# UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

School of Mathematics


**Generic Planar Lattice Patterns In Liquid Crystals**

by


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ABSTRACT

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In this thesis we will be studying symmetries and pattern formation within a planar layer of liquid crystal. Some of the generic equilibrium patterns (steady states) on a square or hexagonal lattice that bifurcate from a homeotropic or planar isotropic state were calculated in Chillingworth and Golubitsky 2003 *J. Mathematical Physics* **44(9)** 4201-4219. Continuing this work we calculate a second set of steady states and go on to calculate the time periodic solutions resulting from Hopf bifurcations in the same planar layer of liquid crystal. We describe the possible symmetries of the system by the group $\Gamma_{\mathcal{L}} \times \mathbf{S}^1$, (or just $\Gamma_{\mathcal{L}}$ in the steady states), $\Gamma_{\mathcal{L}} = (H \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$, where $H$ is the holohedry of the chosen lattice $\mathcal{L}$, that is the finite group of rotations and reflections that preserve the lattice, $\mathbf{T}^2 = \mathbf{R}^2/\mathcal{L}$ is the torus group representing translations on the lattice, $\mathbf{Z}_2$ represents the reflection in the $xy$ plane, and $\mathbf{S}^1$ is the circle group representing time periodicity. We find the equilibrium solutions by applying the Equivariant Branching Lemma and finding isotropy subgroups of $\Gamma_{\mathcal{L}}$ with fixed-point subspaces of dimension 1. We then find the time periodic solutions using the Equivariant Hopf Theorem, finding isotropy subgroups of $\Gamma \times \mathbf{S}^1$ with fixed-point subspaces of dimension 2 by using the group theory methods shown in Dionne *et al* 1995 *Phil. Trans. Physical Sciences and Engineering* **352(1698)** 125-168.

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DECLARATION OF AUTHORSHIP

I, Theresa Lockett, declare that the thesis entitled Generic Planar Lattice Patterns In Liquid Crystals and the work presented in this thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
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- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
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- none of this work has been published before submission.

**Signed:** ...................................................................

**Date:** .......................................................................

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## 0.1 Introduction

A dynamical system describes the changing state of a physical system over time and can often be written as a set of differential equations. It is a common occurrence in these physical systems for some property to be dependent on a particular parameter. As this parameter is increased or decreased a critical value is reached at which a sudden change, called a *bifurcation*, in the behaviour of the system is affected [1]. For example a uniform layer of liquid heated uniformly from below will initially have zero velocity but when a certain temperature is reached the liquid will begin to move under convection. The velocity of the liquid is dependent on the temperature (assuming there are no other external forces at work). The temperature is known as the bifurcation parameter and the critical value at which the onset of convection takes place is the bifurcation point. In many systems at a bifurcation point a pattern will form. Much work has been done to understand these pattern-forming phenomena.

Some of the more commonly studied examples are convection between two horizontal plates, known as Rayleigh-Benard convection [49]; Faraday waves, formed by shaking a layer of fluid or sand up and down [25]; and reaction-diffusion systems where two chemicals are mixed together [64]. Common patterns are stripes, squares, hexagons and spirals.

In this thesis we will be studying pattern formation in a planar layer of liquid crystal. These patterns are formed by the orientation of the molecules within the layer. The probability that a molecule points in a certain direction can be described by a symmetric $3 \times 3$ matrix $Q$ representing an ellipsoid [17], see Section 1.6. We will restrict our research to those patterns that are spatially doubly periodic with respect to a planar lattice and that bifurcate from one of two trivial states, the planar isotropic state in which all the molecules lie flat within the plane but pointing in no particular direction and the homeotropic state in which all the molecules are aligned vertically.

In Chapter 1 we explain the necessary background to find steady state patterns. We start with a system of partial differential equations

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{F}(\mathbf{Q}, \lambda)$$

where $\mathbf{Q}(\mathbf{x}, t)$ is a matrix valued function of time $t \in \mathbf{R}$, and space $\mathbf{x} \in \mathbf{R}^2$, and $\lambda \in \mathbf{R}$ is a bifurcation parameter. We assume there is a solution $Q_0$, (see Section 1.6.1), for all $\lambda$, with full Euclidean symmetry. Let $\mathsf{L} = d\mathbf{F}|_{(Q_0,\lambda)}$ denote the equations linearized about $Q_0$, and suppose that $\mathsf{L}$ is invertible, then by the Implicit Function Theorem [35] there exist solutions to $\mathbf{F}(\mathbf{Q}, \lambda) = 0$ near $Q_0$, see Section (1.4.1).

Any solution that is spatially doubly periodic, i.e. has translational symmetry in two directions, must lie on a planar lattice $\mathcal{L}$ in $\mathbf{R}^2$ such that $\mathbf{u}(\mathbf{x} + l) = \mathbf{u}(\mathbf{x})$ for all $l \in \mathcal{L}$, where $l$ is a basis vector for the lattice [21]. Therefore we can restrict the differential equations to the space of functions that are doubly periodic with respect to a planar lattice. This implies that elements of $\ker \mathsf{L}$ have *plane wave* form $w_{\mathbf{k}}(\mathbf{x}) = e^{2\pi i \mathbf{k}\cdot\mathbf{x}} Q$, $\mathbf{k} \in \mathbf{R}^2$. This is explained in Section 1.5.

Since $\ker \mathsf{L}$ is finite-dimensional we can use *Liapunov-Schmidt Reduction* [8] [58] to simplify the problem of searching for equilibria and periodic orbits bifurcating from a (stable) equilibrium in our system of PDEs to the simpler problem of finding the equilibria and periodic solutions of a reduced system of equations, see Section 1.4.

The *Equivariant Branching Lemma* [65] tells us that generically we will find equilibrium solutions with symmetry group $\Sigma \in \Gamma_{\mathcal{L}}$ where $\Sigma$ is an isotropy subgroup with $\dim \mathrm{Fix}\, \Sigma = 1$, see section 1.7. The calculations for the equilibrium solutions and the resulting patterns are shown in Chapter 2.

The *Equivariant Hopf Theorem* [32] states that generically there exist branches of periodic solutions with period close to $2\pi$ having symmetry group $\Sigma \subset \Gamma \times \mathbf{S}^1$ where $\Sigma$ is an isotropy subgroup with $\dim\mathrm{Fix}(\Sigma) = 2$, explained in Section 3.1. To find these isotropy subgroups and the dimensions of their fixed-point subspaces we use the group theory methods involving *wave pairs* shown in Dionne et al. [21], see Section 3.2. The calculations and some stills of the resulting patterns are shown in Chapter 4, the moving patterns are on the attached CD.

## 0.2 A bit about Liquid Crystals

Before we begin looking at the mathematical methods it is helpful to have a rough idea of what a liquid crystal is.

The existence of liquid crystals was first observed in the late 1800s by Austrian scientist Friedrich Reinitzer who discovered that cholesterol extracted from carrots appeared to have two melting points, the first was a transition from a solid to an opaque liquid and then the second when this liquid became clear, this process was also reversible [51]. Fellow Austrian Otto Lehmann identified this intermediate cloudy stage as crystalline. It was Lehmann who did much of the early research in the field, publishing a book containing literally hundreds of illustrations of liquid crystals observed through his specialist (and rather unusual at the time) polarizing microscope [44]. In 1905 Friedrich Rudolf Schenck presented a paper on liquid crystals at the annual meeting of the Deutsche Bunsengesellschaft in Karlshruhe [57]. The paper had a mixed reception but Schenck went on to publish a book on the subject [56], though this was the end of his contribution to the field. A few years later Daniel Vorländer, a German chemist, realised that a crystalline fluid necessarily had rod-like molecules [66]. It was not until the 1920's that Frenchman Georges Friedel concluded that it was the orientational order of these molecules that was the key factor of liquid crystals, it was Friedel who introduced the terms *nematic* and *smectic* to the field [27]. Translations of many of the papers cited here can be found it the book by Sluckin, Dunmur and Stegemeyer [61], or for a concise history see [63].

The following explanation is a paraphrased version of that given in the book by Michael Fisch [26].

The three classical states of matter are gas, liquid and solid. A useful means of describing the differences between their molecular structure is to talk about the relation between the distance between the individual molecules which we'll call $l$, and the diameter of the molecules which we'll call $d$. The diameter is a very approximate term since the molecules are by no means always spherical in fact we will later on be discussing the orientation of individual molecules according to their major axis, but it will suffice for the moment as we only need a very basic description of the three states.

**Gas**

In a gas $d \ll l$, the size of the molecules is much smaller than the distance between them, and the molecules are free to move around. Gases flow, and will expand to fill a container.

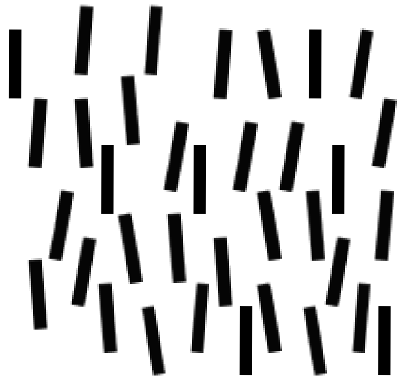
**Liquid**

In a liquid $l \approx d$, the molecules in a liquid are free to move around, it flows and can be poured, in a container it will settle to the bottom. An *isotropic* liquid has a random arrangement of molecules, there is no order of position or orientation.

**Solid**

There are two types of solids, *crystalline* and *amorphous*, though we only need concern ourselves with crystalline solids. These also have $l \approx d$ but there must be a periodic arrangement of the molecules so the molecules are

Figure 1: Diagram of a Nematic Liquid Crystal



not free to move. If the molecules are not spherical there is also orientational order, all the molecules point in the same direction. Solids do not flow, they retain their shape (except when strong forces are applied in which case they may be forced to distort), when placed in a container they will not adjust to fit it in any way. By contrast amorphous solids have no long-range positional or orientational molecular order though they do retain their shape.

**Liquid Crystal**

A liquid crystal is a phase of matter between an isotropic liquid and crystalline solid displaying some properties of both states. The molecules have some degree of long range orientational order and may have some positional order too. Our research is concerned only with *nematic* liquid crystals, which display orientational order but no positional order. However there are other types, for example *smectic* liquid crystals exhibit orientational order and a degree of positional order in layers, and *chiral* liquid crystals appear to have orientational order locally but over a larger area are seen to follow a helical pattern, see Sluckin (2000) [62].

Liquid crystal phases exist in many substances between the solid and liquid phases. For example, as observed by Reinitzer a substance may start in the solid phase, melt to an opaque liquid crystal phase at a certain temperature, and then melt again at a higher temperature to a clear liquid [51]. For a more thorough discussion of the liquid crystal phases see [62]. Within the liquid crystal phase temperature changes can affect the molecule alignment causing pattern formation. Other factors that can affect this pattern formation are electric and magnetic fields and, in the case of some liquid crystal phases that exist only when the main substance is diluted in another substance, changes in the concentration of the solution. It is important to note that liquid crystals are strongly affected by physical boundaries, for example plates containing a thin layer of liquid crystal between them in experimental situations. Commonly the molecules will allign themselves parallel to or perpendicular to the boundaries.

## 0.3 A Brief Overview of Existing Research

The use of group theory to describe symmetries in bifurcation theory has been established for some time, see Ruelle (1973) [54] and Sattinger (1979) [55]. A comprehensive discussion of the subject is given in Golubitsky and Schaeffer (1985) Volumes 1 and 2 [31], and there are some good modern texts on the subject, Golubitsky and Stewart (2003) [30] and Hoyle (2006) [37].

Basic equilibrium lattice patterns appear in many papers applied to different physical models, for generic results see Golubitsky and Stewart (2003) [30]. Superlattice equilibrium patterns on square and hexagonal lattices are covered in Dionne *et al* (1997) [22]. Square lattice solutions resulting from a Hopf bifurcation and their stability are discussed in Silber and Knobloch (1991) [59] and Dawes (2001) [16]. Results for the Hopf bifurcation on a hexagonal lattice are discussed in Roberts *et al* (1986) [53]. A good overview of the Hopf Bifurcation and Its Applications is given in the book by that name by Marsden and McCracken (1976) [45].

The most commonly studied examples of pattern formation are found in fluid dynamics, Crawford and Knobloch (1991) [12]. In Rayleigh-Bénard convection equilibrium solutions displaying rolls and hexagonal patterns are common, see Buzano and Golubitsky (1983) [4] and Golubitsky *et al* (1984) [29]. Also square lattice patterns can be found, as shown in Le Gal and Croquette (1988) [43]. Dionne *et al* (1995) [21] calculated possible time periodic solutions on the square, hexagonal and rhombic lattices.

Turing patterns found in reaction-diffusion systems can be beautiful and often unstable, sometimes oscilating between two or more patterns. Hexagonal superlattice patterns called 'black eye' patterns, as well as rhombs and stripes, are shown in experiments by Gunaratne *et al* (1994) [34]. These Hexagonal black eye patterns also appear in the experiments by Yang *et al* (2002) [68] who found a three-phase oscillating hexagonal lattice pattern. Analysis of basic stripes, squares, hexagons and rhombs as well as superlattice patterns is shown in Judd and Silber (1999) [38]. Three dimensional Turing patterns are discussed in Callahan and Knobloch (1999) [5].

Faraday experiments have shown a wide variety of patterns including stripes, squares and hexagons, as shown in Douady and Fauve (1988) [23] and Kudrolli and Gollub (1996) [41]; triangles are shown in Müller (1993) [46]; and eight and twelve fold quasipatterns are shown in Christiansen *et al* (1992) [9] and Edwards and Fauve (1993) [24] respectively. Complicated 'superlattice' patterns can also be found, see Crawford *et al* [13] for squares, and Kudrolli *et al* (1998) [40] and Silber et al. (2000) [60] for hexagons.

While this thesis makes no attempt to determine the experimental conditions under which the patterns we discuss can be observed, it is nonetheless interesting to comment on some examples of pattern forming phenomena in liquid crystals, see for example Buka (1989) [3]. Observing patterns in liquid crystals is usually done by shining a light onto the liquid crystal, the specific allignment of the molecules that are creating the pattern are not visible but the orientation of the molecules affects their refractive properties and a black and white image is created by some areas reflecting back more light than others.

A liquid crystal light valve consists of a layer of liquid crystal sandwiched between a glass plate on one side and a mirror and photoconductor plate on the other side with clear electrodes covering both plates allowing an electric current to be applied. Light shone onto the photoconductor plate reduces its resistance and thus changes the voltage that is being applied to the liquid crystal layer, this results in a change in the orientation of the molecules of the liquid crystal. Patterns can be seen by shining a light onto the liquid crystal layer that is then refected back off the mirror, see Residori (2005) [52]. Rolls, squares, hexagons and triangles can all be observed, see Neubecker *et al* (1995) [47] and D'Alessandro *et al* (1995) [15].

In electrohydrodynamic convection a thin layer of nematic liquid crystal is held between two parallel glass plates and an alternating voltage is applied, see Cross and Hohenberg (1993) [14]. Roll patterns (called Williams domain) and chevrons are common, see Huh *et al* (2000), though squares and travelling rolls are also possible, see Kai and Hirakawa (1978) [39], and Rehberg *et al* (1988) [50]. This is a more complicated system since the liquid crystal molecules are changing position as well as orientation and is therefore not fully explained by our model.

The use of $3 \times 3$ symmetric matrices to describe molecular alignment in liquid crystals has been established for some time, see deGennes (1974) [17]. Since liquid crystals are greatly affected by boundaries, e.g. the plates on either of the layer described in the examples above, we will be considering the midplane between these plates. A set of steady state patterns bifurcating from the homeotropic and planar isotropic states predicted by the Equivariant Branching Lemma in a planar layer of liquid crystal are shown in Chillingworth and Golubitsky (2003) [7] and Golubitsky (2003) [30]. The paper by Dionne et al (1995) [21] uses group theory methods and the Equivariant Hopf Theorem to find time-periodic patterns in Rayleigh-Bénard convection. These patterns are created by variations in a scalar in $\mathbf{R}$, temperature or vertical velocity, over the plane $\mathbf{R}^2$. This thesis takes the same method shown in the Dionne paper and applies it to the case of a planar layer of liquid crystal that is discussed in Chillingworth and Golubitsky (2003) [7] where patterns are the result of variations in molecular alignment described by a director field in $\mathbf{R}^3$ defined on the plane $\mathbf{R}^2$.

Finally it is worth mentioning that while much of the investigation in pattern formation has focused on the two dimensional models there is also some interesting work on the three dimensional case. Equilibrium solutions for the standard cubic lattices are shown in Dionne and Golubitsky (1992) [20], classification of the other three-dimensional lattices can be found in Dionne (1993) [19]. Hopf bifurcations on 3 dimensional lattices are discussed in Golubitsky and Stewart (1985) [33], Dias and Stewart (1999) [18] and Callahan (2003) [6].

# 1

# Preliminaries

## 1.1 The general strategy

We are looking for generic patterns within a planar layer of liquid crystal that are spatially doubly periodic with respect to a lattice. How do we go about finding these patterns? We begin by assuming a trivial state with approximate Euclidean symmetry in a planar layer of liquid crystal, i.e. it is invariant to all rotations reflections and translations in the plane. Physical practicalities mean this is not actually possible since an infinite layer of liquid crystal does not exist in the physical world, the liquid crystal will have to be contained in something. However we can ignore the boundaries of the vessel since we are looking at patterns on a very local scale. We assume that cooling takes place uniformly across the entire layer, so there are no hot spots. There are then several different aspects of the problem that need to be explained.

- An $\mathbf{R}^3$ director field on $\mathbf{R}^2$ describes the current state of the planar layer of liquid crystal.
- Group theory describes the possible symmetries of the system.
- Restricting our research to those patterns that are spatially doubly

periodic with respect to a planar lattice reduces the infinite dimensional problem to one that is finite dimensional.

- Liapunov-Schmidt Reduction allows us to restrict our attention to a finite dimensional eigenspace.
- Matrices describe the orientation of the molecules
- The Equivariant Branching Lemma tells us how to find patterns close to the homeotropic and isotropic states.

## 1.2 Getting Started

We are looking specifically at patterns in a planar layer of liquid crystal. We will approximate this layer by the plane $\mathbf{R}^2$. We will describe the state of the liquid crystal by a director field defined on $\mathbf{R}^2$ that assigns to each point in the plane a unit vector $\mathbf{n}$ (called a director) in $\mathbf{R}^3$ showing the probable orientation of the molecules at that point. Since the molecules themselves have no positive or negative direction we treat the vector $-\mathbf{n}$ as being the same as the vector $\mathbf{n}$. We need to describe how this director field changes as our bifurcation parameter changes. We start with a system of partial differential equations (1.1) which express the rate of change of the system over time in terms of its current state

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{F}(\mathbf{Q}, \lambda) \tag{1.1}$$

where $\mathbf{Q}(\mathbf{x}, t)$ is a $\mathbf{3} \times \mathbf{3}$ **symmetric matrix** valued function of time, $t \in \mathbf{R}$, and space, $\mathbf{x} \in \mathbf{R}^2$, with trace$=0$, describing the direction of $\mathbf{n}$ (this will be explained in section 1.6), and $\lambda \in \mathbf{R}$ is a bifurcation parameter, (eg. temperature)[7].

An *equilibrium/steady state*, is a solution $\mathbf{Q}$ of this system of PDEs where

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{F}(\mathbf{Q}, \lambda) = 0 \tag{1.2}$$

We describe the symmetry of the solutions using group theory. The Euclidean group $E(2)$ is the group of all translations, rotations and reflections in the plane. Elements take the form $g(\mathbf{x}) = B\mathbf{x} + b$ where $B \in O(2)$ and $b \in \mathbf{R}^2$. We use the group $\mathbf{Z}_2$ to describe the reflection in the $xy$ plane: elements take the form $\psi = \pm 1$. We define the action of $E(2) \times \mathbf{Z}_2$ on our functions $\mathbf{Q}$ by

$$A = \begin{pmatrix} B & \\ & \psi \end{pmatrix}$$

$$((g, \psi) \cdot \mathbf{Q})(\mathbf{x}, t) = A\mathbf{Q}(g^{-1}\mathbf{x}, t)A^{-1} \qquad \forall (g, \psi) \in E(2) \times \mathbf{Z}_2.$$

The same group action applies in Section 1.3 and is explained in Appendix A.

### 1.2.1 Group Theory Definitions

Throughout our discussion there are a few group theory terms that we will be using, as outlined below.

The group of orthogonal matrices $O(n)$ can be regarded as the group of all rotations and reflections of $\mathbf{R}^n$. We use a subgroup of $O(n)$ to describe the symmetries of a particular system. The group element $g \in G$ is a symmetry of the dynamical system if for every solution $\mathbf{x}(t)$ of the system, $g\mathbf{x}(t)$ is also a solution.

#### Representations

A *representation* of a group $G$ over a field $F$ is a homomorphism $\rho : G \to GL(n, F)$ that maps each group element $g \in G$ to an invertible $n \times n$ matrix $\rho(g)$. The matrices act on the vector space $V = F^n$. If $F = \mathbf{R}$ then the representation is a *real representation.* The *natural representation* $\rho : \mathbf{D}_n \to GL(2, \mathbf{R})$ of a dihedral group $\mathbf{D}_n$ (the group of all rotations and reflections of a regular $n$ sided polygon) maps each rotation and reflection of the group to the $2 \times 2$ matrix corresponding to that linear transformation in the plane in coordinate geometry.

**Irreducibility**

A subspace $W$ of the vector space $V$ is said to be $G$-*invariant* (or just invariant if $G$ and $\rho$ are already assumed) under the representation $\rho$ of the group $G$ if

$$\rho(g)w \in W, \qquad \forall g \in G, \qquad \forall w \in W.$$

A representation $\rho(G)$ is said to be *irreducible* if the only $G$-invariant subspaces of $V$ are the origin and the whole space. A representation of $G$ is *absolutely irreducible* if it is irreducible and the only matrices that commute with all the matrices of the representation are scalar multiples of the identity.

**Equivariance**

The function $\mathbf{F}(v, \lambda) : V \times \mathbf{R} \to V$ is equivariant with respect to the group $G$, given a representation $\rho$ if

$$\mathbf{F}(\rho(g)v, \lambda) = \rho(g)\mathbf{F}(v, \lambda)$$

**Isotropy Subgroup**

The *isotropy subgroup* $H_{\mathbf{x}} \subseteq G$ of a point $\mathbf{x} \in V = \mathbf{R}^n$ is defined to be:

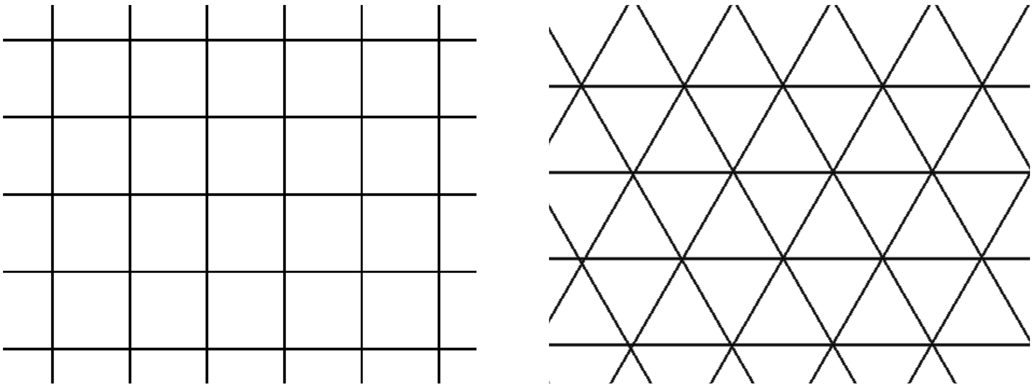
$$\Sigma_{\mathbf{x}} = \{g \in G | gx = x\}.$$

**Fixed-point Subspace**

For a subgroup $H \subseteq G$ the *fixed-point subspace* $\mathrm{Fix}(H)$ is defined as

$$\mathrm{Fix}(H) = \{\mathbf{x} \in \mathbf{R}^n | h\mathbf{x} = \mathbf{x}, \forall h \in H\}.$$

Figure 1.1: The Square and Hexagonal Lattices (left and right respectively).



If $\mathbf{x} \in \mathrm{Fix}(H)$ then $H \subseteq \Sigma_{\mathbf{x}}$.

**Axial**

An isotropy subgroup $\Sigma$ is *axial* if $\dim\mathrm{Fix}(\Sigma) = 1$.

Axial isotropy subgroups will play an important role in finding solutions of Equation (1.2).

## 1.3 The Symmetry Group $\Gamma_{\mathcal{L}}$

The symmetry group for our specific problem will depend on the lattice we choose. We are looking for those solutions that are periodic with respect to either the square or hexagonal lattice. First we choose a lattice and call it $\mathcal{L}$. The symmetry group for our system is

$$\Gamma_{\mathcal{L}} = (H_{\mathcal{L}} \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$$

where

- $H_{\mathcal{L}}$ is the holohedry of the chosen lattice $\mathcal{L}$, that is the finite group of rotations about the origin and reflections in lines through the origin of $\mathbf{R}^2$ that preserve the lattice, in this case the dihedral groups $\mathbf{D}_4$ or $\mathbf{D}_6$

- $\mathbf{T}^2 = \mathbf{R}^2/\mathcal{L}$ is the torus group representing translations of $\mathbf{R}^2$ modulo the lattice
- $\mathbf{Z}_2$ represents the action of the reflection in the $xy$ plane since, while we are looking at a planar layer of liquid crystal where each molecule will have a position vector in $\mathbf{R}^2$, the molecules themselves can point in any direction including out of the plane so each molecule will have a director in $\mathbf{R}^3$ describing which way it points. The $\mathbf{Z}_2$ represents the action of the reflection in the $xy$ plane which takes $z$ to $-z$.

Elements of this group take the form of ordered triples, for example $(sr^2, v_1, \tau)$, where $sr^2$ is a reflection in the holohedry, $v_1$ is a vector in the torus group representing the translations on the lattice, and $\tau$ is the reflection in the $xy$ plane. Since the construction of the group involves indirect products it is important to state an order of action as not all the individual elements of our component groups commute with one another. We will follow the rule that the group element always acts from right to left, so in our example $\tau$ would act first followed by $v_1$ and then by $sr^2$ (which by the same rule is $r^2$ followed by $s$). $\Gamma$ is a subgroup of $E(2) \times \mathbf{Z}_2$ and its elements act on functions $\mathbf{Q}$ in the same way

$$\gamma = (g, \psi), \text{ where } g \in E(2) \text{ and } \psi = \pm 1 \in \mathbf{Z}_2$$

$$g(\mathbf{x}) = B\mathbf{x} + b, \text{ where } B \in H_{\mathcal{L}} \subset 0(2), \text{ and } b \in \mathbf{T}^2$$

$$A = \begin{pmatrix} B & \\ & \psi \end{pmatrix}, \qquad \psi = \pm 1$$

$$\gamma \cdot \mathbf{Q}(\mathbf{x}, t) = A\mathbf{Q}(g^{-1}\mathbf{x}, t)A^{-1} \qquad \forall \gamma \in \Gamma.$$

It is perhaps easiest to understand the group action by following the example shown in Appendix A.

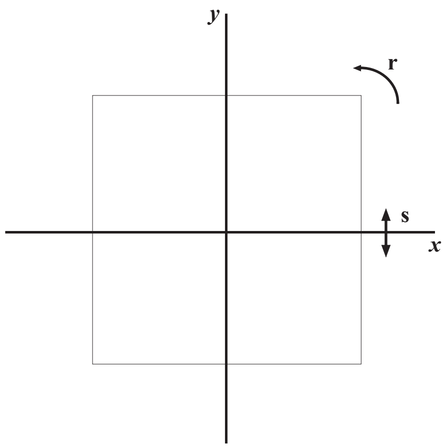
**Definition** A *shifted subgroup* $K$ of $\Gamma_{\mathcal{L}}$ is a subgroup with no elements that are purely translations, so $K \bigcap \mathbf{T}^2 = \{0\}$.

We will be looking only at these shifted subgroups because if we had a pattern relating to a subgroup that had an element that was purely translational i.e. $K \bigcap \mathbf{T}^2 \neq \{0\}$ we could simply use a smaller lattice. The elements of $\mathbf{T}^2$ representing the translations on the lattice will be explained in section 1.5. First we will outline the information we will need about the dihedral groups $\mathbf{D}_4$ and $\mathbf{D}_6$.

### 1.3.1 Group Theory for Square Lattice

The symmetry group of the square is generated by a rotation anticlockwise through $\frac{\pi}{2}$, and a reflection in the $x$ axis, denoted by $r$ and $s$ respectively.

Figure 1.2: The Symmetries of the Square




$$\begin{aligned}\mathbf{D}_4 &= \langle r^4 = s^2 = (rs)^2 = e \rangle \\ &= \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}\end{aligned}$$

Subgroups of $\mathbf{D}_4$

$$\begin{array}{rcl}
1 & = & \{e\} \\
\mathbf{Z}_2[r^2] & = & \{e, r^2\} \\
\mathbf{Z}_2[s] & = & \{e, s\} \\
\mathbf{Z}_2[sr] & = & \{e, sr\} \\
\mathbf{Z}_4[r] & = & \{e, r, r^2, r^3\} \\
\mathbf{Z}_2^2[r^2, s] & = & \{e, r^2, s, sr^2\} \\
\mathbf{Z}_2^2[r^2, sr] & = & \{e, r^2, sr, sr^3\} \\
\mathbf{D}_4[r, s] & = & \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}
\end{array}$$

The natural representation of $\mathbf{D}_4$

$\rho : \mathbf{D}_4 \to GL(2, \mathbf{R})$

$$\rho(\mathbf{e}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(\mathbf{r}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(\mathbf{r^2}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(\mathbf{r^3}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

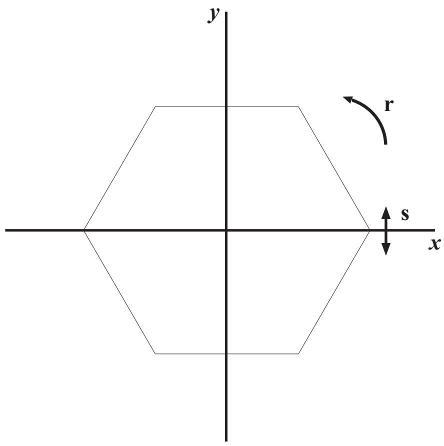
$$\rho(\mathbf{s}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(\mathbf{sr}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\rho(\mathbf{sr^2}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(\mathbf{sr^3}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

### 1.3.2 Group Theory for Hexagonal Lattice

The symmetry group of the hexagon is generated by a rotation $r$ anticlockwise through $\frac{\pi}{3}$, and a reflection $s$ in the $x$ axis.

Figure 1.3: The Symmetries of the Hexagon




$$
\begin{aligned}
\mathbf{D}_6 &= \langle r^6 = s^2 = (rs)^2 = e \rangle \\
&= \{e, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}
\end{aligned}
$$

Subgroups of $\mathbf{D}_6$

$$
\begin{aligned}
1 &= \{e\} \\
\mathbf{Z}_2[r^3] &= \{e, r^3\} \\
\mathbf{Z}_2[s] &= \{e, s\} \\
\mathbf{Z}_2[sr^3] &= \{e, sr^3\} \\
\mathbf{Z}_3[r^2] &= \{e, r^2, r^4\} \\
\mathbf{Z}_6[r] &= \{e, r, r^2, r^3, r^4, r^5\} \\
\mathbf{Z}_2^2[r^3, s] &= \{e, r^3, s, sr^3\} \\
\mathbf{D}_3[r^2, s] &= \{e, r^2, r^4, s, sr^2, sr^4\} \\
\mathbf{D}_3[r^2, sr] &= \{e, r^2, r^4, sr, sr^3, sr^5\} \\
\mathbf{D}_6[r, s] &= \{e, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}
\end{aligned}
$$

The natural representation of $\mathbf{D}_6$
$\rho : \mathbf{D}_6 \to GL(2, \mathbf{R})$

$$\rho(\mathbf{e}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(\mathbf{r}) = \begin{pmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{pmatrix}$$

$$\rho(\mathbf{r^2}) = \begin{pmatrix} -1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -1/2 \end{pmatrix} \qquad \rho(\mathbf{r^3}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rho(\mathbf{r^4}) = \begin{pmatrix} -1/2 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -1/2 \end{pmatrix} \qquad \rho(\mathbf{r^5}) = \begin{pmatrix} 1/2 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 1/2 \end{pmatrix}$$

$$\rho(\mathbf{s}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(\mathbf{sr}) = \begin{pmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -1/2 \end{pmatrix}$$

$$\rho(\mathbf{sr^2}) = \begin{pmatrix} -1/2 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -1/2 \end{pmatrix} \qquad \rho(\mathbf{sr^3}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho(\mathbf{sr^4}) = \begin{pmatrix} -1/2 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{pmatrix} \qquad \rho(\mathbf{sr^5}) = \begin{pmatrix} 1/2 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -1/2 \end{pmatrix}$$

## 1.4 A Reduced Problem

Rather than trying to find equilibrium solutions for our system of PDEs directly we use *Liapunov-Schmidt Reduction* to obtain a system of reduced bifurcation equations whose zeros are in one to one correspondence with those of the original equations. The method is shown here in terms of our specific model and is derived from the more general method described in Golubitsky and Schaeffer [31].

We are trying to find solutions to the set of differential equations (1.1)

$$\mathbf{F}(\mathbf{Q}, \lambda) = 0$$

where

- $\mathbf{Q}(\mathbf{x}, t)$, the unknown to be solved for, is a matrix valued function of space $\mathbf{x} \in \mathbf{R}^2$ and time $t \in \mathbf{R}$
- $\mathcal{Q}$ is a suitable space of such functions
- $\lambda$ is the bifurcation parameter
- $\mathbf{F} : \mathcal{Q} \times \mathbf{R} \to \mathcal{Q}$ is a smooth mapping that is $s$-times differentiable where $0 \leq s \leq \infty$.

We assume that our system of PDEs has a trivial equilibrium solution with full Euclidean symmetry, we call this solution $Q_0$. Let $\mathsf{L} = d\mathbf{F}|_{(Q_0,\lambda)}$ denote the linearization of these equations about $Q_0$. We attempt to describe solutions to this system locally near this solution.

### 1.4.1 Implicit Function Theorem

Let $\mathbf{F}$ be as above. Suppose that $\mathbf{F}(Q_0, 0) = 0$ and $d\mathbf{F}|_{(Q_0,0)}$ is invertible Then there exist neighbourhoods $U$ of $Q_0$ in $\mathcal{Q}$ and $V$ of $\lambda_0 = 0$ in $\mathbf{R}$ and a function $\mathbf{X} : V \to U$ such that for every $\lambda \in V$ the set of differential equations has a unique solution $Q = \mathbf{X}(\lambda)$ in $U$.
Moreover, if $\mathbf{F}$ is of class $C^s$ so is $\mathbf{X}$. Thus

$$\mathbf{F}(\mathbf{X}(\lambda), \lambda) = 0, \qquad \mathbf{X}(\lambda_0) = Q_0$$

For proof of the implicit function theorem see Chow and Hale (1982) [8].

### 1.4.2 Liapunov-Schmidt Reduction

Let

$$\mathbf{F} : \mathcal{Q} \times \mathbf{R} \to \mathcal{Q}, \qquad \mathbf{F}(Q_0, 0) = 0$$

be a smooth mapping. We want to solve the equation

$$\mathbf{F}(\mathbf{Q}, \lambda) = 0$$

for $\mathbf{Q}$ as a function of $\mathbf{x}$ and $t$ near $(Q_0, 0)$. Let $\mathsf{L}$ be the derivative of $\mathbf{F}$ at $(Q_0, 0)$. We assume that $\mathsf{L}$ is Fredholm of index zero.
Since $\mathsf{L}$ is Fredholm [31] we know that

- $\ker \mathsf{L} \neq 0$ is a finite-dimensional subspace of $\mathcal{Q}$
- $\operatorname{range} \mathsf{L}$ is a closed subspace of $\mathcal{Q}$ of finite codimension.

Also, index zero means $\dim \ker \mathsf{L} = \operatorname{codim} \operatorname{range} \mathsf{L}$ [31].
Since $\mathsf{L} : \mathcal{Q} \to \mathcal{Q}$ is Fredholm we can choose vector space complements $M$ and $N$ to $\ker \mathsf{L}$ and $\operatorname{range} \mathsf{L}$ respectively

$$\mathcal{Q} = \ker \mathsf{L} \oplus M$$

$$\mathcal{Q} = N \oplus \operatorname{range} \mathsf{L}.$$

Let $E$ denote the projection of $\mathcal{Q}$ onto $\operatorname{range} \mathsf{L}$ with $\ker E = N$

$$E : \mathcal{Q} \to \operatorname{range} \mathsf{L}.$$

There is a complementary projection

$$(I - E) : \mathcal{Q} \to N$$

with $\ker(I - E) = \operatorname{range} \mathsf{L}$.
If $\mathbf{Q} \in \mathcal{Q}$ then $\mathbf{Q} = 0$ iff $E\mathbf{Q} = 0$ and $(I - E)\mathbf{Q} = 0$. Thus the system of

equations $\mathbf{F}(\mathbf{Q},\lambda)=0$ may be expanded to an equivalent pair of equations

$$E\mathbf{F}(\mathbf{Q},\lambda)=0$$

$$(I-E)\mathbf{F}(\mathbf{Q},\lambda)=0.$$

We can write $\mathbf{Q}=\mathbf{Q}_1+\mathbf{Q}_2$ where $\mathbf{Q}_1\in\ker\mathsf{L}$ and $\mathbf{Q}_2\in M$. Apply the implicit function theorem to solve $E\mathbf{F}(\mathbf{Q},\lambda)=0$ for $\mathbf{Q}_2$ as a function of $\mathbf{Q}_1$ and $\lambda$. We write this solution as a function $W:\ker\mathsf{L}\times\mathbf{R}\to M$ such that

$$E\mathbf{F}(\mathbf{Q}_1+W(\mathbf{Q}_1,\lambda),\lambda)\equiv 0$$

$$W(0,0)=0.$$

We then substitute $W$ into the second of our two equations to obtain the reduced mapping $f:\ker\mathsf{L}\times\mathbf{R}\to N$

$$f(\mathbf{Q}_1,\lambda)=(I-E)\mathbf{F}(\mathbf{Q}_1+W(\mathbf{Q}_1,\lambda),\lambda).$$

The zeros of this reduced mapping $f(\mathbf{Q}_1,\lambda)$ are in one to one correspondence with the zeros of $\mathbf{F}(\mathbf{Q},\lambda)$:

$$f(\mathbf{Q}_1,\lambda)=0 \iff \mathbf{F}(\mathbf{Q}_1+W(\mathbf{Q}_1,\lambda),\lambda)=0.$$

For a full explanation of Liapunov-Schmidt reduction see Golubitsky and Schaeffer (1985) Volume 1 [31].

## 1.5 Lattices

Liapunov-Schmidt reduction gives us a much more manageable task of finding solutions to the reduced problem. However, in order for us to be able to apply Liapunov-Schmidt reduction to our initial problem the kernel of the linearization must be finite dimensional whereas in our case rotation symmetry implies that the eigenspace of $\ker \mathsf{L}$ is infinite dimensional, since for each eigenvalue there is a circle of infinitely many eigenfunctions. We solve this problem by restricting ourselves to the problem of finding those solutions that are doubly periodic with respect to a planar lattice thus making $\ker \mathsf{L}$ finite dimensional.

We define a *plane wave* as a complex-valued function of the form

$$W_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $\mathbf{k} \in \mathbf{R}^2$ is a *wave vector* and $k = |\mathbf{k}|$ is the *wave number*, [20].

Translation symmetry means that our eigenfunctions must be $\mathcal{L}$ periodic functions, so they will be linear combinations of matrices in plane wave form:

$$e^{i\mathbf{k}\cdot\mathbf{x}}Q + c.c.$$

For each wave number $k = |\mathbf{k}|$ there is a smallest $\lambda_k$ at which the trivial solution loses stability to a disturbance with this wave number. Dispersion curves often have a unique absolute minimum at $k_c$, the *critical wave number.* The first instability is assumed to occur with wave number equal to $k_c$, [30].

In the case with full Euclidean symmetry there are infinitely many vectors starting at the origin with length $k_c$: they define a *critical circle* centred at the origin with radius$=k_c$. However, when we restrict ourselves to the lattice we need only consider those vectors where the critical circle intersects the vertices of the lattice. It is these vectors that we need to find in order to find the appropriate eigenfunctions that will enable us to plot the patterns associated with the solutions to equation (1.2), these patterns are called the *planforms*. We need to find the critical vectors of each lattice, ie. those vectors that lie on a critical circle.

### 1.5.1 Lattice Generating Vectors

First we need to find those vectors that define the individual lattices. Each lattice is generated by a pair of linearly independent vectors $l_1, l_2 \in \mathbf{R}^2$. We define this lattice by the set

$$\mathcal{L} = \{n_1 l_1 + n_2 l_2\}.$$

In this model we are looking at the square and hexagonal lattices.

Let $\mathcal{X}_{\mathcal{L}}$ denote the space of $\mathcal{L}$-periodic functions on $\mathbf{R}^2$. The symmetries of $\mathcal{X}_{\mathcal{L}}$ have the form $H_{\mathcal{L}} \ltimes \mathbf{T}^2 \subset E(2) = O(2) \ltimes \mathbf{R}^2$, where $H_{\mathcal{L}}$ is the holohedry of the lattice, and $\mathbf{T}^2 = \mathbf{R}^2/\mathcal{L}$ is the torus of translations modulo the lattice.

The size of the lattice is chosen so that a plane wave with critical wave number $k_c$ is an eigenfunction of the space $\mathcal{Q}_{\mathcal{L}}$ of matrix functions that are periodic with respect to $\mathcal{L}$, [7]. Those $\mathbf{k} \in \mathbf{R}^2$ for which the scalar plane wave $e^{2\pi i \mathbf{k}\cdot\mathbf{x}}$ is $\mathcal{L}$-periodic are dual wave vectors. We consider only those lattice sizes where the critical dual wave vectors are of shortest length in $\mathcal{L}^*$.

We then define the *dual lattice* $\mathcal{L}^*$ to be

$$\mathcal{L}^* = \{\mathbf{k} \in \mathbf{R}^2 | e^{2\pi i \mathbf{k}\cdot\mathbf{x}} \text{ is } \mathcal{L} \text{ periodic}\}$$

$$\mathcal{L}^* = \{n_1 \mathbf{k}_1 + n_2 \mathbf{k}_2\}.$$

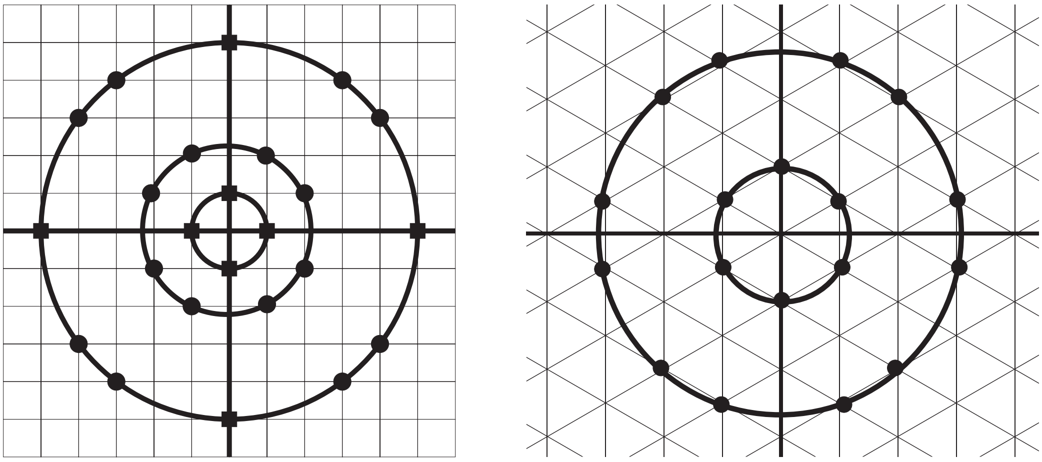
Table 1.1: Generators for Lattices and Dual Lattices

| lattice | holohedry $H_{\mathcal{L}}$ | basis of $\mathcal{L}$ | basis of $\mathcal{L}^*$ |
|---|---|---|---|
| square | $\mathbf{D}_4$ | $l_1 = (1,0)$ | $\mathbf{k}_1 = (1,0)$ |
| | | $l_2 = (0,1)$ | $\mathbf{k}_2 = (0,1)$ |
| hexagonal | $\mathbf{D}_6$ | $l_1 = \left(\frac{1}{\sqrt{3}}, 1\right)$ | $\mathbf{k}_1 = (0,1)$ |
| | | $l_2 = \left(\frac{2}{\sqrt{3}}, 0\right)$ | $\mathbf{k}_2 = \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$ |

### 1.5.2 Critical Circles and Wave Vectors

We can find the irreducible representations of the group $H_{\mathcal{L}} + \mathbf{T}^2$ (the $\mathbf{Z}_2$ will not be relevant at this stage) by looking at where the critical circles intersect the lattice, shown in figure (1.4).

Figure 1.4: Critical Circles Intersecting the Dual Lattices



If we restrict our solutions to those that are doubly periodic with respect to a planar lattice all wave vectors $\mathbf{k}$ contributing to the pattern must lie at the vertices of a dual lattice [37, p138].

There is a countable infinity of irreducible representations of the symmetry group for both lattices. These can be grouped into two types for each lattice. The first type is four-dimensional for the square lattice and six-dimensional for the hexagonal lattice. The smallest example in both cases is when the lattice size is chosen to make the critical wavenumber $k_c = 1$, shown by the inner cirle in each picture in figure (1.4).

$$u(x, y, t) = z_1(t)e^{i\mathbf{K}_1} + z_2(t)e^{i\mathbf{K}_2} + c.c. \in \mathbf{C}^2$$

$$u(x, y, t) = z_1(t)e^{i\mathbf{K}_1} + z_2(t)e^{i\mathbf{K}_2} + z_3(t)e^{i\mathbf{K}_3} + c.c. \in \mathbf{C}^3$$

where the $\mathbf{K}_j$ s are given in table 1.2.

The second type of irreducible representation for the square lattice is eight-dimensional and is given by

$$u(x, y, t) = z_1(t)e^{i\mathbf{K}_1} + z_2(t)e^{i\mathbf{K}_2} + z_3(t)e^{i\mathbf{K}_3} + z_4(t)e^{i\mathbf{K}_4} + c.c. \in \mathbf{C}^4$$

The first of these representations occurs when the lattice size is chosen with $k_c = \sqrt{5}$, shown by the middle circle in the left hand picture of figure (1.4).

The second type of absolutely irreducible representation for the hexagonal lattice is twelve-dimensional and is given by

$$u(x, y, t) = z_1(t)e^{i\mathbf{K}_1} + z_2(t)e^{i\mathbf{K}_2} + z_3(t)e^{i\mathbf{K}_3} + z_4(t)e^{i\mathbf{K}_4} + z_5(t)e^{i\mathbf{K}_5} + z_6(t)e^{i\mathbf{K}_6} + c.c. \in \mathbf{C}^6$$

The first of these representations occurs when the lattice size is chosen making $k_c = \sqrt{7}$, shown by the largest circle in the right hand picture of figure (1.4). [11]

### 1.5.3 Half Lattice

As stated in Section 1.3 we are only interested in shifted subgroups of $\Gamma_{\mathcal{L}}$. As a result of this the only elements of $\mathbf{T}^2$ that will be relevent are the half lattice points since they are the only elements that can be combined with elements of the holohedry to create an element of $\Gamma_{\mathcal{L}}$ such that the subgroup generated by that element has a non-trivial fixed-point subspace.

We introduce the idea of a *half lattice* $\frac{1}{2}\mathcal{L} = \frac{1}{2}\mathcal{L}[v_1, v_2] \approx \mathbf{Z}_2^2$ generated by the vectors $v_1 = \frac{1}{2}l_1$ and $v_2 = \frac{1}{2}l_2$, where $l_1$ and $l_2$ are the generators of $\mathcal{L}$. The vectors $v_1$ and $v_2$ generate a subgroup of $\mathbf{T}^2$, the other non-trivial element being $v_d = v_1 + v_2$, [21].

We will use the notation $\Gamma_{\frac{1}{2}\mathcal{L}} = (H_{\mathcal{L}} \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2$.

Table 1.2: Irreducible Representations

| lattice | $dimV$ | $\mathbf{K}_s$ |
|---|---|---|
| square | 4 | $\mathbf{K}_1 = \mathbf{k}_1$<br>$\mathbf{K}_2 = \mathbf{k}_2$ |
| square | 8 | $\mathbf{K}_1 = \alpha\mathbf{k}_1 + \beta\mathbf{k}_2$<br>$\mathbf{K}_2 = -\beta\mathbf{k}_1 + \alpha\mathbf{k}_2$<br>$\mathbf{K}_3 = \beta\mathbf{k}_1 + \alpha\mathbf{k}_2$<br>$\mathbf{K}_1 = -\alpha\mathbf{k}_1 + \beta\mathbf{k}_2$ |
| hexagonal | 6 | $\mathbf{K}_1 = \mathbf{k}_1 + \mathbf{k}_2$<br>$\mathbf{K}_2 = -\mathbf{k}_2$<br>$\mathbf{K}_3 = -\mathbf{k}_1$ |
| hexagonal | 12 | $\mathbf{K}_1 = \alpha\mathbf{k}_1 + \beta\mathbf{k}_2$<br>$\mathbf{K}_2 = (-\alpha + \beta)\mathbf{k}_1 - \alpha\mathbf{k}_2$<br>$\mathbf{K}_3 = -\beta\mathbf{k}_1 + (\alpha - \beta)\mathbf{k}_2$<br>$\mathbf{K}_4 = \alpha\mathbf{k}_1 + (\alpha - \beta)\mathbf{k}_2$<br>$\mathbf{K}_5 = -\beta\mathbf{k}_1 - \alpha\mathbf{k}_2$<br>$\mathbf{K}_6 = (-\alpha + \beta)\mathbf{k}_1 + \beta\mathbf{k}_2$ |

Where $\alpha$ and $\beta$ are integers, and greatest common divisor $gcd(\alpha, \beta) = 1$.
In the square case: $\alpha > \beta > 0$, and $\alpha + \beta$ is odd.
In the hexagonal case: $\alpha > \beta > \alpha/\beta > 0$, and $gcd(3, \alpha + \beta) = 1$.

## 1.6 The representations of $\Gamma_{\mathcal{L}}$

The material in this section follows the paper by Chillingworth and Golubitsky [7], using the Landau - de Gennes model [17] to describe molecular orientation. We are looking for patterns created by the orientation of molecules within a planar layer of liquid crystal. Each molecule has a position vector $\mathbf{x} \in \mathbf{R}^2$, and a director $\mathbf{n} = -\mathbf{n} \in \mathbf{R}^3$. We can use an ellipsoid to approximate the probability that the molecule at position $\mathbf{x}$ points in a certain direction. The more elongated the ellipsoid, the higher the probability that the molecule points in the direction of the elongation. An ellipsoid can be defined by a real symmetric $3 \times 3$ matrix.

$$Q = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

The space of $3 \times 3$ symmetric matrices is 6 dimensional, but this is not absolutely irreducible since those matrices that are scalar multiples of the identity representing spheres will themselves commute with matrices that are not scalar multiples of the identity. We are concerned with the elongation of the ellipsoid, and the direction of the elongation, which describes how the ellipsoid differs from the sphere, therefore we can take out the 1 dimensional space of scalar multiples of the identity and look at the five dimensional space of $3 \times 3$ symmetric matrices with trace$= 0$, equivalent to $\mathbf{R}^5$. Thus the matrix represents the standard deviation of the probability that the director points in a certain direction.

$$Q = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & -a-b \end{pmatrix}$$

### 1.6.1 The Trivial States

We assume that the crystal is in an initial equilibrium state that is $\mathbf{E}(2)$-invariant, where $\mathbf{E}(2)$ is the Euclidean group of all symmetries of $\mathbf{R}^2$. There are two such possible states, the homeotropic state where all the molecules align vertically, and the planar isotropic state where all the molecules lie flat within the $xy$ plane but with no propensity to point in any particular direction. These states have the form:

$$Q_0 = \alpha \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{1.3}$$

If $\alpha > 0$ then there is a single largest eigenvalue with eigenvector pointing in the $z$ direcetion meaning the matrix represents the homeotropic state. If $\alpha < 0$ there are joint largest eigenvalues with eigenvectors pointing in the $x$ and $y$ directions meaning the matrix represents the planar isotropic state. The state $Q_0$ is also invariant under the reflection in the $xy$ plane, actioned by conjugating by the matrix:

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This gives us the symmetry group $\Gamma = \mathbf{E}(2) \times \mathbf{Z}_2(\tau)$.

### 1.6.2 Q Matrices

From Section 1.5 recall that planar translation symmetry implies that eigenfunctions of $\mathsf{L}$ are linear combinations of matrices that have the plane wave form:

$$e^{2\pi i \mathbf{k}\cdot\mathbf{x}}Q + c.c.$$

Here $Q \in V_{\mathbf{C}}$ is a constant matrix where $V_{\mathbf{C}}$ denotes the space of complex $3 \times 3$ symmetric matrices with trace$= 0$, and $\mathbf{k} \in \mathbf{R}$ is a wave vector.

For fixed $\mathbf{k}$ let

$$W_{\mathbf{k}} = \{e^{2\pi i \mathbf{k}\cdot\mathbf{x}}Q + c.c. | Q \in V_{\mathbf{C}}\}$$

be the ten-dimensional real linear subspace consisting of such functions. Since we are always adding the complex conjugate this will ensure we always end up with a $3 \times 3$ symmetric matrix with real entries and trace$= 0$ representing a real ellipsoid.

$W_{\mathbf{k}}$ can be broken down into four $\mathsf{L}$-invariant subspaces, limiting the possible forms of $Q$, as follows.

The symmetries of the system place restriction on the possible forms of $Q$.

$\mathsf{L}$ commutes with $\rho$, the rotation through $\pi$ in the $xy$ plane, described by the matrix.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By looking at the action of $\rho$ on $Q$ we can simplify the form of $Q$ as follows:

$$\rho(e^{2\pi i\mathbf{k}\cdot\mathbf{x}}Q) = e^{2\pi i\rho\mathbf{k}\cdot\mathbf{x}}\rho Q\rho^{-1} = e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\rho Q\rho^{-1} = e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\overline{\rho Q\rho^{-1}}$$

$$\rho(Q) = \overline{\rho Q\rho^{-1}}$$

The kernel of $\mathsf{L}$ can be divided into two $\mathsf{L}$-invariant subspaces, the first with $\rho(Q) = Q$ and the second with $\rho(Q) = -Q$. Also, translation by $\frac{1}{4}\mathbf{k}$ implies that if $e^{2\pi i\mathbf{k}\cdot\mathbf{x}}Q$ is an eigenfunction then $ie^{2\pi i\mathbf{k}\cdot\mathbf{x}}Q$ is a (symmetry related) eigenfuntion. It follows that if $\rho$ acts as minus the identity on $Q$ then it acts as the identity on $iQ$ [7]. If $\rho$ acts as the identity it has no effect on the form of $Q$. This implies that $Q$ has the form

$$Q = \begin{pmatrix} a & d & ie \\ d & b & if \\ ie & if & -a-b \end{pmatrix}$$

The reflection $\kappa : y \to -y$ divides $W_{\mathbf{k}}$ into two subspaces: $W_{\mathbf{k}}^{+}$ where $\kappa$ acts trivially, this contains even functions in $y$; and $W_{\mathbf{k}}^{-}$ where $\kappa$ acts as minus the identity, this contains odd functions in $y$. Bifurcations based on even eigenfunctions are called *scalar* and bifurcations based on odd functions are called *pseudoscalar*.

$$\kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\kappa Q \kappa^{-1} = \begin{pmatrix} a & -d & e \\ -d & b & -f \\ e & -f & -a-b \end{pmatrix}$$

If $\kappa$ acts trivially then $Q$ has the form:

$$\begin{pmatrix} a & 0 & e \\ 0 & b & 0 \\ e & 0 & -a-b \end{pmatrix} = Q^{+}.$$

If $\kappa$ acts non-trivially then $Q$ has the form:

$$\begin{pmatrix} 0 & d & 0 \\ d & 0 & f \\ 0 & f & 0 \end{pmatrix} = Q^{-}.$$

So eigenfunctions in $W_{\mathbf{k}}$ lie in one of the two-dimensional subspaces $V_{\mathbf{k}}^{+}, V_{\mathbf{k}}^{-}$ of $W_{\mathbf{k}}^{+}, W_{\mathbf{k}}^{-}$ that have the form

$$V_{\mathbf{k}}^{+} = \{e^{2\pi i \mathbf{k}\cdot\mathbf{x}} Q^{+}\}$$

$$V_{\mathbf{k}}^{-} = \{e^{2\pi i \mathbf{k}\cdot\mathbf{x}} Q^{-}\}.$$

Also, $\mathsf{L}$ commutes with the reflection in the $xy$ plane

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\tau Q \tau^{-1} = \begin{pmatrix} a & d & -e \\ d & b & -f \\ -e & -f & -a-b \end{pmatrix}$$

If $\tau$ acts trivially then $Q$ has the form:

$$\begin{pmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & -a-b \end{pmatrix}.$$

If $\tau$ acts non-trivially then $Q$ has the form:

$$\begin{pmatrix} 0 & 0 & e \\ 0 & 0 & f \\ e & f & 0 \end{pmatrix}.$$

This further decomposes the $\mathsf{L}$-invariant subspaces according to whether $\tau$ acts trivially or non-trivially.

$$V_{\mathbf{K}}^{+} = V_{\mathbf{K}}^{++} + V_{\mathbf{K}}^{+-}$$

$$V_{\mathbf{K}}^{-} = V_{\mathbf{K}}^{-+} + V_{\mathbf{K}}^{--}$$

By combining these properties we get four possible forms for $Q$. We identify the individual forms by double indices $Q^{\epsilon\psi}$ where the first index $\epsilon = \pm 1$ describes the action of $\kappa$, and the second index $\psi = \pm 1$ describes the action of $\tau$.

The four different forms of $Q$ are as follows:

$$Q^{++} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \quad Q^{+-} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$Q^{-+} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q^{--} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{1.4}$$

We are looking at rotations and reflections of ellipsoids, in matrix terms these linear transformations result from conjugating the ellipsoid matrix by the appropriate rotation or reflection matrix, hence the elements of the group $\Gamma$ will act on the matrices $Q$ by conjugation, $\gamma Q \gamma^{-1}$.

## 1.7 Finding the patterns

Now that we have gathered all the components for the eigenfunctions we can put them together to see that the generalized eigenspace is generated by expressions of the form

$$\tilde{Q} = \sum_{j=1}^{s} z_j e^{2\pi i \mathbf{K}_j \cdot \mathbf{x}} Q_j + c.c.$$

where the $\mathbf{K}_j$s are given in Table 1.2. Here $Q_j = K_j Q^{\pm\pm}(K_j)^{-1}$, $K_j$ is the linear transformation matrix from the natural representation of $H_{\mathcal{L}}$ giving the rotation through $\phi$ in the $xy$ plane, and $\phi$ is the angle $\mathbf{K}_j$ makes with the positive $x$ axis.

It is these eigenfunctions we will be plotting so we need to find the complex coefficients $z_j$ to input into $\tilde{Q}$ in order to plot the planforms. The *Equivariant Branching Lemma* tells us that we will find solutions generically with symmetry group $\Sigma \in \Gamma_{\mathcal{L}}$ whenever $\dim \mathrm{Fix}\, \Sigma = 1$

### 1.7.1 The Equivariant Branching Lemma

**Lemma 1.7.1. (Equivariant Branching Lemma)***[65] [10]*
*Let $G$ be a compact Lie group acting absolutely irreducibly on $\mathbf{R}^n$.*
*Let $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \lambda)$ be a $G$-equivariant set of differential equations.*
*Then it follows that:*

$$\mathbf{F}(\mathbf{0}, \lambda) = \mathbf{0}, \forall\lambda$$

$$D\mathbf{F}|_{(\mathbf{0},\lambda)} = c(\lambda)I_n$$

*Also assume that:*

| | |
|---|---|
| $c(0) = 0$ | *Bifurcation occurs at $\lambda = 0$.* |
| $\frac{dc}{d\lambda}\|_{\lambda=o} \neq 0$ | *The eigenvalues cross the imaginary axis with non-zero speed.* |

*Then: For each axial* isotropy subgroup $\Sigma \subseteq G$ there exists generically a unique branch of solutions $\mathbf{x}(\lambda)$ satisfying $\mathbf{F}(\mathbf{x}(\lambda), \lambda) = \mathbf{0}$ branching from the origin and having symmetry $\Sigma$.*
**Recall that an isotropy subgroup $\Sigma$ is axial if* $\mathrm{dimFix}(\Sigma) = 1$.

### 1.7.2 The Method

In order to plot the equilibrium solutions we must first find all the isotropy subgroups of $\Gamma_{\mathcal{L}}$ with fixed-point subspace of dimension 1. We will follow the steps shown below.

**Step 1** Calculate all the shifted subgroups $K$ of $\Gamma_{\mathcal{L}}$.

**Step 2** Calculate the action of each group element on $\tilde{Q}$ by applying each element of the group to $\tilde{Q}$ and record how it permutes the complex coefficients $z_i$.

**Step 3** Use these group actions to find the fixed-point subspaces of all the shifted subgroups and their dimensions.

**Step 4** Of those cases where $\dim\mathrm{Fix}(K) = 1$ check for equivalent cases.

**Step 5** Plot the planforms. This will be done in Matlab.

# 2

# Equilibrium Solutions from 2nd Representations

In this chapter we will use the method shown in Chapter 1 to find equilibrium solutions close to isotopic or homeotropic states. There are four possible lattice representations in total, two for each of the square and hexagonal lattices, shown by the two smallest critical circles on both pictures in Figure 1.4. In Chillingworth and Golubitsky [7] the equilibria for the smaller representation (ie. where $k_c = 1$) for both the square and hexagonal lattices are shown. However, the second possible representation in each case is not covered and we will show the results for these cases here. The Equivariant Branching Lemma tells us that to find these solutions we need to find all possible axial isotropy subgroups and their corresponding fixed-point subspaces.

## 2.1 The Shifted Subgroups

We begin by listing all the shifted subgroups $K$ of $\Gamma_{\mathcal{L}}$ for both the square and hexagonal cases. Listing all these subgroups may seem like a daunting task at first but much of the work has already been done for us. In Dionne et al. [21] the subgroups by conjugacy class of the groups $\mathbf{D}_4 \ltimes \mathbf{T}^2$ and $\mathbf{D}_6 \ltimes \mathbf{T}^2$ are already calculated; these are shown in the left hand column of

Tables 2.1 and 2.2. Since $(e, e, \tau)$ commutes with everything it is a simple task to expand these lists to include all possible conjugacy classes of our group $\Gamma_{\mathcal{L}}$ for both cases. The extra subgroups of $\Gamma_{\mathcal{L}}$ are shown in the right hand column of Tables 2.1 and 2.2.

Table 2.1: Shifted Subgroups of $\Gamma_{\mathcal{L}} = (\mathbf{D}_4 \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$.

| | **Shifted Subgroups of $\mathbf{D}_4 \ltimes \mathbf{T}^2$** | **Extra subgroups of $\Gamma_{\mathcal{L}}$** |
|---|---|---|
| 1 | $\mathbf{1}$ | |
| 2 | $\mathbf{Z}_2[(r^2, e, e)]$ | |
| 3 | | $\mathbf{Z}_2[(e, e, \tau)]$ |
| 4 | | $\mathbf{Z}_2[(r^2, e, \tau)]$ |
| 5 | $\mathbf{Z}_2[(s, e, e)]$ | |
| 6 | $\mathbf{Z}_2[(s, v_1, e)]$ | |
| 7 | $\mathbf{Z}_2[(sr, e, e)]$ | |
| 8 | | $\mathbf{Z}_2[(s, e, \tau)]$ |
| 9 | | $\mathbf{Z}_2[(s, v_1, \tau)]$ |
| 10 | | $\mathbf{Z}_2[(sr, e, \tau)]$ |
| 11 | $\mathbf{Z}_4[(r, e, e)]$ | |
| 12 | | $\mathbf{Z}_4[(r, e, \tau)]$ |
| 13 | $\mathbf{Z}_2^2[(r^2, e, e), (s, e, e)]$ | |
| 14 | $\mathbf{Z}_2^2[(r^2, e, e), (s, v_1, e)]$ | |
| 15 | $\mathbf{Z}_2^2[(r^2, e, e), (s, v_d, e)]$ | |
| 16 | | $\mathbf{Z}_2^2[(r^2, e, e), (s, e, \tau)]$ |
| 17 | | $\mathbf{Z}_2^2[(r^2, e, e), (s, v_1, \tau)]$ |
| 18 | | $\mathbf{Z}_2^2[(r^2, e, e), (s, v_d, \tau)]$ |
| 19 | | $\mathbf{Z}_2^2[(r^2, e, \tau), (s, e, e)]$ |
| 20 | | $\mathbf{Z}_2^2[(r^2, e, \tau), (s, v_1, e)]$ |
| 21 | | $\mathbf{Z}_2^2[(r^2, e, \tau), (s, v_d, e)]$ |
| 22 | $\mathbf{Z}_2^2[(r^2, e, e), (sr, e, e)]$ | |
| 23 | | $\mathbf{Z}_2^2[(r^2, e, e), (sr, e, \tau)]$ |
| 24 | | $\mathbf{Z}_2^2[(r^2, e, \tau), (sr, e, e)]$ |
| 25 | | $\mathbf{Z}_2^2[(r^2, e, e), (e, e, \tau)]$ |
| 26 | | $\mathbf{Z}_2^2[(s, e, e), (e, e, \tau)]$ |
| 27 | | $\mathbf{Z}_2^2[(s, v_1, e), [(e, e, \tau)]$ |
| 28 | | $\mathbf{Z}_2^2[(sr, e, e), (e, e, \tau)]$ |
| 29 | | $\mathbf{Z}_4[(r, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$ |
| 30 | | $\mathbf{Z}_2^3[(r^2, e, e), (s, e, e), (e, e, \tau)]$ |

Table 2.1: Shifted Subgroups of $\Gamma_{\mathcal{L}} = (\mathbf{D}_4 \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$.

| | **Shifted Subgroups of $\mathbf{D}_4 \ltimes \mathbf{T}^2$** | **Extra subgroups of $\Gamma_{\mathcal{L}}$** |
|---|---|---|
| 31 | | $\mathbf{Z}_2^3[(r^2,e,e),(s,v_1,e),(e,e,\tau)]$ |
| 32 | | $\mathbf{Z}_2^3[(r^2,e,e),(s,v_d,e),(e,e,\tau)]$ |
| 33 | | $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)]$ |
| 34 | $\mathbf{D}_4[(r,e,e),(s,e,e)]$ | |
| 35 | $\mathbf{D}_4[(r,e,e),(s,v_d,e)]$ | |
| 36 | | $\mathbf{D}_4[(r,e,e),(s,e,\tau)]$ |
| 37 | | $\mathbf{D}_4[(r,e,e),(s,v_d,\tau)]$ |
| 38 | | $\mathbf{D}_4[(r,e,\tau),(s,e,e)]$ |
| 39 | | $\mathbf{D}_4[(r,e,\tau),(s,v_d,e)]$ |
| 40 | | $\mathbf{D}_4[(r,e,\tau),(s,e,\tau)]$ |
| 41 | | $\mathbf{D}_4[(r,e,\tau),(s,v_d,\tau)]$ |
| 42 | | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 43 | | $\mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |

Table 2.2: Shifted Subgroups of $\Gamma_{\mathcal{L}} = (\mathbf{D}_6 \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$.

| | **Shifted Subgroups of $\mathbf{D}_6 \ltimes \mathbf{T}^2$** | **Extra subgroups of $\Gamma_{\mathcal{L}}$** |
|---|---|---|
| 1 | $\mathbf{1}$ | |
| 2 | $\mathbf{Z}_2[(r^3,e,e)]$ | |
| 3 | | $\mathbf{Z}_2[(e,e,\tau)]$ |
| 4 | | $\mathbf{Z}_2[(r^3,e,\tau)]$ |
| 5 | $\mathbf{Z}_2[(s,e,e)]$ | |
| 6 | $\mathbf{Z}_2[(sr^3,e,e)]$ | |
| 7 | | $\mathbf{Z}_2[(s,e,\tau)]$ |
| 8 | | $\mathbf{Z}_2[(sr^3,e,\tau)]$ |
| 9 | $\mathbf{Z}_3[(r^2,e,e)]$ | |
| 10 | | $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)]$ |
| 11 | $\mathbf{Z}_2^2[(r^3,e,e),(s,e,e)]$ | |
| 12 | | $\mathbf{Z}_2^2[(r^3,e,e),(s,e,\tau)]$ |
| 13 | | $\mathbf{Z}_2^2[(r^3,e,\tau),(s,e,e)]$ |
| 14 | | $\mathbf{Z}_2^2[(r^3,e,\tau),(s,e,\tau)]$ |
| 15 | | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ |
| 16 | | $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$ |
| 17 | $\mathbf{Z}_6[(r,e,e)]$ | |

Table 2.2: Shifted Subgroups of $\Gamma_{\mathcal{L}} = (\mathbf{D}_6 \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$.

| | Shifted Subgroups of $\mathbf{D}_6 \ltimes \mathbf{T}^2$ | Extra subgroups of $\Gamma_{\mathcal{L}}$ |
|---|---|---|
| 18 | | $\mathbf{Z}_6[(r, e, \tau)]$ |
| 19 | | $\mathbf{Z}_6[(r^2, e, \tau)]$ |
| 20 | $\mathbf{D}_3[(r^2, e, e), (s, e, e)]$ | |
| 21 | $\mathbf{D}_3[(r^2, e, e), (sr, e, e)]$ | |
| 22 | | $\mathbf{D}_3[(r^2, e, e), (s, e, \tau)]$ |
| 23 | | $\mathbf{D}_3[(r^2, e, e), (sr, e, \tau)]$ |
| 24 | | $\mathbf{Z}_2^3[(r^3, e, e), (s, e, e), (e, e, \tau)]$ |
| 25 | | $\mathbf{Z}_6[(r, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$ |
| 26 | $\mathbf{D}_6[(r, e, e), (s, e, e)]$ | |
| 27 | $\mathbf{D}_6[(r, e, e), (s, e, \tau)]$ | |
| 28 | $\mathbf{D}_6[(r, e, \tau), (s, e, e)]$ | |
| 29 | $\mathbf{D}_6[(r, e, \tau), (s, e, \tau)]$ | |
| 30 | $\mathbf{D}_6[(r^2, e, \tau), (s, e, e)]$ | |
| 31 | $\mathbf{D}_6[(r^2, e, \tau), (sr, e, e)]$ | |
| 32 | $\mathbf{D}_6[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$ | |

## 2.2 The Action of the Elements of $\Gamma_{\frac{1}{2}\mathcal{L}}$

Next it is necessary to calculate the action of each group element on $\tilde{Q}$. We apply each element of the group to $\tilde{Q}$ and record how it permutes the complex coefficients $z_i$: an example of this calculation is show in Appendix A. Since we are only interested in the shifted subgroups we need only look at the generating elements of the group $\Gamma_{\frac{1}{2}\mathcal{L}}$. The actions of the generating elements are shown in Tables 2.3 and 2.4. Due to the complicated nature of calculating the fixed-point subspaces in the next step the actions of all relevant group elements are shown in Appendix B to make these calculations easier.

Table 2.3: The Action of the Generators of $(\mathbf{D}_4 \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2$

| $g$ | **Action on** $\mathbf{C}^2$ $g(z_1, z_2)$ | **Action on** $\mathbf{C}^4$ $g(z_1, z_2, z_3, z_4)$ |
|---|---|---|
| $(r, e, e)$ | $(\overline{z_2}, z_1)$ | $(\overline{z_2}, z_1, \overline{z_4}, z_3)$ |
| $(s, e, e)$ | $\epsilon(z_1, \overline{z_2})$ | $\epsilon(\overline{z_4}, \overline{z_3}, \overline{z_2}, \overline{z_1})$ |
| $(e, v_1, e)$ | $(-z_1, z_2)$ | $(-z_1, z_2, z_3, -z_4)$, $\alpha$ odd |
| | | $(z_1, -z_2, -z_3, z_4)$, $\beta$ odd |
| $(e, v_2, e)$ | $(z_1, -z_2)$ | $(z_1, -z_2, -z_3, z_4)$, $\alpha$ odd |
| | | $(-z_1, z_2, z_3, -z_4)$, $\beta$ odd |
| $(e, v_d, e)$ | $(-z_1, -z_2)$ | $(-z_1, -z_2, -z_3, -z_4)$ |
| $(e, e, \tau)$ | $\psi(z_1, z_2)$ | $\psi(z_1, z_2, z_3, z_4)$ |

Table 2.4: The Action of the Generators of $(\mathbf{D}_6 \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2$

| $g$ | **Action on** $\mathbf{C}^3$ $g(z_1, z_2, z_3)$ | **Action on** $\mathbf{C}^6$ $g(z_1, z_2, z_3, z_4, z_5, z_6)$ |
|---|---|---|
| $(r, e, e)$ | $(\overline{z_2}, \overline{z_3}, \overline{z_1})$ | $(\overline{z_2}, \overline{z_3}, \overline{z_1}, \overline{z_5}, \overline{z_6}, \overline{z_4})$ |
| $(s, e, e)$ | $\epsilon(\overline{z_2}, \overline{z_1}, \overline{z_3})$ | $\epsilon(z_6, z_5, z_4, z_3, z_2, z_1)$ |
| $(e, v_1, e)$ | $(-z_1, z_2, -z_3)$ | $(-z_1, -z_2, z_3, -z_4, z_5, -z_6)$, $\alpha$ odd |
| | | $(z_1, -z_2, -z_3, z_4, -z_5, -z_6)$, $\beta$ odd |
| | | $(-z_1, z_2, -z_3, -z_4, -z_5, z_6)$, $\alpha$ and $\beta$ odd |
| $(e, v_2, e)$ | $(-z_1, -z_2, z_3)$ | $(z_1, -z_2, -z_3, -z_4, -z_5, z_6)$, $\alpha$ odd |
| | | $(-z_1, z_2, -z_3, -z_4, z_5, -z_6)$, $\beta$ odd |
| | | $(-z_1, -z_2, z_3, z_4, -z_5, -z_6)$, $\alpha$ and $\beta$ odd |
| $(e, e, \tau)$ | $\psi(z_1, z_2, z_3)$ | $\psi(z_1, z_2, z_3, z_4, z_5, z_6)$ |

## 2.3 Fixed-point Subspaces

Having calculated both the actions of the group elements and the full list of possible subgroups by conjugacy class we combine the two to discover the fixed-point subspace of each subgroup and its dimension. This is shown in Tables 2.5 and 2.6.

Table 2.5: Shifted Subgroups and their Fixed Point Subspaces for the Square Lattice $\mathbf{C}_4$

| | **Shifted Subgroup** | **Fixed point subspace** | dimFix |
|---|---|---|---|
| 1 | $\mathbf{1}$ | $\mathbf{C}^4$ | 8 |
| 2 | $\mathbf{Z}_2[(r^2,e,e)]$ | $z=\overline{z}$ | 4 |
| 3 | $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^4$ when $\psi=+1$ | 8 |
| | | 0 when $\psi=-1$ | 0 |
| 4 | $\mathbf{Z}_2[(r^2,e,\tau)]$ | $z=\overline{z}$ when $\psi=+1$ | 4 |
| | | $Re(z)=0$ when $\psi=-1$ | 4 |
| 5 | $\mathbf{Z}_2[(s,e,e)]$ | $z_1=\epsilon\overline{z}_4, z_2=\epsilon\overline{z}_3$ | 4 |
| 6 | $\mathbf{Z}_2[(s,v_1,e)]$ | $z_1=-\epsilon\overline{z}_4, z_2=\epsilon\overline{z}_3$ | 4 |
| 7 | $\mathbf{Z}_2[(sr,e,e)]$ | $z_1=\epsilon\overline{z}_3, z_2=\epsilon z_4$ | 4 |
| 8 | $\mathbf{Z}_2[(s,e,\tau)]$ | $z_1=\epsilon\psi\overline{z}_4, z_2=\epsilon\psi\overline{z}_3$ | 4 |
| 9 | $\mathbf{Z}_2[(s,v_1,\tau)]$ | $z_1=-\epsilon\psi\overline{z}_4, z_2=\epsilon\psi\overline{z}_3$ | 4 |
| 10 | $\mathbf{Z}_2[(sr,e,\tau)]$ | $z_1=\epsilon\psi\overline{z}_3, z_2=\epsilon\psi z_4$ | 4 |
| 11 | $\mathbf{Z}_4[(r,e,e)]$ | $z_1=z_2, z_3=z_4, z=\overline{z}$ | 2 |
| 12 | $\mathbf{Z}_4[(r,e,\tau)]$ | $z_1=\psi z_2, z_3=\psi z_4, z=\overline{z}$ | 2 |
| 13 | $\mathbf{Z}_2^2[(r^2,e,e),(s,e,e)]$ | $z_1=\epsilon z_4, z_2=\epsilon z_3, z=\overline{z}$ | 2 |
| 14 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_1,e)]$ | $z_1=-\epsilon z_4, z_2=\epsilon z_3, z=\overline{z}$ | 2 |
| 15 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_d,e)]$ | $z_1=-\epsilon z_4, z_2=-\epsilon z_3, z=\overline{z}$ | 2 |
| 16 | $\mathbf{Z}_2^2[(r^2,e,e),(s,e,\tau)]$ | $z_1=\epsilon\psi z_4, z_2=\epsilon\psi z_3, z=\overline{z}$ | 2 |
| 17 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_1,\tau)]$ | $z_1=-\epsilon z_4, z_2=\epsilon z_3, z=\overline{z}$ | 2 |
| 18 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_d,\tau)]$ | $z_1=-\epsilon z_4, z_2=-\epsilon z_3, z=\overline{z}$ | 2 |
| 19 | $\mathbf{Z}_2^2[(r^2,e,\tau),(s,e,e)]$ | $z_1=\epsilon z_4, z_2=\epsilon z_3, z=\overline{z}$ when $\psi=+1$ | 2 |
| | | $z_1=\epsilon z_4, z_2=\epsilon z_3, Re(z)=0$ when $\psi=-1$ | 2 |
| 20 | $\mathbf{Z}_2^2[(r^2,e,\tau),(s,v_1,e)]$ | $z_1=-\epsilon z_4, z_2=\epsilon z_3, z=\overline{z}$ when $\psi=+1$ | 2 |
| | | $z_1=-\epsilon z_4, z_2=\epsilon z_3, Re(z)=0$ when $\psi=-1$ | 2 |
| 21 | $\mathbf{Z}_2^2[(r^2,e,\tau),(s,v_d,e)]$ | $z_1=-\epsilon z_4, z_2=-\epsilon z_3, z=\overline{z}$ when $\psi=+1$ | 2 |
| | | $z_1=-\epsilon z_4, z_2=-\epsilon z_3, Re(z)=0$ when $\psi=-1$ | 2 |
| 22 | $\mathbf{Z}_2^2[(r^2,e,e),(sr,e,e)]$ | $z_1=\epsilon z_3, z_2=\epsilon z_4, z=\overline{z}$ | 2 |
| 23 | $\mathbf{Z}_2^2[(r^2,e,e),(sr,e,\tau)]$ | $z_1=\epsilon\psi z_3, z_2=\epsilon\psi z_4, z=\overline{z}$ | 2 |
| 24 | $\mathbf{Z}_2^2[(r^2,e,\tau),(sr,e,e)]$ | $z_1=\epsilon z_3, z_2=\epsilon z_4, z=\overline{z}$ when $\psi=+1$ | 2 |
| | | $z_1=\epsilon z_3, z_2=\epsilon z_4, Re(z)=0$ when $\psi=-1$ | 2 |
| 25 | $\mathbf{Z}_2^2[(r^2,e,e),(e,e,\tau)]$ | $z=\overline{z}$ when $\psi=+1$ | 4 |

Table 2.5: Shifted Subgroups and their Fixed Point Subspaces for the Square Lattice $\mathbf{C}_4$

| | **Shifted Subgroup** | **Fixed point subspace** | dimFix |
|---|---|---|---|
| | | 0 when $\psi = -1$ | 0 |
| 26 | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $z_1 = \epsilon\overline{z}_4, z_2 = \epsilon\overline{z}_3$ when $\psi = +1$ | 4 |
| | | 0 when $\psi = +1$ | 0 |
| 27 | $\mathbf{Z}_2^2[(s,v_1,e),[(e,e,\tau)]$ | $z_1 = -\epsilon\overline{z}_4, z_2 = \epsilon\overline{z}_3$ when $\psi = +1$ | 4 |
| | | 0 when $\psi = +1$ | 0 |
| 28 | $\mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)]$ | $z_1 = \epsilon\overline{z_3}, z_2 = \epsilon z_4$ when $\psi = +1$ | 4 |
| | | 0 when $\psi = -1$ | 0 |
| 29 | $\mathbf{Z}_4[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $z_1 = z_2, z_3 = z_4, z = \overline{z}$ when $\psi = +1$ | 2 |
| | | 0 when $\psi = -1$ | 0 |
| 30 | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)]$ | $z_1 = \epsilon z_4, z_2 = \epsilon z_3, z = \overline{z}$ when $\psi = +1$ | 2 |
| | | 0 when $\psi = -1$ | 0 |
| 31 | $\mathbf{Z}_2^3[(r^2,e,e),(s,v_1,e),(e,e,\tau)]$ | $z_1 = -\epsilon z_4, z_2 = \epsilon z_3, z = \overline{z}$ when $\psi = +1$ | 2 |
| | | 0 when $\psi = -1$ | 0 |
| 32 | $\mathbf{Z}_2^3[(r^2,e,e),(s,v_d,e),(e,e,\tau)]$ | $z_1 = -\epsilon z_4, z_2 = -\epsilon z_3, z = \overline{z}$ when $\psi = +1$ | 2 |
| | | 0 when $\psi = -1$ | 0 |
| 33 | $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)]$ | $z_1 = \epsilon z_3, z_2 = \epsilon z_4, z = \overline{z}$ when $\psi = +1$ | 2 |
| | | 0 when $\psi = -1$ | 0 |
| 34 | $\mathbf{D}_4[(r,e,e),(s,e,e)]$ | $z_1 = z_2 = \epsilon z_3 = \epsilon z_4, z = \overline{z}$ | 1 |
| 35 | $\mathbf{D}_4[(r,e,e),(s,v_d,e)]$ | $z_1 = z_2 = -\epsilon z_3 = -\epsilon z_4, z = \overline{z}$ | 1 |
| 36 | $\mathbf{D}_4[(r,e,e),(s,e,\tau)]$ | $z_1 = z_2 = \epsilon\psi z_3 = \epsilon\psi z_4, z = \overline{z}$ | 1 |
| 37 | $\mathbf{D}_4[(r,e,e),(s,v_d,\tau)]$ | $z_1 = z_2 = -\epsilon\psi z_3 = -\epsilon\psi z_4, z = \overline{z}$ | 1 |
| 38 | $\mathbf{D}_4[(r,e,\tau),(s,e,e)]$ | $z_1 = \psi z_2 = \epsilon\psi z_3 = \epsilon z_4, z = \overline{z}$ | 1 |
| 39 | $\mathbf{D}_4[(r,e,\tau),(s,v_d,e)]$ | $z_1 = \psi z_2 = -\epsilon\psi z_3 = -\epsilon z_4, z = \overline{z}$ | 1 |
| 40 | $\mathbf{D}_4[(r,e,\tau),(s,e,\tau)]$ | $z_1 = \psi z_2 = \epsilon z_3 = \epsilon\psi z_4, z = \overline{z}$ | 1 |
| 41 | $\mathbf{D}_4[(r,e,\tau),(s,v_d,\tau)]$ | $z_1 = \psi z_2 = -\epsilon z_3 = -\epsilon\psi z_4, z = \overline{z}$ | 1 |
| 42 | $\mathbf{D}_4[(r,e,e),(s,e,e)]$ | $z_1 = z_2 = \epsilon z_3 = \epsilon z_4, z = \overline{z}$ when $\psi = +1$ | 1 |
| | $\times\mathbf{Z}_2[(e,e,\tau)]$ | 0 when $\psi = -1$ | 0 |
| 43 | $\mathbf{D}_4[(r,e,e),(s,v_d,e)]$ | $z_1 = z_2 = -\epsilon z_3 = -\epsilon z_4, z = \overline{z}$ when $\psi = +1$ | 1 |
| | $\times\mathbf{Z}_2[(e,e,\tau)]$ | 0 when $\psi = -1$ | 0 |

Table 2.6: Shifted Subgroups and their Fixed Point Subspaces for the Hexagonal Lattice $\mathbf{C}_6$.

| | **Shifted Subgroup** | **Fixed Point Subspace** | dimFix |
|---|---|---|---|
| 1 | $\mathbf{1}$ | $\mathbf{C}^6$ | 12 |
| 2 | $\mathbf{Z}_2[(r^3,e,e)]$ | $z=\overline{z}$ | 6 |
| 3 | $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^6$ when $\psi=+1$ | 12 |
| | | 0 when $\psi=-1$ | 0 |
| 4 | $\mathbf{Z}_2[(r^3,e,\tau)]$ | $z=\overline{z}$ when $\psi=+1$ | 6 |
| | | $Re(z)=0$ when $\psi=-1$ | 6 |
| 5 | $\mathbf{Z}_2[(s,e,e)]$ | $z_1=\epsilon z_6, z_2=\epsilon z_5, z_3=\epsilon z_4$ | 6 |
| 6 | $\mathbf{Z}_2[(sr^3,e,e)]$ | $z_1=\epsilon\overline{z}_6, z_2=\epsilon\overline{z}_5, z_3=\epsilon\overline{z}_4$ | 6 |
| 7 | $\mathbf{Z}_2[(s,e,\tau)]$ | $z_1=\epsilon\psi z_6, z_2=\epsilon\psi z_5, z_3=\epsilon\psi z_4$ | 6 |
| 8 | $\mathbf{Z}_2[(sr^3,e,\tau)]$ | $z_1=\epsilon\psi\overline{z}_6, z_2=\epsilon\psi\overline{z}_5, z_3=\epsilon\psi\overline{z}_4$ | 6 |
| 9 | $\mathbf{Z}_3[(r^2,e,e)]$ | $z_1=z_2=z_3, z_4=z_5=z_6$ | 4 |
| 10 | $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)]$ | $z=\overline{z}$ when $\psi=+1$ | 6 |
| | | 0 when $\psi=-1$ | 0 |
| 11 | $\mathbf{Z}_2^2[(r^3,e,e),(s,e,e)]$ | $z_1=z_2=z_3, z_4=z_5=z_6, z=\overline{z}$ | 2 |
| 12 | $\mathbf{Z}_2^2[(r^3,e,e),(s,e,\tau)]$ | $z_1=\epsilon\psi z_6, z_2=\epsilon\psi z_5, z_3=\epsilon\psi z_4, z=\overline{z}$ | 3 |
| 13 | $\mathbf{Z}_2^2[(r^3,e,\tau),(s,e,e)]$ | $z_1=z_2=z_3, z_4=z_5=z_6, z=\overline{z}$ when $\psi=+1$ | 2 |
| | | $z_1=z_2=z_3, z_4=z_5=z_6, Re(z)=0$ when $\psi=-1$ | 2 |
| 14 | $\mathbf{Z}_2^2[(r^3,e,\tau),(s,e,\tau)]$ | $z_1=\epsilon\psi z_6, z_2=\epsilon\psi z_5, z_3=\epsilon\psi z_4, z=\overline{z}$ when $\psi=+1$ | 3 |
| | | $z_1=\epsilon\psi z_6, z_2=\epsilon\psi z_5, z_3=\epsilon\psi z_4, Re(z)=0$ when $\psi=-1$ | 3 |
| 15 | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $z_1=\epsilon z_6, z_2=\epsilon z_5, z_3=\epsilon z_4$ when $\psi=+1$ | 6 |
| | | 0 when $\psi=-1$ | 0 |
| 16 | $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$ | $z_1=\epsilon\overline{z_6}, z_2=\epsilon\overline{z_5}, z_3=\epsilon\overline{z_4}$ when $\psi=+1$ | 6 |
| | | 0 when $\psi=-1$ | 0 |
| 17 | $\mathbf{Z}_6[(r,e,e)]$ | $z_1=z_2=z_3, z_4=z_5=z_6, z=\overline{z}$ | 2 |
| 18 | $\mathbf{Z}_6[(r,e,\tau)]$ | $z_1=z_2=z_3, z_4=z_5=z_6, z=\overline{z}$ when $\psi=+1$ | 2 |
| | | 0 when $\psi=-1$ | 0 |
| 19 | $\mathbf{Z}_6[(r^2,e,\tau)]$ | $z_1=z_2=z_3, z_4=z_5=z_6$ when $\psi=+1$ | 4 |
| | | 0 when $\psi=-1$ | 0 |
| 20 | $\mathbf{D}_3[(r^2,e,e),(s,e,e)]$ | $z_1=z_2=z_3=\epsilon z_4=\epsilon z_5=\epsilon z_6$ | 2 |
| 21 | $\mathbf{D}_3[(r^2,e,e),(sr,e,e)]$ | $z_1=z_2=z_3=\epsilon\overline{z}_4=\epsilon\overline{z}_5=\epsilon\overline{z}_6$ | 2 |
| 22 | $\mathbf{D}_3[(r^2,e,e),(s,e,\tau)]$ | $z_1=z_2=z_3=\epsilon\psi z_4=\epsilon\psi z_5=\epsilon\psi z_6$ | 2 |
| 23 | $\mathbf{D}_3[(r^2,e,e),(sr,e,\tau)]$ | $z_1=z_2=z_3=\epsilon\psi\overline{z}_4=\epsilon\psi\overline{z}_5=\epsilon\psi\overline{z}_6$ | 2 |
| 24 | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $z_1=\epsilon z_6, z_2=\epsilon z_5, z_3=\epsilon z_4, z=\overline{z}$ | 3 |
| 25 | $\mathbf{Z}_6[(r,e,e)]\times\mathbf{Z}_2[(e,e,\tau)]$ | $z_1=z_2=z_3, z_4=z_5=z_6, z=\overline{z}$ when $\psi=+1$ | 2 |

Table 2.6: Shifted Subgroups and their Fixed Point Subspaces for the Hexagonal Lattice $\mathbf{C}_6$.

| | **Shifted Subgroup** | **Fixed Point Subspace** | dimFix |
|---|---|---|---|
| | | $0$ when $\psi = -1$ | 0 |
| 26 | $\mathbf{D}_6[(r,e,e),(s,e,e)]$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ | 1 |
| 27 | $\mathbf{D}_6[(r,e,e),(s,e,\tau)]$ | $z_1 = z_2 = z_3 = \epsilon\psi z_4 = \epsilon\psi z_5 = \epsilon\psi z_6, z = \overline{z}$ | 1 |
| 28 | $\mathbf{D}_6[(r,e,\tau),(s,e,e)]$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ when $\psi = +1$ | 1 |
| | | $0$ when $\psi = -1$ | 0 |
| 29 | $\mathbf{D}_6[(r,e,\tau),(s,e,\tau)]$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ when $\psi = +1$ | 1 |
| | | $0$ when $\psi = -1$ | 0 |
| 30 | $\mathbf{D}_6[(r^2,e,\tau),(s,e,e)]$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6$ when $\psi = +1$ | 2 |
| | | $0$ when $\psi = -1$ | 0 |
| 31 | $\mathbf{D}_6[(r^2,e,\tau),(sr,e,e)]$ | $z_1 = z_2 = z_3 = \epsilon\overline{z_4} = \epsilon\overline{z_5} = \epsilon\overline{z_6}$ when $\psi = +1$ | 2 |
| | | $0$ when $\psi = -1$ | 0 |
| 32 | $\mathbf{D}_6[(r,e,e),(s,e,e)]$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ when $\psi = +1$ | 1 |
| | $\times\mathbf{Z}_2[(e,e,\tau)]$ | $0$ when $\psi = -1$ | 0 |

## 2.4 Equivalent Cases

It can easily been seen in Tables 2.5 and 2.6 that there are 10 subgroups in the square lattice case with a one dimensional fixed-point subspace and 5 in the hexagonal case. However, while the subgroups may be different, Tables 2.7 and 2.8 show that many of the fixed-point subspaces associated with these subgroups are actually the same. In the square case there are two possible patterns for each of the four representations, while it may at first glance seem as though there are more than this for the two representations where $\psi = -1$ all cases with any combination of two $+1$s and two $-1$s are actually equivalent and will produce translations of the same pattern. In the hexagonal case it is easy to see that there is only one pattern for the two cases where $\psi = +1$ and two patterns for each of the cases where $\psi = -1$.

Table 2.7: Fixed-point Subspaces for the Square Lattice

| **Case** | $Q^{++}$ | $Q^{+-}$ | $Q^{-+}$ | $Q^{--}$ |
|---|---|---|---|---|
| 34 | 1,1,1,1 | 1,1,1,1 | 1,1,-1,-1 | 1,1,-1,-1 |
| 35 | 1,1,-1,-1 | 1,1,-1,-1 | 1,1,1,1 | 1,1,1,1 |
| 36 | 1,1,1,1 | 1,1,-1,-1 | 1,1,-1,-1 | 1,1,1,1 |
| 37 | 1,1,-1,-1 | 1,1,1,1 | 1,1,1,1 | 1,1,-1,-1 |
| 38 | 1,1,1,1 | 1,-1,-1,1 | 1,1,-1,-1 | 1,-1,1,-1 |
| 39 | 1,1,-1,-1 | 1,-1,1,-1 | 1,1,1,1 | 1,-1,-1,1 |
| 40 | 1,1,1,1 | 1,-1,1,-1 | 1,1,-1,-1 | 1,-1,-1,1 |
| 41 | 1,1,-1,-1 | 1,-1,-1,1 | 1,1,1,1 | 1,-1,1,-1 |
| 42 | 1,1,1,1 | | 1,1,-1,-1 | |
| 43 | 1,1,-1,-1 | | 1,1,1,1 | |

## 2.5 Phase Portraits

All that remains is to plot the phase portraits. We will plot a director field that for each point in a square subset of $\mathbf{R}^2$ will draw a unit length line in the direction of the eigenvector associated with the largest eigenvalue of the following real matrix.

$$Q = Q_0 + \delta \sum_{j=1}^{s} z_j e^{2\pi i \mathbf{K}_j \cdot \mathbf{x}} K_j Q^{\pm\pm} (K_j)^{-1} + c.c. \qquad \delta \text{ small}$$

where $Q_0$ is one of the two matrices representing the trivial states, either homeotropic or planar isotropic, and $\delta$ is small since we are looking for patterns close to these two trivial states. Given that we are plotting lines of unit length in $\mathbf{R}^3$ on a 2 dimensional plane those directors that lie flat within the $xy$ plane will appear to be of unit length, those directors that in any way point out of the plane will appear proportionally shorter according to how much they deviate from the $xy$ plane and vertical lines will appear simply as

Table 2.8: Fixed-point Subspaces for the Hexagonal Lattice

| **Case** | $Q^{++}$ | $Q^{+-}$ | $Q^{-+}$ | $Q^{--}$ |
|---|---|---|---|---|
| 26 | 1,1,1,1,1,1 | 1,1,1,1,1,1 | 1,1,1,-1,-1,-1 | 1,1,1,-1,-1,-1 |
| 27 | 1,1,1,1,1,1 | 1,1,1,-1,-1,-1 | 1,1,1,-1,-1,-1 | 1,1,1,1,1,1 |
| 28 | 1,1,1,1,1,1 | | 1,1,1,-1,-1,-1 | |
| 29 | 1,1,1,1,1,1 | | 1,1,1,-1,-1,-1 | |
| 32 | 1,1,1,1,1,1 | | 1,1,1,-1,-1,-1 | |

points. With this in mind it is immediately obvious that we will not get any pictures for the two cases where $\psi = +1$ bifurcating from the homeotropic state since all the lines are pointing straight up to start with and as $Q^{\epsilon+}$ is invariant under conjugation by $\tau$ they will stay pointing straight up, any deviation from the vertical is only possible when $\psi = -1$. We will however get pictures for these two cases bifurcating from the isotropic state since all the molecules will stay flat within the $xy$ plane. It is important to note that not all points are necessarily vertical lines, a point can also mean that there is a double maximum eigenvalue and therefore no definite direction.

Figure 2.1: Square Lattice Patterns for $\epsilon = +1$; patterns shown when $\alpha = 2$ and $\beta = 1$

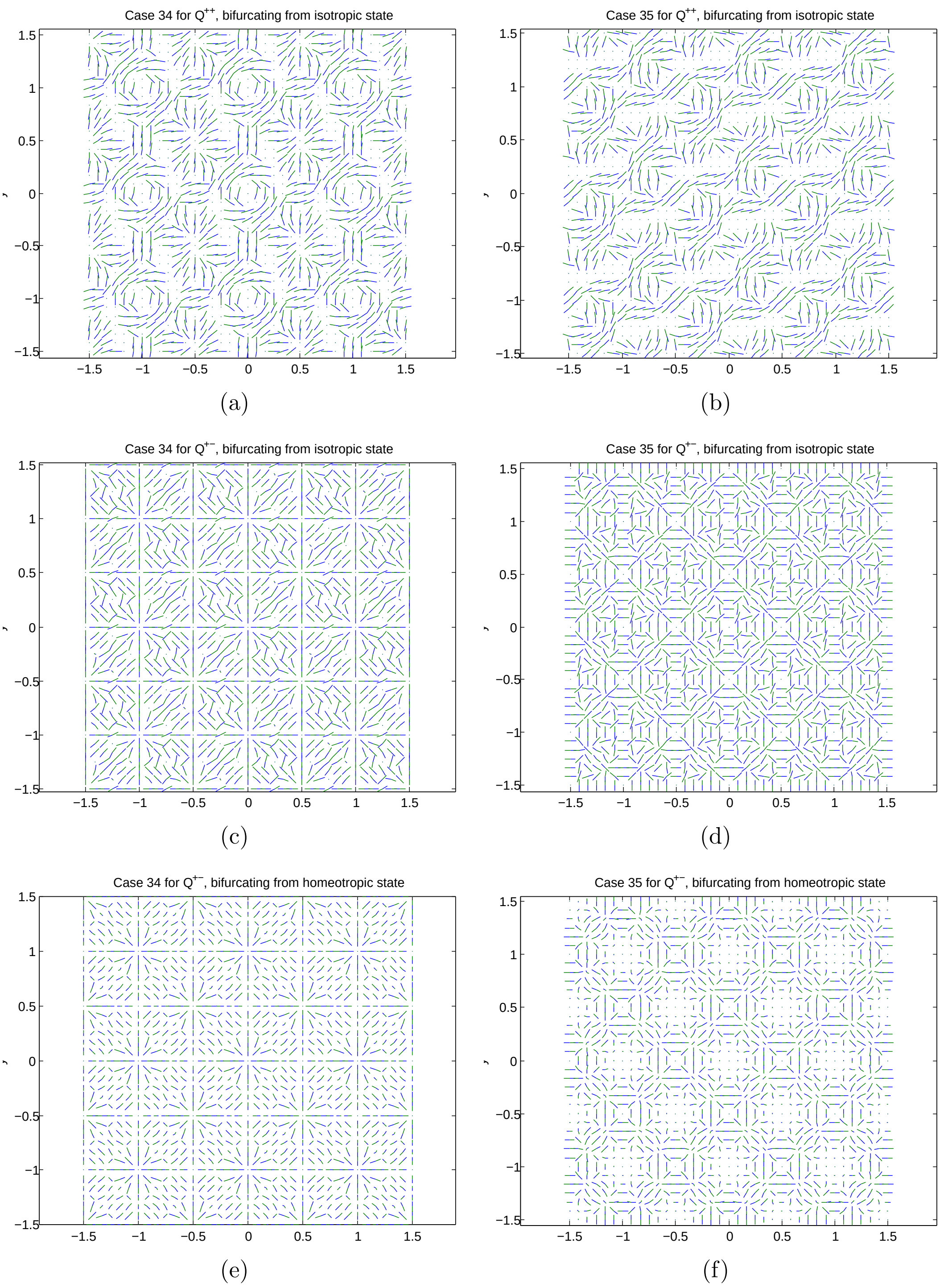
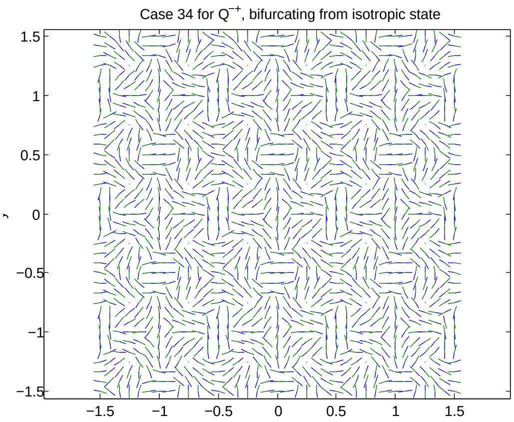
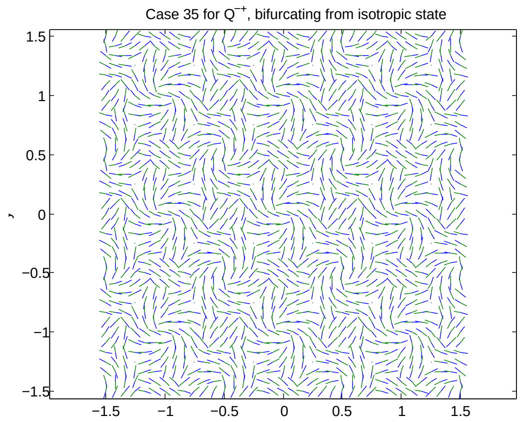


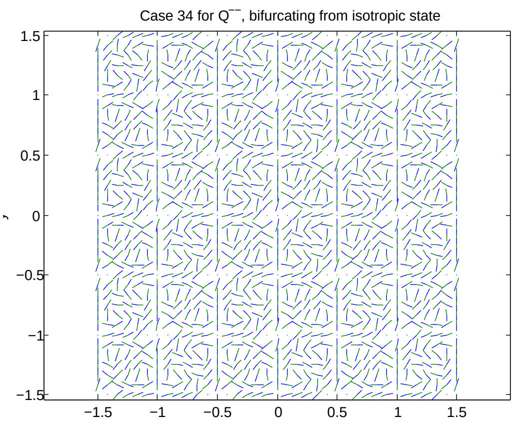
Figure 2.2: Square Lattice Patterns for $\epsilon = -1$; patterns shown when $\alpha = 2$ and $\beta = 1$



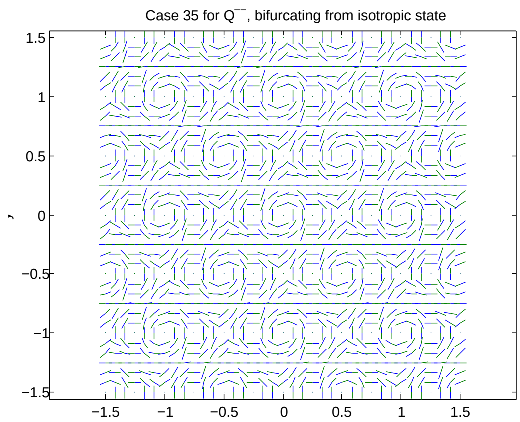

(a)



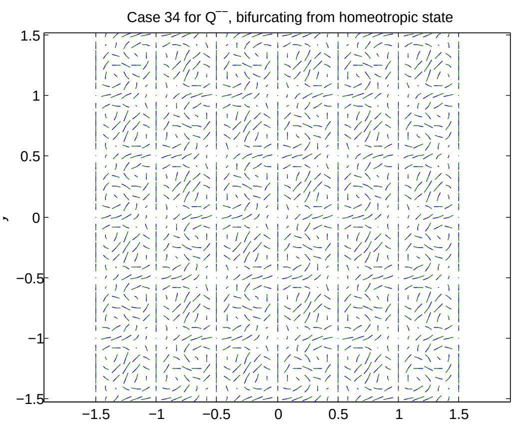

(b)



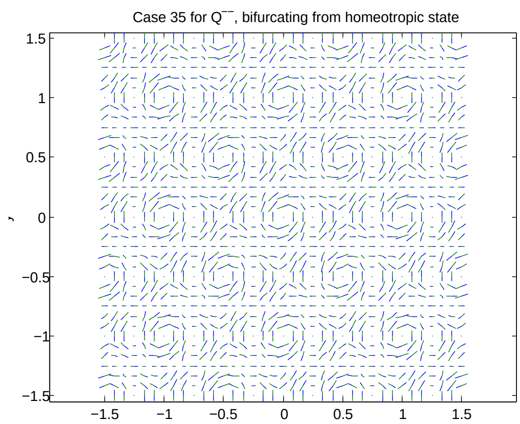

(c)




(d)




(e)




(f)

Figure 2.3: Hexagonal Lattice Patterns for $\epsilon = +1$; patterns shown when $\alpha = 3$ and $\beta = 2$

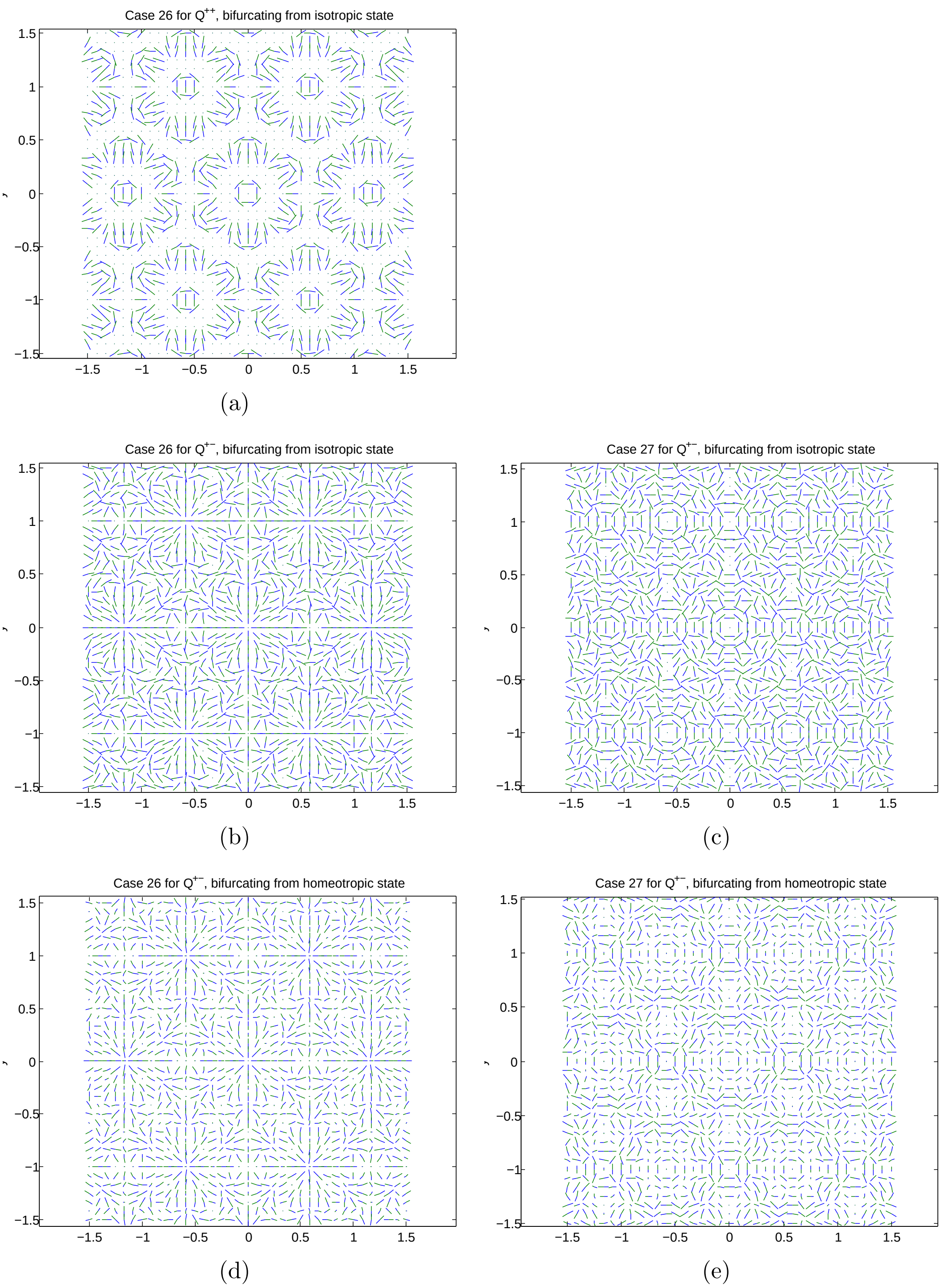
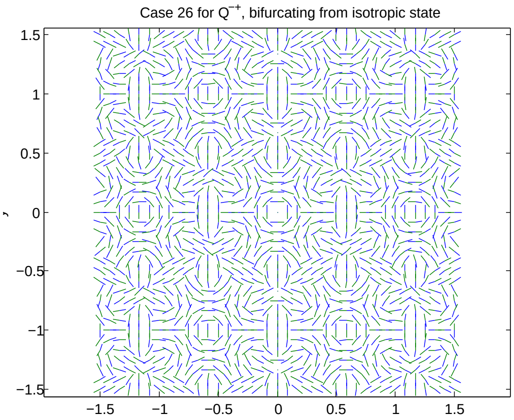
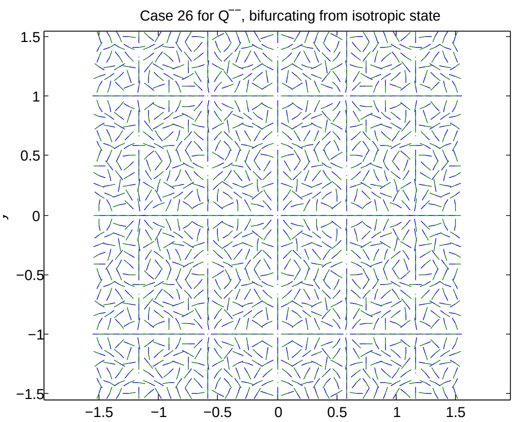


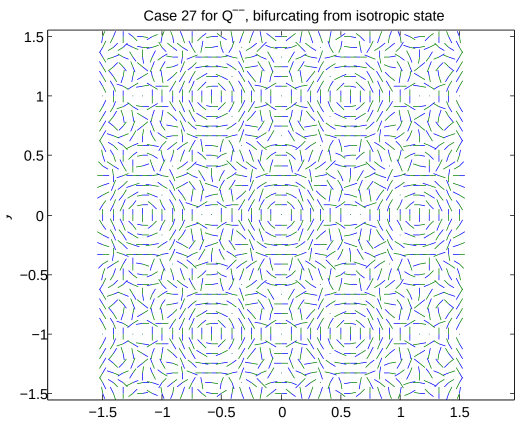
Figure 2.4: Hexagonal Lattice Patterns for $\epsilon = -1$; patterns shown when $\alpha = 3$ and $\beta = 2$



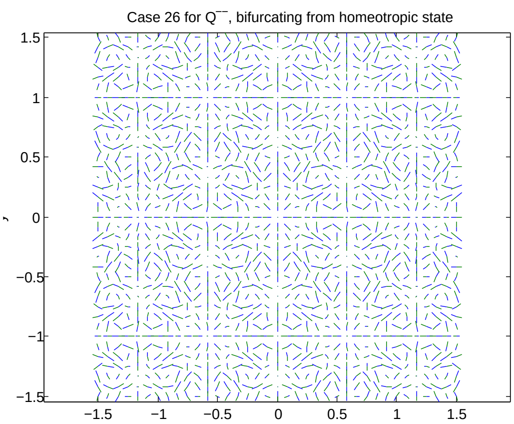

(a)



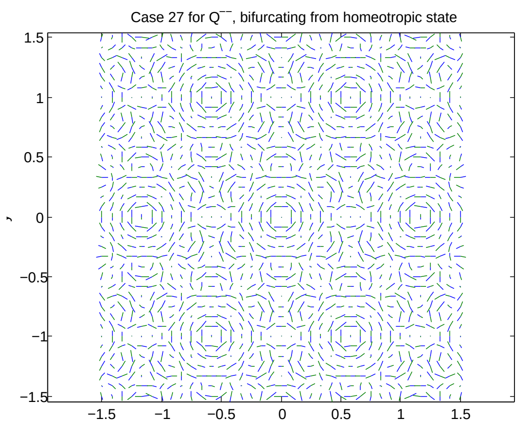

(b)




(c)




(d)




(e)

# 3

# Hopf Bifurcation and Group Theory Methods

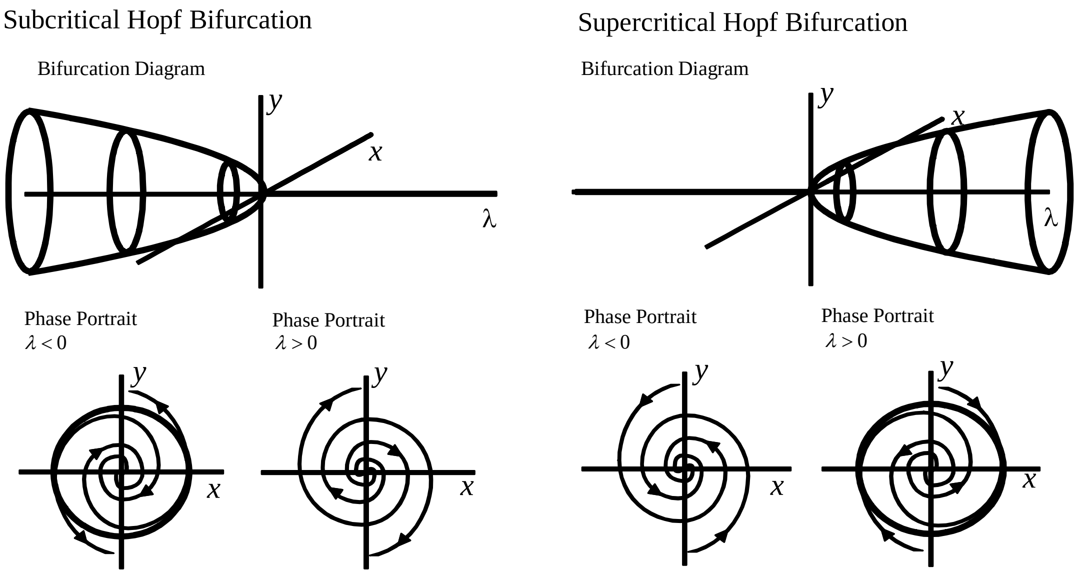
In Chapters 3 and 4 we will look for patterns that are not only spatially doubly periodic but also time periodic. Again we start with our system of differential equations:

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{F}(\mathbf{Q}, \lambda) \tag{3.1}$$

A solution of (3.1) is periodic of period $p$ if there exists $p > 0$ such that $\mathbf{Q}(t, \lambda_0) = \mathbf{Q}(t + p, \lambda_0)$ for all $t \in \mathbf{R}$ [67].

A *Hopf Bifurcation* is a bifurcation where a family of periodic orbits bifurcates from a path of equilibria. A Hopf bifurcation is called *subcritical* when an unstable periodic orbit shrinks toward a stable fixed-point at the origin and vanishes with the origin becoming unstable as $\lambda$ passes through zero, and *supercritical* when the origin is a stable fixed-point that becomes unstable and throws off a stable periodic orbit as $\lambda$ passes through zero [42], see Figure 3.1. The Hopf bifurcation is explained in more detail over the next few pages.

Figure 3.1: Hopf Bifurcation Diagrams




In a similar way to finding the equilibrium solutions by finding isotropy subgroups with fixed-point subspaces of dimension one, when looking for Hopf bifurcations the *Equivariant Hopf Theorem* tells us that generically we will find a periodic solution with symmetry group $\Sigma$ whenever $\dim \mathrm{Fix}\, \Sigma = 2$. However, in this case rather than looking for subgroups of $\Gamma$ we need to find isotropy subgroups of the group $\Gamma \times \mathbf{S}^1$ where the $\mathbf{S}^1$ accounts for the time periodicity.

## 3.1 The Equivariant Hopf Theorem

The following explanation is in terms of our particular model; for a more general version see [32]

### 3.1.1 The Hopf Bifurcation Theorem

Consider our system of differential equations (3) $\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{F}(\mathbf{Q}, \lambda)$
where $\mathbf{F} : \mathcal{Q} \times \mathbf{R} \to \mathcal{Q}$ is $C^\infty$ and $\lambda$ is the bifurcation parameter.
Suppose that:

$$\mathbf{F}(Q_0, \lambda) \equiv 0$$

so $\mathbf{Q} = Q_0$ is a steady state solution for all $\lambda$

A one-parameter family of periodic solutions to (3.1) emanating from $(\mathbf{Q}, \lambda) = (Q_0, 0)$ can be found if two hypotheses on $\mathbf{F}$ are satisfied.

Let $\mathsf{L}_\lambda = (d\mathbf{F})_{(Q_0,\lambda)}$ be the linearization of $\mathbf{F}$ along steady state solutions.

**First Hopf Assumption**

$\mathsf{L}_0$ has simple eigenvalues $\pm i$
$\mathsf{L}_0$ has no other eigenvalues lying on the imaginary axis

$\mathsf{L}_\lambda$ has simple eigenvalues of the form $\sigma(\lambda) \pm i\omega(\lambda)$, where $\sigma(0) = 0$, $\omega(0) = 1$, and $\sigma$ and $\omega$ are smooth.

**Second Hopf Assumption**

We assume that $\sigma'(0) \neq 0$, that is the imaginary eigenvalues of $\mathsf{L}_\lambda$ cross the imaginary axis with non-zero speed as $\lambda$ crosses zero.

**Theorem 3.1.1. (The Hopf Bifurcation Theorem)** *[48] [2] [36]*
*If the first and second Hopf assumptions both hold, then there is a one-parameter family of periodic solutions to (3.1) bifurcating from $Q_0$.*

For a translation of Hopf's 1943 paper [36] see [45].

### 3.1.2 The Equivariant Hopf Theorem

The Equivariant Hopf Theorem enables us to find periodic solutions with symmetry. First we introduce the idea of a space being $\Gamma$-simple.

**Definition** The vector space $W$ is $\Gamma$-*simple* if either:
1) $W = V \oplus V$ where $V$ is an absolutely irreducible representation of $\Gamma$, or
2) $G$ acts irreducibly but not absolutely irreducibly on $W$ [30].

The $\Gamma$-simple representation for Hopf bifurcation is the equivalent of the absolutely irreducible representation for steady-state bifurcation. This leads to the following lemma:

**Lemma 3.1.2.** *Generically a Hopf bifurcation is supported by a finite-dimensional $\Gamma$-simple representation. See [30].*

In a Hopf bifurcation with no symmetry a pair of simple eigenvalues $\pm\omega i$ cross the imaginary axis at the bifurcation point. In a Hopf bifurcation with symmetry we expect these eigenvalues to each have multiplicity $m$, so the $\pm\omega i$ eigenspace has dimension $2m$. Again, without a lattice the eigenspace would be infinite dimensional, hence we restrict ourselves to a lattice to ensure a finite dimensional eigenspace.

As with the steady-state bifurcation we can reduce the original system of PDEs $\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{F}(\mathbf{Q}, \lambda)$ to a system of ODEs $\frac{d\mathbf{Q}}{dt} = f(\mathbf{Q}_1, \lambda)$. In this case $\mathbf{Q}_1 \in \mathcal{Q}_1$ where $\mathcal{Q}_1$ is the $\pm\omega i$ eigenspace of $\mathsf{L}_\lambda = (d\mathbf{F})_{(Q_0,\lambda)}$. This is different from the steady-state case where $\mathbf{Q}_1 \in \ker \mathsf{L}$.

**Theorem 3.1.3. (Equivariant Hopf Theorem)**
*Let $\Gamma$ be a compact Lie group acting $\Gamma$-simply on $\mathcal{Q}_1$*
*Assume that $f : \mathcal{Q}_1 \times \mathbf{R} \to \mathcal{Q}_1$ is $\Gamma$-equivariant. Then $f(Q_0, \lambda) = 0$ and there exist real functions $\sigma(\lambda)$ and $\omega(\lambda)$ such that the eigenvalues of $df|_{(Q_0,0)}$ are $\sigma(\lambda) \pm i\omega(\lambda)$ each of multiplicity $m$ and, after time rescaling, $\omega(0) = 1$. We also assume that there is a non-degenerate bifurcation at $\lambda = 0$, meaning*

$\sigma(0) = 0$ *and* $\sigma\prime(0) \neq 0$ .
*Then there exist branches of periodic solutions of period close to* $2\pi$ *having isotropy subgroups* $\Sigma \subset \Gamma \times S^1$ *whenever* $\mathrm{dimFix}(\Sigma) = 2$ .

For proof of the Equivariant Hopf Theorem see [30, p91].

In our case we are dealing with the group $\Gamma = (H \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$ acting on the space $V$ , where $V$ is the real part of the eigenspace corresponding to the eigenvalue $i$ , so for a Hopf bifurcation we look at the group $\Gamma \times \mathbf{S}^1$ acting on the space $V \oplus V = \mathcal{Q}_1$ . This action can be more easily imagined if it is described as the action of $\Gamma \times \mathbf{S}^1$ on $V \otimes \mathbf{C}$ defined by

$$(\gamma, \theta)(v \otimes z) = (\gamma v) \otimes (e^{-i\theta} z)$$

where $v \in V, z \in \mathbf{C}, \gamma \in \Gamma, \theta \in \mathbf{S}^1$ [32]. This eigenspace is generated by expressions of the form

$$\tilde{Q} = \sum_{j=1}^{s} z_j e^{2\pi i(\mathbf{K}_j\cdot\mathbf{x}+t)} Q_j + w_j e^{2\pi i(-\mathbf{K}_j\cdot\mathbf{x}+t)} Q_j + c.c.$$

where the $\mathbf{K}_j$ s are given in Table 1.2. Here $Q_j = K_j Q^{\pm\pm}(K_j)^{-1}$ , $K_j$ is the matrix giving the rotations through $\phi$ in the $xy$ plane and $\phi$ is the angle $\mathbf{K}_j$ makes with the positive $x$ axis.

## 3.2 Wave Pairs and Trace Formulae

In Chapter 2 we managed to find the axial isotropy subgroups of $\Gamma$ by inspection, simply listing all the possible subgroups by conjugacy class and checking which had $\mathrm{dimFix}(\Sigma) = 1$ . See Section 1.2.1 for a reminder of the group theory definitions. Finding the isotropy subgroups of $\Gamma \times \mathbf{S}^1$ with $\mathrm{dimFix}(\Sigma) = 2$ by inspection would be an arduous task and would most likely result in several cases being overlooked. However, there is a group theory method we can use to make the problem much more manageable.

This method is taken from the paper by Dionne et al [21], though the actual calculations will differ from those in that paper given that $\Gamma$ in our case has an extra copy of $\mathbf{Z}_2$.

### 3.2.1 Wave Pair

The symmetries of a time-periodic solution in $\chi_{\mathcal{L}}$ are described by a pair of subgroups $K \subset G$ of $\Gamma$, where the elements of $G$ map the periodic trajectory in phase space onto itself and the elements of $K$ fix the periodic trajectory pointwise [21]. The Subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ can be identified with this pair of subgroups as follows.

- $K = \Sigma \cap \Gamma$
- $G = \pi_\Gamma(\Sigma)$ where $\pi_\Gamma : \Gamma \times \mathbf{S}^1 \to \Gamma$ is the projection.

$\Sigma$ has the form of a *twisted subgroup* [32] $G^\Theta$ meaning, since $\mathbf{S}^1$ acts irreducibly on $\mathbf{C}^n$, there exists a unique homomorphism $\Theta : G \to \mathbf{S}^1$ such that

$$\Sigma = G^\Theta \equiv \{(g, \Theta(G)) \in \Gamma \times \mathbf{S}^1 \mid g \in G\}$$

which gives us

- $K = \ker(\Theta)$
- $\Theta(G) \cong G/K$.

We call $(G, K)$ in each case a *wave pair* [21] and by finding these wave pairs we can find the relevant isotropy subgroups. To summarize:

**Definition** The *normalizer* of a subgroup $H$ in $G$, written $N_G(H)$ or just $N(H)$ if $G$ is assumed, is defined as

$$N_G(H) = \{g \in G | g^{-1}Hg = H\}$$

**Definition** The pair of subgroups $(G, K)$ of $\Gamma$ forms a *wave pair* if

- $K$ is a shifted subgroup of $\Gamma$
- $K \subset G$ and $G/K$ is a Lie subgroup of $\mathbf{S}^1$
- $G/K$ is a maximal Abelian subgroup of $N_\Gamma(K)/K$ [28]

where $N_\Gamma(K) = \{\gamma \in \Gamma \mid \gamma K \gamma^{-1} = K\}$ is the *Normalizer* of $K$ in $\Gamma$

If $G/K$ is cyclic the corresponding solution is called a *discrete* wave, often, though not always, producing a standing wave pattern. If $G/K \cong \mathbf{S}^1$ the solution is called a rotating wave: these solutions correspond to travelling waves in $\mathbf{R}^2$ and rotating waves in $\mathbf{T}^2$ [21].

### 3.2.2 Trace Formulae

We use these wave pairs to find where $\mathrm{dimFix}\,\Sigma = 2$ by applying the trace formula.

**Theorem 3.2.1.** *Let $G$ be a compact Lie group acting on $V$ and let $\Sigma \in G$ be a Lie subgroup. Then*

$$\mathrm{dimFix}(\Sigma) = \int_\Sigma tr(\sigma), \sigma \in \Sigma$$

*where $\int$ denotes the normalized Haar integral on $\Sigma$ and $tr(\sigma)$ is the trace of $\rho_\sigma$ where $\rho$ is the linear mapping $\rho_g : x \to gx$ [32].*

From Theorem 3.2.1 the list of individual formulae shown in Table 3.1 can be derived, [32]. We can apply these formulae to the dimension of the fixed-point subspaces of the wave pairs to calculate $\mathrm{dimFix}(\Sigma)$.

## 3.3 Extra Lattice Notation

We introduce some extra lattice notation that will simplify our efforts to describe the normalizers of the shifted subgroups.

Table 3.1: Trace Formulae

| $G/K$ | $\dim\mathrm{Fix}(G\Theta)$ |
|---|---|
| 1 | $2\dim\mathrm{Fix}(G)$ |
| $\mathbf{Z}_2$ | $2(\dim\mathrm{Fix}(K) - \dim\mathrm{Fix}(G))$ |
| $\mathbf{Z}_3$ | $\dim\mathrm{Fix}(K) - \dim\mathrm{Fix}(G)$ |
| $\mathbf{Z}_4$ | $\dim\mathrm{Fix}(K) - \dim\mathrm{Fix}(M)$ |
| | where $K \subset M \subset G$ and $\lvert G/M\rvert = 2$ |
| $\mathbf{Z}_6$ | $\dim\mathrm{Fix}(K) - \dim\mathrm{Fix}(M) - \dim\mathrm{Fix}(L) + \dim\mathrm{Fix}(G)$ |
| | where $K \subset M \subset G, \lvert G/M\rvert = 2, K \subset L \subset G$ and $\lvert G/L\rvert = 3$ |

For the hexagonal lattice we will use the vector $v_t = \frac{1}{3}l_1 + \frac{1}{3}l_2$ that generates $\mathbf{Z}_3$ [21].

Other notation we will use, also from [21] is as follows. If $g$ is a reflection belonging to the holohedry $H$, then the eigenvalues of $\rho(g)$ given by the natural representation are $+1$ and $-1$. We define two circles in $\mathbf{T}^2$

$$E^+(g) = \text{the projection of the } +1 \text{ eigenspace into } \mathbf{T}^2$$

$$E^-(g) = \text{the projection of the } -1 \text{ eigenspace into } \mathbf{T}^2.$$

For each vector $w \in \mathbf{R}^2$ we can write $w = w^+ + w^-$ where $w^+ \in E^+(g)$ and $w^- \in E^-(g)$.
Next we define subsets of $\mathbf{T}^2$ by

$$F^+(g) = \{v \in \mathbf{T}^2 | gv = v\}$$

$$F^-(g) = \{v \in \mathbf{T}^2 | gv = -v\}.$$

Calculating these subsets for the square lattice gives us:
When $g = s$, the reflection in the $x$ axis,

$$E^+(s) = \{\alpha l_1 | \alpha \in \mathbf{R}\}$$

$$E^-(s) = \{\alpha l_2 | \alpha \in \mathbf{R}\}$$

and

$$F^+(s) = \{(x, y) \in \mathbf{T}^2 | 0 \leq x < 1, y = 0 \text{ or } y = \frac{1}{2}\} = E^+(s) \oplus \mathbf{Z}_2[v_2]$$

$$F^-(s) = \{(x, y) \in \mathbf{T}^2 | 0 \leq y < 1, x = 0 \text{ or } x = \frac{1}{2}\} = E^-(s) \oplus \mathbf{Z}_2[v_1].$$

When $g = rs$, the reflection in the diagonal line $x = y$,

$$E^+(sr) = \{\alpha(l_1 + l_2) | \alpha \in \mathbf{R}\} = F^+(sr)$$

$$E^-(sr) = \{\alpha(l_1 - l_2) | \alpha \in \mathbf{R}\} = F^-(sr).$$

Next we consider the same for the hexagonal lattice, where $g = s$, the reflection in the $x$ axis.

$$E^+(s) = \{\alpha l_2 | \alpha \in \mathbf{R}\} = F^+(s)$$

$$E^-(s) = \{-2\alpha l_1 + \alpha l_2 | \alpha \in \mathbf{R}\} = F^+(s).$$

## 3.4 The Method

We now have all the tools we need to find those isotropy subgroups that will give a Hopf bifurcation.

**Step 1** List all shifted subgroups $K \in \Gamma$ up to conjugacy in $\Gamma$ and find their normalizers.

**Step 2** Find all subgroups $G \in \Gamma$ such that subgroups $K$ and $G$ form a wave pair in $\Gamma$.

**Step 3** For each translation-free irreducible representation of $\Gamma$, determine those wave pairs that correspond to twisted subgroups $G^{\Theta}$ such that $\dim \mathrm{Fix}(G^{\Theta}) = 2$. Recall from Chapter 1 that if the representation is not translation free we can use a smaller lattice.

**Step 4** For each wave pair find the action of the generators of $\Sigma$ and its fixed-point subspace.

**Step 5** Plot the phase portraits. This will be done in Matlab.

# 4

# Periodic Solutions

## 4.1 Shifted Subgroups and their Normalizers

We have already found all the shifted subgroups of $\Gamma_{\mathcal{L}} = (H_{\mathcal{L}} \ltimes \mathbf{T}^2) \times \mathbf{Z}_2$ by conjugacy class in Chapter 1; these results are shown in tables 2.1 and 2.2. Recall from Section 3.3 we will be using some extra lattice notation to describe the normalizers. It is helpful to calculate the action of $v_t$, shown in Table 4.1, note that the action of $v_t$ commutes with every other element in $\mathbf{C}^3$ but in $\mathbf{C}^6$ it commutes only with the other translation elements $v_1$ and $v_2$.

Table 4.1: The Action of $(e, v_t, e)$

| $g$ | **Action on $\mathbf{C}^3$**<br>$g(z_1, z_2, z_3)$ | **Action on $\mathbf{C}^6$**<br>$g(z_1, z_2, z_3, z_4, z_5, z_6)$ |
|---|---|---|
| $(e, v_t, e)$ | $e^{(2\pi i)/3}(z_1, z_2, z_3)$ | $e^{(2\pi i)/3}(e^{-(\alpha+\beta)}z_1, e^{(2\alpha-\beta)}z_2, e^{(-\alpha+2\beta)}z_3,$<br>$e^{(-2\alpha+\beta)}z_4, e^{(\alpha+\beta)}z_5, e^{(\alpha-2\beta)}z_6)$ |

From here it is easy to find the normalizers for them by referring to Dionne *et al*, [21]. The normalizers in $\mathbf{H}_{\mathcal{L}} \times \mathbf{T}^2$ of cases 1,2,5,6,7,11,13,14,15,22,34 and 35 for the square lattice and cases 1,2,5,6,9,11,17,20,21 and 26 for the hexagonal lattice are shown in [21]. Since the only difference between the Dionne case and our case is the extra copy of $\mathbf{Z}_2$ generated by $\tau$, and since $\tau$ commutes with every other element of the group, it is obvious that the normalizer in $\Gamma_{\mathcal{L}}$ for each of these cases will be the direct product of the equivalent normalizer in the Dionne case and $\mathbf{Z}_2[\tau]$. Also, the normalizer for any case involving $\tau$ will be the same as the normalizer for the case acquired by simply ignoring any occurrences of $\tau$. The shifted subgroups and their normalizers are shown in Tables 4.2 and 4.3.

Table 4.2: Shifted Subgroups and their Normalizers in $\Gamma$ for the Square Lattice.

| | **Shifted Subgroup** | **Normalizer** |
|---|---|---|
| 1 | $\mathbf{1}$ | $\Gamma$ |
| 2 | $\mathbf{Z}_2[(r^2,e,e)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 3 | $\mathbf{Z}_2[(e,e,\tau)]$ | $\Gamma$ |
| 4 | $\mathbf{Z}_2[(r^2,e,\tau)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 5 | $\mathbf{Z}_2[(s,e,e)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \ltimes F^+(s,e,e)$ |
| 6 | $\mathbf{Z}_2[(s,v_1,e)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \ltimes F^+(s,e,e)$ |
| 7 | $\mathbf{Z}_2[(sr,e,e)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)] \ltimes E^+(sr,e,e)$ |
| 8 | $\mathbf{Z}_2[(s,e,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \ltimes F^+(s,e,e)$ |
| 9 | $\mathbf{Z}_2[(s,v_1,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \ltimes F^+(s,e,e)$ |
| 10 | $\mathbf{Z}_2[(sr,e,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)] \ltimes E^+(sr,e,e)$ |
| 11 | $\mathbf{Z}_4[(r,e,e)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 12 | $\mathbf{Z}_4[(r,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 13 | $\mathbf{Z}_2^2[(r^2,e,e),(s,e,e)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 14 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_1,e)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \times \frac{1}{2}\mathcal{L}$ |
| 15 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_d,e)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 16 | $\mathbf{Z}_2^2[(r^2,e,e),(s,e,\tau)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 17 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_1,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \times \frac{1}{2}\mathcal{L}$ |
| 18 | $\mathbf{Z}_2^2[(r^2,e,e),(s,v_d,\tau)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 19 | $\mathbf{Z}_2^2[(r^2,e,\tau),(s,e,e)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 20 | $\mathbf{Z}_2^2[(r^2,e,\tau),(s,v_1,e)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \times \frac{1}{2}\mathcal{L}$ |
| 21 | $\mathbf{Z}_2^2[(r^2,e,\tau),(s,v_d,e)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 22 | $\mathbf{Z}_2^2[(r^2,e,e),(sr,e,e)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 23 | $\mathbf{Z}_2^2[(r^2,e,e),(sr,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 24 | $\mathbf{Z}_2^2[(r^2,e,\tau),(sr,e,e)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 25 | $\mathbf{Z}_2^2[(r^2,e,e),(e,e,\tau)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 26 | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \ltimes F^+(s,e,e)$ |
| 27 | $\mathbf{Z}_2^2[(s,v_1,e),[(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \ltimes F^+(s,e,e)$ |
| 28 | $\mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)] \ltimes E^+(sr,e,e)$ |
| 29 | $\mathbf{Z}_4[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 30 | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 31 | $\mathbf{Z}_2^3[(r^2,e,e),(s,v_1,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)] \times \frac{1}{2}\mathcal{L}$ |
| 32 | $\mathbf{Z}_2^3[(r^2,e,e),(s,v_d,e),(e,e,\tau)]$ | $(\mathbf{D}_4[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 33 | $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 34 | $\mathbf{D}_4[(r,e,e),(s,e,e)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |

Table 4.2: Shifted Subgroups and their Normalizers in $\Gamma$ for the Square Lattice.

| | **Shifted Subgroup** | **Normalizer** |
|---|---|---|
| 35 | $\mathbf{D}_4[(r,e,e),(s,v_d,e)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 36 | $\mathbf{D}_4[(r,e,e),(s,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 37 | $\mathbf{D}_4[(r,e,e),(s,v_d,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 38 | $\mathbf{D}_4[(r,e,\tau),(s,e,e)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 39 | $\mathbf{D}_4[(r,e,\tau),(s,v_d,e)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 40 | $\mathbf{D}_4[(r,e,\tau),(s,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 41 | $\mathbf{D}_4[(r,e,\tau),(s,v_d,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 42 | $\mathbf{D}_4[(r,e,e),(s,e,e)]\times$ $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 43 | $\mathbf{D}_4[(r,e,e),(s,v_d,e)]\times$ $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |

Table 4.3: Shifted Subgroups and their Normalizers in Γ for the Hexagonal Lattice.

| | **Shifted Subgroup** | **Normalizer** |
|---|---|---|
| 1 | $\mathbf{1}$ | $\Gamma$ |
| 2 | $\mathbf{Z}_2[(r^3,e,e)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 3 | $\mathbf{Z}_2[(e,e,\tau)]$ | $\Gamma$ |
| 4 | $\mathbf{Z}_2[(r^3,e,\tau)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 5 | $\mathbf{Z}_2[(s,e,e)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)] \ltimes E^+(s,e,e)$ |
| 6 | $\mathbf{Z}_2[(sr^3,e,e)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)] \ltimes E^-(s,e,e)$ |
| 7 | $\mathbf{Z}_2[(s,e,\tau)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)] \ltimes E^+(s,e,e)$ |
| 8 | $\mathbf{Z}_2[(sr^3,e,\tau)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)] \ltimes E^-(s,e,e)$ |
| 9 | $\mathbf{Z}_3[(r^2,e,e)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 10 | $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 11 | $\mathbf{Z}_2^2[(r^3,e,e),(s,e,e)]$ | $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ |
| 12 | $\mathbf{Z}_2^2[(r^3,e,e),(s,e,\tau)]$ | $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ |
| 13 | $\mathbf{Z}_2^2[(r^3,e,\tau),(s,e,e)]$ | $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ |
| 14 | $\mathbf{Z}_2^2[(r^3,e,\tau),(s,e,\tau)]$ | $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ |
| 15 | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)] \ltimes E^+(s,e,e)$ |
| 16 | $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)] \ltimes E^-(s,e,e)$ |
| 17 | $\mathbf{Z}_6[(r,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 18 | $\mathbf{Z}_6[(r,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 19 | $\mathbf{Z}_6[(r^2,e,\tau)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 20 | $\mathbf{D}_3[(r^2,e,e),(s,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 21 | $\mathbf{D}_3[(r^2,e,e),(sr,e,e)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 22 | $\mathbf{D}_3[(r^2,e,e),(s,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 23 | $\mathbf{D}_3[(r^2,e,e),(sr,e,\tau)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 24 | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ |
| 25 | $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 26 | $\mathbf{D}_6[(r,e,e),(s,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 27 | $\mathbf{D}_6[(r,e,e),(s,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 28 | $\mathbf{D}_6[(r,e,\tau),(s,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 29 | $\mathbf{D}_6[(r,e,\tau),(s,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 30 | $\mathbf{D}_6[(r^2,e,\tau),(s,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 31 | $\mathbf{D}_6[(r^2,e,\tau),(sr,e,e)]$ | $(\mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 32 | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ |

## 4.2 Wave Pairs

In order for a wave pair to exist in each case we need to find a G that satisfies the conditions described in 3.2 for each shifted subgroup K. We do this by looking for maximal abelian subgroups of $N(K)/K$ that are either cyclic or isomorphic to $S^1$. In some cases there is more than one possibility for $N(K)/K$, but since they all result in the same wave pair only one is shown.

**Rule 1** Since $\tau$ commutes with every other element of $\Gamma$, any case where $\mathbf{Z}_2[(e,e,\tau)] \subsetneq N(K)/K$ will not produce any wave pairs because for every other abelian subgroup $H \neq \mathbf{Z}[(e,e,\tau)]$, $H \subset H \times \mathbf{Z}_2[(e,e,\tau)]$ which will also be abelian and therefore $H$ cannot be maximal abelian.

### 4.2.1 Square Lattice Wave Pairs

The quotient group $\mathbf{D}_4[r,s]/\mathbf{Z}_2[r^2]$ is not a subgroup of $\mathbf{D}_4[r,s]$, we introduce the following notation to allow for this.
$H_{41} = \mathbf{D}_4[r,s]/\mathbf{Z}_2[r^2] = \{\{e,r^2\},\{r,r^3\},\{s,sr^2\},\{sr,sr^3\}\} \cong \mathbf{Z}_2^2$
$H_{42} = \mathbf{D}_4[r,s]/\mathbf{Z}_2^2[r^2,s] = \{\{e,r^2,s,sr^2\},\{r,r^3,sr,sr^3\}\} \cong \mathbf{Z}_2$
$H_{43} = \mathbf{D}_4[r,s]/\mathbf{Z}_2^2[r^2,sr] = \{\{e,r^2,sr,sr^3\},\{r,r^3,sr^2,s\}\} \cong \mathbf{Z}_2$

**Cases without wave pairs** The following cases do not have corresponding wave pairs because of Rule 1 above.

| Case | $N(K)/K$ |
|---|---|
| 1 | $\Gamma$ |
| 2,4 | $(H_{41} \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 5,6,8,9 | $\mathbf{Z}_2^2[(r^2,e,e),(e,e,\tau)] \ltimes F^+(s,e,e)$ |
| 7,10 | $\mathbf{Z}_2^2[(r^2,e,e),(e,e,\tau)] \ltimes E^+(sr,e,e)$ |
| 11,12 | $\mathbf{Z}_2^3[(s,e,e),(e,v_d,e),(e,e,\tau)]$ |
| 13,15,16,18,19,21 | $(H_{42} \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 14,17,20 | $\frac{1}{2}\mathcal{L} \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 22,23,24 | $H_3 \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |
| 34,35,36,37,38,39,40,41 | $\mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ |

**Case 3**

$N(K)/K = \mathbf{D}_4 \ltimes \mathbf{T}^2$

The element $(e, v_d, e)$ in $\mathbf{T}^2$ in commutes with everything so for any abelian cyclic subgroup $H \neq \mathbf{Z}[(e, v_d, e)]$, $H \subset H \times \mathbf{Z}_2[(e, v_d, e)]$ which will also be abelian and therefore $H$ cannot be maximal abelian, in the same way as rule 1 works for the element $(e, e, \tau)$.

**Case 25**

$N(K)/K = H_{41} \ltimes \frac{1}{2}\mathcal{L}$

$H_{41} \cong \mathbf{Z}_2^2$ and $\frac{1}{2}\mathcal{L}$ are both maximal abelian but not cyclic therefore there are not wave pairs for this case.

**Cases 26 and 27**

$N(K)/K = \mathbf{Z}_2[(r^2, e, e)] \ltimes F^+(s, e, e) = \mathbf{Z}_2[(r^2, e, e)] \ltimes (E^+(s, e, e) \times \mathbf{Z}_2[(e, v_2, e)])$

$(e, v_2, e)$ commutes with $(r^2, e, e)$ and $E^+(s, e, e)$, but $(r^2, e, e)$ and $E^+(s, e, e)$ do not commute with each other. This give us two maximal abelian subgroups, $\mathbf{Z}_2^2[(r^2, e, e), (e, v_2, e)]$ and $E^+(s, e, e) \times \mathbf{Z}_2[(e, v_2, e)]$ neither of which is cyclic, therefore there are no wave pairs for this case.

**Case 28**

$N(K)/K = \mathbf{Z}_2[(r^2, e, e)] \ltimes E^+(sr, e, e)$

This is not abelian. However both $\mathbf{Z}_2[(r^2, e, e)]$ and $E^+(sr, e, e)$ are maximal abelian and cyclic, therefore there are two possibilities for $G$ and two wave pairs for this case.

**Case 29**

$N(K)/K = \mathbf{Z}_2^2[(s, e, e), (e, v_d, e)]$

This is maximal abelian but not cyclic, therefore there are no wave pairs for this case.

Table 4.4: Wave Pairs for the Square Lattice.

| | K | G | G/K |
|---|---|---|---|
| 28a | $\mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2[(r^2,e,e)]$ |
| 28b | $\mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)] \ltimes E^+(sr,e,e)$ | $E^+(sr,e,e)$ |
| 30 | $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $H_{42} \cong \mathbf{Z}_2$ |
| 32 | $\mathbf{Z}_2^3[(r^2,e,e),(s,v_d,e),(e,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $H_{42} \cong \mathbf{Z}_2$ |
| 42 | $\mathbf{D}_4[(r,e,e),(s,e,e)]\times$ $\mathbf{Z}_2[(e,e,\tau)])$ | $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ | $\mathbf{Z}_2[(e,v_d,e)]$ |
| 43 | $\mathbf{D}_4[(r,e,e),(s,v_d,e)]\times$ $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ | $\mathbf{Z}_2[(e,v_d,e)]$ |

**Cases 30 and 32**

$N(K)/K = H_{42} \ltimes \frac{1}{2}\mathcal{L}$

$(e,v_1,e)$ and $(e,v_2,e)$ commute with each other but not with $H_{42}$. This gives us two maximal abelian subgroups, $\frac{1}{2}\mathcal{L}$ which is not cyclic, and $H_{42} \cong \mathbf{Z}_2$ which is cyclic, giving us one wave pair for each case.

**Case 31**

$\frac{1}{2}\mathcal{L}$

This is maximal abelian but not cyclic, therefore there are no wave pairs for this case.

**Case 33**

$N(K)/K = H_{43} \times \mathbf{Z}_2[(e,v_d,e)]$

This is maximal abelian but not cyclic, therefore there are no wave pairs for this case.

**Cases 42 and 43**

$N(K)/K = \mathbf{Z}_2[(e,v_d,e)]$

which is maximal abelian and cyclic.

### 4.2.2 Hexagonal Lattice Wave Pairs

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The quotient groups $\mathbf{D}_6[r,s]/\mathbf{Z}_3[r^2]$ and $\mathbf{D}_6[r,s]/\mathbf{Z}_2[r^3]$ are not subgroups of $\mathbf{D}_4[r,s]$, we introduce the following notation to allow for this.

$H_{61} = \mathbf{D}_6[r,s]/\mathbf{Z}_2[r^3] = \{\{e,r^3\},\{r,r^4\},\{r^2,r^5\},\{s,sr^3\},\{sr,sr^4\},\{sr^2,sr^5\}\} \cong \mathbf{D}_3$

$H_{62} = \mathbf{D}_6[r,s]/\mathbf{Z}_3[r^2] = \{\{e,r^2,r^4\},\{r,r^3,r^5\},\{s,sr^2,sr^4\},\{sr,sr^3,sr^5\}\} \cong \mathbf{Z}_2^2$

$H_{63} = \mathbf{D}_6[r,s]/\mathbf{D}_3[r^2,s] = \{\{e,r^2,r^4,s,sr^3,sr^4\},\{r,r^3,r^5,sr,sr^3,sr^5\}\} \cong \mathbf{Z}_2$

$H_{64} = \mathbf{D}_6[r,s]/\mathbf{D}_3[r^2,sr] = \{\{e,r^2,r^4,sr,sr^3,sr^5\},\{r,r^3,r^5,sr^2,sr^4,s\}\} \cong \mathbf{Z}_2$

**Cases without wave pairs** The following cases do not have corresponding wave pairs because of Rule 1 above.

| Case | $N(K)/K$ |
|---|---|
| 1 | $\Gamma$ |
| 2,4 | $(H_{61} \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 5,7 | $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)] \ltimes E^+(s,e,e)$ |
| 6,8 | $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)] \ltimes E^-(s,e,e)$ |
| 9 | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)] \ltimes \mathbf{Z}_3[(e,v_t,e)]$ |
| 11,12,13,14 | $\mathbf{Z}_2^2[(e,v_2,e),(e,e,\tau)]$ |
| 17,18 | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ |
| 20,22 | $H_{63} \times \mathbf{Z}_2[(e,e,\tau)]$ |
| 21,23 | $H_{64} \times \mathbf{Z}_2[(e,e,\tau)] \times \mathbf{Z}_3[(e,v_t,e)]$ |

**Case 3**

$N(K)/K = \mathbf{D}_6[(r,e,e),(s,e,e)] \ltimes \mathbf{T}^2$

$\mathbf{T}^2$ is maximal abelian but not cyclic; $(r^3,e,e)$ commutes with every reflection in $\mathbf{D}_6$ so none of them can generate a maximal abelian subgroup, which leaves only one maximal abelian subgroup that is cyclic, $\mathbf{Z}_6[r]$, giving us one possible wave pair.

**Case 10**

$N(K)/K = H_{61} \ltimes \frac{1}{2}\mathcal{L}$

$(e, v_1, e)$ and $(e, v_2, e)$ commute with each other but not with elements of $H_{61}$, therefore $\frac{1}{2}\mathcal{L}$ is maximal abelian, but since it is not cyclic it will not produce a wave pair. $H_{61} \cong \mathbf{D}_3[(r^2, e, e), (s, e, e)]$ is not abelian but the groups $\mathbf{Z}_3[(r^2, e, e)]$ and $\mathbf{Z}_2[(s, e, e)]$ are both abelian and cyclic, and since they commute with nothing else in the group they are both maximal abelian, hence there are two possible wave pairs for this case.

**Case 15**

$N(K)/K = \mathbf{Z}_2[(r^3, e, e)] \ltimes E^+(s, e, e)$

This is not abelian, but the subgroups $\mathbf{Z}_2[(r^3, e, e)]$ and $E^+(s, e, e)$ are both maximal abelian and cyclic therefore there are two possibilities for $G$ and two wave pairs for this case.

**Case 16**

$N(K)/K = \mathbf{Z}_2[(r^3, e, e)] \ltimes E^-(s, e, e)$

This is not abelian, however both $\mathbf{Z}_2[(r^3, e, e)]$ and $E^-(s, e, e)$ are maximal abelian and cyclic therefore there are two possibilities for $G$ and two wave pairs for this case.

**Case 19**

$N(K)/K = H_{62} \ltimes \mathbf{Z}_3[(e, v_t, e)])$

$H_{62}$ and $\mathbf{Z}_3[(e, v_t, e)]$ are both maximal abelian, $H_{62} \cong \mathbf{Z}_2^2$ is not cyclic but $\mathbf{Z}_3[(e, v_t, e)])$ is cyclic, so there is one possible wave pair for this case.

**Case 24**

$N(K)/K = \mathbf{Z}_2[(e, v_2, e)]$

This is maximal abelian and cyclic so $G = N(K)$.

**Case 25**

$N(K)/K = \mathbf{Z}_2[(s, e, e)]$

This is maximal abelian and cyclic, therefore $G = N(K)$.

**Cases 26,27,28, and 29**

$N(K)/K = \mathbf{Z}_2[(e, e, \tau)]$

This is maximal abelian and cyclic, therefore $G = N(K)$.

**Case 30**

$N(K)/K = H_{63}$

This is maximal abelian and cyclic, therefore $G = N(K)$.

**Case 31**

$N(K)/K = H_{64} \ltimes \mathbf{Z}_3[(e, v_t, e)]$

$H_{64}$ and $\mathbf{Z}_3[(e, v_t, e)]$ are both maximal abelian and are also both cyclic so there are two possible wave pairs for this case.

**Case 32**

$N(K)/K = \mathbf{1}$

therefore $G = N(K) = K$.

Table 4.5: Wave Pairs for the Hexagonal Lattice

| | **K** | **G** | **G/K** |
|---|---|---|---|
| 3 | $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_6[(r,e,e)]$ |
| 10a | $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)]$ | $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_3[(r^2,e,e)]$ |
| 10b | $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2[(s,e,e)]$ |
| 15a | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2[(r^3,e,e)]$ |
| 15b | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)] \ltimes E^+(s,e,e)$ | $E^+(s,e,e)$ |
| 16a | $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2[(r^3,e,e)]$ |
| 16b | $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)] \ltimes E^-(s,e,e)$ | $E^-(s,e,e)$ |
| 19 | $\mathbf{Z}_6[(r^2,e,\tau)]$ | $\mathbf{Z}_6[(r^2,e,\tau)] \ltimes \mathbf{Z}_3[(e,v_t,e)]$ | $\mathbf{Z}_3[(e,v_t,e)]$ |
| 24 | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ | $\mathbf{Z}_2[(e,v_2,e)]$ |
| 25 | $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)])$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_2[(s,e,e)]$ |
| 26 | $\mathbf{D}_6[(r,e,e),(s,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_2[(e,e,\tau)]$ |
| 27 | $\mathbf{D}_6[(r,e,e),(s,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_2[(e,e,\tau)]$ |
| 28 | $\mathbf{D}_6[(r,e,\tau),(s,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_2[(e,e,\tau)]$ |
| 29 | $\mathbf{D}_6[(r,e,\tau),(s,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{Z}_2[(e,e,\tau)]$ |
| 30 | $\mathbf{D}_6[(r^2,e,\tau),(s,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $H_{63} \cong \mathbf{Z}_2$ |
| 31a | $\mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $H_{64} \cong \mathbf{Z}_2$ |
| 31b | $\mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)]$ | $\mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]]$ | $\mathbf{Z}_3[(e,v_t,e)]$ |
| 32 | $\mathbf{D}_6[(r,e,e),(s,e,e)]\times$ $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{1}$ |

## 4.3 Wave Pairs corresponding to Twisted Subgroups

Now that we have found all the wave pairs for each lattice we need to check which correspond to twisted subgroups $G^\Theta$ such that $\dim\mathrm{Fix}(G^\Theta) = 2$. In the case of the standing waves we will calculate the dimension of the fixed-point subspace of each $G$ and $K$ and then apply the trace formula. In the case of the rotating waves we will calculate the dimension of the fixed-point subspace of the twisted subgroup $G^\Theta$ directly.

It is important to remember that the groups $G$ and $K$ are subgroups of $\Gamma_{\mathcal{L}}$ whereas $G^\Theta$ is a subgroup of $\Gamma_{\mathcal{L}} \times \mathbf{S}^1$. The group $\Gamma \times \mathbf{S}^1$ acts on $V \oplus V$. As we are looking only at shifted subgroups and their normalizers we will only need the half lattice points and $v_t$, the action of the generating elements of $\Gamma_{\frac{1}{2}\mathcal{L}} \times \mathbf{S}^1$ and $v_t$ on $V \oplus V$ are shown in Tables 4.6 and 4.7.

Table 4.6: The Action of the Generators of $((\mathbf{D}_4 \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2) \times \mathbf{S}^1$

| $g$ | **Action on** $\mathbf{C}^2$ $g(z_1, z_2, w_1, w_2)$ | **Action on** $\mathbf{C}^4$ $g(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$ |
|---|---|---|
| $((r, e, e), 0)$ | $(w_2, z_1, z_2, w_1)$ | $(w_2, z_1, w_4, z_3, z_2, w_1, z_4, w_3)$ |
| $((s, e, e), 0)$ | $\epsilon(z_1, w_2, w_1, z_2)$ | $\epsilon(w_4, w_3, w_2, w_1, z_4, z_3, z_2, z_1)$ |
| $((e, v_1, e), 0)$ | $(-z_1, z_2, -w_1, w_2)$ | $(-z_1, z_2, z_3, -z_4, -w_1, w_2, w_3, -w_4)$, $\alpha$ odd |
| | | $(z_1, -z_2, -z_3, z_4, w_1, -w_2, -w_3, w_4)$, $\beta$ odd |
| $((e, v_2, e), 0)$ | $(z_1, -z_2, w_1, -w_2)$ | $(z_1, -z_2, -z_3, z_4, w_1, -w_2, -w_3, w_4)$, $\alpha$ odd |
| | | $(-z_1, z_2, z_3, -z_4, -w_1, w_2, w_3, -w_4)$, $\beta$ odd |
| $((e, v_d, e), 0)$ | $(-z_1, -z_2, -w_1, -w_2)$ | $(-z_1, -z_2, -z_3, -z_4, -w_1, -w_2, -w_3, -w_4)$ |
| $((e, e, \tau), 0)$ | $\psi(z_1, z_2, w_1, w_2)$ | $\psi(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$ |
| $((e, e, e), \Theta)$ | $e^{2\pi i\Theta}(z_1, z_2, w_1, w_2)$ | $e^{2\pi i\Theta}(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$ |

Table 4.7: The Action of the Generators of $((\mathbf{D}_6 \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2) \times \mathbf{S}^1$

| $g$ | **Action on $\mathbf{C}^3$** $g(z_1, z_2, z_3, w_1, w_2, w_3)$ | **Action on $\mathbf{C}^6$** $g(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |
|---|---|---|
| $((r, e, e), 0)$ | $(w_2, w_3, w_1, z_2, z_3, z_1)$ | $(w_2, w_3, w_1, w_5, w_6, w_4, z_2, z_3, z_1, z_5, z_6, z_4)$ |
| $((s, e, e), 0)$ | $\epsilon(w_2, w_1, w_3, z_2, z_1, z_3)$ | $\epsilon(z_6, z_5, z_4, z_3, z_2, z_1, w_6, w_5, w_4, w_3, w_2, w_1)$ |
| $((e, v_1, e), 0)$ | $(-z_1, z_2, -z_3, -w_1, w_2, -w_3)$ | $(-z_1, -z_2, z_3, -z_4, z_5, -z_6,$ $-w_1, -w_2, w_3, -w_4, w_5, -w_6)$, $\alpha$ odd |
| | | $(z_1, -z_2, -z_3, z_4, -z_5, -z_6,$ $w_1, -w_2, -w_3, w_4, -w_5, -w_6)$, $\beta$ odd |
| | | $(-z_1, z_2, -z_3, -z_4, -z_5, z_6,$ $-w_1, w_2, -w_3, -w_4, -w_5, w_6)$, $\alpha$ and $\beta$ odd |
| $((e, v_2, e), 0)$ | $(-z_1, -z_2, z_3, -w_1, -w_2, w_3)$ | $(z_1, -z_2, -z_3, -z_4, -z_5, z_6,$ $w_1, -w_2, -w_3, -w_4, -w_5, w_6)$, $\alpha$ odd |
| | | $(-z_1, z_2, -z_3, -z_4, z_5, -z_6,$ $-w_1, w_2, -w_3, -w_4, w_5, -w_6)$, $\beta$ odd |
| | | $(-z_1, -z_2, z_3, z_4, -z_5, -z_6,$ $-w_1, -w_2, w_3, w_4, -w_5, -w_6)$, $\alpha$ and $\beta$ odd |
| $((e, v_t, e), 0)$ | $e^{(2\pi i)/3}(z_1, z_2, z_3, w_1, w_2, w_3)$ | $e^{(2\pi i)/3}(e^{-(\alpha+\beta)}z_1, e^{(2\alpha-\beta)}z_2, e^{(-\alpha+2\beta)}z_3,$ $e^{(-2\alpha+\beta)}z_4, e^{(\alpha+\beta)}z_5, e^{(\alpha-2\beta)}z_6$ $e^{(\alpha+\beta)}w_1, e^{(-2\alpha+\beta)}w_2, e^{(\alpha-2\beta)}w_3,$ $e^{(2\alpha-\beta)}w_4, e^{-(\alpha+\beta)}w_5, e^{(-\alpha+2\beta)}w_6)$ |
| $((e, e, \tau), 0)$ | $\psi(z_1, z_2, z_3, w_1, w_2, w_3)$ | $\psi(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |
| $((e, e, e), \Theta)$ | $e^{2\pi i\Theta}(z_1, z_2, z_3, w_1, w_2, w_3)$ | $e^{2\pi i\Theta}(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |

### 4.3.1 Square Lattice, Standing Waves

There are five wave pairs for the square lattice where $G/K$ is cyclic, these are cases 25, 28a, 30, 32, 42, and 43. We have already calculated the action of $(\mathbf{D}_4 \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2$ in section 2.2 for finding the equilibria. Since every subgroup involved in these calculations includes $\mathbf{Z}_2[(e, e, \tau)]$ it is important to

Table 4.8: Square Lattice Standing Waves: Fixed-point Subspaces by Subgroup for $\psi = +1$

| Group | C | Fix($K$) | dimFix($K$) |
|---|---|---|---|
| $\mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)]$ | $\mathbf{C}^2$ | $z_1 = \epsilon\overline{z_2}$ | 2 |
| $\mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)]$ | $\mathbf{C}^4$ | $z_1 = \epsilon\overline{z_3}, z_2 = \epsilon z_4$ | 4 |
| $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)]$ | $\mathbf{C}^2$ | $z_1 = \epsilon z_2, z = \overline{z}$ | 1 |
| $\mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)]$ | $\mathbf{C}^4$ | $z_1 = \epsilon z_3, z_2 = \epsilon z_4, z = \overline{z}$ | 2 |
| $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{C}^2$ | $z = \overline{z}$ for $\epsilon = +1$ | 2 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{C}^4$ | $z_1 = \epsilon z_4, z_2 = \epsilon z_3, z = \overline{z}$ | 2 |
| $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^2$ | $z_1 = z_2, z = \overline{z}$ for $\epsilon = +1$ | 1 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^4$ | $z_1 = z_2 = \epsilon z_3 = \epsilon z_4, z = \overline{z}$ | 1 |
| $\mathbf{Z}_2^3[(r^2,e,e),(s,v_d,e),(e,e,\tau)]$ | $\mathbf{C}^2$ | 0 for $\epsilon = +1$ | 0 |
| | | $z = \overline{z}$ for $\epsilon = -1$ | 2 |
| $\mathbf{Z}_2^3[(r^2,e,e),(s,v_d,e),(e,e,\tau)]$ | $\mathbf{C}^4$ | $z_1 = -\epsilon z_4, z_2 = -\epsilon z_3, z = \overline{z}$ | 2 |
| $\mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^2$ | 0 for $\epsilon = +1$ | 0 |
| | | $z_1 = z_2, z = \overline{z}$ for $\epsilon = -1$ | 1 |
| $\mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^4$ | $z_1 = z_2 = -\epsilon z_3 = -\epsilon z_4, z = \overline{z}$ | 1 |
| $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ | $\mathbf{C}^2$ | 0 | 0 |
| $\mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ | $\mathbf{C}^4$ | 0 | 0 |
| $\mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ | $\mathbf{C}^2$ | 0 | 0 |
| $\mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$ | $\mathbf{C}^4$ | 0 | 0 |

note that whenever $\psi = -1$ the fixed-point subspace will equal zero since $(e,e,\tau)$ sends $z$ to $-z$ for all $z$ and the only thing that this fixes is 0. Therefore table 4.8 shows results for when $\psi = +1$ only.

We use the trace formulae shown in Section 3.2 to calculate the dimension of the fixed-point subspace of the twisted subgroup. In each of our five cases $G/K = \mathbf{Z}_2$ so we use the trace formula:

$$\mathrm{dimFix}(G^{\Theta}) = 2(\mathrm{dimFix}(K) - \mathrm{dimFix}(G))$$

**Case 28a**

$K = \mathbf{Z}_2^2[(sr,e,e),(e,e,\tau)], G = \mathbf{Z}_2^3[(r^2,e,e),(sr,e,e),(e,e,\tau)]$

28a.2 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$

28a.4 $\mathbf{C}^4$ $\dim\mathrm{Fix}(G^\Theta) = 2(4-2) = 4$

**Case 30**

$K = \mathbf{Z}_2^3[(r^2,e,e),(s,e,e),(e,e,\tau)], G = \mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$

30.21 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$ when $\epsilon = +1$

30.22 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(0-0) = 0$ when $\epsilon = -1$

30.4 $\mathbf{C}^4$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$

**Case 32**

$K = \mathbf{Z}_2^3[(r^2,e,e),(s,v_d,e),(e,e,\tau)], G = \mathbf{D}4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2[(e,e,\tau)]$

32.21 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(0-0) = 0$ when $\epsilon = +1$

32.22 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$ when $\epsilon = -1$

32.4 $\mathbf{C}^4$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$

**Case 42**

$K = \mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)], G = \mathbf{D}_4[(r,e,e),(s,e,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$

42.21 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(1-0) = 2$ when $\epsilon = +1$

42.22 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(0-0) = 0$ when $\epsilon = -1$

42.4 $\mathbf{C}^4$ $\dim\mathrm{Fix}(G^\Theta) = 2(1-0) = 2$

**Case 43**

$K = \mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2[(e,e,\tau)], G = \mathbf{D}_4[(r,e,e),(s,v_d,e)] \times \mathbf{Z}_2^2[(e,v_d,e),(e,e,\tau)]$

43.21 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(0-0) = 0$ when $\epsilon = +1$

43.22 $\mathbf{C}^2$ $\dim\mathrm{Fix}(G^\Theta) = 2(1-0) = 2$ when $\epsilon = -1$

43.4 $\mathbf{C}^4$ $\dim\mathrm{Fix}(G^\Theta) = 2(1-0) = 2$

### 4.3.2 Hexagonal Lattice, Standing Waves

There are 12 wave pairs for the hexagonal lattice where $G/K$ is cyclic in both $\mathbf{C}^3$ and $\mathbf{C}^6$, and an extra three cases just in $\mathbf{C}^6$. Here every sub-

Table 4.9: Hexagonal Lattice Standing Waves: Fixed Point Subspaces by Subgroup

| **Group** | **C** | Fix$(K)$ | dimFix$(K)$ |
|---|---|---|---|
| $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^3$ | $\mathbf{C}^3$ | 6 |
| $\mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^6$ | $\mathbf{C}^6$ | 12 |
| $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3, z = \overline{z}$ | 1 |
| $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3, z_4 = z_5 = z_6, z = \overline{z}$ | 2 |
| $\mathbf{Z}_6[(r^2,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3$ | 2 |
| $\mathbf{Z}_6[(r^2,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3, z_4 = z_5 = z_6$ | 4 |
| $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)]$ | $\mathbf{C}^3$ | $z = \overline{z}$ | 3 |
| $\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)]$ | $\mathbf{C}^6$ | $z = \overline{z}$ | 6 |
| $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2, z = \overline{z}$ for $\epsilon = +1$ | 2 |
| | | $z_1 = -z_2, z_3 = 0, z = \overline{z}$ for $\epsilon = -1$ | 1 |
| $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = \epsilon z_6, z_2 = \epsilon z_5, z_3 = \epsilon z_4, z = \overline{z}$ | 3 |
| $\mathbf{Z}_2^3[(r^3,e,e),(sr^3,e,e),(e,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2, z = \overline{z}$ for $\epsilon = +1$ | 2 |
| | | $z_1 = -z_2, z_3 = 0, z = \overline{z}$ for $\epsilon = -1$ | 1 |
| $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = \epsilon z_6, z_2 = \epsilon z_5, z_3 = \epsilon z_4, z = \overline{z}$ | 3 |
| $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = \overline{z_2}, z_3 = \overline{z_3}$ for $\epsilon = +1$ | 3 |
| | | $z_1 = -\overline{z_2}, Re(z_3) = 0$ for $\epsilon = -1$ | 3 |
| $\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = \epsilon z_6, z_2 = \epsilon z_5, z_3 = \epsilon z_4$ | 6 |
| $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2$ for $\epsilon = +1$ | 4 |
| | | $z_1 = -z_2, z_3 = 0$ for $\epsilon = -1$ | 2 |
| $\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = \epsilon\overline{z_6}, z_2 = \epsilon\overline{z_5}, z_3 = \epsilon\overline{z_4}$ | 6 |
| $\mathbf{Z}_6[(r^2,e,\tau)] \ltimes \mathbf{Z}_3[(e,v_t,e)]]$ | $\mathbf{C}^3$ | 0 | 0 |
| $\mathbf{Z}_6[(r^2,e,\tau)] \ltimes \mathbf{Z}_3[(e,v_t,e)]]$ | $\mathbf{C}^6$ | 0 | 0 |
| $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = 0, z = \overline{z}$ for $\epsilon = +1$ | 1 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = \epsilon z_6, z_2 = z_3 = z_4 = z_5 = 0$ | 1 |
| | | $z = \overline{z}$ for $\alpha$ odd | |
| | | $z_2 = \epsilon z_5, z_1 = z_3 = z_4 = z_6 = 0$ | 1 |
| | | $z = \overline{z}$ for $\beta$ odd | |
| | | $z_3 = \epsilon z_4, z_1 = z_2 = z_5 = z_6 = 0$ | 1 |
| | | $z = \overline{z}$ for $\alpha$, $\beta$ odd | |
| $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3, z = \overline{z}$ for $\epsilon = +1$ | 1 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ | 1 |
| $\mathbf{D}_6[(r,e,e),(s,e,e)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3, z = \overline{z}$ for $\epsilon = +1$ | 1 |

Table 4.9: Hexagonal Lattice Standing Waves: Fixed Point Subspaces by Subgroup

| **Group** | **C** | Fix($K$) | dimFix($K$) |
|---|---|---|---|
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{D}_6[(r,e,e),(s,e,e)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ | 1 |
| $\mathbf{D}_6[(r,e,e),(s,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3, z = \overline{z}$ for $\epsilon = \psi$ | 1 |
| | | 0 for $\epsilon \neq \psi$ | 0 |
| $\mathbf{D}_6[(r,e,e),(s,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3 = \epsilon\psi z_4 = \epsilon\psi z_5 = \epsilon\psi z_6$ | 1 |
| | | $z = \overline{z}$ | |
| $\mathbf{D}_6[(r,e,\tau),(s,e,e)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3, z = \overline{z}$ for $\epsilon = +1$ | 1 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{D}_6[(r,e,\tau),(s,e,e)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ | 1 |
| $\mathbf{D}_6[(r,e,\tau),(s,e,\tau)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3, z = \overline{z}$ for $\epsilon = +1$ | 1 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{D}_6[(r,e,\tau),(s,e,\tau)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, z = \overline{z}$ | 1 |
| $\mathbf{D}_6[(r^2,e,\tau),(s,e,e)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3, z = \overline{z}$ for $\epsilon = +1$ | 1 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{D}_6[(r^2,e,\tau),(s,e,e)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6$ | 2 |
| $\mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)]$ | $\mathbf{C}^3$ | $z_1 = z_2 = z_3$ for $\epsilon = +1$ | 2 |
| | | 0 for $\epsilon = -1$ | 0 |
| $\mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)]$ | $\mathbf{C}^6$ | $z_1 = z_2 = z_3 = \epsilon\overline{z_4} = \epsilon\overline{z_5} = \epsilon\overline{z_6}$ | 2 |
| $\mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]]$ | $\mathbf{C}^3$ | 0 | 0 |
| $\mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]]$ | $\mathbf{C}^6$ | 0 | 0 |

group includes the action of $(e,e,\tau)$ though not necessarily in the form of $\mathbf{Z}_2[(e,e,\tau)]$. With the single exception of $\mathbf{D}_6[(r,e,e),(s,e,\tau)]$ used in case 27, whenever $\psi = -1$ the fixed-point subspace will equal zero. Therefore the table below shows results for when $\psi = +1$ only except for the one case where $\psi = \epsilon$.

In case 3 $G/K = \mathbf{Z}_6$ so we use the trace formula

$$\begin{array}{c} \mathrm{dimFix}(G^{\Theta}) = \mathrm{dimFix}(K) - \mathrm{dimFix}(M) - \mathrm{dimFix}(L) + \mathrm{dimFix}(G) \\ \text{where } K \subset M \subset G, |G/M| = 2 \text{ and } K \subset L \subset G \text{ and } |G/L| = 3 \end{array}.$$

**Case 3**

$K = \mathbf{Z}_2[(e, e, \tau)], G = \mathbf{Z}_6[(r, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$

$L = \mathbf{Z}_2^2[(r^3, e, e), (e, e, \tau)], M = \mathbf{Z}_6[(r^2, e, \tau)]$

3.3 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 6 - 2 - 3 + 1 = 2$

3.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 12 - 4 - 6 + 2 = 4$

The following cases have $G/K = \mathbf{Z}_2$ so we use the trace formula:

$$\dim\mathrm{Fix}(G^\Theta) = 2(\dim\mathrm{Fix}(K) - \dim\mathrm{Fix}(G)).$$

**Case 10b**

$K = \mathbf{Z}_2^2[(r^3, e, e), (e, e, \tau)], G = \mathbf{Z}_2^3[(r^3, e, e), (s, e, e), (e, e, \tau)]$

10b.31 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(3 - 2) = 2$ when $\epsilon = +1$

10b.32 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(3 - 1) = 4$ when $\epsilon = -1$

10b.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 2(6 - 3) = 6$

**Case 15a**

$K = \mathbf{Z}_2^2[(s, e, e), (e, e, \tau)], G = \mathbf{Z}_2^3[(r^3, e, e), (s, e, e), (e, e, \tau)]$

15a.31 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(3 - 2) = 2$ when $\epsilon = +1$

15a.32 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(3 - 1) = 4$ when $\epsilon = -1$

15a.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 2(6 - 3) = 6$

**Case 16a**

$K = \mathbf{Z}_2^2[(sr^3, e, e), (e, e, \tau)], G = \mathbf{Z}_2^3[(r^3, e, e), (sr^3, e, e), (e, e, \tau)]$

16a.31 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(4 - 2) = 4$ when $\epsilon = +1$

16a.32 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(2 - 1) = 2$ when $\epsilon = -1$

16a.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 2(6 - 3) = 6$

**Case 24**

$K = \mathbf{Z}_2^3[(r^3, e, e), (s, e, e), (e, e, \tau)], G = \mathbf{Z}_2^4[(r^3, e, e), (s, e, e), (e, v_2, e), (e, e, \tau)]$

| | | | |
|---|---|---|---|
| 24.31 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(2-1) = 2$ | when $\epsilon = +1$ |
| 24.32 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(1-0) = 2$ | when $\epsilon = -1$ |
| 24.6 | $\mathbf{C}^6$ | $\dim\text{Fix}(G^\Theta) = 2(3-1) = 4$ | |

**Case 25**

$K = \mathbf{Z}_6[(r, e, e)] \times \mathbf{Z}_2[(e, e, \tau)], G = \mathbf{D}_6[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$

| | | | |
|---|---|---|---|
| 25.31 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(1-1) = 0$ | when $\epsilon = +1$ |
| 25.32 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(1-0) = 2$ | when $\epsilon = -1$ |
| 25.6 | $\mathbf{C}^6$ | $\dim\text{Fix}(G^\Theta) = 2(2-1) = 2$ | |

**Case 26**

$K = \mathbf{D}_6[(r, e, e), (s, e, e)], G = \mathbf{D}_6[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$

| | | | |
|---|---|---|---|
| 26.31 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(1-1) = 0$ | when $\epsilon = +1$ |
| 26.32 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(0-0) = 0$ | when $\epsilon = -1$ |
| 26.6 | $\mathbf{C}^6$ | $\dim\text{Fix}(G^\Theta) = 2(1-1) = 0$ | |

**Case 27**

$K = \mathbf{D}_6[(r, e, e), (s, e, \tau)], G = \mathbf{D}_6[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$

| | | | |
|---|---|---|---|
| 27.31 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(1-1) = 0$ | when $\epsilon = \psi = +1$ |
| 27.32 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(1-0) = 2$ | when $\epsilon = \psi = -1$ |
| 27.6 | $\mathbf{C}^6$ | $\dim\text{Fix}(G^\Theta) = 2(1-1) = 0$ | |

**Case 28**

$K = \mathbf{D}_6[(r, e, \tau), (s, e, e)], G = \mathbf{D}_6[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$

| | | | |
|---|---|---|---|
| 28.31 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(1-1) = 0$ | when $\epsilon = +1$ |
| 28.32 | $\mathbf{C}^3$ | $\dim\text{Fix}(G^\Theta) = 2(0-0) = 0$ | when $\epsilon = -1$ |
| 28.6 | $\mathbf{C}^6$ | $\dim\text{Fix}(G^\Theta) = 2(1-1) = 0$ | |

**Case 29**

$K = \mathbf{D}_6[(r,e,\tau),(s,e,\tau)], G = \mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$

29.31 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(1-1) = 0$ when $\epsilon = +1$

29.32 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(0-0) = 0$ when $\epsilon = -1$

29.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 2(1-1) = 0$

**Case 30**

$K = \mathbf{D}_6[(r^2,e,\tau),(s,e,e)], G = \mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$

30.31 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(1-1) = 0$ when $\epsilon = +1$

30.32 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(0-0) = 0$ when $\epsilon = -1$

30.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$

**Case 31a**

$K = \mathbf{D}_6[(r^2,e,\tau),(sr^3,e,e)], G = \mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$

31a.31 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$ when $\epsilon = +1$

31a.32 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2(0-0) = 0$ when $\epsilon = -1$

31a.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 2(2-1) = 2$

In cases 10a, 19 and 31b $G/K = \mathbf{Z}_3$ so we use the trace formula:

$$\dim\mathrm{Fix}(G^\Theta) = \dim\mathrm{Fix}(K) - \dim\mathrm{Fix}(G)$$

**Case 10a**

$\mathbf{Z}_2^2[(r^3,e,e),(e,e,\tau)], G = \mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$

10.3 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 3-1 = 2$

10.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 6-2 = 4$

**Case 19**

$K = \mathbf{Z}_6[(r^2,e,\tau)], G = \mathbf{Z}_6[(r^2,e,\tau)] \ltimes \mathbf{Z}_3[(e,v_t,e)]$

19.3 $\mathbf{C}^3$ $\dim\mathrm{Fix}(G^\Theta) = 2-0 = 2$

19.6 $\mathbf{C}^6$ $\dim\mathrm{Fix}(G^\Theta) = 4-0 = 4$

**Case 31b**

$K = \mathbf{D}_6[(r^2, e, \tau), (sr^3, e, e)], G = \mathbf{D}_6[(r^2, e, \tau), (sr^3, e, e)] \ltimes \mathbf{Z}_3[(e, v_t, e)]]$

| | | | |
|---|---|---|---|
| 31b.31 | $\mathbf{C}^3$ | $\mathrm{dimFix}(G^\Theta) = 2 - 0 = 2$ | when $\epsilon = +1$ |
| 31b.32 | $\mathbf{C}^3$ | $\mathrm{dimFix}(G^\Theta) = 0 - 0 = 0$ | when $\epsilon = -1$ |
| 31b.6 | $\mathbf{C}^6$ | $\mathrm{dimFix}(G^\Theta) = 2 - 0 = 2$ | |

In case 32 $G/K = \mathbf{1}$ so we use the trace formula:

$$dimFix(G^\Theta) = 2dimFix(G)$$

**Case 32**

$K = \mathbf{D}_6[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)], G = \mathbf{D}_6[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$

| | | | |
|---|---|---|---|
| 32.31 | $\mathbf{C}^3$ | $\mathrm{dimFix}(G^\Theta) = 2 \cdot 1 = 2$ | when $\epsilon = +1$ |
| 32.32 | $\mathbf{C}^3$ | $\mathrm{dimFix}(G^\Theta) = 2 \cdot 0 = 0$ | when $\epsilon = -1$ |
| 32.6 | $\mathbf{C}^6$ | $\mathrm{dimFix}(G^\Theta) = 2 \cdot 1 = 2$ | |

### 4.3.3 Square Lattice, Rotating Waves

There is only one wave pair for square lattice where $G/K \cong \mathbf{S}^1$.

**Case 28b**

K=$\mathbf{Z}_2^2[(sr, e, e), (e, e, \tau)]$, $G = \mathbf{Z}_2^2[(sr, e, e), (e, e, \tau)] \ltimes E^+(sr, e, e)$

$G$ is generated by $((sr, e, e), 0)$, $((e, e, \tau), 0)$ and $((e, \theta(l_1 + l_2), e), \theta)$,
$0 \le \theta < 1$, $\theta \in \mathbf{R}$

The actions of these elements on $V \oplus V$ are as follows:

$\chi = e^{2\pi\theta i}$

| $g$ | $g(z_1, z_2, w_1, w_2)$ |
|---|---|
| $((sr, e, e), 0)$ | $\epsilon(w_2, w_1, z_2, z_1)$ |
| $((e, e, \tau), 0)$ | $\psi(z_1, z_2, w_1, w_2)$ |
| $((e, \theta(l_1 + l_2), e), \theta)$ | $(z_1, z_2, \chi^2 w_1, \chi^2 w_2)$ |

| | |
|---|---|
| $g$ | $g(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4)$ |
| $((sr,e,e),0)$ | $\epsilon(w_3,z_4,w_1,z_2,z_3,w_4,z_1,w_2)$ |
| $((e,e,\tau),0)$ | $\psi(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4)$ |
| $((e,\theta(l_1+l_2),e),\theta)$ | $\chi(\chi^{-(\alpha+\beta)}z_1,\chi^{(-\alpha+\beta)}z_2,\chi^{-(\alpha+\beta)}z_3,\chi^{-(\alpha-\beta)}z_4,$ $\chi^{(\alpha+\beta)}w_1,\chi^{(\alpha-\beta)}w_2,\chi^{(\alpha+\beta)}w_3,\chi^{(-\alpha+\beta)}w_4)$ |

For both the $\mathbf{C}^2$ and $\mathbf{C}^4$ representations this gives us $\dim\mathrm{Fix}(G^\Theta)=0$ so there is no Hopf Bifurcation for this case.

### 4.3.4 Hexagonal Lattice, Rotating Waves

There are two wave pairs for hexagonal lattice where $G/K \cong \mathbf{S}^1$.

**Case 15b**

$K=\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]$, $G=\mathbf{Z}_2^2[(s,e,e),(e,e,\tau)]\ltimes E^+(s,e,e)$

$G$ is generated by $((s,e,e),0)$, $((e,e,\tau),0)$ and $((e,\theta l_2,e),\theta)$, $0\le\theta<1$, $\theta\in\mathbf{R}$

**Case 16b**

$K=\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]$, $G=\mathbf{Z}_2^2[(sr^3,e,e),(e,e,\tau)]\ltimes E^-(s,e,e)$

$G$ is generated by $((sr^3,e,e),0)$, $((e,e,\tau),0)$ and $((e,-2\theta l_1+\theta l_2),e),\theta)$, $0\le\theta<1$, $\theta\in\mathbf{R}$

The actions of these elements on $V\oplus V$ are as follows

| | |
|---|---|
| $g$ | $g(z_1,z_2,z_3,w_1,w_2,w_3)$ |
| $((s,e,e),0)$ | $\epsilon(w_2,w_1,w_3,z_2,z_1,z_3)$ |
| $((sr^3,e,e),0)$ | $\epsilon(z_2,z_1,z_3,w_2,w_1,w_3)$ |
| $((e,e,\tau),0)$ | $\psi(z_1,z_2,z_3,w_1,w_2,w_3)$ |
| $((e,\theta l_2,e),\theta)$ | $(z_1,e^{4\pi\theta i}z_2,e^{2\pi\theta i}z_3,e^{4\pi\theta i}w_1,w_2,e^{2\pi\theta i}w_3)$ |
| $((e,-2\theta l_1+\theta l_2),e),\theta)$ | $(e^{4\pi\theta i}z_1,e^{4\pi\theta i}z_2,e^{-2\pi\theta i}z_3,w_1,w_2,e^{6\pi\theta i}w_3)$ |

| | |
|---|---|
| $g$ | $g(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |
| $((s, e, e), 0)$ | $\epsilon(z_6, z_5, z_4, z_3, z_2, z_1, w_6, w_5, w_4, w_3, w_2, w_1)$ |
| $((sr^3, e, e), 0)$ | $\epsilon(w_6, w_5, w_4, w_3, w_2, w_1, z_6, z_5, z_4, z_3, z_2, z_1)$ |
| $((e, e, \tau), 0)$ | $\psi(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |
| $((e, \theta l_2, e), \theta)$ | $\chi(\chi^{-\beta} z_1, \chi^{\alpha} z_2, \chi^{-\alpha+\beta} z_3, \chi^{-\alpha+\beta} z_4, \chi^{\alpha} z_5, \chi^{-\beta} z_6,$ $\chi^{\beta} w_1, \chi^{-\alpha} w_2, \chi^{\alpha-\beta} w_3, \chi^{\alpha-\beta} w_4, \chi^{-\alpha} w_5, \chi^{\beta} w_6)$ |
| $((e, -2\theta l_1 + \theta l_2), e), \theta)$ | $\chi(\chi^{2\alpha-\beta} z_1, \chi^{-\alpha+2\beta} z_2, \chi^{-\alpha-\beta} z_3, \chi^{\alpha+\beta} z_4, \chi^{\alpha-2\beta} z_5, \chi^{-2\alpha+\beta} z_6,$ $\chi^{-2\alpha+\beta} w_1, \chi^{\alpha-2\beta} w_2, \chi^{\alpha+\beta} w_3, \chi^{-\alpha-\beta} w_4, \chi^{-\alpha+2\beta} w_5, \chi^{2\alpha-\beta} w_6)$ |

Both of these cases produce fixed-point subspaces of dimension 2 for both the $\mathbf{C}^3$ and $\mathbf{C}^6$ representations. The fixed-point subspaces are shown along with those for the standing waves in table 4.13.

## 4.4 Fixed-point Subspaces of $\Sigma = G^{\Theta}$

### 4.4.1 Action of $\Sigma$

In order to find the fixed-point subspace of each twisted subgroup we must first find the action of each of the generating elements of $\Sigma = G^{\Theta} = \{(g, \Theta(g)) | g \in G\}$ where $\Theta : G \to \mathbf{S}^1$. It is the quotient $G/K$ that maps on to $S^1$ so the generators of $G^{\Theta}$ will be the generators of $K$ plus the element $(\hat{g}, \Theta(\hat{g}))$ where $< \hat{g} >= G/K$. The generators of $G^{\Theta}$ are shown in Tables 4.10 and 4.11. An example of how to find the generators and the fixed-point subspace of $G^{\Theta}$ is shown in Appendix C.

Table 4.10: Generators of Twisted Subgroups for the Square Lattice

| **Case** | **G** | **Generators of** $G^{\Theta}$ |
|---|---|---|
| 28a | $\mathbf{Z}_2^3[(r^2, e, e), (sr, e, e), (e, e, \tau)]$ | $((r^2, e, e), \frac{1}{2}), ((sr, e, e), 0), ((e, e, \tau), 0)$ |
| 30 | $\mathbf{D}_4[(r, e, e), (s, e, e)] \times \mathbf{Z}_2[(e, e, \tau)]$ | $((r, e, e), \frac{1}{2}), ((s, e, e), 0), ((e, e, \tau), 0)$ |
| 32 | $\mathbf{D}_4[(r, e, e), (s, v_d, e)] \times \mathbf{Z}_2[(e, e, \tau)]$ | $((r, e, e), \frac{1}{2}), ((s, v_d, e), 0), ((e, e, \tau), 0)$ |
| 42 | $\mathbf{D}_4[(r, e, e), (s, e, e)]$ $\times \mathbf{Z}_2^2[(e, v_d, e), (e, e, \tau)]$ | $((r, e, e), 0), ((s, e, e), 0), ((e, v_d, e), \frac{1}{2}), ((e, e, \tau), 0)$ |
| 43 | $\mathbf{D}_4[(r, e, e), (s, v_d, e)]$ $\times \mathbf{Z}_2^2[(e, v_d, e), (e, e, \tau)]$ | $((r, e, e), 0), ((s, v_d, e), 0), ((e, v_d, e), \frac{1}{2}), ((e, e, \tau), 0)$ |

### 4.4.2 Fixed-point Subspaces

We use the action of the generating elements to calculate the fixed-point subspace of each twisted subgroup, shown in Tables 4.12 and 4.13. In each case $\dim \mathrm{Fix}\, G^{\Theta} = 2$.

Table 4.11: Generators of Twisted Subgroups for the Hexagonal Lattice

| **Case** | **G** | **Generators of** $G^\Theta$ |
|---|---|---|
| 3 | $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $((r,e,e)\frac{1}{6}), ((e,e,\tau),0)$ |
| 10a | $\mathbf{Z}_6[(r,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $((r,e,e),\frac{1}{3}), ((e,e,\tau),0)$ |
| 10b | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $((r^3,e,e),0), ((s,e,e),\frac{1}{2}), ((e,e,\tau),0)$ |
| 15a | $\mathbf{Z}_2^3[(r^3,e,e),(s,e,e),(e,e,\tau)]$ | $((r^3,e,e),\frac{1}{2}), ((s,e,e),0), ((e,e,\tau),0)$ |
| 16a | $\mathbf{Z}_2^3[(r^3,e,e),(sr^3,e,e),(e,e,\tau)]$ | $((r^3,e,e),\frac{1}{2}), ((sr^3,e,e),0), ((e,e,\tau),0)$ |
| 19 | $\mathbf{Z}_6[(r^2,e,\tau)] \ltimes \mathbf{Z}_3[(e,v_t,e)]$ | $((r^2,e,\tau),0), ((e,v_t,e),\frac{1}{3})$ |
| 24 | $\mathbf{Z}_2^4[(r^3,e,e),(s,e,e),(e,v_2,e),(e,e,\tau)]$ | $((r^3,e,e),0), ((s,e,e),0), ((e,v_2,e),\frac{1}{2}), ((e,e,\tau),0)$ |
| 25 | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $((r,e,e),0), ((s,e,e),\frac{1}{2}), ((e,e,\tau),0)$ |
| 27 | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $((r,e,e),0), ((s,e,e),0), ((e,e,\tau),\frac{1}{2})$ |
| 30 | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $((r,e,e),\frac{1}{2}), ((s,e,e),0), ((e,e,\tau),0)$ |
| 31a | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $((r,e,e),\frac{1}{2}), ((s,e,e),0), ((e,e,\tau),0)$ |
| 31b | $\mathbf{D}_6[(r^2,e,\tau),(sr,e,e)] \ltimes \mathbf{Z}_3[(e,v_t,e)]]$ | $((r^2,e,\tau),0), ((sr,e,e),0), ((e,v_t,e),\frac{1}{3})$ |
| 32 | $\mathbf{D}_6[(r,e,e),(s,e,e)] \times \mathbf{Z}_2[(e,e,\tau)]$ | $((r,e,e),0), ((s,e,e),0), ((e,e,\tau),0)$ |

Table 4.12: Fixed-point Subspaces for Standing Waves on the Square Lattice

| **Case** | $\mathbf{C}^2$ | $\mathbf{C}^4$ |
|---|---|---|
| 28a | $z_1 = -\epsilon z_2 = -w_1 = \epsilon w_2$ | N/A |
| 28b | 0 | 0 |
| 30 | $z_1 = -z_2 = w_1 = -w_2, \epsilon = +1$ | $z_1 = -z_2 = -\epsilon z_3 = \epsilon z_4 = w_1 = -w_2 = -\epsilon w_3 = \epsilon w_4$ |
| 32 | $z_1 = -z_2 = w_1 = -w_2, \epsilon = -1$ | $z_1 = -z_2 = \epsilon z_3 = -\epsilon z_4 = w_1 = -w_2 = \epsilon w_3 = -\epsilon w_4$ |
| 42 | $z_1 = z_2 = w_1 = w_2, \epsilon = +1$ | $z_1 = z_2 = \epsilon z_3 = \epsilon z_4 = w_1 = w_2 = \epsilon w_3 = \epsilon w_4$ |
| 43 | $z_1 = z_2 = w_1 = w_2, \epsilon = -1$ | $z_1 = z_2 = -\epsilon z_3 = -\epsilon z_4 = w_1 = w_2 = -\epsilon w_3 = -\epsilon w_4$ |

Table 4.13: Fixed-point Subspaces on the Hexagonal Lattice.
$\chi = e^{\frac{2\pi i}{3}}$, and $\psi = +1$ unless otherwise stated

| **Case** | $\mathbf{C}^3$ | $\mathbf{C}^6$ |
|---|---|---|
| 3 | $z_1 = \chi^2 z_2 = \chi z_3, w = z$ | N/A |
| 10a | $z_1 = \chi z_2 = \chi^2 z_3, w = -z$ | N/A |
| 10b | $z_1 = -z_2, z_3 = 0, w = z, \epsilon = +1$ | N/A |
| 15a | $z_1 = -z_2, z_3 = 0, w = -z, \epsilon = +1$ | N/A |
| 15b | $z_1 = \epsilon w_2, z_2 = z_3 = w_1 = w_3 = 0$ | $z_3 = \epsilon z_4, z_i = w_j = 0$ otherwise, |
| | | when $\alpha - \beta = 1$ |
| | | 0 otherwise |
| 16a | $z_1 = z_2, z_3 = 0, w = -z, \epsilon = -1$ | N/A |
| 16b | $w_1 = \epsilon w_2, z_1 = z_2 = z_3 = w_3 = 0$ | $z_2 = \epsilon w_5, z_i = w_j = 0$ otherwise, |
| | | when $\alpha - 2\beta = 1$ |
| | | $z_5 = \epsilon w_2, z_i = w_j = 0$ otherwise, |
| | | when $-\alpha + 2\beta = 1$ |
| | | 0 otherwise |
| 19 | $z_1 = z_2 = z_3 = 0$ | N/A |
| | $w_1 = w_2 = w_3$ | |
| 24 | $z_1 = \epsilon z_2, z_3 = 0, w = z$ | N/A |
| 25 | $z_1 = z_2 = z_3, w = z, \epsilon = -1$ | $z_1 = z_2 = z_3 = -\epsilon z_4 = -\epsilon z_5 = -\epsilon z_6, w = z$ |
| 27 | $z_1 = -z_2, z_3 = 0, w = z, \epsilon = -1, \psi = -1$ | N/A |
| 30 | N/A | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, w = -z$ |
| 31a | $z_1 = z_2 = z_3, w = z, \epsilon = +1$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, w = -z$ |
| 31b | $z_1 = z_2 = z_3 = 0$ | $z_1 = z_2 = z_3 = \epsilon w_4 = \epsilon w_5 = \epsilon w_6$ |
| | $w_1 = w_2 = w_3$ | $z_4 = z_5 = z_6 = w_1 = w_2 = w_3 = 0$ |
| | | if $(\alpha + \beta) \equiv 1 (mod 3)$ |
| | | $z_4 = z_5 = z_6 = \epsilon w_1 = \epsilon w_2 = \epsilon w_3$ |
| | | $z_1 = z_2 = z_3 = w_4 = w_5 = w_6 = 0$ |
| | | if $(\alpha + \beta) \equiv 2 (mod 3)$ |
| 32 | $z_1 = z_2 = z_3 = w_1 = w_2 = w_3, \epsilon = +1$ | $z_1 = z_2 = z_3 = \epsilon z_4 = \epsilon z_5 = \epsilon z_6, w = z$ |

## 4.5 The Planforms

The planforms are plotted using the program shown in Appendix E. In each case we set $z_1 = e^{2\pi i \frac{t}{T}}$, or another of the coefficients if $z_1 = 0$, and then write each other coefficient in terms of this, usually either $\pm z_1$ or zero. As with the equilibrium states shown in Chapter 2 we plot a director field that for each point in a square subset of $\mathbf{R}^2$ will draw a unit length line in the direction of the eigenvector associated with the largest eigenvalue. In this case the equation is:

$$Q = Q_0 + \delta \sum_{j=1}^{s} z_j e^{2\pi i(\mathbf{K}_j \cdot \mathbf{x} + t)} Q_j + w_j e^{-2\pi i(\mathbf{K}_j \cdot \mathbf{x} - t)} \overline{Q_j} + c.c. \qquad \delta \text{ small}$$

The patterns are time periodic and the movies show one full period in each case. The time interval $0 < t \leq 1$ is divided into 360 individual frames, this number is chosen to make it easy to spot when major changes occur.

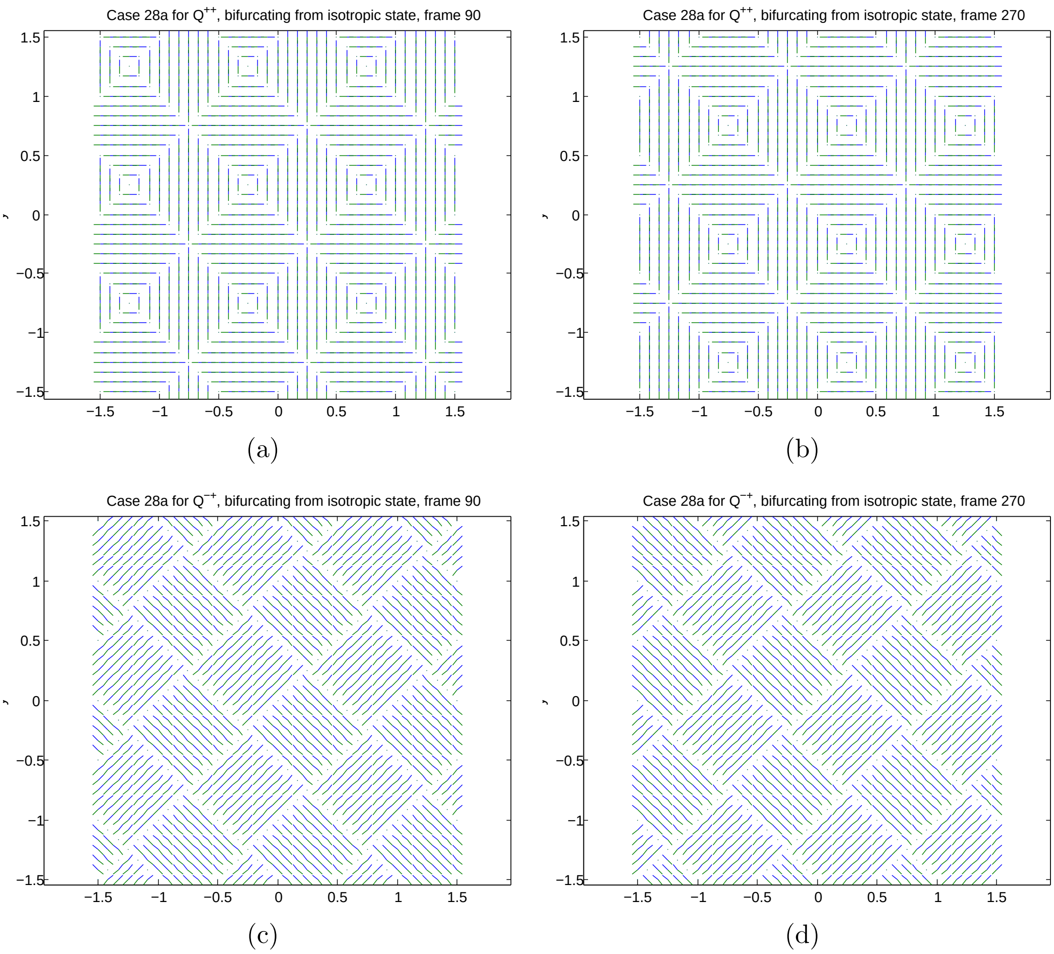
### 4.5.1 Planforms for the Square Lattice

The patterns on the square lattice are fairly straightforward to describe: each movie switches between two copies of the same pattern, translations of each other, passing through the isotropic state in between the two copies of the pattern. The points at which the isotropic state appears, twice in each time interval $0 < t \leq 1$, are dependent on the individual case.

#### The $\mathbf{C}^2$ cases

In the $\mathbf{C}^2$ cases there are three cases that produce patterns for each of the $Q^{++}$ and $Q^{-+}$ representations but there is only one distinct pattern for each representation and each case shows two different translations of the same pattern.

Figure 4.1: Frames from Square Lattice $\mathbf{C}^2$ Time Periodic Patterns




**Case 28a**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 180, 0/360$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 180, 0/360$

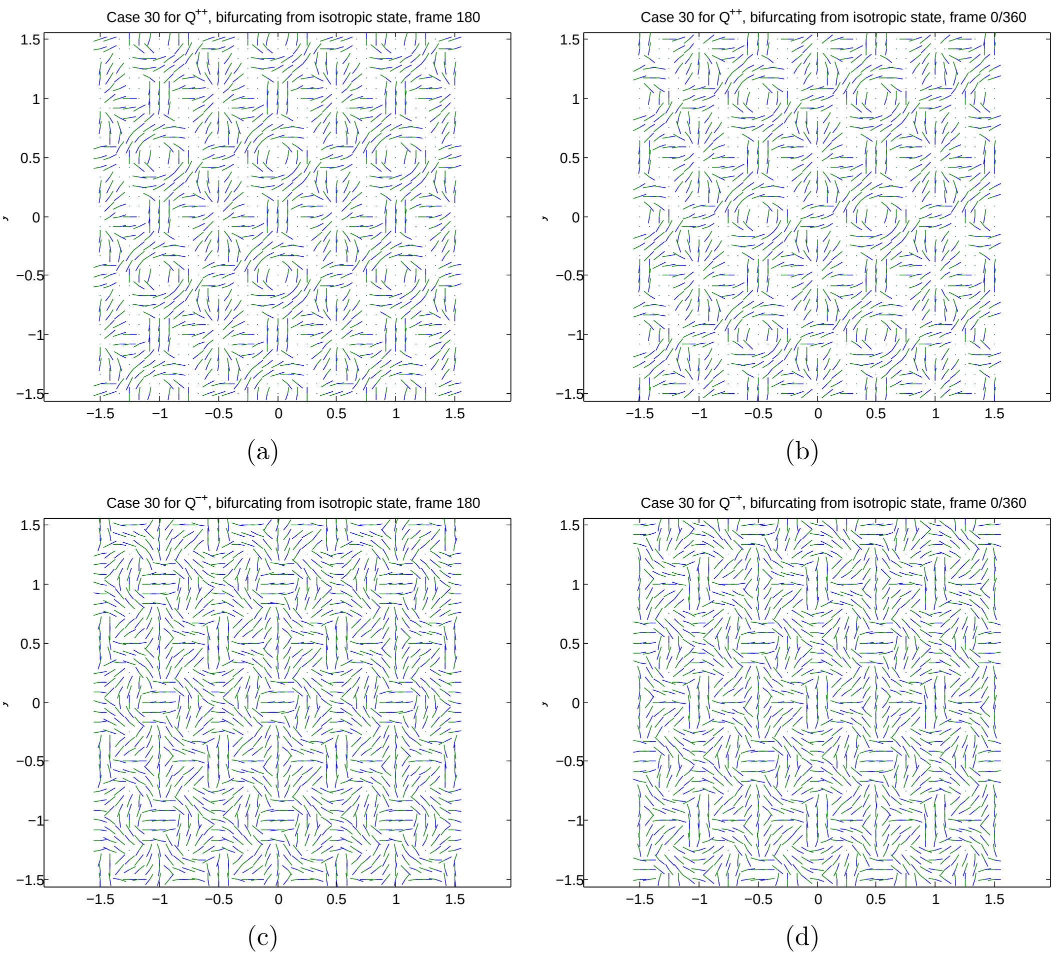
**Case 30**

$\mathbf{Q}^{++}$, same pattern as Case 28a, isotropic state appears at $t = 90, 270$

**Case 32**

$\mathbf{Q}^{-+}$, same pattern as Case 28a, isotropic state appears at $t = 90, 270$

Figure 4.2: Frames from Square Lattice $\mathbf{C}^4$ Time Periodic Patterns




**Case 42**

$\mathbf{Q}^{-+}$, same pattern as Case 28a, isotropic state appears at $t = 90, 270$

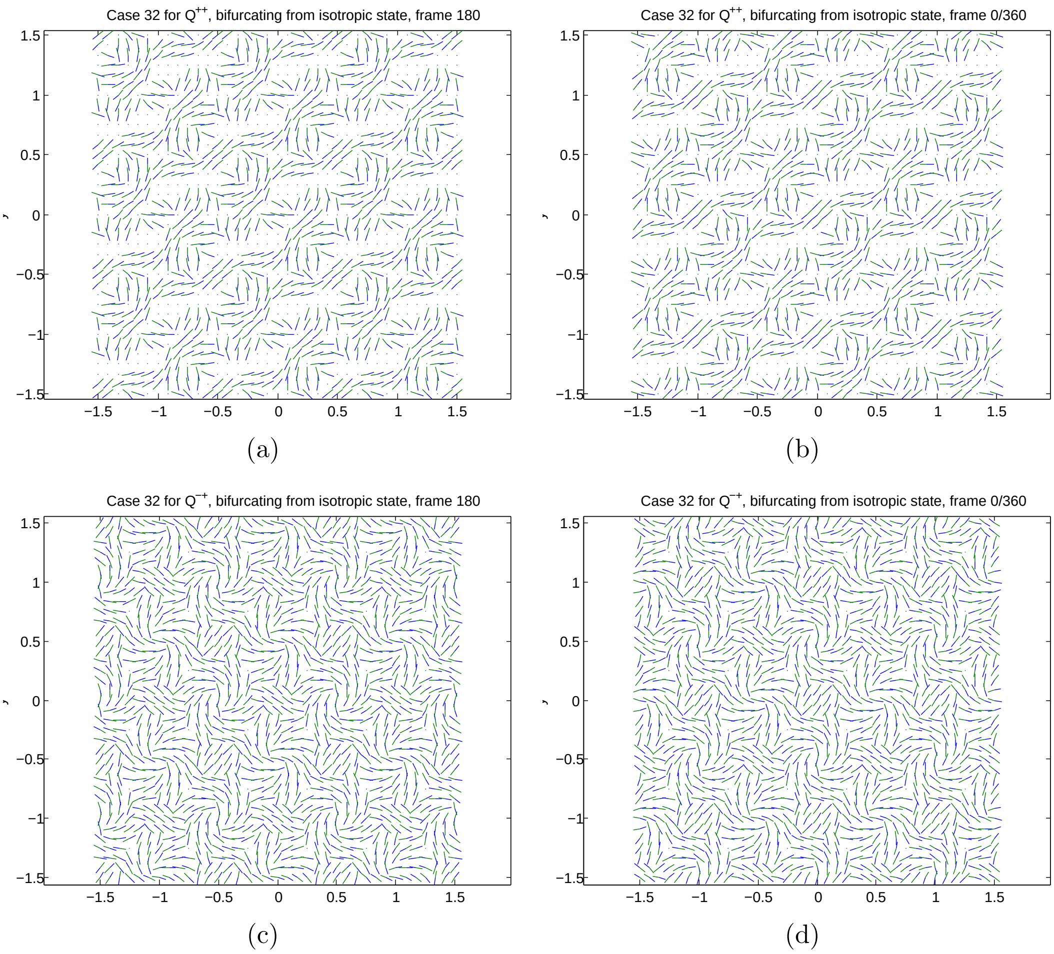
**Case 43**

$\mathbf{Q}^{-+}$, same pattern as Case 28a, isotropic state appears at $t = 90, 270$

### The $\mathbf{C}^4$ cases

In the $\mathbf{C}^4$ case there are two different patterns, two cases for each.

Figure 4.3: Frames from Square Lattice $\mathbf{C}^4$ Time Periodic Patterns




**Case 30**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 90, 270$

**Case 32**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 90, 270$

**Case 40**

$\mathbf{Q}^{++}$, same pattern as Case 30, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, same pattern as Case 30, isotropic state appears at $t = 90, 270$

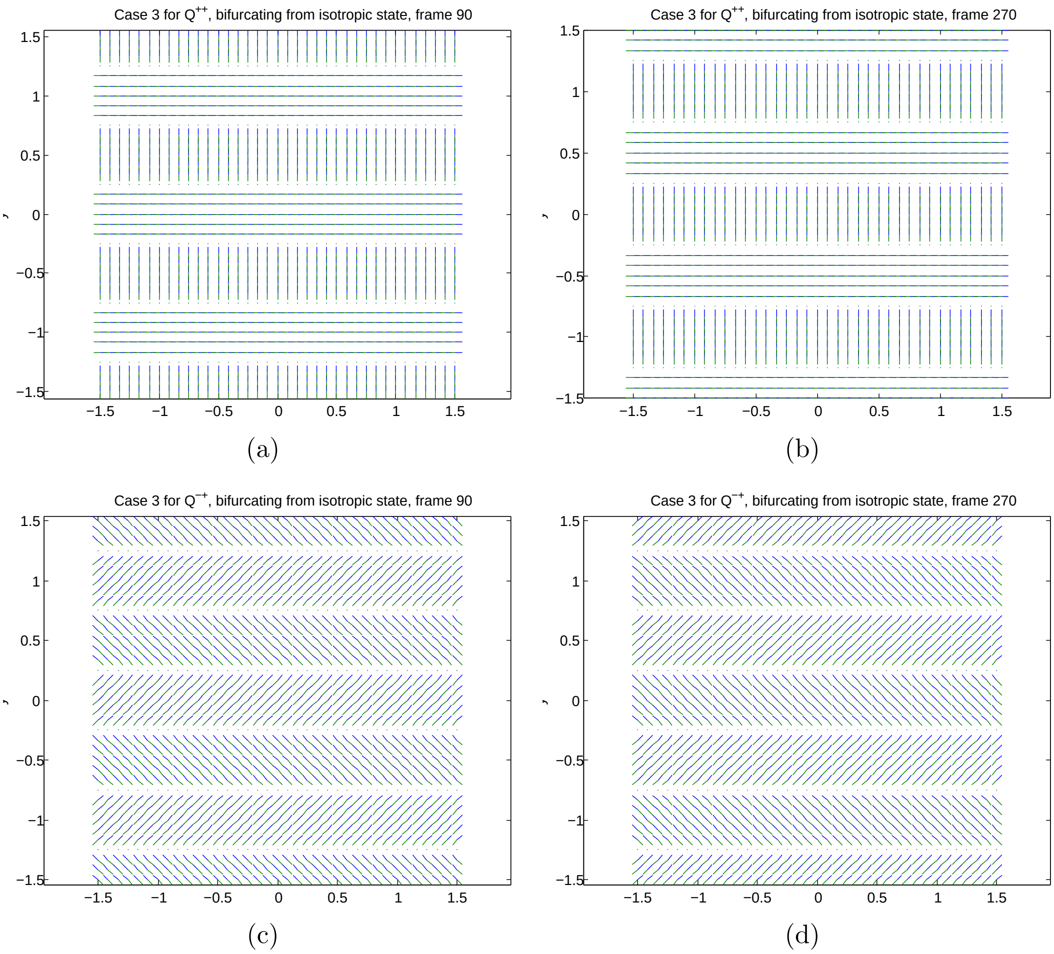
**Case 42**

$\mathbf{Q}^{++}$, same pattern as Case 32, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, same pattern as Case 32, isotropic state appears at $t = 90, 270$

### 4.5.2 Planforms for the Hexagonal Lattice

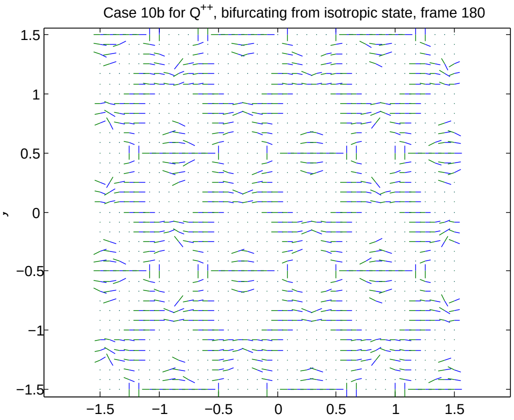
Figure 4.4: Frames from Hexagonal Lattice $\mathbf{C}^3$ Time Periodic Patterns



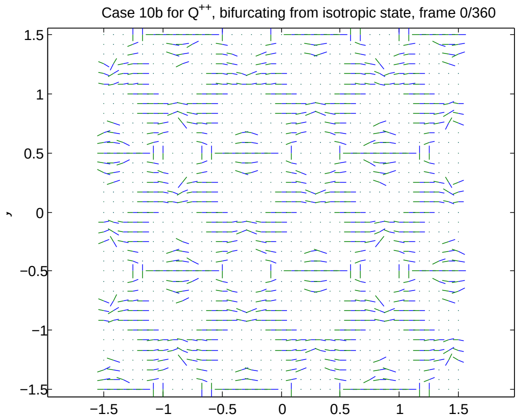

(a) (b) (c) (d)

The patterns for the hexagonal lattice are more complicated to describe than those for the square lattice. While some do switch between two copies of the same pattern, there is often much more movement in between these two stages. Some cases display only one copy of the same pattern between occurrences of the isotropic or homeotropic state, some switch between two different patterns, and some are rotating waves so we will see one pattern moving across the plane.

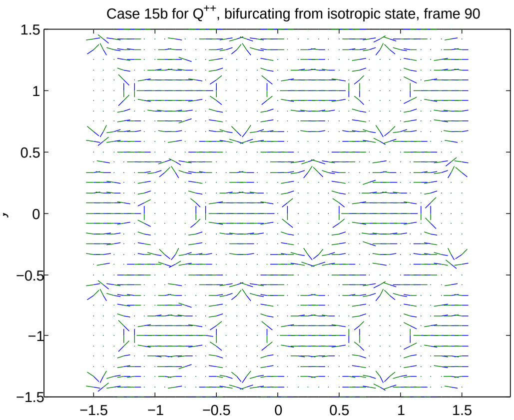
Figure 4.5: Frames from Hexagonal Lattice $\mathbf{C}^3$ Time Periodic Patterns



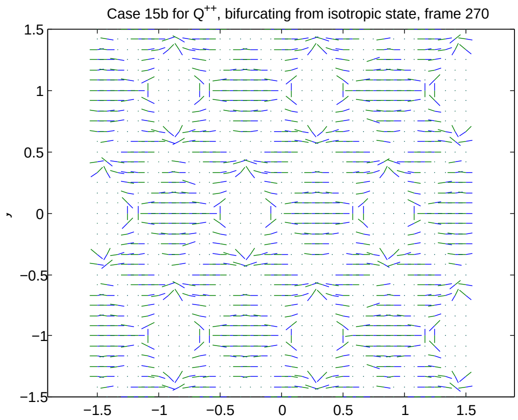

(a)



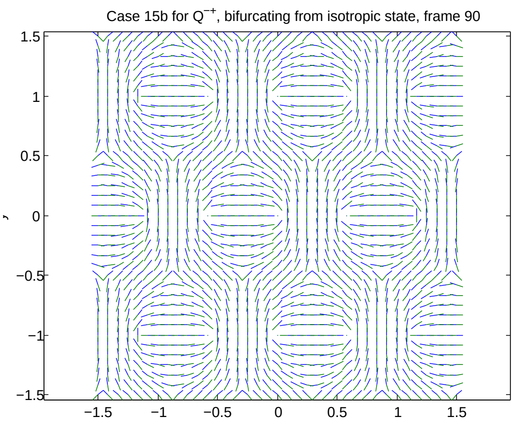

(b)



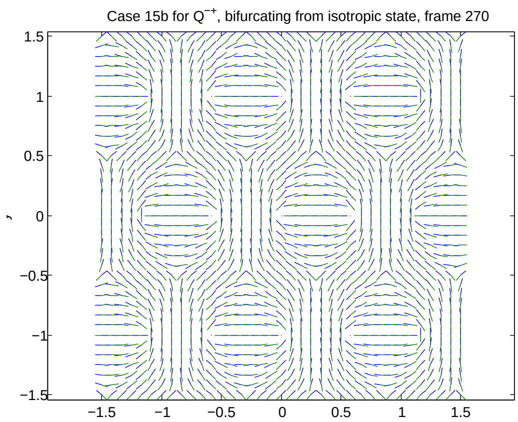

(c)




(d)

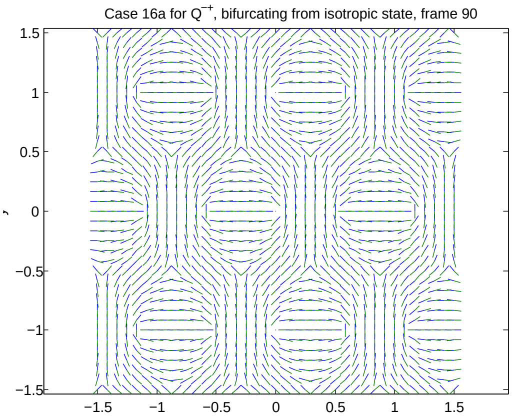



(e)

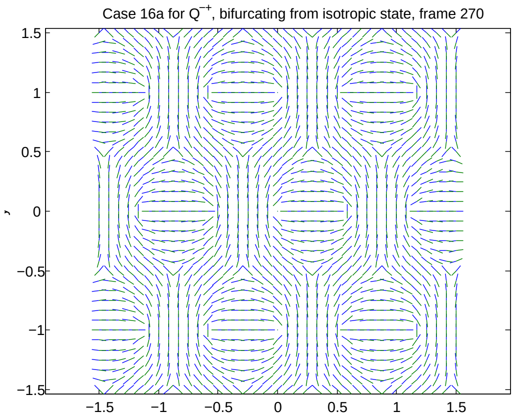



(f)

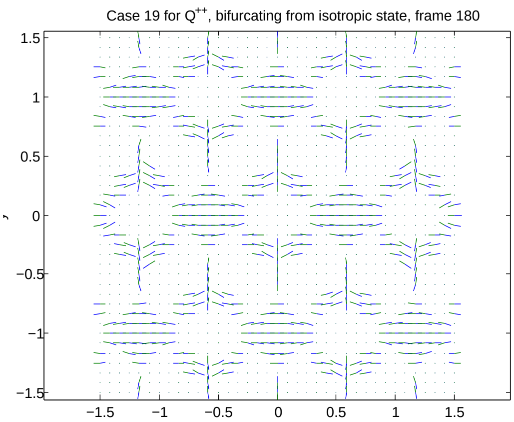
Figure 4.6: Frames from Hexagonal Lattice $\mathbf{C}^3$ Time Periodic Patterns



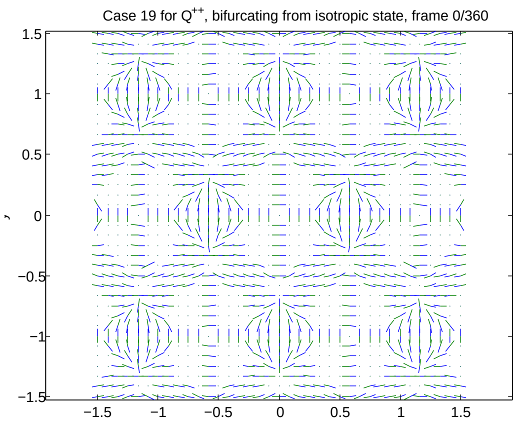

(a)



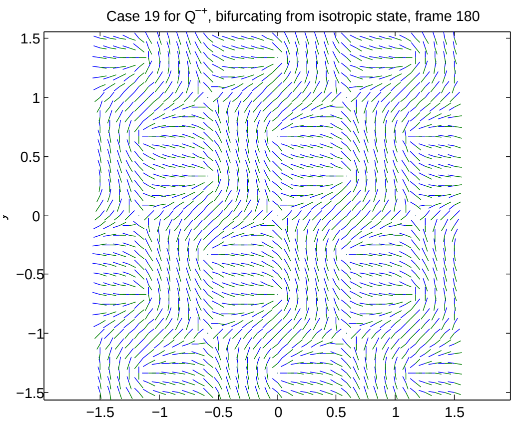

(b)



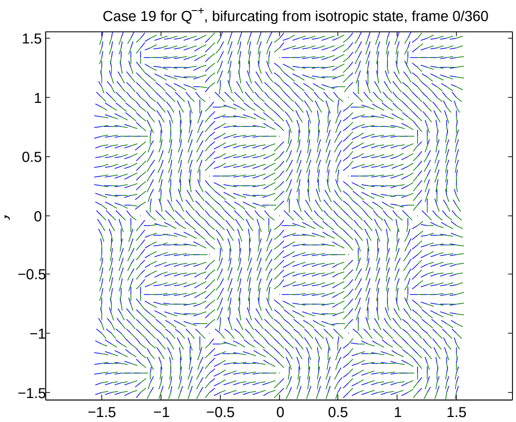

(c)




(d)

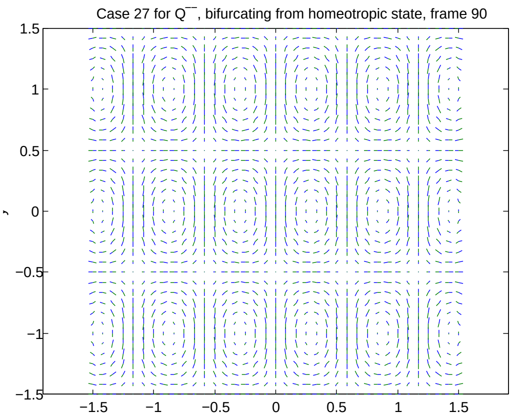



(e)

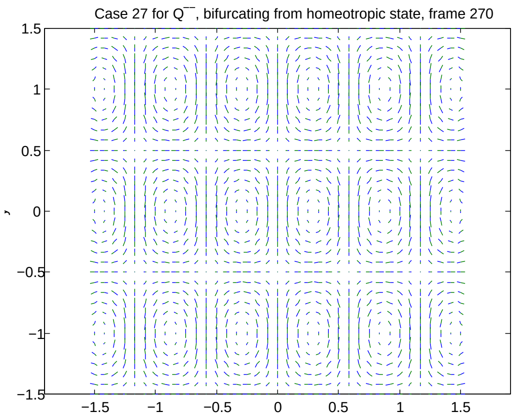



(f)

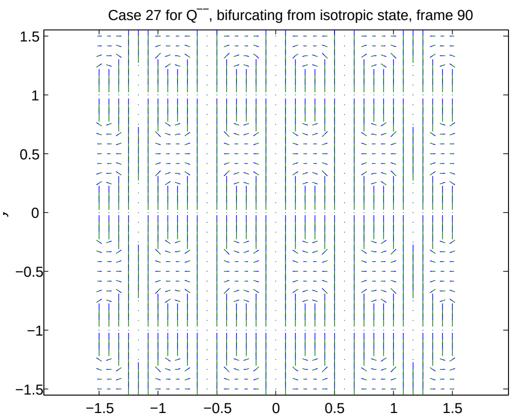
Figure 4.7: Frames from Hexagonal Lattice $\mathbf{C}^3$ Time Periodic Patterns



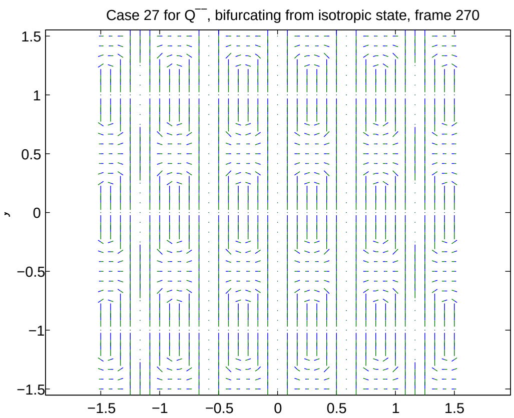

(a)



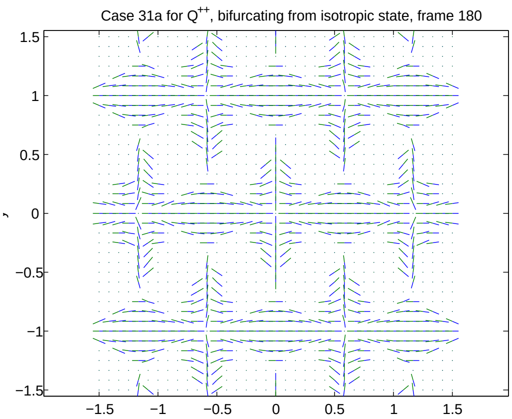

(b)



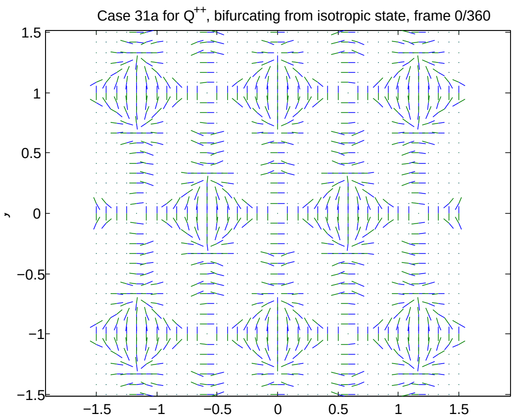

(c)




(d)

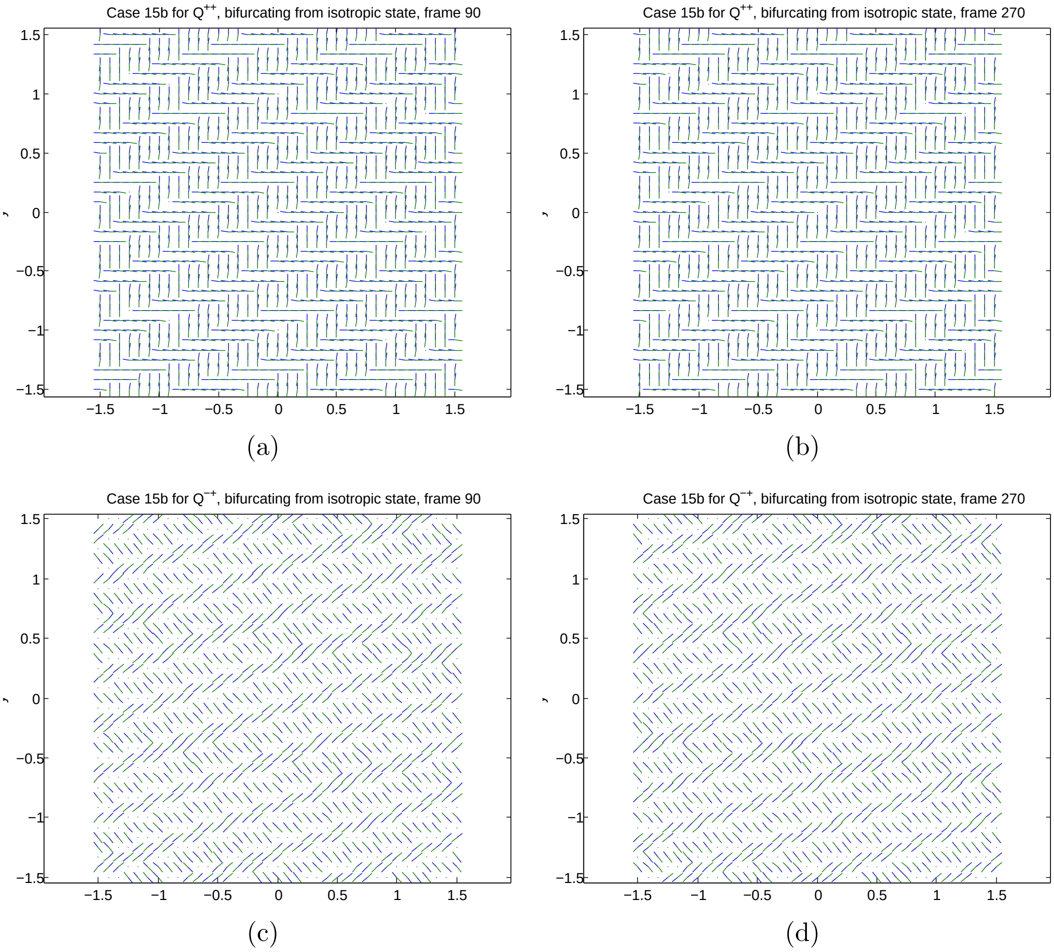



(e)




(f)

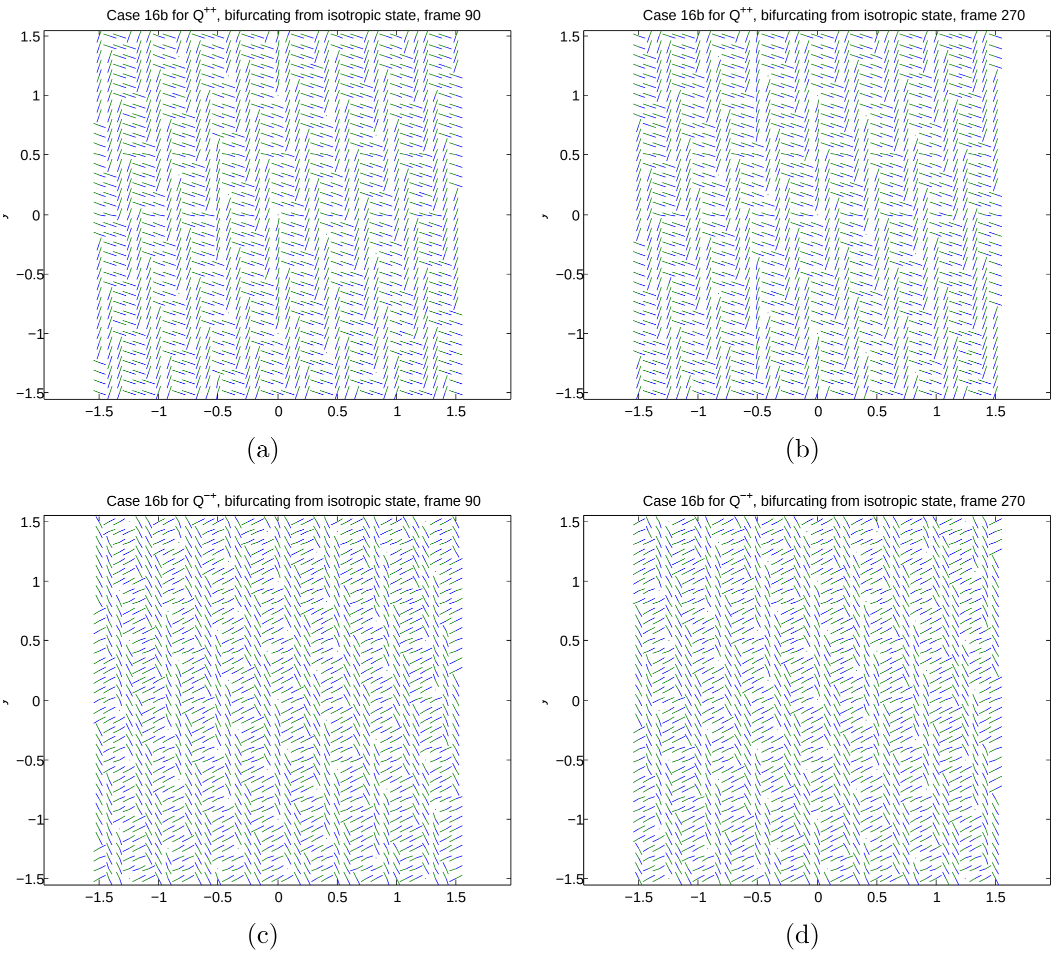
Figure 4.8: Frames from Hexagonal Lattice $\mathbf{C}^6$ Time Periodic Patterns




### The $\mathbf{C}^3$ cases

While quite a few cases switch between two copies of the same pattern passing through the isotropic state in between, for the $Q^{-+}$ representation it is often necessary to reduce the value $\delta$ dramatically in order for the isotropic state to actually appear. This results in the patterns in between being largely isotropic as well with only small sections of the plane showing non-isotropic behaviour. Because of this the stills shown in this chapter are produced using higher values of $\delta$ for these cases.

Figure 4.9: Frames from Hexagonal Lattice $\mathbf{C}^6$ Time Periodic Patterns



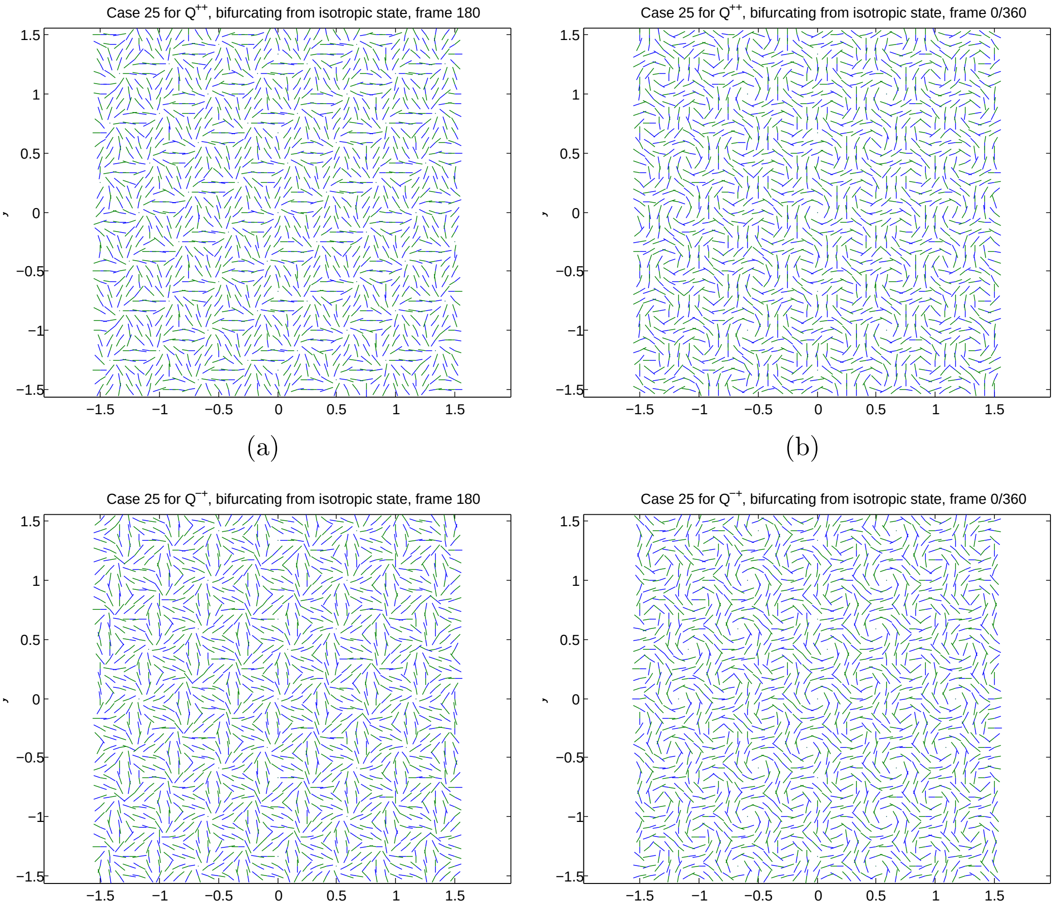

(a) (b) (c) (d)

**Case 3**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 150, 330$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 120, 300$

Figure 4.10: Frames from Hexagonal Lattice $\mathbf{C}^6$ Time Periodic Patterns




(a) (b) (c) (d)

**Case 10a**

$\mathbf{Q}^{++}$, same pattern as Case 3, isotropic state appears at $t = 120, 300$

$\mathbf{Q}^{-+}$, same pattern as Case 3, isotropic state appears at $t = 160, 340$

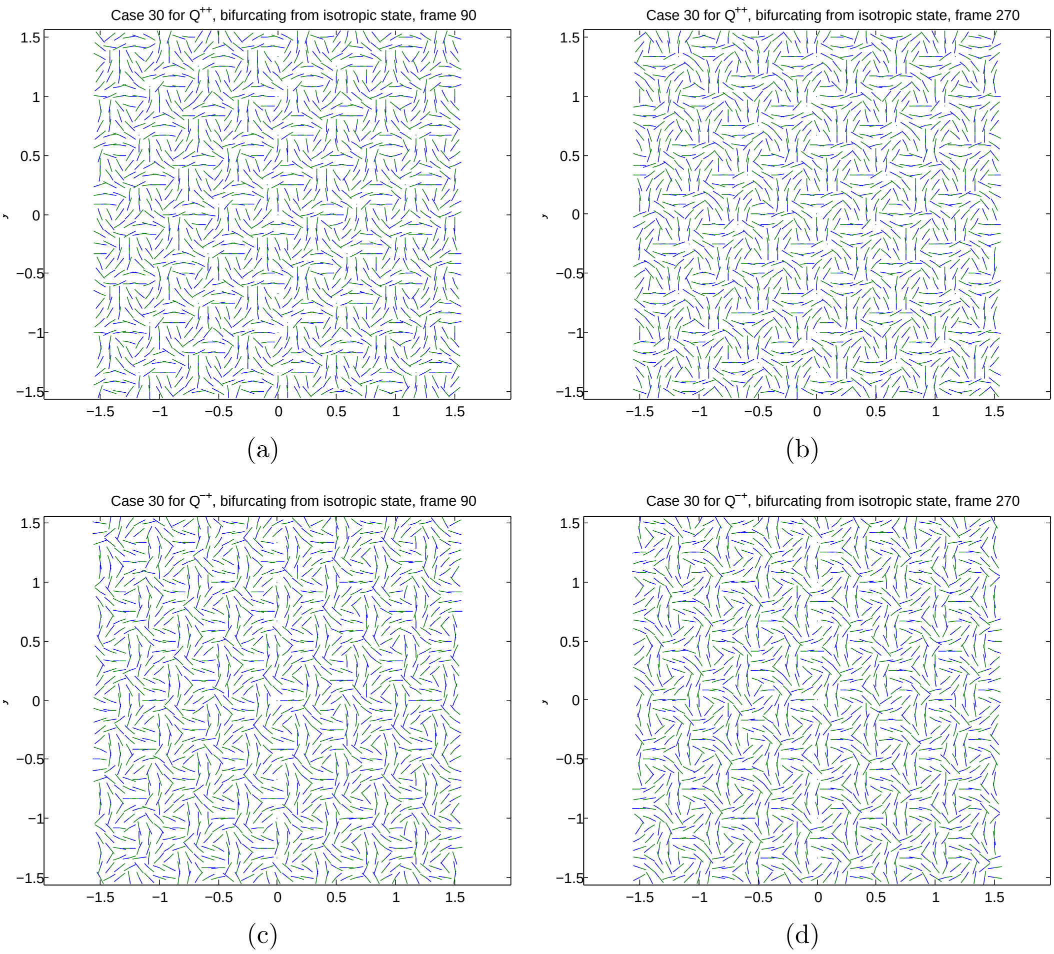
**Case 10b**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 90, 270$

**Case 15a**

$\mathbf{Q}^{++}$, same pattern as Case 10b, isotropic state appears at $t = 180, 0/360$

Figure 4.11: Frames from Hexagonal Lattice $\mathbf{C}^6$ Time Periodic Patterns




**Case 15b**

$\mathbf{Q}^{++}$, rotating wave travelling from right to left.

$\mathbf{Q}^{-+}$, rotating wave travelling from right to left.

**Case 16a**

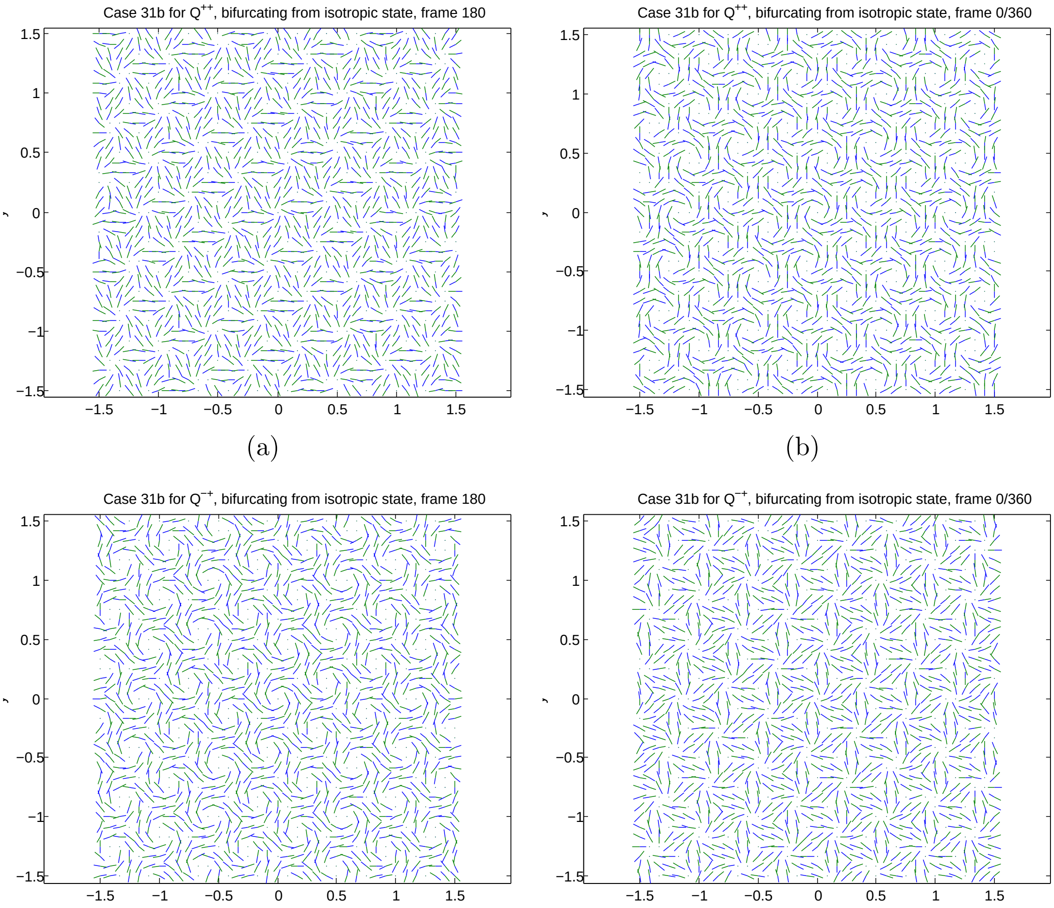
$\mathbf{Q}^{-+}$, isotropic state appears at $t = 180, 0/360$

**Case 19**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 30, 90, 150, 210, 270, 330$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 30, 90, 150, 210, 270, 330$

Figure 4.12: Frames from Hexagonal Lattice $\mathbf{C}^6$ Time Periodic Patterns




(a) (b) (c) (d)

**Case 24**

$\mathbf{Q}^{++}$, same pattern as Case 10b, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, same pattern as Case 16a, isotropic state appears at $t = 90, 270$

**Case 27**

$\mathbf{Q}^{--}$, bifurcating from isotropic state, isotropic state appears at $t = 180, 0/360$

$\mathbf{Q}^{--}$, bifurcating from homeotropic state, homeotropic state appears at $t = 180, 0/360$

**Case 31a**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 90, 270$

**The $\mathbf{C}^6$ cases**

**Case 15b**

$\mathbf{Q}^{++}$, rotating wave travelling from right to left.

$\mathbf{Q}^{-+}$, rotating wave travelling from right to left.

**Case 16b**

$\mathbf{Q}^{++}$, rotating wave travelling from bottom to top.

$\mathbf{Q}^{-+}$, rotating wave travelling from bottom to top.

**Case 25**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 90, 270$

**Case 30**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 90, 270$

**Case 31b**

$\mathbf{Q}^{++}$, isotropic state appears at $t = 30, 90, 150, 210, 270, 330$

$\mathbf{Q}^{-+}$, isotropic state appears at $t = 30, 90, 150, 210, 270, 330$

**Case 32**

$\mathbf{Q}^{++}$, same patterns as Case 31b, isotropic state appears at $t = 90, 270$

$\mathbf{Q}^{-+}$, same patterns as Case 31b, isotropic state appears at $t = 90, 270$

# 5

# Conclusion

Continuing the work of Chillingworth and Golubitsky [7] who calculated the set of patterns on the hexagonal and square lattices that can bifurcate from a homeotropic or planar isotropic state in a planar layer of nematic liquid crystal found from the standard representations of the symmetry groups, I have classified a second set of such patterns found from a second larger representation of each symmetry group. I have also classified sets of generic time periodic square and hexagonal patterns bifurcating from the same homeotropic or planar isotropic state. This is an extension of the work by Dionne *et al* [21] using the group theory methods applied in that paper to Rayleigh-Bénard convection, where the pattern is described by a scalar, to find patterns in the more complicated liquid crystal model where the pattern is described by a director in $\mathbf{R}^3$ defined by a $3 \times 3$ symmetric matrix with trace$= 0$. It is noted that the liquid crystal model produces a larger set of results than the Rayleigh-Bénard convection model shown in the Dionne paper. This is partly due to the symmetry group $\Gamma_{\mathcal{L}}$ containing a copy of $\mathbf{Z}_2$ to represent the reflection in the $xy$ plane that is not present in the Dionne paper, resulting in a larger set of possible subgroups of $\Gamma_{\mathcal{L}}$ and hence a larger number of wave pairs which then produce a larger number of twisted subgroups $\Sigma$ with $\dim \mathrm{Fix}(\Sigma) = 2$. Also, the four possible representations $Q^{\pm\pm}$ of $\Gamma_{\mathcal{L}}$ give the potential for four possible results for each wave pair. In actuality this does not occur, there are no patterns at all for the $Q^{+-}$ representation and only one set of patterns for the $Q^{--}$ representation, hexagonal case 27. However many of the other cases do produce patterns for both the $Q^{++}$ and $Q^{-+}$ representations. It is important to emphasize that this work does not attempt to predict experimental circumstances in which these patterns may be found. Instead it provides a catalogue of all those possible patterns that arise generically where the assumptions we have made about symmetries on lattices hold, and is intended as a useful reference for identification. Therefore any square or hexagonal lattice pattern observed in a suitable experimental set-up can be expected to be found in the resutls given here.

# Appendix A

# Action of the group $\Gamma_{\mathcal{L}}$

The action of the group $\Gamma_{\mathcal{L}}$ is as follows:

$$\text{Let } \gamma = (g, \psi), \text{ where } g \in E(2) \text{ and } \psi = \pm 1 \in \mathbf{Z}_2$$

$$g(\mathbf{x}) = B\mathbf{x} + b, \text{ where } B \in H_{\mathcal{L}} \subset 0(2), \text{ and } b \in \mathbf{T}^2$$

$$A = \begin{pmatrix} B & \\ & \psi \end{pmatrix}, \qquad \psi = \pm 1$$

The definition of the action of $\gamma$ on the function $\mathbf{Q}$ is:

$$\gamma \cdot \mathbf{Q}(\mathbf{x}, t) = A\mathbf{Q}(g^{-1}\mathbf{x}, t)A^{-1} \qquad \forall \gamma \in \Gamma.$$

To see that this is indeed a group action we check that

$$\gamma_1 \cdot (\gamma_2 \cdot \mathbf{Q}) = (\gamma_1 \gamma_2) \cdot \mathbf{Q}$$

where $\gamma_1 = (g_1, \psi_1)$ and $\gamma_2 = (g_2, \psi_2)$.

To simplify matters we will look at this in two stages, first we consider just the action $*$ of $g$ on the space of functions $\mathbf{Q} : \mathbf{R}^2 \times \mathbf{R} \to X$ where $X \approx \mathbf{R}^5$

is the space of real $3 \times 3$ symmetric matrices with trace$= 0$.

$$g * \mathbf{Q} : (x,t) \mapsto \mathbf{Q}(g^{-1}x, t)$$

This gives us

$$\begin{aligned} g_1 * (g_2 * \mathbf{Q}) : (x,t) &\mapsto (g_2 * \mathbf{Q})(g_1^{-1}x, t) \\ &= \mathbf{Q}(g_2^{-1}g_1^{-1}x, t) \\ &= \mathbf{Q}((g_1 g_2)^{-1}x, t) \\ &= ((g_1 g_2) * \mathbf{Q})(x,t). \end{aligned}$$

Now we look at the group action as a whole.

$$\begin{aligned} \gamma_1 \cdot (\gamma_2 \cdot \mathbf{Q}) = ((g_1,\psi_1)((g_2,\psi_2)\cdot \mathbf{Q}))(\mathbf{x},t) &= (g_1,\psi_1)(A_2((g_2 * \mathbf{Q})(x,t))A_2^{-1}) \\ &= A_1 A_2((g_1 * (g_2 * \mathbf{Q}))(x,t))A_2^{-1}A_1^{-1} \\ &= A_1 A_2(((g_1 g_2) * \mathbf{Q})(x,t))(A_1 A_2)^{-1} \\ &= ((g_1,\psi_1)(g_2,\psi_2))\mathbf{Q}(\mathbf{x},t) = (\gamma_1\gamma_2)\cdot \mathbf{Q}. \end{aligned}$$

## A.1 Example of a group action calculation

This example of the calculation of the group action on the elements of the kernel of $\mathsf{L}$ is from the standing waves $\mathbf{C}^2$ case, $Q^{+-}$ representation. The standard form of the elements of the kernel written in full is as follows.

$$\begin{aligned} \tilde{Q}^{+-} &= z_1 e^{2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} K_1 Q^{+-}(K_1)^{-1} + z_2 e^{2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} K_2 Q^{+-}(K_2)^{-1} \\ &\quad + \overline{z_1} e^{-2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} \overline{K_1 Q^{+-}(K_1)^{-1}} + \overline{z_2} e^{-2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} \overline{K_2 Q^{+-}(K_2)^{-1}} \end{aligned}$$

$$Q^{+-} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**The action of the rotation $(r, e, e)$ on $\tilde{Q}^{+-}$**

$$r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 1 in the bottom right hand corner of $R$ shows that in this case the reflection in the $xy$ plane $\tau$ acts as the identity.

$$\begin{aligned}(r,e,e)\tilde{Q}^{+-} = \; & z_1 e^{2\pi i(\mathbf{K_1}\cdot r^{-1}\mathbf{x}+t)} R K_1 Q^{+-}(K_1)^{-1}R^{-1} + z_2 e^{2\pi i(\mathbf{K_2}\cdot r^{-1}\mathbf{x}+t)} R K_2 Q^{+-}(K_2)^{-1}R^{-1} \\ & + \overline{z_1} e^{-2\pi i(\mathbf{K_1}\cdot r^{-1}\mathbf{x}+t)} R\overline{K_1 Q^{+-}(K_1)^{-1}}R^{-1} + \overline{z_2} e^{-2\pi i(\mathbf{K_2}\cdot r^{-1}\mathbf{x}+t)} R\overline{K_2 Q^{+-}(K_2)^{-1}}R^{-1} \\ = \; & z_1 e^{2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} K_2 Q^{+-}(K_2)^{-1} + z_2 e^{-2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} \overline{K_1 Q^{+-}(K_1)^{-1}} \\ & + \overline{z_1} e^{-2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} \overline{K_2 Q^{+-}(K_2)^{-1}} + \overline{z_2} e^{2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} K_1 Q^{+-}(K_1)^{-1}\end{aligned}$$

This gives the result

$$(r,e,e)(z_1, z_2) = (\overline{z_2}, z_1)$$

**The action of the translation $(e, v_1, e)$ on $\tilde{Q}^{+-}$**

$$v_1 = \left(\frac{1}{2}, 0\right)$$

$$\begin{aligned}(e,v_1,e)\tilde{Q}^{+-} = \; & z_1 e^{2\pi i(\mathbf{K_1}\cdot(\mathbf{x}-v_1)+t)} K_1 Q^{+-}(K_1)^{-1} + z_2 e^{2\pi i(\mathbf{K_2}\cdot(\mathbf{x}-v_1)+t)} K_2 Q^{+-}(K_2)^{-1} \\ & + \overline{z_1} e^{-2\pi i(\mathbf{K_1}\cdot(\mathbf{x}-v_1)+t)} \overline{K_1 Q^{+-}(K_1)^{-1}} + \overline{z_2} e^{-2\pi i(\mathbf{K_2}\cdot(\mathbf{x}-v_1)+t)} \overline{K_2 Q^{+-}(K_2)^{-1}} \\ = \; & z_1 e^{2\pi i(\mathbf{K_1}\cdot\mathbf{x}-\frac{1}{2}+t)} K_1 Q^{+-}(K_1)^{-1} + z_2 e^{2\pi i(\mathbf{K_2}\cdot\mathbf{x}-0+t)} K_2 Q^{+-}(K_2)^{-1} \\ & + \overline{z_1} e^{-2\pi i(\mathbf{K_1}\cdot\mathbf{x}-\frac{1}{2}+t)} \overline{K_1 Q^{+-}(K_1)^{-1}} + \overline{z_2} e^{-2\pi i(\mathbf{K_2}\cdot\mathbf{x}-0+t)} \overline{K_2 Q^{+-}(K_2)^{-1}} \\ = \; & z_1 e^{-\pi i} e^{2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} K_1 Q^{+-}(K_1)^{-1} + z_2 e^{2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} K_2 Q^{+-}(K_2)^{-1} \\ & + \overline{z_1} e^{\pi i} e^{-2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} \overline{K_1 Q^{+-}(K_1)^{-1}} + \overline{z_2} e^{-2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} \overline{K_2 Q^{+-}(K_2)^{-1}} \\ = \; & z_1(-1) e^{2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} K_1 Q^{+-}(K_1)^{-1} + z_2 e^{2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} K_2 Q^{+-}(K_2)^{-1} \\ & + \overline{z_1}(-1) e^{-2\pi i(\mathbf{K_1}\cdot\mathbf{x}+t)} \overline{K_1 Q^{+-}(K_1)^{-1}} + \overline{z_2} e^{-2\pi i(\mathbf{K_2}\cdot\mathbf{x}+t)} \overline{K_2 Q^{+-}(K_2)^{-1}}\end{aligned}$$

Which gives

$$(e, v_1, e)(z_1, z_2) = (-z_1, z_2)$$

**The action of the** $(r, v_1, e)$ **on** $\tilde{Q}^{+-}$

We can then combine these two elements to find the action of $(r, v_1, e)$ on $\tilde{Q}^{+-}$. The group elements act from right to left so we need to calculate $(r, e, e)((e, v_1, e)(z_1, z_2))$

$(r, e, e)(z_1, z_2) = (\overline{z_2}, z_1)$
$(e, v_1, e, )(z_1, z_2) = (-z_1, z_2)$

$$\begin{aligned} (r, e, e)((e, v_1, e)(z_1, z_2) &= (r, e, e)(-z_1, z_2) \\ &= (\overline{z_2}, -z_1) \end{aligned}$$

So

$$(r, v_1, e)(z_1, z_2) = (\overline{z_2}, -z_1).$$

# Appendix B

# Full group action tables

Table B.1: The Action of the Generators of $(\mathbf{D}_4 \ltimes \frac{1}{2}\mathsf{L}) \times \mathbf{Z}_2$

| $g$ | **Action on** $\mathbf{C}^2$ <br> $g(z_1, z_2)$ | **Action on** $\mathbf{C}^4$ <br> $g(z_1, z_2, z_3, z_4)$ |
|---|---|---|
| $(r, e, e)$ | $(\overline{z_2}, z_1)$ | $(\overline{z_2}, z_1, \overline{z_4}, z_3)$ |
| $(s, e, e)$ | $\epsilon(z_1, \overline{z_2})$ | $\epsilon(\overline{z_4}, \overline{z_3}, \overline{z_2}, \overline{z_1})$ |
| $(e, v_1, e)$ | $(-z_1, z_2)$ | $(-z_1, z_2, z_3, -z_4)$, $\alpha$ odd |
| | | $(z_1, -z_2, -z_3, z_4)$, $\beta$ odd |
| $(e, v_2, e)$ | $(z_1, -z_2)$ | $(z_1, -z_2, -z_3, z_4)$, $\alpha$ odd |
| | | $(-z_1, z_2, z_3, -z_4)$, $\beta$ odd |
| $(e, v_d, e)$ | $(-z_1, -z_2)$ | $(-z_1, -z_2, -z_3, -z_4)$ |
| $(e, e, \tau)$ | $\psi(z_1, z_2)$ | $\psi(z_1, z_2, z_3, z_4)$ |
| $(r^2, e, e)$ | $(\overline{z_1}, \overline{z_2})$ | $(\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4})$ |
| $(sr, e, e)$ | $\epsilon(\overline{z_2}, \overline{z_1})$ | $\epsilon(\overline{z_3}, z_4, \overline{z_1}, z_2)$ |
| $(sr^2, e, e)$ | $\epsilon(\overline{z_1}, z_2)$ | $\epsilon(z_4, z_3, z_2, z_1)$ |
| $(r, v_1, e)$ | $(\overline{z_2}, -z_1)$ | $(\overline{z_2}, -z_1, -\overline{z_4}, z_3)$, $\alpha$ odd |
| | | $(-\overline{z_2}, z_1, \overline{z_4}, -z_3)$, $\beta$ odd |
| $(r^2, v_1, e)$ | $(-\overline{z_1}, \overline{z_2})$ | $\epsilon(-\overline{z_1}, \overline{z_2}, \overline{z_3}, -\overline{z_4})$, $\alpha$ odd |
| | | $\epsilon(\overline{z_1}, -\overline{z_2}, -\overline{z_3}, \overline{z_4})$, $\beta$ odd |
| $(s, v_1, e)$ | $\epsilon(-z_1, \overline{z_2})$ | $\epsilon(-\overline{z_4}, \overline{z_3}, \overline{z_2}, -\overline{z_1})$, $\alpha$ odd |

Table B.1: The Action of the Generators of $(\mathbf{D}_4 \ltimes \frac{1}{2}\mathsf{L}) \times \mathbf{Z}_2$

| $g$ | **Action on** $\mathbf{C}^2$ <br> $g(z_1, z_2)$ | **Action on** $\mathbf{C}^4$ <br> $g(z_1, z_2, z_3, z_4)$ |
|---|---|---|
| | | $\epsilon(\overline{z_4}, -\overline{z_3}, -\overline{z_2}, \overline{z_1})$, $\beta$ odd |
| $(sr, v_1, e)$ | $\epsilon(\overline{z_2}, -\overline{z_1})$ | $\epsilon(\overline{z_3}, -z_4, -\overline{z_1}, z_2)$, $\alpha$ odd |
| | | $\epsilon(-\overline{z_3}, z_4, \overline{z_1}, -z_2)$, $\beta$ odd |
| $(sr^2, v_1, e)$ | $\epsilon(-\overline{z_1}, z_2)$ | $\epsilon(z_4, -z_3, -z_2, z_1)$, $\alpha$ odd |
| | | $\epsilon(-z_4, z_3, z_2, -z_1)$, $\beta$ odd |
| $(r, v_2, e)$ | $(-\overline{z_2}, z_1)$ | $(-\overline{z_2}, z_1, \overline{z_4}, -z_3)$, $\alpha$ odd |
| | | $(\overline{z_2}, -z_1, -\overline{z_4}, z_3)$, $\beta$ odd |
| $(r^2, v_2, e)$ | $(\overline{z_1}, -\overline{z_2})$ | $\epsilon(\overline{z_1}, -\overline{z_2}, -\overline{z_3}, \overline{z_4})$, $\alpha$ odd |
| | | $\epsilon(-\overline{z_1}, \overline{z_2}, \overline{z_3}, -\overline{z_4})$, $\beta$ odd |
| $(s, v_2, e)$ | $\epsilon(z_1, -\overline{z_2})$ | $\epsilon(\overline{z_4}, -\overline{z_3}, -\overline{z_2}, \overline{z_1})$, $\alpha$ odd |
| | | $\epsilon(-\overline{z_1}, \overline{z_2}, \overline{z_3}, -\overline{z_4})$, $\beta$ odd |
| $(sr, v_2, e)$ | $\epsilon(-\overline{z_2}, \overline{z_1})$ | $\epsilon(-\overline{z_3}, z_4, \overline{z_1}, -z_2)$, $\alpha$ odd |
| | | $\epsilon(\overline{z_3}, -z_4, -\overline{z_1}, z_2)$, $\beta$ odd |
| $(sr^2, v_2, e)$ | $\epsilon(\overline{z_1}, -z_2)$ | $\epsilon(z_4, -z_3, -z_2, z_1)$, $\alpha$ odd |
| | | $\epsilon(-z_4, z_3, z_2, -z_1)$, $\beta$ odd |
| $(r, v_d, e)$ | $(-\overline{z_2}, -z_1)$ | $(-\overline{z_2}, -z_1, -\overline{z_4}, -z_3)$ |
| $(r^2, v_d, e)$ | $(-\overline{z_1}, -\overline{z_2})$ | $(-\overline{z_1}, -\overline{z_2}, -\overline{z_3}, -\overline{z_4})$ |
| $(s, v_d, e)$ | $\epsilon(-z_1, -\overline{z_2})$ | $\epsilon(-\overline{z_4}, -\overline{z_3}, -\overline{z_2}, -\overline{z_1})$ |
| $(sr, v_d, e)$ | $\epsilon(-\overline{z_2}, -\overline{z_1})$ | $\epsilon(-\overline{z_3}, -z_4, -\overline{z_1}, -z_2)$ |
| $(sr^2, v_d, e)$ | $\epsilon(-\overline{z_1}, z_2)$ | $\epsilon(-z_4, -z_3, -z_2, -z_1)$ |

Table B.2: The Action of the Generators of $(\mathbf{D}_6 \ltimes \frac{1}{2}\mathsf{L}) \times \mathbf{Z}_2$

| $g$ | **Action on** $\mathbf{C}^3$ <br> $g(z_1, z_2, z_3)$ | **Action on** $\mathbf{C}^6$ <br> $g(z_1, z_2, z_3, z_4, z_5, z_6)$ |
|---|---|---|
| $(r, e, e)$ | $(\overline{z_2}, \overline{z_3}, \overline{z_1})$ | $(\overline{z_2}, \overline{z_3}, \overline{z_1}, \overline{z_5}, \overline{z_6}, \overline{z_4})$ |
| $(s, e, e)$ | $\epsilon(\overline{z_2}, \overline{z_1}, \overline{z_3})$ | $\epsilon(z_6, z_5, z_4, z_3, z_2, z_1)$ |
| $(e, v_1, e)$ | $(-z_1, z_2, -z_3)$ | $(-z_1, -z_2, z_3, -z_4, z_5, -z_6)$, $\alpha$ odd |

Table B.2: The Action of the Generators of $(\mathbf{D}_6 \ltimes \frac{1}{2}\mathsf{L}) \times \mathbf{Z}_2$

| $g$ | **Action on $\mathbf{C}^3$** $g(z_1, z_2, z_3)$ | **Action on $\mathbf{C}^6$** $g(z_1, z_2, z_3, z_4, z_5, z_6)$ |
|---|---|---|
| | | $(z_1, -z_2, -z_3, z_4, -z_5, -z_6)$, $\beta$ odd |
| | | $(-z_1, z_2, -z_3, -z_4, -z_5, z_6)$, $\alpha$ and $\beta$ odd |
| $(e, v_2, e)$ | $(-z_1, -z_2, z_3)$ | $(z_1, -z_2, -z_3, -z_4, -z_5, z_6)$, $\alpha$ odd |
| | | $(-z_1, z_2, -z_3, -z_4, z_5, -z_6)$, $\beta$ odd |
| | | $(-z_1, -z_2, z_3, z_4, -z_5, -z_6)$, $\alpha$ and $\beta$ odd |
| $(e, v_t, e)$ | $e^{(2\pi i)/3}(z_1, z_2, z_3)$ | $e^{(2\pi i)/3}(e^{(\alpha+\beta)}z_1, e^{(2\alpha-\beta)}z_2, e^{(-\alpha+2\beta)}z_3,$ |
| | | $e^{(-2\alpha+\beta)}z_4, e^{(\alpha+\beta)}z_5, e^{(\alpha-2\beta)}z_6)$ |
| $(e, e, \tau)$ | $\psi(z_1, z_2, z_3)$ | $\psi(z_1, z_2, z_3, z_4, z_5, z_6)$ |
| $(r^2, e, e)$ | $(z_3, z_1, z_2)$ | $(z_3, z_1, z_2, z_6, z_4, z_5)$ |
| $(r^3, e, e)$ | $(\overline{z_1}, \overline{z_2}, \overline{z_3})$ | $(\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}, \overline{z_5}, \overline{z_6})$ |
| $(sr, e, e)$ | $\epsilon(z_3, z_2, z_1)$ | $\epsilon(\overline{z_4}, \overline{z_6}, \overline{z_5}, \overline{z_1}, \overline{z_3}, \overline{z_2})$ |
| $(sr^2, e, e)$ | $\epsilon(\overline{z_1}, \overline{z_3}, \overline{z_2})$ | $\epsilon(z_5, z_4, z_6, z_2, z_1, z_3)$ |
| $(sr^3, e, e)$ | $\epsilon(z_2, z_1, z_3)$ | $\epsilon(\overline{z_6}, \overline{z_5}, \overline{z_4}, \overline{z_3}, \overline{z_2}, \overline{z_1})$ |
| $(r, v_1, e)$ | $(\overline{z_2}, -\overline{z_3}, -\overline{z_1})$ | $(-\overline{z_2}, \overline{z_3}, -\overline{z_1}, \overline{z_5}, -\overline{z_6}, -\overline{z_4})$, $\alpha$ odd |
| | | $(-\overline{z_2}, -\overline{z_3}, \overline{z_1}, -\overline{z_5}, -\overline{z_6}, \overline{z_4})$, $\beta$ odd |
| | | $(\overline{z_2}, -\overline{z_3}, -\overline{z_1}, -\overline{z_5}, \overline{z_6}, -\overline{z_4})$, $\alpha$ and $\beta$ odd |
| $(r^2, v_1, e)$ | $(-z_3, -z_1, z_2)$ | $(z_3, -z_1, -z_2, -z_6, -z_4, z_5)$, $\alpha$ odd |
| | | $(-z_3, z_1, -z_2, -z_6, z_4, -z_5)$, $\beta$ odd |
| | | $(-z_3, -z_1, z_2, z_6, -z_4, -z_5)$, $\alpha$ and $\beta$ odd |
| $(r^3, v_1, e)$ | $(-\overline{z_1}, \overline{z_2}, -\overline{z_3})$ | $(-\overline{z_1}, -\overline{z_2}, \overline{z_3}, -\overline{z_4}, \overline{z_5}, -\overline{z_6})$, $\alpha$ odd |
| | | $(\overline{z_1}, -\overline{z_2}, -\overline{z_3}, \overline{z_4}, -\overline{z_5}, -\overline{z_6})$, $\beta$ odd |
| | | $(-\overline{z_1}, \overline{z_2}, -\overline{z_3}, -\overline{z_4}, -\overline{z_5}, \overline{z_6})$, $\alpha$ and $\beta$ odd |
| $(s, v_1, e)$ | $\epsilon(\overline{z_2}, -\overline{z_1}, -\overline{z_3})$ | $\epsilon(-z_6, z_5, -z_4, z_3, -z_2, -z_1)$, $\alpha$ odd |
| | | $\epsilon(-z_6, -z_5, z_4, -z_3, -z_2, z_1)$, $\beta$ odd |
| | | $\epsilon(z_6, -z_5, -z_4, -z_3, z_2, -z_1)$, $\alpha$ and $\beta$ odd |
| $(sr, v_1, e)$ | $\epsilon(-z_3, z_2, -z_1)$ | $\epsilon(-\overline{z_4}, -\overline{z_6}, \overline{z_5}, -\overline{z_1}, \overline{z_3}, -\overline{z_2})$, $\alpha$ odd |
| | | $\epsilon(\overline{z_4}, -\overline{z_6}, -\overline{z_5}, \overline{z_1}, -\overline{z_3}, -\overline{z_2})$, $\beta$ odd |
| | | $\epsilon(-\overline{z_4}, \overline{z_6}, -\overline{z_5}, -\overline{z_1}, -\overline{z_3}, \overline{z_2})$, $\alpha$ and $\beta$ odd |

Table B.2: The Action of the Generators of $(\mathbf{D}_6 \ltimes \frac{1}{2}\mathsf{L}) \times \mathbf{Z}_2$

| $g$ | **Action on $\mathbf{C}^3$** $g(z_1, z_2, z_3)$ | **Action on $\mathbf{C}^6$** $g(z_1, z_2, z_3, z_4, z_5, z_6)$ |
|---|---|---|
| $(sr^2, v_1, e)$ | $\epsilon(-\overline{z_1}, -\overline{z_3}, \overline{z_2})$ | $\epsilon(z_5, -z_4, -z_6, -z_2, -z_1, z_3)$, $\alpha$ odd |
| | | $\epsilon(-z_5, z_4, -z_6, -z_2, z_1, -z_3)$, $\beta$ odd |
| | | $\epsilon(-z_5, -z_4, z_6, z_2, -z_1, -z_3)$, $\alpha$ and $\beta$ odd |
| $(sr^3, v_1, e)$ | $\epsilon(z_2, -z_1, -z_3)$ | $\epsilon(-\overline{z_6}, \overline{z_5}, -\overline{z_4}, \overline{z_3}, -\overline{z_2}, -\overline{z_1})$, $\alpha$ odd |
| | | $\epsilon(-\overline{z_6}, -\overline{z_5}, \overline{z_4}, -\overline{z_3}, -\overline{z_2}, \overline{z_1})$, $\beta$ odd |
| | | $\epsilon(\overline{z_6}, -\overline{z_5}, -\overline{z_4}, -\overline{z_3}, \overline{z_2}, -\overline{z_1})$, $\alpha$ and $\beta$ odd |
| $(r, v_2, e)$ | $(-\overline{z_2}, \overline{z_3}, -\overline{z_1})$ | $(-\overline{z_2}, -\overline{z_3}, \overline{z_1}, -\overline{z_5}, \overline{z_6}, -\overline{z_4})$, $\alpha$ odd |
| | | $(\overline{z_2}, -\overline{z_3}, -\overline{z_1}, \overline{z_5}, -\overline{z_6}, -\overline{z_4})$, $\beta$ odd |
| | | $(-\overline{z_2}, \overline{z_3}, -\overline{z_1}, -\overline{z_5}, -\overline{z_6}, \overline{z_4})$, $\alpha$ and $\beta$ odd |
| $(r^2, v_2, e)$ | $(z_3, -z_1, -z_2)$ | $(-z_3, z_1, -z_2, z_6, -z_4, -z_5)$, $\alpha$ odd |
| | | $(-z_3, -z_1, z_2, -z_6, -z_4, z_5)$, $\beta$ odd |
| | | $(z_3, -z_1, -z_2, -z_6, z_4, -z_5)$, $\alpha$ and $\beta$ odd |
| $(r^3, v_2, e)$ | $(-\overline{z_1}, -\overline{z_2}, \overline{z_3})$ | $(\overline{z_1}, -\overline{z_2}, -\overline{z_3}, -\overline{z_4}, -\overline{z_5}, \overline{z_6})$, $\alpha$ odd |
| | | $(-\overline{z_1}, \overline{z_2}, -\overline{z_3}, -\overline{z_4}, \overline{z_5}, -\overline{z_6})$, $\beta$ odd |
| | | $(-\overline{z_1}, -\overline{z_2}, \overline{z_3}, \overline{z_4}, -\overline{z_5}, -\overline{z_6})$, $\alpha$ and $\beta$ odd |
| $(s, v_2, e)$ | $\epsilon(-\overline{z_2}, -\overline{z_1}, \overline{z_3})$ | $\epsilon(z_6, -z_5, -z_4, -z_3, -z_2, z_1)$, $\alpha$ odd |
| | | $\epsilon(-z_6, z_5, -z_4, -z_3, z_2, -z_1)$, $\beta$ odd |
| | | $\epsilon(-z_6, -z_5, z_4, z_3, -z_2, -z_1)$, $\alpha$ and $\beta$ odd |
| $(sr, v_2, e)$ | $\epsilon(z_3, -z_2, -z_1)$ | $\epsilon(-\overline{z_4}, \overline{z_6}, -\overline{z_5}, \overline{z_1}, -\overline{z_3}, -\overline{z_2})$, $\alpha$ odd |
| | | $\epsilon(-\overline{z_4}, -\overline{z_6}, \overline{z_5}, -\overline{z_1}, -\overline{z_3}, \overline{z_2})$,, $\beta$ odd |
| | | $\epsilon(\overline{z_4}, -\overline{z_6}, -\overline{z_5}, -\overline{z_1}, \overline{z_3}, -\overline{z_2})$,, $\alpha$ and $\beta$ odd |
| $(sr^2, v_2, e)$ | $\epsilon(-\overline{z_1}, \overline{z_3}, -\overline{z_2})$ | $\epsilon(-z_5, -z_4, z_6, -z_2, z_1, -z_3)$, $\alpha$ odd |
| | | $\epsilon(z_5, -z_4, -z_6, z_2, -z_1, -z_3)$, $\beta$ odd |
| | | $\epsilon(-z_5, z_4, -z_6, -z_2, -z_1, z_3)$, $\alpha$ and $\beta$ odd |
| $(sr^3, v_2, e)$ | $\epsilon(-z_2, -z_1, z_3)$ | $\epsilon(\overline{z_6}, -\overline{z_5}, -\overline{z_4}, -\overline{z_3}, -\overline{z_2}, \overline{z_1})$, $\alpha$ odd |
| | | $\epsilon(-\overline{z_6}, \overline{z_5}, -\overline{z_4}, -\overline{z_3}, \overline{z_2}, -\overline{z_1})$, $\beta$ odd |
| | | $\epsilon(-\overline{z_6}, -\overline{z_5}, \overline{z_4}, \overline{z_3}, -\overline{z_2}, -\overline{z_1})$, $\alpha$ and $\beta$ odd |

It is worth noting that the action of $v_t$ commutes with every other group element in the $\mathbf{C}^3$ case but in the $\mathbf{C}^6$ case it only commutes with the other translation elements $v_1$ and $v_2$.

# Appendix C

# Example of $\mathrm{Fix}(G^{\Theta})$ calculation

This example shows how to find the generating elements of the twisted subgroup $\Sigma = G^{\Theta}$ and its fixed-point subspace. The example shown is case 28a on the square lattice.

$K = \mathbf{Z}_2^2[(sr, e, e), (e, e, \tau)]$
$G = \mathbf{Z}_2^3[(r^2, e, e), (sr, e, e), (e, e, \tau)]$
$G/K = \mathbf{Z}_2[(r^2, e, e)]$

There are three generators of $\Sigma = G^{\Theta}$. The first two are the generators of $K$, that is $((sr, e, e), 0)$ and $((e, e, \tau), 0)$. The third element is found by taking the generator of $G/K$, which is the element $(r^2, e, e)$ and mapping it onto $\mathbf{S}^1$, which gives us the element $((r^2, e, e), \frac{1}{2})$. The actions of these elements are found by combining the actions of their component parts, recall that the group acts from right to left.

| Element | Action on $\mathbf{C}^3$ | Action on $\mathbf{C}^6$ |
|---|---|---|
| $g$ | $g(z_1, z_2, z_3, w_1, w_2, w_3)$ | $g(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |
| | | |
| $((r, e, e), 0)$ | $(w_2, z_1, z_2, w_1)$ | $(w_2, z_1, w_4, z_3, z_2, w_1, z_4, w_3)$ |
| $((s, e, e), 0)$ | $\epsilon(z_1, w_2, w_1, z_2)$ | $\epsilon(w_4, w_3, w_2, w_1, z_4, z_3, z_2, z_1)$ |
| $((sr, e, e), 0)$ | $\epsilon(w_2, w_1, z_2, z_1)$ | $\epsilon(w_3, z_4, w_1, z_2, z_3, w_4, z_1, w_2)$ |
| | | |
| $((e, e, \tau), 0)$ | $\psi(z_1, z_2, w_1, w_2)$ | $\psi(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$ |
| | | |
| $((r, e, e), 0)$ | $(w_2, z_1, z_2, w_1)$ | $(w_2, z_1, w_4, z_3, z_2, w_1, z_4, w_3)$ |
| $((r^2, e, e), 0)$ | $(w_1, w_2, z_1, z_2)$ | $(w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4)$ |
| $((e, e, e), \Theta)$ | $e^{2\pi i\Theta}(z_1, z_2, z_3, w_1, w_2, w_3)$ | $e^{2\pi i\Theta}(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |
| $((e, e, e), \frac{1}{2})$ | $(-1)(z_1, z_2, z_3, w_1, w_2, w_3)$ | $(-1)(z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3, w_4, w_5, w_6)$ |
| $((r^2, e, e), \frac{1}{2})$ | $(-1)(w_1, w_2, z_1, z_2)$ | $(-1)(w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4)$ |

Next we find the individual elements' fixed-point subspaces and from there it is easy to see what their common fixed-point subspace is.

| Element | Fixed-point subspace in $\mathbf{C}^2$ | Fixed-point subspace in $\mathbf{C}^4$ |
|---|---|---|
| $((sr, e, e), 0)$ | $z_1 = \epsilon w_2, z_2 = \epsilon w_1$ | $z_1 = \epsilon w_3, z_2 = \epsilon z_4, z_3 = \epsilon w_1, w_2 = \epsilon w_4$ |
| $((e, e, \tau), 0)$ | $\mathbf{C}^2$ when $\psi = +1$ | $\mathbf{C}^6$ when $\psi = +1$ |
| | 0 when $\psi = -1$ | 0 when $\psi = -1$ |
| $((r^2, e, e), \frac{1}{2})$ | $z_1 = -w_1, z_2 = -w_2$ | $z_1 = -w_1, z_2 = -w_2, z_3 = -w_3, z_4 = -w_4$ |

From this we can work out the fixed-point subspace of the group $G^\Theta$ as a whole.

| Fix($G^{\Theta}$) in $\mathbf{C}^2$ | dimFix($G^{\Theta}$) in $\mathbf{C}^2$ |
|---|---|
| $z_1 = -\epsilon z_2 = -w_1 = \epsilon w_2$ when $\psi = +1$ | 2 |
| 0 when $\psi = -1$ | 0 |

| Fix($G^{\Theta}$) in $\mathbf{C}^4$ | dimFix($G^{\Theta}$) in $\mathbf{C}^4$ |
|---|---|
| $z_1 = -\epsilon z_3 = -w_1 = \epsilon w_3, z_2 = \epsilon z_4 = -w_2 = -\epsilon w_4$ when $\psi = +1$ | 4 |
| 0 when $\psi = -1$ | 0 |

This confirms the result from Section 4.3.1, there will be a Hopf bifurcation for the $\mathbf{C}^2$ case when $\psi = +1$ because that gives dimFix$(G^{\Theta}) = 2$, but not otherwise.

# Appendix D

# Index of Notation

| | |
|---|---|
| $l_j$ | Base vector of lattice |
| $\mathcal{L}$ | Lattice |
| $\mathcal{L}^*$ | Dual lattice |
| $\mathbf{k}_j$ | base vector for Dual lattice |
| $\mathbf{K}_j$ | A wave vector |
| $Q$ | Generic $3 \times 3$ symmetric matrix with trace$= 0$ representing an ellipsoid |
| $Q_0$ | $3 \times 3$ matrix of the homeotropic or isotropic trivial states |
| $\Gamma_{\mathcal{L}}$ | The symmetry group $H_{\mathcal{L}} \ltimes \mathbf{T}^2 \times \mathbf{Z}_2$ |
| $\Gamma_{\frac{1}{2}\mathcal{L}}$ | The group $(H_{\mathcal{L}} \ltimes \frac{1}{2}\mathcal{L}) \times \mathbf{Z}_2$ |
| $H_{\mathcal{L}}$ | The holohedry (group of symmetries) of the lattice $\mathcal{L}$ |
| $\mathbf{T}^2$ | The torus group $\mathbf{R}^2/\mathcal{L}$ of translations on the lattice |
| $\mathbf{S}^1$ | The circle group |
| $\frac{1}{2}\mathcal{L}$ | The group generated by the half lattice points, isomorphic to $\mathbf{Z}_2^2$ |
| $\Sigma$ | An isotropy subgroup of $\Gamma$ |
| $g$ | An element of a group $G$ |
| $G$ | A group |
| $E(2)$ | The Euclidean group of all translations, rotations and reflections in the plane |
| $O(n)$ | The group of orthogonal matrices |
| $\mathcal{F}_{\mathcal{L}}$ | The space of matrix functions periodic with respect to $\mathcal{L}$ |
| $k$ | wave number, the length of a wave vector |

| | |
|---|---|
| $k_c$ | critical wave number |
| $E^+(g)$ | The projection of the +1 eigenspace of the natural representation of $g$ into $\mathbf{T}^2$ |
| $E^+(g)$ | The projection of the +1 eigenspace of the natural representation of $g$ into $\mathbf{T}^2$ |
| $F^+(g)$ | $\{v \in \mathbf{T}^2 \vert gv = v\}$ |
| $F^-(g)$ | $\{v \in \mathbf{T}^2 \vert gv = -v\}$ |
| $\lambda$ | bifurcation parameter |
| $\mathbf{u}$ | a matrix valued function of $\mathbf{x}$ and $t$ |
| $\mathbf{x}$ | A position vector in $\mathbf{R}^2$ |
| $t$ | Time |
| $\mathbf{F}$ | System of partial differential equations, $\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}, \lambda)$ |
| $\mathsf{L}$ | The system of differential equations linearized about $Q_0$, $\mathsf{L} = d\mathbf{F}\vert_{(Q_0,\lambda)}$ |
| $V$ | Chapters 1 and 2, The kernel of $\mathsf{L}$ |
| $V$ | Chapters 3 and 4, The $i$ eigenspace of $\mathsf{L}$ |
| $\tilde{Q}$ | Generic element of $ker\mathsf{L}$ |
| $\tau$ | reflection $z \to -z$ |
| $\kappa$ | reflection $y \to -y$ |
| $\times$ | direct product |
| $H_{41}$ | $\mathbf{D}_4[r,s]/\mathbf{Z}_2[r^2] = \{\{e,r^2\},\{r,r^3\},\{s,sr^2\},\{sr,sr^3\}\} \approx \mathbf{Z}_2^2$ |
| $H_{42}$ | $\mathbf{D}_4[r,s]/\mathbf{Z}_2^2[r^2,s] = \{\{e,r^2,s,sr^2\},\{r,r^3,sr,sr^3\}\} \approx \mathbf{Z}_2$ |
| $H_{43}$ | $\mathbf{D}_4[r,s]/\mathbf{Z}_2^2[r^2,sr] = \{\{e,r^2,sr,sr^3\},\{r,r^3,sr^2,s\}\} \approx \mathbf{Z}_2$ |
| $H_{61}$ | $\mathbf{D}_6[r,s]/\mathbf{Z}_2[r^3] = \{\{e,r^3\},\{r,r^4\},\{r^2,r^5\},\{s,sr^3\},\{sr,sr^4\},\{sr^2,sr^5\}\} \approx \mathbf{D}_3$ |
| $H_{62}$ | $\mathbf{D}_6[r,s]/\mathbf{Z}_3[r^2] = \{\{e,r^2,r^4\},\{r,r^3,r^5\},\{s,sr^2,sr^4\},\{sr,sr^3,sr^5\}\} \approx \mathbf{Z}_2^2$ |
| $H_{63}$ | $\mathbf{D}_6[r,s]/\mathbf{D}_3[r^2,s] = \{\{e,r^2,r^4,s,sr^3,sr^4\},\{r,r^3,r^5,sr,sr^3,sr^5\}\} \approx \mathbf{Z}_2$ |
| $H_{64}$ | $\mathbf{D}_6[r,s]/\mathbf{D}_3[r^2,sr] = \{\{e,r^2,r^4,sr,sr^3,sr^5\},\{r,r^3,r^5,sr^2,sr^4,s\}\} \approx \mathbf{Z}_2$ |
| $\mathcal{Q}$ | A space of matrix valued functions of space $\mathbf{x} \in \mathbf{R}^2$ and time $t \in \mathbf{R}$ |
| $\mathcal{Q}_{\mathcal{L}}$ | A space of matrix valued functions $\mathbf{Q}(\mathbf{x}, t)$ periodic with respect to the lattice $\mathcal{L}$ |
| $V_{\mathbf{C}}$ | The space of complex $3 \times 3$ symmetric matrices with trace= 0 |

# Appendix E

# Matlab Program

```
clear
T = 360
for t = 1:T

a  =  0.5;
b  =  0.1;
ex = 0.05;

Q  = [ a 0 0; 0 b 0; 0 0 -a-b];
%Q  = [ 0 0 i; 0 0 0; i 0 0];
%Q  = [ 0 1 0;  1 0 0; 0 0 0];
%Q  = [ 0 0 0;  0 0 i; 0 i 0];

%R0 = diag([-1,-1,2]);
R0 = diag([1,1,-2]);
%R0 = diag([0,0,0]);

z1 = exp(2*pi*i*t/T); z2 = exp(2*pi*i*t/T);
w1 = exp(2*pi*i*t/T); w2 = exp(2*pi*i*t/T);
```

```
%resolution of pictures
dx = 1/12;

[x,y] = meshgrid(-1.5: dx : 1.5);
quivscale = 0.5;

N = length(x);

k1 = [1; 0];
k2 = [0; 1];

k1dot = k1(1)*x + k1(2)*y;
e1 = exp(2*pi*i*k1dot);

k2dot = k2(1)*x + k2(2)*y;
e2 = exp(2*pi*i*k2dot);

C2K1 = [ 1 0 0; 0 1 0; 0 0 1];
C2K2 = [ 0 -1 0; 1 0 0; 0 0 1];

Q1 = C2K1*Q*inv(C2K1);
Q2 = C2K2*Q*inv(C2K2);

clf

for m = 1:N
for n = 1:N
Rp = real(z1*e1(m,n)*Q1 + z2*e2(m,n)*Q2
     + w1*(e1(m,n))^(-1)*Q1 + w2*(e2(m,n))^(-1)*Q2);
R = R0 + ex*Rp;
E = eig(R);
```

```
[V,D] = eig(R);
[Y,I] = max(E);
VL = norm(V(:,I));
u(m,n) = V(1,I)/VL;
v(m,n) = V(2,I)/VL;
w(m,n) = V(3,I)/VL;
if w(m,n) < 0
u(m,n) = -u(m,n); v(m,n) = -v(m,n);
end
YY = sort(E);
% if YY(end) == YY(end-1)
if abs(YY(end) - YY(end-1)) < .0058
w(m,n) = 0;
else w(m,n) = 1;
end
end
end

%quiver(x,y,w.*u,w.*v,quivscale)
quiver(x,y,w.*u,w.*v,quivscale,'.')
hold on
quiver(x,y,-w.*u,-w.*v,quivscale,'.')

axis('equal')
xlabel('x')
ylabel('y')

M(t) = getframe;
end
movieview(M,10)
```

# Appendix F

# Index of programs

Table F.1: Index of Programs for $\mathbf{C}^2$

| **Case** | **Program** | **Representation** | **Standing or Rotating** | **Isotropic or Homeotropic** |
|---|---|---|---|---|
| 28a | pp2228ai | $Q^{++}$ | Standing | Isotropic |
| | mp2228ai | $Q^{-+}$ | Standing | Isotropic |
| 28b | pp2228bi | $Q^{++}$ | Rotating | Isotropic |
| | mp2228bi | $Q^{-+}$ | Rotating | Isotropic |
| 30 | pp2230i | $Q^{++}$ | Standing | Isotropic |
| 32 | mp2232i | $Q^{-+}$ | Standing | Isotropic |
| 42 | pp2242i | $Q^{++}$ | Standing | Isotropic |
| 43 | mp2243i | $Q^{-+}$ | Standing | Isotropic |

Table F.2: Index of Programs for $\mathbf{C}^4$

| **Case** | **Program** | **Representation** | **Standing or Rotating** | **Isotropic or Homeotropic** |
|---|---|---|---|---|
| 30 | pp4230i | $Q^{++}$ | Standing | Isotropic |
| | mp4230i | $Q^{-+}$ | Standing | Isotropic |
| 32 | pp4232i | $Q^{++}$ | Standing | Isotropic |
| | mp4232i | $Q^{-+}$ | Standing | Isotropic |
| 42 | pp4242i | $Q^{++}$ | Standing | Isotropic |
| | mp4242i | $Q^{-+}$ | Standing | Isotropic |
| 43 | pp4243i | $Q^{++}$ | Standing | Isotropic |
| | mp4243i | $Q^{-+}$ | Standing | Isotropic |

Table F.3: Index of Programs for $\mathbf{C}^3$

| **Case** | **Program** | **Representation** | **Standing or Rotating** | **Isotropic or Homeotropic** |
|---|---|---|---|---|
| 3 | pp323i | $Q^{++}$ | Standing | Isotropic |
| | mp323i | $Q^{-+}$ | Standing | Isotropic |
| 10a | pp3210ai | $Q^{++}$ | Standing | Isotropic |
| | mp3210ai | $Q^{-+}$ | Standing | Isotropic |
| 10b | pp3210bi | $Q^{++}$ | Standing | Isotropic |
| 15a | pp3215ai | $Q^{++}$ | Standing | Isotropic |
| 15b | pp3215bi | $Q^{++}$ | Rotating | Isotropic |
| | mp3215bi | $Q^{-+}$ | Rotating | Isotropic |
| 16a | mp3216ai | $Q^{-+}$ | Standing | Isotropic |
| 16b | pp3216bi | $Q^{++}$ | Rotating | Isotropic |
| | mp3216bi | $Q^{-+}$ | Rotating | Isotropic |
| 19 | pp3219i | $Q^{++}$ | Standing | Isotropic |
| | mp3219i | $Q^{-+}$ | Standing | Isotropic |
| 24 | pp3224i | $Q^{++}$ | Standing | Isotropic |
| | mp3224i | $Q^{-+}$ | Standing | Isotropic |
| 25 | mp3225i | $Q^{-+}$ | Standing | Isotropic |
| 27 | mm3227h | $Q^{--}$ | Standing | Homeotropic |
| | mm3227i | $Q^{--}$ | Standing | Isotropic |
| 31a | pp3231ai | $Q^{++}$ | Standing | Isotropic |

Table F.4: Index of Programs for $\mathbf{C}^6$

| **Case** | **Program** | **Representation** | **Standing or Rotating** | **Isotropic or Homeotropic** |
|---|---|---|---|---|
| 15b | pp6215bi | $Q^{++}$ | Rotating | Isotropic |
| | mp6215bi | $Q^{-+}$ | Rotating | Isotropic |
| 15b | pp6216bi | $Q^{++}$ | Rotating | Isotropic |
| | mp6216bi | $Q^{-+}$ | Rotating | Isotropic |
| 25 | pp6225i | $Q^{++}$ | Standing | Isotropic |
| | mp6225i | $Q^{-+}$ | Standing | Isotropic |
| 30 | pp6230i | $Q^{++}$ | Standing | Isotropic |
| | mp6230i | $Q^{-+}$ | Standing | Isotropic |
| 31b | pp6231bi | $Q^{++}$ | Standing | Isotropic |
| | mp6231bi | $Q^{-+}$ | Standing | Isotropic |
| 32 | pp6232i | $Q^{++}$ | Standing | Isotropic |
| | mp6232i | $Q^{-+}$ | Standing | Isotropic |

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