

Fig. 3.1.1 Arithmetic progression question without real-world contextual framing

Chris saves money regularly each week. In the first week, he saves £7. Each week after that, he saves £3 more than the previous week.

(i) In which week does he save £73?

(ii) Find his total savings after 30 weeks.

Fig. 3.1.2 Arithmetic progression question with real-world contextual framing

In the version in Figure 3.1.1, the language used, ('arithmetic progression', 'common difference', 'term', 'sum', 'progression) may be identified as from the mathematics 'register' (Pimm, 1987): their meaning is defined through mathematical, rather than common-sense, convention and negotiation. In the second version, the language utilises an everyday context of money and saving. Although some mathematical terms are used (for example whole numbers, 'more than', 'total'), the same mathematics is framed in terms of a narrative involving 'Chris', 'saves', 'weeks'. In both parts of this version, a 'real-world' context, namely finance and savings, is used to frame the mathematics.

The use of the phrase 'real-world' here presupposes that a distinction can be made between a 'real world' and a 'mathematical world'. The 'reality' of mathematical concepts is a problem of mathematical philosophy which is beyond the scope of this study to consider in detail. It suffices for this study, in determining the meaning of the term 'real-world contextual framing', to be able to classify questions according to whether it is present or absent.

The distinction between questions with and without RWCF can, in a number of respects, be rather subtle. In Fig. 3.1.3, the mathematical content (arithmetic progressions) is expressed through a secondary context, that of spirals. The issue this raises is whether this constitutes a real-world context.

A spiral is formed with sides of lengths 7 cm, 10 cm, 13 cm, …

which are in arithmetic progression:

(i) How many sides does the spiral have if its longest side is 73 cm?

(ii) Find the total length of the spiral with 30 sides.

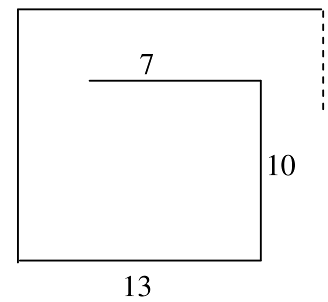


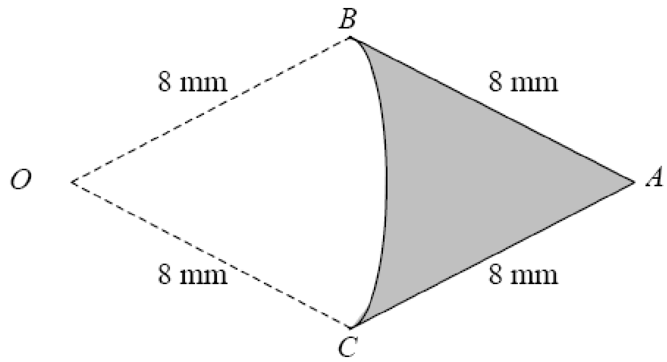
Fig. 3.1.3 Arithmetic progression question - spiral version

In a sense, the term ‘spiral’ may be defined as a geometrical - and therefore mathematical - object, similar to ‘circle’, ‘square’, ‘straight line’, etc. Nevertheless, the context suggests a spatial ‘realisation’ of an algebraic concept (viz arithmetic progression). The relationship between ‘spirals’ and ‘arithmetic progressions’ is indirect, and requires the solver to engage in a process of transfer from one context to another. For the purposes of this study, ‘pattern’ contexts like this one are classified as ‘real-world’ contexts. This is because the transfer from one mathematical concept to another is taken as equivalent to transfer from a ‘real-world’ context to a mathematical context.

Now consider the question in Fig. 3.1.4. Here, the question is presented predominantly in the mathematical register. However, a real-world context is hinted at through the word ‘badge’. There seems to be a qualitative difference in the way context is deployed in this question compared to that in Fig. 3.1.2: the word ‘badge’ could be replaced by ‘shape’ without changing the mathematical task. In this example, the real-world context serves the function of suggesting an image for the shape, but there is no requirement for the solver to match the context with the mathematics, or to model knowledge of ‘badges’ through mathematics. Although the diagram, as in the ‘spiral’ example (Fig. 3.1.3), provides a picture, this is integral to understanding the content of the question, and does not introduce a secondary context that requires a process of transfer. I shall therefore discount this type of question from the analysis of RWCF.

In a somewhat similar way to Fig. 3.1.4, Fig. 3.1.5 utilises the names of two students in an integration task. It can be surmised that the purpose of this is to humanise the content of the

5. **Figure 1**



The shaded area in Fig. 1 shows a badge ABC, where AB and AC are straight lines, with $AB = AC = 8$ mm. The curve BC is an arc of a circle, centre O, where $OB = OC = 8$ mm and O is in the same plane as ABC. The angle BAC is 0.9 radians.

(a) Find the perimeter of the badge.

(2)

(b) Find the area of the badge.

(5)

Fig. 3.1.4 ‘Badge’ question (Edexcel)

question. However, the ‘real-world context’ is superficial, in the sense that it has no effect on the task, and the question is therefore classified as without RWCF.

6 Two students are trying to evaluate the integral $\int_1^2 \sqrt{1+e^{-x}}\,dx$.

Sarah uses the trapezium rule with 2 strips, and starts by constructing the following table.

x	1	1.5	2
$\sqrt{1+e^{-x}}$	1.1696	1.1060	1.0655

(i) Complete the calculation, giving your answer to 3 significant figures. [2]

Anish uses a binomial approximation for $\sqrt{1+e^{-x}}$ and then integrates this.

(ii) Show that, provided e^{-x} is suitably small, $\left(1+e^{-x}\right)^{\frac{1}{2}} \approx 1 + \frac{1}{2}e^{-x} - \frac{1}{8}e^{-2x}$. [3]

(iii) Use this result to evaluate $\int_1^2 \sqrt{1+e^{-x}}\,dx$ approximately, giving your answer to 3 significant figures. [3]

Fig. 3.1.5 Superficial real-world context (MEI)

To sum up the above discussion, it is possible to formulate a working definition of real-world contextual framing. For the purposes of this thesis, real-world contextual framing (RWCF) refers to *questions which utilise, within their narrative, either an everyday real-world context, or an unfamiliar context, which requires a process of translation to the primary mathematical content*. There may, in practice, be some overlap between the domains of 'everyday' and 'mathematical' language, and it is likely that the degree to which the everyday context impinges on the mathematical content of the question may vary from question to question.

3.2 Theoretical assessment framework

In the preceding section, I clarified what is meant by real-world contextual framing in mathematics questions. In this section, I outline the key ideas from assessment theory that are applied in this study, and indicate how these ideas might relate to the A/AS Mathematics curriculum, and, more specifically, the use of real-world contexts.

3.2.1 Primary purposes of assessment

Traditionally, assessment theory subdivides assessments according to their primary purposes (Marsh, 2004 p. 51)

Formative assessment is used to describe assessment (and testing), predominantly carried out during a course of study, to assist student learning. It is often characterised as assessment *for* learning (Black and Wiliam, 1998). For example, formative assessment might be a teacher providing feedback on a student's work, and would not necessarily be used for grading purposes.

Diagnostic assessment measures a student's current knowledge and skills for the purpose of identifying a suitable program of learning, or in order to inform a teacher's teaching.

Summative assessment is generally carried out at the end of a course or project. In an educational setting, summative assessments are typically used to assign students a course grade. Such assessment (or testing) is sometimes characterised as assessment *of* learning (Black and Wiliam, 1998).

In practice, assessment can serve both a formative and a summative purpose. The focus of this research is A/AS pure mathematics examination questions that are designed to be summative in nature, and that contribute to the grading of candidates. However, questions set in A/AS examinations are frequently used in A/AS courses with a formative purpose: questions therefore have an important 'backwash' effect in the classroom, contributing to candidates' perceptions of the subject. As timed written papers form the basis of all assessment schemes for A/AS level mathematics, the predominant aim of teaching and learning is to enable students to achieve success in answering examination questions.

For this reason, the nature of examination questions and, in particular, the extent to which they are framed in real-world contexts, plays an important role in determining or influencing the *construct* A/AS mathematics.

3.2.2 Validity

Quality in assessment is conventionally considered using the concept of *validity*. All assessment requires the selection of specific items, and measuring outcomes on these items. Validity measures the extent to which one can generalise from these results and draw justifiable conclusions. Wiliam (2007) gives the example of a spelling test comprising 20 items, selected from 40 words written on the board the day before the test. Anita spells all 20 correctly, and Robin spells 10 out of 20. Can we conclude that Anita is better at spelling than Robin?

> 'Anita may have got 20 out of 20 on the test because she is a good speller in general, or she may have carefully prepared for the test by working very hard to learn the spelling of those 40 words. Some conclusions are warranted on the basis of the results of the assessment, and others are not. The process of establishing which kinds of conclusions are warranted and which are not is called *validation* and is, quite simply, *the* central concept in assessment' (Wiliam, 2007 p 125)

Wiliam then goes on to trace the evolution of the concept over the last 50 years. Until the 1980s, different aspects or notions of validity were distinguished, in particular *content-related validity* and *criterion-related validity*.

Content-related validity refers to the extent to which an assessment measures what it claims to be measuring. In the case of A/AS level, this would be the mathematical content specified by the syllabus. *Criterion-related validity* refers to the extent to which an assessment succeeds in measuring a stated criterion. This can itself be classified as *concurrent*, or *predictive*, depending upon when the criterion is to be measured. For example, an examination at age 11 may be designed to assess students' ability to benefit from selective secondary education (predictive validity); a short test of dyslexia may be designed to assess the degree of dyslexia before referral to a one-to-one interview (concurrent validity). Similarly, A/AS level is designed to measure students' ability to succeed with a course in higher education.

Wiliam then outlines the development of a third notion of validity, *construct validity*:

> 'For many years, these two forms of validity, content validity and criterion-related validity dominated thinking about how to validate assessments. However, the validation of some forms of assessment, particularly in the area of personality psychology, didn't fit easily into either category. For example, if we have a questionnaire that was meant to measure someone's neuroticism, how could we check this? There is no clearly defined domain of questions that we could draw from, nor is it clear that neuroticism predicts anything. For that reason, interest focused on a third kind of validity - *construct validity.*'

Construct validity is determined by measuring its correlation with other measures designed to measure the construct (*convergent* evidence) and its relative lack of correlation with measures of differing constructs (*divergent* evidence). Over time, construct validity was seen to include and subsume the two other forms of validity, and Messick developed a unified theory of validity:

> 'Although there are different sources and mixes of evidence for supporting score-based inferences, validity is a unitary concept. Validity always refers to the degree to which empirical evidence and theoretical rationales support the adequacy and appropriateness of interpretations and actions based on test scores.' (Messick, 1989, p. 13)

In A/AS Mathematics, therefore, the construct validity of the assessment may be taken to measure not just the extent to which examination questions are seen to assess the mathematical content outlined in the syllabus, or fulfil the technical assessment objectives specified, but also wider aspects of predictive validity, such as success with higher-order skills required in university degree courses.

Messick points out that construct validity is not a property of a test or assessment, but rather a property of the ways in which test data are used or interpreted:

> 'The emphasis is on scores and measurements as opposed to tests or instruments because the properties that signify adequate assessment are properties of scores, not tests. Tests do not have reliabilities and validities, only test responses do. This is an important point because test responses are a function not only of the items, tasks or stimulus conditions but of the *persons* responding and the *context* of measurement. This latter context includes factors in the environmental background as well as the assessment setting.

> As a consequence, the social psychology of the assessment setting requires careful attention, as do relevant aspects of the examinee's environmental background and experiential history. For example, is the presented task meaningful to the respondent in terms of prior life experience, or does it assume a special or different meaning in the assessment context.' (p. 14)

Thus, for example, while a test might be perceived to have a certain content-related validity, its construct validity may differ from one group of students to another. To quote from Wiliam (2007):

> 'If we have a history test that has a high reading demand, then how do we make sense of the results? For students with good reading skills, we might reasonably conclude that low scores on the test indicate that these students don't know much about history. But for students with poor reading skills, we don't know whether low scores mean poor history knowledge, poor reading skills, or both.'

Messick also points out that tests can serve different purposes:

> 'Tests often do double duty, as in the example of an essay examination on the causes of the Civil War, which provides samples of writing and thinking as well as signs of analytical and critical skills. In educational measurement, the latter are usually acknowledged as *constructs* or inferences about underlying processes or structures, whereas the former often are not. This leads, as we shall see, to considerable contention over what makes for a valid score.' (Messick, 1989, p.15)

In the case of RWCF, questions set in real-world contexts can be seen in a similar way to serve the double purpose of providing samples, for example, of algebraic skills and processes, and of a modelling 'construct'.

3.2.3 Requirements for validity

Although there is no such thing as 'test validity', in the sense that tests in themselves are inherently valid, there are nevertheless certain requirements for a test which can enhance or jeopardise the validity of test outcomes.

Reliability

A reliable test is one in which the score that a student gets on different occasions, or with slightly different questions, or with different markers, does not change very much. It can be argued (Ahmed and Pollitt, 2007) that this criterion for validity applied to A/AS level mathematics questions might be jeopardised by the introduction of real-world context, which adds a dimension of variability to students' responses. However, it is equally possible to argue that tests which contain questions which are relatively routine variations of each other from one assessment to another may be prone to 'teaching to the test', so that problem solving capability becomes a learned outcome (Little and Jones, 2008).

Ensuring the construct is adequately represented

If problem solving is an element of the construct A/AS mathematics, then, as argued above, this may be inadequately represented by a test consisting of relatively routine questions. Equally, if an ability to translate real-world situations into mathematical model is part of this construct, then this would imply that some questions should be presented using real-world contextualisations.

There is therefore always a tension between schemes of assessment which, as in the case of A/AS mathematics, contain assessment objectives which may require a range of assessment methods such as coursework, and the necessity to maintain a high level of reliability.

Eliminating irrelevant factors

If, in contrast, the assessment includes factors which are deemed to be irrelevant to the construct, then this can introduce an element of *construct-irrelevant variance*. For example, if real-world scientific contexts are used in questions which require students to have specialised scientific knowledge not required by the mathematics syllabus, then this would jeopardise the validity of the question.

Wiliam (2007) points out that the relationship between reliability, construct under-representation and construct-irrelevant variance is a complex one. He uses the analogy with stage lighting:

> 'For a given power of illumination, we can either focus this as a spotlight or a floodlight. The spotlight brings real clarity to a small part of the stage, but the rest of the stage is in darkness. This is analogous to a highly reliable multiple choice test, in which the scores on the actual matter tested are highly reliable, but we know nothing about the other aspects of the domain that were not tested (construct under-representation). A floodlight, on the other hand, illuminates the whole stage. We may not be able to make quite such accurate distinctions in the small part of the domain assessed by the multiple choice test, but what we can say about the other areas will be *more* accurate. However, if the floodlight is cast too wide, we will illuminate parts of the theatre, such as the orchestra pit, that we did not want to illuminate (construct irrelevant variance).' (p. 131)

3.2.4 Consequential basis for validity

Another important aspect of construct validity is to consider the consequences of testing. Messick therefore divides construct validity into two facets, its evidential basis and its consequential basis:

> ‘The consequential basis of test interpretation is the appraisal of the value implications of the construct label, of the theory underlying test interpretation, and of the ideologies in which the theory is embedded. A central issue is whether or not the theoretical implications and the value implications of the test interpretation are commensurate, because value implications are not ancillary but, rather, integral to score meaning. Finally, the consequential basis of test use is the appraisal of both potential and actual social consequences of the applied testing.’ (Messick, 1989, p. 20)

Thorndike, commenting on Messick’s unified theory, puts the case for consequential validity thus:

> ‘Historically the central question in test use has been ‘Does the test do what it is employed to do, does it serve its intended purpose?’ Messick argues that the *intended* outcomes of testing do not, in and of themselves, provide sufficient justification for a particular test use. To assess the functional worth of testing in a certain context, we must consider *all* of its effects – intended and unintended, positive and negative – to determine whether or not a proposed test use is justified. This is especially true if adverse consequences have a basis in test invalidity.’ (Thorndike, 2005 p. 187)

How does this relate to RWCF in A/AS Mathematics questions? These, in addition to their summative role in providing evidence of achievement in mathematics, for example for the purpose of gaining access to higher education, also play a major role in influencing teaching and learning in the classroom. Thus, in addition to considering the evidential basis provided by the results of examinations, an equally important aspect of their construct validity is considering the consequential validity of these assessments in the classroom context. Harlen puts it as follows:

> ‘What is assessed, and how, will always have an impact on teaching. The impact can be positive if the assessment coves the full range of intended goals, when the assessment criteria often help to clarify the meaning of the goals. However, the impact on learning experiences can be restrictive if there is a mismatch between the intended curriculum and the scope of the assessment. The consequences are likely to be more severe when results are used for accountability of teachers and schools. It is these uses that raise the ‘stakes’ of pupil assessment and lead to summative assessment having a narrowing influence on the curriculum and teaching methods.’ (Harlen, 2007, p. 145)

In the UK, A/AS results are increasingly used to assess not just students, but teachers and schools. The consequential basis of evidence from A/AS mathematics assessments therefore needs to be considered in assessing the validity of real-world contextual framing.

3.2.5 Construct fidelity

So far, the theoretical concepts discussed have derived from general assessment theory, which is designed to apply to all forms of assessment. However, some research has been done which adapts these notions to questions used in UK public examinations. Ahmed and Pollitt (2007) , in their study of context in GCSE science questions, define the *construct fidelity* of examination questions in terms of whether the students' minds are doing things the examiner wants them to show they can do.

> ‘Anything that reduces the examiners' level of control over the process occurring in students' minds when they are answering a question will get in the way of measuring what we want to measure. Setting questions in real-world contexts is therefore a threat to validity. The effects of a real-world context on the processes that occur in students' minds when they are answering a question are in some ways unpredictable: a context will have different effects on different students since it will differ in familiarity to them. It is therefore much more difficult for examiners to be in control with a contextualised question, and much harder to say that we are measuring understanding of a particular topic.’

While Ahmed and Pollitt acknowledge that real-world context in science or mathematics may assess students’ abilities to apply their knowledge to new, real-life situations, and serve to make abstract concepts more concrete, relevant and motivating, they argue that this is more important in teaching than in high stakes assessment, where test reliability is the over-riding concern.

They identify the following factors as added demands of context:

Language - we would not wish to penalise poor reading skills in a science exam;

Familiarity - varies from student to student. They might have difficulty in using their everyday knowledge of the context rather than the science. Unfamiliarity with the context may put students off answering the question. Some students confuse context with content.

Attention - irrelevant information requires students to select what is relevant.

Ahmed and Pollitt define the degree of *focus* of the context as the extent to which the most salient features of the context correspond to the main issues in the question. A focused content helps to activate relevant concepts rather than interfering with comprehension and scientific thinking. It will bias students into thinking about the context in the right way.

They investigated this by varying the degree of 'focus' in variations of science questions, and found that questions were in all cases improved when placed in more natural and focused contexts, designed to provoke the same schemas in the students' minds as the science or mathematics in the question.

3.2.6 Summary of key theoretical ideas

I now summarise the key theoretical notions discussed in this section, and their relationship to the research questions.

- A/AS Mathematics examination questions serve a *summative* role, in the evidence they provide for certification of students for purposes of university entrance and more generally, careers requiring evidence of mathematics attainment;
- A/AS Mathematics questions also serve a *formative* role, through their extensive use in the mathematics classroom during teaching sessions;
- In assessing the function of real-world contextual framing in such questions, it is necessary to assess the *construct validity* of the evidence from assessments used in A/AS mathematics. This is a concept which unifies all aspects of validity, such as content-related and criterion-related validity;
- The *reliability* of assessment relates to the consistency of evidence from assessments. This may be affected by the deployment of real-world context in questions;
- Validity may be jeopardized by *construct under-representation*. Thus, if real-world contextual framing plays an important role in assessing part of the agreed A/AS Mathematics construct, then removing it from questions might influence the validity of the assessment.
- Yet, validity may equally be affected by the introduction of *construct-irrelevant variances* to the assessment, in other words factors which affect the evidence of the assessment but which are not part of the A/AS mathematics construct, for example, specific knowledge of real-world contexts, or facility with the English language.
- The *consequential validity* of the assessments needs to be considered. What are the effects of real-world contextual framing on the curriculum? What are the social

effects? Are its effects on A/AS students beneficial to their perception of mathematics?

When used as summative assessments contributing to high stakes national qualifications, additional considerations apply to the validity of assessment:

- Are questions accessible to candidates in timed written examinations? Is the language, familiarity and focus of the context in the questions appropriate?

3.3 Towards a theoretical framework for evaluating the validity of RWCF

Silver and Herbst (2007) distinguish between three types of theories, *grand theories*, *middle-range theories* (which concern subfields of study), and *local theories* (which help to mediate specific connections among practices, research and problems). Palm (2009), in proposing what he describes as a local theory of authentic task situations (see Section 2.4), quotes Lester (2005), who suggests that:

> '…we should focus our efforts on using smaller, more focused theories and models of teaching, learning and development. This position is best accommodated by making use of conceptual frameworks to design and conduct our inquiry. I propose that we view the conceptual frameworks we adopt for our research as sources of ideas that we can appropriate and modify for our purposes as mathematics educators.'

In a similar fashion, I propose a 'local' theoretical framework which is designed to address the issues of real-world contextual framing as outlined in chapter 2.

The first issue raised in the literature review in chapter 2 was that of misunderstandings and misinterpretations of real-world contexts in questions. If real-world contextualisation were to mislead solvers, or cause misinterpretations of questions, then this would be a possible source of construct-irrelevant variance, and a consequential threat to validity. Here, Ahmed and Pollitt's (2007) notion of *construct fidelity* appears to be relevant. With regard to real-world context, one might surmise the following factors which might be the cause of misinterpretation:

(a) Language and comprehensibility

If the linguistic demands of the question are too great, then the question might no longer test the mathematical construct, but instead test the linguistic abilities of students. This would then add a source of construct irrelevant variance and threaten validity. Similarly, given the examination context, it may be that excessive wordiness in questions might detract from it comprehensibility.

(b) Familiarity with the context

If real-world contexts are used which require specialist knowledge outside the mathematical subject domain, then this again would appear to threaten validity. On the other hand, the use of routine contexts, or no real-world contexts at all, might threaten validity by under-representing the A/AS Mathematics construct, which requires the assessment of students' ability to translate the real-world into mathematics.

(c) The match between the real-world context and the mathematics

Solving questions utilising RWCF would appear to require a matching process between the real-world context and the mathematics. This matching process would seem to be relevant to the difficulty of the question, and is therefore a factor in assessing validity.

Clearly, there may be an element of judgment necessary in assessing the linguistic, comprehension and translation demands of questions. For example, if a student misinterprets a question presented in a real-world context, then this does not automatically render the question invalid, if the requirement to interpret the question is part of the construct. On the other hand, this misinterpretation might be caused by faults in the design of the question, which would indeed threaten its item validity.

In order to apply these ideas to the analysis of questions, they can be fruitfully be combined into a notion of *accessibility*, which attempts to assess the extent to which language, familiarity and translation affects the demands of questions using RWCF.

An issue raised in chapter 2 was that of the artificial use of contexts, which may be summarised by a lack of authenticity or realism. If the real-world context used simply camouflages the mathematics, then this may affect its consequential validity in the classroom, by undermining the perception of mathematics as a useful and practical subject.

Consider the criticisms levelled both by Wiliam (1997) and Cooper and Dunne (2000) of the following Key Stage 2 question shown in Fig. 3.3.1.

The criticisms concern the artificiality of modelling the real-world context of a lift using division with remainder. Clearly, the model is not realistic: 269 people are not going to patiently wait for 20 lifts and, moreover, it is unlikely that the lift will be filled to full

This is the sign in a lift at an office block: In the morning rush, 269 people want to go up in this lift. How many times must it go up?	**This lift can carry up to 14 persons**

Fig. 3.3.1 Key Stage 2 SAT 'lift' question

capacity each time. However, if the task is simply an invitation to find how many full lifts are required to take up 269 passengers, then the modelling is valid enough. In other words, as a metaphor for division, a lift with a number of passengers would appear to be appropriate and not unreasonable.

A further example, Fig. 3.3.2, taken from an A-level paper, illustrates the issue of realism. Here, the lack of realism of the model in (b) is perhaps less overt than in the 'lift' example, until one solves the differential equation, and finds that the model predicts a stain of area of 100 m^2 in 5 seconds! Does this result threaten the validity of the item? This would seem to be worthy of further study. Thus, the second aspect of real-world contexts as deployed in questions is *realism*: how *realistic* is the mathematical modelling implied by the question?

There is, however, another aspect of authenticity which is worth distinguishing from realism: notwithstanding the realism of the context, is the *task* itself relevant to the context? On the one hand, in the 'lift' question (Fig. 3.3.1), the question asks the solver to find out something which is relevant to the context, namely the number of lifts required. On the other hand, in the 'stain' example, the task in part (b) requires the solver to solve a differential equation, but this result is not used meaningfully within the context itself. Indeed, a request to calculate the area after 5 seconds, and then comment on the result, might be a way of improving the authenticity of the task.

I therefore propose a third notion when considering the validity of questions using RWCF, which I call *task authenticity*. This measures the degree to which the task set relates meaningfully to the real-world context, and the extent which the result is evaluated which regard to the real-world context.

8. A circular stain grows in such a way that the rate of increase of its radius is inversely proportional to the square of the radius. Given that the area of the stain at time t seconds is A cm^2,

(a) show that $\frac{dA}{dt} \propto \frac{1}{\sqrt{A}}$.

(6)

Another stain, which is growing more quickly, has area S cm² at time t seconds. It is given that

$$\frac{dS}{dt} = \frac{2e^{2t}}{\sqrt{S}}.$$

Given that, for this second stain, $S = 9$ at time $t = 0$,

(b) solve the differential equation to find the time at which $S = 16$. Give your answer to 2 significant figures.

(7)

Fig 3.3.2 AS Mathematics ‘stain’ question (Edexcel)

A further aspect which would seem to be relevant to the authenticity of the task is the intrinsic interest of the real-world context to the solver. One can surmise that tasks which provide answers to potentially interesting questions would appear to be more authentic to solvers. The consequential validity of such tasks in the classroom would also appear to be enhanced by this.

The concepts of accessibility, realism and task authenticity defined above clearly relate to aspects of mathematical modelling discussed in section 2.5. However, it would perhaps be unrealistic to expect short, examination questions to test the full mathematical modelling cycle. The extent to which such questions might fulfil the requirements of AOs 4 and 5 quoted in Section 1.1 might be indirectly measured by considering a sample of questions using these notions.

I now propose a ‘local’ (Palm, 2009) theoretical framework based on the above ideas. I call this the ARTA framework (see Fig. 3.3.3) to be used as an evaluative tool for analysing the construct validity of questions. This consists of a checklist of questions which is applied in considering the role of real-world context in questions (Little and Jones, 2007). The framework is used in my study to analyse qualitatively a sample of questions using RWCF.

The qualitative application of the ARTA framework on questions provides a theoretical tool

Accessibility

- How accessible is the script implied by the real-world context?
- How familiar is the real-world context to students?
- How accessible is the match between the structure of the real-world context and the mathematical model?
- How explicit is the match between the real-world context and the mathematical model?
- How accessible is the language used in the question?
- What comprehension demands are made of candidates in explaining the real-world context?
- How many words are used?

Realism

- How realistic is the real-world context used?
- How realistic are the assumptions made when applying the mathematical model to the real-world context?
- How realistic is the data generated by the mathematical model?

Task **A**uthenticity

- How authentic is the task in relation to the real-world context?
- Does the task include evaluation of the appropriateness of the model?
- How interesting is the task to the solver?

Fig. 3.3.3 The ARTA Framework

for evaluating the function and effect of real-world context. However, it is also necessary to back this up with assessment data. For example, is there evidence that real-world context makes questions harder, or, by providing a 'model to think with', perhaps easier? Such a question can be decided by collecting data on how questions are answered. It is also germane to enquire about students' perceptions of real-world context in questions. Do they believe that RWCF enhances the validity of questions? These methodological issues are considered in the next chapter.

Another potential function of RWCF, which emerged from the analysis of chapter 2, is that of the formative role which real-world contexts might play in reifying mathematical concepts, and the potential function of providing a 'mental scaffolding' for thinking with in mathematics questions. These aspects of real-world context may be seen to add to the potential for consequential validity by enhancing students' understanding of mathematical concepts. On the other hand, in post-16 mathematics at least, it could be argued that the content validity of questions, for example those which require students to translate from the real world to an algebraic model, and use algebraic techniques to solve the question, might be undermined if the 'mental scaffolding' supplied by the context bypasses the need to mathematise the problem algebraically. This issue is taken up in later chapters of this study.

CHAPTER 4

METHODOLOGY

Overview

The focus of the last two chapters has been establishing the theoretical background to the study by considering theories of teaching and learning mathematics and assessment. The aim of this chapter is to outline and discuss the methods to be used to address the research questions.

Anderson (1998) classifies educational research methods as being of eight types: historical, descriptive, experimental, correlational, qualitative, program evaluation, case study, policy research and organisational evaluation. In investigating the rationale, scope, function and effect of real-world contextual framing, this chapter presents the argument for adopting a range of methods, considered to be appropriate to the nature of the enquiry.

Table 4.0.1 summarises the types of method this study adopts in order to investigate the research questions developed in chapter 1.

RQ1: What has led to the introduction of real-world context and mathematical modelling in A-level mathematics?	Historical, descriptive
RQ2: To what degree are 'pure' mathematics questions in A/AS level examinations capable of being framed within real-world contexts, and what is the nature of these contexts?	Descriptive, experimental
RQ3: What functions and effects are served by real-world contextual framing (RWCF) of pure A-level mathematics questions?	Descriptive, experimental, qualitative

Table 4.0.1 Research questions with types of method

As Chapter 2 indicates, there appears to be a relative lack of research directly relevant to these questions, and the research methods adopted are therefore not directly dictated in advance by earlier studies. Nevertheless, a number of studies suggest possible approaches, and these are discussed in this chapter.

The structure of the chapter is as follows: the first three sections discuss possible methodological approaches to each of the three main research questions, and report on the methods adopted in this study; section 4.4 discusses ethical issues raised by the choice of methods and collection of data; finally, section 4.5 summarises the methods used to collect the data for this study.

It is convenient to divide the research findings of the study into three parts. Part I deals with the origins and degree of RWCF; part II considers the function and effect of RWCF by developing an evaluative framework for analysing examination questions; finally, part III continues the investigation of function and effect by using the results in part II to develop tests, together with a questionnaire, which are administered to a sample of students, and analysing the results.

4.1 Research methods for Part I: the origins and degree of RWCF in A/AS-level mathematics

Part I of the research considers the origins and degree of real-world contextual framing in A/AS Mathematics. A number of different methodological approaches to investigating this question are possible. One approach might be to consider appropriate contemporary sources of data. For example, it would be possible to survey opinion from QCDA and examining board personnel, Principal Examiners from different examining bodies, as well as classroom practitioners.

The introductory chapter quoted the assessment objectives which appear to underlie the 'official' rationale for including real-world modelling in A-level mathematics specifications. A 'contemporary' approach of interviewing current or recent personnel, however, would fail to provide a historical context for how and why current practice has become established. Moreover, as criteria for approval of A/AS specifications are centralised through QCDA, such an approach would not be likely to adequately explain variances in practice between different specifications, and how these have arisen.

A survey of current practitioners would establish the rationale for selecting the approach to context adopted by one syllabus over another. However, these rationales, or pre-dispositions, are likely to have their roots in the educational experiences of the teachers surveyed, which were influenced by forces of curriculum development in the subject since A-level mathematics was established in 1951.

Another methodology which would shed a different light on the question is comparative education theory. Cummings (2003), in his comparative study of the educational development, outlines the educational traditions of six core nations, and claims they provide templates for the development of educational traditions throughout the world. If this were the case, one might hypothesise that these cultural traditions have resulted, or at least influenced, current public examining practices in different countries. This leads naturally to the question of the extent to which real-world contextual framing is manifested in examinations internationally. However, such an approach, while interesting, would likely constitute a substantial study, which is beyond the scope of one component of the present study.

McCulloch and Richardson (2000) make the case for investigating contemporary educational issues historically:

> '(Historical research) can illuminate the structures and the taken for granted assumptions of our contemporary world, by demonstrating that these have developed historically, that they were established for particular purposes that were often social, economic and political in nature, and that in many cases they are comparatively recent in their origin.' (p. 6)

This indicates that in order to develop an understanding of why real-world contextual framing has become the widespread practice in some contemporary A-level pure mathematics papers, it is important to consider its historical roots, and this is the approach adopted in this study.

A comprehensive study of the development of A-level mathematics would require a range of sources of primary and secondary data; for example:

- studies of curriculum development in mathematics;
- official government reports;
- scrutiny of syllabuses and textbook materials;
- reports of conferences which influenced curriculum change.

Such a study would likely provide an interesting doctoral thesis in its own right. However, as the question is but one component of the study (which is primarily to investigate function and effect), it is not feasible to conduct such comprehensive historiographic research.

In investigating the roots of RWCF, I use a number of key secondary sources, as outlined in the literature review, each of which uses a different methodological approach. Cooper's study of the roots of the mathematics curriculum changes of the 1960s, *Renegotiating Secondary School Mathematics* (Cooper, 1985), adopts a sociological theoretical position. Cooper attempts to capture the reasons for the radical changes to the mathematics curriculum in the 1960s through analysis of key conferences, which enabled the principal actors to energise and muster resources. His analysis includes focusing on the composition of attendees at these conferences, and their status and professional background, and quotations for the proceedings which he perceives to have galvanised action for change, and in particular the influence of two curriculum projects, SMP and MME. He also traces the influence of teaching bodies such as the Mathematical Association and the Association of Teachers of Mathematics.

Another source, Griffiths and Howson's *Mathematics: Society and Curricula* (1974) was based on the experience of the two authors in developing a course on curriculum development theory for undergraduates. Howson, as a key member of the original SMP team, was responsible for editing the influential original texts, and is therefore in a key position in commenting on the 'modern mathematics' movement of the 1960s.

Two secondary sources of data relate directly to SMP. Bryan Thwaites's account of the development of SMP, *The School Mathematics Project: the First Ten Years* (1972), includes annual reports of the project, and a commentary from the prime initiator of the project. *Challenges and Responses* (Howson, 1987) is a volume of essays reflecting on the contributions of SMP to curriculum development.

In order to provide some triangulation and test the case for the influence exerted by curriculum development projects in the 1960s onwards, I use some primary data in the form of examination papers and sample questions. These examples are drawn from the Archives of Cambridge Assessment, and my own archive of question papers. In particular, syllabuses and examinations drawn from one source, the University of Cambridge Local Examinations Syndicate (UCLES), are sampled at ten-year intervals (1951, 1961, 1971, 1981, 1991 and 2001), in order to determine developments in the curriculum and styles of question. The ethics of this approach is discussed in Section 4.5.

Unlike the analysis provided by accounts such as Cooper, Howson and Thwaites, examination papers may be classified as official documents whose purpose is not associated with social theory, and may be regarded as *unobtrusive measures* (Jupp, 1996), defined by Denzin as follows:

> 'An unobstrusive measure of observation is any method of observation that directly removes the observer from the set of interactions and events being studied.' (p. 299)

Bearing in mind the plethora of different A-level syllabuses throughout its period of evolution, the above sample is necessarily highly selective. However, given the status of the UCLES syllabus as a 'traditional' syllabus which has been influenced by the success of the SMP A-level syllabus, it might be claimed that this represents a 'case study' which would need further research to confirm its generalisability to other A-level syllabuses.

My study investigates the degree of real-world contextualisation in A/AS-Mathematics examinations, by a detailed analysis of papers from two A/AS specifications. A number of studies have sought to make comparisons between different A-level syllabuses. These have included (see Pollitt et al., 2007) using panels of ‘experts’ to make qualitative judgements, rating specific demands or aspects of demand. For example, Christie and Forrest (1980) used experienced examiners to scrutinise archived scripts from A-level papers from 1963 and 1973, to make judgements of the standard of these papers. Quinlan’s comparability study of 1994 A-level Mathematics syllabuses (Quinlan, 1995) utilised three strands, a statistical review, a syllabus review and a cross-moderation exercise, conducted by experienced scrutineers from the different examination boards. The syllabus review, following methodology developed by McLone and Patrick (1990) consisted of a question review, in which questions were rated according to demand in interpretation, structure, intermediate/final answers, routine processes and manipulation, and a question paper/mark scheme review, which used assessments of syllabus coverage, evenness of demand, formulae booklet, user friendliness, year-on-year demand, time demand and mark scheme. It is perhaps surprising, given the different extent to which real-world context is used in questions from these different syllabuses reported in Chapter 5, that this was not one of the factors considered in McLone and Patrick’s study.

These types of study start by scrutinising questions from past papers, and I adopt this approach in evaluating the degree of RWCF in A/AS examination questions. My initial assumption, based on experience as a teacher and examiner, is that there are quantitative differences in the degree of contextualisation used in different syllabuses. In order to test this, the study selects a sample of pure mathematics papers from two current specifications from the same examining body (OCR), identifies questions in which a real-world context is mentioned, and counts the number of such questions, and the total number of marks allocated to these questions. As discussed in section 3.1, some questions do mention a real-world context, but superficially, and it was decided to exclude these from this analysis.

As well as establishing a difference in the number of questions utilising RWCF in these two syllabuses, my research aims to determine what types of syllabus content appear to be amenable to RWCF. The questions with RWCF from each syllabus are therefore categorised according to their syllabus content, in order to investigate this question. The set of mathematical content considered by this approach is effectively determined by the content categorised as ‘pure mathematics’ in the two syllabuses studied; however, this is largely prescribed by QCDA for all A/AS specifications. There has been some change in this content over the period of existence of the A-level Mathematics qualification, although this, unlike the content of applied mathematics within the qualification, has not changed radically.

The methods outlined above could be extended to include other specifications, and backwards in time by analysing papers from earlier A-level Mathematics syllabuses. They may then be used to establish real-world context as a discriminating factor between two classes of A-level syllabus, which may loosely be described as 'modern', and 'traditional', deriving from the modern mathematics movement of the 1960s. If extended to a consideration of mathematical content, such an enquiry would reinforce the understanding of the relationship between mathematics and the real world, and could be extended to consider the nature of the difference between 'pure' and 'applied' mathematics, and their overlap. Such a larger-scale study is beyond the scope of this research.

4.2 Survey of existing research methods on context

Before discussing the methods adopted in parts II and III of this study (to investigate the functions and effects of RWCF), I consider the range of methods adopted by existing research to investigate context.

Verschaffel et al (1994) administered pencil-and-paper tests to large groups of 11-13 year old pupils. The tests comprised Standard (S) problems, which could be properly modelled and solved by a straightforward application of one or more arithmetic operation, and Problematic (P) modelling problems, where realities of the context call into question a routine solution of the problem. Similar studies (Verschaffel L et al., 1997) have explored the responses of teachers in training to similar tasks.

Ahmed and Pollitt (2000), in a paper studying context in action, used video recording of students while answering questions, with immediate playback to prompt recall of their though processes. The same authors, in a study of contextual 'focus', (2007) adapted three science and one mathematics question, all of which were originally developed for national science assessments at age 14, into three or four versions, with varying degrees of contextual 'focus', which, together with other common questions, were made up into 12 different versions of a test, so that each version of the manipulated questions occurred in a paper with every other version of the other questions. This was given to a sample of 405 children in year 9 (aged 14) from two comprehensive schools. Data from the tests was subjected to a Rasch analysis. In addition, fourteen students were interviewed in pairs immediately after the test.

Another study, albeit a small scale one, in which facility levels of differently contextualised versions of essentially the same mathematics are compared is that of Shannon (2007).

However, no details of the sample are given in this study, and the conclusions drawn are given in general terms only.

Boaler's study of different contexts (1994) utilised a sample of 50 students from two schools, comparable in socio-economic terms, but with different pedagogical approaches. The students were given a test comprising six questions. Three of these tested equivalent fractions in different contexts, one an abstract calculation, one in a football context (number of penalties scored out of number taken in a season) and one a 'plants' context (fraction of plants grown out of seeds planted). The other three were abstract, a wood-cutting, and a fashion context.

Nickson and Green's (1996) study on the effect of context in tasks for 10 and 11 year old pupils identified five elements of context, and developed parallel sets of contextualised and non-contextualised questions. They then interviewed 10 pupils who had taken part in trialling of the versions. The aim of the interview analysis was to identify ways in which the contexts may have intervened in the problem solving process.

Vappula and Clausen-May (2006) utilise a similar parallel-version method with a large sample of 1795 primary pupils. In their study, parallel versions of arithmetic questions were developed and trialled, for example fraction questions with and without a graphic, and subtraction questions with and without a real-world context, and percentages of successful students from years 6 – 9 were compared. Fig. 4.2.1 gives examples of parallel test items.

Cooper and Dunne's (2000) major study utilised a sample of 136 primary and 473 secondary school pupils from three primary and secondary schools. In addition to completing national assessment test items, all the primary school and one third of the secondary school pupils were interviewed, some more than once. Pupils were classified by social class (service, intermediate and working). Their study contains a detailed analysis of a number of the test items.

They describe their basic strategy as follows.

> 'We have employed both quantitative and qualitative methods. The basic strategy has been to use initially statistical analysis of children's performance on items in test situations to generate insights concerning broad classes of test items, e.g. items that embed mathematical operations in 'realistic' and 'esoteric' contexts respectively. This has involved coding items on a number of dimensions. These have included type of contextualisation, 'wordiness', difficulty levels, attainment target, type of response required, and use of pictorial

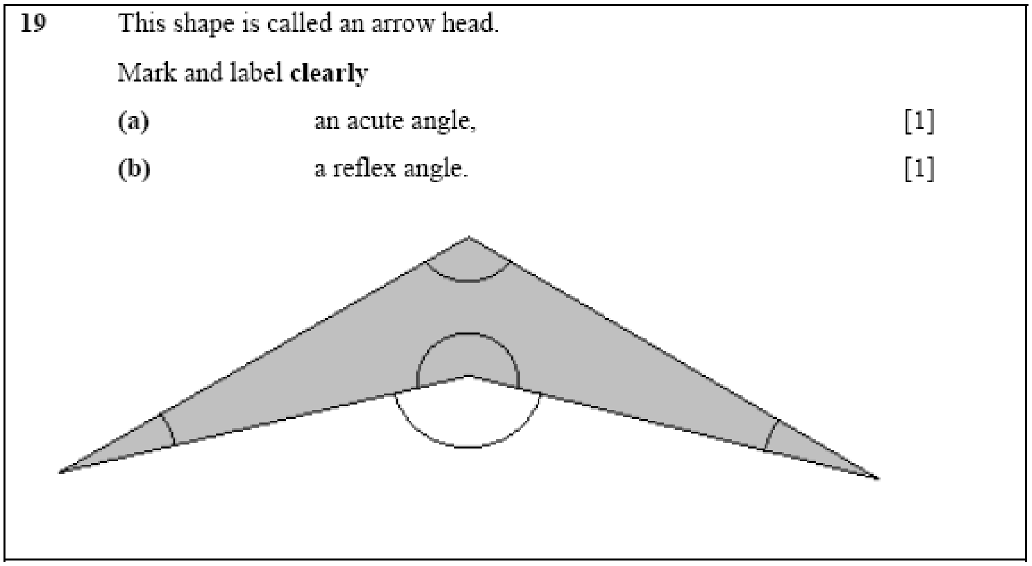
$\frac{3}{5}$ of this rectangle is shaded.  What fraction is not shaded?	Mrs Jenkins' car must be serviced after it has gone 6000 miles. Service after 6000 miles The car has gone 2369 miles. How many more miles can it go before it is serviced?
$1-\frac{3}{5}=$ ______	6000 – 2369 = _____

Fig. 4.2.1 Parallel test items from Vappula and Clausen-May (2006)

> representation. Analyses of the relationships between social class, gender, measured 'ability', item type and performance have been carried out. Some of these use the child as the case for analysis, others use the item itself. Alongside this approach we have used more qualitative analyses of children's responses to particular items in both the tests and the subsequent individual interviews to generate understanding of *why*, for example, 'realistic' and 'esoteric'[1] items seem to be differentially difficult for children from particular socio-cultural backgrounds.' (p.12)

Fisher-Hoch et al (1997), in an investigation of GCSE examination difficulty in subjects (including mathematics), analysed results from a sample of 600 scripts from a GCSE Mathematics syllabus (SMP 11-16), to identify and codify Sources of Difficulty (SoDs). These were then used to manipulate questions into alternative versions. For example, Fig. 4.2.2 shows an original question, in which candidates had to mark already marked angles, and a manipulated version, in which the angles are not marked.

[1] defined respectively as items involving 'everyday' objects and people or not.

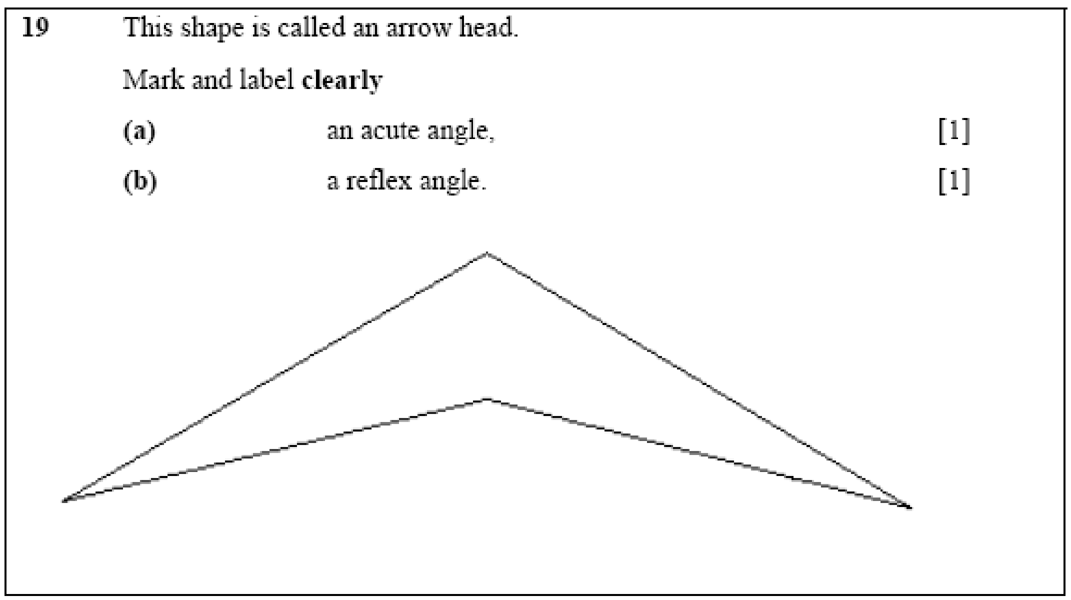

19 This shape is called an arrow head.

Mark and label **clearly**

(a) an acute angle, [1]

(b) a reflex angle. [1]

Fig. 4.2.2a Original version of an SMP 11-16 GCSE question (Fisher-Hoch et al (1997))


19 This shape is called an arrow head.

Mark and label **clearly**

(a) an acute angle, [1]

(b) a reflex angle. [1]

Fig. 4.2.2b Manipulated version of an SMP 11-16 GCSE question (Fisher-Hoch et al (1997)

Results of the different versions were analysed, and changes in facility calculated.

In a study of authentic assessment, Maclellan (2004) used interviews with academic staff to investigate the extent to which higher education tutors' perceptions of assessment were consistent with the construct of authenticity.

Summarising the methods used in these studies of context, it is clear that the predominant methodologies for investigating the degree, function and effect of real-world context is the analysis of tasks, manipulation of questions into parallel versions, and analysis of performance on these items, and interviews or questionnaires to explore more qualitative aspects, such as authenticity, misinterpretations and attitudes.

4.3 Research methods for Parts II and III: the functions and effects of RWCF

My starting point for part II of the study, following Cooper and Dunne's (2000) codification of national test items, is to reflect upon and analyse a sample of recent examples of A/AS questions, using theoretical ideas developed from research and theory. These ideas are used to develop an evaluative framework, which is used as a unit of analysis for these questions, together with a 'task analysis', which outlines the relationship between the real-world context and the mathematics used to model this context. Although this analysis is essentially subjective, the 'word-to-mark ratio', calculated by dividing the number of words by the number of marks, is used to give a quantitative measure of the density of words in each question.

The initial sample of questions used for evaluation are selected across the complete range of mathematical content amenable to RWCF. In order to test the effect of real-world context, the focus of the investigation then narrows to one topic, namely arithmetic (AP) and geometric (GP) sequences, selected for its amenability to a variety of approaches, both with and without real-world context. The study then evaluates a larger sample of AP and GP questions, and classifies them into four broad types, 'explicit', 'mathematical', 'word' and 'pattern'.

As the design of part III of the study builds upon the analysis in part II, it is not possible to give a complete account of the methodology adopted in part III without forestalling the results of part II. A more detailed account of the methodology of this final part of the study is therefore given in Chapter 9. For the moment it is possible to say that the focus changes from scrutiny of the questions themselves to the performance of students in solving questions with and without real-world context. The categorisation of AP and GP questions conducted in part II enables the construction of parallel questions, one of each type, but with the same, or similar, solutions. Questions are then compiled into four parallel test papers, each of which contains one version of each question. After piloting these tests, they are given to a suitable sample of students, large enough to enable robust statistical comparisons to be made from the results of each version of the questions. The analysis calculates mean scores for each part question in its four versions, and uses two-tailed difference of two means tests to compare these. Questions are then re-analysed, and the results accounted for using the theoretical ideas and results developed in part II of the research.

Although this AP/GP study (reported in Part III) generates data on the effect of real-world context on responses to questions, it does not provide insight into the more qualitative and affective aspects of the research topic. One approach favoured by some of the above studies would be to interview students in order to develop a clearer understanding of how context affects the solution process, and how students themselves feel about real-world context and its use in questions. However, the studies which have utilised this method have often been concerned with misunderstandings of questions, or interpretation, of context. In my study, it is not possible to conduct sufficient interviews to generate the large-scale data set necessary to establish the effect of real-world context statistically. This is not to exclude the use of interviews with both students and teachers in follow-up work in order to provide evidence for findings of this study, especially those which have arisen from a subjective analysis of question content.

Specific questions which arise in the assessment of AS/A level mathematics relate to the issue of modelling. Do the models utilised in these questions appear to be realistic or authentic? Do they motivate the mathematics? Would students prefer to be assessed using context –free questions? More generally, do students prefer applied mathematics to pure mathematics?

In order to survey students' opinions on these issues, the study uses a short questionnaire to be completed soon after the AP/GP test. This questionnaire was adapted from a similar design to one which probed student's opinions of coursework assessment in A-level Mathematics (see Little, 2007).

Cooper and Dunne's (2000) study reported social class as a factor in responses to pre-16 national assessments. In this study, it is conjectured that social class might be less of a factor in post-16 assessment than skills of comprehension, which might differentially affect the accessibility of contexts. Some pilot work on responses to a comprehension paper suggested that this might be an issue worth exploring. For this reason, the questionnaire asks students to declare whether English is their first or second language, in order to investigate whether this might affect their opinions on RWCF. Clearly, this question does not give an entirely scientific measure of English language skills, but it was considered, given a sufficiently large sample of students, that it would provide some useful initial data in investigating this question. Evidence provided by this method would, however, require further, more detailed, research to become an authoritative research finding.

Another aspect of students' opinions on real-world context which emerged after analysis of the questionnaire data is that these appear to be gender dependent. Although this was not anticipated when the questionnaire was constructed, the analysis by gender provided an interesting, and unexpected research finding, which is discussed in Chapter 9.

4.4 Ethical issues

This section discusses ethical issues relevant to the study.

Cohen et al (2007) propose the following set of initial considerations to be addressed in planning research:

- informed consent
- gaining access and acceptance in the research setting
- the nature of ethics in social research
- sources of tension in the ethical debate
- problems and dilemmas confronting the researcher, including matters of privacy, anonymity, confidentiality, betrayal and deception
- ethical problems endemic in particular research methods
- regulatory ethical frameworks, guidelines and codes of practice for research
- sponsored research, and responsibilities to the research community (p.51)

This research, as it is conducted under the auspices of the University of Southampton School of Education, is required to satisfy the procedures laid down by the school. This includes the completion of an Ethics Protocol Guidance Form. A copy of the completed form is provided in Appendix 6.

(a) Informed consent

This, according to Cohen et al (p.52), involves four elements: competence, voluntarism, full information and comprehension.

In pursuing this study, permission was sought from Examination Boards to reproduce examination questions. These are governed by copyright laws. Access to UCLES archives was sought through the archivist and director of research. Schools and colleges participating in the AP/GP study were sent an initial letter (see Appendix 7) to the Head of Department, or personal responsible for mathematics, outlining the nature of the research and the study and seeking permission for their students to undertake the test. This made clear that participation was entirely voluntary. A more detailed pack of information, with procedural instructions, and a copy of the student questionnaire, test versions and mark schemes, was sent to participating centres. Thus, informed consent was sought from competent professionals, who contributed voluntarily, and with full information and comprehension of the aims of the research.

With a sample of over 500 individual students, it was judged not have been feasible to seek individual consent from each student. Aside from its research purposes, the test was considered to be appropriate as revision material for AS level Mathematics, and therefore a useful additional resource in its own right. However, students were given the option to opt out of the test and the questionnaire. Prior to the test, teachers were asked to read out the following statement to students:

> 'This test, as well as helping you to revise for AS Maths, will be used for research purposes, to improve our understanding of testing maths in examinations. Any data from your participation will be stored securely and will not be divulged to anyone outside the research team in a way that might identify you. However, if you have particular reasons for not wishing your work to be used in the study, you should write the word 'object' on your script. Your test will then not be forwarded to the researcher.'

With regard to information and comprehension, participating institutions were informed of the purposes of the research in general terms. It was hoped that this would be of interest to teachers participating in the research, and indeed might stimulate some debate about the issues. However, the research was not directed primarily at classroom practice, but theory of assessment, and it was not judged to be necessary or desirable to discuss the research in detail, prior to reporting results to participating institutions.

Certain information was sought in the student questionnaire. In particular, students were asked if English was their first language, in order to determine whether this factor affected results, through additional demands of comprehension placed on students in contextualised questions. The ethics of this were considered: it was felt that alerting students to the rationale for this question might interfere with results, and that answering such a question was not a source of sensitivity for students.

(b) Gaining access and acceptance

As Cohen et al point out (p. 55)

> 'Investigators cannot expect access to a nursery, school, college or university as a matter of right. They have to demonstrate that they are worthy, as researchers and human beings, of being accorded the facilities needed to carry out their investigations.'

The researcher in this study was able to use contacts built professionally as a former head of mathematics in schools and colleges, a Principal Examiner for A/AS level mathematics, and through writing, curriculum development and professional development activities, both locally and nationally. This facilitated access to local colleges, and to UCLES archives in Cambridge.

The anonymity of participants, both at school / college and individual student level, when reporting the research was guaranteed.

(c) The nature of ethics

Again, quoting Cohen et al (p. 58)

> 'What ever the specific nature of their work, social researchers must take into account the effects of research on participants. Such is ethical behaviour. Indeed, ethics has been defined as 'a matter of principled sensitivity to the rights of others, and that 'while truth is good, respect for human dignity is better' (Cavan).'

If, for example, the AP/GP test was found to be too difficult or demanding for average A/AS students, then completing this test might be a negative experience for students. Piloting with a small sample of students in one college suggested that the level of the test was appropriate, and would indeed help students in preparing for their AS examination.

(d) Sources of tension

One source of tension proposed by Cohen et al (p. 58) is between a belief in the value of free scientific enquiry in pursuit of truth and knowledge, and belief in the dignity of individuals and their right to those considerations that follow from it. On consideration of the nature of the research in this study, it would not seem to be an issue here, as it is not likely that individual dignity would be jeopardised by research into assessment of this kind. Similarly, there would appear to be little conflict between the research agenda in this study and an absolutist ethical position.

However, one possible source of tension is the 'insider' position of the researcher. Most research into this appears to consider the position of a researcher researching the places they work (see, for example, Mercer, 2007). Some of the research reported in Chapter 2 is sponsored directly by examining bodies, which have their own research departments;

however, to my knowledge, the ethics of such research is not discussed in the research literature.

These working relationships do not appear to apply to the current study: although I do currently contribute to the work of an Examining Group as a Principal Examiner, I am not a full-time employee. However, the study does draw upon my experience as a Principal Examiner for SMP and MEI A-levels, and my classroom experience in teaching and curriculum development since 1974, as listed in Table 4.4.1. If, for example, the research were to conclude that utilising RWCF in A/AS pure mathematics questions was necessary, or desirable, then this conclusion might be influenced by my professional work and role as an examiner.

1974-9	Teaching A-level Mathematics and Further Mathematics, using Joint Matriculation Board A-level syllabuses
1979-83	Head of mathematics department in a grammar school, teaching A-level using Oxford and Cambridge Board MEI Mathematics
1983-85	Head of mathematics at a comprehensive school, teaching University of London Schools Examination Board A-level Mathematics
1985-92	Executive Director, School Mathematics Project. Editor of revision of SMP A-level text. Contributor to SMP 11-16 and SMP 16-19 Mathematics projects. Chair of SMP teaching committee scrutinising SMP A-level papers.
1994 – 2006	Head of Mathematics in a sixth form college. Edexcel A-level (1994 – 1998) MEI Structured Mathematics A-level (1998 – 2006).
1993 – present	Principal Examiner for Oxford, Cambridge and RSA Examining Group, setting and marking papers for SMP and MEI A-levels.

Table 4.4.1 Researcher's 'insider' experience (relevant to this study)

It has to be recognised that research, of necessity, is influenced by the professional and life experiences of the researcher. Indeed, the questions posed by this study arise from my own experiences as an examiner and the question setting process. Clearly, any predisposition towards bias in considering the functions and effects of RWCF in questions needs to be carefully considered and minimised if the outcomes of the research are to be considered to be robust.

The validity of the research outcomes are open to scrutiny by a wider research community, and while the research was ongoing, the researcher took opportunities to present and publish papers based upon the research, at conferences of the British Society for Research in Learning Mathematics (BSRLM) , in order to allow such scrutiny (Little and Jones, 2007, Little, 2007, Little, 2008a, Little, 2008b, Little, 2009, Little and Jones, 2008). Furthermore, the motivation for undertaking this research was intellectual curiosity, and not professional advancement in the examining world.

With regard to the examining community at large, the intention of the research was not to favour one style of examining over another, but rather to offer insights into the examining process in mathematics. Nevertheless, the possibility of experience jeopardising the necessary neutrality of the observer needs to be borne in mind.

Another tension which may require to be addressed is caused by the source of the questions chosen for analysis in chapter 8, which in turn lead to the AP/GP tests developed in chapter 9. The majority of these were developed by myself, albeit moderated and validated by a question paper evaluation committee. On the one hand, this personal involvement with the questions may be regarded as a strength: I was able to draw upon the experience of constructing the questions, and seeing how the questions faired in operational examinations. Moreover, reflecting upon the experience of constructing AP and GP questions both with and without real-world context was invaluable in constructing parallel versions, and in the analysis and classification of questions.

On the other hand, it is possible that this classification is subjective, and depends on the limits of one person's imagination: there may be a range of other contexts, entirely different to these, which could potentially be utilised for this topic. Scrutiny of questions from other sources suggests that this is not the case, but the possibility of bias in the selection of questions needs to be kept open.

(e) Ethical dilemmas

Potential sources for ethical dilemmas proposed by Cohen et al (p.62-69) are privacy, anonymity, confidentiality, betrayal and deception.

With regard to privacy, there would appear to be no ethical issues of substance with regard to the study's use of question paper material, as this is all open to public scrutiny. Indeed, much of this material, albeit subject to laws of copyright, is accessible via the examining boards' websites. It may be taken as read that all the questions printed in the study come from past papers! While examining bodies were, in the past, somewhat secretive organisations, this has become less so in recent times, and public examination papers may be regarded as public, or even government, documents.

Students, schools and colleges have a right to privacy with regard to the results of assessments made in the classroom context. The privacy of individual students is protected by ensuring anonymity in reporting; the privacy of individual institutions s protected by anonymous reporting of aggregate results, whilst providing access to these results through individual school reports. All scripts and questionnaires were treated as confidential documents, and kept securely after analysis.

Finally, it is important that results of questionnaires and tests are open to scrutiny and verification, to ensure that the data upon which the research is based is genuine and soundly based. For this reason, the researcher's supervisor was given free access to this data, subject to the needs of anonymity and confidentiality.

(f) Regulatory ethical frameworks

In order to comply with University of Southampton regulations regarding ethics, a UoS School of Education Ethics protocol was completed – see Appendix 6.

(g) Sponsored research and responsibilities to the research community

This research is not sponsored, and is undertaken as pure academic research, under the auspices of the School of Education of the University of Southampton. The researcher has a responsibility not to jeopardise the academic reputation of the institution through which the research is carried out. It is important, therefore, that correspondence with schools and colleges which uses the university logo, materials, academic papers etc. are of high quality and reflect the high academic standards of the sponsor.

The main mechanism for ensuring these standards are upheld is the research supervisor. It is therefore important that all materials, correspondence, academic papers etc. are monitored by him to ensure that they are appropriate in standard. The researcher had the opportunity to present at a number of conferences, including regular meetings of BSRLM, and the International Congress of Mathematics Education in Mexico 2008, and was personally aware of his responsibilities as a representative of the university on these occasions (see list of publications referenced in (*d*) above).

4.5 Summary

Table 4.5.1 summarises the research methods used in this study. It is intended that the account in this chapter gives sufficient detail to describe the methodological approaches adopted in this study. It is difficult to provide a more detailed picture at this stage of the enquiry, as some of these methods emerge organically as the work progresses, rather than suggesting themselves at the outset. Section 8.1 of Chapter 8 gives a more detailed account of the methods adopted in part III of the research, and the methods utilised in the study are re-visited and evaluated in the concluding chapter (section 9.3).

	Research question	**Methodology**	**Type of data**
Part I	(1) Origins of RWCF	Historical survey of A-level Mathematics, focusing on project developments in the 1960s-90s (ch 6)	literary sources, past papers from OCR archives
	(2) Degree of RWCF	Comparative survey of past papers from two contrasting A/AS specifications (OCR Specs A and B) (ch 6)	past papers
Part II	(3) Functions and effects of RWCF	Analysis of sample questions (from OCR and Edexcel) using task analysis and evaluative framework developed from theory and research (ch 7)	past paper questions utilising RWCF
		Further analysis on a larger sample of AP/GP questions (OCR Spec B P2 questions); classification into broad categories (explicit, mathematical, word and pattern (ch 8)	Sequence questions from past papers.
Part III	(3) Functions and effects of RWCF (continued)	Development of four AS tests using parallel version of AP/GP questions, together with student questionnaire. (ch 9)	Questions based on sequence questions above.
		Analysis of data from a sample of approximately 600 students. (ch 9)	Scripts and completed questionnaires

Table 4.5.1 Summary of research methods

PART I

CHAPTER 5

FINDINGS: THE ORIGINS AND DEGREE OF REAL-WORLD CONTEXTUAL FRAMING IN A/AS MATHEMATICS

Overview

This chapter addresses Research Questions 1 and 2, which focus on the historical development of real-world context in A-level mathematics and the degree to which current A/AS pure mathematics questions use real-world contextual framing. Section 5.1 traces the historical context which has led to the introduction of context and modelling into A/AS mathematics. The Appendix reproduces sample questions and examination papers from the University of Cambridge Local Examination Syndicate in 1951, and from the School Mathematics Project in 1966, 1995 and 1998. Section 5.2 provides an account of the current content of A/AS pure mathematics, and the extent to which this is amenable to RWCF. Section 5.3 briefly summarises current developments of Free-Standing Mathematics Qualifications and Use of Mathematics syllabuses. Finally, section 5.4 analyses two current A/AS syllabuses, to investigate the degree to which they utilise real-world context in their pure mathematics questions. The findings of the chapter are summarised in section 5.5.

5.1 The development of A-level mathematics since the 1950s

Griffiths and Howson (1974) trace the roots of public examinations back to the mandarinate of ancient China. Systems of public examinations have evolved in different countries in disparate ways. Many countries use a model based upon the Prussian Abitur or French Baccalaureate, both developed towards the end of the 18th century. In England[2], the General Certificate of Education at Advanced level (shortened to GCE A-level) is our longest-standing qualification (the Ordinary or 'O' level was superceded in 1988). This GCE A-level qualification was developed in 1951 out of the Higher School Certificate, whose origins lie in

[2] In the UK, Scotland has a separate educational system to England, Wales and Northern Ireland, and is excluded for this discussion. For the sake of simplicity, I shall use the term 'England' to refer to 'England, Wales and Northern Ireland'.

university matriculation examinations, developed initially by the universities of Oxford, Cambridge and London (Kingdon, 1991). Public examination systems grew largely through pressure on higher education to provide a fairer qualification for selecting students for mathematics courses in universities.

Since the 1950s, the A-level Mathematics qualification in England has developed in response to societal, technological and cultural changes. The 'modern mathematics' movement of the 1960s, which spawned high profile projects such as SMSG in the US and SMP in the UK, developed from the expansion in the industrial applications of mathematics such as statistics, operational research, linear programming and numerical analysis, coupled with a somewhat contradictory movement towards the inclusion of more abstract mathematics such as functions, matrices, vectors, group theory and linear algebra (Cooper, 1985, Thwaites, 1972, Griffiths and Howson, 1974).

A scrutiny of early examples of A-level single mathematics papers from 1950 – 1980 shows that these typically comprised two 3 hour examinations, consisting of short, 10- to 15- minute questions in pure mathematics and mechanics. The pure mathematical questions were uniformly 'pure' in character, with no reference to 'real-world' contexts. Some required students to prove standard textbook results; all were predominantly tests of algebraic, geometric and analytical technique. Mechanics questions, which have changed little over the 60 years of A-level, tested Newtonian mechanics through idealised models of, for example, coplanar forces. Early examples of A-level mechanics contain no diagrams – students were required to construct these from verbal descriptions of the model. Calculations were assisted by four figure tables, and required high levels of fluency in pencil and paper work with decimals and fractions. A typical pair of A-level mathematics papers from 1951 is reproduced in Appendix 1.

The emphasis in early A-level questions on technique, as opposed to understanding, attracted considerable criticism by the end of the 1960s (Cooper, 1985). In particular, the needs of industry and technology for applied mathematicians led to the development of syllabuses such as SMP A-level with a much greater emphasis on mathematical modelling, numerical methods, computing and statistics. Hammersley, an Oxford statistician who was also a Principal Scientific Officer at Harwell, organised a seminal conference in Oxford in 1957 of mathematicians and teachers from prestigious public schools, which was eventually to lead to the establishment of SMP (Cooper 1985, p 96). In his opening address, he criticised current A-level papers on the following grounds:

> 'Mathematical examination problems are usually considered unfair if insoluble or improperly described; whereas the mathematical problems of real life are almost invariably insoluble and badly stated, at least in the first instance. In real life, the mathematician's main task is to formulate problems by building an abstract mathematical model consisting of equations, which shall be simple enough to solve without being so crude as to fail to mirror reality…'
>
> 'At school, statics and dynamics is frequently the only example of applied mathematics; and even then is generally emasculated by the removal of the model-building side, for the pupil is rarely left to make his own assumptions on the weightlessness of rods or the smoothness of planes, say. Further, at school and university, there is too much preoccupation with the detailed techniques of mathematics and far too little thinking about mathematics, about its uses, its values and about its meaning.' (quoted in Cooper 1985, p 99)

Following this and a number of further conferences, SMP set about writing and trialling a radically new O-level course, followed by A-level courses in Mathematics and Further Mathematics, for first examination in 1966 (Thwaites, 1972). This new course represented the most radical change in the nature of A-level Mathematics in its history. Instead of separate syllabuses in pure mathematics and mechanics, the SMP A-level syllabus was an integrated syllabus, consisting of pure mathematics, dynamics, statistics, electricity and computing. The approach to pure mathematics was a hybrid of 'modern' abstract algebraic concepts, such as functions and mappings, vectors and groups, together with a 'modelling' approach to functions, which were introduced as far as possible through a real-life context: chapter 2 of the first A-level text produced (School Mathematics Project, 1967) contained a chapter entitled 'Mathematical Models and Functions'. The novelty of the course, compared to 'traditional' syllabuses, which universities were thoroughly familiar with, caused considerable controversy at the time. Thwaites comments:

> 'The rumours attending the SMP plan for A-level had been flying thick and fast during 1963 and the traumatic experience through which we passed in obtaining general agreement to our plans ended with their publication in April 1964; there is no point in dwelling upon this experience except to hope that other experimental projects may succeed in avoiding it. The final result … has been liberally praised and equally bitterly attacked in the Press.' (Thwaites 1972 p 49).

SMP's plans were so radical that it took considerable negotiation and consultation with universities before the new course was accepted for university entrance (Thwaites 1972 p 98).

Although the SMP A-level text emphasised functions as models for real-life situations, early examination papers – see Appendix 2 for the 1966 papers – show evidence of real-life context only in the applied questions, the pure questions being in style not dissimilar to 'traditional' papers, albeit on more modern curriculum content. There is, however, a greater emphasis on understanding at the expense of technique. This can be seen from the relatively more open-ended character of some the demands, for example to 'justify' in Paper I A4, to 'explain carefully' in Paper I A12, 'give a rough sketch' (Paper I A13), 'criticise the argument' (Paper I, A19), construct a flow diagram (Paper I, A20), 'state carefully what is meant by' a statement of differentiability (Paper II, A5), 'discuss the formula' (Paper II, A18). A greater degree of open-endedness can be seen in Paper II, B28, which invites the candidate to model a cricketing context using projectiles, with no mathematical variables provided in the question. Other questions of a similar nature are quoted in Howson (1987):

> 'As the sun was setting in a clear African sky, it was noticed in a Super VC10 flying north that the outline of the westward windows was projected on the other side of the cabin about 6 inches above the windows on that side. Estimate roughly the height of the aircraft. (p 49)
>
> 'Make as good an estimate as you can of the total work which a champion high jumper expends in making one jump.' (p 63).

It is instructive in hindsight to quote the comments of two SMP authors on these types of question. On the first of these, Colin Goldsmith, comments:

> 'This imaginative question, set in one of the early SMP A-level examinations, epitomises the desire at that time to breathe new life into sixth-form courses and to emphasise applicability, not merely of the branches traditionally designated 'applied mathematics'. It must be admitted that subsequent questions have not been as unstructured or as memorable; that just shows the constraints which operated then and continue to operate.' (p. 49)

Commenting on the 'high jump' and 'cricket' questions, Douglas Quadling, having admitted that the questions were 'overambitious', refers to SMP's panels of teachers who monitored examinations in commenting:

> 'One imagines that nowadays questions as open-ended as these … would prompt a vigorous reaction from the SMP's panel of examination watchdogs.' (p.63).

These retrospective comments highlight the difficult balance to be struck between the construct validity of testing open-ended mathematical modelling skills and the reliability of such questions as short written examination questions. Setting challenging questions such as these may have beneficent effects on classroom instruction, encouraging students to discuss aspects of modelling, and in doing so simulating more closely the work of applied mathematicians such as Hammersley in taking account of real life problem solving constraints. However, they equally clearly failed to operate successfully as assessment instruments within a high-stakes summative timed written examination, and in the course of time, and criticism from teachers, were replaced by ‘safer’, more carefully structured tasks of proven reliability.

Another influential curriculum project of the 1960s, which emphasised real-life applications, was the Mathematics for Education and Industry project, or MEI. In 1962, B. T. Bellis, then head of mathematics at Highgate School, carried out an investigation into the mathematics used in industry during a schoolmaster fellowship at Balliol College, Oxford. This led to new syllabuses being developed, with the support of the Mathematical Association and BP, for first examination in 1967 (Mathematics for Education and Industry Project, 2008). Early MEI papers show a similar interest to SMP in developing elements of real-life application and modelling into A-level questions.

The success of projects such as SMP and MEI, which were taken up by influential independent schools, led to other examination boards being forced to introduce ‘modern’ syllabuses at O and A-level, which incorporated many of the novel features of the ‘modern’ courses, including transformation geometry, the formal language of functions, and vector methods, and considerably more emphasis on statistics, at the expense of geometrical topics such as the detailed study of conic sections, and some mechanics.

Thus, in 1981, UCLES was offering two alternative A-level syllabuses, Syllabus A, which had changed little in content from earlier syllabuses (albeit with a larger statistics component), and a ‘modern’ Syllabus B, which included functions and relations, matrices, vector methods and numerical analysis. Although these syllabuses offered alternative mathematical content, the style of examining remains ‘traditional’ in its emphasis on technique and relative lack of contextualisation. By the 1990s, the number of syllabuses having expanded substantially to accommodate both ‘modern’ and ‘traditional’ approaches, hybrid syllabuses, which incorporated elements of the modern topics, were being developed, the UCLES 1991 syllabus being an example.

By the 1990s, SMP A-level papers maintained their commitment to ‘modelling’ by presenting as many questions as possible in a real-life context; however, the requirement that candidates formulate the model had been lifted. Thus, questions presented the candidate with an explicit model - see sample questions in Appendix 3 from the 1994 examination. It is immediately noticeable how much longer these questions are on the page: the need to carefully set up the model relies heavily on words and diagrams, which clearly test the candidates’ comprehension skills, even though the mathematical content of these questions, taken from an examination aimed at candidates achieving lower grades, is not particularly demanding.

The 1990s saw the development, following the Cockcroft Report ‘Mathematics Counts’ (Cockcroft, 1982), with its influential paragraph 343 encouraging a broadening of classroom styles, and the introduction in 1988 of the GCSE examination, of alternative forms of assessment, such as practical and investigational project work and comprehension papers. These assessment tools provide greater scope for students to engage in tasks which, unlike timed written paper questions, require the full modelling cycle, including formulating a mathematical model from a real-life situation. Comprehension papers require students to understand a mathematical model presented in greater detail in the form of a short article, and then answer questions which test their level of comprehension.

Examples of influential syllabus developments in the 1990s were SMP 16-19 (Dolan, 1994) and MEI Structured Mathematics (Mathematics for Education and Industry Project, 2008). Both were modularised schemes emphasising applications, and with schemes of assessment which included coursework and comprehension papers.

5.2 The current A/AS Mathematics curriculum

At the time of writing, the most recent major overhaul of the A/AS examination system was ‘Curriculum 2000’, which incorporated the introduction of the Advanced Subsidiary (AS) examination (as distinct from the Advanced Supplementary examination), designed as a qualification after one year of post-16 study, as a stepping stone towards Advanced level. Students typically select four or five AS subjects in year one of their sixth-form course, and then reduce these to three A-levels in their second year. The major structural change was that *all* A/AS specifications were required to have a six module structure, which in A-level single mathematics comprised three pure modules and three applied modules. An influential pre-cursor of the six-module A-level was the MEI Structured Mathematics syllabus developed in 1990. Elements of the ‘modelling’ approach, incorporating real-life contexts into questions, were incorporated into mainstream six-module schemes.

A perceived imbalance in the demands of AS Mathematics compared to other A-levels, and the resulting decline in the number of students studying mathematics led, following the publication of the Smith Enquiry (Smith, 2004), to a revision of the mathematics specifications, in which the number of applied mathematics modules was reduced from three to two, and the content of the three pure modules was redistributed over four modules, thus reducing the overall demand of the syllabuses.

A question arises as to the extent to which modularisation affected the incorporation of contextualised questions and modelling into A-level syllabuses. The division of the syllabus content into smaller units has severely limited the cross-disciplinary approach of the original SMP A-level, which sought to emphasise the connections between pure and applied mathematics. The shorter examination papers also offered less scope for extended questions – the SMP examinations included 25 mark questions, allowing considerable scope for developing a context – see Appendix 4 for examples from a 1998 SMP A-level paper. Currently, the highest tariff is 18 marks, in the current MEI syllabus, and the other syllabuses have no questions above 14 marks.

However, current subject criteria for A/AS Mathematics (Qualifications and Curriculum Authority, 2002) do require all specifications to test mathematical modelling, by including the following assessment objectives (AOs):

> AO3 Recall, select and use their knowledge of standard mathematical models to represent situations in the real-world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinements of such models. (10%)
>
> AO4 Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications. (5%)

Thus, contextualisation and modelling are embedded in the current A/AS mathematics construct. The extent to which these assessment objectives can be tested validly using timed written paper questions is addressed in future chapters. The rest of this chapter considers the current A/AS mathematics curriculum, and the use of RWCF in two contrasting specifications

Within A/AS mathematics, there has been a traditional division of the mathematical content into 'pure mathematics' and 'applied mathematics'. As the previous section has outlined, the pure mathematical content of A/AS level has evolved during its history, but has comprised topics from the following areas of mathematics:

- Algebra, for example algebraic manipulation, theory of quadratic and polynomial equations, algebraic fractions;

- Trigonometry, for example the sine, cosine and tangent functions and their inverses, trigonometric identities, compound and double angle formulae, solution of trigonometric equations;
- Sequences and series, for example arithmetic and geometric progressions, the binomial series, McLaurin series;
- Two- and three-dimensional coordinate geometry, for example the equations of lines, planes, length, midpoint, circles, conic sections;
- Exponential and logarithmic functions, for example, use in reduction to linear form, exponential growth and decay;
- Differential and integral calculus, for example analytical methods of differentiation and integration;
- Numerical methods, for example iteration, Newton Raphson.

The applied mathematical content has been more variable. Traditionally, this focused on mechanics, but increasingly students have had the option to choose to study statistics or discrete mathematics.

The role of real-world context can be seen to differ in pure mathematics and applied mathematics. Applied mathematics relates to topics which are naturally situated in the ‘real world’:

- mechanics deals with physical concepts such as displacement, velocity and acceleration, force, momentum and energy, and questions deploy real-world contexts which employ these physical concepts;
- statistics deals with probability models for data, and employs real-world contexts which involve data, probability and uncertainty;
- discrete mathematics deals with applications of the theory of networks, linear programming and simulation to real-world contexts.

It would be difficult (though not impossible) to treat these areas of applied mathematics as pure mathematics: it is possible, for example, to develop the laws of probability as an axiomatic system. However, at the level of mathematical sophistication expected for A/AS level, it would be perverse to take such an approach. For this reason, this study does not consider the use of context in applied mathematics questions.

Within the pure mathematical content of A/AS level, questions can be broadly classified according to the following criteria:

- questions which have no ‘real-world’ framing, but utilise language and concepts that are inherently pure mathematical;
- questions which frame pure mathematical content in an extra-mathematical context.

Within these extremes, pure mathematical questions may vary in the balance of elements from the ‘mathematical’ and ‘real’ world. Moreover, the level of abstraction from the real world exhibited by mathematical concepts may also vary. For example, geometrical concepts such as ‘circle’ evoke an image of real-world objects such as wheels, CDs, etc. which are circular in shape. By way of contrast, the concept of ‘function’ as a mapping from one set of objects would appear to be less immediately accessible to real-world contexts.

Theoretically, it may be argued that mathematical artefacts such as graphs of functions used in questions are pictorial representations of mathematical objects which belong to, or exist in, the ‘real world’. However, for the purposes of this research, such artefacts are not regarded as examples of real-world contextualisation.

5.3 Free-Standing Mathematics Qualifications and Use of Mathematics

Another recent development which, although not directly impinging upon ‘mainstream’ A/AS Mathematics syllabuses, has been that of ‘Free-Standing Mathematics Qualifications’ (FSMQs) (Assessment and Qualifications Alliance, 2010). These are mathematics units developed at three levels (Foundation, Intermediate and Advanced); the Advanced units can be aggregated into an AS qualification entitled ‘Use of Mathematics’. The emphasis of these units is to develop ‘real-world mathematical understanding’:

> ‘FSMQ units can support a wide variety of other courses, for example providing algebraic and graphical support for science, 3D and spatial awareness for technology, statistics for geography and psychology, or decision maths for business and IT’.

The two current pure mathematics units are Working with Algebraic and Graphical Techniques and Modelling with Calculus. 50% of the assessment of each of these units is through a portfolio assessment, and the remaining 50% is a written paper, which contains questions which are exclusively framed in real-world contexts. An innovation of these papers is to provide candidates with preliminary materials in the form of a ‘data sheet’, which gives information about the real-world context used in the examination questions in advance of the examination. Students can in this way familiarize themselves with the context, whilst not being given specific information about the questions. Detailed consideration of these papers is beyond the scope of this thesis. At the time of writing, the candidature for these qualifications is small, and it is too early to assess the impact of this work in relation to mainstream A/AS Mathematics qualifications.

5.4 Analysis of OCR Specification A and B (MEI) questions with RWCF

This section analyses the pure mathematics papers of two current A/AS specifications, in order to investigate the degree of real-world contextual framing used. These two specifications, called 'A' and 'B (MEI)' are both administered by OCR.

Tables 5.4.1 and 5.4.2 give details of the questions from the pure mathematics (C1-4) papers in OCR specifications A and B (MEI) utilising real-world contexts in the period 2005 – 2008.

Each of these papers has a maximum mark of 72. The total number of papers set in each specification in the period studied is 22 (six C1, six C2, five C3 and five C4). Thus the percentage of marks from questions set in context is 5.2% for Specification A and 31.0% for Specification B: 31.0%. This shows clearly that specification B has a very significantly higher proportion of the question papers set in real-world contexts.

Spec A questions	Marks	
C1 Jan 05	7	Quadratic function modelling a children's playground – solving inequality for area.
C1 June 06 7	10	Minimising surface area of a cuboid.
C1 June 07	6	Maximising area of a rectangular enclosure with a wall.
C2 Jan 05	7	Surveying a landmark – trigonometry.
C2 June 05	9	GP modelling oil production from a well.
C2 June 06	9	APs and GPs to model savings.
C2 Jan 07	10	GP to model coal consumption of a steam train.
C3 June 07	7	Exponential decay of a substance.
C4 June 06	8	Area of a forest fire modelled by a differential equation.
C4 June 07 8	10	Height of a shrub modelled by a differential equation.
Total	*83*	

Table 5.4.1: OCR Specification A questions with RWCF

Spec B (MEI) questions	Marks	
C1 June 05 8	5	Quadratic equation for area of rectangular enclosure.
C2 Jan 05 9	12	Quadratic used to model cross section of tunnel. Area by integration and trapezium rule.
C2 Jan 05 11	13	Reduction to linear form used on temperature of cooling drink.
C2 June 05	12	Sector length and area applied to an arrowhead logo.
C2 June 05	11	GP modelling flower-head pattern. .
C2 Jan 06	13	APs and GPs modelling pocket money.
C2 June 06 11	11	Lengths and areas of triangles and sectors modelling motion of a ship.
C2 June 06 12	12	Reduction to linear form on a population of bats.
C2 Jan 07 11	12	Lengths and areas of triangles and sectors modelling shape of village green.
C2 Jan 07 13	12	Reduction to linear form on profits of a business.
C2 June 07 10	10	Velocity – time graph of a car - trapezium rule and integration of quadratic to estimate distance travelled.
C2 June 07 11	12	APs and GPs applied to game with counters, dice throw probability.
C2 Jan 08 10	12	Differentiation to find minimum surface area of a cuboid.
C2 Jan 08 11	12	Lengths and areas of triangles and sectors modelling yacht race.
C2 June 08 12	12	Trapezium rule and integration to estimate area of cross section of a trough.
C2 June 08 13	12	Reduction to linear form on cinema data.
C3 Jan 06 2	6	Exponential function on population.
C3 Jan 06 4	7	Chain rule on water poured into a cone.
C3 June 06 4	6	Chain rule on water poured into a pond.
C3 June 06 6	8	Exponential decay applied to radioactive substance.
C3 Jan 07 3	7	Exponential function on value of a car.
C3 Jan 07 6	8	Chain rule on connected points moving on axes.
C3 June 07 4	8	Exponential decay applied to cooling water.
C3 Jan 08 3	8	Exponential function on profit made by a company.
C3 Jan 08 4	7	Chain rule on pressure / volume of gas in a balloon (Boyle's Law).
C3 June 08 6	6	Exponential function on mass of substance in chemical reaction.
C4 Jan 06 7	17	Calculus used to maximise angle between posts in rugby.
C4 Jan 06 8	19	Differential equations to model populations of red and grey squirrels.
C4 June 06 4	8	Differential equations for bacteria colony.
C4 June 06 5	7	Volume of revolution of a vase shape.
C4 June 06 6	18	Parametric equations for cycloid to model bridge.
C4 June 06 7	18	Vector geometry applied to a house.
C4 Jan 07 7	20	Parametric equations, volume of revolution on an egg shape.
C4 Jan 07 8	16	Vector geometry applied to a pipeline under a river.
C4 June 07 7	20	Differential equations to model oscillating infection cases.
C4 June 07 8	16	Parametric equations modelling a theme park ride.
C4 Jan 08 7	18	Vector geometry applied to a glass ornament shape.
C4 Jan 08 8	18	Differential equations applied to a mountain stream.
C4 June 08 8	18	Vector geometry applied to coal seems.
C4 June 08 9	19	Differential equation used to model motion of a sky diver.
Total	*486*	

Table 5.4.2: OCR Specification B questions with RWCF (continued)

The following syllabus content items from these pure mathematics modules have questions which are framed in real-world contexts (Table 5.4.3).

Specification A content items	Specification B (MEI) content items
Quadratic equations (C1) Maxima and minima (C1) Trigonometry (C2) APs and GPs (C2) Exponential growth and decay (C3) Differential equations to model change in time (C4)	Quadratic equations and functions (C1) Reduction to linear form (C2) Area and length of sector formulae (C2) Trigonometry (C1) Integration (analytical or approximate) to estimate areas or distance from velocity-time graph (C2) Exponential growth and decay (C3) Chain rule for related rates of change (C3) Maxima and minima (C4) Differential equations to model change in time (C4) Volumes of revolution to model shapes (C4) Parametric equations to model shapes (C4) Three-dimensional vector geometry to model real-world geometry (C4)

Table 5.4.3 Syllabus items using RWCF from OCR Specs A and B papers

These applications may be classified further into:

- Geometrical models – trigonometry, shapes of functions, volumes of revolution, three dimensional vector geometry

In these questions, mathematical models are used to model two- or three-dimensional physical configurations or objects (bridges, vases, tunnels, balloons, eggs, etc.) Thus, the real-world context is used to provide a pictorial context to the solver. Questions of this type can then apply the results of mathematical calculations (lengths, distances, angles, areas, volumes) to the script.

- Models of growth or change in time

These questions use calculus or discrete functions (e.g. arithmetic and geometric progressions) to model discrete or continuous change. Examples include differential equations, related rates of change, maxima and minima, APs and GPs.

- Mathematical models of patterns

Arithmetic and geometric progressions can also be used to model patterns in space and time (stacking cards, 'Pascal's triangle' generalised).

Thus, as suggested in section 1.2, there appears to be a spectrum of 'contextualisability' in the pure mathematical content of A/AS Mathematics, and RWCF appears to be confined to a specific subset of the pure mathematics in these syllabuses. Moreover, the degree to which this subset of syllabus content is contextualised in examination questions has been seen to vary substantially between these two specifications, one of which derives from 'traditional' syllabuses and the other from 'alternative' or 'modern' syllabuses, as differentiated by the account of the history of the subject given in section 5.1.

5.5 Summary

This chapter has presented a brief account of the evolution of the A level mathematics curriculum since its inception in the 1950s, and traced the development of real-world contextual framing to project syllabuses developed in the 1960, 70s and 80s by projects such as SMP and MEI. It then outlined the pure mathematics content of A/AS syllabuses, and found considerable differences in the degree to which RWCF is utilised in questions from two contrasting specifications. Examples of real-world contextual framing were then classified broadly into geometrical models, models of growth and change, and patterns.

In the next chapter, I look more closely at the function of real-world context in A/AS questions, by analysing a sample of questions, using an evaluative framework derived from theoretical ideas developed in chapter 3.

PART II

CHAPTER 6

THE ROLE OF REAL-WORLD CONTEXT IN A/AS MATHEMATICS QUESTIONS

Overview

Having established in chapter 5 that a quantitative difference in the degree of RWCF exists between A/AS syllabuses, and outlined some of the areas of pure mathematics content in which it is currently utilised, this chapter looks in more detail at the use of context in questions, drawing on the theoretical ideas developed in Chapter 3. Section 6.1 starts by re-capping the ARTA framework, using the notions of accessibility, realism and task authenticity, developed in section 3.3. Section 6.2 then applies this framework to a selection of A/AS questions. The results of this analysis are discussed in section 6.3, and conclusions drawn from the analysis are given in section 6.4.

6.1 The ARTA framework: a tool for evaluating RWCF

At the end of chapter 3, three aspects of the use of RWCF in assessment were highlighted. First, it was suggested that real-world context may affect the *accessibility* of a question. The research on pre-16 mathematics has suggested that students vary in their responses to real-world contexts, due to the accessibility of the script implied by this context. Some contexts may be less familiar than others, and a novel context might add to the demand of questions.

Another aspect affecting accessibility is the match between the real-world context and the mathematics intended to model it. This requires the solver to transfer between context and mathematics, and this process may involve selecting and rejecting relevant aspects of the context. In some questions, this match may be assisted by specifying the model required explicitly in the question, whereas in others it may require to be established by the solver.

Questions utilising RWCF would appear to be longer than those without context, and this may make greater demands of comprehension on solvers. Thus, the linguistic structure of the question, in terms of the level and quantity of language used, would also seem to be relevant.

The other two aspects of real-world contextualisation which emerged from Chapter 2 are *realism* and *authenticity*. The research literature (see, for example, Boaler, 1994, Wiliam, 1997, Verschaffel et al., 1994) has criticised the lack of realism or artificiality of contexts. This may be affected by the assumptions made by the question designer in applying a given model, and whether real-world data used in the question are realistic.

The *authenticity* of the task may be taken to mean the extent to which the task itself, as well as testing mathematical techniques, is meaningful and germane to the real-world context. In order to satisfy a test of authenticity, the solution should provide useful insights into the real-world context, and ideally encourage the solver to evaluate the results given by the model, or models, in the light of this context. Finally, the authenticity of the task may be enhanced by the intrinsic interest of the context to the solvers: the more interesting a task appears, the more likely it will appear to be worthwhile and valid. These ideas were used in chapter 3 to formulate the ARTA framework, a checklist of questions relating to accessibility, realism and task authenticity of individual questions which utilise RWCF, as set out in Fig. 6.1.1. In the next section, I use this ARTA framework to analyse a sample of A/AS questions which utilise RWCF. Each question analysis is accompanied by a task analysis which clarifies the connection between the real-world context and the intended mathematical model.

Accessibility

- How accessible is the script implied by the real-world context?
- How familiar is the real-world context to students?
- How accessible is the match between the structure of the real-world context and the mathematical model?
- How explicit is the match between the real-world context and the mathematical model?
- How accessible is the language used in the question?
- What comprehension demands are made of candidates in explaining the real-world context?
- How many words are used?

Realism

- How realistic is the real-world context used?
- How realistic are the assumptions made when applying the mathematical model to the real-world context?
- How realistic is the data generated by the mathematical model?

Task **A**uthenticity

- How authentic is the task in relation to the real-world context?
- Does the task include evaluation of the appropriateness of the model?
- How interesting is the task to the solver?

Fig. 6.1.1 The ARTA Framework

6.2 Applying the ARTA framework

This section presents an analysis of a sample of ten A/AS mathematics questions with RWCF using the ARTA framework. The questions were selected to illustrate differences in the function of real-world context. They come from specimen papers for the Edexcel (2004) and OCR 'B' (MEI) syllabuses. For each sample question, a task analysis outlines the real-world context, and the mathematics used to model this in solving the question. The ARTA analysis evaluates the role of the context in each of the questions using the framework in Fig. 6.1.1.

Example 1 (Edexcel C1 paper)

> **7.** Ahmed plans to save £250 in the year 2001, £300 in 2002, £350 in 2003, and so on until the year 2020. His planned savings form an arithmetic sequence with common difference £50.
>
> (a) Find the amount he plans to save in the year 2011.
> **(2)**
>
> (b) Calculate his total planned savings over the 20 year period from 2001 to 2020.
> **(3)**
>
> Ben also plans to save money over the same 20 year period. He saves £A in the year 2001 and his planned yearly savings form an arithmetic sequence with common difference £60.
>
> Given that Ben's total planned savings over the 20 year period are equal to Ahmed's total planned savings over the same period,
>
> (c) calculate the value of A.
> **(4)**

Task Analysis (Example 1)

	Real-world context (script)	**Mathematical model**
	Saving money at regular time intervals	Sequences
(a)	Ahmed's savings increase at constant rate	Arithmetic sequence
	His initial amount is £250	a = 250
	His increment is £50	d = 50
	2011 is 10th year of saving	n = 10
	How much would he save in year 10?	a + (n − 1)d = 250 + 9×50 = 700
	what is the total savings in 20 years?	n = 20, Sn =10(500+19×50) = 14500
(b)	Ben's initial amount is £A	a = A
	His increment is £60	d = 60
	period is 20 years	n = 20
	total savings equal Ahmed's	Sn = 10(2A + 19×60) = 14500 ⇒ A = 155

ARTA Analysis (Example 1)

Accessibility

The script in this question is readily accessible and familiar: financial contexts such as this are quite commonly used in school textbooks and in teaching to illustrate sequences. In order to match the context to the mathematical model, the 'year of investment' requires translation into 'term number' of sequence by deducting 2000.

The connection with the model is made partially explicit through use of the terms 'arithmetic sequence' and 'common difference'. The language used is everyday. The question uses 120 words, or 13 words per mark.

Realism

It is feasible that Ahmed and Ben might model their savings plans using arithmetic progressions – it is reasonable to expect that they might increase the amount they save per year. However, this is unlikely to happen in practice. The model is inflexible, and fails to take account of interest payments.

Task Authenticity

The question refers back to the context, and comparing two savings plans which yield the same amount is a valid task within the context of savings. However, the lack of the realism of the models impinges on its authenticity, and there is no invitation to evaluate the models.

Example 2 (Edexcel C2 paper)

6. At the beginning of the year 2000 a company bought a new machine for £15 000. Each year the value of the machine decreases by 20% of its value at the start of the year.

(a) Show that at the start of the year 2002, the value of the machine was £9600. **(2)**

When the value of the machine falls below £500, the company will replace it.

(b) Find the year in which the machine will be replaced. **(4)**

To plan for a replacement machine, the company pays £1000 at the start of each year into a savings account. The account pays interest at a fixed rate of 5% per annum. The first payment was made when the machine was first bought and the last payment will be made at the start of the year in which the machine is replaced.

(c) Using your answer to part (b), find how much the savings account will be worth immediately after the payment at the start of the year in which the machine is replaced. **(4)**

Task Analysis (Example 2)

	Real-world context (script)	Mathematical model
	Depreciation of machine and compound interest on investment	Sequences
(a)	Machine decreases in value by 20%	Geometric sequence with r = 0.8
	Value multiplied by 0.8 each year	$15000 \times 0.8^2 = 9600$
	Year 2016	Term $15000 \times 0.8^{n-1} < 500$ $\Rightarrow (n - 1) \ln 0.8 < \ln(0.033..) \Rightarrow n = 16$
(b)	Interest at fixed rate of 5%	Multiplier 1.05
	Total savings after 16 years?	Sum of GP with a = 1000, r = 1.05, n = 16 $= 1000(1.05^{15} - 1)/0.05 = 21580$

ARTA Analysis (Example 2)

Accessibility

The script, as with example 1, is considered to be accessible and familiar: depreciation and savings are familiar contexts for growth and decay. In order to match the contexts with geometric sequences, solvers need to translate the percentages given into ratios, and match the value of *n* to the year of depreciation and investment respectively. The model (GPs) is not given explicitly in the question, and solvers are required to deduce the values of *a*, *r* and *n* from the real-world contexts. The question is quite wordy, with 170 words, 17 words per mark.

Realism

Unlike APs in example 1, GPs provide realistic models of depreciation and compound interest, although depreciation in practice is likely to be greater in the first few years.

Task Authenticity

Both parts ask questions which are valid and interesting in the real-world context. However, there is no evaluation of the model.

Example 3 (Edexcel C3 question)

6. As a substance cools its temperature, T °C, is related to the time (t minutes) for which it has been cooling. The relationship is given by the equation

$$T = 20 + 60e^{-0.1t}, \quad t \geq 0.$$

(a) Find the value of T when the substance started to cool. **(1)**

(b) Explain why the temperature of the substance is always above 20°C. **(1)**

(c) Sketch the graph of T against t. **(2)**

(d) Find the value, to 2 significant figures, of t at the instant $T = 60$. **(4)**

(e) Find $\frac{dT}{dt}$. **(2)**

(f) Hence find the value of T at which the temperature is decreasing at a rate of 1.8 °C per minute. **(3)**

Task Analysis (Example 3)

	Real-world context (script)	**Mathematical model**
	Cooling of a substance	Exponential decay model
(a)	'…started to cool'	t = 0
(b)	Temperature always above 20	$e^{-0.1t}$ tends to zero.
(c)	Substance cooling at decreasing rate	Graph of exponential function
(d)	-	Solving exp equation
(e)	-	dT/Dt
(f)	Rate of decrease	Derivative

ARTA Analysis (Example 3)

Accessibility

Cooling laws is a naturally occurring scientific context for exponential functions, which will be familiar to solvers. The model is given explicitly in the question, 'starting to cool' implies 't = 0', and 'rate of cooling' implies 'derivative', though the 'hence' in the question hints at this. Also 'decreasing at a rate of 1.8° C' implies 'dT/dt = −1.8'. The context requires relatively few words to set up for this length of question – 98 words, 7.5 words per mark.

Realism

Newton's Law of Cooling provides a scientific basis for modelling with an exponential function, and the model can therefore be regarded as realistic.

Task Authenticity

The mathematical model predicts the graph of temperature against time. Question (b) hints at 'room' temperature. Other questions use the model to make specific predictions about temperature and rate of cooling which are valid, albeit not significant in themselves. There is no evaluation.

Example 4 (Edexcel C4 Paper)

8. A circular stain grows in such a way that the rate of increase of its radius is inversely proportional to the square of the radius. Given that the area of the stain at time t seconds is A cm^2,

(a) show that $\frac{dA}{dt} \propto \frac{1}{\sqrt{A}}$.

(6)

Another stain, which is growing more quickly, has area S cm² at time t seconds. It is given that

$$\frac{dS}{dt} = \frac{2e^{2t}}{\sqrt{S}}.$$

Given that, for this second stain, $S = 9$ at time $t = 0$,

(b) solve the differential equation to find the time at which $S = 16$. Give your answer to 2 significant figures.

(7)

Task Analysis (Example 4)

	Real-world context (script)	**Mathematical model**
	Circular stain growing in time	differential equations
(a)	rate inversely proportional to square of radius	$dr/dt = k/r^2$
	Circular stain	$A = \pi r^2$
	-	$dA/dt = dA/dr \cdot dr/dt$ = … etc
(b)	rate of change of S	dS/dt

ARTA Analysis (Example 4)

Accessibility

The context is accessible and fairly familiar – it is not unusual to present related rates of change questions in terms of the growth of areas or volumes. The context is not hard to understand, though the growth of a 'stain' is not a routine idea. The match in (a) requires translating 'circular' to '$A = \pi r^2$'; in (b) the match is explicit as the differential equation is given, so the context is not required to solve the question. The word ‘stain’ may be unfamiliar to some solvers, but the language is everyday. There are 44 words, 3.4 words per mark, which is low.

Realism

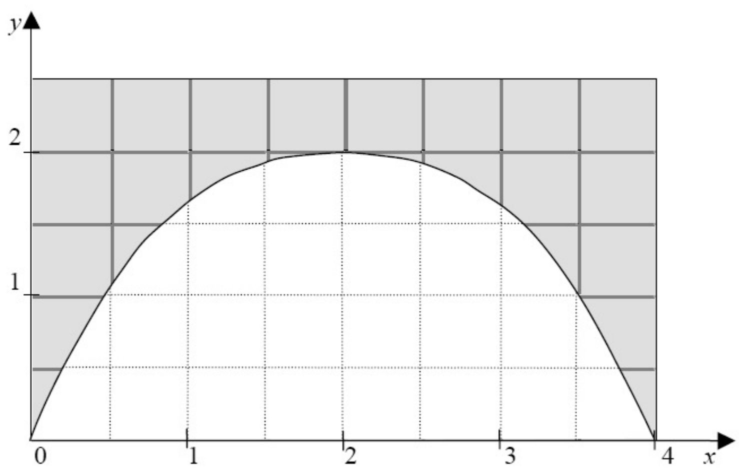
Part (a) is a feasible model – one would expect the rate of growth to slow down as r and A increase. However, the differential equation in (b) is unrealistic, producing a stain of area 100 m^2 in 10 seconds. See also the discussion in section 3.3.

Task Authenticity

Although differential equations can be used to model growth, there is no reference back to the context in either part of the question, and no evaluation of the models.

Example 5 (MEI C1 question)

5 The diagram shows a bridge.
The units are metres.



It is suggested that the curved underside of the bridge can be modelled by the curve $y = \frac{1}{2}x(4-x)$ for $0 \le x \le 4$.

(i) Give two different reasons why this is a good model. **[2]**

Task Analysis (Example 5)

	Real-world context (script)	Mathematical model
	A ‘bridge’.	Quadratic equation to model its shape.
(i)	Why is the model ‘good’? Because it fits the curve	f(0) = f(4) = 0; f(2) = 2, so fits at ends and middle
(ii)	Why is it not a ‘perfect’ model?	e.g. f(1) = 1.5, bridge higher at this point.

ARTA Analysis (Example 5)

Accessibility

Although bridges are everyday objects, the concept of modelling their shape with functions may be unfamiliar. It is possible that solvers might confuse the diagram as referring to the quadratic function rather than the shape of the bridge, which then makes the task meaningless. The questions imply that solvers need to match points on the diagram with points calculated using the function.

The vocabulary is everyday, using 50 words for 3 marks, or 16.7 words per mark, which is quite high. However, the tariff for the question is low, so overall the question is easy to read.

Realism

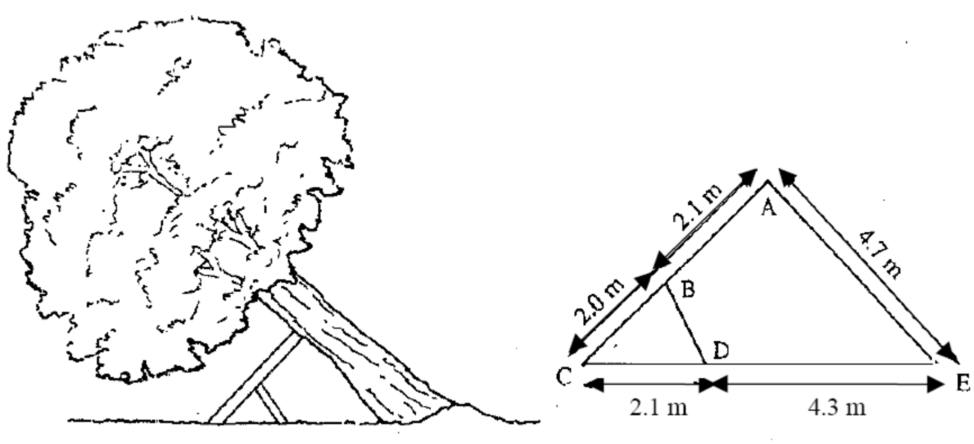
Functions can be used to model the shapes of curves found in the real world. However, modelling the underside of a bridge with a quadratic appears somewhat contrived, as most bridge undersides are likely to be arcs of circles.

Task Authenticity

The real-world context is essential to the question, and there is an element of evaluation implied by comparing the function and the bridge shape. The task may therefore be regarded as authentic, albeit at a simple level.

Example 6 (MEI C2 Paper)

8



In the gales last year, a tree started to lean and needed to be supported by struts that were wedged as shown above. There is also a simplified diagram giving dimensions.
Calculate the angle the tree makes with the vertical, giving your answer to the nearest degree. **[5]**

Task Analysis (Example 6)

	Real-world context (script)	**Mathematical model**
	A leaning tree.	A triangle ACE with a line BD, B on AC and D on CE
	Angle which tree makes with vertical.	Angle AEC, calculated using trigonometry in the triangle ACE.

ARTA Analysis (Example 6)

Accessibility

This question is borderline in its use of RWCF, as the task is virtually unaltered if the 'tree' context is removed. The tree context serves to motivate the trigonometrical solution of the triangle; the only matching required is 'angle of tree' with 'angle AEC', which is trivial. The context is unlikely to affect the accessibility of the question.

Realism

The measurements are reasonable, although the position of 'E' is ill defined, and might not justify distances to the nearest 10 cm.

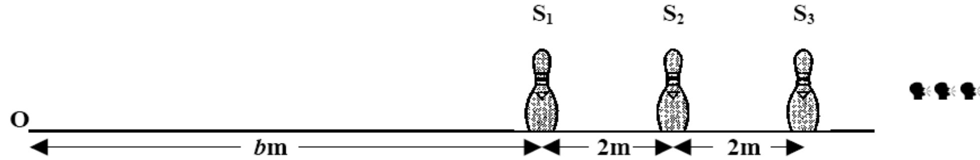
Task Authenticity

In practice, trigonometry would not be an appropriate method for estimating the angle of the tree, and there is no reason why we should want to calculate this information. This would appear to be an example of 'mathematics in search of a context'.

Example 7 (MEI C2 Paper)

9 In a race, skittles S_1, S_2, S_3, …are placed in a line, spaced 2 metres apart.

Contestants run from the starting point O, b metres from the first skittle. They pick up the skittles, one at a time and in order, returning them to O each time.



(i) Show that the total distance of a race with 3 skittles is $6(b+2)$ metres. **[1]**

(ii) Show that the total distance of a race with n skittles is $2n(b+n-1)$ metres. **[4]**

(iii) With $b=5$, the total distance is 570 metres. Find the number of skittles in this race. **[3]**

A football coach uses this race for training the team. The total distance for each contestant is exactly 1000 metres. The skittles are still 2 metres apart and the value of b is a whole number less than 20.

(iv) How many skittles are there in this form of the race? **[3]**

Task Analysis (Example 7)

	Real-world context (script)	**Mathematical model**
	A 'skittles' race	Arithmetic sequences
(i)	Add distances to pick up 1st, 2nd, 3rd skittles	$2b + 2(b + 2) + 2(b + 4) = 6b + 12 = 6(b + 2)$
(ii)	Add distances to pick up n skittles	$2b + (2b + 4) + \dots$ is an AP with $a = 2b$ and $d = 4$, so total distance is $S_n = ½\, n(4b + (n-1)4) = 2n(b + n - 1)$
(iii)	b is 5 and S_n is 570	$2n(4 + n) = 570 \Rightarrow n^2 + 4n - 285 = 0$ $\Rightarrow (n - 15)(n + 19) = 0 \Rightarrow n = 15$
(iv)	Total distance 1000, 4 m extra per skittle	$1000 = 2n(b + n - 1)$ $\Rightarrow n(b+ n-1) = 500 = 20 \times 25$ so $n = 20$, $b = 6$

ARTA Analysis (Example 7)

Accessibility

This is an unusual context for sequences (unlike Example 1, which is common). The match between context and model is also less straightforward, as the skittles race requires a doubling of the distance between O and each skittle, giving $d = 2$ rather than 4. The connection is implicit, as arithmetic sequence is not mentioned. The context is non-standard, and needs to be established carefully, using 144 words at 13.1 words per mark, which is high.

Realism

The laps of this sort of relay race are likely to form an arithmetic sequence, so the modelling is appropriate, albeit in an artificial context. The context is contrived to model the sequence, rather than vice-versa

Task Authenticity

The total distance for the race is a natural result within the race context, but the modelling is essentially artificial.

Example 8 (MEI C2 Paper)

10 A virus is spreading through a population and so a vaccination programme is introduced.

Thereafter, the numbers of new cases are as follows:

Week number, x	1	2	3	4	5
Number of new cases, y	240	150	95	58	38

The number of new cases, y, in week x is to be modelled by an equation of the form $y = pq^x$, where p and q are constants.

(i) Copy and complete this table of values.

x	1	2	3	4	5
$\log_{10} y$					

[1]

(ii) Plot a graph of $\log_{10} y$ against x, taking values of x from 0 to 8. **[2]**

(iii) Explain why the graph confirms that the model is appropriate. **[2]**

(iv) Use the graph to predict the week in which the number of new cases will fall below 20.
Explain why you should treat your answer with caution. **[3]**

(v) Estimate the values of p and q.
Use your values of p and q, and the equation $y = pq^x$, to calculate the value of y when $x = 3$.
Comment on your answer. **[5]**

Task Analysis (Example 8)

	Real-world context (script)	Mathematical model
	Number of new cases of a virus	Reduction to linear form, $y = pq^x$
(i)	-	Calculating logarithms
(ii)	-	Plotting graph
(iii)	-	$\log y = \log p + x \log q$ $\Rightarrow$ straight line, gradient $\log q$, intercept $\log p$
(iv)	Week 7, extrapolation	$\log 20 = 1.3, \Rightarrow x = 6.4$
(v)	Good agreement	$p = 380$, $q = 0.63$, $y = 95$

ARTA Analysis (Example 8)

Accessibility

The growth of a virus is a commonly used context for exponential growth, and is likely to be familiar to AS students. The accessibility of the match between context and model is enhanced by the use of the tables. The vocabulary is everyday: ‘virus’ and ‘vaccination’ are commonly used words. The question uses 164 words, at 12.6 per mark, but the familiarity of the context, together with the layout using tables, aids the accessibility of the context.

Realism

As in Example 3, exponential decay is an appropriate model for the growth of a virus, assuming that the rate of increase is proportional to the number of carriers. The figures will no doubt have been created to ensure a good fit – in reality, this fit is unlikely to be as perfect!

Task Authenticity

The question uses the model to predict an existing datum. This ‘verifies’ the appropriateness of the model, which could then perhaps have been used to extrapolate a future value.

Example 9 (MEI C4 Question)

7 The population of a city is P millions at time t years. When $t = 0$, $P = 1$.

(i) A simple model is given by the differential equation: $\frac{dP}{dt} = kP$ where k is a constant.

(*A*) Verify that $P = Ae^{kt}$ satisfies this differential equation, and show that $A = 1$.
Given that $P = 1.24$ when $t = 1$, find k. **[5]**

(*B*) Why is this model unsatisfactory in the long term? **[1]**

(ii) An alternative model is given by the differential equation: $4\frac{dP}{dt} = P(2 - P)$.

(*A*) Express $\frac{4}{P(2-P)}$ in partial fractions. **[3]**

(*B*) Hence, by integration, show that: $\frac{P}{2-P} = e^{\frac{1}{2}t}$. **[5]**

(*C*) Express P in terms of t.
Verify that, when $t = 1$, P is approximately 1.24. **[3]**

(*D*) According to this model, what happens to the population of the city in the long term? **[1]**

Task Analysis (Example 9)

	Real-world context (script)	**Mathematical model**
	Population of a city	Differential equations
(i)		(*A*) Verification by differentiation
	(*B*) Grows without limit	e^{kt} tends to infinity as t tends to infinity
(ii)		(*A*) partial fractions
		(*B*) Integration by separating variables
		(*C*) Re-arrange formula
	(*D*) Long term population	Limit of P as t tends to infinity

ARTA Analysis (Example 9)

Accessibility

Population growth is an accessible and familiar context for differential equations. The match is explicit, as the differential equations are given, although 'in the long term' needs to be interpreted as $t \to \infty$, and the unit (millions) needs to be noted. The question uses 115 words for 18 marks, or 6.4 words per mark, which is low.

Realism

Exponential and logistical models are commonly used for population growth. The fact that the question offers two alternative models adds to the realism, suggesting that there is not one single appropriate model.

Task Authenticity

The question uses models to predict long-term growth. The first model fails to account for long term population, the second improves on the first. Again, the fact that two models are used to make long-term predictions adds to the authenticity of the task.

Example 10 (MEI C4 question)

8 Fig. 8 illustrates the flight path of a helicopter H taking off from an airport.
Coordinate axes Oxyz are set up with the origin O at the base of the airport control tower. The x-axis is due east, the y-axis due north, and the z-axis vertical. The units of distance are kilometres throughout.
The helicopter takes off from the point G. The position vector r of the helicopter t minutes after take-off is given by: $\mathbf{r} = (1+t)\mathbf{i} + (0.5+2t)\mathbf{j} + 2t\mathbf{k}$.

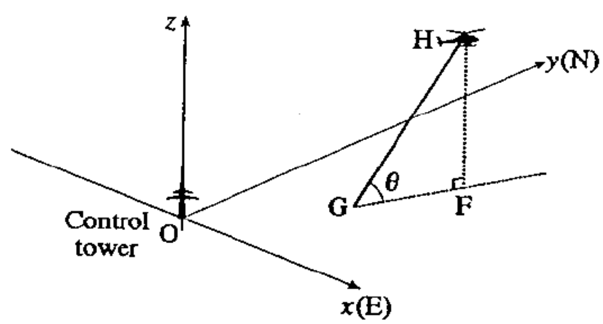


Fig. 8

(i) Write down the coordinates of G. **[1]**

(ii) Find the angle the flight path makes with the horizontal. (This angle is shown as θ in Fig. 8). **[3]**

(iii) Find the bearing of the flight path. (This is the bearing of the line GF shown in Fig. 8). **[2]**

(iv) The helicopter enters a cloud at a height of 2 km.
Find the coordinates of the point where the helicopter enters the cloud. **[3]**

(v) A mountain top is situated at M (5, 4.5, 3).
Find the value of t when HM is perpendicular to the flight path GH.
Find the distance from the helicopter to the mountain top at this time. **[5]**

(vi) Find, in vector form, the equation of the line GM.
Find also the angle between the line from G to the mountain top and the helicopter's flight path. **[4]**

Task Analysis (Example 10)

	Real-world context (script)	**Mathematical model**
	Helicopter flight	3 dimensional vectors
(i)	-	Substitute $t = 0$ into position vector.
(ii)	Angle	Use trig in triangle shown
(iii)	Bearing is angle with N line.	Find direction of GF from position vector, then use trig to find the bearing.
(iv)	z-coordinate is 2	So $t = 1$, etc.
(v)	Perpendicular	Use of scalar product
(vi)		vector eqn of line, angle between two vectors

ARTA Analysis (Example 10)

Accessibility

Modelling helicopter flight as a position vector might be more familiar to students of mechanics than those studying other applied disciplines. The model is explicit, in the sense that the position vector at time t is given. The tasks are framed within the context (angle with horizontal, bearings, perpendicular, height), and these need to be matched to the three-dimensional geometry techniques. The question uses 208 words for 18 marks, ratio 11.6 words per mark. This is quite high, so the question is wordy.

Realism

It is unlikely that a helicopter would take off in a straight line.

Task Authenticity

The tasks are artificial, though one could argue that finding out whether the helicopter hits the mountain is interesting!

6.3 Discussion

Table 6.3.1 summarises the topics from the sample A/AS mathematics questions which were analysed in the last section.

In Example 6 (MEI C2 Q6), the context is not exploited in the question beyond providing a ‘setting’ for the mathematics: little or no reference is made to the context except in the stem of the question. To all intents and purposes, this question is not altered, except superficially, by removing reference to the context altogether; for example by omitting the picture of the

Example	Edexcel questions	Marks	
1	C1 Q7 Arithmetic series applied to savings	9	APs
2	C2 Q6 Depreciation and financing replacement of a machine	10	GPs
3	C3 Cooling of a substance	13	Exponential growth and decay
4	C4 Growth of a stain	13	Chain rule and differential equations
	MEI questions		
5	C1 Q5 Shape of a bridge	3	Quadratic functions
6	C2 Q8 Tree held up by struts	5	Sine and cosine rules
7	C2 Q9 Skittles race	11	APs
8	C2 Q10 Spread of a virus	13	Reduction to linear form
9	C4 Q7 Population modelling	18	Differential equations
10	C4 Q8 Helicopter flight	18	Vector geometry

Table 6.3.1: Sample questions analysed using the ARTA Framework

leaning tree in Example 6. This is similar to the 'badge' question shown in Section 3.1, which was discounted as a question utilising RWCF. Whether this tree question should be discounted as well is open to debate.

The mathematical models in these questions may be classified into two categories:

- Geometrical models – bridge, tree, skittles, helicopter
- Growth/decay models – savings, monetary value, cooling, growth of stain, spread of virus, change in population

Accessibility

It would appear that all these questions utilise commonly held and readily understood scripts. However, this does not necessarily imply that all questions have scripts of equal accessibility to all students. For example, the SMP 1995 question on canoeing in Appendix 4 referred to in the previous chapter would appear to utilise a context which may not be equally familiar to all students. Some of these contexts are familiar to students from the classroom – for example, APs and GPs are commonly applied to finance and savings, and population growth is a common context for modelling with differential equations. Other contexts, such as the 'skittles' race, although based on familiar ideas to students, will be more novel. It is also possible that some contexts, such as the kinematics in Example 10, might be more familiar to students who are studying mechanics modules.

In some contexts, the match between the structure of the real-world context and the mathematical model would appear to be more explicit than others. For example, investing amounts at regular intervals produces a sequence of numbers which is readily matched with terms of a mathematical sequence. However, the structure of the context can add complexity to the transfer from real world to mathematical model. For example, the 'skittles' context requires a doubling of terms of the sequence.

The contexts all appear to utilise everyday language, though words like 'strut', 'stain', or 'skittles' may be unfamiliar to some students with limited English. However, if, as in the 'tree' question, knowledge of the context is not required, and it is there simply to provide an image or metaphor for a geometrical diagram, then this may not affect the question's facility. Some questions would appear to establish the link between the context and the intended mathematical model more explicitly than others, and this may also affect the facility of the question.

Candidates are required to read between 6 and 13 words per mark, and it would seem to be logical that the higher this figure, the greater will be the demands of comprehension on candidates (although the familiarity of the context, and the overall length of the question, will also play a part here).

In chapter 4, it was suggested that real-world context might be a source of construct-irrelevant variance in questions, by testing knowledge of the context rather than the mathematics. None of these questions would appear to assume any detailed knowledge of the context; but excessively wordy questions might discriminate against students whose English is not strong.

In all these questions, there is a degree of matching required between a real-world context and a mathematical model, although in Example 6 this was minimal. Whether this requirement is irrelevant to the A/AS Mathematics construct, however, depends upon how the construct is defined. As the assessment objectives for A/AS Mathematics require the ability to translate between real-world and mathematics, this would appear to embrace this matching process.

Realism

How realistic are these models? It would be more accurate to ask how real they *appear* to be: none of them are genuinely realistic, since the data in the questions will have been carefully created for the purposes of the question. In the exponential growth and decay questions, the models appear to follow naturally from applying scientific principles, for example Newton's Law of Cooling. Arithmetic and geometric progressions are, in a sense, the simplest models to apply to any sequence, although the extent to which assumptions which lead to such sequences are fulfilled by real-world contexts is debatable. Even if the data provided by the models is appropriate, this does not necessarily imply realistic modelling – the application of trigonometry to solve the 'tree' question, and the application of three-dimensional geometry to the flight of a helicopter, both appear to be contrived, since the mathematical techniques seem inappropriate for these contexts. They are therefore certainly candidates as 'McGuffins' (Wiliam, 1997).

Does a lack of realism matter? Section 4.2 discussed of the 'stain' question and its lack of realism. However, from a candidate's perspective, he or she is unlikely to be worried by this, as the question does not require candidates to consider the appropriateness of the given model.

Task authenticity

Some of these questions have a spirit of genuine 'modelling' by posing questions which are worthwhile and genuine questions in the real-world context. For example, it is authentic to ask what a model predicts about the future number of cases of a virus. If the context is contrived, such as the 'skittles' race, then the questions may still be meaningful within the context (and in this sense authentic), but the usefulness is jeopardised by the artificiality of the model.

There is perhaps a danger that in requiring utility, contexts are required to be 'serious': the skittles race is clearly a playful context for doing mathematical tasks, and thereby to make connections between reality and mathematics. Does a lack of utility matter? How do students perceive the artificiality of such questions?

The most 'authentic' of these questions would appear to be Example 9, which offers two alternative models, and invites, albeit relatively superficially, some evaluation and comparison of the two models, which lies closer to the spirit of the modelling cycle.

6.4 Conclusions

In this chapter, an evaluative framework has been developed deployed to analyse a sample of ten questions which utilise real-world contexts. This analysis suggested the following:

1. Accessibility may be affected by familiarity of the context, the language, the word-to-mark ratio, the explicitness and structural isomorphism of the match between real-world and mathematical model.

2. The realism of some questions derives from a scientific or real-world rationale for the mathematical model used, for example Newton's Law of Cooling, Newton's Law of Impact, or application of compound interest to investment growth. These may be described as *natural* models. In other questions, the modelling may be described as empirical: there is no natural basis for the model other than a superficial match between data generated by the mathematical model and the real world. The context is contrived to embody the mathematics. I call this use of real-world contextual framing *synthetic*: the context is developed to fit the mathematics, rather than vice versa.

What is the *function* (in connection with RQ3) of the RWCF in these questions, and does this function differ between natural and synthetic models?

In all these questions, the primary function of the RWCF is to embed the mathematical models in 'real', or 'realistic' non-mathematical contexts. Note that 'realistic' does not, in this regard, imply any *utility*, but rather realism in the RME sense (see section 2.2), that is relating to concepts in a non-mathematical world that is 'real' to the solver (for example, a skittles race).

In the case of natural contexts, the mathematical models used have, in addition to establishing a match between a real and a mathematical world, a degree of *utility* in describing the real world, because of their non-arbitrary basis.

3. The tasks may be described as *authentic* when the mathematical answers provide data which is relevant within the context. In the case of *natural* tasks, the results can be accepted as being *useful*, notwithstanding the simplifications required in presenting a context in a 'short' question. In the case of *synthetic* contexts, the tasks set may answer authentic questions within the context, albeit without any utility. Thus, in the 'skittles' example, the total length of the race is an authentic application of the sum of an AP formula, which makes sense within the real-world context; but, due to the synthetic nature of the context, the answers to the question are not useful.

These findings now need to be considered in relation to the concept of utility. Given that synthetic contexts provide no practically useful information about the real world, are they therefore 'useless'? Are they examples of Wiliam's (1997) 'McGuffins', of 'mathematics looking for somewhere to happen'? Recall the A/AS assessment objectives relating to real-world modelling:

> AO3 Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving

standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinements of such models.

AO4 Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.

In AO3, the word ‘standard’ is significant. There is a heuristical utility in applying standard mathematical techniques and models to a variety of real-world contexts, since first-principles thinking within each context requires different problem solving methods (for more on this, see Little, 2008). The utility of ‘standard’ mathematical methods and results is that they can be applied equally to a variety of real-world contexts. Finding uses for the term and sum formulae of arithmetic and geometric sequences, albeit in synthetic contexts, reinforces the *mathematical* utility of these formulae. Equally, it is possible to derive maxima and minima of functions within a real-world context without the use of differentiation; but applying calculus to the solution of problems in a variety of contexts reinforces the general utility of such methods (even though there may be little practical utility in particular applications).

To what extent do these sample questions test mathematical modelling skills? Real-world contextual framing requires solvers to abstract features of the context and map these into the world of pure mathematics. However, referring to the modelling cycle (Fig. 6.4.1), the solver is not required to make assumptions in selecting the model, the information content of the context is selected and constructed by the task designer, and there is rarely any opportunity for the solver to review results or assumptions.

Thus, the ‘pseudo-modelling’ required to solve these questions is, at best, represented in Fig. 6.4.2.

Examining context in A/AS questions in applied mathematics, (for example in statistics, mechanics or discrete mathematics) is beyond the scope of this study. However, my experience as teacher and examiner suggests that there is equally little choice of data, model or reviewing of results or assumptions in their solution, since these questions are closed in nature, and constructed in order to have a unique solution. ARTA analysis of such questions would be required to confirm this.

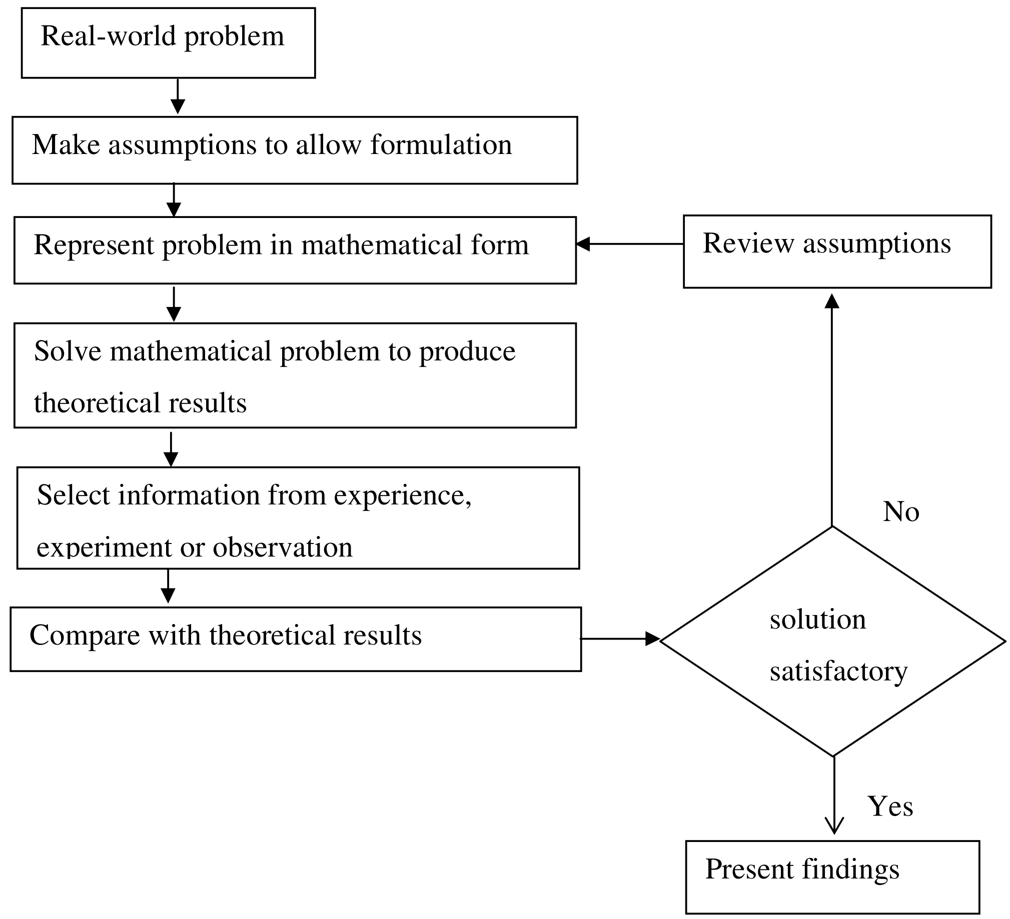


Fig 6.4.1 The mathematical modelling cycle

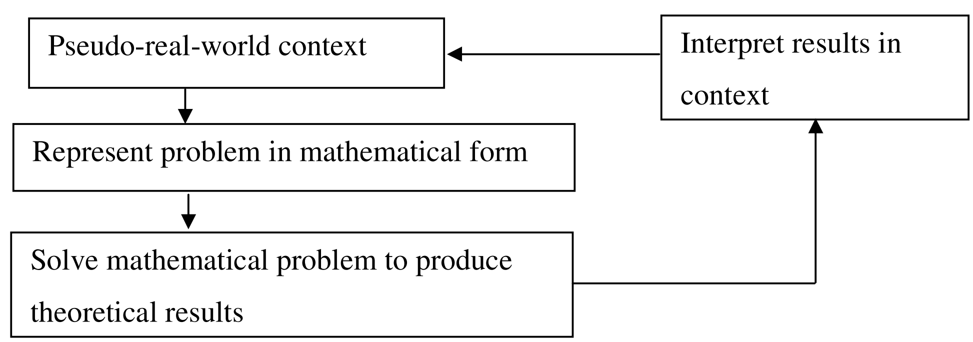


Fig 6.4.2 ‘Pseudo-modelling’ in questions with RWCF

The analysis in this chapter focuses on a sample of A/AS questions to classify the use of real-world context and investigate the scope and nature of its implementation in A/AS pure mathematics questions. However, in order to assess the effect of RWCF, it is necessary to compare questions on a topic set both with and without real-world context. In order to conduct this comparison, the next chapter focuses on questions on mathematical sequences, a topic that lends itself to a variety of questioning approaches.

CHAPTER 7

FINDINGS: THE ROLE OF CONTEXT IN SEQUENCE QUESTIONS

Overview

In order to consider RQ3, regarding the *effect* of RWCF, it is necessary to compare questions on a mathematical topic both with and without real-world contextual framing. Analysis of past paper questions suggests that the topic of *sequences and series* provides scope for a variety of question types. This chapter therefore looks in greater detail at A/AS questions designed to test sequences, in particular arithmetic and geometric progressions. Section 7.1 classifies a sample of sequence questions into four basic types, explicit, algebraic, word and pattern. The questions themselves are reproduced in Appendix 5. Section 7.2 uses the ARTA framework developed in chapter 3 to analyse and compare these questions. Finally, section 7.3 summarises the results of this analysis.

7.1 Methodology and classification

The sample

The analysis is based on a cluster sample of 27 questions from 20 examination papers in pure mathematics, each paper designed to assess one module of a six-module GCE A/AS Level qualification in mathematics. The unit, entitled Pure Mathematics 2, was part of the OCR MEI specification, one of whose aims is, wherever possible, to emphasise the application of mathematics to real-life contexts or situations. The full set of questions considered is provided in Appendix 5. Most of the items were one of four single complete questions worth approximately a quarter of the marks for the paper (total 60). The parts of each of these whole questions were designed to be thematically linked. In addition, there were some shorter questions which comprised parts of questions which tested other syllabus items.

The syllabus content covered by the questions comprises sequences and series, defined either using formula for the nth term or recursively, types of sequence, arithmetic and geometric progressions, the formulae for the nth term and sum of n terms of such sequences, and sums to infinity of geometric sequences. The sample questions included all those set on this topic for this syllabus from January 1997 to January 2006, during which the specification was in operation.

Classification by type of context

Scrutiny of this sample of questions suggests a classification into a number of broad categories. Some questions treat the topic as ‘pure mathematics’, without any real-world contextual framing. Considering these questions further, it appears that a subset of these define the sequences directly, giving the type – arithmetic or geometric – the first term and the common difference or ratio. Others utilise mathematical notation to define the sequence or its sum, for example giving the *n*th term as a formula or using a recurrence relation to express u_{n+1} in terms of u_n. The distinction between these types is not entirely clear-cut: some questions include algebraic notation in some parts of the question, but not others. However, the two classes of question are sufficiently distinct to be treated separately, and would appear to require different skills for their solution.

Considering now questions which include RWCF, these appear to split naturally into two types, those which describe the real-world context to be modelled by sequences using words alone, and those which describe the context by means of a pattern, usually accompanied by a diagram. This analysis suggests that questions may be divided into the following broad categories or types:

Explicit (e)	questions which predominantly define the sequences explicitly;
Algebraic (a)	questions which predominantly use mathematical notation to define sequences;
Word (w)	questions which use word descriptions to develop a real-world context;
Pattern (p)	questions which define a real-world context using a spatial pattern.

Table 7.1.1 gives the classification of the questions by context type, a brief description of the context, the number of marks (m), the number of words (w) and the word-to-mark ratio (w/m). The actual questions are given in Appendix 5.

Type	Question	Description	marks	words	w/m
e1	Jan 1999 Q2	(a) AP term and sum, (b) GP – find *n* and *r* given S_3 and sum to infinity.	15	132	8.8
e2	May 1999 Q2(a)	Term, sum and sum to infinity of a GP.	5	32	
e3	Jan 2001 Q2	(a) Two AP sums equated, (b) u_n defined using a sine, investigate sums of terms.	15	148	9.9
e4	Jan 2003 Q1(a)	AP term and sum.	4	24	6.0
e5	June 2005 Q2	APs and GPs – algebraic derivation of parameters from properties of terms and sums.	15	153	10.2
a1	Jan 1998 Q4	Sequence defined recursively, classify type for differing first terms.	14	116	8.3
a2	Jan 2002 Q3(a)	AP defined recursively.	7	51	7.3
a3	June 2002 Q2(a)	GP defined with negative index. Find first 3 terms and sum to infinity.	8	50	6.3
a4	June 2003 Q3	Investigate and classify sequences defined in various ways.	15	135	9.0
a5	Nov 2003 Q2	Sequence defined as function of *r*. Investigate sequence for various values of parameters.	14	137	9.8
a6	Jan 2004 Q3	Sequence defined as function of e, and its ln. Various mathematical requests.	15	122	8.1
a7	Jan 2006 Q2	Sequence defined recursively. Investigate and classify when parameters are varied.	15	155	10.3
Totals without RWCF			142	1255	8.8
w1	Jan 1997 Q1	Two gardeners spreading sand in a garden, one in AP one in GP.	15	181	12.1
w2	May 1997 Q2	Borrowing £50 000 at fixed interest rate to buy a house, paying back a fixed amount a year.	15	234	15.6
w3	Jan 2000 Q3	Investments with simple / compound interest.	15	271	18.1
w4	June 2001 Q1	Populations of oaks, beeches and pines defined recursively, two giving AP and GP.	14	179	12.8
w5	Jan 2003 Q2(i)	Cases of virus infection modelled by GP.	7	92	13.1
w6	Jan 2005 Q2	Phasing in and out of 'widget' production using GPs with $r > 1$ and $r < 1$.	14	184	13.1
w7	Spec paper Q2	Height of a rebounding ball.	15	186	12.4
w8	Jan 2002 Q3(b)	GP applied to times between rebounds of a ball travelling horizontally.	8	130	16.3
w9	June 2002 Q2(b)	AP applied to legs of a 'skittles' race.	6	123	20.5
Total word			109	1580	14.5
p1	May 1999 Q2	Building 'houses' from stacks of cards.	10	155	12.6
p2	June 2000 Q2	Spirals with sides defined recursively.	15	207	13.8
p3	Nov 2002 Q4	Array of numbers defined as in Pascal's triangle.	16	181	11.3
p4	June 2004 Q4	APs applied to matchstick patterns.	15	243	16.2
p5	Nov 2004 Q2	Division of circle into sectors whose angles are in AP or GP.	14	167	11.9
Total Pattern			75	987	13.2
Total with RWCF			184	2567	14.0

Table 7.1.1 Summary of AP and GP questions from OCR MEI P2 papers (1997 – 2006)

7.2 ARTA Analysis

In this section, the ARTA framework is used to analyse this set of sequence questions in greater detail.

Accessibility

Referring first to the 'e' and 'a' questions, although no real-world context is involved, some aspects of the notion of 'accessibility' may be applied to these questions. There is no extra-mathematical 'script' involved here, but the 'a' questions may involve an element of transfer or matching between sequences or series defined mathematically, and the corresponding arithmetic and geometric progression. The algebraic notation needs to be interpreted and then matched to the appropriate type of sequence. One can conjecture that this additional step in the problem-solving strategy might reduce the accessibility of 'a' questions, and make these more difficult for solvers.

Most of the contextualised questions develop the context 'from first principles', and rely upon familiar cultural constructs, for example gardening, card patterns, trees, etc. However, the modelling in some of the contexts, for example house buying, investment and bouncing balls, may utilise financial and scientific knowledge which will vary from candidate to candidate. In particular, the 'growth' contexts frequently use percentage increase or decrease, which needs to be converted to a ratio of a geometric progression. They may also involve conversion of units. These two aspects would appear to be potential sources of error which are not present in 'e' or 'a' questions.

Candidates for A/AS Mathematics are required to study applied mathematics, and there is an element of choice here, currently between statistics, mechanics and discrete mathematics. Those students who study mechanics will be more used to questions which specify a kinematical context, and may well be familiar with Newton's Law of Impact, as applied to collisions of particles. It is likely, therefore, that the kinematics contexts used in questions w7 and w8 will be more familiar to them.

Another issue which has a bearing on accessibility is whether students are likely to have met similar questions in a classroom context. Here, the more 'natural' contexts of finance and population growth are commonly used, and likely to be familiar to students, whereas other synthetic contexts such as those used in p1 and p2 may well appear to be less accessible to candidates in an examination by dint of their novelty and unexpectedness.

The accessibility of the question would also seem to depend upon the match between the structure of the real-world context and the sequence models to be applied. In some questions, this match may be described as *isomorphic*: features of the context map naturally onto the mathematical concepts. However, in other questions, the context has additional features that need to be taken account of before applying a sequence model. Examples of these are:

- w2: this produces a sequence which is a combination of an arithmetic and a geometric progression.
- w7: the initial height of the ball needs to treated separately for the distances to form a geometric progression.
- w9: Susan runs ‘there and back’, so the sequence needs to be doubled at some stage.
- p2: there are two lines of equal length in each part of the spiral.

Thus the real-world context can provide additional elements of complexity to the de-coding of the problem, which may act as additional sources of error.

This appears to be consistent with Shannon’s (2007) observations on formulating linear functions. She tested three different contextual representations of a linear function task – stacking supermarket carts (trolleys), shopping baskets and paper cups – and found differences in facility levels. These she explains not in terms of the familiarity of the items used to the task solvers, but by analysing the ease with which the salient features of the geometry of the stacking diagram could be abstracted into the variables required by the mathematics. She claims that the modelling of everyday objects with mathematics as a motivational tool is relatively unimportant in these tasks, compared to the opportunities they provide for mathematical abstraction and justification.

It is instructive in this regard to compare two of the ‘growth’ questions in the sample which deal with compound interest, questions w2 and w3. In question w2, the candidate is required to construct the second term of the relevant series, whereas in w3, the terms of the series are tabulated in the question, thus making the modelling of the task much easier. Moreover, the formulation of the series in w2 is more demanding, as the expression for the amount owed after each year is a ‘hybrid’ involving the difference of two sequences, the interest on the loan minus the amount paid back.

In some ‘w’ and ‘p’ questions, the sequence to be used to model the context is explicitly named in the question, whereas in others the solver needs to ‘spot’ the correct sequence. It might be conjectured that this will affect the accessibility of the question.

One source of construct-irrelevant variance appears to be the wordiness of the question. The more words the candidate has to assimilate, the more the question becomes a test of their verbal comprehension skills, rather than their mathematical skills. The word-to-mark ratio for the different contextual types were as follows (Table 7.2.1):

Type	w/m
e and a	8.8
w	14.5
p	13.2
All contexts	14.0

Table 7.2.1 Word-to-mark ratios for different AP/GP question contextual types

For this sample, the words per mark ratio was about 60% higher for questions using RWCF than for questions without. This offers strong evidence for the conjecture that candidates have to assimilate substantially more words in contextualised questions than in context-free questions to earn the same marks.

Realism

This aspect clearly does not apply to the 'e' and 'a' categories of question.

The contexts for w2, w3 and w6 are financial, and deal will simple and compound interest on payments, for which arithmetic and geometric progressions provide natural mathematical models – indeed, the mathematics effectively defines these financial models. Question w5, involving a growth pattern of a virus, again provides a natural context for an exponential, or geometric model, as this model is implied by the assumption that the rate of increase of the infected population is proportional to the number of people infected.

In contrast, contexts w1 and w6 are examples of entirely synthetic contexts which have no scientific or financial basis. Indeed, these questions actively play down the 'realism' of the context. For example, the gardeners in w1 are described respectively as 'eccentric' and 'priding himself on his fitness', both descriptions intended to add a justification for their artificially-manufactured planting patterns. The 'widgets' in w6 suggest an unreal, fictitious object, which might imply that the production plans are equally fictitious or, at least, not to be taken too seriously.

The context in w4 may be regarded as partly synthetic and partly natural. It is feasible that the growth patterns of trees might, as with the virus example, be exponential, as in part (ii), or exponential with an added constant representing new planting, as in part (i). However, it is hard to provide scientific justification for the recurrence relation in part (iii), which is clearly designed to elicit an arithmetic series. In practice, it is unlikely that tree planting is managed according to mathematical recurrence relations, and more likely to follow pragmatic laws of supply and demand! In general terms, however, one could argue that recurrence relations can provide authentic mathematical models for population growth over discrete intervals of time.

Turning to the kinematics contexts, questions w7 and w8 have contexts which arise naturally from Newton's Law of Impact, which states that the ratio of the speed of separation to the speed of approach is constant. On the other hand, the 'skittles' context is purely synthetic, as a vehicle for modelling with arithmetic sequences. It is of course entirely possible, even natural, to place the skittles an even distance apart, but no physical or financial laws dictate that this should be done.

Considering the 'pattern' contexts, p1 and p4 suggest activities which are realistic to many children, and illustrate the way in which arithmetic sequences arise naturally from patterns made from objects such as playing cards and matchsticks. Questions p3 and p4 apply arithmetic and geometric sequences to spiral and number patterns which, one could argue, are themselves not 'real-world' but 'mathematical' in nature. Section 3.1 debated whether this type of context should be regarded as 'real-world', and adjudicated in favour of this, on the grounds that the solver is still required to match the appropriate model to the context. Nevertheless, the realism of the application of one aspect of mathematics to another is perhaps a different issue.

What does the foregoing analysis tell us about the concept of 'realism' applied to this sample of questions testing one post-16 mathematics topic? The data in all these questions may be regarded as being *synthetic*, in the sense that the primary purpose of these the questions is to test arithmetic and geometric sequences, and they are 'made up' to achieve this purpose. However, pursuing the distinction suggested in section 6.4, it does appear that some of the contexts are *natural* vehicles for modelling with this particular mathematics, in the sense that some extra-mathematical justification can be provided for this. In these cases, the utility of model and modelled appears to be two-way: not only does the context embody the mathematics in a meaningful way, but the mathematics models the context in a useful way. On the other hand, in a purely synthetic context, such as a skittles race, the context provides an interesting way of 'looking at' arithmetic sequences; but there is little or no practical utility in modelling such races in this way.

Of course, on the one hand, not all candidates will be familiar with exponential growth laws or the physics of Newton's Law of Impact and its mathematical consequences. On the other hand, compound interest and constructing patterns are likely to be familiar to most candidates. Thus, the extent to which these contexts appear 'natural' to candidates may depend upon their prior knowledge.

Whether such prior knowledge confers an advantage to candidates, however, is open to question. Boaler (1994) found that specialist knowledge of the context of 'fashion' effectively handicapped girls by side-tracking them into thinking non-mathematically about the question. It is perhaps less likely that this would occur with the older students sitting these examinations, who are more expert at playing the examination 'game'. Nevertheless, it is possible that students who are well versed in biology or physics might 'miss the point' of these questions, especially when invited to interpret results in these familiar contexts[3].

What is the effect of using synthetic contexts on candidates? Are these questions in some sense less 'valid' in their use of these contexts? Are these examples of 'McGuffins' (Wiliam, 1997) which reinforce the notion that classroom mathematics has little to do with reality? Students are, however, well used in the classroom to reality being manipulated in order to develop mathematical concepts (see, for example, the uses in Realistic Mathematics Education of contexts such as 'Gulliver' (Treffers, 1987)). It is perhaps significant that three questions in the sample hinted at their synthetic nature to candidates by their use of language.

Scrutiny of these *synthetic* contexts suggests that they do not intend to present genuine applications of the mathematics to candidates, but to provide a 'real', albeit artificial, situation and challenge candidates to formulate this in mathematical language. However, there is little pretence of genuine practical utility in such questions: their utility lies, as proposed in section 6.4, in presenting a range of 'realistic' (RME) problems which can be translated into standard mathematical models, which can then be solved using standard algebraic methods.

Moreover, while *natural* contexts reinforce the utility of the mathematics by providing genuine applications, there may be risks in using them. Firstly, there is the problem of candidates' prior knowledge leading them to misunderstand the intention of the question (as

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[3] An example of this is provided by a reduction to linear form question, set by the author, which involved a population of cockroaches. The final question invited candidates to interpret a mathematical result from the question. This elicited many responses from candidates which were cast in terms of their knowledge of the life cycle of cockroaches!

in the ‘cockroach’ example – see footnote). Secondly, there can be problems of accessibility caused by realistically modelling the natural situation. With an artificially defined context, these modelling difficulties can be more readily controlled by the question setter.

Task Authenticity

The third component of the proposed framework is the notion of task authenticity, which measures the extent to which the questions asked in the task are relevant to the context. Is the modelling cycle closed by asking candidates to reflect back on the meaning of their mathematical solutions, and in doing so provide insight into the context? This must surely be an important purpose behind framing questions in context, by pointing to the utility of the mathematical modelling process.

Perhaps the most fundamental difference between contextual questions and pure context-free questions lies in the provision of a subtext, or narrative, within which the mathematics is embedded. In an authentic task, the questions asked have a non-mathematical meaning within this narrative framework. Examples from the sample questions are shown in Table 7.2.2.

Although the degree of realism varies from question to question, from natural to synthetic, the goals of all the contextualised questions are presented in contextual terms. The mathematics serves a purpose other than deriving a mathematical result from the theory of sequences and series. There is a sense in which the mathematical tasks in the question move the narrative forward.

Does this contribute to a sense that mathematics does indeed serve purposes beyond its own horizons? Or does the artificiality of the context undermine any utility value, and merely present, to reiterate Wiliam, ‘mathematics looking for somewhere to happen’? As discussed in the previous chapter, it is perhaps the diversity of the contexts to which the ‘term’ and ‘sum’ formulae for APs and GPs can be applied which enhances the mathematical functionality of these results.

One further point pertaining to the relationship between context and mathematics as exemplified by our sample of questions is that, as has already been pointed out, this relationship may be two-way. Not only can the mathematics serve the context, but vice-versa: the context can serve to illustrate and elucidate aspects of the mathematics. An interesting example of this is the ‘spiral’ question (p2), which encourages students to provide a geometrical image of the structure of arithmetic and geometric sequences. Other examples

Question	Subtext or narrative	Question posed
w1	Two gardeners are spreading 4000 kg of sand over a garden.	How long will it take?
w2	Mr and Mrs Brown are borrowing £50 000 to buy a house	How much are the re-payments?
w3	Anne and Brian have £100 to invest.	Which plan is more profitable?
w4	Oaks, beeches and pines are growing in a forest.	Are there numbers declining, stable or growing?
w5	A virus is spreading.	How long before there are 5000 cases?
w6	Production of new widgets is replacing old widgets.	How long before the new overtakes the old?
w7	A ball is bouncing.	Can we predict its bounce? How long before it stops?
w8	A ball is rebounding between walls.	Can we predict how long it will take? How many rebounds in 15 minutes?
w9	Setting out a skittles race.	How long is the total race?
p1	Building houses of cards	How many cards does it take?
p2	Spiral patterns	Can we predict their length, and what they look like?
p3	Arrays of numbers	What is the sum of the numbers in the array?

Table 7.2.2 Questions posed relevant to the real-world context

are the questions which apply the sum to infinity result to reinforce the notion implicit in Zeno's paradox of Achilles and the tortoise, that an infinite process can have a finite sum.

Although all of these questions with RWCF pose tasks which relate to the context, in none of these does the task include any evaluation of the appropriateness of the model. They therefore conform to the truncated form of the modelling cycle, as proposed in Fig. 6.4.2. Finally, it would require further research to establish whether the tasks are of interest to the solver, although Chapter 8 gives some indication, through a questionnaire of student opinion, of how students feel about RWCF in sequence questions.

7.3 Summary

I now summarise what has been learnt from scrutinising this sample of sequence questions with and without RWCF, classifying them, and applying the ARTA framework, and relate these findings to the issues raised in the introduction.

Accessibility

1. There is strong evidence that contextualised questions are more wordy than non-contextualised questions, and consequently impose greater tests of comprehension.

2. The structure dictated by the context may present different levels of complexity in the modelling process, which may affect accessibility. In the case where the match is *isomorphic*, the transfer from context to mathematical model is relatively straightforward. In other cases, the solver is required to take account of features in the context in mapping the context to the mathematical model.

3. Some contexts, especially naturally occurring ones, are more familiar to students than others. The novelty of the context is likely to affect its accessibility.

Realism

1. Contexts may be classified as *natural* in cases where extra-mathematical justification exists for modelling them with the mathematical content being tested, or *synthetic*, in cases where the context is chosen and manipulated to fit the mathematical content. Synthetic contexts are designed to fit the mathematics, but in these cases the mathematics is less likely to be of practical utility in modelling the context.

2. The perceived reality of the context need not be less in synthetic contexts than in naturally-occurring contexts, and may depend on the knowledge of the solver.

Task authenticity

1. In all the contexts considered here, the questions posed are relevant to the context. This furnishes the questions with a sense of purpose which is absent from the pure mathematical questions.

2. However, artificially constructed contexts may have a negative effect on the perceived utility value of mathematics to candidates.

3. Some of the contexts used contribute to the understanding of the mathematics, by requiring students to think of the mathematical ideas in novel and unexpected ways.

4. None of these questions include any evaluation of the mathematical model.

Returning to the issues raised in the earlier chapters of this thesis, it appears that the negative effects of context reported by some researchers who have questioned their validity and value need to be weighed carefully against some of the potential benefits proposed above. More research is clearly needed to ascertain whether evidence of some of these effects, derived from detailed analysis of a sample of questions, can be verified using students' responses and attitudes to the questions.

Messick (1989) has emphasised that the validity of test items is a function not of the task but of the way evidence accrued from it is used. The validity of contextualised pure mathematical questions depends crucially on one's construct of mathematical ability. If this embraces the notion of modelling, albeit in the relatively restricted form (see Fig. 6.4.2) required to negotiate short questions in high-stakes timed written examinations, then one may be inclined to overlook, or at least limit, the risk of construct-irrelevant variance caused by wordy, novel, complicated questions, in favour of questions which represent the construct effectively.

Little has been said so far in this research of the use made by examination questions in the classroom. The consequential validity (see section 3.2.4) derived from their use in teaching sessions cannot be under-emphasised (Niss, 1993, Cockcroft, 1982). Some may argue that context-rich questions should be preserved for the classroom, where the students' interests can be engaged, but that they should be sacrificed in summative, high-stakes end of course examinations in favour of more reliable and controllable tasks.

An alternative viewpoint is that it is vital that questions set in these examinations, especially without the additional evidence of coursework, need to fully reflect the construct of mathematics to be assessed, as articulated by agreed assessment objectives. In setting questions which have an intrinsic interest and novelty, one may risk losing some degree of test validity – as a Principal Examiner I have constructed a number of questions which I know have failed to work well in the examination room. However, examination papers wield such a powerful influence on how the subject is taught, learned and perceived, that this may be a sacrifice worth making. Further research into understanding the relationship between context and content in questions may help to achieve the appropriate balance.

This theoretical analysis, using the ARTA model, employed on A/AS questions has served to throw light on the function of RWCF in questions at this level. However, in order to research its effect on questions (RQ3), it is necessary to collect and analyse data on solvers' responses to questions, to enable comparison to be made of outcomes for questions with and without real-world contextual framing. The next chapter describes a study designed to collect and analyse such data.

PART III

CHAPTER 8

FINDINGS: THE EFFECT OF CONTEXT IN ARITHMETIC AND GEOMETRIC SEQUENCE QUESTIONS

Overview

In the previous two chapters, A/AS pure mathematics questions were analysed in order to develop an understanding of the function of RWCF. In chapter 6, a selection of sample questions were analysed. In chapter 7, the focus of the analysis was narrowed to consider one topic, that of sequences and series, in order to compare and classify questions with and without real-world contexts. Each chapter used the ARTA framework (section 3.3) as an evaluative tool in this analysis.

However, in order to assess the *effect* of RWCF in questions, it is necessary to study not just the inputs to the assessment process – the questions – but also outputs – how students respond to answering questions with and without RWCF. The aim of this part of the study is to explore the effect of RWCF on the facility of questions, using the topic of sequences. Section 8.1 discusses the methods adopted for this study, including the categories of question types adopted, the design of the tests, and a questionnaire to investigate students' opinions of RWCF, a pilot study, details of the sample of students used, and finally further discussion of ethical considerations specific to the study. Section 8.2 presents the analysis of test data collected from the study, and Section 8.3 discusses these results. Sections 8.4 and 8.5 analyse and discuss the results of the student questionnaire. The Instructions to Centres, test versions and mark schemes are provided in Appendix 7.

8.1 Methodology

Versioning of the test

Chapter 8 analysed the role of context in sequence questions, and categorized the type of context as follows:

- *Explicit* (e): non-contextualised questions which explicitly use the terms 'arithmetic progression' or 'geometric progression, 'term', 'sum', without employing mathematical notations u_n and $\sum_{r=1}^{n} u_n$;

- *Algebraic* (a): non-contextualised questions which employ the mathematical notations u_n and $\sum_{r=1}^{n} u_n$, and sequences defined inductively;
- *Word* (w): questions which describe in words arithmetic or geometric sequences in growth contexts;
- *Pattern* (p): questions which contextualise arithmetic or geometric sequences using a pattern context, described in words and through a diagram.

The tests utilise four versions (e, a, w and p) of four arithmetic (AI-IV) and four geometric (GI-IV) questions, each asking the same (or as similar as possible) part questions, with the same tariffs, about each sequence.

Four versions (A, B, C and D) of a test were constructed, according to the scheme in Table 8.1.1.

Test	A	B	C	D
Qu 1	AIe	AIIe	AIIIe	AIVe
Qu 2	GIIIw	GIVw	GIw	GIIw
Qu 3	GIIa	GIIIa	GIVa	GIa
Qu 4	AIVp	AIp	AIIp	AIIIp
Qu 5	AIIIw	AIVw	AIw	AIIw
Qu 6	GIVp	GIp	GIIp	GIIIp
Qu 7	GIe	GIIe	GIIIe	GIVe
Qu 8	AIIa	AIIIa	AIVa	AIa

Table. 8.1.1 Make-up of AP/GP test versions

The test versions, together with mark schemes, are listed in Appendix 7.

The student questionnaire

As well as collecting data on the performance of students on questions of varying contextual types, it is relevant to RQ3 on the function and effect of RWCF to ask the students themselves about their attitudes and opinions on real-world context and its use in A/AS questions. In addition to the test, a short student questionnaire is used, with the objective of investigating student attitudes to pure and applied mathematics, and the use of real-world contexts in questions. The questionnaire is included in Appendix 7.

The pilot study

The tests were piloted in June 2008 using a sample of 40 AS students from two AS mathematics classes from a local sixth-form college. Students were given 55 minutes to complete the tests, which were administered one week before they sat their AS level examinations. The quality of the students who piloted the test was very variable, with one AS group being considerably weaker than the other. The tests were allocated to the students in the order A, B, C, D, A, … etc.

The lessons learned and issues raised from this pilot were as follows:

- Many students failed to complete the test in the time available.

In order to address this issue, the test was shortened to 40 marks instead of 50 marks.

- Students' attempts at later questions in the test proved to be weaker than in the first few questions.

This second point may have been caused by the initial structure of the tests, which started with shorter 'e' and 'a' questions, and finished with longer 'w' and 'p' questions (see Table 8.1.2).

Test	A	B	C	D
Qu 1	AIe	AIIe	AIIIe	AIVe
Qu 2	GIe	GIIe	GIIIe	GIVe
Qu 3	GIIa	GIIIa	GIVa	GIa
Qu 4	AIIa	AIIIa	AIVa	AIa
Qu 5	AIIIw	AIVw	AIw	AIIw
Qu 6	GIVp	GIp	GIIp	GIIIp
Qu 7	GIIIw	GIVw	GIw	GIIw
Qu 8	AIVp	AIp	AIIp	AIIIp

Table 8.1.2 Initial structure of pilot tests

For this reason the order of questions in the tests was altered, by interchanging questions 2 and 7, and questions 4 and 8, so that the longer 'w' and 'p' versions were not at the end of the test. This resulted in the structure shown in Table 8.1.1.

- Students were in the middle of their examination period, and had done variable amounts of revision on this topic.

Given the practical difficulties of timetabling in such a test, a degree of flexibility was allowed to centres participating in the study, either as a topic test following the teaching of a module on APs and GPs, or as an examination revision test, later in the AS course.

Students who had not revised the topic might be unable to make any progress with the test. Students in the AS Examination are provided with a booklet with mathematical formulae. For these reasons, formulae for arithmetic and geometric progressions were printed on the test, as they appear in the OCR Formula book for A/AS Mathematics. It was hoped that the availability of these formulae, together with some prior warning, would avoid the test being a negative experience for students through lack of preparation.

The sample

The sample of students taking the test and the questionnaire needs to be sufficiently large to detect differences in performance on four different test versions. For this reason, a large sample was desirable, which allows detailed analysis of results for each question version. The final sample comprised 625 year 12 students from four centres, 594 of whom completed tests and 525 questionnaires, with one very large centre supplying the bulk of students. The make-up of the sample was as in Table 8.1.3.

	Centre type	Total
A	Sixth form college	531
B	Sixth form college	71
C	Independent day school	14
D	comprehensive school	9

Table 8.1.3 Structure of sample (AP/GP study)

As all AS students were invited to participate from each centre, the sample might be characterised statistically as a ‘cluster sample’. However, the sample may not be representative of the AS Mathematics population as a whole, for a number of reasons, for example:

- the preponderance of results from one large college;
- the lack of a balance of centre types (independent, maintained, etc.);
- different syllabuses being used;
- unrepresentative achievement/ability levels of students.

This lack of sample representativeness needs to borne in mind when considering the validity of results, and the interpretation of findings. However, the large sample of students used partially validates this; it is also possible that variations caused by, for example, a different balance of centre types, are unlikely to be systematic when applied to the hypotheses and investigations considered.

Nevertheless, in order to validate the generality of results for the AS mathematics population taken as a whole, the study may need replication in a balanced range of centres.

Ethical considerations

The study was run in accordance with the ethical protocol guidance of the University of Southampton School of Education, in order to ensure responsibilities to the participants – schools, teachers and students – the sponsors – the University of Southampton – and the educational research community were considered and respected. In particular:

It was made clear to participating centres that:

- students were free to withdraw from allowing the test to be analysed by writing ‘object’ on the script;
- students were not obliged to complete the questionnaire;
- strict anonymity of schools and students was observed in reporting the research;
- photocopies of scripts, and completed questionnaires, would be kept securely, and made available to the research supervisor for verification of results.

A copy of the completed ethical protocol guidance form is included in Appendix 6.

8.2 Analysis of the AP/GP test data

The mean number of marks scored per question is shown in Table 8.2.1 and Fig. 8.2.1.

From the table and the figure, it can be seen that the ‘e’ (explicit) versions, as might be expected, gained higher marks than the ‘a’, ‘w’ and ‘p’ versions. Overall, the ‘e’ versions scored on average approximately 14% higher than the ‘w’ versions, and 12% higher than the ‘p’ versions. However, this is not true of all questions – see GII, for example, where the ‘w’ version (word) scored slightly higher than the ‘e’ version. The causes of these variations require more detailed investigation of the results for each question. This is done below.

	AI	AII	AIII	AIV	GI	GII	GIII	GIV	totals
Max score	5	6	5	4	5	3	6	6	40
explicit	4.36	4.41	4.22	2.99	3.13	1.01	3.82	4.49	28.43
algebraic	3.33	2.25	4.08	1.68	2.83	0.83	3.27	3.64	21.91
word	3.37	4.03	3.90	2.02	2.51	1.20	3.22	3.92	24.17
pattern	4.04	3.51	3.92	2.43	2.08	0.80	3.67	4.48	24.91

Table 8.2.1: Mean number of marks per question (AP/GP study)

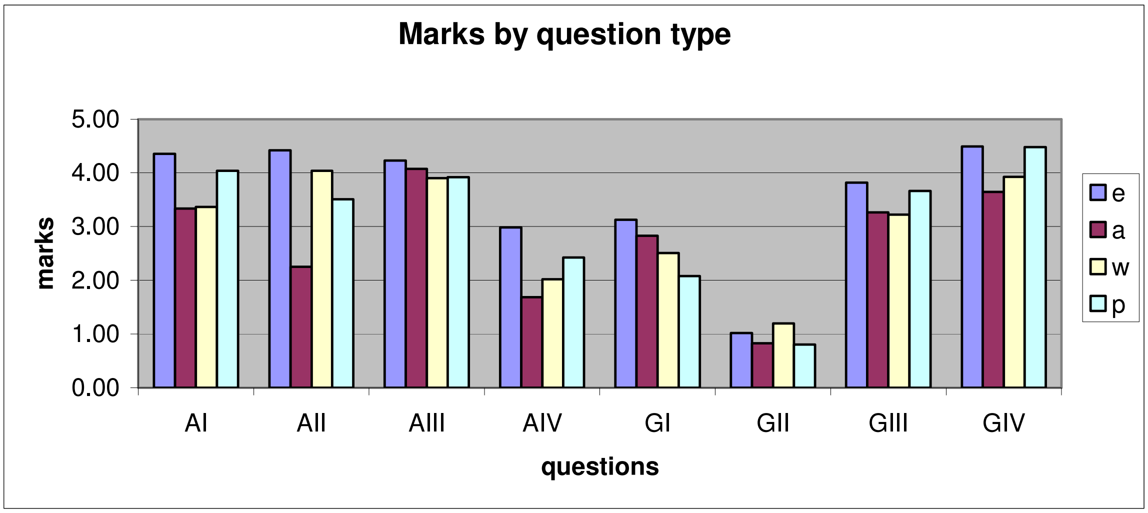


Fig. 8.2.1: Mean marks per question. (AP/GP study)

I now compare the facility of different versions (e, a, w and p) using difference of two means tests. Two-tailed test are used, which measure whether the mean scores for two versions are significantly different. For ease of reference, the four versions are reproduced first. For each part question, the mean mark for the ‘e’ version is then compared with the mean mark for the ‘a’, ‘w’ and ‘p’ versions, and the ‘w’ version with the ‘p’ version. An analysis of the questions then proposes reasons for the differences in mean scores.

In conducting a large number of comparisons in this way, one must clearly be aware of the increased probability of Type I errors, and this should be borne in mind in the following analysis, especially in cases where the probability of such an error is relatively high. Nevertheless, it is revealing to consider each individual part question in order to formulate possible reasons for differences in facility.

A1e (A1) An arithmetic progression has first term 7 and common difference 3.

(i) Which term of the progression equals 73?
[3]

(ii) Find the sum of the first 30 terms of the progression.
[2]

AIa (D8) The *n*th term of an arithmetic progression is denoted by u_n. $u_1 = 7$, $u_2 = 10$ and $u_3 = 13$.

(i) If $u_n = 73$, find n.
[3]

(ii) Find $\sum_{r=1}^{30} u_r$.

[2]

AIw (C5) Chris saves money regularly each week. In the first week, he saves £7. Each week after that, he saves £3 more than the previous week.

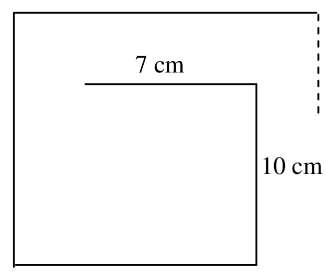
(i) In which week does he save £73?
[3]

(ii) Find his total savings after 30 weeks.
[2]

AIp (B4) A spiral is formed with sides of lengths 7 cm, 10 cm, 13 cm, … which are in arithmetic progression.

(i) How many sides does the spiral have if its longest side is 73 cm? [3]

(ii) Find the total length of the spiral with 30 sides. [2]



x	y	$\overline{x}$	$\overline{y}$	s_x	s_y	z	p
(i)e	(i)a	2.60	2.35	0.91	1.16	-2.08	0.0094
(i)e	(i)w	2.60	2.19	0.91	1.27	-3.19	0.0004
(i)e	(i)p	2.60	2.52	0.91	0.93	-0.74	-
(ii)e	(ii)a	1.76	0.99	0.57	0.98	-8.41	0.0000
(ii)e	(ii)w	1.76	1.18	0.57	0.95	-6.47	0.0000
(ii)e	(ii)p	1.76	1.52	0.57	0.80	-2.95	0.0008
(i)w	(i)p	2.19	2.52	1.27	0.93	2.54	0.0028
(ii)w	(ii)p	1.18	1.52	0.95	0.80	3.40	0.0002

Table 8.2.2 A1 difference of two means test data

Analysis

Both ‘a’ versions proved to be significantly harder than the ‘e’ version. In particular, the use of sigma notation in part (ii) reduced the mean mark from 1.76 to 0.99. Similarly, the ‘w’ version proved significantly harder than the ‘e’ version, with a more significant difference (z

= –6.47) in part (ii). Comparing the ‘e’ with the ‘p’ versions, there was no significant difference in scores for part (i), but part (ii) was significantly harder in the ‘p’ version. Finally, comparison of the ‘w’ and ‘p’ versions shows the ‘w’ version to be significantly harder than the ‘p’ version.

The ‘e’ v ‘a’ results are easily explained in terms of the additional demand of interpreting the question when set in algebraic notation, with the use of sigma notation adding substantially to the demand. In the ‘e’ version, the explicit reference to arithmetic progression’, ‘term’ and ‘sum’ leads the solver directly to the appropriate formulae (given on the question paper). These cues are not present in the ‘w’ version, in which solvers are required to translate elements from the real-world context to the algebraic model (‘first week £7’ = a, ‘£3 more’ = d). This would account for the extra difficulty of the ‘w’ version compared to the ‘e’ version.

On the other hand, the ‘p’ version of part (i) proved to be no harder than the ‘e’ version. This might be because the first three terms of the sequence are stated explicitly (7 cm, 10 cm, 13 cm, …), thus making the match to an AP model easier than in the ‘w’ version. Indeed, it is possible to think within the context to derive the number of terms (length increased by $67 = 3 \times 29$, so 30 sides). This type of ‘first principles’ thinking is not available in part (ii), which perhaps explains why this proved harder than the ‘e’ version.

The explicit statement of the first three terms in the ‘p’ version, thus hinting at an arithmetic sequence model, might also explain why this version proved to be easier than the ‘w’ version, where this cue was not given.

AIIe (B1) An arithmetic progression has first term 2 and common difference 3.

(i) Prove that the sum of n terms of the arithmetic progression is ½ $(3n^2 + n)$. [3]

(ii) Given that the sum of n terms is 1855, find n. [3]

AIIa (A8) A sequence u_r is defined by $u_1 = 2$, $u_{n+1} = u_n + 3$.

(i) Prove that $\sum_{r=1}^{n} u_r = \frac{1}{2}(3n^2 + n)$. [3]

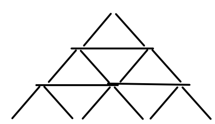
(ii) Given that $\sum_{r=1}^{n} u_r = 1855$, find n. [3]

AIIw (D5) The number of new cases of infection from a virus goes up by three each day. On the first day, there were 2 cases, on the second day there were 5 new cases, on the third day 8 new cases, and so on.

(i) Prove that the total number of cases after n days is ½ $(3n^2 + n)$. [3]

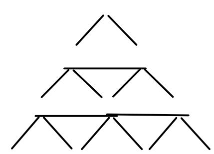
(ii) After how many days has the total number of cases reached 1855? [3]

AIIp (C4) Some people use playing cards to build 'houses'. A house with 3 layers is illustrated below.



The diagram below shows the separate layers of the house. Each line represents one card. The layers are numbered from the top downwards. Further layers are built in the same way.

Layer 1 has 2 cards
Layer 2 has 5 cards
Layer 3 has 8 cards



(i) Prove that there are ½ $(3n^2 + n)$ cards in a house with n layers. [3]

(ii) A house is made with exactly 1855 cards. How many layers does it have? [3]

x	y	$\overline{x}$	$\overline{y}$	s_x	s_y	z	p
(i)e	(i)a	2.48	0.95	1.00	1.36	-10.86	0.0000
(i)e	(i)w	2.48	2.01	1.00	1.35	-3.37	0.0002
(i)e	(i)p	2.48	1.52	1.00	1.49	-6.52	0.0000
(ii)e	(ii)a	1.94	1.33	1.13	1.40	-4.08	0.0000
(ii)e	(ii)w	1.94	2.02	1.13	1.22	0.60	-
(ii)e	(ii)p	1.94	1.99	1.13	1.21	0.36	-
(i)w	(i)p	2.01	1.52	1.35	1.49	-3.04	0.0006
(ii)w	(ii)p	2.02	1.99	1.22	1.21	-0.24	-

Table 8.2.3 AII difference of two means test data

Analysis

In both parts, the algebraic versions proved to be substantially harder than the explicit versions, no doubt because of the deployment of sigma notation.

In part (i), both the ‘w’ and ‘p’ versions proved harder. This can be accounted for by a larger number of instances where the ‘term’ formula was used instead of the ‘sum’ formula. In the ‘w’ version, the word ‘total’ is easily missed, leading to this error; in the ‘p’ formula, solvers must realise that it is the total number of cards required, not the cards in the n^{th} layer. The (i)w v (i)p result suggests that the latter difficulty is more pronounced. This may be because there is no explicit word such as ‘total’ to cue the sum: the pattern context needs to be understood clearly before selecting the appropriate formula.

The results for part (ii) in versions e, w and p were not significantly different. This may be explained by the fact that the ‘sum’ result is given in part (i), and equated to 1855 in each case. Thus no context-specific thought is required to generate the required equation.

AIIIe (C1) An arithmetic progression starts 7, 11, 15, …

(i) Write down the next term, and find the nth term, simplifying your answer. [3]

(ii) The progression ends with the term 175. How many terms are there? [2]

AIIIa (B8) The sequence u_n is an arithmetic progression. $u_1 = 7$, $u_2 = 11$ and $u_3 = 15$.

(i) Write down u_4 and u_n, simplifying your answer. [3]

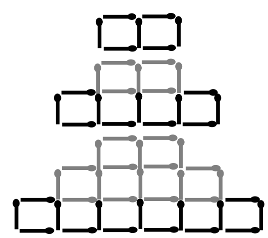
(ii) Given that $u_n = 175$, find n. [2]

AIIIw(A5) A factory makes cars. In its first week, it completes 7 cars. In the second week, it completes 11 cars, and in the third week 15 cars. Production continues to rise by four additional cars each week.

(i) Write down how many cars are completed in the fourth week and the nth week, simplifying your answer. [3]

(ii) Find the week number in which 175 cars are made. [2]

AIIIp (D4) Jenny is making a pattern consisting of rows of matchstick squares.



She uses 7 matches to complete a first row of 2 squares.

She uses 11 matches to complete a second row of 4 squares.

She uses 15 matches to complete a third row of 6 squares.

She continues adding rows to the pattern in this way.

(i) Find how many additional matches are needed to complete
(A) the fourth row,
(B) the nth row, simplifying your answer. [3]

(ii) Which row of the pattern needs 175 matches to complete? [2]

x	y	$\overline{x}$	$\overline{y}$	s_x	s_y	z	p
(i)e	(i)a	2.53	2.43	0.79	0.92	-0.99	-
(i)e	(i)w	2.53	2.27	0.79	1.00	-2.50	0.0031
(i)e	(i)p	2.53	2.35	0.79	0.95	-1.81	0.0176
(ii)e	(ii)a	1.69	1.65	0.69	0.74	-0.46	-
(ii)e	(ii)w	1.69	1.64	0.69	0.74	-0.62	-
(ii)e	(ii)p	1.69	1.56	0.69	0.77	-1.53	-
(i)w	(i)p	2.27	2.35	1.00	0.95	0.73	-
(ii)w	(ii)p	1.64	1.56	0.74	0.77	-0.87	-

Table 8.2.4 AIII difference of two means test data

Analysis

In this question, there were no significant differences between the explicit and algebraic versions, though the ‘a’ versions were slightly harder on average. This suggests that the u_n notation for term number is not as difficult to understand as the sigma notation deployed in questions AI and AII.

Comparing the ‘e’, ‘w’, and ‘p’ results for part (i), the explicit version proved significantly easier than the ‘w’ and ‘p’ questions, which were of similar difficulty. In each of the latter forms, the terms 7, 11, 15 were given but without explicitly describing the sequence as ‘arithmetic’, as in the ‘e’ version.

In part (ii), marks for the ‘e’ and ‘w’ versions were not significantly different. This may be because it is possible to think within the context here, for example $175 - 7 = 168 = 4 \times 42$, so week 43. Comparing the ‘e’ and ‘p’ versions, though the difference does not quite attain significance at 5%, there is evidence that the ‘p’ version is a little harder, which might be caused by using the ‘sum’ formula, thinking that 175 represents the total number of matches instead of the row number.

AIVe (D1) An arithmetic progression has common difference 0.5.

The sum of 30 terms of this progression is 360. Find the first term. [4]

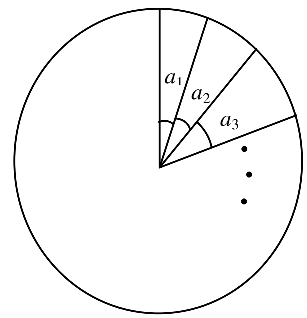
AIVa (C8) An arithmetic progression v_n has common difference 0.5, and $\sum_{r=1}^{30} v_r = 360$. Find v_1. [4]

AIVw (B5) Beth invests an amount on the first day of each month, starting in January. She increases the amount she invests each month by 50p, and finds that she has invested £360 after 30 months.

What was her initial investment? [4]

AIVp(A4) In this question, a circle consists of a sequence of sectors with angles $a_1, a_2, a_3, \ldots$ as shown in the diagram.
The angles are measured in degrees, and form an arithmetic progression with common difference is 0.5°.

Given that 30 sectors fill the circle exactly, find a_1. [4]



x	y	$\bar{x}$	$\bar{y}$	s_x	s_y	z	p
e	a	2.99	1.68	1.58	1.88	-6.56	0.0000
e	w	2.99	2.02	1.58	1.93	-4.72	0.0000
e	p	2.99	2.44	1.58	1.85	-2.75	0.0015
w	p	2.02	2.44	1.93	1.85	1.87	0.0154

Table 8.2.5 AIV difference of two means test data

Analysis

The algebraic version uses sigma notation, which, as before, makes it substantially more difficult. Both the ‘w’ and ‘p’ versions are significantly harder than the ‘e’ version, with the ‘w’ version harder than the ‘p’ version. In both, the solver is required to recognise the implicit requirement to sum 30 terms of an AP within the context, whereas this is explicitly cued in the ‘e’ version. The difference between the ‘w’ and ‘p’ versions may be ascribed to the change of units required in the ‘w’ context (50p = £0.5).

GIe (A7) A geometric progression starts 9, 12, 16, …

(i) Verify that the first three terms of this sequence are in geometric progression, and find the common ratio. [2]

(ii) Find the 10th term of the sequence. [2]

(iii) Describe the behaviour of the nth term of the sequence as n gets larger and larger. [1]

GIa (D3) A geometric sequence u_n is defined by $u_1 = 9$, $u_{r+1} = u_r \times 4 \div 3$.

(i) Write down u_2 and u_3, and state its common ratio. [2]

(ii) Find $u_{10.}$ [2]

(iii) Describe the behaviour of u_n as n gets larger and larger. [1]

GIw (C2) The mass of a substance grows in geometric progression. It is initially 9 grams, and increases by 1/3 each hour.

(i) Write down the mass of the substance after 1 hour and after 2 hours, and the common ratio of the geometric progression. [2]

(ii) Find the mass of the substance after 9 hours. [2]

(iii) Describe the behaviour of the mass of substance after n hours as n gets larger and larger. [1]

GIp (B6) Figures 1, 2, and 3 show a sequence of patterns created from an equilateral triangle of side 3 cm. To get the next pattern in the sequence, each side 'grows' a triangular 'spike' as illustrated below:

is replaced by

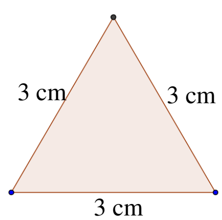


Fig. 1

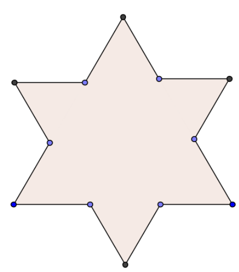


Fig. 2

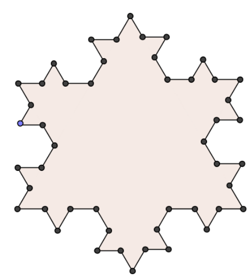


Fig. 3

(i) Write down the perimeters of figures 1 and 2. Given that the perimeters of the figures are

in geometric progression, find the common ratio. [2]

(ii) Find the perimeter of the 10th figure. [2]

(iii) Describe the behaviour of the perimeter of Figure n as n gets larger and larger. [1]

x	y	$\overline{x}$	$\overline{y}$	s_x	s_y	z	p
(i)e	(i)a	1.45	1.39	0.74	0.88	-0.63	-
(i)e	(i)w	1.45	1.49	0.74	0.74	0.41	-
(i)e	(i)p	1.45	0.90	0.74	0.91	-5.60	0.0000
(ii)e	(ii)a	1.53	1.24	0.78	0.91	-2.90	0.0001
(ii)e	(ii)w	1.53	0.89	0.78	0.72	-7.25	0.0000
(ii)e	(ii)p	1.53	1.05	0.78	0.79	-5.19	0.0000
(iii)e	(iii)a	0.16	0.20	0.37	0.40	0.82	-
(iii)e	(iii)w	0.16	0.13	0.37	0.33	-0.85	-
(iii)e	(iii)p	0.16	0.12	0.37	0.35	-0.84	-
(i)w	(i)p	1.49	0.90	0.74	0.91	-6.03	0.0000
(ii)w	(ii)p	0.89	1.05	0.72	0.79	1.75	0.0201
(iii)w	(iii)p	0.13	0.12	0.33	0.35	-0.02	-

Table 8.2.6 GI difference of two means test data

Analysis

The versions of this question differ more substantially than in previous questions, and differences in facility may therefore be explained by these differences. (i)e asks for a verification of the GP, whereas (i)a, (i)w and (i)p asks for the second and third terms to be calculated. In this part, the pattern version proved to be substantially more difficult. This was caused by a misinterpretation of the figures, in which, notwithstanding the explanation in the preamble, the length of the side and perimeter of the star in Fig. 2 was taken to be 3 cm and 27 cm., giving a common ratio of 3 rather than 4/3. This misinterpretation was unintended and may be regarded as a fault in the design of the question.

Although some follow-through was allowed for $r = 2$ in question (ii), the mark for (ii)p suffered from this misinterpretation. Another common error, which may account for the difference between the mean score for (ii)e and (ii)a, was calculating u_{10} as $9 \times (4/3)^{10}$.

The results for (iii) were poor, as only the answers ‘tends to infinity’, or ‘grow exponentially’ were allowed. Other answers are, arguably, worthy of credit, for example ‘grows without limit’. For this reason, the results for this question are unreliable and may be discounted for the purposes of this research.

Nevertheless, unintended ambiguities such as that described in the ‘p’ version above are relevant to the validity of questions utilising real-world contextualisation.

GIIe (B7) A geometric progression is such that its 20th term is three times its 10th term.

The first term is not zero, and the common ratio is positive.

Find the common ratio, giving your answer to 3 significant figures. [3]

GIIa(A3) The *n*th term of a geometric progression with common ratio *r* is denoted by u_n.

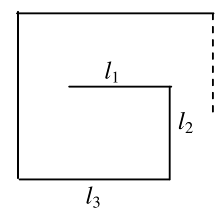
Given that $u_{20} = 3\ u_{10}$, $u_1 \neq 0$ and $r > 0$, find *r*, giving your answer to 3 significant figures. [3]

GIIw (D2) Chris saves money regularly each week. In the first week, he saves £*a*, where *a* is greater than zero. Each week after that, he saves *r* times what he saves in the previous week.

Given that in week 20 he saves three times what he saves in week 10, find *r*, giving your answer to 3 significant figures. [3]

GIIp (C6) A spiral is formed with sides whose lengths l_1, l_2, l_3, … are in geometric progression, with common ratio *r* (see diagram).

Given that the length of the 20th side is three times the length of the 10th side, find *r*, giving your answer to 3 significant figures. [3]



x	y	$\bar{x}$	$\bar{y}$	s_x	s_y	z	p
e	a	1.01	0.84	1.25	1.18	-1.22	-
e	w	1.01	1.20	1.25	1.28	1.25	-
e	p	1.01	0.80	1.25	1.16	-1.51	-
w	p	1.20	0.80	1.28	1.16	-2.81	0.0013

Table 8.2.7 GII difference of two means test data

Analysis

Scores for all versions of the question were low, the order from lowest to highest being p < a < e < w.

Only the comparison between p and w reaches significance at 5%. There was some evidence of ‘thinking within the context’ of the ‘w’ version, which may account for the slightly higher mean mark.

GIIIe (C7) A geometric progression has first term 90 and common ratio ¾.

(i) How many terms of the progression are greater than one? [4]

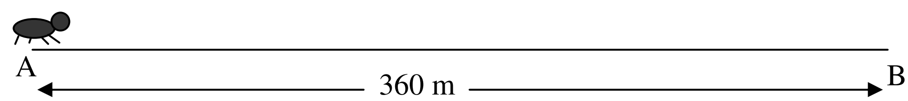
(ii) Find the sum to infinity of the progression. [2]

GIIIa (B3) A sequence u_n is defined by $u_1 = 90$, $u_{r+1} = ¾\, u_r$.

(i) How many terms of the sequence are greater than one? [4]

(ii) Find the sum to infinity of the sequence. [2]

GIIIw(A2) A beetle starts at point A and moves in a straight line towards point B, 360 metres from A.

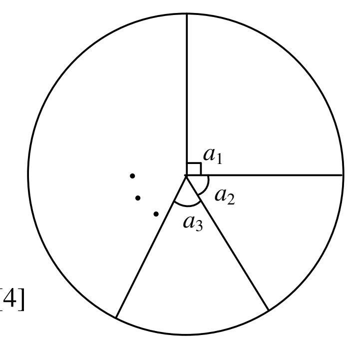


In the first minute, the beetle covers 90 metres. In each minute thereafter, the distances it covers form a geometric progression with common ratio ¾.

(i) Find for how many minute intervals the beetle covers at least 1 metre. [4]

(ii) Show that the beetle never reaches B. [2]

GIIIp (D6) In this question, a circle consists of a sequence of sectors with angles a_1, a_2, a_3, … as shown in the diagram.



The angles are measured in degrees, and form a geometric progression with $a_1 = 90°$ and common ratio ¾.

(i) Find how many sectors have an angle greater than 1°. [4]

(ii) Show that no matter how many sectors are used they will always fit into the circle. [2]

x	y	$\overline{x}$	$\overline{y}$	s_x	s_y	z	p
(i)e	(i)a	2.16	1.79	1.81	1.85	-1.78	0.0188
(i)e	(i)w	2.16	2.13	1.81	1.76	-0.19	-
(i)e	(i)p	2.16	2.45	1.81	1.78	1.39	-
(ii)e	(ii)a	1.65	1.48	0.72	0.83	-1.94	0.0131
(ii)e	(ii)w	1.65	1.12	0.72	0.79	-6.04	0.0000
(ii)e	(ii)p	1.65	1.22	0.72	0.89	-4.72	0.0000
(i)w	(i)p	2.13	2.45	1.76	1.78	1.59	0.0280
(ii)w	(ii)p	1.12	1.22	0.79	0.89	1.00	-

Table 8.2.8 GIII difference of two means test data

Analysis

For both parts, the algebraic version proved to be significantly harder than the explicit. This can be accounted for by the use of a recurrence relation to define the geometric sequence.

The results for (i)e and (i)w are not significantly different. As (i)w explicitly defines the sequence as in (i)e, the versions are not dissimilar. The sequence is also defined explicitly in (i)p, and the results for (i)p are in fact slightly better than (i)e, though not attaining significance at the 5% level (z = 1.39). There was some evidence of first principles calculator work on the size of the sectors, which may account for this slightly better average mark.

In part (ii), the lower w and p scores are explained by a difference in mark schemes: for the ‘e’, marks, it was sufficient to give the sum to infinity, but for the ‘w’ and ‘p’ marks, solvers needed to compare this to 360 to achieve the final mark.

GIVe (D7) A sequence starts 5, 10, 20, 40, …

(i) Assuming the sequence continues with the same pattern, write down the next term. [1]

(ii) Describe the sequence. [2]

(iii) Find the sum of the first 20 terms of the sequence. [3]

GIVa (C3) A sequence u_n starts $u_1 = 5$, $u_2 = 10$, $u_3 = 20$, $u_4 = 40$, …

(i) Assuming the sequence continues with the same pattern, write down u_5. [1]

(ii) What type of sequence is u_n? Write down a formula for u_{n+1} in terms of u_n. [2]

(iii) Find $\sum_{r=1}^{20} u_r$. [3]

GIVw (B2) James records his expenditure in £ each week as follows:

Week 1: £5 Week 2: £10 Week 3: £20 Week 4: £40

(i) If he continues this unlikely pattern of expenditure, write down how much he spends in week 5. [1]

(ii) Describe the sequence formed by the amounts he spends. [2]

(iii) Assuming he carries on spending according to this sequence, find out his total expenditure after 20 weeks. [3]

GIVp (A6) The diagram below shows an array of numbers. Each row starts with a 3 and ends with a 2. Each of the other numbers is formed, as in Pascal's triangle, by adding two numbers from the row above.

Row 1					3		2				
Row 2				3		5		2			For example, 8 = 3 + 5
Row 3			3		8		7		2		
Row 4		3		11		15		9		2	

(i) Write down the next row of the table. [1]

(ii) Write down the sum of the numbers in (a) row 1, (b) row 2, (c) row 3 and (d) row 4. Describe the sequence formed by these four numbers. [2]

(iii) Find the sum of all the numbers in an array of 20 rows. [3]

x	y	$\overline{x}$	$\overline{y}$	s_x	s_y	z	p
(i)e	(i)a	0.93	1.00	0.26	0.00	3.44	0.0002
(i)e	(i)w	0.93	0.99	0.26	0.12	2.52	0.0030
(i)e	(i)p	0.93	0.92	0.26	0.30	-0.35	-
(ii)e	(ii)a	1.15	1.36	0.86	0.59	2.43	0.0038
(ii)e	(ii)w	1.15	0.97	0.86	0.88	-1.83	0.0168
(ii)e	(ii)p	1.15	1.44	0.86	0.59	3.46	0.0002
(iii)e	(iii)a	2.41	1.29	1.08	1.38	-7.92	0.0000
(iii)e	(iii)w	2.41	1.97	1.08	1.37	-3.07	0.0006
(iii)e	(iii)p	2.41	2.13	1.08	1.29	-2.07	0.0096
(i)w	(i)p	0.99	0.92	0.12	0.30	-2.59	0.0024
(ii)w	(ii)p	0.97	1.44	0.88	0.59	5.44	0.0000
(iii)w	(iii)p	1.97	2.13	1.37	1.29	0.97	-

Table 8.2.9 GIV difference of two means test data

Analysis

Part (i) can be discounted from the analysis as virtually all students scored this mark.

In part (ii), the algebraic version asked for a recurrence formula, which proved more difficult than describing the sequence. (ii)w was significantly less well answered: this may be because the real-world context perhaps suggests real-world descriptions such as ‘doubling’, rather than algebraic formulations such as ‘geometric sequence’. The ‘p’ version was easier since it included a mark for simply summing the numbers in the first 4 rows.

Part (iii)a again confirmed the increased demand of sigma notation, and the ‘w’ and ‘p’ versions were significantly harder than the ‘e’ version, with ‘w’ harder than ‘e’. In the ‘w’ formulation, the sum is implied by ‘total’, whereas in the ‘p’ formulation, the GP is dependent on a correct answer to part (ii).

8.3 Discussion of test results

Detailed analysis of these questions suggests a number of factors which may affect the facility of questions, as follows.

- The effect of sigma notation on sequence questions is to add substantially to the difficulty. Definition of sequences using iteration formulae also adds to the difficulty, though to a lesser degree.
- The requirement to identify the nature of sequences and to use the appropriate term or sum formula from data given in a real-world context also adds to the demand of questions, compared to explicit formulations.
- Some real-world contextual framing requires solvers to interpret text carefully in order to select the appropriate match between context and model. This can lead to semantic ambiguities causing unintended errors – see, for example GIIIp.
- In contrast, a real-world context can make some questions easier by enabling solvers to 'think within the context' and derive results using 'first principles' strategies, rather than utilising algebraic formulae or results.

How do these results inform the research questions on function and effect of RWCF? They seem to provide considerable evidence that setting questions in real-world contexts does indeed add to the overall demand, though a context can on occasions provide 'mental scaffolding' (see section 2.2) to help the solver to use context-specific heuristic strategies. The 'term' formula from an arithmetic progression [$u_n = a + (n - 1)\, d$] is not as essential to solving problems involving term calculations, as such questions are amenable to calculations using first principles (e.g. nth term = 1st term + $(n - 1) \times$ the 'step'). However, using the 'sum' formulae for both APs and GPs is a pre-requisite to the efficient solutions of problems involving summation (though students do occasionally succeed, with considerable expenditure of time and effort, in adding together large numbers of terms by calculator).

One could argue about the merit of such solutions, which effectively side-track the application of standard algebraic formulae to model realistic situations. The potency of algebraic formulae lies in their universality and blindness to individual contexts (Little, 2008), and, in resorting to context-bound thinking to solve these questions, students are avoiding the necessity to transfer and abstract from context to mathematical model, which is, arguably, the heuristic strategy intended by the questions.

However, questions with RWCF need to be carefully constructed to avoid unwanted distractors and ambiguities. They can require much greater interpretative acuity from the solver, in order to correctly match the context with the mathematical (in this case algebraic) model. Questions with RWCF may therefore disadvantage students with dyslexia, or non-native language speakers. It would, however, require further research to establish this without doubt.

It is important that questions should avoid ambiguity caused by inaccurate use of language, and careful revision of examination questions should ensure that the language used is clear and unequivocal. It is perhaps relevant here to note that the ‘p’ and ‘w’ versions of questions constructed for the tests were all based on past examination paper questions, with the exception of the ‘snowflake’ context (Gip), which proved to be open to ambiguity of interpretation which a question paper scrutinising committee may have spotted.

It is also important to consider the overall length of questions in relation to the time allowed to answer them: asking students to read and comprehend complex, novel contexts in a timed written examination clearly adds to the stress of the experience, and may place too much emphasis on comprehension skills which lie beyond the mathematical goals of the assessment. These comprehension skills would seem to be valid goals for an A/AS qualification in mathematics, but may be better tested in a separate comprehension paper. This is discussed further in the final chapter.

What is gained by presenting questions in real-world context? A test which utilises explicit, non-contextualised versions of these questions may be criticised for testing algebraic routines attached to arithmetic and geometric sequences, without testing understanding of what an arithmetic or geometric sequence represents or stands for. Forcing solvers to make the transfer between real-world, albeit artificially constructed, situations into mathematical models may require relational, rather than instrumental understanding (Skemp, 1971).

8.4 Analysis of the questionnaire data

Questions 1 – 6

The results of questions 1 – 6 of the questionnaire are shown in Figure 8.4.1. Two thirds of the students believed that questions set in real-world context are harder than those without context. In terms of whether or not real-world context makes questions more interesting, 33% agreed, and 30% disagreed. 55% agreed, and 30% disagreed, with the statement that real-world context shows how mathematics is useful. Over half of the students preferred pure mathematics to applied mathematics, and felt that pure mathematics is interesting in its own right.

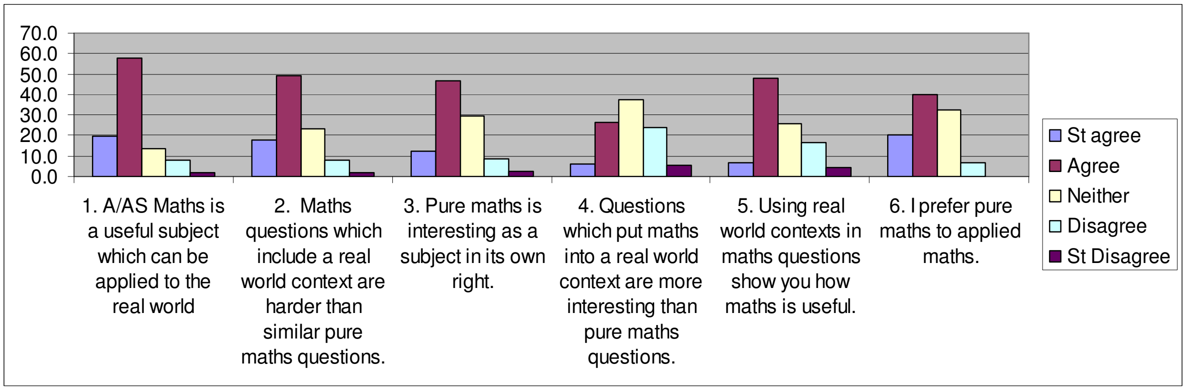


Fig. 8.4.1 Questionnaire results

In order to investigate differences in responses by gender, a random variable is defined to measure 'degree of disagreement' using a scale 1 for strongly agree to 5 for strongly disagree, and the mean and standard deviation of this random variable calculated for males and females. A difference of two means test on these values gives the values in Table 8.3.1.

	n	Q1	Q2	Q3	Q4	Q5	Q6
females	191	2.18	2.24	2.32	3.10	2.52	2.11
s		0.82	0.89	0.80	0.91	0.86	0.84
males	326	2.18	2.38	2.48	2.85	2.61	2.48
s		0.90	0.94	0.96	1.02	1.02	0.92
z		0.01	-1.69	-1.98	2.93	-1.06	-4.68

Table 8.3.1 Difference of two means tests on questionnaire scores by gender

This table suggests significant differences by gender for questions 2, 3, 4 and 6. These may be interpreted as follow:

- Girls agree more with the statement that real-world contexts make questions harder.
- Girls agree more with the statement that pure maths is interesting as a subject in its own right.

- Boys agree more with the statement that questions with real-world context are more interesting.
- Girls agree more with the statement that they prefer pure maths to applied maths.

Overall, these results show a consistent pattern of girls preferring pure maths questions without real world contextual framing to boys.

Table 8.3.2 shows a similar analysis to compare responses for students for whom English was or was not their first language.

	n	Q1	Q2	Q3	Q4	Q5	Q6
English not first language	49	2.29	2.53	2.51	2.86	2.49	2.53
	s	0.00	0.71	0.71	1.41	0.71	2.83
English first language	474	2.17	2.30	2.41	2.95	2.59	2.33
	s	0.88	0.93	0.91	1.00	0.96	0.90
z		2.94	2.07	0.87	-0.46	-0.90	0.50

Table 8.3.2 Difference of two means test for students with/without English as first language

This shows significant differences in response to questions 1 and 2, which may be interpreted as follows:

- Students who declared English as their first language agreed more strongly with the statement that A/AS maths is a useful subject which can be applied to the real world.
- Students who declared English as their first language agreed more strongly with the statement that maths questions set in real-world context are harder.

The first result might be interpreted as showing cultural differences concerning the nature of mathematics. The second result is perhaps surprising, as one might have expected non-native speakers to find contextualised questions harder to comprehend. However, as the number of students in the first category was relatively small (49), the sample may not be large enough to be representative.

Open question responses

22% of students volunteered additional comments on their questionnaires, some of which were quite detailed and articulate. This suggests that the issue of real-world context in mathematics questions is of interest to many students.

Students' comments related to the added difficulty of transfer from real-world context to mathematics:

> 'I prefer pure maths because the questions are easier to understand.'
>
> 'Pure maths is well laid out and simple to understand.'
>
> 'I strongly dislike real world context questions as they turn maths that I can do into something I can barely understand.'
>
> 'Sometimes when it is in context it is really difficult to understand what the question is asking.'

Some students found real-world contextual framing 'confusing', for example:

> 'The wording is always confusing in 'real world' questions.'
>
> 'Real world (context) just makes it more confusing and harder to put into formulas'
>
> 'Applied maths confuses you as you have to pick out the correct numbers to begin with.'

Others referred to 'ambiguity', for example:

> 'Some of the real-world questions are ambiguous, meaning that it can be taken several ways.'
>
> 'It is more difficult when the questions in applied maths are worded ambiguously.'
>
> 'With applied maths it is harder to recognise which rules or methods apply.'
>
> 'I found the questions with more words make the question a lot harder as you had to pick the correct information out of the question.'

Some comments referred to the difficulty of 'decoding' the context in order to apply the correct formulae:

> 'I found making the questions more wordy makes them harder and I find it less interesting as I prefer to be just given a question and then use my knowledge to find the answer, not to have to decode a problem.'
>
> 'Making questions apply to real life complicates the question, testing you more on interpretation and not what it should be testing, i.e. maths.'
>
> 'Putting questions in a real-world context only makes it harder to find what the question is asking you and which formulas to use.'

'I found the real life maths questions hard to distinguish which information I needed to use, e.g. which number was the common difference, etc.'

'I enjoy pure maths more than applied because it is easier to identify the method/formula that needs to be applied to solve a problem.'

Other comments refer to the linguistic demands of real-world context:

'Applied maths requires English skill and other skills to understand the question which you don't learn in maths.'

'I prefer the questions which are worded similar to the box 1 question ('e' version). I believe this is due to dyslexia, which means I find box 2 ('w' version) questions harder to understand.'

Do students believe that RWCF makes questions more interesting? Opinion is divided. Some students agree, and believe that RWCF does show that mathematics can be useful:

'Having real world questions is more interesting to see how it applies to life but it can make the question more confusing and therefore harder.'

'Using it in 'real life' context makes it more rewarding rather than just having a number that doesn't mean anything.'

'In applied maths you have to sometimes think outside the numbers which can be more challenging, but I think getting the right answer is more rewarding because you can link it to a possible 'real life' situation.'

'Sometimes applied maths questions are harder but they are more interesting. They take longer to process, good to have a bit of variation. Pure maths is simple already in the way to answer the question.'

'Applying maths to real situations is certainly more difficult but are a lot more interesting and satisfying to complete rather than straight pure maths questions.'

'It's more confusing but helps me in understanding how maths could be used in the real world.'

'It is harder questions that are wordy but I prefer them.'

'Real world contexts are more difficult but make it a bit more interesting.'

'Although putting maths into real world context is more interesting the questions can sometimes seem harder.'

On the other hand, some students are not convinced by the 'realism' of contexts:

> 'The sort of maths we're doing isn't really applicable to real world situations - it just shows how pointless doing it is.'

> 'The sort of questions asked in the real world context do not necessarily apply to the sort of things we'd get asked in real life. Therefore it shouldn't make a difference what sort of questions we get asked at AS level: all the real world questions do is to make it harder.'

> 'Maths questions put into a real world context are generally based on mundane aspects of life, and so are less interesting and/or inspiring. It would be more interesting to have questions based on things in life that are more inspired, such as the formulas for geometric arrangements of flowers, as opposed to how much pocket money your stereotypical adolescent receives on a weekly basis.'

> 'Questions using real world context can sometimes be a bit patronising / childish.'

> 'Sometimes they help if you don't know terms or helps you to get an idea of what the question requires which is more comprehendable but otherwise they are just plain patronising!'

> 'I think that maths is good and interesting but some parts of the course seem pointless because there is hardly any chance of being faced with a situation like it in the real world.'

While many students commented on the usefulness of mathematics, others appeared content to study mathematics as a subject in its own right:

> 'It is unnecessary to ask questions relating to the 'real world' or trying to apply maths to the 'real world' as people have obviously chosen maths because they like it as a subject, and for other subjects like economics which adds a social dimension to the maths. The maths itself does not need to shape itself to its use in the 'real world', as in itself it is already a useful and interesting subject.'

8.5 Discussion of questionnaire results

How do these questionnaire results resonate with the results of the test data, and inform the research questions? The test data confirmed that RWCF in general increases the demand of questions, and this triangulates well with the students' views that these questions are harder. For some students, real-world context does indeed re-affirm the utility of mathematics, and adds interest to the questions. On the other hand, not all students are convinced of this utility, and would side with Wiliam and Boaler in finding some contexts artificial, even 'patronising'. This finding reinforces the analysis reported in section 8.3 which suggests that artificial contexts may have a negative effect on students' perception of the utility of mathematics.

It is, perhaps, somewhat surprising that the majority of these students, even though they recognise the power of applied mathematics, appreciate pure mathematics more, though this view may be influenced by the perception that applied mathematics is more difficult.

The criticism of RWCF on the grounds of artificiality (see section 2.4) may be a result which is specific to the topic of the study. It is possible that realistic contexts for APs and GPs are hard to come by. I have argued elsewhere (Little, 2008) that the role of real-world context in linear equation contexts is not utilitarian but formative: the process of transfer from real-world context to mathematical model plays a role in enriching the understanding of the mathematics, as, for example, in Treffers'(1987) use of the 'Gulliver' metaphor to develop the ratio concept in younger children The distinction made in chapter 6 between natural and synthetic contexts is perhaps one which should be made explicit to students: if real-world contexts are 'sold' to students on grounds of utility, then criticisms of artificiality would be hard to refute. However, if the utility of algebraic models such as arithmetic and geometric progressions in modelling a wide range of contexts, both natural and synthetic, and providing a standard method of solution which is independent of the context, then students may learn to appreciate the relationship between pure mathematics and the real world, and be in a stronger position to develop more genuinely useful modelling skills, which often require students to have strong relational understanding of linear, exponential, logarithmic and trigonometric functions.

CHAPTER 9

CONCLUSIONS, SUMMARY AND IDEAS FOR FUTURE RESEARCH

Overview

This chapter draws together, and interprets the results of the study. In section 9.1, I re-visit the research questions, and discuss the overall findings of the research. The findings are summarised in section 9.2, together with some implications for the examination of A/AS Mathematics. Section 9.3 reviews the methodology of the study, and section 9.4 proposes areas for future research. Finally, section 9.5 presents a more personal coda to the study.

9.1 The Research themes re-visited

Chapter 1 of the study posed the following research questions concerning the use of real-world contextual framing in A/AS pure mathematics questions.

Research question 1	What has led to the introduction of real-world context and mathematical modelling in A-level mathematics?
Research question 2	To what degree are 'pure' mathematics questions in A/AS level examinations capable of being framed within real-world contexts, and what is the nature of these contexts?
Research question 3	What functions are served by real-world contextual framing (RWCF) of pure A-level mathematics questions, and what are its effects?

I now summarise the findings of the study on these questions.

Origins and degree of real-world context in A/AS Mathematics

Chapter 5 reports the origins of the introduction of real-world context in A-level Mathematics examinations, and concludes that current practices can be traced to curriculum development carried out by projects such as SMP and MEI from the 1960s onwards. The stimulus for this development may be summarised as coming from the need to increase the number of students in higher education with skills in applying mathematics, following the expansion of applications in statistics, discrete mathematics and numerical methods stimulated by the development of computer technology. New syllabuses were developed with the aim of broadening the range of applications, and motivating students through emphasising the applicability of mathematics in the modelling of real-world problems.

The A-level Mathematics syllabus has undergone numerous revisions since its introduction in 1951. The current specification for all A/AS syllabuses includes assessment objectives which require students to use mathematical models to represent real-world situations. Nevertheless, it is still possible to identify two broad types of syllabus, one derived from a 'traditional' approach, in which comparatively greater emphasis is placed on pure mathematics processing skills (such as algebraic manipulation), and a 'modern' approach (whose lineage may be traced to project syllabuses such as SMP and MEI) which emphasises real-world modelling, not just in applied mathematics questions, but in pure mathematics papers. This difference of approach is confirmed by the comparison of two current OCR syllabuses (reported in section 5.4) that shows a substantial difference in the number of questions utilising RWCF in a sample of pure mathematics papers.

This component of the analysis also examined the extent to which syllabus content is amenable to RWCF. This analysis revealed three broad categories of content: *geometrical models* (which apply trigonometry, functions, volumes of revolution, and three-dimensional vector geometry), *models of growth or change in time* (which utilise calculus or discrete functions to model discrete or continuous change), and *mathematical models of patterns* (examples being the application of sequences to spirals or matchstick puzzles).

The ARTA Framework

In section 3.3, a 'local' (Silver and Herbst, 2007) theory of the validity of RWCF in post-16 mathematics questions was developed, drawing upon issues drawn from the literature review in Chapter 2, and notions from the theory of measurement discussed in section 3.2. This 'ARTA' framework utilises the key ideas of *accessibility*, *realism* and *task authenticity*, and is then used in chapters 6 and 7 to analyse a sample of A/AS mathematics questions. Each of these three key ideas is worthy of specific consideration, in the light of this analysis.

Accessibility

Some of the existing research, as reviewed in section 2.2, suggests that real-world context might, on the one hand, enhance the accessibility of questions by providing a 'mental scaffolding' to the solver. On the other hand, others suggest that the introduction of real-world contexts can make questions less accessible, by dint of the added demands of comprehension required by solvers (e.g. Pollitt et al., 2000). Indeed, some research has argued that real-world contexts in questions can add an element of construct-irrelevant variance to questions, and in so doing jeopardise their validity as assessment items (e.g. Ahmed and Pollitt, 2007).

The analysis reported in chapters 6 and 7, taken together with the data collected in the AP/GP study reported in chapter 8, supports the notion that real-world context, as deployed in the A/AS questions investigated in this study, in general adds to the difficulty of questions, especially where the match between the real-world context and the mathematical model is implicit, and non-isomorphic.

Unlike real-world context as deployed in more elementary mathematics (such as in arithmetic 'word' problems), the process of translation from real-world context to abstract mathematical model is a requirement placed upon the efficient solution of questions at A/AS level. When questions appear to be easier with real-world context, this may be because they offer ad-hoc 'within context' methods of solution, which obviate the need to transfer from context to mathematical model.

Turning now to the validity of real-world contextualisation in A-level Mathematics questions, the evidence of this study suggests that adding real-world context does indeed add a degree of variability to the question setting, and problem solving, process. There is greater scope for misunderstanding in questions which are required to set up a real-world context or scenario, as well as pose a mathematical problem. Hence, such questions require more words, and this in turn demands careful attention by solvers to the meaning of these words.

Taking account of the high-stakes nature of the public examinations that are the primary purpose of setting these questions, it follows that, if their validity is not to be compromised, the length and complexity of the question and its language must be carefully considered in the design of these tasks. There is much greater scope for ambiguity of language, and careful revision of such questions is essential to ensure that misunderstanding is not the fault of the question setter, but its interpreter.

It is possible to argue that the relative transparency and straightforwardness of questions set in the mathematical register are more appropriate to the demands of the examination hall than questions which add the requirement on candidates to understand a non-mathematical context as well as solve a mathematical problem. However, the strong backwash which these summative assessment tasks have in the classroom cannot be underestimated: if the requirement to recognise and apply mathematical models to real-world situations is mandatory, as current assessment objectives for A/AS Mathematics confirm, then such demands cannot be fulfilled using assessment tasks which are set within the mathematics register alone, without compromising the consequential validity (Messick, 1989) of the assessment.

A further argument that can be made against the validity of real-world contextualisation in questions is that it is *unfair*, because it differentiates against certain classes of candidate. This study was not designed to utilise the sociological methodology of Cooper and Dunne (2000) to investigate the effect of social class. In any case, it is arguable that students who study A/AS Mathematics have already proved their ability to play the ‘assessment game’ successfully, and that the effect of social class might therefore be less pronounced at this level. My experience suggests that it is those students whose English is less secure who might find real-world contextualisation more to their disadvantage than native English-speakers. However, these conclusions would require further study, as the numbers of such students in the study reported in chapter 9 were insufficient to reach any firm conclusions.

Realism

The enquiry into the origins of real-world context in A/AS mathematics reported in chapter 5 suggests that RWCF may serve the function of motivating the learner by reinforcing the notion that mathematics is *useful* in solving real-world problems. Yet, as section 2.4 related, there is considerable criticism of the use of real-world context use on the grounds that it is *artificial* . In this section I propose a way of addressing this issue.

First, in terms of utility, it is clear that none of the real-world situations described in these questions are genuinely *real*: they are contrived to fit in with, and be capable of solution using, a tightly constrained set of mathematical techniques. No matter how interesting or useful the real-world context deployed in an A/AS mathematics question might be, if it fails to test the appropriate mathematical techniques, it is unusable as a short, closed item in a timed, written summative examination. My experience of the question-setting, and revision, process, confirms this: often the most imaginative and interesting ideas for questions deploying real-world context have failed to survive the examination - setting process because they are too wordy, too complicated, or fail to test the appropriate mathematics efficiently.

It follows that such questions can only hint at the possible utility of mathematics, for example by suggesting how a population might be modelled using a differential equation, or a roof might be modelled using three-dimensional vector geometry, or a series formula might be used to find the length of a skittles race. Notwithstanding this general lack of practical utility, it seems that the mathematical model should at least fit reasonably well with the real world, and it would be desirable that, in using mathematical techniques to model a context, questions should invite students to reflect upon this fit. This, on the evidence of the questions analysed in this study, rarely happens.

Moreover, the modelling credentials of a question are enhanced if more than one possible mathematical model is presented in a question. In practice, there is no unique mathematical model which captures a real-world situation: the assumptions that underlie the application of the mathematics need to be clarified, and these assumptions can never be regarded as absolute truths. Questions which propose more than one mathematical model emphasise the hypothetical relationship between the real world and mathematics.

These conclusions have implications for the length of questions in mathematics examinations. First, the real-world context requires to be explained, and this, as this study has demonstrated, requires more words. Secondly, if more than one model is presented, this means even more words. Thirdly, if an element of evaluation is required of the solver, then this requires a high enough tariff (or mark per question) to cope, not just with the mathematical solution, but also the evaluation. The OCR Specification B (MEI) papers have 18-mark questions which, in my experience, is the minimum length which allows two models to be explored, albeit briefly.

Students in mathematics classrooms can speculate, implicitly or out loud, 'Why are we doing this?'(see, for example, Boaler and Greeno, 2000), although this perhaps occurs less frequently in more advanced mathematics lessons, in which many (but not all) the students may find enough intrinsic interest and fascination in the mathematics per se. Evidence from the questionnaire reported in section 8.4 suggests that students can see through a claim that real-world contextual framing shows how mathematics is *useful* if this is predicated upon examples which are manifestly impractical, artificial or whimsical. However, if these examples are used to discuss how *mathematical modelling* is useful, then even the most artificial examples, such as the 'lift' question (see Fig. 3.3.1), may contribute valuable insight.

The fundamental concepts in enhancing students' understanding of the relationship between the 'real world' and the 'mathematical world' may be stated as follows:

- mathematical ideas and concepts originate, and are abstracted from, the real world;
- the real world can, in turn, be modelled by mathematics, but these models are not unique;
- the utility of mathematical models depends upon the 'fit' between the mathematics and the real-world, and this needs to be considered.

If *all* real-world contextual framing addressed this more complex relationship between mathematics and the real world, then the arguments over utility are but the result of a misunderstanding of this relationship. Rather than artificially-contrived contexts being regarded as 'McGuffins' (Wiliam, 1997), or mathematics 'looking for somewhere to happen', they become inefficient mathematical models of reality.

The data from the student questionnaire (see section 8.4) are consistent with the above findings. Students are clearly ambivalent about the use of real-world context, some liking it for adding interest and purpose to questions, others doubting its authenticity and relevance. However, most agree that the introduction of a real-world context adds to the demand of questions, by making the questions 'confusing', even 'annoying'. Of course, for students entering an examination which may determine their future prospects, education or career, the best questions are, naturally, the ones they personally find easy to solve! However, the added demands which RWCF makes on candidates need to be recognised and kept in balance. It is worth noting from the questionnaire results that girls voice a greater tolerance of, or interest in, pure mathematics, and a greater indifference to its applicability.

Is this an argument for removing real-world contextual framing from pure mathematics examination questions? From *some* questions, yes: a relentless search for contexts to frame all mathematics would be pressing the case for mathematical modelling to extremes. This study has found that opportunities for real-world contextual framing appear to be confined to particular aspects of mathematics, in particular vector geometry, calculus, functions and sequences. Moreover, there is such a thing as pure mathematics, whose utility transcends that of mathematical models, and advanced mathematics courses should reflect and celebrate that world. Although some students (often girls – from the evidence presented in section 8.4) embrace this pure mathematics world more readily than others, there needs to be an appropriate balance in the mathematics classroom between mathematics and mathematical modelling. This balance varies from one A/AS specification to another, as reported in section 5.4 (which compared the two OCR syllabuses). There does appear to be a trade-off here between teaching purely mathematical algebraic manipulation skills and what might be termed 'proto-modelling' skills in matching real-world context to mathematical concepts necessary to solve questions utilising real-world contexts.

It is possible to apply these conclusions to more elementary mathematics classrooms. For instance, a contextualised approach to Pythagoras' theorem could, quite readily, be adapted to embrace the 'modelling' agenda above. For example, how appropriate is the model of a ladder resting against a vertical wall as a right-angled triangle? What assumptions does this make about the dimensions and shape of the ladder? Why do painters and decorators not use

Pythagoras' theorem in real life? Can we think of a situation where Pythagoras' theorem might be more useful?

Yet students need to appreciate the need for proof in mathematics, and this has little or nothing to do with mathematical modelling. Right-angled triangles can be measured (or evidence can be drawn from interactive geometry software), to suggest that $a^2 = b^2 + c^2$ is a good model; but this relationship is more than a model: it expresses a mathematical *truth* about right angled triangles, and this raises the question of why is it true, and under what conditions.

Task authenticity

Another aspect of utility and artificiality is the nature of tasks set within a context. I have called *task authenticity* the extent to which the tasks themselves are meaningful and relevant to the real-world context. At one extreme, the task(s) might not be expressed at all in terms of the real-world context. In such cases, the use of a real-world context at all would seem to be superfluous. In other cases, even if the solution of the task might follow from an explicit mathematical model, the relating of this solution to the real-world context provides a question with a purpose which enhances its authenticity and utility. Equally, the contextual embodiment of a mathematical result may help to reify the mathematical concept, for example using a sum to infinity to express an upper bound, or presenting an asymptotic value as a terminal velocity.

I would argue that a minimal requirement of a question with RWCF is that the task should relate back to the context. Without this, it seems to me, the context might as well be omitted from the question. Beyond this minimal requirement, it is desirable, though not essential, that the task asks useful and interesting questions with respect to the context.

Types of real-world context

Another conclusion which can be drawn from the 'ARTA' analysis reported in chapters 6 and 7 relates to types of context. A distinction has been made between *natural* and *synthetic* contexts. Examples of *natural* contexts are compound interest for savings, linear models for calculating the cost of petrol at a pump, exponential decay for the cooling of a liquid, and Newton's law of restitution implying a geometric sequence of times or heights for the bounce of a ball. Examples of *synthetic* contexts are a skittles race modelled by an arithmetic progression, eccentric gardeners spreading fertiliser in arithmetic or geometric progression, and a savings plan in which money is invested in arithmetic progression.

In the case of natural contexts, reality is sufficiently 'well-behaved' to be capable of being modelled by a particular piece of mathematics. In my teaching, I remember one lesson on differential equations in which I started by presenting a beaker of boiling water to the class, measuring its temperature at five minute intervals, whilst simultaneously solving the differential equation which modelled this through Newton's Law of Cooling. The solution was then used to predict further temperature, and these predictions proved to be remarkably accurate! Here, I was pleasantly surprised to find how well, almost perfectly, reality was behaving mathematically. In other real-world situations, the mathematics dictates the reality, for example in the cases of interest rates or petrol costs. We could impose different models – for example, offering discounted petrol for larger purchases – but we can effectively decide which model to apply by 'fiat'.

In practice, reality tends to be less well-behaved mathematically, and does not follow deterministic, or even stochastic laws. The occurrence of 'natural' contexts is therefore somewhat limited. In the case of *synthetic* contexts, it is as if we are configuring real-world contexts to behave according to mathematical laws, for example instructing gardeners to spread fertiliser in arithmetic or geometric progression!

'Pattern' contexts appear to occupy the ground somewhere between 'natural' and 'synthetic'. It is true that, on the one hand, patterns occur in nature which conform to mathematical laws and relationships, (for example Fibonacci sequences and the golden ratio); on the other hand, matchstick patterns or card stacks would seem to be more synthetic than natural.

The utility of these different types of context, however, would appear to be different. On the one hand, natural contexts suggest how single mathematical models can sometimes describe realistic situations accurately. On the other hand, synthetic contexts can be effective in embodying mathematical concepts in a tangible way. It seems that this distinction might be a valuable one to share with students, in particular with those who view the utility of mathematics sceptically.

In some cases, the real-world contexts would seem to be fundamentally serious in character, for example modelling growth of a virus, or financial savings, or depreciation, or modelling a scientific enquiry. In some of these, a 'natural' mathematical model might be suggested: for example, compound interest is applied by fiat to savings, the laws of physics may be applied to collisions or to rates of cooling. On the other hand, other contexts are not just artificial constructions based on naturally occurring events, but are artificial constructions based on artificial events! In this category are eccentric gardeners, widget manufacturers, skittles races, etc. Should these contexts be dismissed as ‘McGuffins’ which reinforce stereotypes of school mathematics as being devoid of practical value?

My instinct is to reject this criticism, and propose that there is a place for such ‘whimsy’, even in important mathematics examination questions. There is, after all, a danger that mathematical modelling might be taken too seriously, or literally, and mathematics itself is regarded as offering solutions to all human problems. Moreover, although such contexts are not useful in the real world, they may be seen to be useful in illustrating, even reifying, the underlying mathematical structure, as ‘Gulliver’ can be used to model concepts of scale and ratio (see Treffers, 1987). Thus, a skittles race provides a vivid picture of an arithmetic progression, and in doing so, embeds an abstract mathematical idea into an action ‘schema’ which is memorable and revealing.

De Lange (1999) voices a caution by pointing out that whimsical contexts do need to be used sparingly. There, is, I believe, an important need for teachers to discuss the role of context with their students, to ensure that they have a clearer understanding of what mathematical models can and cannot do in solving genuine problems.

Pseudo-modelling

I turn now to the assessment objectives of A/AS level relating to mathematical modelling. The following have been quoted on a number of occasions in this study, and are worth re-visiting:

> AO3 Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinements of such models.
>
> AO4 Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.

The first of these assessment objectives requires students of A/AS Mathematics to learn modelling skills, including *interpreting*, *discussing assumptions*, and *refining*. The analysis of questions in this study (and, I predict, applied mathematics questions of a similar length) suggests that RWCF of short, timed written questions can at best test skills of *pseudo-modelling* (see Fig. 9.1.1), which does not require students to formulate or select appropriate models, or make and review assumptions.

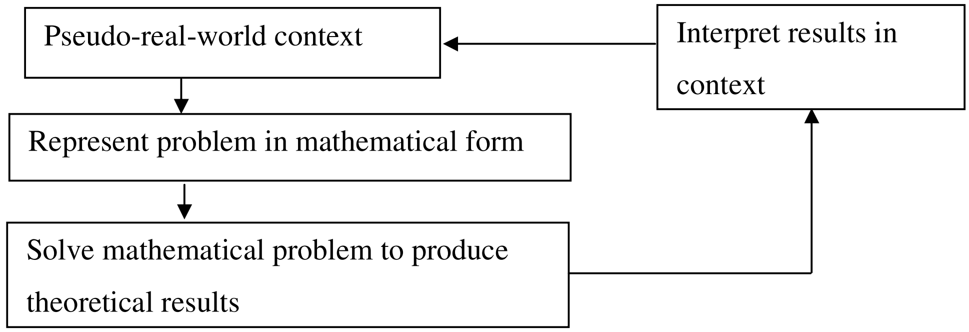


Fig. 9.1.1 The 'Pseudo-modelling' cycle

AO3 requires candidates not just to comprehend translations of realistic situations into mathematics, but to *comment* on the context, *read critically* and comprehend *longer arguments*. Again, the pseudo-modelling found in the questions analysed in this study cannot be said to satisfy this assessment objective. It is my conclusion that A/AS specifications which rely exclusively on timed written papers are not meeting these assessment objectives.

What sort of assessment tools fit the purposes of these assessment objectives? AO3 requires students to start from a real-world problem, preferably one of their own devising, and then select appropriate mathematical models. This is quite different to configuring a real-world problem so that it is amenable to modelling with a given piece of mathematics, and implies an element of open-endedness. I cannot see how this objective can be met without some form of coursework project.

Coursework has, in recent years, all but disappeared from public examinations in the UK. The reasons for this are considered in Porkess (2006), but may be summarised as being caused by the difficulty of teachers' assessing such work consistently, and the conditions on students to conduct the work fairly (see also Little, 2006). The conclusions of this study suggest that removing the requirement of A/AS specification to include an element of open-ended project work from the assessment loses this element of the A/AS Mathematics construct.

As for AO4, which refers to the comprehension skills of students, this, in my experience, can be assessed without the requirement for coursework through the inclusion of a comprehension paper, in which students are required to read critically an article which presents a mathematical model, and answer questions which test their understanding. Extracts from such a paper are shown in Appendix 8.

Section 5.3 reported on current UK post-16 qualifications (FSMQs) which are indeed utilising a portfolio approach to assessment, and experimenting with novel approaches to presenting richly contextualised examination questions, for example providing candidates with data sheets in advance of the examination. However, the candidatures for units such as Use of Mathematics are at the time of writing small (less than 1000), and the impact of such examinations is consequently small compared to A/AS Mathematics qualifications. Although it is possible that these relatively new qualifications might grow in future, it seems likely that universities will still require many students for mathematics-related course to have an A/AS level in mathematics.

RWCF and Realistic Mathematics Education

What are the implications of this study of RWCF with regard to classroom pedagogy? Section 2.2 gave an account of RME as a pedagogical approach which utilises real-world contexts to develop mathematical concepts. However, a distinction needs to be made between using real-world context as an organising activity (Gravemeijer, 1997), which requires the learner to engage actively with the context, and the process of transfer from context to mathematical model required to solve A/AS problems with RWCF.

Thus, the process of solving problems with RWCF, while it is consistent with an RME approach to mathematics in the classroom, does not in itself constitute Realistic Mathematics Education, which requires the student to engage is a process of mathematical 'reinvention' (Freudenthal, 1991). Moreover, RME envisages a secondary process of 'vertical mathematising', or developing the mathematics from within, which involves 'pure' mathematical concepts such as proof.

A classroom approach which is consistent with the ideas of RME would, in the initial stages of establishing mathematical concepts, eschew formal definitions, and seek out real-world contexts which would enable the student to draw out, or 'reinvent', these concepts. This study suggests that not all mathematical concepts appear to be readily amenable to real-world contextualisation (see sections 5.2 and 5.4), and for this reason that, at post-16 level, such an RME approach might need to be used selectively.

Some pilot work on RME is being conducted in the UK (MMU Realistic Mathematics Project, 2009), but as yet this has not been extended into a sixth-form course. In the 1990s, the School Mathematics Project's 16-19 Mathematics course (Dolan, 1994) adopted a 'guided reinvention' approach which contained some of the elements of RME, and this proved quite popular at this time. However, the predominant influence of assessment and examinations in high-stakes courses such as A/AS level suggests that reforms in pedagogy will only take place if they are seen to be directly linked to examination success. My conclusion is that RWCF in examination questions is not in itself a vehicle for radical classroom reform.

Construct Validity

I shall now pick up on the discussion in Chapter 3 on assessment theory, and, in particular, on construct validity (see section 3.2). This study has highlighted a difference in approach required by solvers in tackling questions with and without RWCF. While questions without RWCF restrict the problem-solving process to the mathematics register (Pimm, 1987), questions with RWCF usually require the solver to engage in a process of transfer from real-world context to mathematical model. If the A/AS Mathematics examination, as it applies to the pure mathematical content of syllabuses, requires that this process of *pseudo-modelling* is to be tested, then RWCF would appear to be a *valid* means of testing this part of the construct.

Does the introduction of real-world context constitute a threat to examination reliability, as Ahmed and Pollitt (2007) propose? The evidence of the AP/GP study suggests that it is clearly possible to use RWCF in questions without jeopardising reliability, although more detailed analysis of the data may be needed to establish this with more certainty. Certainly, there was little evidence in scrutinising the solutions to these questions of the sorts of misunderstanding found in National Curriculum test items by Cooper and Dunne (2000). It appears that real-world contextual framing does not necessarily entail a reduction in reliability.

However, as already reported above, there was one question in the AP/GP study (the 'snowflake' item – see Gip in section 8.2) where RWCF led to a possible ambiguity which compromised its reliability. Moreover, as questions set in a timed written examination get longer, the greater the demands these questions place on the comprehension skills of candidates. Given the proposal above that effective pseudo-modelling questions should present alternative models and some opportunity to evaluate them, assessment schemes will need to balance the requirement to test modelling in the construct, with the potential construct-irrelevant variance of comprehensibility. There is clearly a limit to the fitness of purpose of timed written papers in assessment schemes, and consideration needs to be given to finding an appropriate balance between construct representation and reliability.

9.2 Summary of findings and implications for examination practice

Before considering the implications for examination practice, it is important to summarise the main findings of this study.

(1) Real - world contextual framing (RWCF) of pure mathematics questions in A/AS Mathematics varies in degree. Its roots in current practice can be traced to curriculum development projects of the 1960s. Syllabuses differ in the balance between assessment of mathematical manipulation and assessment of modelling skills.

(2) RWCF is found in A-level pure mathematics questions on *geometrical models*, *models of growth or change in time*, and *mathematical models of patterns*.

(3) RWCF in general adds to the difficulty of questions, by requiring solvers to understand and match the context to the appropriate model. However, it can ease questions which can be solved by providing mental scaffolding through thinking ‘within the context’.

(4) The *accessibility* of questions with RWCF is a function of comprehensibility of language, the explicitness of the match between context and mathematical model.

(5) Real-world contexts can be *natural* or *synthetic*, according to the degree to which the context matches reality, or reality is configured to match the mathematics. Natural contexts are more *realistic*; but synthetic contexts can provide realistic embodiments of abstract mathematical ideas, which reify them and illustrate how mathematics can be used to model reality.

(6) Questions with RWCF should set tasks that are *authentic* within the context.

(7) There is a *gender* difference in students’ attitudes to RWCF, with boys in general expressing more favourable views about its use in pure mathematics questions.

(8) At best, RWCF in examination questions require solvers to engage in *pseudo-modelling*. Questions cannot test aspects of the modelling cycle such as discussing assumptions, refining, and critical reading of longer arguments, although those that present alternative models give some scope for comparison.

(9) Criticism of questions utilising RWCF on the grounds of artificiality represents a misunderstanding of the role of context: it is not the context that is artificial, but the modelling of the context. Solvers should be encouraged to evaluate this modelling.

These findings suggest a number of implications for examination and assessment practice:

(1) A/AS syllabuses which use short, closed, timed, written questions, notwithstanding their use of RWCF, cannot satisfy the current QCDA assessment objectives on mathematical modelling (AOs 4 and 5). For these AOs to be adequately addressed, open-ended project work and comprehension papers need to be a requirement of schemes of assessment (as is the case in some current FSMQ units such as Use of Mathematics).

(2) Questions utilising RWCF need careful revision to ensure that language is precise and un-ambiguous, and comprehension demands are manageable.

(3) Students should be encouraged to discuss and compare the appropriateness of models, and hence appreciate more deeply the relationship between reality and mathematical models. Questions which present more than one model, and which invite the comparison of models, are therefore desirable, and this requires longer questions with higher tariffs to achieve than are used in current specifications.

(4) Tasks set in questions utilising RWCF need to pose authentic questions about the context. If not, then the real-world context is redundant and should not be used.

9.3 Reflections on methodology

In this section, I consider the effectiveness and appropriateness of the methodology adopted by the study.

I have, of necessity, sampled specific sources for investigating the origins of RWCF in A-level Mathematics, as a historical account of the development of styles of question in the qualification that has changed continually within its current lifetime of over 50 years would require a thesis of its own. In selecting these sources, I have leaned upon my own experience of teaching the subject over 30 years. The conclusions must therefore be regarded with a suitable degree of caution, pending a larger study.

The selection of the two OCR specifications to compare the degree of contextualisation of pure mathematics papers was also dictated by personal experience – both these specifications are familiar to me through the experience studying papers at joint awarding meetings. However, as the object of this comparative exercise was to confirm quantitatively that syllabuses vary in the degree to which pure mathematics questions are contextualised, then the existence of two syllabuses which differ in this regard in sufficient. Clearly, such a study could be readily extended to embrace other current specifications in order to achieve a more complete picture of the degree of variability.

My claim is that there is some trade-off in syllabuses between the use of real-world context and mathematical manipulation skills. This would require further detailed analysis of questions to establish, but my experience of setting questions for OCR specification B (MEI), and comparing questions on the same topics from specifications A and B, lead me to believe that this is indeed the case.

The 'ARTA' framework (section 3.3) was developed from ideas adapted from current assessment theory, but is, to my knowledge, a novel approach to analysing questions which use real-world context. It seems to reflect what is expected of real-world context in questions, namely that the context should not make questions inaccessible or obscure to solvers, the context should appear realistic, and the question should present authentic tasks within the context. However, the analysis I have conducted in chapters 6 and 7 is essentially subjective in nature. Indeed, it is possible to argue that individuals' reactions to real-world contexts are socially constructed, and therefore generalisations about the accessibility or realism of contexts require a wider, and perhaps different, study to establish them. Nevertheless, the ARTA framework does, I claim, establish a mechanism for systematically reflecting on context in mathematics questions – a framework which I have found valuable in thinking about the relation between the context and the mathematics. The framework can be revised or refined through further research.

Finally, the design of the AP/GP study reported in chapter 8 builds on methods used in earlier studies. The idea of versioning questions in this particular way also stemmed from my own experience of developing contexts for many of these sorts of questions. Indeed, it is revealing that the one 'pattern' context which was developed ab initio for the test – that of the 'snowflakes' question (Gip) - proved to have ambiguities which threatened its validity as a test item. I have no doubt that the design fault in this question would have been spotted by question paper revisers; but the question serves to highlight the importance of detailed scrutiny of questions, which is itself a highly skilled task which requires experienced practitioners. It is, despite thorough scrutiny of this kind, still extremely difficult to predict how solvers might react to contextualised questions. Nevertheless, some variability is, I would argue, desirable to avoid over-'routinisation' of questions (see Little and Jones, 2008).

Reflecting on this versioning method, it could be readily adapted to other topics, for example calculus, or indeed more elementary mathematics, in order to enhance understanding of the effect of context in mathematics more generally.

9.4 Future Research

A number of avenues for future research have already been hinted at in the previous section. First, there is a need for more research into the history of curriculum development, in the UK and internationally. Without this, there is a danger that new ideas are proposed which have been tried before, gone out of fashion, and been forgotten. Chapter 5 of this study has presented an account of developments in A-level Mathematics, but this could fruitfully be expanded into a larger study.

This study has considered Realistic Mathematics Education and its relationship with real-world context in assessment. However, there is scope for a wider consideration of current constructivist and socio-cultural theories of learning, in order to develop a clearer understanding of the role of real-world context in teaching and learning. For example, the role of context in the reification of mathematical concepts (Sfard, 1991) is worthy of more study. Further, theories of situated learning suggest that real-world context can play an important role in how problems in mathematics are solved in practice (Lave, 1988, Taylor, 1989). The real-world contexts used in the UK post-16 mathematics ‘community of practice’ are socio-cultural constructs, and it is likely that other cultures will generate alternative contexts. A cross-cultural approach to real-world context would, I believe, provide some insights into the relationship between ‘reality’ and mathematics.

The ARTA framework developed in this study has been applied as a means of evaluating questions with RWCF. Further research would be required to establish the wider applicability of using accessibility, realism and task authenticity of questions, for example through a study involving students and teachers which might investigate how they perceive, and respond to, real-world contexts used in questions.

The results reported in chapters 7 and 8 relied upon the selection of one topic – sequences and series – which might not be representative of pure mathematics set within and without real-world context. Other topics could be chosen to repeat this design with the aim of triangulating the findings of this study.

More generally, this study has concentrated its attention on real-world context in post-16 pure mathematics. Whether the conclusions drawn, for example about the nature of contexts, apply equally to applied mathematics questions remains an open question. Further, a study of context in pre-16 examinations such as the English General Certificate of Secondary Education, utilising some of the methods adopted in this study, would be of value.

Finally, the different attitudes and opinions on context caused by gender and linguistic competence requires further research. It is only possible to speculate why, for example, girls appear to prefer their pure mathematics presented ‘plain’, but boys seem to prefer it dressed in real-world contexts. Also, the effect of real-world context deployed in questions on students whose first language is not English, or on dyslexic students, is clearly worthy of further study.

9.5 Coda

The stimulus for this study has been my own experience of setting examination questions, coupled with the work of Cooper and Dunne (2000) on national test questions, who indeed espouse the need for more research on context (p. 204). However, the apparent lack of such research, in particular in post-16 mathematics, leads me to ask myself why I believe it is important.

First, research into the forms of questions used in public examinations seems to me to be vitally important. In a course such as A/AS Mathematics, the influence of examination questions on how mathematics is perceived by students cannot be underestimated: for students, especially in a qualifications-driven society, mathematics is, to a large extent, characterised by the questions they are asked to solve in examinations, especially if this is the sole means by which their talents are assessed.

Second, without a greater insight into the role of real-world context in these questions, there is a danger that students are left with an inadequate understanding of how and why mathematics works, and why it is important. By encouraging a more sophisticated understanding amongst teachers and their students of the role of real-world context, and the mathematical modelling process, and indeed the importance of pure mathematics, there is a chance that we can attract more young people to share our fascination with the subject.

Third, there is a need to scrutinise schemes of assessment, and the assessment tools used in these schemes, to ensure that they are indeed fit for purpose. Laying down ambitious assessment objectives cannot in itself ensure that these objectives are met: they require assessment tasks which test them effectively. In public examinations, there is a balance to be struck between reliability and consistency of assessment and validity (Little, 1993), a balance which, through the virtual abandonment of teacher-assessed coursework in the UK, is, at the time of writing, firmly tilted towards the former (Little, 2006). This study proves that there are limits to what can be effectively assessed through short, closed, written examination questions.

Finally, I hope that this study encourages a clearer understanding of what makes a good examination question, and thereby improve the standard of examination papers, and examination practice.

APPENDICES

Appendix 1 UCLES 1951 GCE A-level Papers

186
ADV. LEVEL
MATH. I
Tuesday
5 JUNE 1951
3 *hours*
A

UNIVERSITY OF CAMBRIDGE
LOCAL EXAMINATIONS SYNDICATE
GENERAL CERTIFICATE OF EDUCATION

MATHEMATICS

ADVANCED LEVEL

PAPER I.

(*Three hours*)

Full marks can be obtained by complete answers to about **ten** *questions, but a pass mark can be obtained by good answers to about* **four** *questions or their equivalent.*

Write on one side of the paper only.

Mathematical tables and squared paper may be obtained from the Supervisor.

1. The pth term of a progression is P, the qth term is Q, and the rth term is R. Show that, if the progression is arithmetical,

$$P(q-r)+Q(r-p)+R(p-q)=0,$$

and that, if it is geometrical,

$$(q-r)\log P+(r-p)\log Q+(p-q)\log R=0.$$

2. If α and β are the roots of the equation $2x^2-3x-1=0$,

(i) find the value of $\alpha^3+\beta^3$,

(ii) form an equation with integral coefficients which has the roots $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

3. Expand $(x-y)^6$ by the binomial theorem and use the result to evaluate $(19\frac{3}{4})^6$ correct to the nearest thousand.

4. Prove that the number of selections of a group of r things that can be made from n things, all of which are different, is

$$\frac{n!}{r!\,(n-r)!}.$$

A pack of 52 playing cards, all different, contains 20 honours. Show that a selection of three cards from the pack can be made in 22,100 different ways, and that 4960 of these contain *no* honour. Deduce the probability that a particular selection of three cards contains at *least* one honour.

5. Prove that in the triangle ABC, in which the angle A may be assumed to be acute,

$$a^2=b^2+c^2-2bc\cos A.$$

A quadrilateral $ABCD$ is such that $AB=3$ in., $BC=CD=5$ in., $DA=6$ in. and the diagonal $AC=7$ in. Find, as accurately as the tables allow, the angle BCD and the length of BD.

6. (i) Prove that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$

where the angles A, B and $A+B$ are all acute.

(ii) By projecting the sides of an equilateral triangle on to a certain line, or otherwise, prove that

$$\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)=0,$$

and find the value of the expression

$$\sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right).$$

7. The angular elevation of the summit of a mountain is measured from three points on a straight level road. From a point due south of the summit the elevation is α, from a point due east of it the elevation is β, and from the point of the road nearest to the summit the elevation is γ. If the direction of the road makes an angle θ east of north, prove that

(i) $\tan\theta=\tan\alpha\cot\beta$,

(ii) $\tan^2\gamma=\tan^2\alpha+\tan^2\beta$.

Find γ, if $\theta=31^\circ$ and $\alpha=8^\circ$.

8. Find the values of x between 0° and 180° inclusive for which

(i) $\sin 3x=\sin x$,

(ii) $2\cos^2 x-\sin^2 x=1$,

(iii) $\sin 2x+\cos x=0$.

9. Show that the gradient of the curve $y=x(x-3)^2$ is zero at the point $P(1, 4)$, and sketch the curve.

The tangent at P cuts the curve again at Q. Calculate the area contained between the chord PQ and the curve.

10. (i) Differentiate, with respect to x,

$$\sqrt{\left(\frac{1-x}{x}\right)} \quad \text{and} \quad x^2e^{-x}.$$

(ii) Find the differential coefficient of $\log_{10} x$ with respect to x. Deduce the value, correct to four decimal places, of $\log_{10} 1{\cdot}002$, given that $\log_{10} e=0{\cdot}4343$.

11. (i) Use Maclaurin's series to expand $\tan x$ in ascending powers of x as far as the term in x^3.

(ii) Evaluate

$$\int_0^3 \frac{3x+4}{\sqrt{x+1}}\,dx \quad \text{and} \quad \int_0^1 \cos^2\left(\frac{\pi x}{2}\right)dx.$$

12. Draw the graph of $\sqrt{\sin x}$ for values of x from 0 to $\frac{\pi}{2}$, and hence obtain an approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$, giving your answer to two significant figures.

(A scale of 6 in. to $\frac{\pi}{2}$ radians along the x-axis, and 5 in. to 1 unit along the y-axis is suggested.)

187
ADV. LEVEL
MATH. II
Thursday
7 JUNE 1951
3 *hours*
C

UNIVERSITY OF CAMBRIDGE
LOCAL EXAMINATIONS SYNDICATE
GENERAL CERTIFICATE OF EDUCATION

MATHEMATICS

ADVANCED LEVEL

PAPER II

(*Three hours*)

Answer the whole of Section A *and* **four** *questions from Section* B *which may include any of the questions on statistics.*

Full marks can be obtained by complete answers to about **ten** *questions, but a pass mark can be obtained by good answers to about* **four** *questions, or their equivalent.*

Write on one side of the paper only.

Mathematical tables and squared paper may be obtained from the Supervisor.

SECTION A.

1. Write down the equations of the two straight lines through the point (h, k) which are respectively parallel and perpendicular to the straight line $ax+by+c=0$.

The vertices of a triangle are $A\ (-2, 3)$, $B\ (3, 7)$, and $C\ (4, 0)$. Find the coordinates of the point D on the same side of AC as B such that the triangle ACD is right angled at C and equal in area to the triangle ABC. Calculate the area of the triangle ACD.

2. Show that the perpendicular distance of the point (h, k) from the straight line $x\cos\alpha+y\sin\alpha=p$ is the numerical value of

$$(h\cos\alpha+k\sin\alpha-p).$$

Calculate the coordinates of the centres of the two circles of radius 5 which pass through the point (4, 4) and touch the straight line $3x-4y-28=0$.

3. Prove the formula $s = ut + \frac{1}{2}at^2$ for uniformly accelerated motion in a straight line.

The motion of a train between two stations A and B is in three stages. In the first stage the train starts from rest at A and moves with constant acceleration. In the second stage it moves with constant speed and in the last stage it has constant retardation and comes to rest at B. If the times taken over the three stages are in the ratio 6 : 15 : 4, show that the average speed is four-fifths of the maximum speed and that the distance travelled with constant speed is three-quarters of the distance AB.

4. A car of mass 2000 lb. can descend an incline of $\sin^{-1}\frac{1}{20}$ at a steady speed of 20 m.p.h. with the engine shut off. Assuming that the resistance to the motion of the car is proportional to the square of the speed, find the horse-power necessary to drive the car up the same incline at a steady speed of 30 m.p.h.

Find also the maximum speed that can be maintained on the level if the engine is working at 15 horse-power.

5. A uniform rod of length $2a$ and weight w, smoothly hinged at one end to a vertical wall, slopes upward from the wall at an inclination α to the horizontal. It is supported by a string tied to the other end, sloping towards the wall at α to the horizontal. Prove that the tension in the string is $\frac{1}{4}w \operatorname{cosec} \alpha$, and find the horizontal and vertical components of the reaction at the hinge.

The string is now transferred to the mid-point of the rod, the angles of inclination of the rod and string being the same as before. Find the moment and sense of the couple which must be applied if the rod remains in equilibrium without alteration in the tension of the string.

6. A particle is projected with a velocity whose horizontal and vertical components are u and v respectively. Show that the range on a horizontal plane through the point of projection is $2uv/g$.

If α is the angle of projection, show that after a time $v/2g$ the particle is moving at an angle β to the horizontal given by

$$\tan \beta = \frac{1}{2} \tan \alpha.$$

Calculate α if the greatest height attained by the particle is one-fifth of the horizontal range.

7. A uniform ladder of length 24 ft. and weight 95 lb. rests on a rough horizontal floor and against a rough vertical wall, the ladder being in a vertical plane perpendicular to the wall. If the ladder is inclined at 60° to the horizontal and the coefficient of friction at both points of contact is $\frac{1}{4}$, find how far up the ladder a man of weight 160 lb. may ascend before the ladder begins to slip.

Section B.

8. One end of a uniform rod of length $2a$ and mass m is freely hinged to a fixed point and a particle of equal mass m is attached to the other end of the rod. If the rod is released from rest when horizontal, prove that its angular velocity when it reaches the vertical is $\sqrt{9g/8a}$.

Immediately after passing through the vertical position the rod is acted upon by a constant friction couple of moment $9mga/2\pi$. Verify that it will ascend through 60° before coming to instantaneous rest.

9. Prove by integration that the centre of mass of a uniform solid hemisphere of radius r is at a distance $3r/8$ from the base of the hemisphere.

From a uniform solid hemisphere of radius r a hemispherical portion of radius x is removed. The portion removed has the same centre as, and its base is coplanar with, that of the original hemisphere. Find the distance of the centre of mass of the remainder from the centre of the hemisphere.

By increasing the portion removed towards equality with the original hemisphere, deduce the position of the centre of mass of a thin hemispherical bowl.

10. Explain Newton's method for the approximate solution of an equation.

Obtain the real root of the equation

$$x^3+x-4=0$$

correct to three decimal places.

11. The foci of an ellipse are S (4, 0) and S' (-4, 0), and any point P on the ellipse is such that $SP+S'P=10$. Find the equation of the ellipse.

If Q is the point in the first quadrant which lies on this ellipse and has its x-coordinate equal to 3, find the equation of the tangent to the ellipse at Q.

Find also the coordinates of the point in which this tangent meets the directrix associated with the focus S.

Questions on Statistics.

12. Explain the meaning of the term "binomial distribution".

A certain newspaper racing correspondent can forecast the winner of a race 3 times out of 10 on the average. Assuming that a race meeting consists of 6 races, and that there are 380 race meetings in a season, how many times in a season would you expect him to forecast at least two winners at a meeting?

13. The weight of a young pig is shown below for eight successive weeks of an experimental period. Plot these numbers on a graph. Calculate the average growth-rate in pounds per week, to one decimal place. Draw on the graph the regression line showing how weight changes with time, and write down its equation.

Week	1	2	3	4	5	6	7	8
Weight (lb.)	48	54	60	67	76	86	94	104

14. The Arithmetic and English marks of 20 candidates for an Entrance Examination to Secondary Schools were:

Arithmetic	41	23	33	45	19	14	56	54	28	34
English	81	56	54	66	70	41	77	74	62	38
Arithmetic	53	57	71	23	21	47	37	50	19	55
English	54	68	82	33	48	76	52	48	42	78

Calculate the coefficient of correlation between the marks for the two subjects, and comment on the result.

15. An event may happen or fail, and in n independent occurrences the proportion of happenings is p. Prove that the standard error of this proportion is $\sqrt{(pq/n)}$, where $q=1-p$.

A battered penny tossed 20 times gave 14 heads and 6 tails. Discuss whether this provides convincing evidence that the coin is biased.

OXFORD AND CAMBRIDGE
SCHOOLS EXAMINATION BOARD

(on behalf of the G.C.E. Examining Boards)

General Certificate Examination

Advanced Level

SCHOOL MATHEMATICS PROJECT

MATHEMATICS I

MONDAY 4 JULY 1966. 3 HOURS

Candidates must not attempt more than **twelve** *questions from Section* A, *or more than* **four** *questions from Section* B.

Candidates are strongly advised not to spend more than half the time on Section A.

SECTION A

A 1. Find the general solution of the equation

$$\cos 2x = \tfrac{1}{2},$$

giving your answer in radians.

A 2. The roots of the equation

$$x^3+ax^2+bx+c = 0$$

are three times the roots of the equation

$$x^3+2x^2+3x+4 = 0.$$

Find a and c.

A 3. Express in partial fractions

(i) $\dfrac{1}{(x+1)(x+2)}$,

(ii) $\dfrac{1}{(x+1)(x^2+2)}$.

A 4. State for each of the following functions whether it is odd, even, periodic, or none of these, justifying your answers:

(i) $x \sin x$,

(ii) $\sin x + \cos x$,

(iii) $x + \cos x$.

A 5. Starting from the approximation

$$\sqrt{17} \simeq 4,$$

apply Newton's method to the equation

$$x^2 = 17$$

to obtain first the approximation $x \simeq 4{\cdot}125$ and then the approximation $x \simeq 4{\cdot}123$.

A 6. Show that the set of matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is closed under the operation of matrix multiplication.

A 7. Prove that

$$\int_0^4 \frac{x\,dx}{\sqrt{(x^2+1)}} = \sqrt{17} - 1.$$

Apply Simpson's rule, with ordinates at $x = 0, 2, 4$, to this integral to estimate the value of $\sqrt{17}$.

A 8. Given that

$$3x + 2y = u,$$
$$4x + 3y = v,$$

express x, y in the form

$$x = au + bv,$$
$$y = cu + dv.$$

Write down the product of the matrices

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A 9. The position vector **r** of a particle can be expressed in terms of the constant orthogonal unit vectors **i**, **j** by means of the relation

$$\mathbf{r} = \mathbf{i}\cos nt + \mathbf{j}\sin nt.$$

Find the speed of the particle at time t.

Show that its acceleration at time t is the vector $-n^2\mathbf{r}$.

A 10. Express the complex number

$$2+j\sqrt{3}$$

in the form

$$r(\cos\theta + j\sin\theta).$$

[Your value of θ may be given to the nearest degree.]

A 11. Differentiate

(i) $x^2 \sin 3x$ with respect to x,

(ii) $\sin^2 2x$ with respect to u, where $u = 4x$.

A 12. Prove that

$$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n} = 2-\frac{1}{2^n}.$$

Use this finite summation to explain carefully what you mean when you say that 'the sum of the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}+\ldots$$

is 2.'

A 13. Give a rough sketch of the curve

$$y = (x-1)(x-2)(x-3).$$

Find the equation of the tangent to the curve at the point (1, 0).

A 14. Solve the differential equation

$$\frac{ds}{dt}+s = 1,$$

given that $s = 2$ when $t = 1$.

A 15. Perform the following operations in binary arithmetic:

(i) $101+10110+1111$,

(ii) 1011×1101.

Decompose the following binary numbers into prime factors:

$$1111 \quad \text{and} \quad 11001.$$

A 16. A truck of mass 3 tons moving at 6 ft. per sec. catches up and collides with one of 2 tons moving at 4 ft. per sec., the two trucks remaining together after the collision. With what speed may they be expected to move off, and what is the corresponding loss of energy in foot-poundals.

A 17. A man tosses a coin six times. Find the probability that he will throw two heads and four tails

(i) in any order,

(ii) the two heads being thrown consecutively.

A 18. In a class of 10 children the marks in an examination are 14, 18, 21, 25, 29, 29, 30, 31, 35, 38. Find the mean and the standard deviation, giving your answers to the nearest integer.

A 19. A particle moves in a straight line so that, at time t, its distance from a fixed origin O is x, its velocity is $v = \dot{x}$ and its acceleration is $f = \ddot{x}$. Prove that

$$f = v\frac{dv}{dx}.$$

A particle is released from rest under gravity in a medium which offers a resistance kv per unit mass, where k is constant. Obtain the equation of motion

$$\ddot{x} = g - k\dot{x},$$

where x is the depth at time t below the point of release.

Criticise the argument:

This equation is

$$v\frac{dv}{dx} = g - kv.$$

But $v = 0$ *initially. Hence* $g = 0$.

A 20. It is required to find the highest factor common to two given numbers. Give a flow diagram for the general solution of such a problem and follow the steps through to obtain the answer in the particular case when the numbers are 231 and 1078.

Section B

B 21. Determine (giving reasons) which of the following properties define equivalence relations, and state the corresponding equivalence classes:

(i) parallelism among the set of straight lines;

(ii) possession of a factor in common among the set of positive integers;

(iii) equality of modulus among the set of complex numbers;

(iv) equality of radius among the set of circles.

B 22. Solve the simultaneous equations

$$\begin{aligned} x+y+z &= 3, \\ x+2y+3z &= 6, \\ x+3y+kz &= 4+k, \end{aligned}$$

(i) when $k \neq 5$;

(ii) when $k = 5$, giving the general solution.

B 23. The operation of *multiplication modulo* 15 is defined on the set of integers by defining the product 'mn' of two integers m and n to be the remainder after dividing the ordinary arithmetical product by 15. Prove that the set $\{3, 6, 9, 12\}$, together with this rule of multiplication modulo 15, is a group.

Is this group isomorphic with the group consisting of the four numbers $1, -1, j, -j$ subject to the operation of ordinary multiplication of complex numbers?

B 24. Find in Cartesian coordinates the equation of the plane through the three points $(1, 1, 1)$, $(4, 1, 0)$, $(3, 0, 1)$.

Find also the coordinates of the foot of the perpendicular from the origin to this plane.

B 25. Solve the differential equations:

(i) $2x\frac{dy}{dx}+y^2 = 1$,

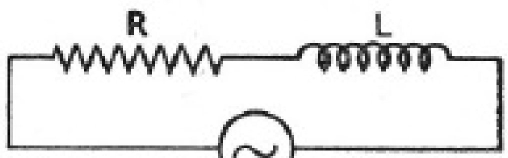
given that $y = 0$ when $x = 1$;

(ii) $\frac{dy}{dx}+3y = x$,

given that $y = 0$ when $x = 0$.

B 26. In the circuit illustrated, current flows through the resistance R and inductance L as a result of the application of the alternating E.M.F. equal to $E\sin pt$ at time t, where E is constant. Prove that, after the circuit has been oscillating for a long time, the current alternates with a frequency $2\pi/p$ and maximum amplitude $(R^2+p^2L^2)^{-\frac{1}{2}}E$.

If R and L may be adjusted to take various values in such a way that this peak current is constant, sketch a graph showing the relationship between R and L.



B 27. A plank AB of length l and mass M is at rest on a smooth horizontal floor. A 'man' of mass m is originally at rest on the plank at the end A. He 'walks' along the plank from A to B, without slipping, at constant speed V relative to the plank, stopping just as he reaches B. Find

(i) the time of the motion,

(ii) the distance moved by the plank,

(iii) the kinetic energy (man *plus* plank) at any instant during the motion.

B 28. A standard pack contains 52 cards. In a defective pack, the King and Queen of Spades are missing but the King and Queen of Diamonds are duplicated. What is the probability that four cards removed one after another will produce the King of Hearts, the King of Diamonds, the Queen of Diamonds, the King of Clubs in that order?

Find the corresponding answers if (i) each card is replaced in the pack after identification; (ii) cards are not replaced (as in the original problem) but the four named cards may be drawn in any order.

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD

(on behalf of the G.C.E. Examining Boards)

General Certificate Examination

Advanced Level

SCHOOL MATHEMATICS PROJECT

MATHEMATICS II

TUESDAY 5 JULY 1966. 3 HOURS

Candidates must not attempt more than **twelve** *questions from Section* A, *or more than* **four** *questions from Section* B.

Candidates are strongly advised not to spend more than half the time on Section A.

SECTION A

A 1. In three-dimensional Euclidean space with Cartesian coordinates (x, y, z), points A and B are given by $(2, 1, 1)$ and $(1, -4, 2)$. Prove that the sphere with AB as diameter passes through the origin.

A 2. Sketch the curve which, in the (x, y)-plane, is given by $y(x^2-1) = x$.

A 3. The complex number $x+jy$ is mapped into the complex number $X+jY$ where X, Y are given by the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Which numbers are invariant under the mapping?

A 4. Draw up a flow diagram for calculating, to a prescribed degree of accuracy, the real root of the equation $x^3 - x^2 + x + 1 = 0$.

A 5. State carefully what is meant by the statement that '$f(x)$ is differentiable at $x = a$'. Use your definition to evaluate the derivative of $g(x)$ at $x = 1$, where

$$\begin{aligned} g(x) &= x - 1, \quad x \leqslant 1 \\ &= x^2 - x, \quad x > 1. \end{aligned}$$

A 6. Find the set of values of x which satisfy

$$2 \sin x + \sin 2x > 0.$$

A 7. A cylindrical water trough is 6 ft. long. Its cross-section is of the same shape as the graph (drawn in rectangular co-ordinates) of $y = \sin x$, $\frac{1}{2}\pi \leqslant x \leqslant \frac{5}{2}\pi$. When it is arranged so as to hold the greatest quantity of water, the greatest depth is 6 in. Prove that the volume of water is then about 2·35 cu. ft.

A 8. Draw two number lines, parallel and in the same sense. By sketching straight arrows which go from one line to the other, illustrate the mapping $f\colon R \to R$ defined by $f(x) = \sinh x$.

Say whether or not two such arrows can intersect, and explain your answer.

A 9. Early experimenters on the flow of fluids through capillary tubes proposed a formula of the form

$$Q \propto \frac{1}{\mu}\left(\frac{dp}{dx}\right) r^n,$$

where Q is the volume flow per unit time,
μ is the coefficient of viscosity (of dimensions $ML^{-1}T^{-1}$),
dp/dx is the pressure gradient along the tube,
r is the radius of the tube, and
n is a numerical index to be found.

Suggest a suitable value for n.

A 10. If θ may take any real value, prove that the maximum value of $(3\cos\theta - \cos^2\theta)$ is 2.

Explain whether, by putting $\cos\theta = x$, it is correct to deduce that the maximum value of $(3x - x^2)$, for all x, is 2.

A 11. 'x real $\Rightarrow (x^2 > 1 \Rightarrow x > 1)$.'

Is this assertion true or false? If it is true, prove it; if it is false, give a counter example.

A 12. Two unequal electrical resistances combine in series to give a resistance R_s and in parallel to give a resistance R_p. Prove that $R_s > R_p$.

A 13. Prove that the set-operation of union is associative. (A demonstration by Venn diagram is unacceptable.)

A 14. Prove that $[\sqrt{(n^2+1)} - n].[\sqrt{(n^2+1)} + n] = 1$, and hence find $\lim\limits_{n\to\infty}[\sqrt{(n^2+1)} - n]$.

A 15. How many distinct linear functions $y = f(x)$ are there which map $0 \leqslant x \leqslant 1$ onto $-1 \leqslant y \leqslant 3$?

For *one* such function, find the value of x which satisfies $x = f(x)$.

A 16. The mean survival period of daisies after being sprayed with a certain make of weed killer is 24 days. If the probability of survival after 27 days is $\frac{1}{4}$, estimate the standard deviation of the survival period.

A 17. Evaluate

$$\int_{-2}^{2} \frac{2dt}{4+t^2} \quad \text{and} \quad \int_{-2}^{2} \frac{2t\,dt}{4+t^2}.$$

A 18. By multiplying the series $\left[\frac{1}{2} + \sum_{n=1}^{N} \cos nx\right]$ by $\sin\frac{1}{2}x$, or otherwise, find a formula for its sum. Discuss the formula when $x = 0$.

A 19. If the function $f(x)$ satisfies $x\,df/dx = 1$, find $f(-2)$ given that $f(-3) = -3$.

A 20. If x is real and positive, $\sqrt{x}$ is defined as the positive square root of x. Explain briefly whether this definition can be generalised for complex numbers.

Find the two complex numbers whose squares equal $-5-12j$.

Section B

B 21. Two pendulums of the same length—the first, with a concentrated mass at one end of a light string whose other end is held fast: the second, a uniform rod pivoted at one end—are released simultaneously from horizontal positions.

Neglecting all frictional effects, express the time of swing of each pendulum in terms of the integral $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}$, and hence prove that there is no subsequent time at which the two pendulums are simultaneously horizontal.

B 22. S is the set of numbers $a+b\sqrt{c}$, where a and b are any members of the set of positive rationals and c is a fixed positive irrational. Under which of the four arithmetic operations is S closed?

Is it closed under the operation of extracting the positive square root?

B 23. The binary operation of 'multiplication' $\otimes$ is defined on the set of ordered pairs (z, w) of complex numbers z and w by setting

$$(z_1, w_1)\otimes(z_2, w_2) = (z_1z_2-\bar{w}_1w_2,\ z_1w_2+w_1\bar{z}_2),$$

where the bar denotes the complex conjugate. Write out the multiplication table for the subset of elements

$$(1, 0),\ (j, 0),\ (0, 1) \text{ and } (0, j),$$

where $j^2 = -1$.

On the original set of all pairs (z, w),

(i) prove that the operation $\otimes$ is not commutative;

(ii) obtain the 'identity element' for the operation $\otimes$.

B 24. $$x^2+2y^2-2x+4y-2 = 0$$
$$\text{and } x^2+2y^2-2x+4y-17 = 0$$

are the equations of two ellipses expressed in Cartesian co-ordinates. Show that they have a common centre.

Show also that one is an enlargement of the other and find the centre of enlargement.

If the point (x_1, y_1) on the first ellipse goes, under the enlargement, to the point (x_2, y_2) of the second, find a linear equation connecting the two matrices

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ and } \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Is there a pair of corresponding points which lie on a line through the origin? If so, what is the equation of the line?

B 25. A set has n members. How many distinct functions exist which map the set onto itself?

For a set of three members A, B and C, enumerate in full the six functions which map the set onto itself.

Under the operation of combination, these six functions form a group; enumerate its subgroups.

B 26. Use a step-by-step method, with 0·2 as the step-length, to estimate the value of $y(1)$, given that

$$\frac{dy}{dx} = 1+y^2, \text{ and } y(0) = 0.$$

Lay your work out in tabular form after the pattern:

x	y	y^2	$1+y^2$	Δy
0	0	0	1	0·2
0·2				

Compare your value of $y(1)$ with the exact value.

B 27. A sequence a_i $(i = 1, 2, 3, \ldots)$ is defined by

$$a_1 = a_2 = 2,$$
$$a_n = a_{n-1}+a_{n-2} \qquad (n = 3, 4, 5, \ldots).$$

Prove that $a_i \equiv 0 \pmod{10}$ when i is a multiple of 5.

B 28. You have positioned yourself on the boundary so as to catch a ball hit by the batsman. Show that, if it were not for the effects of perspective, air resistance and a few other realities, the ball would appear to be rising vertically with a constant speed, all the time.

Describe, as carefully as you can, how air resistance would affect this result.

Appendix 3 1994 SMP A-level questions

1

Fig. 1.1

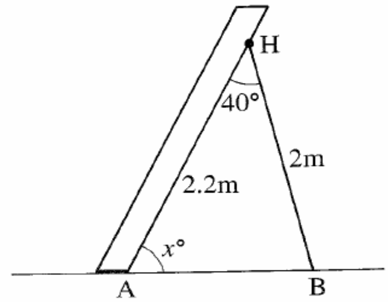


Fig. 1.2

A step-ladder stands on a level floor. A diagrammatic side view is shown in Fig. 1.2. When the steps are fully opened out, the support HB makes an angle of 40° with the edge HA of the steps. The length HA = 2.2 m, and HB = 2 m.

(i) If the angle HAB is denoted by $x°$, show that

$$1.1 \sin x° = \sin (140 - x)°.$$

Give a reason why this equation can be written as

$$1.1 \sin x° = \sin (40 + x)°. \quad [3]$$

(ii) Expand the right side of this last equation; then rearrange the terms to calculate $\tan x°$. Show all the figures given by your calculator. [5]

(iii) Find, to the nearest degree, the angle which the edge of the steps makes with the floor. [2]

at right angles to the wall BCRQ, and the z-axis is vertical.

4

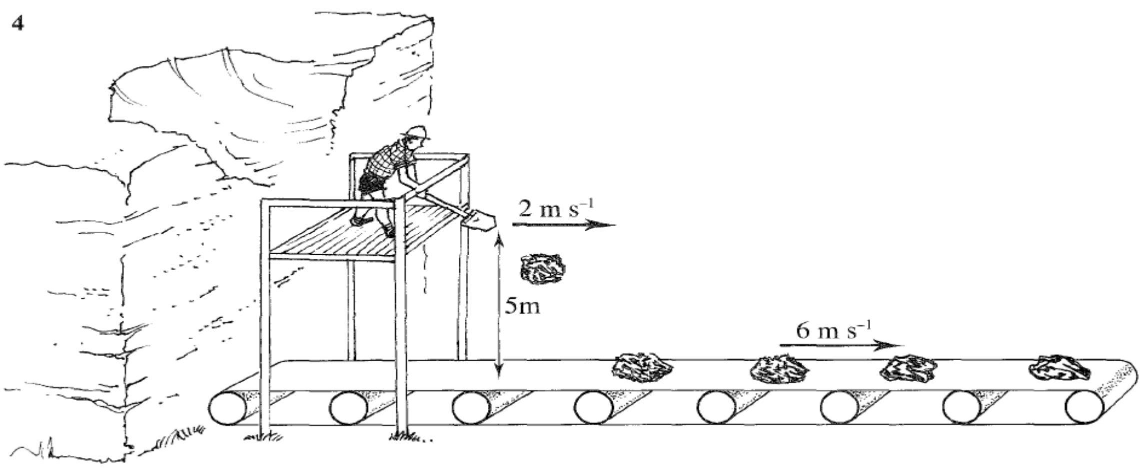


Fig. 4

In a quarry a horizontal conveyor belt carries the rock away. A worker standing on a platform 5 m directly above the belt shovels a piece of rock horizontally at a speed of 2 m s^{-1}, so that it lands on the belt.

(i) Find the horizontal and vertical components of the velocity of the rock just before it lands on the belt. [3]

[Use the approximation $g = 10$ m s^{-2}.]

(ii) The belt is moving at 6 m s^{-1}, and the mass of the rock is 8 kg. The rock lands on the belt without bouncing, and friction causes it to take up the speed of the belt almost instantaneously.

(*a*) Find the horizontal and vertical components of the impulse of the belt on the rock, stating the unit in which your answers are measured. [4]

(*b*) Find the magnitude and direction of the resultant impulse of the belt on the rock. [3]

Appendix 4   1998 SMP A-level questions

2 A dartboard has radius 20 cm, and a 'bull' region of radius 1 cm (see Fig. 2). A dart is thrown at the board and its distance from the centre is measured.

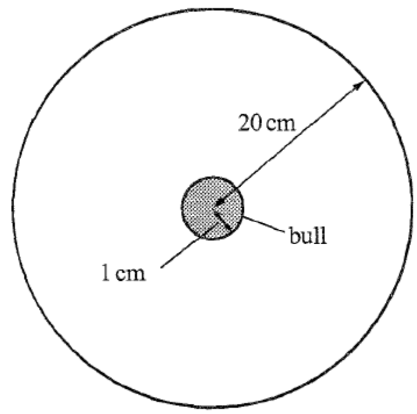


Fig. 2

(a) In a simple model, the dart always lands on the dartboard, and 'each part of the dartboard has the same chance of being hit'. Stated more accurately, the probability of landing in a given region is proportional to its area.

(i) Show that for this model the probability of throwing a dart within x cm (where $0 \leqslant x \leqslant 20$) of the centre of the dartboard is $\frac{x^2}{400}$, and calculate the probability of hitting the bull with one dart. [4]

(ii) Deduce the probability density function for the distance from the centre of the dartboard, and calculate its mean and variance. [9]

(iii) Three darts are now thrown at the dartboard, and the distance of the closest dart to the centre is measured. Assuming independence, find the probability that all three darts are more than x cm from the centre. Hence show that the probability of the closest dart being within x cm (where $0 \leqslant x \leqslant 20$) is

$$1 - \left(1 - \frac{x^2}{400}\right)^3.$$

Calculate the probability of getting at least one bull from three darts. [6]

(iv) Criticise the assumptions made by this model. [2]

(b) An alternative model for the probability of throwing a dart within x cm ($0 \leqslant x \leqslant 20$) of the centre of the dartboard is

$$\frac{cx^2 e^{-x/10}}{400}$$

where c is a constant. Assuming as before that the dart always hits the board, find the value of c, and use this model to calculate the probability of hitting the bull with one dart. [4]

5 This question models the start of a motor race.

(i) Initially a car A of mass 600 kg accelerates from rest along a straight track. For the first T seconds, where $T > 0$, the driving force is a constant 6000 N, and the car travels a distance X m until it reaches a speed of U m s^{-1}. Neglecting air resistance, find X and U in terms of T. [4]

(ii) For the **next** t seconds, the car runs on maximum power, generating a driving force of $\frac{100\,000}{v}$ N, where v m s^{-1} is the velocity. If the car travels a **further** x metres after t seconds, show that, neglecting resistance forces,

$$\frac{dv}{dx} = \frac{500}{3v^2}.$$

Solve this equation to find x in terms of v and U. [7]

(iii) If the car reaches a speed of 50 m s^{-1} after covering a **total** distance of 250 m, find the value of T. [5]

(iv) The car then decelerates before it reaches the first corner, which is flat with no banking. The radius of the bend is 300 m and the maximum force the tyres can exert towards the centre of the bend is 2000 N. Find the maximum speed at which the car can corner safely. [3]

(v) Another car B of mass 500 kg enters the first corner, and crashes into the first car. Relative to x- and y-axes parallel and at right angles to the track side opposite the impact, the velocities immediately before impact are $\begin{pmatrix} 30 \\ 0 \end{pmatrix}$ m s^{-1} for car A and $\begin{pmatrix} 20 \\ 25 \end{pmatrix}$ m s^{-1} for car B. The velocity of car B after impact is $\begin{pmatrix} 12 \\ 10 \end{pmatrix}$ m s^{-1}. Find the impulse vector on car A in the collision, and the velocity vector of this car immediately afterwards. [6]

6 A 'ferry glide' is a technique used by canoeists for moving sideways across a river (see Fig. 6.1). If the speed of the current is u m s^{-1}, then the canoeist points the canoe at an angle α to the current and paddles with speed v m s^{-1} relative to the water.

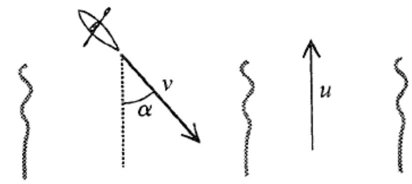


Fig. 6.1

(i) Show that for the canoe to move at right angles to the current, then

$$v > u \text{ and } \cos\alpha = \frac{u}{v}.$$

Calculate the speed of the canoe in terms of u and v. [5]

A novice canoeist attempting a ferry-glide is in trouble. He is 20 m from each bank and some distance upstream of a waterfall. He is paddling upstream with speed $\frac{1}{2}u$ m s^{-1} relative to the water, at an angle α to the stream, aiming for the right-hand bank (see Fig. 6.2).

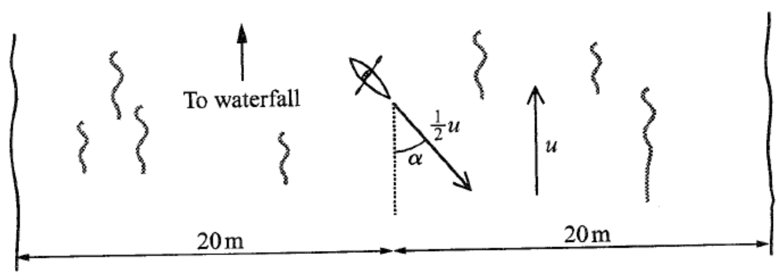


Fig. 6.2

(ii) Show that he would need $\dfrac{40}{u\sin\alpha}$ seconds to reach the right-hand bank, and that in that time he would travel a distance d metres downstream, where

$$d = \frac{20(2-\cos\alpha)}{\sin\alpha}.$$ [5]

(iii) If $\alpha = 60°$ show that the canoeist will be unable to reach the bank if the waterfall is less than 34 metres downstream. [2]

An instructor is 20 m upstream of the novice, at the left-hand bank of the river. He sees the novice in trouble, and paddles to him (see Fig. 6.3).

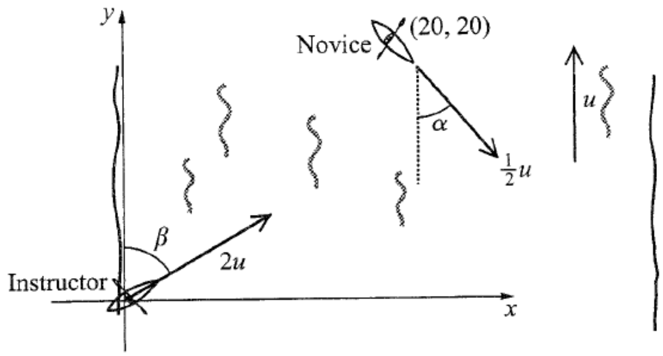


Fig. 6.3

Relative to fixed axes drawn as shown through the initial position of the instructor, the coordinates (in metres) of the novice are (20, 20). The instructor then paddles with speed $2u$ m s^{-1} relative to the water, at an angle β to the stream.

(iv) Suppose that relative to the fixed x- and y-axes the velocities of the instructor and novice are $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Find $\mathbf{v}_1$ and $\mathbf{v}_2$ in terms of u, α and β. Show that if the position vectors of instructor and novice at time t s after the instructor starts paddling are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively, then

$$\mathbf{r}_1 = \begin{bmatrix} 2ut\sin\beta \\ ut(1+2\cos\beta) \end{bmatrix} \text{ and } \mathbf{r}_2 = \begin{bmatrix} 20+\frac{1}{2}ut\sin\alpha \\ 20+ut(1-\frac{1}{2}\cos\alpha) \end{bmatrix}.$$ [4]

(v) Hence or otherwise, show that if $\alpha = 60°$, and the instructor reaches the novice, then

$$\sin(\beta - 45°) = \frac{1+\sqrt{3}}{8\sqrt{2}}$$

and solve this equation for β. [9]

Appendix 5: Sequence questions analysed in chapter 8

'Explicit' Questions

e1 January 1999 Question 2

(a) An arithmetic progression has first term 7 and common difference 3.

(i) Write down a formula for the kth term of the progression.

Which term of the progression equals 73? [3]

(ii) Write down a formula for the sum of the first n terms of the progression. How many terms of the progression are required to give a sum equal to 6300? [4]

(b) A geometric progression has first term a and common ratio r. The sum of the first three terms is 4.88 and the sum to infinity is 10.

(i) Write down two equations involving a and r.

Show that $1 - r^3 = 0.488$.

Hence find the values of a and r. [5]

(ii) The kth term of the progression is u_k.

Calculate the value of $\sum_{k=1}^{\infty}(u_k)^2$. [3]

e2 May 1999 Question 2(a)

A geometric progression has first term 100 and common ratio 0.9.

Calculate **(i)** the fifteenth term, [2]

(ii) the sum of the first 20 terms, [2]

(iii) the sum to infinity. [1]

e3 Jan 2001 question 2

(a) The sum of the first n terms of the arithmetic progression 50, 52, 54, 56 … is denoted by S. The sum of the first n terms of the arithmetic progression 100,99,98,97, … is denoted by T.

(i) Express each of S and T in terms of n, simplifying your answers. [3]

(ii) Deduce the least value of n for which $S > T$. [4]

(b) The sequence u_n is defined by $u_n = n \sin(a + 180n)°$, $n = 1, 2, 3, 4, \ldots$ where a is a constant, and $0 < a < 90$.

(i) Write down and simplify the first 4 terms of the sequence. Find the sum of these 4 terms, giving your answer in terms of sin a. [5]

(ii) Deduce the value of $\sum_{n=1}^{100} u_n$, giving your answer in terms of sin a. [3]

e4 Jan 2003 question 1(a)

An arithmetic progression has first term 4 and common ratio 3.

Find the 50^{th} term, and the sum of the first 50 terms.

[4]

e5 June 2005 question 2

(i) An arithmetic progression has first term −8. The 20th term is three times the 10th term. Find the common difference. [3]

(ii) Another arithmetic progression has common difference 2. The sum of the first 20 terms is three times the sum of the first 10 terms. Find the first term. [4]

(iii) A geometric progression is such that its 20th term is three times its 10th term. The first term is not zero, and the common ratio is positive. Find the common ratio, giving your answer to 3 significant figures. [3]

(iv) Another geometric progression has non-zero first term and common ratio r, where $r > 0$ and $r \neq 1$. The sum of the first 20 terms of this progression is three times the sum of the first 10 terms. Show that $u^2 - 3u + 2 = 0$, where $u = r^{10}$.

Hence find the value of r. [5]

‘Algebraic’ questions

a1 January 1998 Question 4

A sequence of numbers t_1, t_2, t_3, t_4, … is formed by taking a starting value for t_1 and using the rule

$$t_{k+1} = t_k^2 - 2, \quad \text{for } k = 1, 2, 3, \ldots \, .$$

(i) If $t_1 = \sqrt{2}$, calculate t_2, t_3 and t_4. Show that $t_5 = 2$, and write down the value of t_{100}. [4]

(ii) If $t_1 = 2$, show that all terms of the sequence are the same.

Find the other value of t_1 for which all terms of the sequence are the same. [4]

(iii) Determine whether the sequence converges, diverges or is periodic in the cases when

(A) $t_1 = 3$,

(B) $t_1 = 1$

(C) $t_1 = \dfrac{\sqrt{5}-1}{2}$. [6]

a2 Jan 2002 question 3(a)

A sequence is defined by $u_{r+1} = u_r - 3$, $u_1 = 102$.

(i) Find u_2 , u_3 and u_{100} . [3]

(ii) Find an expression for $\sum_{r=1}^{n} u_r$ in terms of n, simplifying this as far as possible.

Find the value of n for which $\sum_{r=1}^{n} u_r = 0$. [4]

a3 June 2002 question 2

A geometric progression is defined by $u_i = 3 \times 1.25^{-i}$, $i = 1, 2, 3, \ldots$.

(i) Calculate u_1, u_2 and u_3. What is the common ratio of the geometric progression? [4]

(ii) Calculate $\sum_{i=1}^{20} u_i$. [2]

(iii) Find the sum to infinity of the geometric progression. [2]

a4 June 2003 question 3

(i) For each of the following sequences, state whether they are arithmetic, geometric or neither of these. For those that are arithmetic or geometric, find the sum of the first 20 terms of the corresponding series.

(A) 50, 52, 54, 56, …

(B) $u_n = 2 \times 0.8^n$, $n = 1, 2, 3, \ldots$

(C) $u_n = 2n + 3$, $n = 1, 2, 3, \ldots$

(D) $u_n = n^2$, $n = 1, 2, 3, \ldots$

(E) $u_{n+1} = -u_n$, $u_1 = 2$, $n = 1, 2, 3, \ldots$

(F) $u_{n+1} = 2u_n + 1$, $u_1 = 1$, $n = 1, 2, 3, \ldots$ [13]

(ii) In the case of one of these sequences, the corresponding series has a sum to infinity.

Calculate the sum to infinity of this series. [2]

a5 Nov 2003 question 2

A sequence u_r is defined for $r = 1, 2, 3, \ldots$ by $u_r = a + b\,r + c\,d^{\,r}$, where a, b, c and d are constants.

(i) In each of the three cases below, find the first three terms u_1, u_2 and u_3, state the type of sequence produced, and calculate the sum of the first 30 terms of the sequence.

(A) $a = 2, b = 3, c = 0.$

(B) $a = 0, b = 0, c = 2, d = 1.1$.

(C) $a = 3, b = 0, c = 2, d = -1$. [12]

(ii) Hence find the sum of the first 30 terms of the sequence in the case where $a = 2$, $b = 3$, $c = 2$ and $d = 1.1$. [2]

a6 Jan 2004 question 3

A geometric progression u_i is defined by

$$u_i = e^{-i},\ i = 1, 2, 3, \ldots$$

where e is the base of natural logarithms.

(i) Calculate u_1 and u_2, and show that $u_3 = 0.050$, correct to 3 decimal places. Write down, in terms of e, the common ratio of the progression. [3]

(ii) Find the least value of i for which $u_i < 10^{-12}$. [4]

(iii) Show that the geometric series $u_1 + u_2 + u_3 + \ldots$ is convergent.

Show that the sum to infinity of this series is $\dfrac{1}{e-1}$. [3]

(iv) The sequence v_i is defined by $v_i = \ln u_i$. Show that v_i is an arithmetic progression, stating its first term and common difference. Hence calculate $\sum_{i=1}^{100} v_i$. [5]

a7 Jan 2006 question 2

A sequence u_r is defined for $r = 1, 2, 3, \ldots$ by $u_1 = a$, $u_{r+1} = b\, u_r + c$, where a, b and c are constants.

(i) In the case where $a = 3$, $b = -1$ and $c = 8$, write down the values of u_1, u_2, u_3 and u_4. State what type of sequence this is. [4]

(ii) Find the values of a, b and c which produce the sequence 1, 3, 5, 7, … .

State what type of sequence this is, and show that the sum of the first n terms of the sequence is n^2. [6]

(iii) In the case where a and b are non-zero and $c = 0$, write down u_1, u_2, and u_3 in terms of a and b. State what type of sequence is produced. Given that the sum to infinity of this sequence is $3\, u_1$, find the value of b. [5]

‘Word’ questions

w1 Jan 1997 question 1

(a) An eccentric and rather unfit gardener needs to spread 4000 kg of sand over his garden. He spreads 5 kg during the first day then increases the amount he spreads each subsequent day by 2 kg, so that he spreads 7 kg during the second day, 9 kg during the third day, and so on.

Find an expression for the mass of sand he has spread by the end of n days. How many days will it take him to spread all 4000 kg? [7]

(b) His neighbour, who prides himself on his fitness, also needs to spread 400 kg of sand over his garden. He decides to spread 200 kg each day but discovers that, after spreading 200 kg during the first day, during each subsequent day he can only spread 95% of the amount he spread during the previous day.

- **(i)** Show that, after n days, he has spread $4000(1 - 0.95^n)$ kg. [4]
- **(ii)** How many days will it take him to spread 3900 kg? [3]
- **(iii)** Explain why he will never spread all 4000 kg. [1]

w2 May 1997 Question 2

Mr and Mrs Brown have found a house they wish to buy which is valued at £50 000. In order to buy it they borrow £50 000 from the bank, intending to pay back the loan over a period of 30 years. At the end of each year the bank charges interest of 8% on the amount still owing and the Browns then pay back a fixed amount £P, so that the amount owed after their first payment is $(50\,000 \times 1.08) - P$.

- **(i)** Write down an expression for the amount owed after their second payment and show that the amount in pounds owed to the bank after their third payment is

$$(50\,000 \times 1.083 - P(1 + 1.08 + 1.082) \quad [4]$$

- **(ii)** Generalise your answers to (i) to write down a formula for the amount owed to the bank after their nth payment ($n \leq 30$). Use the formula for the sum of a geometric progression to simplify this formula. Use this simplified formula with $n = 30$ to find the amount £P, giving your answer to the nearest penny. [7]
- **(iii)** The value of the Browns’ house increase by k% each year, where k is a constant. They discover that, after 30 years, the value of their house equals the total amount

they paid to buy it. Find the value of k, giving your answer to three significant figures. [4]

w3 Jan 2000 question 3

Anne invests £100 at the start of each year. Each £100 earns £10 interest for every year it has been invested, as illustrated in the following table:

n	*Value of Anne's investment at start of year n* (£)
1	100
2	(100 + 10) + 100
3	(100 + 20) + (100 + 10) + 100
…	…

(i) Give the next line of the table, for $n = 4$, and calculate the value of Anne's investment at the start of year 4. [2]

(ii) Show that the value £V_A of Anne's investment at the start of year n is given by

$$V_A = 95n + 5n^2.$$ [3]

Brian also invests £100 at the start of each year. Each investment of £100 grows at a rate of 5% per year. It is thus multiplied by 1.05 each year, as shown in the following table.

n	*Value of Brian's investment at start of year n* (£)
1	100
2	$(100 \times 1.05) + 100$
3	$(100 \times 1.05^2) + (100 \times 1.05) + 100$
…	…

(iii) Show that the value £V_B of Brian's investment at the start of year n is given by

$$V_B = 2000(1.05^n - 1).$$ [3]

(iv) Verify that Brian's investment overtakes Anne's investment in value at the start of the 39th year. [3]

(v) Clyde invests £100 at the start of each year, in a similar way to Brian, but at a rate of interest of p% per year. The value of Clyde's investment is the same as the value of Anne's at the start of the 20th year. Show that

$$\left(1+\frac{p}{100}\right)^{20}=1+0.39p$$
. [4]

w4 June 2001 question 1

A forest contains oaks, beeches and pine trees.

(i) The number of oak trees at the end of n years is modelled by u_n, where $u_{n+1} = 0.8\,u_n + 25$, and u_0 represents the initial number of oak trees.

Calculate u_1, u_2 and u_3 in the cases when

(A) $u_0 = 250$, (B) $u_0 = 125$.

Comment briefly on your results. [5]

(ii) The number of beeches at the end of n years is modelled by v_n, where $v_{n+1} = rv_n$ and r is a constant. The initial number of beeches is v_0, where $v = 1000$.

Show that the numbers of beeches at the end of 1, 2, 3, … years form a geometric progression. If the number of beeches halves after 10 years, find the value of r, giving your answer correct to 2 decimal places. [4]

(iii) The number of pines at the end of n years is modelled by w_n, where $w_{n+1} = w_n + 10(n + 1)$ and initially there are no pine trees, so $w_0 = 0$.

Using this model, $w_1 = 0 + 10 \times 1 = 10$,

$w_2 = 10 + 10 \times 2 = 10 + 20$

$w_3 = 10 + 20 + 10 \times 3 = 10 + 20 + 30$.

Write down a similar expression for w_4. Show that $w_n = 5n(n + 1)$.

At the end of Y years, the number of pines first exceeds 1000. Find Y. [5]

w5 Jan 2003 question 2(i)

The number of new cases of infection from a virus in a week is modelled by a geometric progression with first term 32 and common ratio 1.25. The number in week 1 is 32

(A) Find the number of new cases predicted by the model in each of week 2, week 3 and week 10. [3]

(B) Find an expression in terms of n for the *total* number of cases in the first n weeks, simplifying your answer. After how many weeks would the total number of cases first exceed 5000? [4]

w6 Jan 05 question 2

A factory makes widgets. From January 2006, the production manager plans to phase out the production of old widgets and phase in production of new widgets. He models this process as follows.

For old widgets, the monthly production will form a geometric sequence with common ratio 0.9. The production in month 1 (January 2006) will be 5000.

(i) Find the production figures predicted by the model for months 2, 3 and 12. [3]

(ii) Find the total production of old widgets for the 24 months from the start of January 2006. [3]

(iii) Show that the total production of old widgets from the start of January 2006 will not exceed 50 000. [2]

For new widgets, the production in month 1 (January 2006) will be 500. Production will increase by 10% per month, forming a geometric sequence.

(iv) Write down an expression for the production of new widgets in month n. Hence show that monthly production of new widgets will first exceed that of old widgets in month n, where n is the smallest integer for which

$$\left(\frac{11}{9}\right)^{n-1} > 10.$$

Find this value of n. [6]

w7 Specimen Paper, question 2

The first three terms of a geometric series are 6, 2.4, 0.96

(i) (A) Write down an expression for the nth term.

(B) Find the sum of the first eight terms. [4]

A ball is dropped from a height of 15 m on to a concrete path and bounces repeatedly. After each impact it rebounds to a height 0.4 times the height from which it has just fallen.

(ii) To what height does it rebound after the nth impact? After how many impacts does it rebound to height of less than 1 cm? [5]

(iii) Write down a series expression for the total distance travelled by the ball from the instance when it is dropped until the nth impact.

Hence find the total distance travelled by the ball before it comes to rest. [4]

Another ball is dropped from height h m and bounces repeatedly. After each impact it rebounds to a height r times the height from which it has just fallen ($0 < r < 1$). This ball travels a total distance d m before it comes to rest.

(iv) Prove that $r = \dfrac{d-h}{d+h}$. [2]

w8 Jan 2002 question 3(b)

A ball rebounds backwards and forwards between two walls A and B (see Fig. 3). Each time it rebounds from a wall, its speed is reduced. The times between successive rebounds form a geometric progression. Each term is 20% larger than the previous term. The particle starts at A and takes 2 seconds to reach B for the first time.

Fig. 3

(i) State the first term and the common ratio of the geometric progression, and show that the ball rebounds for the third time after a total time of 7.28 seconds. [2]

(ii) Find the total time taken when the ball rebounds for the 20th time, giving your answer to the nearest second. [2]

(iii) Find how many rebounds the ball makes in the first fifteen minutes. [4]

w9 June 2002 question 2

(b) In a race, skittles S_1, S_2, S_3, … are placed in a line, spaced 2 metres apart. Susan runs from the starting point O, b metres from the first skittle. She picks up the skittles, one at a time and in order (S_1, S_2, S_3, …), returning them to O each time (see Fig. 2).

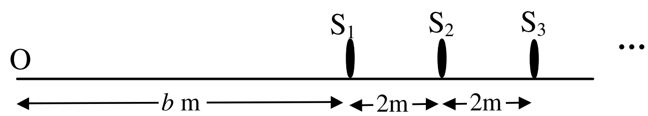


Fig. 2

(i) Show that the total distance Susan runs in a race with 3 skittles is $6(b + 2)$ metres. [1]

(ii) Show that the total distance she runs in a race with n skittles is $2n\,(b + n - 1)$ metres. [2]

(iii) With $b = 5$, the total distance she runs is 570 metres.

Find the number of skittles in this race. [3]

‘Pattern’ questions

p1 May 1999 Question 2(b)

Some people use playing cards to build ‘houses’. A house with 3 layers is illustrated in Fig. 2.1.

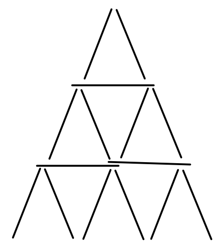


Fig. 2.1

Fig. 2.2 shows the separate layers of the house. Each line represents one card. The layers are numbered from the top downwards. Further layers are built in the same way.

Layer 1 has 2 cards

Layer 2 has 5 cards

Layer 3 has 8 cards

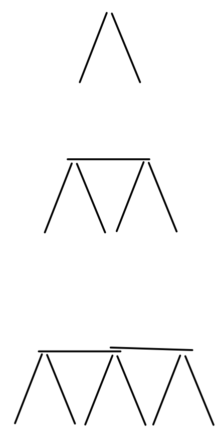


Fig. 2.2

(i) A house is built with 10 layers. How many cards are there in layer 10? [2]

(ii) Prove that there are ½ $(3n^2 + n)$ cards in a house with n layers. [2]

(iii) Jane has built a complete house. She calculates that she would need 44 cards to add one more layer. How many cards has she used already? [3]

(iv) For an exhibition a house is built using all of the cards in 91 packs of 52 cards. How many layers does the house have? [5]

p2 June 2000 question 2

Fig. 2 shows a rectangular spiral. It starts at O, has *two* sides of length u_1 cm, *two* sides of length u_2 cm, etc.

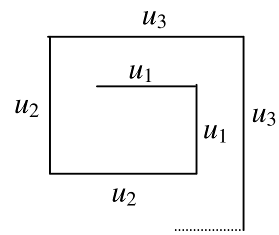


Fig. 2

(a) In one type of rectangular spiral, $u_1 = 10$ and $u_{r+1} = u_r + d$, where d is constant.

(i) Write down u_2 and u_3 in terms of d, and find the total length of a spiral with 6 sides. [2]

(ii) Show that the total length T_1 of a spiral with $2n$ sides is given by

$$T_1 = dn^2 + (20 - d)n.$$

If $d = 1$, and the total length of the spiral is 8100 cm, find the number of sides. [5]

(b) In another type of rectangular spiral, $u_1 = 10$ and $u_{r+1} = u_r \times c$, where c is a positive constant.

(i) Sketch roughly the shape of the spiral in the following cases, marking the starting point O.

(A) $c < 1$ (B) $c = 1$ (C) $c > 1$. [3]

(ii) When $c \neq 1$, find in terms of c the total length T_2 cm of a spiral with $2n$ sides. [3]

(iii) The total length of a spiral with an infinite number of sides is 1 metre.

Find the value of c. [2]

p3 Nov 2002 question 4

Fig. 4 shows an array of numbers. Each row starts with a 3 and ends with a 2. Each of the other numbers is formed, as in Pascal's triangle, by adding two numbers from the row above.

Row 1				3		2				
Row 2			3		5		2			For example, 8 = 3 + 5
Row 3		3		8		7		2		
Row 4	3		11		15		9		2	

Fig. 4

(i) Write down the next row of the table. [1]

(ii) Taking the second number of each row gives the sequence 2, 5, 8, 11, … .

Find the sum of the first 50 terms of this sequence.

Similarly, find the sum of the first 50 terms of the sequence formed by the last but one number of each row. [5]

(iii) Row r starts 3 392 … and ends … t 2 .

Find r and t. [5]

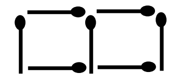
(iv) The sums of the numbers in row 1, row 2, row 3, … form a geometric progression.

Find the sum of all the numbers in the first 20 rows of the triangle. [5]

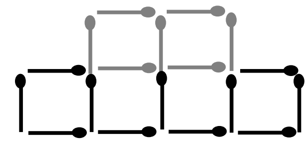
p4 June 2004 question 4

Jenny is making a pattern consisting of rows of matchstick squares.

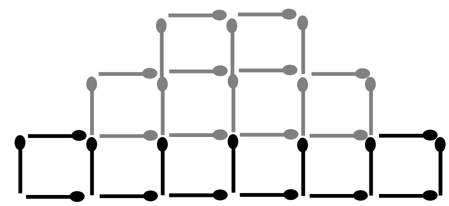
She uses 7 matches to complete a first row of 2 squares.



She uses 11 matches to complete a second row of 4 squares.



She uses 15 matches to complete a third row of 6 squares.



She continues adding rows to the pattern in this way.

You may assume that the number of matches used to complete successive rows of the pattern form an arithmetic progression.

(i) Find how many additional matches are needed to complete

(A) the fourth row,

(B) the nth row, simplifying your answer. [4]

(ii) Show that the total number of matches used in making a pattern with n rows is $n(5 + 2n)$. Hence verify that, with 1000 matches, it is not possible to make more than 21 complete rows. [5]

Jenny, not surprisingly, runs out of matches after a certain number of complete rows of her pattern are made. She decides to leave in place all the matches forming the perimeter, but to remove all the matches inside the pattern.

(iii) Find in terms of n the number of matches in the perimeter of the pattern with n rows. Hence or otherwise show that the number of matches inside a pattern with n rows is $n(2n - 1)$. [3]

(iv) Jenny counts the number of matches she has removed, and finds there are 276. Find how many rows she made before she removed the matches. [3]

p5 Nov 2004 question 2

In this question, a circle consists of a sequence of sectors with angles a_1, a_2, a_3, …as shown in Fig. 2. All angles are measured in degrees. Four cases are considered.

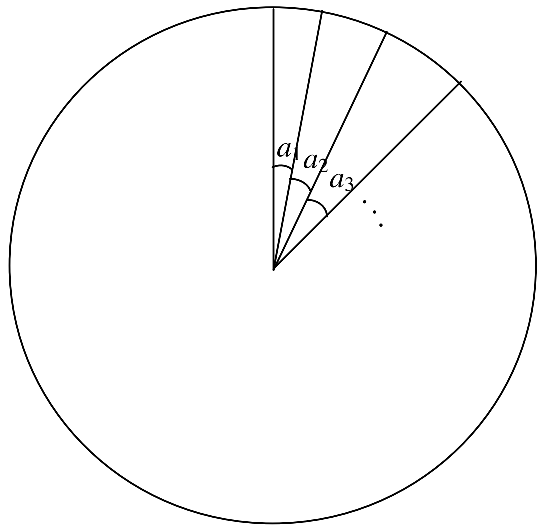


Fig. 2

(i) In the first case, the angles a_1, a_2, a_3, … form a periodic sequence

1°, 2°, 3°, 1°, 2°, 3°, … . How many sectors will fill the circle exactly? [2]

(ii) In the second case, $a_1 = 8.5°$, and the angles form an arithmetic progression with common difference 1°. Verify that 20 sectors fill the circle exactly. [3]

(iii) In the third case, the angles form an arithmetic progression with common difference 0.5°, and 30 sectors fill the circle exactly. Find a_1. [3]

(iv) In the fourth case, the angles form a geometric progression with $a_1 = 90°$ and common

ratio ¾.

(***A***) Find how many sectors have angle greater than 1°. [4]

(***B)*** Show that no matter how many sectors are used they will always fit into

the circle. [2]

Appendix 6 : UoS Ethics Review Checklist and Protocol

UNIVERSITY OF SOUTHAMPTON SCHOOL OF EDUCATION

STUDENT RESEARCH PROJECT: ETHICS REVIEW CHECKLIST

This checklist should be completed by the researcher (with the advice of the research supervisor/tutor) for every research project which involves human participants. Before completing this form, please refer to the Ethical Guidelines in the School's Research Student Handbook and the British Educational Research Association guidelines (http://www.bera.ac.uk/guidelines.html).

Project Title: PhD in Mathematics Education on Assessment of A level Mathematics

Researcher(s): Chris Little

Supervisor: Keith Jones

A. **Student Research Project: Ethics Review Checklist** **Part One**

	YES	NO
1. Does the study involve participants who are particularly vulnerable or unable to give informed consent? (eg children with special difficulties)		✓
2. Will the study require the co-operation of an advocate for initial access to the groups or individuals? (eg children with disabilities; adults with a dementia)		✓
3. Could the research induce psychological stress or anxiety, cause harm or have negative consequences for the participants (beyond the risks encountered in their normal lifestyles)?		✓
4. Will deception of participants be necessary during the study? (eg covert observation of people)?		✓
5. Will the study involve discussion of topics which the participants would find sensitive (eg sexual activity, drug use)?		✓
6. Will the study involve prolonged or repetitive testing or physical testing? (eg the use of sport equipment such as a treadmill) and will a health questionnaire be needed?		✓
7. Will the research involve medical procedures? (eg are drugs, placebos or other substances (eg foods, vitamins) to be administered to the participants or will the study involve invasive, intrusive or potentially harmful procedures of any kind?)		✓
8. Will financial inducements (other than reasonable expenses or compensation for time) be offered to participants?		✓
9. Will you be able to obtain permission to involve children under sixteen from the school or parents and the children themselves?	N/A	
10. Will it be possible to anonymise participants and/or ensure information they give is non-identifiable?	✓	
11. Is the right of participants to freely withdraw from the study at any time made explicit?	✓	
12. Will the study involve recruitment of patients or staff through the NHS?		✓

13. If you are working in a cross-cultural setting do you know enough about the setting to be sensitive to particular issues in that culture (e.g., sexuality, gender role, language use?	N/A	
14. Are you complying with the Data Protection Act?	✓	
15. Have you considered the potential risks to your own health and safety and, if appropriate, completed a risk assessment form?	✓	

If you have answered NO to all of the above questions and you have discussed this form with your supervisor and had it signed and dated, you may proceed to develop an ethics protocol with the assistance of the Ethical Protocol Guidance Form which must also be completed. If you have answered YES to any of the questions, please complete PART TWO of this form below and adopt a similar procedure of discussion with supervisor, signing and proceeding to develop an actual ethical protocol with the assistance of the Ethical Protocol Guidance Form. Please keep a copy of both forms and protocol for your records. Only in exceptional circumstances will cases need to be referred to the School's Research Ethics Committee.

16. Will the research involve medical procedures? (eg are drugs, placebos or other substances (eg foods, vitamins) to be administered to the participants or will the study involve invasive, intrusive or potentially harmful procedures of any kind?)		✓
17. Will financial inducements (other than reasonable expenses or compensation for time) be offered to participants?		✓
18. Will you be able to obtain permission to involve children under sixteen from the school or parents and the children themselves?	N/A	
19. Will it be possible to anonymise participants and/or ensure information they give is non-identifiable?	✓	
20. Is the right of participants to freely withdraw from the study at any time made explicit?	✓	
21. Will the study involve recruitment of patients or staff through the NHS?		✓
22. If you are working in a cross-cultural setting do you know enough about the setting to be sensitive to particular issues in that culture (e.g., sexuality, gender role, language use?	N/A	
23. Are you complying with the Data Protection Act?	✓	
24. Have you considered the potential risks to your own health and safety and, if appropriate, completed a risk assessment form?	✓	

Part Two For each item answered 'YES' please give a summary of the issue and action to be taken to address it.

19. Anonymity
Students' examination results will be stated anonymously,

20. Right to withdraw
Draft results of surveys will be circulated to centres for comment, and permission will be asked to publish these results in an appropriate form.

23. Any data obtained will be held in accordance with the data protection act. No personal data on subjects contributing examination or questionnaire data will be kept.

24. The research will involve routine visits to schools, colleges, universities, libraries, examination board offices and archives. No hazardous activities are planned.

Please continue on a separate sheet if necessary

Signed
(Researcher) Date:

To be completed by the Supervisor (PLEASE TICK ONE)

Appropriate action taken to maintain ethical standards – no further action necessary
The issues require the guidance of the School of Education's Ethics Committee

COMMENTS:

Signed (supervisor):

Date:

Ethical Protocol Guidance

A ETHICS PROTOCOL GUIDANCE FORM

This guidance has been developed to assist you in drawing up an ethics protocol for a research project or bid for research funding. You are advised to also look at the following materials provided by the School of Education Research Ethics Committee, which are available on the School of Education Website:

- Student Research: Ethics Review Checklist:
- Ethics Review Procedure FlowDiagram
- Staff Research: Ethics Review Checklist:
- Ethics Reading List

The Revised Ethical Guidelines for Educational Research (2004) published by the British Educational Research Association are also useful (available on their website at http://www.bera.ac.uk/guidelines.htm).

A. CHECKLIST

HAVE YOU THOUGHT ABOUT HOW YOU WILL ADDRESS:	YES	NO
your responsibilities to the **participants**	✓	
your responsibilities to the **sponsors** of the research	✓	
your responsibilities to the community of **educational researchers**	✓	

HAVE YOU CONSIDERED HOW YOU WILL:	YES	NO
fully inform participants about the nature of the research;	✓	
ensure participants agree to take part freely and voluntarily;	✓	
inform participants that they can withdraw freely at any time;	✓	
justify deception of participants if this is necessarily involved;	N/A	
offer protection for any vulnerable participants or groups in your study;	N/A	
manage the differential 'power relationships' in the setting;	✓	
avoid any pressure on participants to contribute under duress or against their free will;	✓	
guarantee that any research assistants or support staff involved in the project understand and adhere to the ethical guidelines for the project;	N/A	

HAVE YOU CONSIDERED:	YES	NO
what procedures to set in place to ensure a balance between a participant's right to privacy and access to public knowledge;	✓	

how best to provide anonymity and confidentiality and ensure participants are aware of these procedures?	✓	
the implications of the Data Protection Act (1998) particularly in respect to the storage and availability of the data.	✓	
disclosure of information to third parties and getting permission from the participants to use data in any reports/books/articles.	✓	
how you are going to inform the participants of the outcomes of the research;	✓	
how to handle any conflicts of interest arising from sponsorship of the research e.g. a chocolate company sponsoring research into child nutrition, or your own vested interests if any;	N/A	
how you will protect the integrity and reputation of educational research.	✓	

Having considered these questions draw up specific procedures for how you will handle the collection and dissemination of data in your research study.

B. ETHICS PROTOCOL

TITLE OF PROJECT: The use of real-world contextual framing in UK university entrance level mathematics examinations

Name of Principal Investigator: **Chris Little**

Ethics Protocol (Please provide details here of the ethics protocol for your research)

(a) Centres for the AP/GP study will be invited to participate by letter and email.
(b) Letters of invitation to centres will include the following statement:

'Participating in this study will help education to improve. You do not have to participate and you can withdraw at any time. Any data from your participation will be stored securely and will not be divulged to anyone outside the research team in a way that might identify you. The results of the research will not identify you or your school/college.'

(c) Teachers will be asked to read out the following statement prior to students completing the test:

'This test, as well as helping you to revise for AS Maths, will be used for research purposes, to improve our understanding of testing maths in examinations. Any data from your participation will be stored securely and will not be divulged to anyone outside the research team in a way that might identify you. However, if you have particular reasons for not wishing your work to be used in the study, you should write the word 'object' on your script. Your test will then not be forwarded to the researcher.'

(d) The instructions for the student questionnaire include the following statement.

'If students do not wish to complete it, they may of course choose not to.'

(e) Test and questionnaire results will be reported anonymously.

(f) Test scripts and questionnaires will be kept securely.

(g) Test scripts and questionnaires will be available to the researcher's supervisor for verification of data purposes.

(h) The names of participating centres will be reported anonymously in the research.

(i) The AP/GP test may be used by centres as a topic revision test, or as examination revision. This means that the test will have validity and usefulness as part of the AS mathematics curriculum.

(j) A report of the results of the test will be prepared and sent to each participating centre.

(k) A report of the results of the AP/GP study will be sent to participating centres.

(l) Careful trialling of the questions will ensure that the standard is appropriate for AS students with suitable preparation.

(m) Copies of appropriate formulae provided in the examination will be reproduced in the rubric to the tests.

Appendix 7 AP/GP Study Instructions, Tests and Mark Schemes

UNIVERSITY OF Southampton
School of Education

Real-world Context in A/AS Mathematics Study

Instructions

Thank you for agreeing to participate in this study. The aim is to investigate the effects of using real-world contexts in A/AS level maths questions, using parallel versions of a test on arithmetic and geometric progressions, and a short student questionnaire. Students studying Further Maths may take the test if it is felt to be appropriate in standard.

1. The **test** has 2 sides and 8 questions, and is designed to take **one hour**, but students may take longer if it is convenient to do so. It is important that students, as far as is possible, attempt all the questions, to allow valid comparisons to be made between the questions which appear earlier or later in the test.

2. The **questionnaire** can be completed immediately after the test, or in the next lesson. You will need to allow **10 or 15 minutes** of lesson time. If students do not wish to complete it, they may of course choose not to.

3. *Allocation of versions*. Starting with A, B, C or D chosen at random, allocate the versions in sequence to students in order on your class list, i.e. student 1 test C, student 2 test D, student 3 test A, … etc. This should ensure randomness and equal numbers of each version being completed.

4. The test should be done *under examination conditions*. Students should use lined paper, writing their full name and the version of the test on the first page. They will need calculators, but not formula books – the formulae they need are printed on the test paper. At the end of the test, it would be helpful to staple the pages of their scripts together so they do not get muddled up.

5. *Please read out the following statement before starting the test*:

'This test, as well as helping you to revise for AS Maths, will be used for research purposes, to improve our understanding of testing maths in examinations. Any data from your participation will be stored securely and will not be divulged to anyone outside the research team in a way that might identify you. However, if you have particular reasons for not wishing your work to be used in the study, you should write the word 'object' on your script. Your test will then not be forwarded to the researcher.'

6. For centres local to me, I may be able to arrange to pick up and deliver scripts and questionnaires. Otherwise, please send the scripts (except for 'object' scripts – see above) and completed questionnaires to:

Chris Little, 3 Widlers Lane, Upham, Southampton SO32 1JE.

I shall endeavour to mark them with 2 days of receiving them, and will return them promptly. **All scripts will need to be sent to me before 15th July 2009.**

7. A number of schools and colleges are helping in the study, which I hope will involve a sample of about 1000 students. Once all the results have been analysed, I shall send a report to you, with results for your centre, together with overall results, and a paper outlining the conclusions of the study.

8. If you wish to be reimbursed for the costs of postage or photocopying, then invoice me (but bear in mind that all expenses are coming out of my own pocket!).

Chris Little, University of Southampton School of Education

Instructions

UNIVERSITY OF Southampton School of Education

Real-world Context in A/AS Mathematics Study

Student Questionnaire

Name…………………………………………… *Gender Male / Female*

Is English your first language Yes / No

Which AS Maths applied unit are you doing? S1 D1 M1 *Are you doing Further Maths? Yes / No*

Some maths questions put the maths into a real world context, and others are just about maths.
For example, here is a question which is just about maths…

> An arithmetic progression has first term 7 and common difference 3.
> **(i)** Which term of the progression equals 73?
> **(ii)** Find the sum of the first 30 terms of the progression.

… and here is the same question using a real world context:

> Chris saves money regularly each week. In the first week, he saves £7.
> Each week after that, he saves £3 more than the previous week.
> **(i)** In which week does he save £73?
> **(ii)** Find his total savings after 30 weeks.

This questionnaire is about pure and applied maths, and using real world contexts in maths questions (in general).
Consider the following statements, and tick which option best fits your views

	strongly agree	agree	neither agree nor disagree	don't agree	strongly disagree
1. A/AS Maths is a useful subject which can be applied to the real world					
2. Maths questions which include a real world context are harder than similar pure maths questions.					
3. Pure maths is interesting as a subject in its own right.					
4. Questions which put maths into a real world context are more interesting than pure maths questions.					
5. Using real world contexts in maths questions show you how maths is useful.					
6. I prefer pure maths to applied maths.					

7. Write below any other comments you'd like to make about pure and applied maths, and using maths in real world contexts. Continue overleaf if you need more space.

………..........

………..........

………..........

………..........

Questionnaire

Test on sequences (version A)

Instructions

Answer all the questions on lined paper, writing 'Test A' and your full name on the first page. Set out your solutions clearly. You may use a calculator. The following formulae are given:

Arithmetic series

General (kth) term	$u_k = a + (k-1)d$
last (nth) term, $l =$	$u_n = a + (n-1)d$
Sum to n terms,	$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$

Geometric series

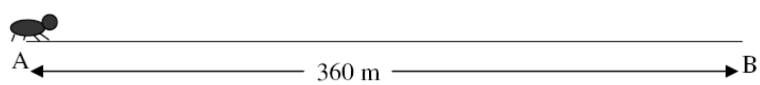
General (kth) term	$u_k = a\,r^{k-1}$
Sum to n terms,	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$
Sum to infinity	$S_\infty = \frac{a}{1-r}$, $-1 < r < 1$

1 An arithmetic progression has first term 7 and common difference 3.

(i) Which term of the progression equals 73? [3]

(ii) Find the sum of the first 30 terms of the progression. [2]

2 A beetle starts at point A and moves in a straight line towards point B, 360 metres from A.



In the first minute, the beetle covers 90 metres. In each minute thereafter, the distances it covers form a geometric progression with common ratio ¾ .

(i) Find for how many minute intervals the beetle covers at least 1 metre. [4]

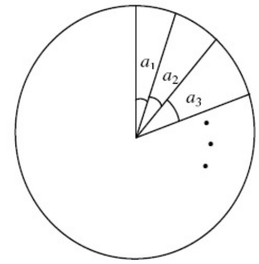
(ii) Show that the beetle never reaches B. [2]

3 The nth term of a geometric progression with common ratio r is denoted by u_n.

Given that $u_{20} = 3\,u_{10}$, $u_1 \neq 0$ and $r > 0$, find r, giving your answer to 3 significant figures. [3]

(turn over)

4 In this question, a circle consists of a sequence of sectors with angles a_1, a_2, a_3, … as shown in the diagram.



The angles are measured in degrees, and form an arithmetic progression with common difference is 0.5°.

Given that 30 sectors fill the circle exactly, find a_1. [4]

5 A factory makes cars. In its first week, it completes 7 cars. In the second week, it completes 11 cars, and in the third week 15 cars. Production continues to rise by four additional cars each week.

(i) Write down how many cars are completed in the fourth week and the nth week, simplifying your answer. [3]

(ii) Find the week number in which 175 cars are made. [2]

6 The diagram below shows an array of numbers. Each row starts with a 3 and ends with a 2. Each of the other numbers is formed, as in Pascal's triangle, by adding two numbers from the row above.

Row 1				3		2				
Row 2			3		5		2			For example, 8 = 3 + 5
Row 3		3		8		7		2		
Row 4	3		11		15		9		2	

(i) Write down the next row of the table. [1]

(ii) Write down the sum of the numbers in (a) row 1, (b) row 2, (c) row 3 and (d) row 4.

Describe the sequence formed by these four numbers. [2]

(iii) Find the sum of all the numbers in an array of 20 rows. [3]

7 A geometric progression starts 9, 12, 16, …

(i) Verify that the first three terms of this sequence are in geometric progression, and find the common ratio. [2]

(ii) Find the 10^{th} term of the sequence. [2]

(iii) Describe the behaviour of the nth term of the sequence as n gets larger and larger. [1]

8 A sequence u_r is defined by $u_1 = 2$, $u_{n+1} = u_n + 3$.

(i) Prove that $\sum_{r=1}^{n} u_r = \frac{1}{2}(3n^2 + n)$. [3]

(ii) Given that $\sum_{r=1}^{n} u_r = 1855$, find n. [3]

version A page 2

Test on sequences (version B)

Instructions

Answer all the questions on lined paper, writing 'Test B' and your full name on the first page. Set out your solutions clearly. You may use a calculator. The following formulae are given:

Arithmetic series

General (kth) term	$u_k = a + (k - 1)d$
last (nth) term, l =	$u_n = a + (n - 1)d$
Sum to n terms,	$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$

Geometric series

General (kth) term	$u_k = a\,r^{k-1}$
Sum to n terms,	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$
Sum to infinity	$S_\infty = \frac{a}{1-r}$, $-1 < r < 1$

1 An arithmetic progression has first term 2 and common difference 3.

(i) Prove that the sum of n terms of the arithmetic progression is ½ $(3n^2 + n)$. [3]

(ii) Given that the sum of n terms is 1855, find n. [3]

2 James records his expenditure in £ each week as follows:

Week 1: £5   Week 2: £10   Week 3: £20   Week 4: £40

(i) If he continues this unlikely pattern of expenditure, write down how much he spends in week 5. [1]

(ii) Describe the sequence formed by the amounts he spends. [2]

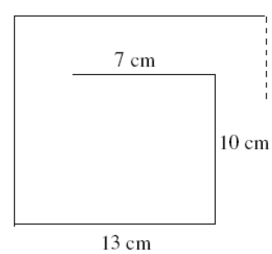
(iii) Assuming he carries on spending according to this sequence, find out his total expenditure after 20 weeks. [3]

3 A sequence u_n is defined by $u_1 = 90$, $u_{r+1} = \frac{3}{4}u_r$.

(i) How many terms of the sequence are greater than one? [4]

(ii) Find the sum to infinity of the sequence. [2]

4 A spiral is formed with sides of lengths 7 cm, 10 cm, 13 cm, … which are in arithmetic progression:



(i) How many sides does the spiral have if its longest side is 73 cm? [3]

(ii) Find the total length of the spiral with 30 sides. [2]

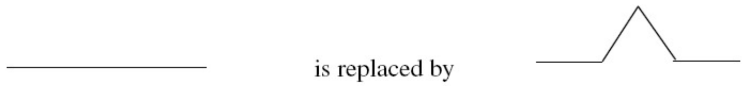
(turn over)

5 Beth invests an amount on the first day of each month, starting in January.

She increases the amount she invests each month by 50p, and finds that she has invested £360 after 30 months. What was her initial investment? [4]

6 Figures 1, 2, and 3 show a sequence of patterns created from an equilateral triangle of side 3 cm.

To get the next pattern in the sequence, each side 'grows' a triangular 'spike' as illustrated below:



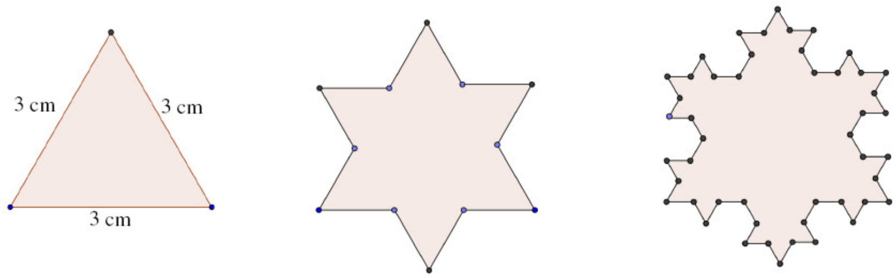


Fig. 1 Fig. 2 Fig. 3

(i) Write down the perimeters of figures 1 and 2. Given that the perimeters of the figures are in geometric progression, find the common ratio. [2]

(ii) Find the perimeter of the 10^{th} figure. [2]

(iii) Describe the behaviour of the perimeter of Figure n as n gets larger and larger. [1]

7 A geometric progression is such that its 20th term is three times its 10th term.

The first term is not zero, and the common ratio is positive.

Find the common ratio, giving your answer to 3 significant figures. [3]

8 The sequence u_n is an arithmetic progression. $u_1 = 7$, $u_2 = 11$ and $u_3 = 15$.

(i) Write down u_4 and u_n, simplifying your answer. [3]

(ii) Given that $u_n = 175$, find n. [2]

Test on sequences (version C)

Instructions

Answer all the questions on lined paper, writing 'Test C' and your full name on the first page. Set out your solutions clearly. You may use a calculator. The following formulae are given:

Arithmetic series

General (kth) term	$u_k = a + (k-1)d$
last (nth) term, l =	$u_n = a + (n-1)d$
Sum to n terms,	$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$

Geometric series

General (kth) term	$u_k = a\,r^{k-1}$
Sum to n terms,	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$
Sum to infinity	$S_\infty = \frac{a}{1-r}$, $-1 < r < 1$

1 An arithmetic progression starts 7, 11, 15, …

(i) Write down the next term, and find the nth term, simplifying your answer. [3]

(ii) The progression ends with the term 175. How many terms are there? [2]

2 The mass of a substance grows in geometric progression. It is initially 9 grams, and increases by 1/3 each hour.

(i) Write down the mass of the substance after 1 hour and after 2 hours, and the common ratio of the geometric progression. [2]

(ii) Find the mass of the substance after 9 hours. [2]

(iii) Describe the behaviour of the mass of substance after n hours as n gets larger and larger. [1]

3 A sequence u_n starts $u_1 = 5$, $u_2 = 10$, $u_3 = 20$, $u_4 = 40$, …

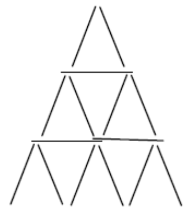
(i) Assuming the sequence continues with the same pattern, write down u_5. [1]

(ii) What type of sequence is u_n? Write down a formula for u_{n+1} in terms of u_n. [2]

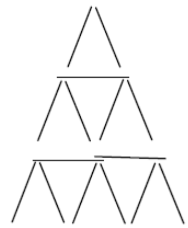
(iii) Find $\sum_{r=1}^{20} u_r$. [3]

(turn over)

4 Some people use playing cards to build 'houses'. A house with 3 layers is illustrated below.



The diagram below shows the separate layers of the house. Each line represents one card. The layers are numbered from the top downwards. Further layers are built in the same way.



(i) Prove that there are ½ $(3n^2 + n)$ cards in a house with n layers. [3]

(ii) A house is made with exactly 1855 cards. How many layers does it have? [3]

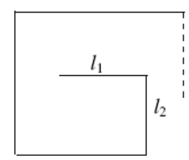
5 Chris saves money regularly each week. In the first week, he saves £7. Each week after that, he saves £3 more than the previous week.

(i) In which week does he save £73? [3]

(ii) Find his total savings after 30 weeks. [2]

6 A spiral is formed with sides whose lengths $l_1, l_2, l_3, \ldots$ are in geometric progression, with common ratio r (see diagram).

Given that the length of the 20th side is three times the length of the 10th side, find r, giving your answer to 3 significant figures. [3]



7 A geometric progression has first term 90 and common ratio ¾.

(i) How many terms of the progression are greater than one? [4]

(ii) Find the sum to infinity of the progression. [2]

8 An arithmetic progression v_n has common difference 0.5, and $\sum_{r=1}^{30} v_r = 360$. Find v_1. [4]

Test on sequences (version D)

Instructions

Answer all the questions on lined paper, writing 'Test D' and your full name on the first page. Set out your solutions clearly. You may use a calculator. The following formulae are given:

Arithmetic series

General (kth) term	$u_k = a + (k - 1)d$
last (nth) term, l =	$u_n = a + (n - 1)d$
Sum to n terms,	$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$

Geometric series

General (kth) term	$u_k = a\,r^{k-1}$
Sum to n terms,	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$
Sum to infinity	$S_\infty = \frac{a}{1-r}$, $-1 < r < 1$

1 An arithmetic progression has common difference 0.5. The sum of 30 terms of this progression is 360. Find the first term. [4]

2 Chris saves money regularly each week.

In the first week, he saves £a, where a is greater than zero. Each week after that, he saves r times what he saves in the previous week.

Given that in week 20 he saves three times what he saves in week 10, find r, giving your answer to 3 significant figures. [3]

3 A geometric sequence u_n is defined by $u_1 = 9$, $u_{r+1} = u_r \times 4 \div 3$.

(i) Write down u_2 and u_3, and state its common ratio. [2]

(ii) Find u_{10}. [2]

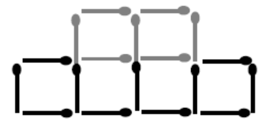
(iii) Describe the behaviour of u_n as n gets larger and larger. [1]

(turn over)

4 Jenny is making a pattern consisting of rows of matchstick squares.

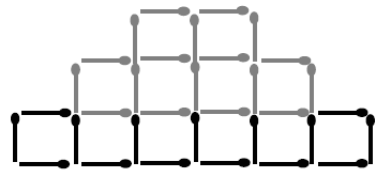
She uses 7 matches to complete a first row of 2 squares.

She uses 11 matches to complete a second row of 4 squares.



She uses 15 matches to complete a third row of 6 squares.

She continues adding rows to the pattern in this way.



(i) Find how many additional matches are needed to complete

(A) the fourth row,

(B) the nth row, simplifying your answer. [3]

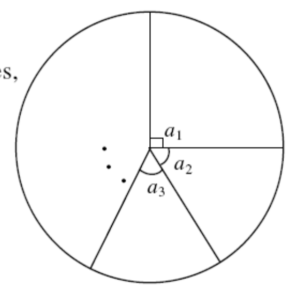
(ii) Which row of the pattern needs 175 matches to complete? [2]

5 The number of new cases of infection from a virus goes up by three each day. On the first day, there were 2 cases, on the second day there were 5 new cases, on the third day 8 new cases, and so on.

(i) Prove that the total number of cases after n days is $\frac{1}{2}(3n^2 + n)$. [3]

(ii) After how many days has the total number of cases reached 1855? [3]

6 In this question, a circle consists of a sequence of sectors with angles a_1, a_2, a_3, … as shown in the diagram. The angles are measured in degrees, and form a geometric progression with $a_1 = 90°$ and common ratio ¾.



(i) Find how many sectors have an angle greater than 1°. [4]

(ii) Show that no matter how many sectors are used they will always fit into the circle. [2]

7 A sequence starts 5, 10, 20, 40, …

(i) Assuming the sequence continues with the same pattern, write down the next term. [1]

(ii) Describe the sequence. [2]

(iii) Find the sum of the first 20 terms of the sequence. [3]

8 The nth term of an arithmetic progression is denoted by u_n. $u_1 = 7$, $u_2 = 10$ and $u_3 = 13$.

(i) If $u_n = 73$, find n. [3]

(ii) Find $\sum_{r=1}^{30} u_r$. [2]

Version A Solutions and Mark Scheme

1. $a = 7$, $d = 3$ **(i)** $a + (n-1)d = 73$ $\Rightarrow 7 + (n-1)\,3 = 73$ $\Rightarrow (n-1) = 66/3 = 22$ $\Rightarrow n = 23$ so 23rd term.	M1 A1 A1	use of $a + (n-1)\,d$ correct equation cao
(ii) sum of 30 terms = 15(14 + 29 . 3) = 1515	M1 A1	
2 (i) $90 \times (3/4)^{n-1} = 1$ $\Rightarrow (3/4)^{n-1} = 1/90$ $\Rightarrow (n-1) \log .75 = \log (1/90)$ $\Rightarrow n - 1 = \log(1/90) / \log(.75) = 15.64$ $\Rightarrow n = 16.64$ so 16 intervals	M1 M1 A1 A1	equation or inequality use of logs or trial and improvement
(ii) sum to infinity = $\dfrac{90}{1-3/4} = 360$, so $S_n < 360$ and beetle never reaches B.	M1 E1	
3 $a\,r^{19} = 3\,a\,r^{9}$ $\Rightarrow r^{10} = 3$ $\Rightarrow r = \sqrt[10]{3} = 1.12$	M1 B1 A1	use of ar^{n-1}
4 $S_{30} = 15(2a_1 + 29 \times .5) = 360$ $\Rightarrow 2a_1 + 14.5 = 24$ $\Rightarrow a_1 = 9.5/2 = 4.75$	M1 A1 A1 A1	

5 (i) 4^{th} week 19 n^{th} week $7 + 4(n-1) = 3 + 4n$	B1 M1 A1	
(ii) $3 + 4n = 175$ $\Rightarrow$ $4n = 172$ $\Rightarrow$ $n = 43$ so 43^{rd} week.	M1 A1	
6 (i) 3 14 26 24 11 2	B1	
(ii) 5, 10, 20, 40 Geometric sequence with $a = 5$, $r = 2$	B1 B1	
(iii) $S_{20} = \dfrac{5(2^{20}-1)}{2-1}$ $= 5242875$	M1 A1 A1	
7. (i) 12/9 = 16/12 = 4/3 common ratio = 4/3	E1 B1	
(ii) 10^{th} term $= 9 \times (4/3)^9$ $= 119.86$	M1 A1	
(iii) Tends to infinity	B1	
8 (i) $\sum_{r=1}^{n} u_r = \frac{n}{2}(2\times 2+[n-1]3)$ $= \frac{n}{2}(1+3n) \quad = \frac{1}{2}(n+3n^2)$ *	M1 A1 E1	
(ii) $\frac{1}{2}(n+3n^2) = 1855$ $\Rightarrow \quad 3n^2 + n - 3710 = 0$ $\Rightarrow \quad n = \dfrac{-1 \pm \sqrt{1+4.3.3710}}{6} = \dfrac{-1 \pm 211}{6} = 35\, or -35.33$ so $n = 35$	M1 M1 A1	equating formula or trial cao

version A mark scheme page 2

Version B Solutions and Mark Scheme

1 (i) sum of n terms = $\frac{n}{2}(2\times 2+[n-1]3)$ $=\frac{n}{2}(1+3n) \quad =\frac{1}{2}(n+3n^2)$ *	M1 A1 E1	
(ii) $\frac{1}{2}(n+3n^2)=1855$ $\Rightarrow \quad 3n^2+n-3710=0$ $\Rightarrow \quad n=\frac{-1\pm\sqrt{1+4.3.3710}}{6}=\frac{-1\pm 211}{6}=35\, or -35.33$ so n = 35	M1 M1 A1	equating formula or trial cao
2 (i) £80	B1	
(ii) Geometric sequence with a = 5, r = 2	B1 B1	
(iii) $S_{20}=\frac{5(2^{20}-1)}{2-1}$ $=£5242875$	M1 A1 A1	
3 (i) $90\times(3/4)^{n-1}=1$ $\Rightarrow \quad (3/4)^{n-1}=1/90$ $\Rightarrow (n-1)\log .75=\log(1/90)$ $\Rightarrow n-1=\log(1/90)/\log(.75)=15.64$ $\Rightarrow n=16.64$ so 16 terms	M1 M1 A1 A1	equation or inequality use of logs or trial and improvement
(ii) Sum to infinity = $\frac{90}{1-3/4}=360$.	M1 A1	
4. $\quad a=7,\ d=3$ **(i)** $\quad a+(n-1)d=73$ $\Rightarrow \quad 7+(n-1)\,3=73$ $\Rightarrow \quad (n-1)=66/3=22$ $\Rightarrow \quad n=23$ so 23 sides.	M1 A1 A1	use of $a+(n-1)\,d$ correct equation cao
(ii) total length = $15(14+29\,.\,3)$ $=1515$	M1 A1	

version B mark scheme page 1

5 $S_{30} = 15(2a + 29 \times .5) = 360$ $\Rightarrow$ $2a + 14.5 = 24$ $\Rightarrow$ $a = 9.5/2 = 4.75$ so initial investment is £4.75	M1 A1 A1 A1	
6 (i) 9, 12 Common ratio is 4/3	B1 B1	
(ii) 10^{th} figure perimeter $= 9 \times (4/3)^9$ $= 119.86$	M1 A1	
(iii) Tends to infinity	B1	
7 $a\,r^{19} = 3\,a\,r^9$ $\Rightarrow$ $r^{10} = 3$ $\Rightarrow$ $r = \sqrt[10]{3} = 1.116$	M1 B1 A1	use of ar^{n-1}
8 (i) $u_4 = 19$ $u_n = 7 + 4(n-1) = 3 + 4n$	B1 M1 A1	
(ii) $3 + 4n = 175$ $\Rightarrow$ $4n = 172$ $\Rightarrow$ $n = 43$	M1 A1	

Version C Solutions and Mark Scheme

1(i) 19 nth term = $7 + 4(n - 1) = 3 + 4n$	B1 M1 A1	
(ii) $3 + 4n = 175$ $\Rightarrow$ $4n = 172$ $\Rightarrow$ $n = 43$ so 43 terms.	M1 A1	
2 (i) 12, 16 Common ratio is 4/3	B1 B1	
(ii) mass = $9 \times (4/3)^9$ = 119.86 grams	M1 A1	
(iii) Tends to infinity	B1	
3 (i) $u_5 = 80$	B1	
(ii) Geometric sequence $u_{n+1} = u_n \times 2$	B1 B1	
(iii) $\sum_{r=1}^{20} u_r = \frac{5(2^{20}-1)}{2-1}$ $= 5242875$	M1 A1 A1	
4 (i) sum of n layers = $\frac{n}{2}(2\times 2+[n-1]3)$ $=\frac{n}{2}(1+3n) = \frac{1}{2}(n+3n^2)$ *	M1 A1 E1	
(ii) $\frac{1}{2}(n+3n^2)=1855$ $\Rightarrow$ $3n^2+n-3710=0$ $\Rightarrow$ $n=\frac{-1\pm\sqrt{1+4.3.3710}}{6}=\frac{-1\pm 211}{6}=35 \text{ or } -35.33$ so n = 35 layers	M1 M1 A1	equating formula or trial cao

5. $a = 7$, $d = 3$ **(i)** $a + (n-1)d = 73$ $\Rightarrow 7 + (n-1)\,3 = 73$ $\Rightarrow (n-1) = 66/3 = 22$ $\Rightarrow n = 23$ so week 23	M1 A1 A1	use of $a + (n-1)\,d$ correct equation cao
(ii) total savings = 15(14 + 29 . 3) = £1515	M1 A1	
6 $a\,r^{19} = 3\,a\,r^{9}$ $\Rightarrow r^{10} = 3$ $\Rightarrow r = {}^{10}\!\sqrt{3} = 1.12$	M1 B1 A1	use of ar^{n-1}
7 (i) $90 \times (3/4)^{n-1} = 1$ $\Rightarrow (3/4)^{n-1} = 1/90$ $\Rightarrow (n-1)\log .75 = \log(1/90)$ $\Rightarrow n - 1 = \log(1/90) / \log(.75) = 15.64$ $\Rightarrow n = 16.64$ so 16 terms	M1 M1 A1 A1	equation or inequality use of logs or trial and improvement
(ii) sum to infinity = $\dfrac{90}{1-3/4} = 360$	M1E1	
8 $\sum_{r=1}^{30} v_r = 15(2a + 29 \times .5) = 360$ $\Rightarrow 2a + 14.5 = 24$ $\Rightarrow a = 9.5/2 = 4.75$, so $v_1 = 4.75$	M1 A1 A1 A1	

version C mark scheme page 2

Version D Solutions and Mark Scheme

1 Sum of 30 terms = $15(2a + 29 \times .5) = 360$ $\Rightarrow$ $2a + 14.5 = 24$ $\Rightarrow$ $a = 9.5/2 = 4.75$	M1 A1 A1 A1	
2 $a\,r^{19} = 3\,a\,r^9$ $\Rightarrow$ $r^{10} = 3$ $\Rightarrow$ $r = {}^{10}\sqrt{3} = 1.12$	M1 B1 A1	use of ar^{n-1}
3 (i) $u_2 = 12$, $u_3 = 16$ $r = 4/3$	B1 B1	
(ii) $u_{10} = 9 \times (4/3)^9$ $= 119.86$	M1 A1	
(iii) Tends to infinity	B1	
4(i) (*A*) 19 matches in fourth row (*B*) *n*th row = $7 + 4(n - 1) = 3 + 4n$ matches	B1 M1 A1	
(ii) $3 + 4n = 175$ $\Rightarrow$ $4n = 172$ $\Rightarrow$ $n = 43$ so 43rd row.	M1 A1	
5 (i) total no of cases = $\frac{n}{2}(2\times2+[n-1]3)$ $=\frac{n}{2}(1+3n) = \frac{1}{2}(n+3n^2)$ *	M1 A1 E1	
(ii) $\frac{1}{2}(n+3n^2)=1855$ $\Rightarrow$ $3n^2+n-3710=0$ $\Rightarrow$ $n=\frac{-1\pm\sqrt{1+4.3.3710}}{6}=\frac{-1\pm211}{6}=35\,or-35.33$ so $n = 35$ days	M1 M1 A1	equating formula or trial cao

version D mark scheme page 1

6 (i) $90 \times (3/4)^{n-1} = 1$ $\Rightarrow (3/4)^{n-1} = 1/90$ $\Rightarrow (n-1) \log .75 = \log (1/90)$ $\Rightarrow n - 1 = \log(1/90) / \log(.75) = 15.64$ $\Rightarrow n = 16.64$ so 16 sectors	M1 M1 A1 A1	equation or inequality use of logs or trial and improvement
(ii) sum to infinity = $\frac{90}{1-3/4} = 360$ so sum always < 360 and will always fill circle	M1E1	
7 (i) 80	B1	
(ii) Geometric with 1st term 5, common ratio 2	B1 B1	
(iii) sum of 20 terms = $\frac{5(2^{20}-1)}{2-1}$ $= 5242875$	M1 A1 A1	
8 $a = 7, d = 3$ **(i)** $u_n = a + (n-1)d = 73$ $\Rightarrow 7 + (n-1)\,3 = 73$ $\Rightarrow (n-1) = 66/3 = 22$ $\Rightarrow n = 23$	M1 A1 A1	use of $a + (n-1)\,d$ correct equation cao
(ii) $\sum_{r=1}^{30} u_r = 15(14 + 29\,.\,3)$ = £1515	M1 A1	

version D mark scheme page 2

Appendix 8: Extract from OCR (MEI) Comprehension Paper (article)

2

Benford's Law

Leading digits

This article is concerned with a surprising property of the leading digits of numbers in various sets. The leading digit of a number is the first digit you read. In the number 193 000 the leading digit is 1. When a number is written in standard form, such as 1.93×10^5 or 2.78×10^{-7}, the leading digit is the digit before the decimal point, in these examples 1 and 2 respectively.

Mathematical sequences

Table 1 shows the integer powers of 2, from 2^1 to 2^{50}. In this table, which digits occur more frequently as the leading digit? You might expect approximately one ninth of the numbers to have a leading digit of 1, one ninth of the numbers to have a leading digit of 2, and so on. In fact, this is far from the truth.

2	2 048	2 097 152	2 147 483 648	2 199 023 255 552
4	4 096	4 194 304	4 294 967 296	4 398 046 511 104
8	8 192	8 388 608	8 589 934 592	8 796 093 022 208
16	16 384	16 777 216	17 179 869 184	17 592 186 044 416
32	32 768	33 554 432	34 359 738 368	35 184 372 088 832
64	65 536	67 108 864	68 719 476 736	70 368 744 177 664
128	131 072	134 217 728	137 438 953 472	140 737 488 355 328
256	262 144	268 435 456	274 877 906 944	281 474 976 710 656
512	524 288	536 870 912	549 755 813 888	562 949 953 421 312
1 024	1 048 576	1 073 741 824	1 099 511 627 776	1 125 899 906 842 624

Table 1

The frequencies of the different leading digits in these powers of 2 are shown in Table 2.

Leading digit	1	2	3	4	5	6	7	8	9
Frequency	15	10	5	5	5	4	1	5	0

Table 2

You can see that, for these data, 1 and 2 appear more frequently than any other numbers as the leading digit. Is this just a peculiarity of the first fifty powers of 2, or is a general pattern emerging?

Here is another example. Imagine you invest £100 in an account that pays compound interest at a rate of 20% per year. Table 3 shows the total amount (in £), after interest is added, at the end of each of the following 50 years.

1 In a certain country, twenty cars are on display in a car showroom. The costs of the cars in the local currency, the zen, are shown below.

10 255	23 250	48 500	25 950	12 340
34 750	5 690	13 580	7 450	9 475
18 890	14 675	6 295	21 225	37 850
51 200	43 340	16 575	8 380	28 880

(i) Complete the table giving the frequencies of the leading digits. [1]

Leading digit	1	2	3	4	5	6	7	8	9
Frequency	6	4	2						

The country joins the European Union and so the costs of the cars are converted to euros. The exchange rate is 1 zen = 3 euros.

(ii) Give the costs of the cars in euros in the space below and then complete the table giving the frequencies of the leading digits in euros. [2]

....

....

....

....

Leading digit	1	2	3	4	5	6	7	8	9
Frequency	7								0

(iii) In the table below, give the frequencies predicted by Benford's Law, in each case correct to **one decimal place**. [2]

Leading digit	1	2	3	4	5	6	7	8	9
Frequency	6.0								

(iv) Compare the results in the three tables. [1]

....

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