UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS
School of Mathematics

Hypermaps: constructions and operations

by

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Thesis for the degree of Doctor of Philosophy

December 2009

UNIVERSITY OF SOUTHAMPTON

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It is conjectured that given positive integers l, m, n with $l^{-1}+m^{-1}+n^{-1}<1$ and an integer $g \geq 0$, the triangle group $\Delta = \Delta(l,m,n) = \langle X, Y, Z | X^l = Y^m = Z^n = XYZ = 1\rangle$ contains infinitely many subgroups of finite index and of genus g. This conjecture can be rewritten in another form: given positive integers l, m, n with $l^{-1}+m^{-1}+n^{-1}<1$ and an integer $g \geq 0$, there are infinitely many nonisomorphic compact orientable hypermaps of type (l,m,n) and genus g. We prove that the conjecture is true, when two of the parameters l, m, n are equal, by showing how to construct those hypermaps, and we extend the result to nonorientable hypermaps.

A classification of all operations of finite order in oriented hypermaps is given, and a detailed study of one of these operations (the duality operation) is developed. Adapting the notion of chirality group, the duality group of $\mathcal{H}$ can be defined as the the minimal subgroup $D(\mathcal{H}) \trianglelefteq Mon(\mathcal{H})$ such that $\mathcal{H}/D(\mathcal{H})$ is a self-dual hypermap. We prove that for any positive integer d, we can find a hypermap of that duality index (the order of $D(\mathcal{H})$), even when some restrictions apply, and also that, for any positive integer k, we can find a non self-dual hypermap such that $|Mon(\mathcal{H})|/d = k$. We call this k the *duality coindex* of the hypermap. Links between duality index, type and genus of a orientably regular hypermap are explored.

Finally, we generalize the duality operation for nonorientable regular hypermaps and we verify if the results about duality index, obtained for orientably regular hypermaps, are still valid.

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I, Daniel Pinto, declare that the thesis entitled

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and the work presented in the thesis are both my own, and have been generated by me as a result of my original research. I confirm that:

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Acknowledgements

I would like to gratefully acknowledge the supervision of Professor Gareth Jones and also the financial support from the Portuguese Foundation for Science and Technology (Fundação para a Ciência e Tecnologia - FCT) and the Department of Mathematics of the University of Coimbra.

Chapter 1

Introduction

There are always many ways of approaching a subject, no matter if one is talking about Algebra, Topology, Graph Theory or any other topic of mathematics. However, this is especially true for research in Theory of Maps and Hypermaps, since this is a field that was developed, from the beginning, in the border of several different areas. Some problems require more Algebra, others more Topology. The background of the researcher and his mathematical intuition is sometimes also important to decide which path to take. In Chapter 3 of this thesis, a more topological approach is followed (based on some methods developed in [31]) to solve a problem in Group Theory, while in Chapter 5, we take the algebraic definition of a hypermap to prove some theorems concerning duality (which is, in a way, a very topological concept). This is possible to be done because, in the last decades, several authors have devoted their time to establish important links between the topological definition of a (hyper)map and its algebraic form. Chapter 2 tries to briefly explain what are the basic ideas behind those two definitions and how they nicely entwine. The aim of the initial chapter is also to make the reader familiar with some graphical representations of a hypermap, namely the Walsh representation [53] and the James representation [27]. Because a (hyper)map is more than a (hyper)graph - it is a (hyper)graph embedded on a surface - one can find, by the end of Chapter 2, a brief but important section about the genus of a (hyper)map, a crucial notion to understand the majority of the results in the main chapters.

In Chapter 3, we present a conjecture, due to J. Wolfart and Gareth Jones:

given positive integers l, m, n with $l^{-1} + m^{-1} + n^{-1} < 1$ and an integer $g \geq 0$, the triangle group : $\Delta = \Delta(l, m, n) = \langle X, Y, Z | X^l = Y^m = Z^n = XYZ = 1 \rangle$ contains infinitely many subgroups of finite index and of genus g. However, we will not try to prove directly this conjecture. We will work with its translation to a more topological version, instead: given positive integers l, m, n with $l^{-1} + m^{-1} + n^{-1} < 1$ and an integer $g \geq 0$, there are infinitely many nonisomorphic compact orientable hypermaps of type (l, m, n) and genus g. Most of the sections of this chapter were written to prove that result for a special family of hypermaps (such that two of the parameters of its type are equal). The proof is divided in several cases, each one involving a different technique, and is constructive, because it also provides a way to explicitly build each one of the hypermaps (and not only proving their existence). The main result of this chapter is a result about orientable hypermaps. However, at the end of the chapter, the constructions are adapted to build infinitely many nonorientable hypermaps for each genus and type. Finally, we discuss some alternatives to prove the conjecture for the general case.

The second part of this thesis starts in Chapter 4, where one can understand how the idea of operations on maps arose [61] and how it was later generalized to hypermaps [26], using a more algebraic approach. In Section 4.5, we classify all operations of order 2 in hypermaps and, by the end of the chapter, we give a definition of *duality group* (a concept very important for the remaining pages of the thesis) and we explain the difference between the two types of duality (the one that just interchanges hypervertices and hyperfaces, and the one that also inverts the orientation).

One can look at the duality group as a way to measure how far a hypermap is from being self-dual, adapting a notion first developed to study chirality [7]. If the duality group has only one element, then the hypermap is self-dual. On the other hand, if its duality group is equal to its monodromy group, we call it *totally dual*. Although it might be interesting to know which is the duality group of a particular hypermap, we are mainly interested in the order of that group. In Chapter 5, one can find several results about the *duality index* (the order of the duality group) in self-dual, non self-dual, totally dual and non totally dual hypermaps. Some of the proofs are also constructive, meaning that we give explicit presentations of the monodromy group of each hypermap and, consequently, a full description of each one of them. In Section 5.1.1 we classify the self-dual and totally dual hypermaps of small order and in Section 5.3 we explain how the choice of the type of duality can affect the duality index of a specific hypermap. The last sections have two main goals: 1) to study for what triples (l, m, n) we can find self-dual and totally dual hypermaps of that type, 2) to relate genus and duality index on hypermaps.

Because, throughout Chapter 5, we are always confined to oriented hypermaps, the last chapter introduces the concept of duality group on nonorientable regular hypermaps, highlighting the differences and exploring the similarities with the case for oriented hypermaps.

We have also included an appendix to show how we can obtain non regular hypermaps of type $(3, 3, 7)$ using subgroups of $PSL_2(13)$. The method also works for other groups $PSL_2(q)$, but we have not proved that, since we were only interested in giving an example of an algebraic way to construct hypermaps, something that might be crucial to prove the conjecture, presented in Chapter 3, for any hyperbolic triple.

Chapter 2

Maps and Hypermaps

2.1 Overview

If we want to trace the roots of the Theory of Maps, we probably have to go back to the ancient Greece and to the study of platonic solids, where some of the main topics of the subject were beginning to take shape. And if we wish to find some examples of our fascination with symmetry, we would not have any trouble to discover them in old mathematical works, many centuries ago. Even if we need a powerful tool to analyse the nature of those symmetries, we can go back almost 200 years, to Galois and the foundations of Group Theory. However, the Theory of Maps, as a field of its own, is very recent and was not entirely emancipated from other areas until Gareth Jones and David Singerman published, in 1978, their paper *Theory of Maps on Orientable Surfaces* [35]. Although that work may have given the subject its proper language, formalization and notation, several authors contributed before to the field. Relevant contributions may be found in the works of Brahana [4], Sherk [52], Garbe [24], Cori [18], Coxeter and Moser [19], Wilson [60], Walsh [53], among others. Besides the growth of the mathematical theory, a few applications have been developed over the last years, specially in molecular chemistry.

The aim of this chapter is to present some basic definitions and results about maps and hypermaps and their connections to Riemann surfaces and the triangle group. There are also some other interesting topics of research (for instance: Belyi functions or links to Galois theory [32]) that will not be mentioned here, despite their relevance to the global view of the theory, since they are not very important to what will follow, in the next chapters.

2.2 Maps, basic concepts

The word *map* is used in many different contexts, with different meanings, even if we restrict ourselves to mathematics. But, here, a *map* $\mathcal{M}$ is a finite, connected graph $\mathcal{G}$ imbedded (without crossings) in a surface $\mathcal{S}$, where the faces of $\mathcal{M}$ (the connected components of $\mathcal{S} \setminus \mathcal{G}$) are homeomorphic to an open disc. A map is said to be *orientable* or *nonorientable* according to whether the underlying surface is orientable or nonorientable. Over these pages, unless a change in these restrictions is explicitly mentioned, a surface will be connected, compact, orientable and without border (we will later extend this definition of map to more general surfaces, namely nonorientable surfaces).

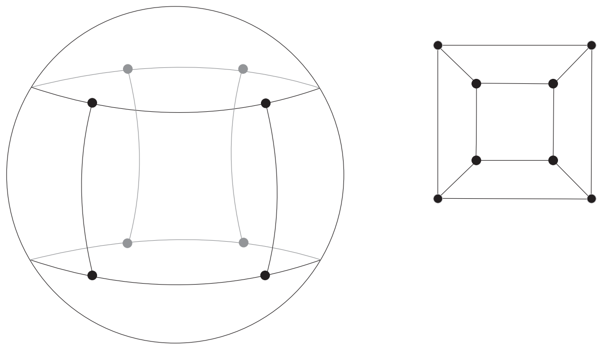

Figure 2.1: Cube map on the sphere.

The orientable surface $\mathcal{S}$ is then homeomorphic to a surface consisting of a sphere with handles attached. The number of these handles (or, roughly, the number of *holes*) is called the *genus* g of the surface and it is defined to be also the genus of the map.

Maps and groups are closely related, as it will be shown in this chapter. The *monodromy group*, the *automorphism group* and the *map subgroup* are the three main concepts that will allow us to establish the connection between the topological and the algebraic approaches. To do this, we will start by exploring the possibility to represent a map on a surface by permutations, the basic idea that brings combinatorial group theory and topology together.

2.3 Orientable maps and permutation groups

Let $D = \{w = (v, e) : v \text{ and } e \text{ are adjacent}\}$, where v is a vertex and e is an edge. We call each element $w \in D$ a *dart* and it can be graphically represented by an arrow:

Figure 2.2: Dart.

To each edge e in $\mathcal{M}$ we can associate two different darts α and β, unless we are dealing with a free edge (which we will not considerer at this stage). Then, it is possible to define a permutation y that transposes these two darts.

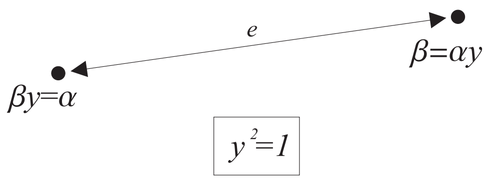

Figure 2.3: Permuting two darts of the same edge.

If the surface $\mathcal{S}$ is orientable and we fix an orientation, this induces cyclic permutations of the darts associated with each vertex[1]. These cyclic permutations are disjoint and together form a permutation of D that we will represent by x. The permutation $z = yx^{-1}$ also acts on the set of the darts, each cycle corresponding to a rotation around a face.

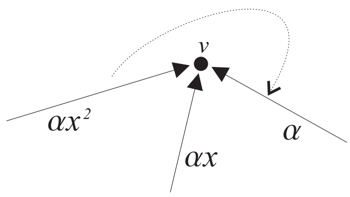

Figure 2.4: Permutation around a vertex.

[1]Hence, for each graph imbedded on an orientable surface (for each orientable map) we can have two *oriented* maps, depending on the orientation we choose.

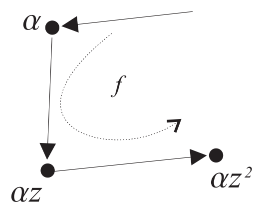

Figure 2.5: Permutation around a face.

It is easy to verify that x, y and z satisfy the following relations:

$$x^l = y^2 = z^n = xyz = 1 \tag{2.1}$$

where l is the least common multiple of the valencies of the vertices of $\mathcal{M}$ and n the least common multiple of the numbers of sides of the faces. Using the Coxeter-Moser notation, we say that a map like this has type $\{n, l\}$ (or type (l, n) if we follow Therefall's notation). An important group, the *monodromy group*, is defined as the group generated by these permutations x, y and z.

$$Mon(\mathcal{M}) = \langle x, y, z\rangle$$

Since $xyz = 1$, we can choose only two of these to generate the group and because $\mathcal{G}$ is connected, $Mon(\mathcal{M}) \leq S^D$ acts transitively on the darts. It is then possible (see [35] for details) to describe an orientable map $\mathcal{M}$ of any type as an *algebraic map*, looking at the transitive permutation representation $\Pi : \Gamma^+ \to Mon((\mathcal{M}))$, $t_0 \mapsto x$, $t_1 \mapsto y$, $t_2 \mapsto z$, of the *cartographic group*:

$$\Gamma^+ = \langle t_0, t_1, t_2 | t_1^2 = t_0 t_1 t_2 = 1\rangle \cong C_\infty * C_2$$

Moreover, the opposite is also true, since every map arises from some algebraic map [35].

2.4 Map subgroups

If instead of dealing with general orientable maps, we are just interested in maps of certain types, it might be useful to substitute the group Γ^+ for the *triangle group* $\Delta(l, 2, n)$:

$$\Delta(l, 2, n) = \langle X, Y, Z | X^l = Y^2 = Z^n = XYZ = 1\rangle.$$

This group is generated by rotations X, Y, Z, through angles $2\pi/l$, π and $2\pi/n$ around a triangle T with angles π/l, $\pi/2$ and π/n. $\Delta(l,2,n)$ is the orientation-preserving subgroup of index 2 in $\Delta[l,2,n]$, the *extended triangle group*:

$$\Delta[l,2,n] = \langle R_0, R_1, R_2 | R_i^2 = (R_1R_2)^l = (R_2R_0)^2 = (R_0R_1)^n = 1\rangle$$

generated by reflections R_0, R_1 and R_2 in the sides of a triangle T with angles π/l, $\pi/2$, π/n in a simply connected Riemann surface. If a map $\mathcal{M}$ has type (a,b) dividing (l,n), there is an obvious epimorphism $\Theta : \Delta(l,2,n) \longrightarrow Mon(\mathcal{M})$ given by $X \mapsto x$, $Y \mapsto y$ and $Z \mapsto z$. Hence, Δ has a transitive action on D. The stabilizer M of a dart in this action of Δ is called a *map subgroup* for $\mathcal{M}$.

$$M = \{g \in \Delta | \alpha g = \alpha\} = \theta^{-1}(Mon(\mathcal{M})_\alpha)$$

(where $Mon(\mathcal{M})_\alpha$) is the stabilizer of α in the monodromy group).

M is determined up to conjugacy in $\Delta(l,2,n)$ by $\mathcal{M}$. Therefore, if M_1, $M_2 \leq \Delta(l,2,n)$ they give rise to isomorphic maps if and only if M_1 and M_2 are conjugate in $\Delta(l,2,n)$. Moreover, D is naturally identified with Δ/M via the bijection $dg \mapsto Mg$, with $d \in D$, so that the action of $\Delta(l,2,m)$ by right multiplication on the cosets Mg is isomorphic to its action on D. $\Delta(l,2,n)$ also acts as a discontinuous group of conformal isometries of a simply connected Riemann surface[2] $\mathcal{U}$, leaving invariant a triangular tessellation of $\mathcal{U}$. This Riemann surface is:

- the hyperbolic plane ($\mathbb{H}$) if $\frac{1}{l} + \frac{1}{n} < \frac{1}{2}$
- the complex plane ($\mathbb{C}$) if $\frac{1}{l} + \frac{1}{n} = \frac{1}{2}$
- the Riemann sphere ($\mathbb{C} \cup \{\infty\}$) if $\frac{1}{l} + \frac{1}{n} > \frac{1}{2}$

It can be proved [57] that $\mathcal{M}$ and $\mathcal{U}/M$ have the same genus and that $\mathcal{U}/M$ carries a map $\widetilde{\mathcal{M}}$ isomorphic to $\mathcal{M}$. This establishes an important link between orientable maps and Riemann Surfaces.

[2]A Riemann surface is a 2-connected 2-manifold endowed with a complex analytic structure (an atlas) that allows local coordinatisation.

If we allow maps whose graphs have free edges[3], loops and multiple edges we can also obtain a one-to-one correspondence between maps of type (l, n) and conjugacy classes of subgroups of finite index in $\Delta(l, 2, n)$.

2.5 The automorphism group

In a sense, the monodromy group $Mon(\mathcal{M}) = \langle x, y, z \rangle$ is nothing but an *instruction guide* that has all the information to build the corresponding map. If we take the edges, vertices and faces to be the cycles of x, y and z, respectively, with incidence given by non-empty intersection, we can, from a concise information about the monodromy group (its presentation, for instance), reconstruct the whole map. It follows that the monodromy group is important to construct the map but not necessarily to study its symmetries. To do that, we need to work with its automorphism group.

In order to give a more general approach and deal with hypermaps that are not oriented, we will use *flags* instead of darts. First, we need to pick a point in the interior of each face and then, in each face, join that point with every vertex and the midpoint of every edge, on the boundary of the face. This process is called the *barycentric subdivision* of the map.

Definition 2.5.1. *Flags* are the cells (topological triangles) of the barycentric subdivsion of the map.

As a consequence of this definition, each dart is made up of two *flags*. An *automorphism* of a map is a permutation of the flags induced by an homeomorphism of the surface on itself that send edges to edges, vertices to vertices and faces to faces, keeping the adjacency relations. These permutations form a group called the *automorphism group* $Aut(\mathcal{M})$. Some of those automorphisms preserve orientation and form another group (a subgroup of $Aut(\mathcal{M})$ and index at most 2), that is usually represented by $Aut^+(\mathcal{M})$. The group $Aut^+(\mathcal{M})$ is a natural generalization of the rotation group of a polyhedron and each of its elements is determined by the effect on any one dart. Algebraically, this *orientation-preserving automorphism group* is the centralizer

[3] A *free-edge* has only one end incident to a vertex. Sometimes, they are called semi-edges or half-edges.

$C_{S^D}(Mon(\mathcal{M}))$ of $Mon(\mathcal{M})$ in S^D and acts faithfully and freely on D. If $N_{\Delta(l,2,n)}(M)$ is the normalizer of the map subgroup M of $\mathcal{M}$ in $\Delta(l,2,n)$, then $Aut^+(\mathcal{M}) \cong N_{\Delta(l,2,n)}(M)/M$. The action of the group $Aut^+(\mathcal{M})$ on the set of darts D can be understood as the action of $N_{\Delta(l,2,n)}/M$ by left multiplication on the set $\Delta(l,2,n)/M$ of M-cosets.

A map is *orientably regular* if the orientation-preserving automorphism group is transitive on D. In that case (but not in every map) $M \trianglelefteq \Delta(l,2,n)$ and the monodromy group and the orientation-preserving automorphism group are both isomorphic to $\Delta(l,2,n)/M$.

2.6 General algebraic theory of maps

The theory of orientable maps can be extended to nonorientable maps, possibly with boundary, by establishing a correspondence between maps and the permutation representation of

$$\Gamma = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = (r_2 r_0)^2 = 1\rangle$$

or between maps of type dividing (l,n), and the group:

$$\Gamma(l,n) = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = (r_1 r_2)^l = (r_2 r_0)^2 = (r_0 r_1)^n = 1\rangle$$

The subgroup $\Gamma^+ = \langle r_1 r_2, r_2 r_0, r_0 r_1\rangle$, is a subgroup of Γ formed by the words of even length and it is called the *even subgroup*. The automorphism group of a map $\mathcal{M}$ is isomorphic to $N_\Gamma(M)/M$, where M is the map subgroup. A map is *regular* if $M \trianglelefteq \Gamma$ and *orientably regular* if $M \trianglelefteq \Gamma^+$. If a map is orientably regular but not regular, we say that it is *chiral*.

2.7 Orientable hypermaps

One of the generators of the triangle group $\Delta(l,2,n)$ is an involution (element of order 2). This restriction can be removed in order to build a more general theory. If we take the triangle group (generated by rotations around the vertices of a triangle)

$$\Delta(l,m,n) = \langle X, Y, Z | X^l = Y^m = Z^n = XYZ = 1\rangle$$

and its subgroups, we can work with a more complicated structure: the *hypermap* (in this case: the orientable hypermaps of type (a, b, c) dividing (l, m, n)). From a topological point of view, this change means that, instead of imbedding a graph on a surface, we are now looking at imbeddings of hypergraphs on surfaces, which is the same that allowing an edge (now called *hyperedge*) to have valency more than 2 (or, in other words, to be adjacent to more than two *hypervertices*).

Some of the results for maps can easily be generalized to hypermaps. For instance, this triangle group $\Delta(l, m, n)$ is the orientation-preserving automorphism group of the universal hypermap[4] $\widetilde{\mathcal{H}}$ of type $\tau = (l, m, n)$ (where l, m and n are the least common multiples of the valencies of the hypervertices, hyperedges and hyperfaces). A hypermap of type (l, m, n) is *uniform* if all hypervertices, all hyperedges and all hyperfaces have valencies l, m and n, respectively

Any hypermap of type (l, m, n) is isomorphic to the quotient of $\widetilde{\mathcal{H}}$ by some subgroup $H \leq \Delta(l, m, n)$, which is unique up to conjugacy. Conversely, any conjugacy class of subgroups H determines a hypermap $\mathcal{H}/H$ of type $\tau' = (l', m', n')$ where l', m' and n' divide l, m and n, respectively, and are the orders of the permutations of the cosets of H induced by X, Y and Z. Two hypermaps are isomorphic if and only if the corresponding subgroups are conjugate in $\Delta(l, m, n)$.

Definition 2.7.1. If H is normal in $\Delta(l, m, n)$ then we say that $\mathcal{H}$ is orientably regular.

The triangle group $\Delta(l, m, n)$ acts on the Riemann sphere, the complex plane or the hyperbolic plane as $l^{-1} + m^{-1} + n^{-1} > 1, = 1$ or < 1.

2.8 James Representation of Hypermaps

One possible way to represent a hypermap (orientable or nonorientable) is using the James representation [27] and take a trivalent graph where the faces are the hypervertices (labelled with 0), the hyperedges (labelled with 1) and the hyperfaces (labelled with 2). The vertices of that graph represent the *(hyper)flags.*

[4]a hypermap on the Riemann surface which Δ acts on as a group of isometries.

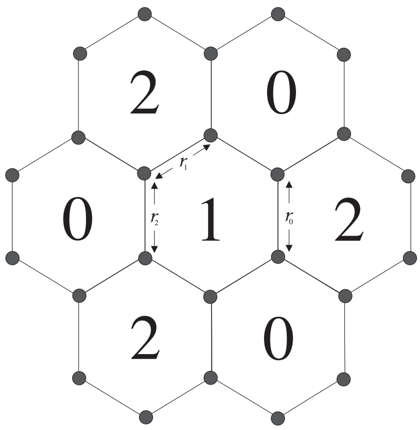

Figure 2.6: James representation of a hypermap.

We can define three permutations of the set of flags, coloring each edge of the graph by the complement of the colors of the faces which it border and transposing each pair of flags that form the ends of an edge of the same color. If we call those permutations r_0, r_1 and r_2 then $r_0^2 = r_1^2 = r_2^2 = 1$ and, if we take a connected graph, they generate a transitive group of permutations of F (the set of flags). A hypermap is a transitive permutation representation $\pi : \Delta \to S^F$ of the group:

$$\Delta = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = 1 \rangle \cong C_2 * C_2 * C_2.$$

This permutation representation is isomorphic to the action of $G = \langle r_0, r_1, r_2 \rangle$ (by right multiplication) on the cosets Hg of a subgroup $H \leq G$ (where H is the hypermap subgroup and G the monodromy group). Algebraically, we can then represent a hypermap as a 4-tuple: $\mathcal{H} = (F; r_0, r_1, r_2)$. The orbits $\langle r_1, r_2 \rangle$, $\langle r_2, r_0 \rangle$ and $\langle r_1, r_0 \rangle$ can be associated, respectively, with the hypervertices, the hyperedges and the hyperfaces (Figure 2.7).

If the surface is orientable and without boundary, then the flags of $\mathcal{H}$ are two-colorable. The orientation of each hypervertex defined by starting at a particular flag and proceeding along an edge of the graph colored 1 and will be positive for the flags of one color and negative for those of the other (Figure 2.8).

Representing each dart by the flag of positive color we can define an action $R = \pi(r_0 r_2)$ and $L = \pi(r_1 r_2)$ on the *darts* of the *oriented hypermap*. Under

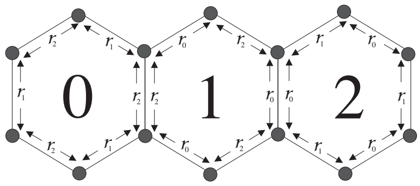

Figure 2.7: Three different orbits.

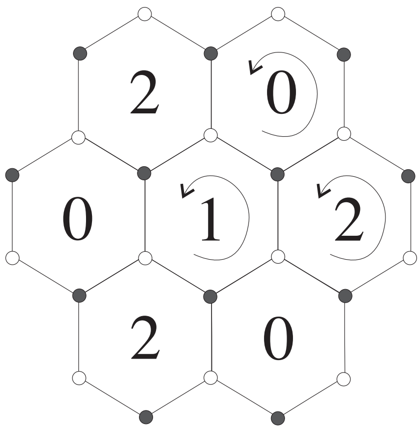

Figure 2.8: 2-colourable flags.

this action the cycles of R and L are the darts bounding the hyperedges and the hypervertices, respectively.

An *oriented hypermap* corresponds then to a transitive permutation of the free group

$$\Delta^{+} = \langle r_0 r_2, r_1 r_2 | - \rangle$$

and can be described as a triple: $\mathcal{H} = (D, R, L)$, where D is a set of darts and R, L are two permutations generating a permutation group $Mon(\mathcal{H}) = \langle R, L \rangle$, called the monodromy group of $\mathcal{H}$, acting transitively on D.

Contracting each edge that is incident to a hypervertex (a face labelled with 0) and a hyperedge (a face labelled with 1) on the trivalent graph of the James representation, we will obtain a graph with vertices of degree 4 that are precisely the (hyper)darts of the oriented hypermap. Hence, if R permutes the darts around the hypervertices and L permutes the darts around the hyperedges, we can call R' and L' the ones that permute the darts, respectively around the hypervertices and around the hyperedges but using the opposite orientation. Hence, from a orientable hypermap, we can have two different oriented hypermaps:

$$\mathcal{H}^{+} = (D^{+}, R, L)$$

$$\mathcal{H}^{-} = (D^{-}, R', L')$$

An *automorphism that preserves the orientation* ϕ of $\mathcal{H}^{+} = (D^{+}, R, L)$ is a permutation of D that commutes with R and L:

$$\phi R = R\phi \quad and \quad \phi L = L\phi.$$

An *automorphism that reverses the orientation* ϕ is a permutation of D such that:

$$\phi R = R^{-1}\phi \quad and \quad \phi L = L^{-1}\phi.$$

The group of automorphisms that preserves the orientation $Aut^{+}(\mathcal{H})$ has order less or equal to the number of darts of the hypermap. If $|Aut^{+}(\mathcal{H})| = D$ then the hypermap is *orientably regular* and $Aut^{+}(\mathcal{H})$ has index 2 in $Aut(\mathcal{H})$.

Definition 2.8.1. If a hypermap is orientably regular (o.r.) but does not allow a automorphism that reverses the orientation, it is called *chiral* (otherwise it is called *reflexible*).

In the nonorientable case there are no real reflections since every symmetry is a product of rotations. As a consequence of that, $\langle R; L\rangle$ has index 1 in $Aut(\mathcal{H})$.

2.9 Walsh representation

Another way to represent an orientable hypermap is to use the Walsh representation [53], that is, modelling the orientable hypermap by an embedding of a bipartite graph on a surface (a bipartite map). The vertices in one partite set represent the hypervertices and the others represent the hyperedges. The faces of the bipartite map represent the hyperfaces and the edges are nothing but the (hyper)darts (see Figure 2.9 for an example). The Walsh representation is a very simple and useful tool to work with hypermaps and will be extremely helpful here, specially in Chapter 3.

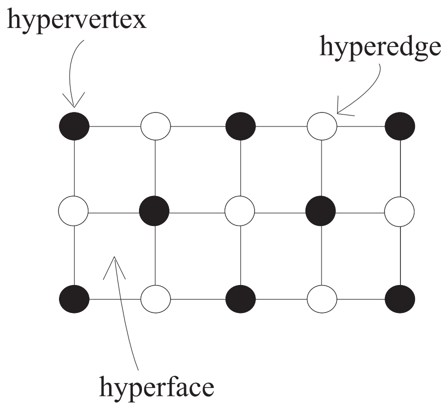

Figure 2.9: Walsh bipartite map.

A face of valency $2n$ in the Walsh bipartite map, represents a hyperface of valency n in the hypermap.

2.10 Hypermap coverings

For $i = 1, 2$, let H_i be the hypermap subgroup for a hypermap $\mathcal{H}_i$. We say that $\mathcal{H}_1$ covers $\mathcal{H}_2$ if there is a branched covering of the underlying surface of $\mathcal{H}_2$ which maps hypervertices and hyperedges of $\mathcal{H}_1$ onto those of $\mathcal{H}_2$. Branching is only permitted over the hypervertices, midpoints of hyperedges and centres of hyperfaces. This definition has a very useful algebraic translation since $\mathcal{H}_1$ covers $\mathcal{H}_2$ if and only if $H_1 \leq H_2$.

2.11 Genus of a hypermap

As we have pointed before, the genus of a map M is the genus of the underlying surface. In terms of its Euler characteristic we have:

$$\chi(\mathcal{M}) = |V| - |E| + |F| = \begin{cases} \text{2-2g}, & \text{if } \mathcal{M} \text{ is orientable;} \\ \text{2-g}, & \text{if } \mathcal{M} \text{ is nonorientable.} \end{cases}$$

For orientably regular maps of type (l, n), by counting the number of flags (number of automorphisms), we will have $|Aut(\mathcal{M})| = 2|E|$ if the map is chiral, or $|Aut(\mathcal{M})| = 4|E|$ if $\mathcal{M}$ is reflexible. Because in any case $l|V| = 2|E| = n|F|$, we have another way of expressing the genus $g(\mathcal{M})$ of a map $\mathcal{M}$, this time in terms of the number of its automorphisms:

$$g(M) = \begin{cases} |Aut(\mathcal{M})|(1/8 - 1/4n - 1/4l) + 1, & \text{if } \mathcal{M} \text{ is o.r. and reflexible;} \\ |Aut(\mathcal{M})|(1/4 - 1/2n - 1/2l) + 1, & \text{if } \mathcal{M} \text{ is o.r. but chiral;} \\ |Aut(\mathcal{M})|(1/4 - 1/2n - 1/2l) + 2, & \text{if } \mathcal{M} \text{ is nonorientable.} \end{cases}$$

The small positive value for the bracketed expression is attained when $(l, n) = (7, 3)$. It follows that regular maps of this type (or type $(3, 7)$) are the ones with largest automorphism groups. These groups (non-trivial quotients of the triangle group $\Delta(2, 3, 7)$) are known as *Hurwitz groups*, since their order, $84(g - 1)$, in the first two cases, correspond to the upper bound for the number of conformal automorphisms of a compact Riemann surface with genus g greater than one, known as the *Hurwitz bound* [25]). For nonorientable maps, that bound is equal to $168(g - 1)$.

Similar Euler formulas can be established for hypermaps.

Let $H = (F; r_0, r_1, r_2)$ be a hypermap on a closed surface S of genus g having $|v|$ hypervertices, $|e|$ hyperedges and $|f|$ hyperfaces. If we take the James representation of the hypermap, we get a 3-valent graph with $3|F|$ vertices, $3|F|/2$ edges and $|v| + |e| + |f|$ faces. Then:

$$|v| + |e| + |f| - |F|/2 = 2 - 2g, \text{ if } \mathcal{H} \text{ is orientable;}$$

$$|v| + |e| + |f| - |F|/2 = 2 - g, \text{ if } \mathcal{H} \text{ is nonorientable.}$$

Therefore:

$$g(M) = \begin{cases} |F|/4 - |v|/2 - |e|/2 - |f|/2 + 1, & \text{if } \mathcal{H} \text{ is orientable} \\ |F|/2 - |v| - |e| - |f| + 2, & \text{if } \mathcal{H} \text{ is nonorientable.} \end{cases}$$

Chapter 3

Infinitely many hypermaps of a given genus and type

3.1 Two conjectures

As we have pointed in the previous chapter, there is a strong link between subgroups of a triangle group and nonoriented hypermaps. In a nutshell, that connection might be stated like this: up to conjugacy, it is possible to associate one of those subgroups to each hypermap and the reverse is also true; if we pick a subgroup of a triangle group, there is only one hypermap (up to isomorphism) that will have it as a hypermap subgroup. The importance of this link is widely shown in several papers about hypermaps, and, here, we will also make use of this connection to translate a conjecture - that was originally written in the language of group theory - to a more topological context. The conjecture, that arose in discussions between Gareth Jones and Jürgen Wolfart, had the following first version:

Conjecture 3.1.1 (A). *Given positive integers l, m, n with*

$$l^{-1} + m^{-1} + n^{-1} < 1$$

and an integer $g \geq 0$, the triangle group :

$$\Delta = \Delta(l, m, n) = \langle X, Y, Z | X^l = Y^m = Z^n = XYZ = 1 \rangle$$

contains infinitely many subgroups of finite index and of genus[1] *g.*

[1]The genus of a subgroup H of a triangle group is the genus of the quotient-surface $\mathcal{S}/H$,

This conjecture is a very strong and general statement and in order to be able to apply some topological methods in the proof, we need to rewrite it in another form:

Conjecture 3.1.2 (B). *Given positive integers l, m, n with*

$$l^{-1} + m^{-1} + n^{-1} < 1$$

and an integer $g \geq 0$, there are infinitely many nonisomorphic compact orientable hypermaps of type (l, m, n) and genus g.

Before studying the relationship between these two conjectures, it is maybe useful to explain the reason why the condition $l^{-1} + m^{-1} + n^{-1} < 1$ was introduced. This inequality is important to avoid trivial cases, where there are none or only a finite number of hypermaps:

i) If $l^{-1} + m^{-1} + n^{-1} > 1$ then $\Delta(l, m, n)$ acts on the Riemann sphere and is finite. Hence, there are only finitely many hypermaps of a given type (l, m, n), all of them having genus 0.

ii) If $l^{-1} + m^{-1} + n^{-1} = 1$ then $\Delta(l, m, n)$ acts on the Euclidean plane and there are infinitely many subgroups and hypermaps of genus 0 or 1, but none of genus $g > 1$.

Therefore, we only need to investigate what happens with the *hyperbolic triples* where $l^{-1} + m^{-1} + n^{-1} < 1$ and $\Delta(l, m, n)$ acts on the hyperbolic plane.

Both conjectures are independent of the ordering of l, m and n and several times, throughout the next sections, that detail will be relevant to reduce the number of cases to be studied. We should also mention that the conjectures are false if we only work with uniform hypermaps (equivalently, torsion-free subgroups of Δ), those for which the hypervertices, hyperedges and hyperfaces all have valencies l, m, n, respectively (and this includes the important family of regular hypermaps corresponding to normal subgroups of Δ). In that case of uniform hypermaps, the size of the hypermap (the index of the corresponding

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where $\mathcal{S}$ is the hyperbolic plane, the Riemann sphere or the complex plane (depending on l, m and n), on which H acts by isometries. It is also the genus of the hypermap $\mathcal{H}$ corresponding to H.

subgroup) is proportional to the Euler characteristic. Hence, for a fixed genus g, there can only be finitely many uniform hypermaps of a given type. So, if we want to prove the conjecture, we will have to use nonuniform hypermaps, where the valencies of the hypervertices, hyperedges and hyperfaces have least common multiples l, m and n, respectively, but not all are necessarily equal to l, m and n.

3.2 Relationship between Conjectures A and B

If Conjecture B, presented in the previous section, is true for a given triple $\tau = (l, m, n)$ and a given genus g, then there are infinitely many nonisomorphic hypermaps of that type and genus, corresponding to mutually nonconjugate subgroups H of finite index in $\Delta(l, m, n)$ and genus g. This means that to prove Conjecture A is enough to prove Conjecture B.

The converse is not true because Conjecture A is a slightly weaker statement than Conjecture B. If Conjecture A is true for type τ and genus g then the triangle group has infinitely many subgroups H of that genus. Each H has finite index in Δ, which means that each one of those has only finitely many conjugates. Hence, among all these subgroups H there are infinitely many which are mutually nonconjugate, corresponding to infinitely many nonisomorphic hypermaps $\mathcal{H}$ of genus g. Each one of these hypermaps has type $\tau' = (l', m', n')$ for some divisors l', m' and n' of l, m and n, namely the orders of the permutations induced by X, Y and Z on the cosets of H. For a given triple τ there are only finitely many such triples τ', so for at least one of them (**but not necessarily for τ itself**) there must be infinitely many nonisomorphic hypermaps of type τ' and genus g.

3.3 Theorem and general method

Instead of trying to prove Conjecture B, in full generality, we will be more modest in our goal and focus our attention on hypermaps of type (l, m, n), where two of the parameters are equal. Since permuting l, m and n (renaming the hypervertices, hyperedges and hyperfaces) does not change the genus of the hypermap, we may also assume that we are dealing with hypermaps of type

$\tau = (m, m, n)$ [41]. Because we are only considering hyperbolic triples, we will have $2m^{-1} + n^{-1} < 1$, i.e. $(m-2)(n-1) > 2$.

Theorem 3.3.1. *Conjectures A and B are true provided at least two of the three parameters l, m and n in τ are equal.*

To prove this theorem, we will construct the Walsh map of each hypermap by joining together an appropriate number of small blocks. This method is based on the one used in [31] and will play a central role in the next sections.

As we have seen before, a Walsh map $W = W(\mathcal{H})$ is a bipartite map on the same surface as $\mathcal{H}$, each hypervertex or hyperedge of $\mathcal{H}$ represented as a black or white vertex, with each incidence between them represented as an edge between corresponding vertices. This gives each black or white vertex the same valency as the hypervertex or hyperedge it represents. However, each face of the Walsh map (bordered by alternating black or white vertices) represents a hyperface of half of its valency, the number of adjacent hyperedges. For instance, a face of valency 8 in the Walsh map represents a hyperface of valency 4 in the hypermap.

Since two of the parameters are equal and we can permute them, we will deal with hypermaps of type $\tau = (m, m, n)$ corresponding to Walsh maps of type $\{2n, m\}$ on the same surface. Then, proving the conjecture for hypermaps of type (m, m, n) is equivalent to proving it for bipartite maps of type $\{2n, m\}$ since $W(\mathcal{H}) \cong W(\mathcal{H}')$ if and only if $\mathcal{H} \cong \mathcal{H}'$.

In order to construct a bipartite map W of type $\mu = \{2n, m\}$ corresponding to a hypermap of type (m, m, n), we will use some *basic pieces*, which are nothing more than three different bipartite maps on three surfaces with boundary: a *2-trisc* $\mathcal{T}$ (a torus minus two discs, with $\mathcal{X}(\mathcal{T}) = -2$), a *closed annulus* $\mathcal{A}$ (with $\mathcal{X}(\mathcal{A}) = 0$) and a *disc* $\mathcal{D}$.

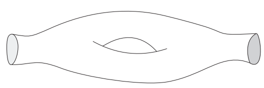


Figure 3.1: 2-Trisc.

Bipartite maps of type $\mu = \{2n, m\}$ on each of these surfaces will be denoted by, respectively, $\mathcal{T}_\mu$, $\mathcal{A}_\mu$ and $\mathcal{D}_\mu$ or, sometimes, $\mathcal{T}_i$, $\mathcal{A}_i$ and $\mathcal{D}_i$ (with $i = n$ or m)

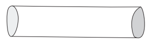


Figure 3.2: Annulus.

if we want to emphasise the valency of the faces or vertices involved in the map (assuming the other parameter is fixed and known). We may also use just $\mathcal{T}$, $\mathcal{A}$ or $\mathcal{D}$ while referring to a map on one of those surfaces if it is clear, from the context, that we are not only dealing with the surface but also with a bipartite graph, of a certain type, imbedded in it.

When a bipartite map exists on $\mathcal{A}$, $\mathcal{D}$ or $\mathcal{T}$, we require that each boundary component of the surface must be a cycle in the map. For instance, in Figure 3.9, the boundary component of the disc is a cycle of order 4.

3.4 Multiplication of an edge

Some of the methods will be applied, with small modifications, several times. One of the operations that will often be used is the *multiplication of an edge* e of the map, by an integer k, and that consists of replacing e with k edges between the same pair of vertices, enclosing $k-1$ new faces of valency 2. If e is a boundary edge then one of these new edges will also be a boundary edge (but not the other ones).

The valencies of the vertices of the boundary components are relevant to describe the pieces and to confirm that a map of a specific type is obtained when they are glued together. We say that a boundary component (denoted by $\partial^i A$, $\partial^i T$ or ∂D for $i = 0, 1$) has *type* $k^{(t)}$ if it has t vertices of valency k. If the vertices have not all the same valency, we will explicitly give those different valencies to the reader.

Figure 3.3: Multiplication of an edge by 3.

Important note: We leave, in the drawing of the graph, the edge that is multiplied. It follows that that edge should not be counted twice. For instance, the number 3, in Figure 3.3, means *exactly* the number of edges between those two vertices.

3.5 Allowed joining

All the boundary components of $\mathcal{T}$, $\mathcal{A}$ and $\mathcal{D}$ will be cycles in the map. If C_1 and C_2 are two cycles of the same length from two different components, an *allowed joining* between the two maps is a homeomorphism $C_1 \to C_2$ which sends vertices to vertices of the same colour so that C_1 and C_2 become a single cycle in the resulting bipartite map and adjacency is preserved.

Example 3.5.1. We can, for instance, build infinitely many hypermaps of genus 2 by conveniently joining, end to end, two 2-triscs, two discs and an arbitrary number of annuli, all carrying suitable hypermaps.

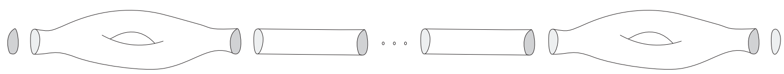


Figure 3.4: Infinitely many hypermaps of genus 2.

3.6 Genus greater than 1

In some cases, we can simplify this general method by constructing a torus minus one disc (1-trisc) and a 2-sheeted unbranched covering of this, in order to obtain a 2-trisc (see Figure 3.5). By joining the boundary components in a suitable way, we can construct bipartite maps on surfaces of genus $g = 2$. Then, if we need maps of genus greater than 2, we use $(g-1)$-sheeted unbranched coverings of those maps of genus 2 (see Figure 3.6).

To solve the remaining cases, for $(g = 0, 1)$ and complete the proof, we have to construct a disc which can be joined to itself (genus 0) or to a 1-trisc (genus 1).

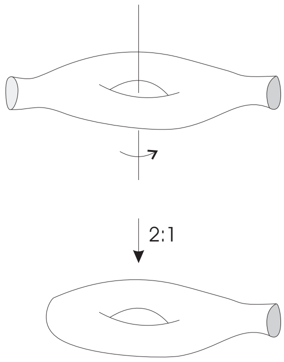


Figure 3.5: 2-sheeted unbranched covering

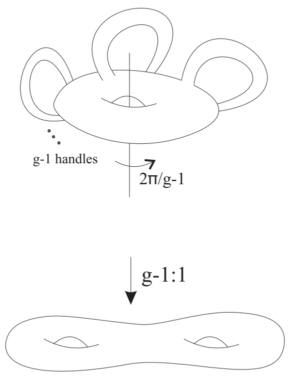


Figure 3.6: $(g-1)$-sheeted unbranched covering.

3.7 The proof

We will divide the proof into several cases for different families of hypermaps. There will be three different main cases:

i) when n is even and the parameters are not too small (if they are not ≤ 3);

ii) when n is odd and the parameters are not too small (if they are not ≤ 4);

iii) the other possibilities, when at least one of the parameters is small.

All possibilities will be covered but we will solve the problem by dealing, in the following order, with families of hypermaps of type:

- (m, m, n) with $m \geq 4$, even $n \geq 4$;
- $(m, m, 2)$ with $m \geq 6$;
- $(5, 5, 2)$;
- $(3, 3, 4)$;
- $(3, 3, n)$ with even $n \geq 6$;
- $(m, m, n+1)$ with $m \geq 5$, odd $n+1 \geq 5$;
- $(m, m, 3)$ with $m \geq 5$;
- $(4, 4, 3)$;
- $(4, 4, n)$ with, odd $n \geq 5$;
- $(3, 3, n)$ with odd $n \geq 5$.

3.8 Hypermaps of type (m, m, n) with n even

3.8.1 Hypermaps of type (m, m, n) with $m \geq 4$, even $n \geq 4$

To introduce the method, we will start by presenting the construction of an infinite number of hypermaps of type $(4, 4, n)$, for $n \geq 4$, since the others are obtained from these by multiplying the suitable edges. We will not give a very precise description at the beginning but a more formal and rigorous explanation of the method will be presented later. This will allow the reader to understand the main ideas before dealing with a more abstract approach and to become familiar with some of the techniques that will appear several times in the proof of the theorem.

Example 3.8.1. One possibility to build hypermaps of type $(4, 4, n)$ is by using the 2-trisc map, the annulus map and the disc map represented in Figures 3.7, 3.8 and 3.9, respectively.

To build the 2-trisc we identify the opposite sides of the rectangle $[0, 4] \times [0, 2n - 6]$ and remove the two shaded squares. There are no edges between vertices $(0, 0)$ and $(0, n - 3)$. Therefore, the faces are 2-gons, 4-gons, and $2n$-gons (denoted by $2n$ in Figure 3.7). Interior vertices have all valency 4 and boundary vertices valency 3.

To build the annulus map, in Figure 3.8, we identify the left and right sides of the rectangle $[0, 4] \times [0, 2n - 6]$ but not the top and bottom, and we do not remove any square faces. All interior vertices have valency 4, all boundary vertices have valency 3.

Finally, the disc map is formed by a cycle of four vertices in the boundary and two more edges, adjacent to opposite pairs of vertices as represented in Figure 3.9.

All boundary components of these pieces have type $3^{(4)}$. Therefore, after identifying any boundary components we will get hypervertices (and hyperedges) of valency 4, as we can see in Figure 3.10.

The hypervertices and hyperedges, that are not on the boundary components, have all valency 4 and the hyperfaces have valency 1, 2 or n (even). So, the final hypermap will have type $(4, 4, n)$.

To build hypermaps of type (m, m, n), for $m, n \geq 4$ and n even, we use similar tessellations, introducing multiple edges (edges incident to the same pair of vertices). For instance, to build a hypermap of type $(5, 5, n)$ for $n \geq 4$, we can use the same torus minus two discs, introducing multiple edges on the *steps of the ladders* by a suitable multiplication (see Figure 3.11).

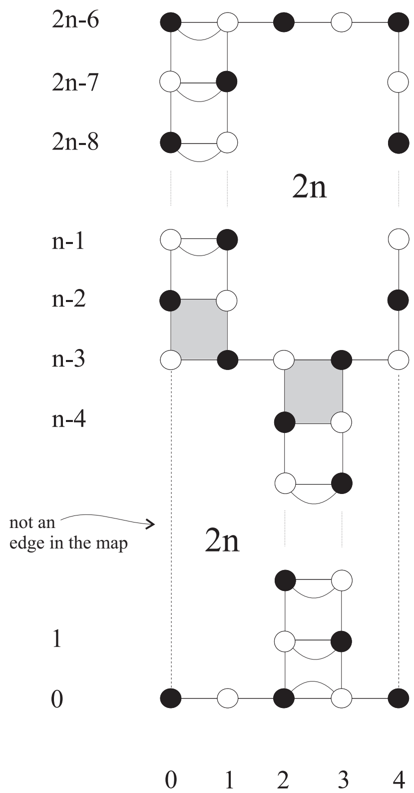


Figure 3.7: 2-trisc to construct hypermaps of type $(4, 4, n)$.

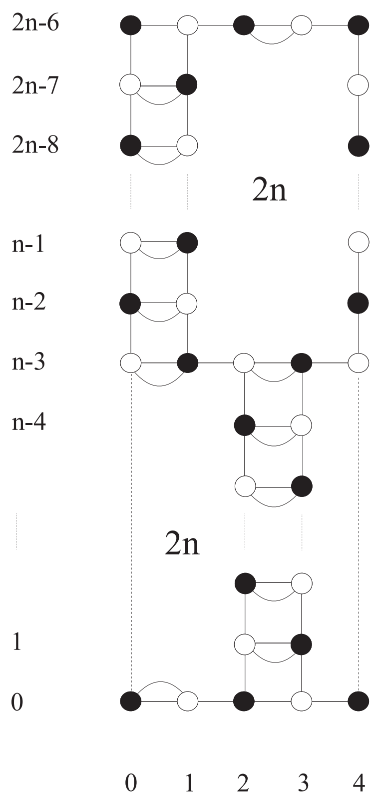


Figure 3.8: annulus to construct hypermaps of type $(4, 4, n)$.

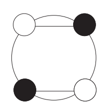


Figure 3.9: disc to construct hypermaps of type $(4, 4, n)$.

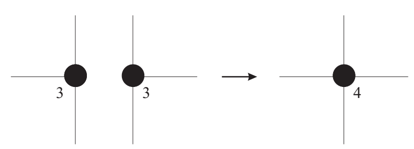


Figure 3.10: Identification of two boundary vertices of valency 3.

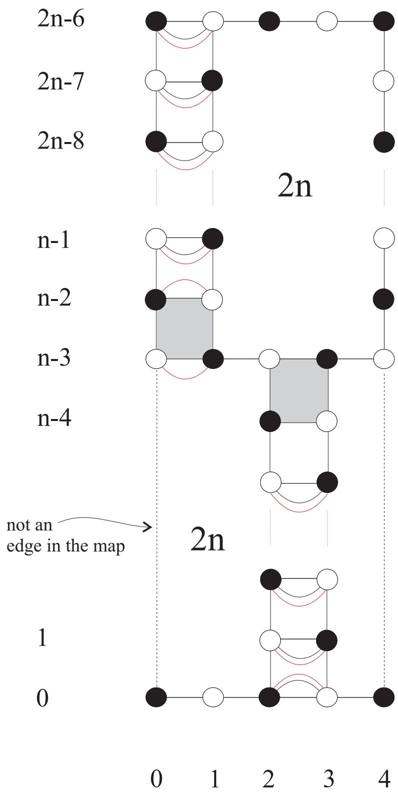


Figure 3.11: 2-trisc to construct hypermaps of type $(5, 5, n)$.

Similar modifications can be made for the annulus in order to have hypervertices of valency 3 in one of the boundary components, of valency 4 in the other boundary component, and of valency 5 anywhere else.

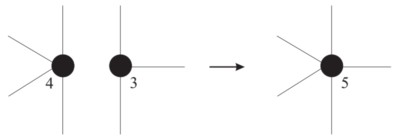


Figure 3.12: identification of two boundary vertices, one of valency 4 and another of valency 3.

We will give now a more formal and general description of a method to solve this particular case: the construction of hypermaps of type (m, m, n) with $m \geq 4$ and even $n \geq 4$. For each even n, let $\mathcal{R}_n$ be a bipartite map on the rectangle $[0, 4] \times [0, 2n - 6] \subset \mathbb{R}^2$.

This bipartite map (see Figure 3.13) has vertices at the points:

$$(0, j), (1, j), \quad \text{for } i \in \{n - 3, ..., 2n - 6\} \cup \{0\}$$

$$(2, j), (3, j), \quad \text{for } j \in \{0, ..., n - 3\} \cup \{2n - 6\}$$

$$(4, j), \quad \text{for } j \in \{n - 3, 2n - 6\} \cup \{0\}$$

The vertices (i, j) are black or white if $i + j$ is even or odd, respectively. Because we want some of them to be adjacent, we introduce some horizontal and vertical edges in the rectangle.

Horizontal:

$$(i, j) \times (i, j + 1) \text{ for } i \in \{0, n - 3, 2n - 6\} \text{ and } j \in \{0, ..., 3\}$$

$$(i, 0) \times (i, 1) \text{ for } i \in \{n - 2, ..., 2n - 7\}$$

$$(i, 2) \times (i, 3) \text{ for } i \in \{1, ..., n - 4\}$$

Vertical:

$$(i, j) \times (i + 1, j) \text{ for } i \in \{n - 3, ..., 2n - 7\} \text{ and } j \in \{0, 1, 4\}$$

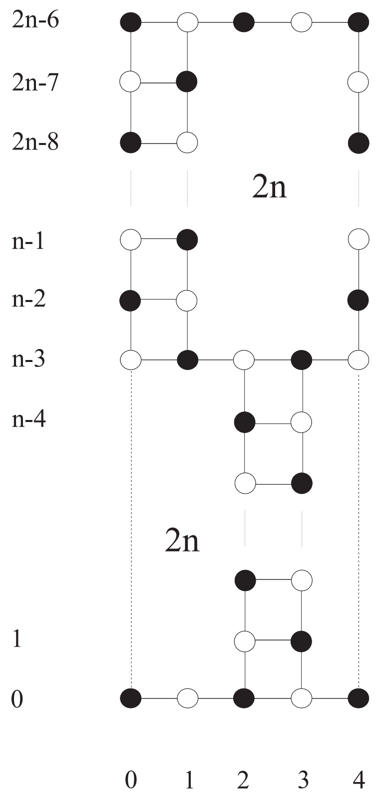


Figure 3.13: Bipartite map on the rectangle $[0,4]\times[0,2n-6]$.

$$(i,j) \times (i+1,j) \text{ for } i \in \{0,...,n-4\} \text{ and } j \in \{2,3\}$$

These edges enclose $2n-4$ faces.

$2n-6$ square faces:

$$0 < x < 1, \quad j < y < j+1 \text{ for } j \in \{n-3,...,2n-7\}$$

$$2 < x < 3, \quad j < y < j+1 \text{ for } j \in \{0,...,n-4\}$$

Two 2n-gons:

$$0 < x < 2 \text{ or } 3 < x < 4, \text{ and } 0 < y < n-3$$

$$1 < x < 4 \text{ and } n-3 < y < 2n-6$$

To obtain a bipartite map on the torus, we identify the opposite sides in the usual way: $(4,y) = (0,y)$ for $0 \leq y \leq 2n-6$ and $(x, 2n-6) = (x,0)$ for $0 \leq x \leq 4$. All the vertices have valency 3 at this stage. To build a 2-trisc $\mathcal{T}$ we need to remove two discs. We can do this by removing two non adjacent square faces (see Figure 3.14). For instance:

$$0 < x < 1, \quad n-3 < y < n-2$$

$$2 < x < 3, \quad n-4 < y < n-3$$

The trivalent map on the 2-trisc, $\mathcal{T}_n$, has now $2n-8$ square faces and two $2n$-gonal faces. The two boundary components of $\mathcal{T}_n$ have both type $3^{(4)}$ (they have 4 vertices of valency 3). We will use this bipartite map as a basis to build blocks of type $\mu = \{2n, m\}$. These are obtained by multiplying by $m-2$ each horizontal edge of the form:

$$(i,0) \times (i,1) \text{ for } i \in \{n-1,...,2n-6\}$$

$$(i,2) \times (i,3) \text{ for } i \in \{0,...,n-5\}$$

Then we choose integers $m_0, m_1 \geq 3$ such that $m_0 + m_1 = m+2$ and, for each $i = 0,1$, we multiply each of the horizontal edges in the boundary components $\partial^i \mathcal{T}_\mu$ by $m_i - 2$. This is a general procedure that always works but we could fix (for instance) $m_0 = 3$ and then take $m_1 = m-1$, multiplying only the edges of one of the boundary components of $\partial^i \mathcal{T}_\mu$.

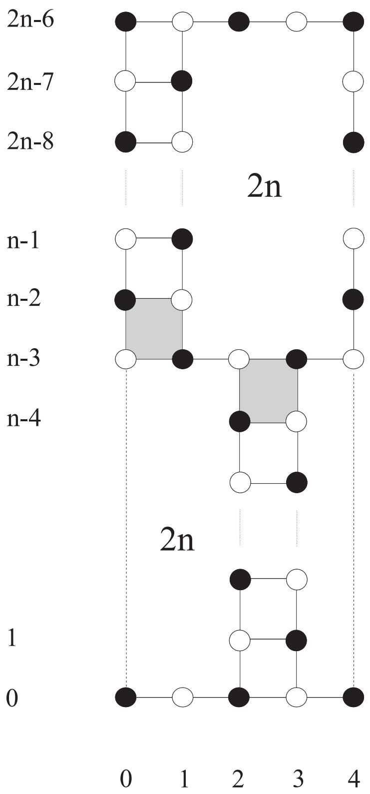


Figure 3.14: 2-trisc.

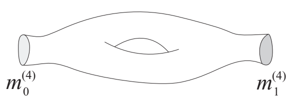


Figure 3.15: 2-trisc with boundary components of type $m_0^{(4)}$ and $m_1^{(4)}$.

Each vertex is incident with exactly one of these multiplied edges. Therefore, every internal vertex has valency m and the vertices on the boundary component $\partial^i \mathcal{T}_\mu$ (i=0,1) have valency m_i, so that this component has type $m_i^{(4)}$.

Hence, this modified map on the 2-trisc (with some edges multiplied) has two faces of valency $2n$ and $2n - 8$ faces of valency 4, just as the first basic bipartite map we have built on this surface, but also:

$$2(m-3)(n-4) + 2(m_0 - 3) + 2(m_1 - 3) = 2(n-3)(m-3)$$

new faces of valency 2. This does not affect the type of the hypermap since they correspond to hyperfaces of valency 1 in the hypermap.

The bipartite map on the annulus is constructed using the same tessellation $\mathcal{R}_n$, identifying, as before, the left and right sides but not the top and bottom sides. We obtain, by this process, a map $\mathcal{A}_\mu$ with two faces of valency $2n$ and $2n - 6$ faces of valency 4. The two boundary components $\partial^i \mathcal{A}_\mu$ $(i = 0, 1)$ are cycles of length 4, like those in $\partial^i \mathcal{T}_\mu$. If we multiply suitable edges, as before, we can create a bipartite map on $\mathcal{A}$ with all internal vertices of valency m, one boundary component of type $m_0^{(4)}$ and the other one of type $m_1^{(4)}$. This new map has two faces of valency $2n$, $2n - 6$ faces of valency 4, and the others of valency 2.

To build the disc for each integer $k \geq 2$, we construct a tessellation $\mathcal{D}_k$ of a closed disc $\mathcal{D}$, with boundary type $k^{(4)}$. We achieve that by starting with a square, regarded as a bipartite map on $\mathcal{D}$ with one face and with four vertices and four edges on $\partial \mathcal{D}$. Then, we multiply a pair of opposite edges by $k - 2$, introducing $2(k - 3)$ extra faces of valency 2, so that all four vertices have valency k.

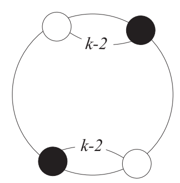


Figure 3.16: Disc D_k

The gluing process is now easy to describe. For a given genus g, we choose an arbitrarily large $h \in \mathbb{N}_0$ and if $g \geq 1$ we take $g - 1$ copies of $\mathcal{T}$ and h

copies of $\mathcal{A}$ in some arbitrary cyclic order. By making allowed joinings between consecutive pieces, we will get a bipartite map $\mathcal{W}_{g,h}$ of genus g and with all vertices of valency m. This map $\mathcal{W}_{g,h}$ has $2(g-1+h)$ faces of valency $2n$, two on each copy of $\mathcal{T}$ or $\mathcal{A}$ and the remaining faces have valency 2 or 4. Hence, $W_{g,h}$ is the Walsh map of a compact orientable hypermap $\mathcal{H}_{g,h}$ of genus g and type $\mu = (m,m,n)$. Because h is as large as we want, we can build in this way an infinite number of nonisomorphic hypermaps of genus g and type μ, as required. If $g = 0$ we do not need to use a 2-trisc, we only need $\mathcal{A}$ and two discs $\mathcal{D}_{m_0}$ and $\mathcal{D}_{m_1}$ (remember $m_0 + m_1 = m + 2$), capping a tube of $h \geq 1$ copies of $\mathcal{A}$ in linear order, by allowing joining at its ends. The resulting map will have $2h$ faces of valency $2n$ and all other faces of valency 2 or 4. Since all the vertices have valency m, the map is a Walsh bipartite map of a hypermap of type $\mu = (m,m,n)$ on the sphere.

3.8.2 Hypermaps of type $(m,m,2)$ with $m \geq 6$

The method used in the previous case does not work for $n = 2$ but we just need to introduce a slight modification to make it right, provided $m \geq 6$. This is achieved by using, first, eight 1×1 square faces to form a tessellation $\mathcal{R}_2$ of the rectangle $[0,4] \times [0,2] \subset \mathbb{R}^2$ with vertices at the points (i,j) colored black or white as $i+j$ is even or odd. By identifying opposite sides of $\mathcal{R}_2$ we obtain a bipartite map of type $\{4,4\}$ on a torus. To build a bipartite map $\mathcal{T}_2$ on a 2-trisc we remove, before identification, two nonadjacent faces: $0 < x < 1, \quad 0 < y < 1$ and $2 < x < 3, \quad 0 < y < 1$.

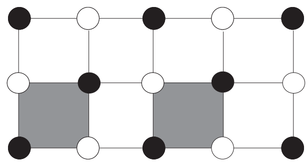


Figure 3.17: Tessellation of $\mathcal{R}_2 = [0,4] \times [0,2]$ with 2 faces removed.

This bipartite map has six square faces and each of the eight vertices lies on a boundary component $\partial^i \mathcal{T}_2 \quad (i = 0,1)$ of type $4^{(4)}$. Then we choose integers $m_i \geq 4 \quad (i = 0,1)$ so that $m_0 + m_1 = m - 2$ and multiply each of the two horizontal edges on $\partial^i \mathcal{T}_2$ by $m_i - 3$ so that $\partial^i \mathcal{T}_2$ has type $m_i^{(4)}$.

The annulus $\mathcal{A}_2$ can be constructed using the same rectangle $\mathcal{R}_2$ but only identifying the vertical sides (not the horizontal ones). On each boundary component $\partial^i \mathcal{A}_2 \quad (i = 0, 1)$ of $\mathcal{A}_2$ we multiply each of two nonadjacent edges by $m_i - 2$ so that this component has type $m_i^{(4)}$. Because we now have internal vertices, we also need to multiply each of two nonadjacent internal edges by $m - 3$. This will transform all the internal vertices into vertices of valency m, as required. With these pieces (together with the discs previously described) we can construct infinitely many hypermaps of type $(m, m, 2)$, provided $m \geq 6$.

3.8.3 Hypermaps of type $(5, 5, 2)$

The method described in the previous subsection does not work for $m = 5$ because we have $m_i \geq 4$ and, consequently, $m = m_0 + m_1 - 2 \geq 6$. To get a hypermap that will work in this particular case, we need to build different blocks whose boundary components have types $3^{(4)}$ and $4^{(4)}$, so that after joining them we will get vertices of valency $3 + 4 - 2 = 5$.

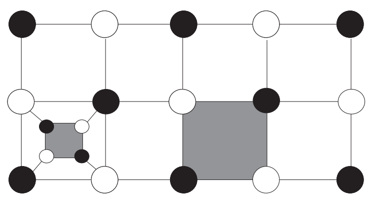


Figure 3.18: 2-trisc map to build hypermaps of type $(5, 5, 2)$.

To build $\mathcal{T}$ we use the same bipartite torus map $\mathcal{T}_2$ described for the previous case but with four extra vertices, a square S, with vertices at $(\frac{1}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{2}{3})$ and $(\frac{1}{3}, \frac{2}{3})$, each joined by a straight edge to the vertex $(0, 0)$, $(1, 0)$, $(1, 1)$ or $(0, 1)$ respectively (see Figure 3.18). This tessellation has five new faces and the new vertices have valency 3. If we remove the face within S, we create a boundary component of type $3^{(4)}$. Removing the face given by $2 < x < 3$ and $0 < y < 1$, we create another boundary component, this one of type $4^{(4)}$ (see Figure 3.19).

We also need to build an annulus satisfying the same conditions, that is, with one boundary component of type $3^{(4)}$ and another one with type $4^{(4)}$. This can be done in the following way: we take the map of the cube on the sphere,

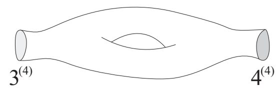


Figure 3.19: 2-trisc map with boundary components of type $3^{(4)}$ and $4^{(4)}$.

removing a pair of opposite faces and we multiply a pair of opposite edges of one of those faces by 2. The resulting bipartite map has four faces of valency 4 and two of valency 2. We just need now two other blocks $\mathcal{D}_3$ and $\mathcal{D}_4$, as described earlier, and proceed as before but this time joining boundary components of type $3^{(4)}$ to boundary components of type $4^{(4)}$.

3.8.4 Hypermaps of type $(3, 3, 4)$

The previous methods used square tessellations of a rectangle $\mathcal{R}_2$ and can not be applied to build hypermaps of type $(3, 3, 4)$. The reason is obvious: if we want to obtain, after joining two blocks, a hypermap with hypervertices of valency $m = m_0 + m_1 - 2$ we will need here $m = 3$. So, $m_0 + m_1 = 5$, giving $m_0 = 2$ and $m_1 = 3$ or vice versa, which is impossible using the strategy we have already introduced. Another method is needed to solve the problem, which means we have to build the blocks following a different idea, an alternative approach.

To build the map $\mathcal{T}$ we take the regular map $\{3, 4 + 4\}$ of type $\{3, 8\}$ and genus 2 (described in [19, Chapter 8] and represented in Figure 3.20, with opposite sides of the octagon identified) and then cut it along a simple closed curve that follows two edges. The map $\{3, 4 + 4\}$, a double cover of the octahedron branched over six vertices, can be constructed by taking a regular octagon, placing vertices at the center, the eight corners and the midpoints of the eight sides; each of these last eight vertices is then joined by straight edges to the central vertex, to the two corner vertices incident with its side, and to the vertices at the midpoints to the two adjacent sides. so that the octagon is tessellated by 16 triangles. If we make orientable identifications of the four pairs of opposite sides of the octagon we obtain the regular map $\{3, 4+4\}$. If we identify just three pairs of opposite sides instead (or, equivalently, we cut the map $\{3, 4 + 4\}$ open along the simple closed path corresponding to the fourth pair, as in Figure 3.21), we obtain a triangular map on a 2-trisc.

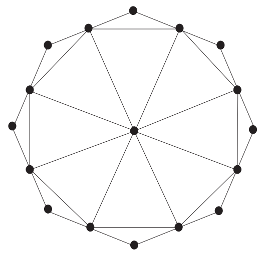


Figure 3.20: Map $\{3, 4+4\}$ of type $\{3, 8\}$ and genus 2.

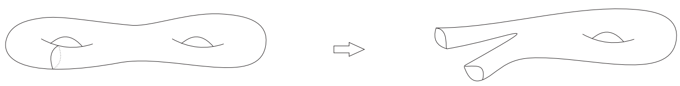


Figure 3.21: The map $\{3, 4+4\}$ after being cut open along a simple closed path.

The boundary components in this block both have type $5^{(2)}$ and all the four interior vertices have valency 8.

To construct the annulus we take $\mathcal{A} = \{z \in \mathbb{C} | 1 \leq |z| \leq 2\}$, with vertices at ± 1, ± 2 and $\pm 3i/2$, and with edges along the boundary components, along $\mathcal{A} \cap \mathbb{R}$, and joining $\pm 3i/2$ each to ± 1 and ± 2. We have then eight triangular faces and the two internal vertices have valency 4. The disc is formed by dividing the closed unit disc D into four triangular faces, with vertices at ± 1 and $\pm i/2$ and edges along the boundary, along $D \cap \mathbb{R}$ and joining each of $\pm i/2$ to ± 1, so that the internal vertices have valency 2. In both of these maps, each boundary component has type $5^{(2)}$.

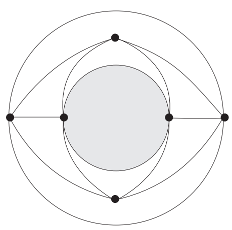


Figure 3.22: Annulus.

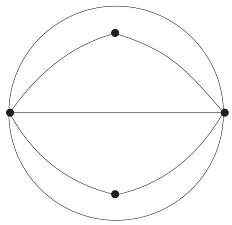


Figure 3.23: Disc.

All the vertices in the boundary components have valency 5, which means that after joining together the pieces these vertices will give rise to vertices of degree $8 = 5 + 5 - 2$. Moreover, all the vertices that are not in the boundary have valency 2, 4 or 8 and all the faces have valency 3. Therefore, the resulting maps will have type $\{3, 8\}$. These maps are also 2-face-colorable, since all the pieces are 2-face-colourable and each of their boundary components has two edges incident with faces of opposite colors.

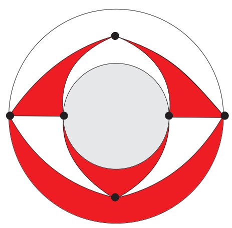


Figure 3.24: 2-face colourable map.

Then, if we take the duals of these 2-face-colourable maps of type $\{3, 8\}$ we get maps which are bipartite and of type $\{8, 3\}$. Hence, these can be understood as the Walsh bipartite maps for hypermaps of type $(3, 3, 4)$. The gluing process of the required pieces, in order to get an infinite number of these with any genus, is exactly as before.

3.8.5 Hypermaps of type $(3, 3, n)$ for even $n \geq 6$

To solve this case we use the same 2-trisc as in Section 3.8.1 but this time removing two rectangles given by $0 < x < 1$, $n - 3 < y < n$ and by $2 < x < 3$, $n - 6 < y < n - 3$ (see Figure 3.25). Consequently, not all the vertices in the boundary components have the same valency, so we need to explicitly write the type of those boundary components. They are cycles of length 8 and type $(2, 2, 3, 3, 2, 2, 3, 3)$ in cyclic order. Both of them are of this type but if we fix an orientation, travelling around each component in the same direction, the first colour of these vertices is black in one of the boundary components and white in the other one. Hence, after the second consecutive vertex of order 2 we will have a black vertex of order 3 in one boundary and a white vertex of order 3 in the other one. Those two boundary components of the same type will be called, respectively: black component and white component.

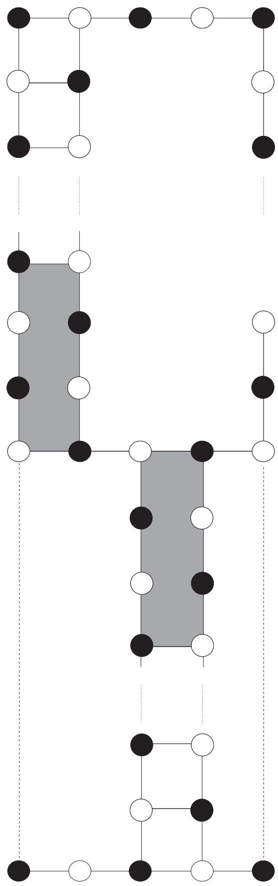


Figure 3.25: 2-trisc map with boundary components of type $(2, 2, 3, 3, 2, 2, 3, 3)$.

To build the map on the annulus we use two copies of the rectangle R_2 of the first case, one for $0 \leq x \leq 4$ and another for $4 \leq x \leq 8$, identifying the two sides of this bigger rectangle. It has boundary components for $y = 0$ and

$y = 2n - 6$, both of type $(2, 2, 3, 3, 2, 2, 3, 3)$, like the map in the 2-trisc, one black and another one white (see Figure 3.26).

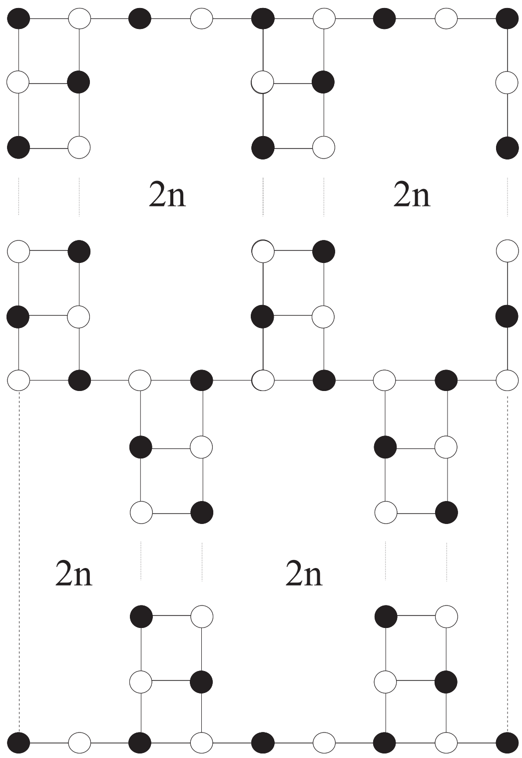


Figure 3.26: 2-trisc map with boundary components of type $(2, 2, 3, 3, 2, 2, 3, 3)$.

Finally, the disc is just an octagon with two edges from two consecutive vertices to their opposite. This gives the disc a white (D_w) or, if we interchange colours, black (D_b) boundary component of the same type $(2, 2, 3, 3, 2, 2, 3, 3)$ (see Figure 3.27). Because all the internal vertices are trivalent, and all faces are of order dividing $2n$ we will obtain hypermaps of the correct type by conveniently gluing the pieces:

- a black boundary is joined to a white boundary;

- a vertex of valency 2 is identified with another of valency 3.

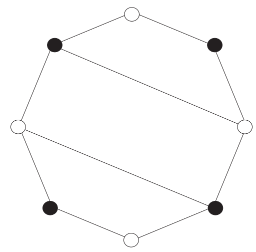


Figure 3.27: A disc map with a boundary component of type $(2, 2, 3, 3, 2, 2, 3, 3)$.

3.9 Hypermaps of type $(m, m, n+1)$ with $n+1$ odd: the general method

If $m, n+1 \geq 5$ we can use the construction presented for n even and introduce some slight but important changes. In the previous cases we could use faces of valency 4 in our pieces because these would correspond to hyperfaces of valency 2, which do not interfere with the type of a hypermap if we require the parameter n (the l.c.m of the valency of the faces) to be even. However, these faces of valency 4 can not appear if we want $n+1$ to be odd. Therefore, other tools must be developed to solve this problem. The general method is the following: to build a hypermap of type $(m, m, n+1)$, $n+1$ odd, we take the pieces that we have built for hypermaps of type (m, m, n) and add new vertices and edges in order to increase by 2 the valency of the old faces of valency $2n$ and transform all the square faces into faces of valency $2n+2$.

The first part is easier because we just need a new edge and a new vertex.

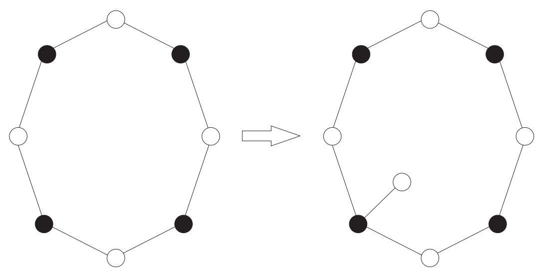


Figure 3.28: A face of valency 8 transformed into a face of valency 10.

For the second, a more delicate procedure, we need to introduce a few *stalks* of length $n-1$: paths of length $(n-1)$ with consecutive vertices $v_0, v_1, ..., v_{n-1}$ alternately black and white and with alternate edges $v_i v_{i+1}$ (i odd) multiplied by $m-1$ so that v_0 and v_{n-1} have valency 1 while the others have valency m.

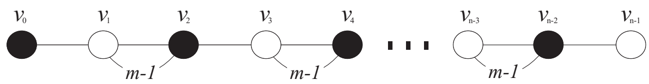


Figure 3.29: A stalk

By attaching a stalk S to a vertex v within a face F we mean identifying v_0 or v_{n-1} with v, as v is black or white, and embedding the rest of the stalk in F without crossings. This raises the valency of the face F by $2(n-1)$ and that of v by 1. It also introduces $(m-2)(n-2)/2$ new faces of valency 2, together with $n-2$ vertices of valency m and one of valency 1. Because these new faces have valency 2 they correspond to hyperfaces of valency 1, so they do not affect the type of the final hypermap. On the other hand, the vertex where the stalk is attached increases its valency by 1, which means that we need to correct this change by modifying the factors by which certain edges are multiplied. We will describe this operation later with more details.

Example 3.9.1. To build hypermaps of type $(m, m, 5)$ we need to start with the pieces that form a hypermap of type $(m, m, 4)$.

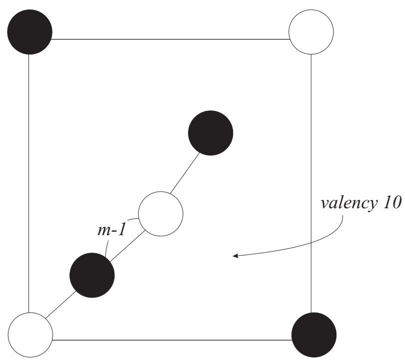


Figure 3.30: Example of a face of valency 4 transformed into a face of valency 10.

3.9.1 Hypermaps of type $(m, m, n+1)$ with $n+1$ odd with $m \geq 5$ and $n+1 \geq 5$

Let $\mathcal{T}_n$ be the trivalent bipartite map constructed in section 3.8.1 by identifying opposite sides of the rectangle R_n. If we remove, as before, the two square faces given by $0 < x < 1$, $n-3 < y < n-2$ and by $2 < x < 3$, $n-4 < y < n-3$, the underlying surface is a 2-trisc. In the face of valency $2n$ given by $1 < x < 4$, $\;n-3 < y < 2n-6$ we insert a white vertex joined by an edge to the black vertex at $(1, n-3)$, and in the other face of valency $2n$ we insert a black vertex joined by an edge to the white vertex at $(2, n-3)$, so that both of these faces now have valency $2(n+1)$. At each white vertex of the form $(i, j) = (0, n-1), (0, n+1), ..., (0, 2n-7)$ or $(2, 1), (2, 3), ..., (2, n-5)$ we attach a stalk of length $n-1$, with all interior vertices of order m, within the incident square face $i < x < i+1$, $j < y < j+1$; at each black vertex of the form $(i, j) = (1, n-1), (1, n+1), ..., (1, 2n-7)$ or $(3, 1,), (3, 3), ..., (3, n-5)$ we also attach a stalk of length $n-1$, with all interior vertices of order m, within the incident square face $i-1 < x < i$, $j-1 < y < j$. The result of this is that each of the $2n-8$ originally square faces now contains a stalk, and hence has valency $2(n+1)$. Since $m \geq 5$, we can choose integers $m_0 \geq 4$ and $m_1 \geq 3$ so that $m_0 + m_1 = m+2$ and then multiply the horizontal boundary edges $0 < x < 1, y = n-3$ and $0 < x < 1, y = n-2$ by $m_0 - 3$ and $m_0 - 2$, respectively, and the other two horizontal boundary edges $2 < x < 3, y = n-4$ and $2 < x < 3, y = n-3$ by $m_1 - 2$, so that the boundary components have types $(m_0 - 1, m_0, m_0, m_0)$ and $(m_1 + 1, m_1, m_1, m_1)$, with both first vertices of these sequences being white vertices. Finally, we multiply each remaining horizontal internal edge of the form $i < x < i+1$, $y = j$ for $i = 0$ or $i = 2$ by $m-2$ or $m-3$ as $i+j$ is even or odd, that is, as the vertex (i, j) is black or white, so that all internal vertices have valency m. This map on the 2-trisc is represented in Figure 3.31 (before multiplication of edges), with stars representing the stalks of length $n-1$.

To construct a map $\mathcal{A}$ on the annulus, before identifying the vertical sides of R_n, we insert a black vertex in the face of valency $2n$ given by $1 < x < 4$, $\;n-3 < y < 2n-6$, joined by an edge to the white vertex at $(3, 2n-6)$, and in the other face of valency $2n$ we insert a white vertex, this time joined

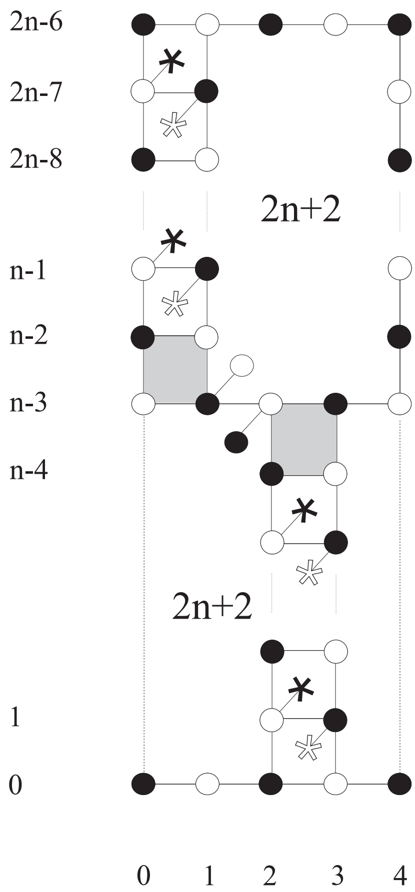


Figure 3.31: 2-trisc map for the odd case (before multiplication of edges).

by an edge to the black vertex at $(0,0)$. At each white vertex of the form $(i,j) = (0,n-3),(0,n+1),...,(0,2n-7)$ or $(2,1,),(2,3),...,(2,n-5)$ we attach a stalk of length $n-1$, with all interior vertices of order m, within the incident square face $i < x < i+1$, $j < y < j+1$; at each black vertex of the form $(i,j) = (1,n-1),(1,n+1),...,(1,2n-7)$ or $(3,1,),(3,3),...,(3,n-3)$ we also attach a stalk of length $n-2$, with all interior vertices of order m, within the incident square face $i-1 < x < i$, $j-1 < y < j$. It follows that each of the $2n-6$ square faces now has valency $2(n+1)$. This annulus is represented in Figure 3.32 (before multiplication of edges), with stars representing the stalks of length $n-1$.

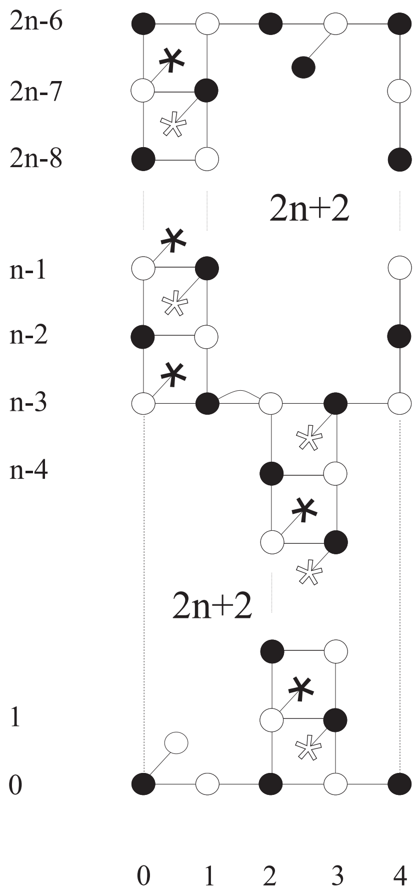


Figure 3.32: Annulus map for the odd case (before multiplication of edges).

We then multiply the boundary edges $0 < x < 1, y = 0$ and $2 < x < 3, y = 0$ by $m_0 - 2$, and the boundary edges $0 < x < 1, y = 2n-6$ and $2 < x < 3, y =$

$2n-6$ by m_1-2 and m_1-1, respectively, so that the two boundary components $y=0$ and $y=2n-6$ have types (m_0-1, m_0, m_0, m_0) and (m_1+1, m_1, m_1, m_1), with both first vertices of these sequences being white vertices. Finally we multiply each horizontal internal edge $i < x < i+1, y=j$ for $i=0$ or $i=2$ by $m-2$ or $m-3$ as $i+j$ is even or odd and we add an extra edge between vertices $(1, n-3)$ and $(2, n-3)$, so that all internal vertices have valency m.

Finally, we will need two discs. To build the disc $\mathcal{D}_a$ we construct a tessellation D_a of a closed disc D, with boundary type (m_1+1, m_1, m_1, m_1). We achieve this by starting with a square, regarded as a bipartite map on D with one face and with four vertices and four edges on $\delta\mathcal{D}$. We multiply each pair of opposite edges by m_1-3, introducing $2(m_1-4)$ extra faces of valency 2, so that all four vertices have valency m_1. Then we introduce, within the face of valency 4, a stalk of length $n-1$ starting at a white vertex. We will get a disc with $2(m_1-4)$ internal faces of valency 2, one face of valency $2n+2$ and with a boundary component of type (m_1+1, m_1, m_1, m_1), with the first vertex of this sequence being white (see Figure 3.33). For the other disc, $\mathcal{D}_b$, we use the same tessellation but with a stalk at a black vertex, and instead of multiplying both opposite edges by m_0-3 we multiply just one of them by m_0-3 (the one that is not adjacent to the black vertex with a stalk) and the other by m_0-4. Then, we will get a disc with $2(m_0-4)$ internal faces of valency 2 and one face of valency $2n+2$, with a boundary component of type (m_0-1, m_0, m_0, m_0), with the first vertex of this sequence being also white (see Figure 3.34).

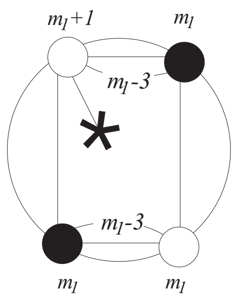


Figure 3.33: Disc a.

Using the bipartite map $\mathcal{A}$, together with $\mathcal{B}$, $\mathcal{D}_a$ and $\mathcal{D}_b$, the proof proceeds as in earlier cases though we have to be careful, this time, to always attach the

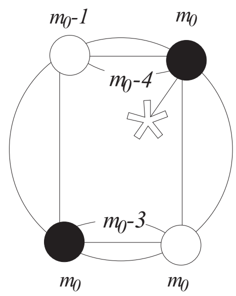


Figure 3.34: Disc b.

boundary white vertices of valency $m_0 - 1$ to the boundary white vertices of valency $m_1 + 1$.

3.9.2 Hypermaps of type $(m, m, 3)$ with $m \geq 5$

To build $\mathcal{A}$ we take the rectangle $[0, 1] \times [0, 6] \subset \mathbb{R}^2$, tesselated by six squares. The vertices, as in previous examples, are the integer points (i, j), coloured black or white as $i + j$ is even or odd, joined by edges along the sides and from $(0, j)$ to $(1, j)$ for $j = 1, ..., 5$. Before identifying the vertical sides $y = 0$ and $y = 6$, we multiply three of the vertical edges: $x = 1, 1 \leq y \leq 2$, $3 \leq y \leq 4$ and $5 \leq y \leq 6$ by $m - 4$. This means that in one of the sides we are multiplying alternate edges leaving the other ones unaltered. To increase the valency of the faces and make them of valency 6 (corresponding to hyperfaces of valency 3) we also need to add a stalk of length 1 (in fact, just an edge and a vertex) at each of the six vertices with $x = 1$, the ones on the right side of the rectangle, in the 4-gonal face below and to the left of the vertices. Thus, we get six faces of valency 6 and the rest of valency 2, as it can be verified with the help of Figure 3.35. All the six internal vertices are of valency 1 and, because of that, they do not interfere with the type of our final hypermap.

On the other hand, the boundary components $x = 0$ and $x = 1$ have types $3^{(6)}$ and $(m - 1)^{(6)}$, respectively.

To construct $\mathcal{T}$ we take another rectangle, this time the rectangle $[0, 2] \times [0, 8] \subset \mathbb{R}^2$, with opposite sides identified. The vertices are again at the integer points (i, j), coloured black or white as $i + j$ is even or odd. There are vertical edges between (i, j) and $(i, j + 1)$ for $i = 0, 1, 2$ and $j = 0, ..., 7$ with those

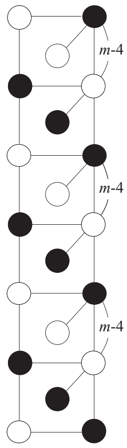


Figure 3.35: Annulus map for hypermaps of type $(m, m, 3)$ with $m \geq 5$.

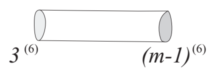


Figure 3.36: Boundary components of the annulus.

between $(i, 1)$ and $(i, 2)$, for $i = 0, 1$, multiplied by $m - 3$. The horizontal edges are the ones between (i, j) and $(i + 1, j)$ for $i = 0$ and j even, and for $i = 1$ and j odd. Some of these, the ones between $(1, j)$ and $(2, j)$ for $j = 3$ and 7, are multiplied by $m - 2$ and the edge between $(0, 0)$ and $(1, 0)$ is multiplied by $m - 3$. We then remove two 6-gonal faces, the ones given by $0 < x < 1$ for $0 < y < 2$ and $4 < y < 6$. This will leave us with boundary components of types $(m - 1)^{(6)}$ and 3^6 respectively (see Figure 3.37).

Six of the faces of $\mathcal{T}$ are 6-gons and the rest are 2-gons. All the four internal vertices are of valency m, which is important to obtain the required final type.

Finally, for the disc we need to place six vertices around the boundary of D, alternately black and white. Three alternate boundary segments are multiplied by $m - 2$ and by this process all the six vertices have valency $m - 1$ and the disc has type $(m - 1)^{(6)}$. All the faces of this piece have valency 2 except one of them that has valency 6 (see Figure 3.39).

We also need another disc of type $3^{(6)}$. That can be achieved by multiplying alternate boundary segments by 2 instead of $m - 2$. To obtain the final hypermaps we just need to join the boundary components of type $(m - 1)^{(6)}$

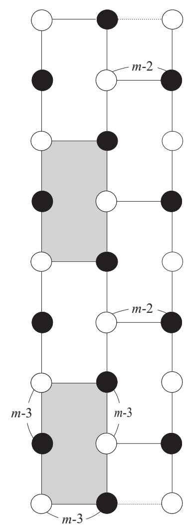


Figure 3.37: 2-trisc map for hypermaps of type $(m, m, 3)$ with $m \geq 5$.

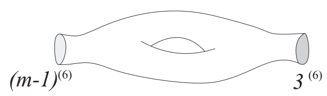


Figure 3.38: Type of the boundary components of the 2-trisc.

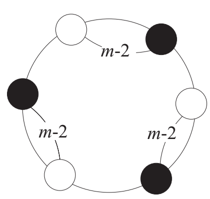


Figure 3.39: Disc map for hypermaps of type $(m, m, 3)$ with $m \geq 5$.

with those of type $3^{(6)}$, proceeding as in earlier cases.

3.9.3 Hypermaps of type $(4, 4, 3)$

This case must be dealt separately because the previous annulus does not work for such a low value for m. If $m = 4$, $m - 4$ would be 0 and that will lead us to a nonconnected graph. However, we can use the same $\mathcal{T}$ and $\mathcal{D}$ as in the previous section and proceed as before, in earlier proofs. The new annulus is the following: we take a rectangle $[0, 2] \times [0, 6] \subset \mathbb{R}^2$, with vertices at the integer points (i, j), colored black or white as $i + j$ is even or odd. The edges are along the sides, and also from (i, j) to $(i + 1, j)$, for $i = 0, 1$ and $j = 0, ..., 6$, so that the rectangle is tessellated by six faces, all of valency 6 (see Figure 3.40).

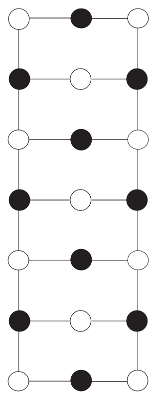


Figure 3.40: Annulus map for hypermaps of type $(4, 4, 3)$.

The side $y = 0$ is identified with the side $y = 6$ and the two boundary components are both of type $3^{(6)}$.

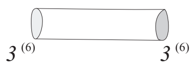


Figure 3.41: Boundary components of the 2-trisc.

3.9.4 Hypermaps of type $(4, 4, m)$, odd $m \geq 5$

If, by a Machi operation [41], we transpose hyperedges and hyperfaces, we will deal with hypermaps of type $(4, m, 4)$ instead, which will be enough to solve this case.

To construct the 2-trisc we take the rectangle $[0,3]\times[0,2] \subset \mathbb{R}^2$, with vertices at the integer points (i,j) coloured black or white as $i+j$ is even or odd. The edges are around the sides and also from $(1,1)$ to $(0,1)$, $(1,0)$, $(2,1)$ and $(1,2)$, and from $(2,1)$ to $(2,2)$. We then remove the square $1 < x < 2$, $1 < y < 2$.

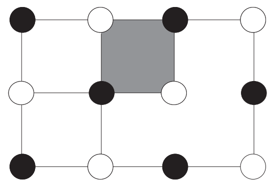


Figure 3.42: 1-trisc map.

The identifications of the sides of the rectangle are slightly different from previous cases: the side $y = 0$ is identified with the side $y = 2$ by putting $(x,0) = (x,2)$ and the side $x = 0$ is identified with the side $x = 3$ by putting $(0,y) = (3,y+1)$ where we take $y+1 \mod (2)$. Within the square $0 < x < 1$, $1 < y < 2$ we draw $m-5$ paths of length 2 between the black vertices at $(0,2)$ and $(1,1)$, each containing a white vertex of valency 2, creating $(m-5)/2+2$ new faces of valency 4. Then, in one of the new faces that shares edges with the old one, we draw two paths of length 1, each joining the black vertex at $(0,2)$ to another white vertex (see example on Figure 3.43). This last step creates a new face of valency 8, representing a hyperface of valency 4). By this process we obtain a torus minus a disc, with boundary having black vertices of order 4 and 2, and white vertices of valencies $m-1$ and 3. This torus minus one disc (1-trisc) has two 8-gonal faces and the other faces are 4-gons. Its unbranched double covering gives us the 2-trisc, as is shown in Figure 3.5. This is equivalent to placing a second copy of the rectangle in Figure 3.42 at $[0,3]\times[2,4]$ (see Figure 3.49).

To construct the annulus we consider the rectangle $[0,1]\times[0,4] \subset \mathbb{R}^2$ with vertices at integer points (i,j) coloured white or black as $i+j$ is even or odd. There are edges around the sides and also from $(0,3)$ to $(1,3)$. The edges from $(0,2)$ to $(0,3)$ and from $(1,0)$ to $(1,1)$ are multiplied by 2. Within the square $0 < x < 1$, $3 < y < 4$ we draw $m-5$ paths of length 2 between the black vertices at $(0,4)$ and $(1,3)$, each containing a white vertex of valency 2. From

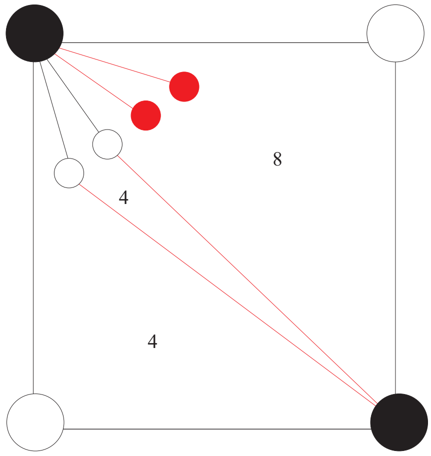


Figure 3.43: A face of valency 8 transformed into two faces of valency 4 and one of valency 8.

each white vertex $(0, 4)$ $(1, 3)$, and inside the same face, we draw two paths of length 1 (Figure 3.47 shows an example of such an annulus for the case $(4, 4, 5)$).

Finally we identify the side $y = 0$ with the side $y = 4$ to form an annulus. Both boundary components have the same type $(m-1, 4, 3, 2)$ as the ones in the 2-trisc but with mutually inverse cyclic orders, so we need to use these annuli in mirror-image pairs to make the identification work properly.

To construct the disc $\mathcal{D}$ we place four vertices around the boundary of $\mathcal{D}$, two black alternating with two white, joined by four edges around the boundary. We then join one black vertex to the two white vertices by edges across the interior, creating one 4-gonal face and two 2-gons. Within one of these 2-gons, we place $k = (m-5)/2 + 1$ black vertices, each joined by a pair of edges to the white in nested fashion, creating one 2-gon and k 4-gons (see Figure 3.44 for an example).

This will make the boundary of the disc have the same type as the boundary components of the 2-trisc and the annulus. In the gluing process, we need to join vertices of order $m-1$, 4, 3 and 2 with vertices of order 3, 2, $m-1$ and 4, respectively, to get black vertices of order m and white vertices of order 4.

Example 3.9.2. To build infinitely many hypermaps of type $(4, 4, 5)$ (which is equivalent, by a Machi operation, to $(4, 5, 4)$) we need the 1-trisc represented in Figure 3.45 and an annulus, as in Figure 3.47.

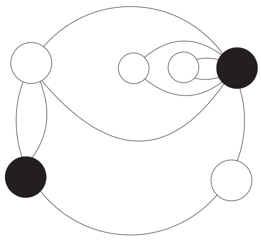


Figure 3.44: Example of a disc with boundary of type $(2, 4, 7, 2)$.

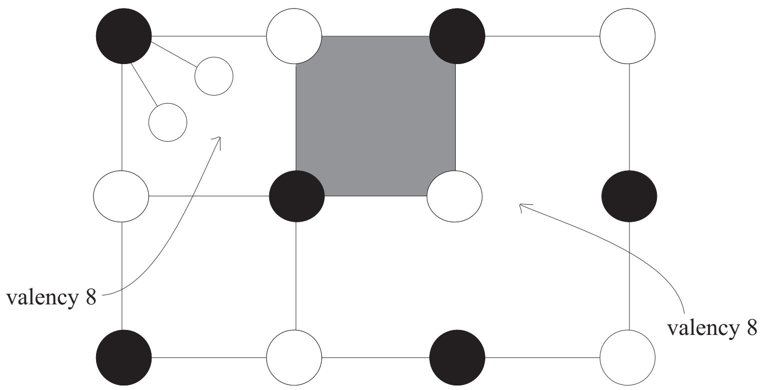


Figure 3.45: 1-trisc map for hypermaps of type $(4, 4, 5)$.

This 1-trisc map has the following properties:

a) the l.c.m. of the valencies of the interior hyperfaces is 4;

b) the l.c.m. of the valencies of the interior hypervertices is 4;

c) the l.c.m. of the valencies of the interior hyperedges is 5.

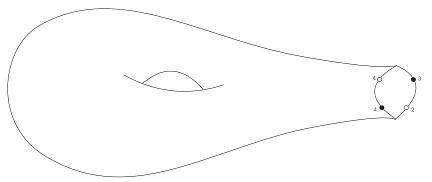


Figure 3.46: 1-trisc.

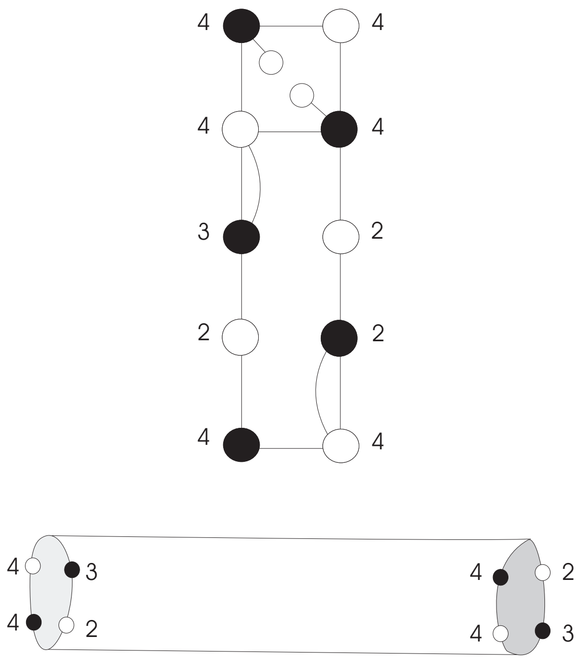


Figure 3.47: Annulus.

Because the cyclic order of the four vertices is reversed on the boundary components we may also need to use the mirror image of this annulus when we glue the pieces together (see Figure 3.48).

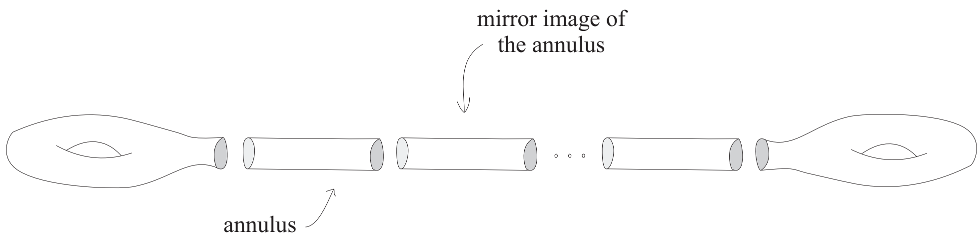


Figure 3.48: Infinitely many hypermaps of genus 2.

If we can construct a torus minus one disc, we can also get a torus minus two discs as a 2-sheeted unbranched covering. Hence, using coverings we can build infinitely many hypermaps of this type and with **any** genus.

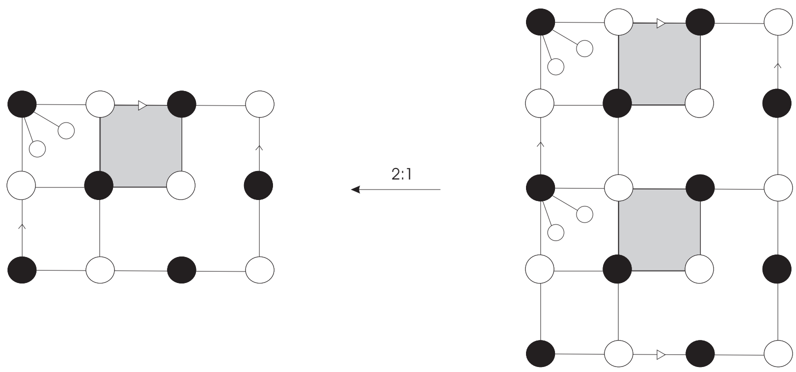


Figure 3.49: 2-sheeted unbranched covering of the 1-trisc.

3.9.5 Hypermaps of type $(3, 3, m)$, for odd $m \geq 5$

In order to build the required hypermaps of type $(3, 3, m)$, for odd $m \geq 5$, we will use 2-face colourable maps of type $\{3, 2m\}$ and then take the duals of these, a method previously used in Section 3.8.4 for the case $(3, 3, 4)$. Let R be the rectangle $[0, 6] \times [0, 2] \subset \mathbb{R}^2$ with vertices at (i, j) (with $i \in \{1, 3, 5\}$ and $j \in \{0, 2\}$) and $(i, 1)$ with $i \in \{0, 2, 4, 6\}$). There are horizontal edges between all consecutive vertices with the same horizontal coordinates and also:

$$(i, 1) \times (i - 1, j) \text{ if } i \in \{2, 4, 6\} \text{ and } j \in \{0, 2\}$$

$$(i, 1) \times (i + 1, j) \text{ if } i \in \{0, 2, 4\} \text{ and } j \in \{0, 2\}$$

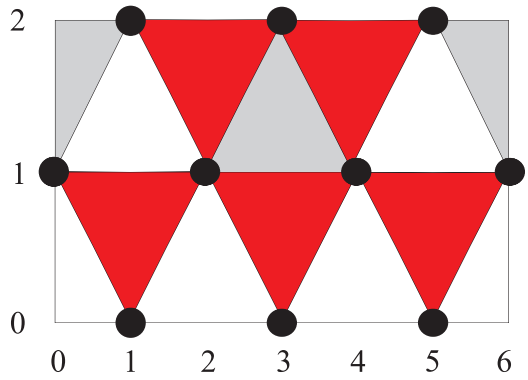


Figure 3.50: 2-Trisc, for $m = 5$ (with no wedges).

We then identify opposite sides of the rectangle to get a torus. This will give rise to 11 triangular faces but we remove two of them (one corresponding to the triangle of vertices $(0,1),(1,2),(5,2)$ and the other to the triangle $(2,1),(3,2),(4,1)$. The face $(2,1),(3,0),(4,1)$ is coloured red and the remaining ones are coloured red or white in such a way that no adjacent faces have the same colour (this operation is possible because this map is 2-face colourable, as it is clear from Figure 3.50). Hence, there are three vertices in each boundary component of the 2-trisc adjacent to three red faces, in one case, and three white faces on the other[2]. All the vertices have valency 6. To build the right map on the 2-trisc, we need to add a suitable amount of *wedges* to some faces (see Figure 3.51 for an example), before identification.

A *wedge* (v_0, w, v_1), attached to vertices v_0, v_1 on a triangular face $f = (v_0, v_1, v_2)$, is constructed by adding a vertex w inside the face f and then joining w to v_0 and v_1, within f, and adding another edge, also within f, between v_0 and v_1 in such a way that (v_0, v_1, v_2) is still a triangular face. Each time we introduce a wedge to a face, we are also adding two more triangular faces to the map. And if the original map is 2-face colourable, so it will be after introducing as many wedges as we want.

---

[2]The colours of the faces adjacent to the boundary components are important because we want the final map to be also 2-face colourable and this is only possible if we have no adjacent faces of the same colour after *gluing* the pieces.

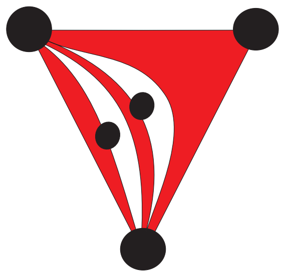


Figure 3.51: 2 wedges attached to the same vertices and on the same face.

In this case, to build our 2-trisc map, we need to add $(2m-10)/4$ wedges, three times: first, between vertices $(2,1),(3,2)$ inside face $((2,1),(3,2),(1,2))$, then between vertices $(3,2),(4,1)$ inside the face $((3,2),(4,1),(5,2))$, and finally between vertices $((2,1),(4,1)$ inside the face $((2,1),(4,1),(3,0)$. We will then obtain two boundary components, one of type $6^{(3)}$ and another one of type $(2m-4)^{(3)}$ (see Figure 3.52, an example of a 2-Trisc for $m=7$).

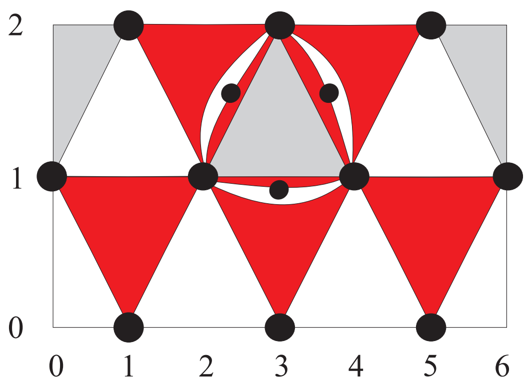


Figure 3.52: 2-Trisc for $m=7$.

To build the annulus we use the same rectangle $[0,6]\times[0,2]\subset\mathbb{R}^2$ with the vertices in the same places as before. However we do not remove any triangular face and we attach $(2m-8)/2$ wedges to each one of the following two pairs of vertices:

$$(1,2),(3,2) \text{ inside face } ((1,2),(3,2),(2,1)),$$

$$(6,1),(5,2) \text{ inside face } ((6,1),(5,2),(1,2));$$

$(2m-6)/2$ wedges to the following pair of vertices:

$$(2,1),(4,1) \text{ inside face } ((2,1),(4,1),(3,2));$$

and one wedge to each of the two following pairs:

$$(3,0),(5,0) \text{ inside face } ((3,0),(5,0),(4,1)),$$

$$(0,1),(1,0) \text{ inside face } ((0,1),(1,0),(2,1));$$

If we identify the vertical sides of the rectangle we get a map on an annulus and, by the way we constructed it, that map is 2-face colourable (we use the opposite colour scheme we have used in the 2-trisc and we use the white colour to each wedge inside a red triangular face, and the red colour to each wedge inside a white triangular face). It follows that one of the boundary components has type $6^{(3)}$ and is adjacent to three red faces, and the other one has type $(2m-4)^{(3)}$ and is adjacent to three white faces (see Figure 3.53 for an example). All the interior vertices have valency $2m$.

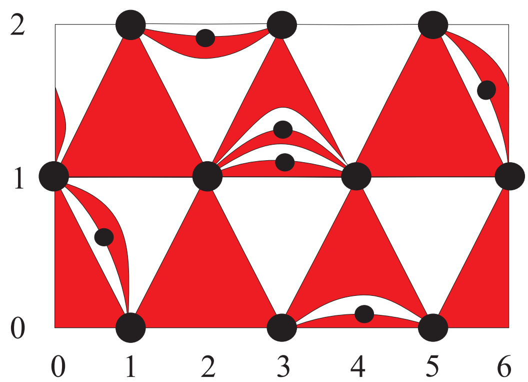


Figure 3.53: Annulus for $m = 5$.

To construct the disc, we take $D = \{z \in \mathbb{C} : |z| \leq 2\}$, with vertices at ± 2, $\pm 1 + i$, 0 and $2i$, with edges along the boundary components and joining the vertex at -2 to 0 and $-1+i$, the vertex at 2 to 0 and $1+i$, and the vertex at $2i$

to $1+i$ and $-1+i$. We have then, at this stage, three triangular faces and one face of valency 6. Inside this hexagonal face we add three more edges, between -2 and 2, between -2 and $2i$, and between 2 and $2i$. Hence, this new Disc, D_1, has 7 triangular faces, is 2-face-colourable and has type $6^{(3)}$ (see Figure 3.54). Depending on the way we coloured the faces, we might have three white faces adjacent to the boundary component or three red faces instead. If we introduce $(2m-6)/4$ wedges attached to each one of the three possible different pairs of boundary vertices (and in each one of the three boundary faces) we will get again a 2-face-colourable disc, D_2, but this time of type $(2m-4)^{(3)}$.

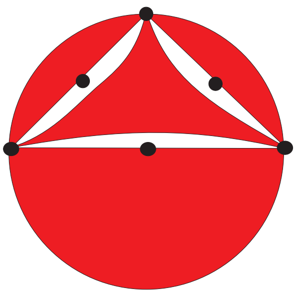


Figure 3.54: Disc D_1 with red boundary.

To build infinitely many hypermaps of type $(3,3,m)$ we need to glue, a suitable number of times, the red boundary component $6^{(3)}$ of an annulus (or a 2-trisc, or a disc) with the white boundary $(2m-4)^{(3)}$ of another annulus (or a 2-trisc, or a disc).

This completes the proof of Theorem 3.3.1. □

3.10 Self-Duality

In Chapter 5, we will introduce the notion of *duality* and we will be interested, among other topics, in self-dual regular hypermaps. Until this point, however, we have used only hypermaps that are not regular and our constructions gave rise to hypermaps that are not (at leat the majority) self-dual (we say that a hypermap is *self-dual* if it is isomorphic to its dual, after interchanging hypervertices and hyperedges). Nevertheless, in a few cases, it is possible to adjust the construction of the pieces in order to assure that all the final hypermaps, after proper gluing, are self-dual. This can be achieved by using pieces that have a symmetry axis (disregarding the colour of the vertices) that crosses the boundary components and so that the corresponding reflection transposes black and white vertices. This means that all the pieces correspond to self-dual hypermaps and that the hypermap that results from the gluing process is still self-dual because it has also a reflection that transposes black and white vertices.

Example 3.10.1. We will now give an example of this method by constructing, for any genus $g \geq 1$, an infinite number of hypermaps of self-dual type $(4, 4, 3)$.

First, we construct a map in the annulus with boundary components of type $3^{(6)}$, six faces of valency 6 and two interior vertices of valency 2, with Walsh map as in Figure 3.55.

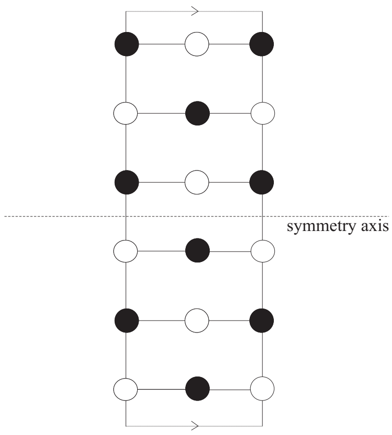


Figure 3.55: Annulus.

Then we construct a 2-trisc map with six faces of valency 6, four interior vertices of order 4 and two boundary components of type $3^{(6)}$, with Walsh map as in Figure 3.56.

Finally we construct a disc with one face of valency 6, three faces of valency 2 and a boundary component also of type $3^{(6)}$ (see Figure 3.57).

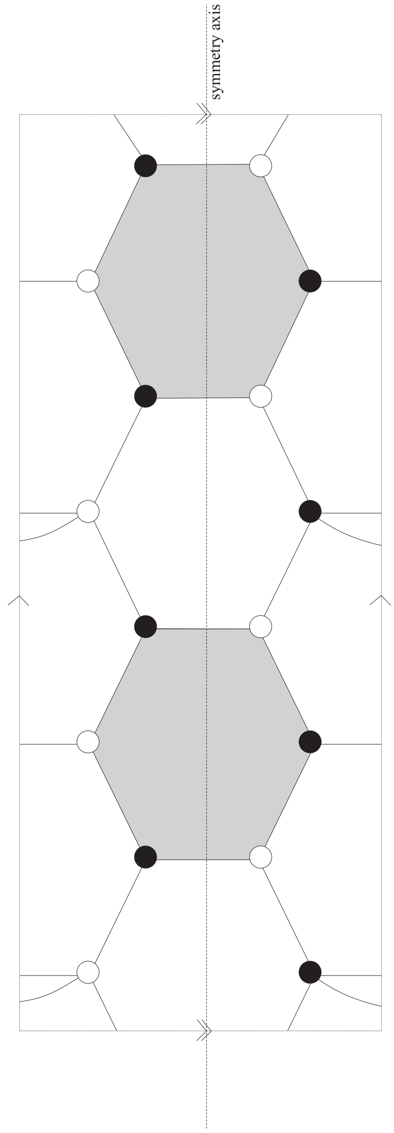


Figure 3.56: 2-trisc.

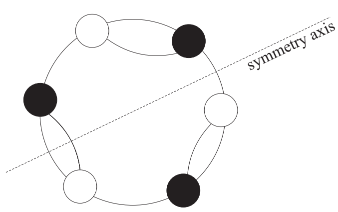


Figure 3.57: Disc.

Hence, in this particular case, we can have a slightly stronger result: given any positive integer $g > 0$, there are infinitely many nonisomorphic compact orientable **self-dual** hypermaps of type $(4, 4, 3)$ and genus g. $\square$

3.11 Nonorientable Surfaces

In all those cases where a boundary component of the 2-trisc or the annulus has a symmetry which reverses the cyclic order of valencies and colors of its vertices, we can generalize these constructions to nonorientable surfaces. For each orientable hypermap of type τ and genus $g \geq 1$, one can reverse the orientation of a boundary component in one of the allowed joins, giving a nonorientable hypermap of type τ and of the same Euler characteristic $2-2g$, that is of nonorientable genus $p = 2g$. This means that we can use this method for p even but it does not work if the joins are in *linear* order as in Fig. 3.4, since reversing a boundary component still gives an orientable surface. We need to take $(g-1)$ 2-triscs, an arbitrary number of annuli and no discs, joined in *cyclic* order, to construct an orientable surface of genus $g \geq 1$ (see Figure 3.58); then reversing one of the joins gives an nonorientable surface of the same Euler characteristic $2 - 2g \leq 0$.

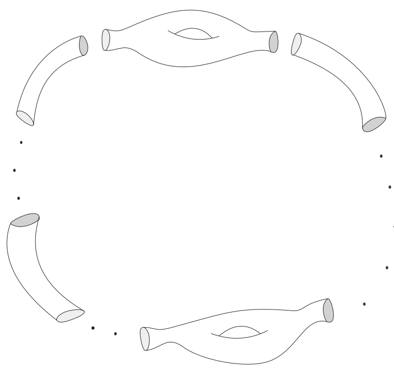


Figure 3.58: Pieces joined in cyclic order.

However, if we want to do it also for p odd, we need to construct a suitable crosscap, for instance an annulus with antipodal points of one boundary identified. The gluing process is identical to the one used in the orientable case but we need to replace one of the discs (see, for instance, Fig. 3.4) with a crosscap.

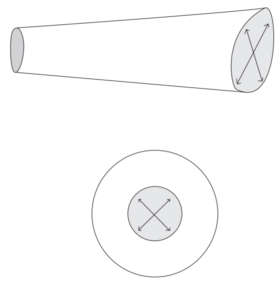


Figure 3.59: How to construct a crosscap from an annulus.

For each nonorientable case, we will use the same annulus that was constructed for the respective orientable case and, after small changes (if required), we will identify antipodal points of one of the boundary components, making sure we are also identifying opposite pairs of vertices in such a way that the type of the map is preserved after this procedure. In some cases, the introduction of new vertices in the boundary components is needed and, as a consequence, that implies also some modification in the 2-trisc and the disc.

3.11.1 Annnuli with boundary components of type $m_0^{(4)}$ and $m_1^{(4)}$

This corresponds to the following cases:

- (m, m, n) with $m \geq 4$, even ;
- $(m, m, 2)$ with $m \geq 6$;
- $(5, 5, 2)$.

We can assume, without loss of generality, that $m_0 \leq m_1$. Hence, we only need to multiply, by $k = m_1 - m_0 + 1$, one of the edges between two adjacent vertices of the boundary component of type $m_0^{(4)}$. These two vertices will then have valency m_1 and after identification of opposites pairs we will get two vertices of type $m = m_1 + m_0 - 2$. The other boundary component of the annulus remains as before and, without any changes, can be glued to any of the other pieces.

Important note: this method also works for annuli with more than 4 vertices, if the number of black (and white) vertices is even. Instead of multi-

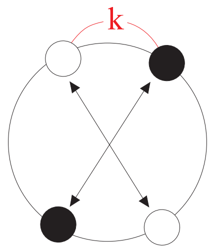


Figure 3.60: Adjustment of one the boundary components of the annulus and identification of opposite vertices.

plying by $k = m_1 - m_0 + 1$ just one of the edges between consecutive vertices, we multiply by k a suitable number of edges between consecutive disjoint pairs of vertices in order to change the valency of half of the boundary vertices (leaving the other half unchanged). Hence, any annulus with vertices of the same valency in one boundary component can be easily adapted to become a crosscap by identifying opposite points on that boundary. We will call this procedure the *standard method*.

3.11.2 Annnuli with boundary components of type $(m_0 - 1, m_0, m_0, m_0)$ and $(m_1 + 1, m_1, m_1, m_1)$

This corresponds to the case:

- $(m, m, n + 1)$ with $m \geq 5$, odd $n + 1 \geq 5$;

The previous method does not work here because not all the vertices on the boundary components have the same valency. So, we need to change the annulus used in the orientable case by taking a slightly different approach.

To construct a new map $\mathcal{A}$ on the annulus, we use a tessellation of $R = [0, 4] \times [0, 1]$ with vertices of the form (i, j) with $i \in \{0, 1, 2.3, 4\}$, $j \in \{0, 1\}$. As before, if $i + j$ is odd, the vertex is white, otherwise it is black. The horizontal edges are of the form $(i, j) \times (i + 1, j)$ $i \in \{0, 1, 2, 3\}$ and $j \in \{0, 1\}$, and there are no vertical edges. At this point, we have a single face of valency 8. We then attach a stalk of length $n - 3$ (with all interior vertices of valency m) to vertex $(1, 0)$ and within the octagonal face $0 < x < 4$, $0 < y < 1$ (that by this process becomes a face of valency $2n + 2$).

Before identifying the vertical sides of the rectangle, we multiply the horizontal edges $(0,1)\times(1,1)$, by (m_0-1), and $(2,1)\times(3,1)$, $(1,0)\times(2,0)$, $(3,0)\times(4,0)$ by m_1-1 (see Figure 3.61). This tessellation will give rise to an annulus with boundary components of types (m_1+1, m_1, m_1, m_1) and (m_1, m_0, m_0, m_1). By identifying the opposite sides (and vertices) of this last boundary component, we can build the required crosscap.

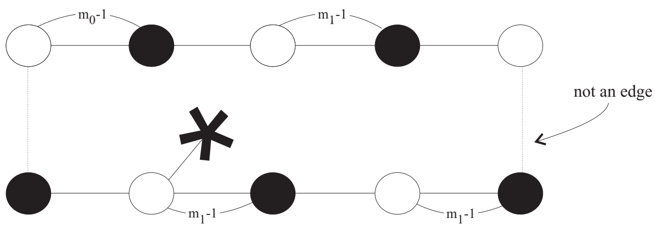


Figure 3.61: Annulus map for the odd nonorientable case.

3.11.3 Annnuli with boundary components of type $3^{(6)}$ and $(m-1)^{(6)}$

This corresponds to the cases:

- $(m, m, 3)$ with $m \geq 5$;
- $(4, 4, 3)$.

Case $(m, m, 3)$ *with* $m \geq 5$:

Because, in the orientable case (see Section 3.9.2) we have six vertices in each boundary component of the pieces we cannot use the same method as before since we can only identify vertices of the same colour and here we have an odd number of black vertices (3 black vertices) and an odd number of white vertices (3 white vertices). However, the construction used in the orientable case can be adapted in order to have eight vertices in the boundary components.

To build $\mathcal{A}$ we take the rectangle $[0,1]\times[0,8]\subset\mathbb{R}^2$, tesselated by eight squares. The vertices, as in previous examples, are the integer points (i,j), coloured black or white as $i+j$ is even or odd, joined by edges along the sides and from $(0,j)$ to $(1,j)$ for $j=1,...,8$. We then identify the vertical sides

$y = 0$ and $y = 8$ and we multiply four of the vertical edges: $x = 1, 1 \leq y \leq 2$, $3 \leq y \leq 4$, $5 \leq y \leq 6$ and $7 \leq y \leq 8$ by $m - 4$. This means that in one of the sides we are multiplying alternate edges leaving the other ones unaltered. To increase the valency of the faces and make them of valency 6 (corresponding to hyperfaces of valency 3) we also need to add a stalk of length 1 (in fact, just an edge and a vertex) at each of the six vertices with $x = 1$, those on the right side of the rectangle, in the 4-gonal face below and to the left to the vertex. Thus we get eight faces of valency 6 and the rest of valency 2, as can be easily checked with the help of Figure 3.62. All the six internal vertices are of valency 1 and because of that they do not interfere with the type of our final hypermap. Hence, all we had to do with the annulus for the orientable case was to introduce two more squares and respective stalks at the bottom (or at the top) and then apply the *standard method* to one of the boundary components. Because we have changed the number of vertices at both boundary components of the annulus, we need to build a new 2-trisc for the nonorientable case.

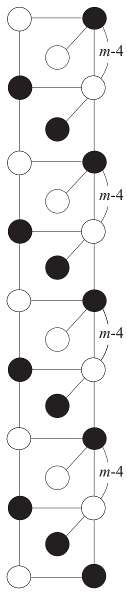
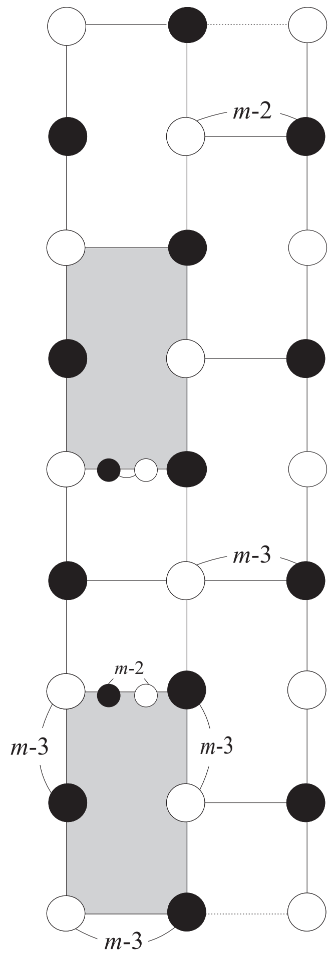


Figure 3.62: Annulus for the nonorientable case.

To construct that $\mathcal{T}$, we take another rectangle $[0, 2] \times [0, 8] \subset \mathbb{R}^2$, with opposite sides identified. The vertices are again at the integer points (i, j), coloured black or white as $i + j$ is even or odd but, this time, we add four more vertices: two black vertices at $(1/3, 2)$ and $(1/3, 4)$. There are vertical edges between (i, j) and $(i, j+1)$ for $i = 0, 1, 2$ and $j = 0, ..., 7$ with those between $(i, 1)$ and $(i, 2)$, for $i \in \{0, 1\}$, multiplied by $m - 3$. The horizontal edges are the ones between (i, j) and $(i+1, j)$ for $i = 0$ and $j = 0, 3, 6, 8$, and for $i = 1$ and j odd. We also have edges between $(0, 2)$ and $(1/3, i)$, $(1/3, i)$ and $(2/3, i)$, $(2/3, i)$ and $(1, i)$, for $i \in \{2, 4\}$. Some of these, the ones between $(1, 7)$ and $(2, 7)$, and the ones between $(1/3, 2)$ and $(2/3, 2)$, are multiplied by $m - 2$, and $(1/3, 4) \times (2/3, 4)$ is multiplied by 2. At the same time, we have edges between $(0, 0)$ and $(1, 0)$, $(0, 1)$ and $(0, 2)$, $(1, 1)$ and $(1, 2)$, $(1, 3)$ and $(1, 4)$, multiplied by $m - 3$. We then remove two 6-gonal faces, the ones given by $1 < x < 2$ for $0 < y < 2$ and $4 < y < 6$, keeping seven faces of valency 6 (see Figure 3.63).

Finally, for the disc, we need to place eight vertices around the boundary of D, alternately black and white. Two alternate boundary segments are multiplied by $m-2$. Then, clockwise, we leave two segments unchanged and multiply by $m - 2$ another two alternate segments. At this point, only two vertices still do not have any multiple edges attached. We then add a new edge between them and we multiply it by $m - 3$ (see Figure 3.64). All the faces of this piece have valency 2 except two of them that have valency 6. There are no interior vertices and the ones in the boundary have all valency $m - 1$.

Remark: These pieces also work in the orientable case but because they are slightly *heavier* (with more vertices and faces) than the ones presented before, we decided to leave the other constructions unchanged.



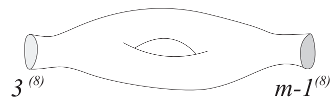


Figure 3.63: 2-trisc for the nonorientable case.

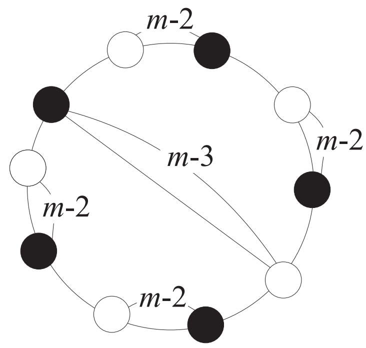


Figure 3.64: Disc for the nonorientable case.

Case $(4, 4, 3)$*:*

We use the same 2-trisc and disc as in the previous case (Figure 3.63), here with $m = 4$, but with a different annulus since the other one does not work for low $m = 4$ (this new annulus, represented in Figure 3.65, is the same as in the orientable case but with two more steps than the original *ladder*): we take a rectangle $[0, 2] \times [0, 8] \subset \mathbb{R}^2$, with vertices at the integer points (i, j), coloured black or white as $i + j$ is even or odd. The edges are along the sides, and also from (i, j) to $(i + 1, j)$, for $i = 0, 1$ and $j = 0, ..., 8$, so that the rectangle is tessellated by six faces, all of valency 6.

3.11.4 Annnuli with both boundary components of type $t^{(2)}$ for some positive integer t

This corresponds to the following case:

- $(3, 3, 4)$ with both boundary components of type $5^{(2)}$ (the orientable case is described in section 3.8.4).

No adaptations are needed. We can identify opposite sides (and opposite vertices) of one of the boundary components of the annulus without making any changes, since we end with a vertex of order $5 + 5 - 2 = 8 = 2 \times 4$.

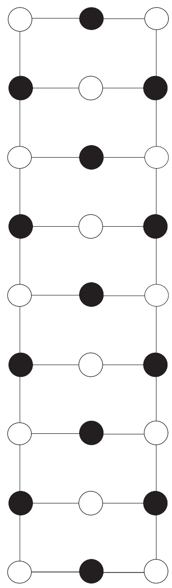


Figure 3.65: Annulus for the nonorientable case.

3.11.5 Annnuli with both boundary components of type $(2, 2, 3, 3, 2, 2, 3, 3)$

This corresponds to the following case:

- $(3, 3, n)$ for even $n \geq 6$;

Although we have already built an annulus with an even number of vertices on a boundary component, it is not possible to adapt it for the nonorientable case (and construct a crosscap) using the standard method because we cannot identify opposite vertices of the same colour and, at the same time, join all vertices of order 2 with vertices of order 3. The reason is that we have to have four vertices in each half of the boundary circle (to identify opposite sides of the boundary) and because the valency sequence must be respected, vertices of valency 2 would be identified with vertices of valency 2, while vertices of valency 3 would be identified with others of valency 3. This problem can be solved by introducing four more vertices in each boundary component of the annulus. Hence, instead of two copies of the rectangle described in Section 3.8.1, we will take three copies of it, one for $0 \leq x \leq 4$, another for $4 \leq x \leq 8$ and finally one for $8 \leq x \leq 12$ (see Figure 3.66). This new annulus has boundary components for $y = 0$ and $y = 2n - 6$ with type $(2,2,3,3,2,2,3,3,2,2,3,3)$. It follows that we can identify opposite sides of one of the boundary components, if the vertices are regularly distributed along the border, to get six vertices of valency 3 in a crosscap. We need now to build a new 2-trisc.

For each even n, let $\mathcal{R}_n$ be a bipartite map on the rectangle $[0,4] \times [0, 2n - 6] \subset \mathbb{R}^2$. This bipartite map (see Figure 3.67) has vertices at the points: $(0,j), (1,j)$ for $i \in \{n-4, ..., 2n-6\}$, at $(2,j), (3,j)$ for $j \in \{0, ..., n-1\}$. The vertices (i,j) are black or white if $i+j$ is even or odd, respectively. Because we want some of them to be adjacent, we introduce some horizontal and vertical edges in the rectangle.

Horizontal:

$$(i,j) \times (i+1,j) \text{ for } j \in \{0, 2n-6\} \text{ and } i \in \{0, ..., 3\}$$

$$(0,j) \times (1,j) \text{ for } j \in \{n+1, ..., 2n-7\} \cup \{n-4\}$$

$$(i,2) \times (i,3) \text{ for } i \in \{0, ..., n-6\} \cup \{n-1\}$$

$$(1,j) \times (2,j) \text{ for } j \in \{n-3, n-2\}$$

$$(3,j) \times (4,j) \text{ for } j \in \{n-3, n-2\}$$

Vertical:

$$(i,j) \times (i,j+1) \text{ for } j \in \{n-4, ..., 2n-7\} \text{ and } i \in \{0,1\}$$

$$(i,j) \times (i,j+1) \text{ for } j \in \{0, ..., n-1\} \text{ and } i \in \{2,3\}$$

These edges enclose $2n-7$ faces: $2n-11$ square faces, two faces of valency $2n$ and two faces of valency 12.

To obtain a bipartite map on the torus, we identify opposite sides in the usual way: $(4,y) = (0,y)$ for $0 \leq y \leq 2n-6$ and $(x, 2n-6) = (x,0)$ for $0 \leq x \leq 4$. All the vertices have valency 3, at this stage. To build a 2-trisc $\mathcal{T}$ we need to remove two discs. We can do this by removing two of those (non-adjacent) faces, in this case, the two faces of valency 12. The map on the 2-trisc, $\mathcal{T}_n$, has now $2n-7$ square faces and two $2n$-gonal faces. The two boundary components of $\mathcal{T}_n$ both have type $(2,2,3,3,2,2,3,3,2,2,3,3)$.

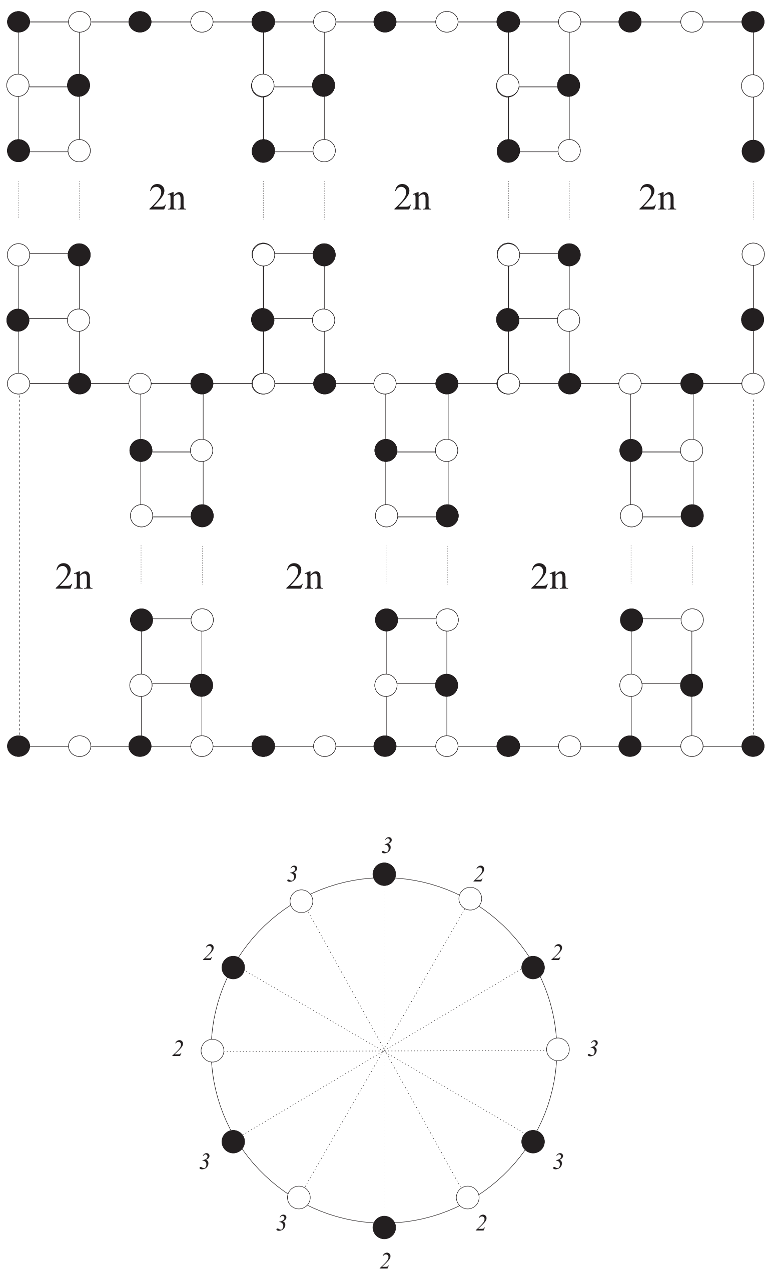


Figure 3.66: Annulus map and the identification of opposite vertices in one of its boundary components.

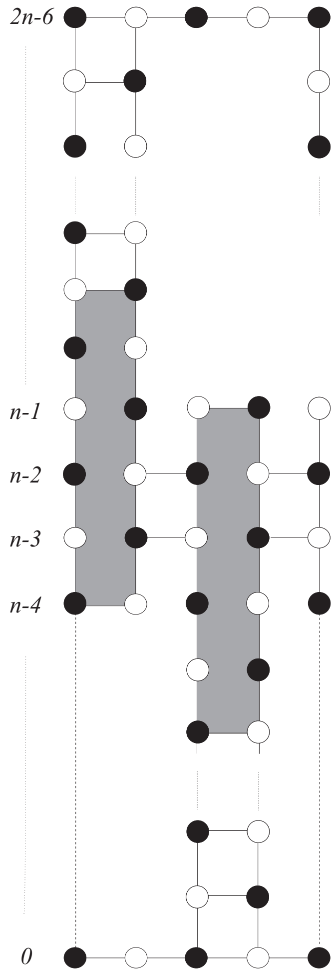


Figure 3.67: 2-trisc with boundary components of type $(2, 2, 3, 3, 2, 2, 3, 3, 2, 2, 3, 3)$.

Finally, to construct the disc $\mathcal{D}_n$, for each even $n \geq 6$, we construct a tessellation D_n of a closed disc D, with boundary type $(2,2,3,3,2,2,3,3,2,2,3,3)$. We achieve this by starting with a dodecahedron, regarded as a bipartite map on D with one face and with 12 vertices and 12 edges on ∂D. Then, we give numbers to those 12 vertices, starting with 1 in a black vertex and following a clockwise direction. Edges are added between vertices 1 and 4, and between vertices 5 and 12. This creates two faces of order 2 and one face of order 8. Inside of this face of order 8 we place a new white vertex adjacent to black vertex number 9 and we built a stalk of length $n-5$ and with all interior vertices of order 3. Hence, that face become of order $8+2+2(n-5) = 2n$ (see figure 3.68 for $n = 8$).

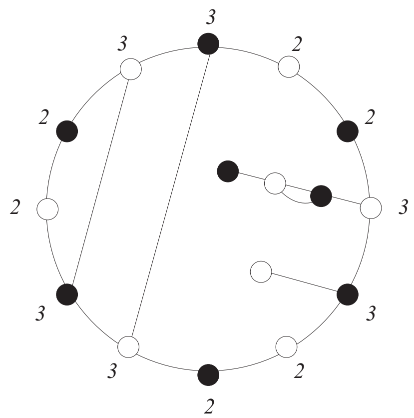


Figure 3.68: Disc map with boundary component of type $(2,2,3,3,2,2,3,3,2,2,3,3)$ for $n = 8$.

3.11.6 Annuli with both boundary components of type $6^{(3)}$

This corresponds to the following case:

- $(3,3,m)$ for odd $m \geq 5$, with both boundary components of type $6^{(3)}$.

To solve this case, we will follow again the idea of building 2-face-colourable maps of type $\{3, 2m\}$. In order to build the new 2-trisc, annulus, disc and crosscap, we will start with the same rectangle and tessellation (before introducing wedges), that we have previously described in section 3.9.5, for the orientable case.

2-Trisc:

To build the new 2-trisc, we need to add the following wedges: $(2m - 10)/4$ wedges between vertices $(2,1),(3,2)$ inside face $((2,1),(3,2),(1,2))$, then between vertices $(3,2),(4,1)$ inside the face $((3,2),(4,1),(5,2))$, $(2m-14)/4$ wedges[3] between vertices $((2,1),(4,1)$ inside the face $((2,1),(4,1),(3,0)$, and one wedge between vertices $(4,1),(5,1)$ inside the face $(4,1),(6,1),(5,0)$. If we remove the same two faces as in the orientable case, we will get then two boundary components, one of type $(6,6,8)$ and another one of type $(2m - 4, 2m-4, 2m-8)$ (see Figure 3.69, an example of a 2-trisc for $m = 7$).

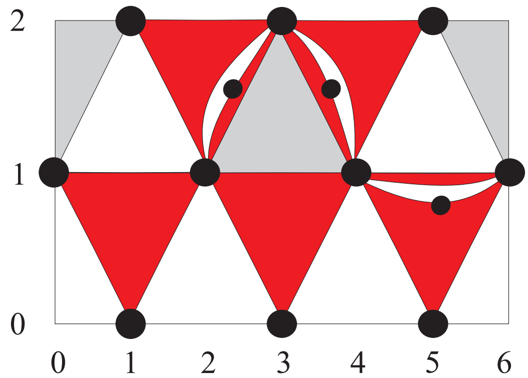


Figure 3.69: 2-Trisc for $m = 7$.

Annulus:

To build the annulus we use the same rectangle $[0,6] \times [0,2] \subset \mathbb{R}^2$ with the vertices in the same places as before. However we do not remove any triangular face and we attach $(2m-8)/4$ wedges to the pair of vertices:

$$(1,2),(3,2) \text{ inside face } ((1,2),(3,2),(2,1));$$

$(2m-10)/2$ wedges to the pair of vertices:

$$(6,1),(5,2) \text{ inside face } ((6,1),(5,2),(1,2));$$

$(2m-6)/2$ wedges to the pair of vertices:

$$(2,1),(4,1) \text{ inside face } ((2,1),(4,1),(3,2));$$

[3] We have to admit that $m \neq 5$ and deal with the case $m = 5$ later.

one wedge to the pair of vertices:

$$(3,0),(5,0) \text{ inside face } ((3,0),(5,0),(4,1)),$$

and two wedges to the pair of vertices:

$$(0,1),(1,0) \text{ inside face } ((0,1),(1,0),(2,1)).$$

If we identify the vertical sides of the rectangle we get a map on an annulus and, by the way we constructed it, that map is 2-face colourable (we use the opposite colour scheme we have used in the 2-trisc and we use the white colour to each wedge inside a red triangular face, and the red colour to each wedge inside a white triangular face). It follows that one of the boundary components has type $6^{(3)}$ and is adjacent to three red faces, and the other one has type $(2m-4)^{(3)}$ and is adjacent to three white faces

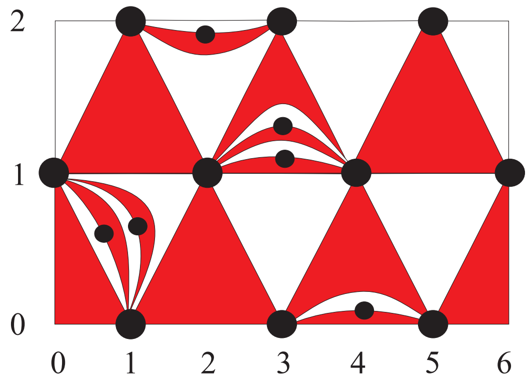


Figure 3.70: Annulus for $m = 7$.

Crosscap on a trisc:

To build the crosscap on a trisc, we need to add the following wedges: $(2m-10)/4$ wedges between vertices $(2,1),(3,2)$ inside face $((2,1),(3,2),(1,2))$, then between vertices $(3,2),(4,1)$ inside the face $((3,2),(4,1),(5,2))$, $(2m-6)/4$ wedges between vertices $((2,1),(4,1)$ inside the face $((2,1),(4,1),(3,0)$, and one wedge between vertices $(4,1),(5,1)$ inside the face $(4,1),(6,1),(5,0)$. Then, we remove the same two faces as in the 2-trisc but introducing three new vertices in the middle of the three boundary edges from the removed face $(2,1),(3,2),(4,1)$. By this process, we get a boundary with six vertices (three old ones and three *new*). If we also add an edge between each pair of old vertices, we will get a piece with two boundary components: one of type $(6, 6, 8)$ and another one of type $(2m, 2, 2m, 2, 2m)$ (see Figure 3.71, an example of a 2-Trisc for $m = 7$). Identifying opposite points (and vertices) of this last boundary component, we get a nonorientable surface of genus 2.

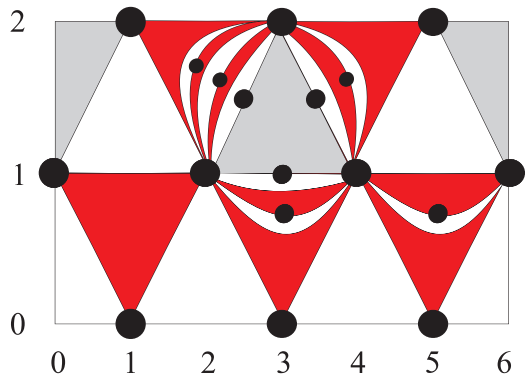


Figure 3.71: Crosscap for $m = 7$.

Disc:

For the Disc, we use the same basic disc that we have described for the orientable case (see Figure 3.54). If we introduce $(2m - 2)/4$ wedges attached to each one of the three possible different pairs of boundary vertices (and in each one of the three boundary faces) we will get again a 2-face-colourable disc, but this time of type $(2m-6)^{(3)}$. Then, we add another wedge between two vertices of the boundary component to get the final disc of type $(2m-6, 2m-4, 2m-4)$, as required.

If $m = 5$, we cannot use the 2-trisc we have previously described. Hence, we need to build new pieces for this type $(3, 3, 5)^4$.

2-Trisc:

To build the new 2-trisc, from the tessellation used in the orientable case (removing the same faces and with the same identifications), we need to add

[4] Again, the low types seem to be the hardest ones to solve or, at least, they need to be dealt separately.

one wedge between vertices $(1,0),(2,1)$ inside face $((1,0),(2,1),(3,0))$, another between vertices $(3,0),(4,1)$ inside the face $((3,0),(4,1),(5,0))$, and finally an wedge between vertices $((5,0),(6,1)$ inside the face $((5,0),(6,1),(1,0)$. We will get then two boundary components of type $(8,8,8)$ (see Figure 3.72).

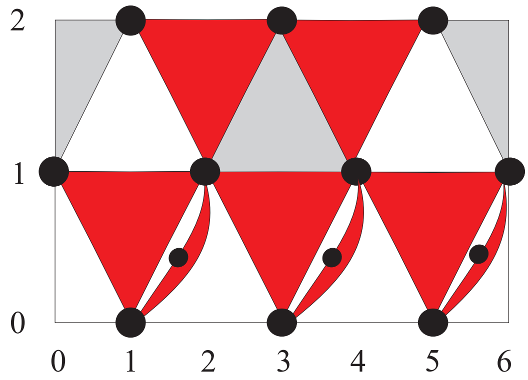


Figure 3.72: 2-Trisc to build nonorientable hypermaps of type $(3,3,5)$.

Annulus:

To build the annulus we use the same rectangle $[0,6]\times[0,2]\subset\mathbb{R}^2$ with the vertices in the same places as before. However we do not remove any triangular face and we attach one wedges to each one of these pair of vertices:

$$(0,1),(2,1) \text{ inside face } ((0,1),(2,1),(1,2)),$$

$$(2,1),(4,1) \text{ inside face } ((2,1),(4,1),(3,2));$$

$$(4,1),(6,1) \text{ inside face } ((4,1),(6,1),(5,2)).$$

If we identify the vertical sides of the rectangle we get a map on an annulus and, by the way we constructed it, that map is 2-face colourable (we use the opposite colour scheme we have used in the 2-Trisc and we use the white colour to each wedge inside a red triangular face, and the red colour to each wedge inside a white triangular face. It follows that both boundary components have type $4^{(3)}$ and is adjacent to three red faces and the interior vertices have valency 10).

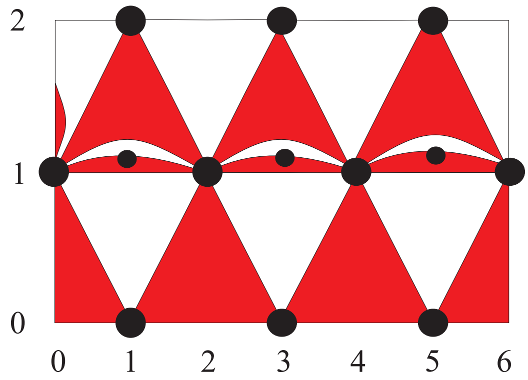


Figure 3.73: Annulus to build nonorientable hypermaps of type $(3, 3, 5)$.

Crosscap on a trisc:

To build the crosscap, we need to add the following wedges: 2 wedges between vertices $(2,1),(3,2)$ inside face $((2,1),(3,2),(1,2))$, then between vertices $(3,2),(4,1)$ inside the face $((3,2),(4,1),(5,2))$, $(2m-6)/4$ wedges between vertices $((2,1),(4,1)$ inside the face $((2,1),(4,1),(3,0)$, and one wedge between vertices $(4,1),(5,1)$ inside the face $(4,1),(6,1),(5,0)$. We will get then two boundary components, one of type $(6,6,8)$ and another one of type $(2m-4, 2m-4, 2m-8)$ (see Figure 3.74, an example of a 2-trisc for $m = 7$).

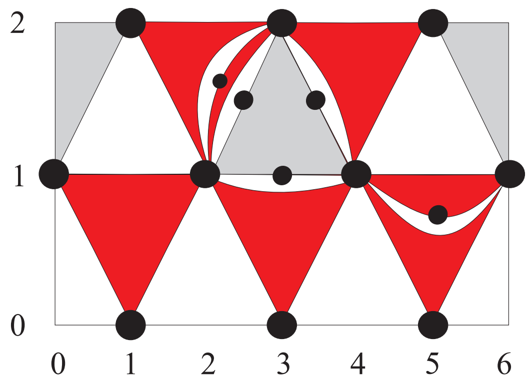


Figure 3.74: Crosscap to build nonorientable hypermaps of type $(3, 3, 5)$.

Disc:

For the disc, we use the same basic disc that we have described for the orientable case (see Figure 3.54). If we introduce $(2m-2)/4$ wedges attached to each one of the three possible different pairs of boundary vertices (and in each one of the three boundary faces) we will get again a 2-face-colourable disc, but this type of type $(2m-6)^{(3)}$. Then, we add another wedge between two vertices of the boundary component to get the final disc of type $(2m-6, 2m-4, 2m-4)$, as required.

For this last case, we have built our nonorientable piece from the 2-trisc, not from the annulus. This is not a problem for the general construction, if we glue the pieces correctly. However, that does not allow us to construct nonorientable hypermaps of genus 1. Hence, we need to build new pieces to complete the construction for this case.

Annulus:

We take the same rectangle and tessellation we have previously build for the annulus in the orientable case but we do not identify the vertical sides. First we make a copy of it (inverting the colours of the faces) and then we glue the points $(6, y)$ of the first rectangle to the points $(6, 2-y)$ of the copy. By this method, we get a rectangle $[0, 12] \times [0, 2]$ with vertices in with vertices at (i, j) (with $i \in \{1, 3, 5, 7, 9, 11\}$ and $j \in \{0, 2\}$) and $(i, 1)$ with $i \in \{0, 2, 4, 6, 8, 10, 12\}$). There are edges between all consecutive vertices with the same horizontal coordinates and also:

$$(i, 1) \times (i-1, j) \text{ if } i \in \{2, 4, 6, 8, 10, 12\} \text{ and } j \in \{0, 2\}$$

$$(i, 1) \times (i+1, j) \text{ if } i \in \{0, 2, 4, 6, 8, 10\} \text{ and } j \in \{0, 2\}$$

.

In this new map we can find $(2m-8)/2$ wedges attached to each one of the following two pairs of vertices:

$$(1, 2), (3, 2) \text{ inside face } ((1, 2), (3, 2), (2, 1)),$$

$$(6, 1), (5, 2) \text{ inside face } ((6, 1), (5, 2), (1, 2));$$

$$(7, 0), (9, 0) \text{ inside face } ((7, 0), (9, 0), (8, 1));$$

$$(11, 0), (12, 1) \text{ inside face } ((11, 0), (12, 1), (10, 1));$$

$(2m-6)/2$ wedges to the following pair of vertices:

$$(2,1),(4,1) \text{ inside face } ((2,1),(4,1),(3,2));$$

$$(8,1),(10,1) \text{ inside face } ((8,1),(10,1),(9,0));$$

and one wedge to each of the two following pairs:

$$(3,0),(5,0) \text{ inside face } ((3,0),(5,0),(4,1)),$$

$$(0,1),(1,0) \text{ inside face } ((0,1),(1,0),(2,1));$$

$$(6,1),(7,2) \text{ inside face } ((6,1),(7,2),(8,1));$$

$$(9,2),(11,2) \text{ inside face } ((9,2),(11,2),(10,1));$$

If we now identify the points $(0, y$ to the points $(12, y)$, of this bigger rectangle, we get an annulus with boundary components of type $(6,6,6,2n-4,2n-4,2n-4)$, one red and another white (see Figure 3.75, an example for $m=5$).

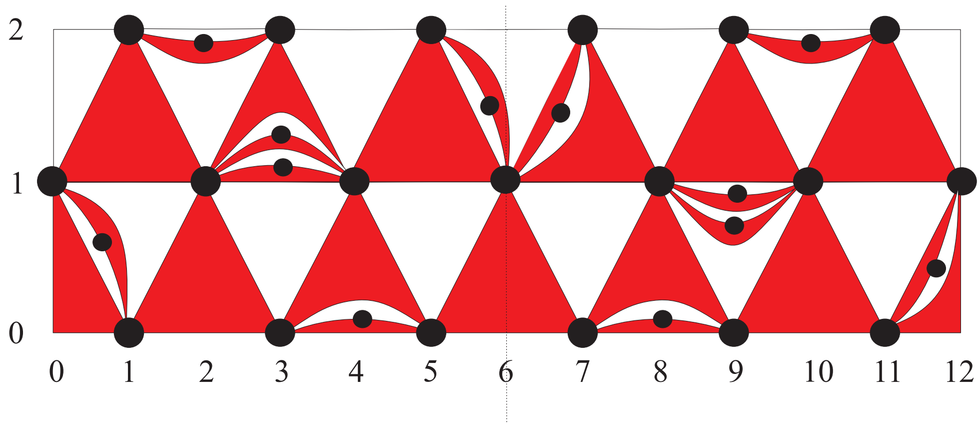


Figure 3.75: Annulus to construct nonorientable hypermaps of genus 1 for $m=5$.

Crosscap: If we the identify opposite vertices in one of the boundary components of the annulus (those vertices will become vertices of valency $2m = (2m-4)+6-2)$, we get a crosscap.

Disc: To construct the disc, we take $D = \{z \in \mathbb{C} : |z| \leq \sqrt{2}\}$, with vertices at ± 1, $\pm 1 \pm i$, $\pm\sqrt{2}i$, with edges along the boundary components and joining the vertex at $-1+i$ to $\sqrt{2}i$, $1+i$, $-1-i$, and $-\sqrt{2}i$; the vertex at $1+i$ to $\sqrt{2}i$, $-\sqrt{2}i$ and $1-i$, and the vertex at $-\sqrt{2}i$ to $-1-i$ and $1-i$. Then, we add six more vertices in the middle of edges $(-1+i,-1-i),(-1+i,\sqrt{2}i),(\sqrt{2}i,1+i),(-\sqrt{2}i,1-i),(-\sqrt{2}i,-1-i),(-1-i,-1+i)$ and six more edges between the same pair of vertices, in such a way that the map on the disc would have only triangular faces (see Figure 3.76). To achieve a suitable valency for the vertices on the boundary, we need to add $(2m-10)/4$ wedges between each pair of these vertices: $-1+i$, $\sqrt{2}i$ and $1+i$. At end of this step we will only have triangular faces, 6 interior vertices of order 2 and a boundary component of type $(2m-4, 2m-4, 2m-4, 6, 6, 6)$, which can be glued to the boundary component of the annulus with the same type (identifying the vertices of valency 6 with vertices of valency $2m-4$.

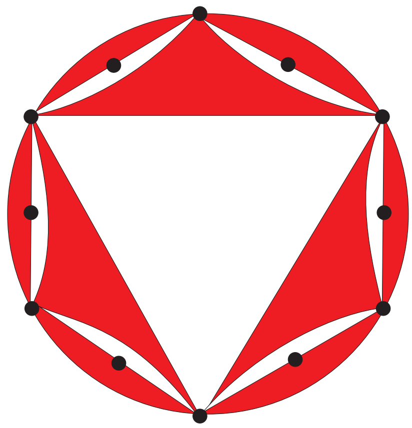


Figure 3.76: Disc to construct nonorientable hypermaps of genus 1 (before introducing wedges).

3.11.7 Annnuli with boundary components of type $(4, m-1, 2, 3)$

This corresponds to the case:

- $(4, 4, n)$ with odd $n \geq 5$.

No adaptations are needed. We can identify opposite sides (identifying opposite vertices of the same colour) of one of the boundary components of the annulus without making any changes, since we get two vertices, each of order 4.

3.11.8 Theorem

This way, adapting the methods used for orientable surfaces, we have proven the following result for nonorientable surfaces:

Theorem 3.11.1. *Given positive integers m, n with*

$$2m^{-1} + n^{-1} < 1$$

and an integer $g \geq 0$, there are infinitely many nonisomorphic compact ***nonorientable*** *hypermaps of type (m, m, n) and genus g.*

3.12 Noncompact Surfaces

Theorem 3.3.1 does not make sense for noncompact surfaces because it is not clear how one might define the genus of a noncompact surface. However, we may want to build an infinite number of nonisomorphic noncompact hypermaps of a given type and such that any compact hypermap that can be imbedded in the surface has maximal genus g. The adaptation of the methods previously explained must be carefully done because if we use an infinite number of pieces in our constructions, we might build an infinite number of isomorphic maps. To solve this problem, in some situations, we can use two different kinds of annuli and somehow control the number of faces of a certain valency.

Example 3.12.1. In the case (m, m, n) for $m \geq 4$ and n even and ≥ 4, we can use a different annulus (see Figure 3.77)

This has faces of order 4 and 2. For a fixed genus (a fixed number of 2-triscs) we can now use an infinite number of copies of this annulus and only a finite number of copies of the old annulus that has faces of valency $2n$. Hence by increasing the number of old annuli we also increase the number of faces of valency $2n$, constructing (for the same maximal genus of an embedded compact surface) an infinite number of nonisomorphic hypermaps. $\square$

If we completely ignore the details related to the *genus* of a surface, Theorem 3.3.1 has an obvious extension that can be stated as follows: for each hyperbolic

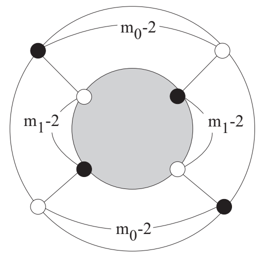


Figure 3.77: New annulus.

triple, there exist infinitely many nonisomorphic noncompact orientably regular hypermaps of that type. This is easily proved by taking different finite numbers of 2-triscs and infinitely many annuli, joined in line.

3.13 The Conjecture

We have not completely proved Conjecture 3.1.2 (B) because we have only dealt with the cases where at least two of the parameter are equal. To tackle the general problem other techniques must be developed, maybe using some algebraic tools instead of just the topological approach that was enough to solve the cases we have already presented. However it is not hard to give examples of infinitely many nonisomorphic hypermaps of a given genus and of type (l, m, n), in some specific cases where l, m and n (the parameters of the type) are all different. We will quickly show an example of a technique that can produce some of those hypermaps.

Example 3.13.1. One possible way to construct hypermaps of type (l, m, n) (when all the parameters are different) is, in some cases, to use the pieces we have built, when two of the parameters are equal, and to modify them. For instance, if we want to build infinitely many hypermaps of type $(7, 6, 8)$ we can start with hypermaps of type $(6, 6, 7)$ and make the required adaptations, increasing the valency of the white vertices together with the valency of the faces (in this case, both have their valency increased by 1). The most difficult problem is to increase the valency of the white vertices of each stalk and have also the right valency for the adjacent faces. There are basically two cases to consider: a) the stalk starts with a white vertex; b) the stalk starts with a black vertex. In Figures 3.78 and 3.79 we show, without going into details, what to do in each case, using the red colour to highlight the changes. The modifications are slightly different in each case but they both lead to the creation of two new faces of valency 16, inside the original face of valency 16. The 2-trisc and the annulus, for the $(6, 6, 7)$ case, can then be adapted to the new type by using those modified stalks together with two more black vertices inside each one of the faces of valency $2n + 2 = 14$. The new 2-trisc and annulus are shown, respectively, in Figure 3.80 and Figure 3.81. The bold red edges represent an altered stalk (the red is also the color used for the new vertices). Hence, all the faces have valency $14 + 2 = 16$ and the boundary components have types (m_0, m_0, m_0, m_0) and $(m_1 + 1, m_1, m_1 + 1, m_1)$, with both first vertices of these sequences being white vertices (and, of course, $m_0 + m_1 - 2 = 6)$[5]. The discs must also be adapted: we introduce in each one a new black vertex inside the face of valency $2n + 2$ (which becomes by this process a face of valency $2n + 4$) adjacent to the proper white vertex such that the boundary components are also of types $(m_1 + 1, m_1, m_1, m_1)$ and (m_0, m_0, m_0, m_0). Then, when we carefully glue any pair of boundary components of these types we will get four vertices of valency $m_0 + m_1 + 1 = m + 1$.

[5]We have chosen the general notation $m_0 + m_1 - 2 = 6$ instead of, for instance, the more specific $m_0 = 4$ and $m_1 = 4$, that would be sufficient for this example, to stress that this technique might also be useful in other cases

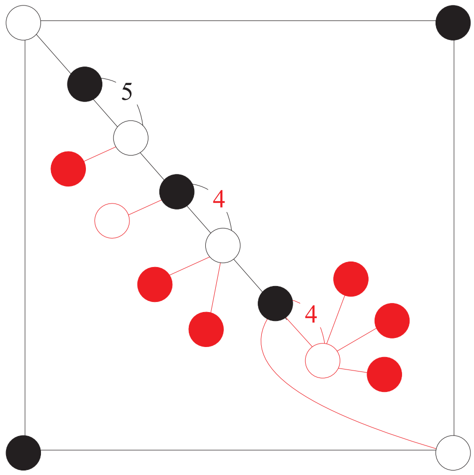


Figure 3.78: case a).

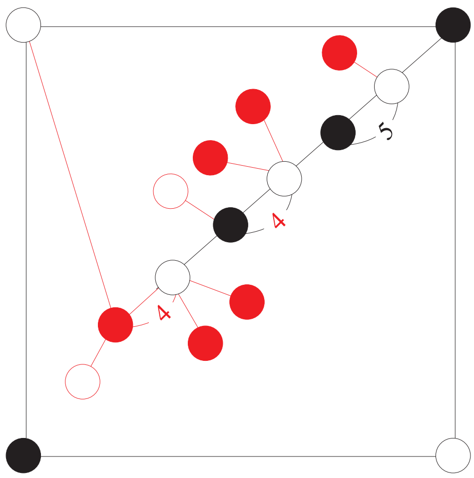


Figure 3.79: case b).

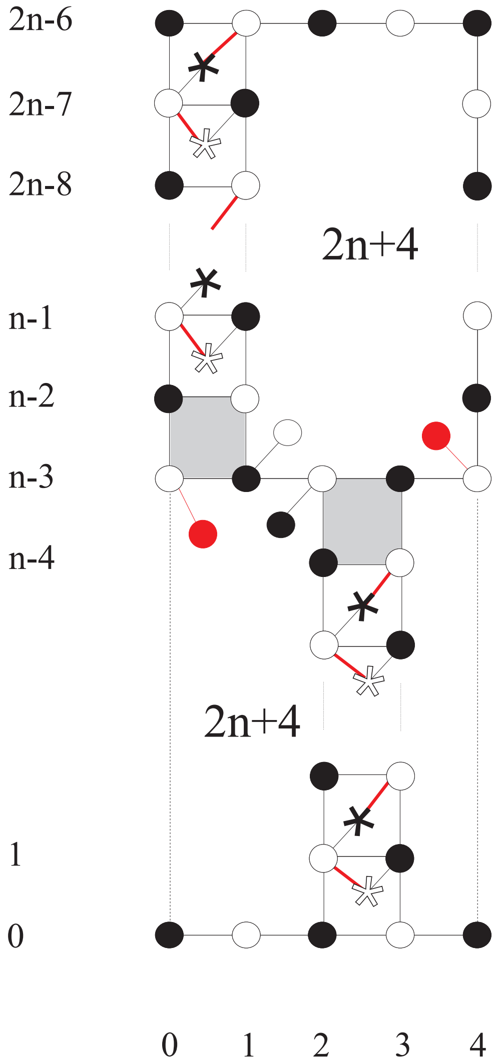


Figure 3.80: 2-Trisc.

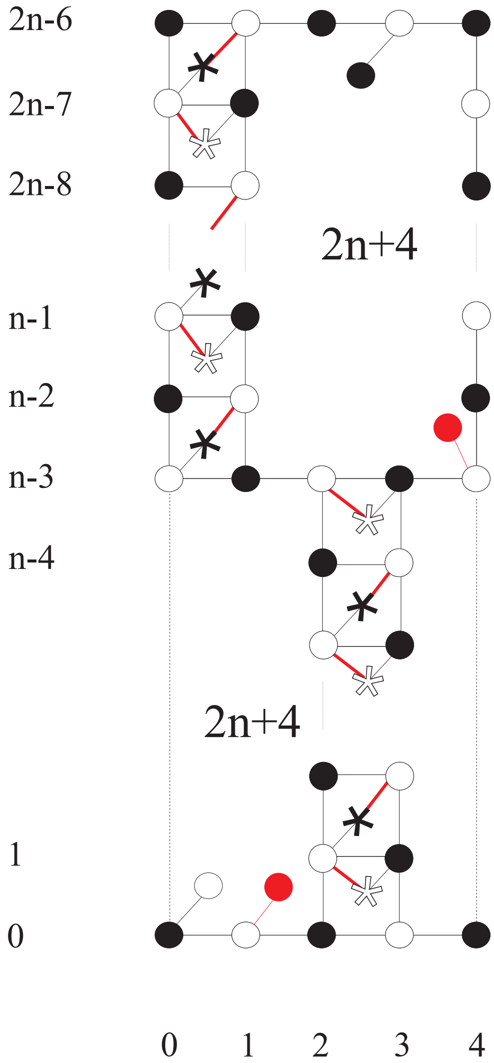


Figure 3.81: Annulus.

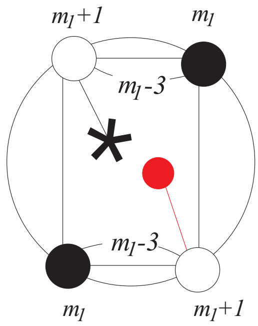


Figure 3.82: Disc 1.

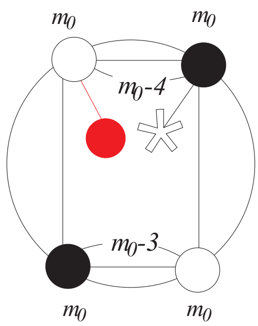


Figure 3.83: Disc 2.

Chapter 4

Duality and other operations on maps and hypermaps

4.1 Introduction

When Steven Wilson published, in 1979, the paper *Operators over Regular Maps* [61], his goal was to study operations that could transform a regular map into another regular map. That would allow, among other things, the construction of new regular maps from old ones. His approach was topological and introduced the following operators (that will be described below): Duality, Petrie, Opposite and Direct Derivatives that generate a group isomorphic to S_3. Jones and Thornton [36], by giving these operations a more algebraic structure, were able to relate them with the outer automorphisms of a certain group. A few years later, Lynn James extended that work to hypermaps [26].

4.2 The operations on Maps

4.2.1 The Duality operator

The general notion of duality is, as Wilson mentioned in [61], an old one that could already be found in the Greek works on polyhedra. If $\mathcal{M}$ is a map on a surface, the duality operator D will interchange vertices and face-centers of M, creating a new map $D(\mathcal{M})$ on the same surface, whose vertices are adjacent if and only if the faces from which the new vertices are formed were adjacent in $\mathcal{M}$.

The duality operator interchanges vertices and face-centers but preserves what are known as *Petrie paths*, a name that comes from the mathematician J.F. Petrie, the first one to use the concept:

Definition 4.2.1. A *Petrie path* is a cyclic sequence of edges, each consecutive sharing a vertex so that, at each vertex, a face is enclosed on the right and on the left alternately.

A Petrie path is, roughly speaking, no more than a *zig-zag path* in the map:

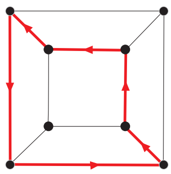


Figure 4.1: One of the Petrie paths of the cube.

4.2.2 The Petrie operator

If we start with a map $\mathcal{M}$, each face of $P(\mathcal{M})$ will be a cycle of edges which forms a Petrie Path in $\mathcal{M}$. This means that: a) $P(\mathcal{M})$ might be a map on a different surface than $\mathcal{M}$.[1]b) the edges of a face in $\mathcal{M}$ are the Petrie paths in $P(\mathcal{M})$.

It is also easy to see that $I = P^2 = D^2 = (PD)^3$ and that P and D generate a group isomorphic to S_3 [61].

4.2.3 The Opposite Operator

This operator is defined using the previous two:

$$opp(\mathcal{M}) = PDP(\mathcal{M}) = DPD(\mathcal{M})$$

[1]In the category of oriented hypermaps, the Petrie operator does not exist since a map on an oriented surface might be sent to a map on a non orientable surface.

and it is obviously an involution because $(PDP)^2 = (DPD)^2 = I$. This algebraic definition has also an interesting topological interpretation. If we label each edge of $\mathcal{M}$ with an arrow running along it on both sides and we cut the map along those edges and then glue it back together again so that all the numbers match but none of the arrows do, we will get *opp*($\mathcal{M}$)

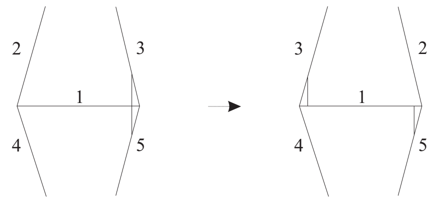


Figure 4.2: Opposite Operator.

This operation sends faces to faces but with all the joining reversed. It also transposes vertices and Petrie paths.

4.2.4 Direct Derivatives

We define the direct derivatives of a map $\mathcal{M}$ to be the maps derivable from it under D or P and their products.

Therefore, no map has more than six different derivatives (preserving certain important features such as the automorphism group and the number of flags).

4.3 An algebraic approach to Operations on Maps

As we have seen before, maps have a topological definition but also an algebraic one. It is somehow a natural path to study these map operations using some of the tools available in Group Theory. That was what G.A. Jones and J.S. Thornton have done in their paper *Operations on Maps and Outer Automorphisms* [36], published in 1982.

Instead of working with a topological map they have used the transitive permutation representation[2] $\pi : \Gamma \to S^F$ of $\Gamma = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = (r_2 r_0)^2 = 1 \rangle$ to reconstruct the map, defining an operation in the following way:

[2]where F is the set of flags.

Definition 4.3.1. An *operation* on maps is any transformation of maps induced by a composition-preserving permutation of Γ, that is, by a group automorphism of Γ.

With this approach it was possible to confirm that there are exactly six operations on maps but also that these are induced by the outer automorphisms of Γ, forming a group isomorphic to S_3:

$$Out(\Gamma) = Aut(\Gamma)/Inn(\Gamma) \cong S_3$$

4.4 Operations on Hypermaps and Outer Automorphisms

If we now take the group $\Delta = \langle r_0, r_1, r_2 | r_0^2 = r_1^2 = r_2^2 = 1 \rangle$ instead of Γ, we can use the transitive permutation representation[3] $\pi : \Delta \to S^F$ of Δ to reconstruct any hypermap. In 1988, Lynne D. James [26] used this notion to describe the group of all operations on hypermaps, giving an important generalization of the previous result for maps, obtained by G.A. Jones and J.S. Thornton [36].

Again, these operations can be understood as elements of a group of outer automorphisms (of Δ, in this case). Since

$$Out(\Delta) = Aut(\Delta)/Inn(\Delta) \cong PGL(2, \mathbb{Z}) \quad [26]$$

we have now an infinite number of operations on hypermaps and not just half a dozen of them like before. Therefore, if instead of working in the restricted class of maps, we extend our study to hypermaps, we will have many more different ways to transform hypermaps preserving, for instance, the automorphism group and the number of (hyper)flags. Moreover, knowing that the group of operations on hypermaps is isomorphic to $PGL(2, \mathbb{Z})$, we may not only assume that its number is infinite but also have complete information about its structure and properties (because $PGL(2, \mathbb{Z})$ is finitely generated, one can restrict attention to finitely many operations).

Similar results [26] were obtained for the group of all operations on oriented hypermaps (induced by outer automorphisms of Δ^+):

$$Out(\Delta^+) = Aut(\Delta^+)/Inn(\Delta^+) \cong GL(2, \mathbb{Z})$$

[3] Where F is the set of (hyper)flags.

(Δ^+ being the free group generated by $x_1 = r_1r_2$ and $x_2 = r_2r_0$).

4.5 Classification of all operations of order 2 (and other finite orders) on hypermaps

Having in mind the results quoted in the previous section, we might ask: what different operations of finite order can we have?

The main purpose of the next two chapters is to study the duality operation on hypermaps that interchanges hypervertices and hyperfaces. This operation, like chirality, has order 2. As will be shown in the following sections, having order 2, instead of higher order, is something that helps the study of several of its properties, namely the way we can deal with some special groups that measure the effects of those operations on a specific hypermap. But are there more operations of order 2 on oriented regular hypermaps besides duality and chirality? To answer this question we need to know the elements of order 2 in $GL(2,\mathbb{Z})$ up to conjugacy. That classification can be found in a paper by Stephen Meskin [43]. The relevance of that work for our study is even greater because Meskin not only lists the elements of order 2 but also all the elements (up to conjugacy) of finite order and one can see the elements of $GL(2,\mathbb{Z})$ as operations on hypermaps [see previous section].

Let

$$p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Up to conjugacy, we have the following non-identity elements of finite order in $GL(2,\mathbb{Z})$ [43]:

- three elements of order 2: p, pt, t^2
- one element of order 3: y^2
- one element of order 4: t
- one element of order 6: y

Let $\Delta^+ = \langle x_1, x_2 | - \rangle$, with $x_1 = r_1 r_2$ and $x_2 = r_0 r_2$ be the monodromy group of the universal oriented hypermap. Lynne D. James [26] defined the following homomorphism[4]:

$$Aut(\Delta^+) \to GL(2, \mathbb{Z})$$

$$\phi \mapsto M$$

where $M = (\phi_{ij})^{-1}$, $i, j \in \{1, 2\}$.

To obtain the four entries of the matrix M, we do $\phi_{ij} = x_j^{\phi\rho\varepsilon_i}$, where $\rho : \Delta^+ \to \mathbb{Z} \times \mathbb{Z}$ is the abelianising homomorphism defined by $x_j^{\rho\varepsilon_i} = \delta_{ij}$ with $\varepsilon_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ the epimorphism that takes the i-th coordinate. We should emphasize that maps here are composed from left to right, so that $\phi\rho\varepsilon_i$ means do ϕ first, then ρ and finally ε_i.

Then, if we consider, for instance, the following automorphism (that interchanges the generators):

$$\phi : \Delta^+ \to \Delta^+$$

$$x_1 \mapsto x_2$$

$$x_2 \mapsto x_1$$

we will have:

$$\phi_{11} = x_1^{\phi\rho\varepsilon_1} = x_2^{\rho\varepsilon_1} = 0$$

$$\phi_{12} = x_2^{\phi\rho\varepsilon_1} = x_1^{\rho\varepsilon_1} = 1$$

$$\phi_{21} = x_1^{\phi\rho\varepsilon_2} = x_2^{\rho\varepsilon_2} = 1$$

$$\phi_{22} = x_2^{\phi\rho\varepsilon_2} = x_1^{\rho\varepsilon_2} = 0$$

It follows that $(\phi_{ij}) = (\phi_{ij})^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Therefore, the element p of order 2 corresponds to the operation that interchanges the hypervertices and hyperfaces (we will call this operation *duality*). Because the extension $Aut(\Delta^+)$ of $Inn(\Delta^+)$ is not split, there is no single subgroup of $Aut(\Delta^+)$ which provides representatives of the required conjugacy

[4]this homomorphism has kernel $Inn(\Delta^+)$ and induces an isomorphism $Out(\Delta^+) \to GL(2, \mathbb{Z})$, as mentioned in the previous section.

classes of outer automorphisms. In this case, instead of p, we could have taken another matrix $\overline{p} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = p^t$ of order 2 in the same conjugacy class as p, so that the operation associated with that matrix would send x_1 to x_2^{-1} and x_2 to x_1^{-1}. We will call this operation *chiral duality* and, in section 5.3, study its connections with the other duality that interchanges the two generators without inverting them.

It is not hard to verify that the matrix $pt = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ corresponds to the automorphism that sends x_1 to x_1^{-1} and fixes x_2 (partial chirality) and that $t^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is associated with the automorphism that reverses both generators (chirality).

If we now consider the element y^2 of order 3, we have:

$$y^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

And then:

$$(y^2)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

To this matrix we can associate the automorphism that sends x_1 to x_2^{-1} and x_2 to $x_1 x_2^{-1}$ (triality). Since

$$t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we have

$$t^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};$$

and this matrix is then associated with the automorphism of order 4 that sends x_1 to x_2^{-1} and x_2 to x_1 (quadrality).

There is only one remaining operation of finite order on oriented hypermaps, corresponding to

$$y = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Because the free group of rank 2 has no automorphisms of order 6 [43] it is not possible to find an automorphism of that same order 6. However, we can find an automorphism γ of Δ^+ of infinite order such that $\gamma^6 \in Inn\Delta^+$. Let $x_3 = r_0 r_1$. Then, we take:

$$\gamma : x_3 \mapsto x_1^{-1},\ x_1 \mapsto x_2^{-1} = x_3 x_1$$

sending hyperfaces to hypervertices and hypervertices to hyperedges, reversing the orientation in both cases. It follows that $\gamma^6 = \iota_{x_2}^{-2}$, where ι_{x_2} is the inner automorphism induced by x_2.

If instead of working with oriented regular hypermaps, we want to study periodic operations on all hypermaps we must use the fact that $Out(\Delta) \cong PGL(2,\mathbb{Z})$ acting as a group of operations on all hypermaps, regardless of their orientation if it exists. These operations are induced by automorphism of Δ which are extensions of the automorphisms of Δ^+ that have been mentioned before (see [33], for details).

4.6 The Duality Group

The aim of this section is to study what we will call the *duality group* of a hypermap. Some work has been done on chirality groups [7] and there is no reason not to extend that notion to duality or other hypermap operations. These operations, as we have mentioned before, come from outer automorphisms of Δ and by choosing the right group Δ^*, containing Δ, we can look at duality as a result of sending a hypermap subgroup to its conjugate in Δ^*. To build this group we should add an element d, of order 2, transposing r_0 and r_2 and fixing r_1. Hence, we can define Δ^* in the following way:

$$\Delta^* = \Delta \rtimes C_2 = \langle r_0, r_1, r_2, d : r_i^2 = d^2 = 1, r_0^d = r_2, r_1^d = r_1 \rangle$$

This also means that Δ is a normal subgroup of index 2 of Δ^*.

Therefore, each conjugacy class of subgroups $H \leq \Delta$ is either a Δ^*-conjugacy class (if the hypermap $\mathcal{H}$ is self-dual) or paired with another Δ-conjugacy class, containing H^d (if the hypermap $\mathcal{H}$ is not self-dual, which is the same as saying that $\mathcal{H} \ncong \mathcal{H}^d$). This last observation is a general one and it is true for every kind of hypermap. However, we will only deal with regular hypermaps and these have normal subgroups as hypermap subgroups, which means that H is conjugate only to itself in Δ.

So, if a hypermap is self-dual, the group H is invariant under that specific outer automorphism of Δ (conjugation in Δ^*).

Theorem 4.6.1. *Let N be a normal subgroup of Δ and let $G = \Delta/N$. Then the following are equivalent:*

i) $N^d = N$

ii) *N is normal in Δ^** $\square$

Because

$$\Delta^* = \langle r_0, r_1, r_2, d : r_i^2 = d^2 = 1, r_0^d = r_2, r_1^d = r_1 \rangle =$$

$$= \langle r_0, r_1, d : r_0^2 = r_1^d = d^2 = 1,\ r_1^d = r_1 \rangle = \langle r_1, d \rangle * \langle r_0 \rangle \cong V_4 * C_2 \cong \Gamma$$

we can build a functor from hypermaps ($H \leq \Delta$) to maps ($H \leq \Delta^* \cong \Gamma$) and depending on the chosen isomorphism between Δ^* and Γ this is the Walsh functor [53], representing a hypermap as a bipartite map or one of its duals.

Some of the key notions for chirality groups [7] can be adapted for duality. First we will do it to regular hypermaps in general and then to regular oriented hypermaps. Although our research work on duality is mainly devoted to orientable hypermaps, some results for nonorientable hypermaps will also be presented, later in Chapter 6.

If $\mathcal{H}$ is a regular hypermap with hypermap subgroup H then H is normal in Δ. The largest normal subgroup of Δ^* contained in H is the group $H_\Delta = H \cap H^d$ and the smallest normal subgroup of Δ^* containing H is the group $H^\Delta = HH^d$. These correspond, respectively, to the smallest self-dual hypermap that covers $\mathcal{H}$, and the largest self-dual hypermap that is covered by $\mathcal{H}$.

We can now adapt, from [7], the following proposition:

Proposition 4.6.1. *The groups H^Δ/H, H/H_Δ, H^Δ/H^d and H^d/H_Δ are all isomorphic to each other.*

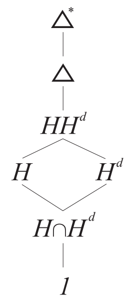


Figure 4.3: H_Δ and H^Δ.

Proof: By the third isomorphism theorem we have

$$H^\Delta/H = HH^d/H \cong H^d//H \cap H^d) = H^d/H_\Delta$$

and also $H^\Delta/H \cong H/H_\Delta$. The other isomorphisms are induced by conjugation by the generator d. Hence $H^\Delta/H \cong H^\Delta/H^d$ and $H/H_\Delta \cong H^d/H_\Delta$. $\square$

This group will be called the *duality group* $D(\mathcal{H})$ of $\mathcal{H}$ and its order the *duality index* d of $\mathcal{H}$. The index is somehow a way to measure how far the hypermap is from being self-dual. If the duality index is 1 then the hypermap is self-dual and the bigger that index, the more distant the hypermap is from being self-dual.

Proposition 4.6.2. *The duality group $D(\mathcal{H})$ of a regular hypermap $\mathcal{H}$ is isomorphic to a normal subgroup of the monodromy group $Mon(\mathcal{H})$.*

Proof:

$$H^\Delta \trianglelefteq \Delta \Rightarrow H^\Delta/H \trianglelefteq \Delta/H$$

Hence

$$D(\mathcal{H}) \trianglelefteq Mon(\mathcal{H}) \quad \square$$

Then, another possible way to understand the duality group is to look at it as the minimal subgroup $D(\mathcal{H}) \trianglelefteq Mon(\mathcal{H})$ such that $\mathcal{H}/D(\mathcal{H})$ is a self-dual hypermap.

If $D(\mathcal{H}) = Mon(\mathcal{H})$ or, equivalently, $H^\Delta = \Delta$ we say that the hypermap is *totally dual*.

First, we will focus our study on orientably regular hypermaps. We have extended Δ to Δ^* by adjoining d such that

$$d^2 = 1, \quad r_0^d = r_2, \quad r_2^d = r_0, \quad r_1^d = r_1.$$

Then,

$$x^d = r_2 r_1 = y^{-1}, \quad y^d = r_1 r_0 = x^{-1}$$

We will denote this kind of duality on orientably regular hypermaps by $\beta-$ *duality* (chiral-duality).

On the other hand, conjugation by $r_1 d$ induces

$$x \mapsto y, \quad y \mapsto x,$$

interchanging generators. This will be called $\alpha-$*duality* (orientation-preserving duality). The relationship between the two (α and β) will be dealt with later. From now on, to simplify the writing, whenever we refer to *duality* we mean $\alpha-$ *duality*, the one that preserves orientation.

From an orientable hypermap we can obtain one or two different oriented hypermaps if the hypermap is regular or chiral, respectively. Hence let $\mathcal{H} = (G, r_0, r_1, r_2)$ and $\mathcal{H}^+ = (G^+, x, y)$, one of the oriented hypermaps associated with $\mathcal{H}$. The duality group $D(\mathcal{H}^+)$ is the minimal normal subgroup of G^+ such that $\mathcal{H}^+/D(\mathcal{H}^+)$ is a self-dual hypermap. It follows that $D(\mathcal{H}^+) \trianglelefteq G^+$ and we say that $\mathcal{H}$ is totally dual if $D(\mathcal{H}^+) = G^+$. (Hence, a hypermap is totally dual if its duality group is equal to its monodromy group).

Classifying all finite simple groups was a task to which several mathematicians, from the 50's until the beginning of the 80's, have devoted their time. Because the proof is long and complex, a program to simplify it and filling its gaps was developed in the next decades, mainly by Gorenstein, Lyons and Solomon (GLS). Among many other important results, that classification has allowed us to conclude that all finite simple groups require at most two generators[5][3]. Therefore, every finite simple group can be taken as a monodromy group of a self-dual or a totally dual o.r. hypermap. If the generators of the

[5] In fact, two random elements of a finite simple group G generate G with probability approaching 1 as $|G|$ goes to infinity [37].

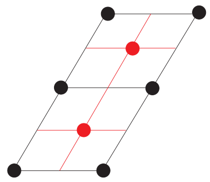


Figure 4.4: Hypermap on the torus (in black) and its dual (in red).

group have distinct orders than the hypermap must be totally dual, otherwise further investigation is needed. With few exceptions, every finite simple group is $(2,3)$-generated (generated by a pair of elements with one member of order 2 and another of order 3). Hence, by choosing the right generators of a finite simple group, we can always - if we avoid those exceptions - build a totally dual hypermap with that group as monodromy group.

However, not all the direct products of finite simple groups can be generated by two elements. For instance, Wiegold [55] proved that A_5^{19} can be generated by two elements but A_5^{20} needs always more then two (meaning that A_5^{20} cannot be a monodromy group of some orientably regular hypermap).

Lemma 4.6.1. *If an oriented hypermap $\mathcal{H}^+ = (G^+, x, y)$ has type (l, m, n), with l and n coprime, then it is totally dual.*

Proof: If $D(\mathcal{H}^+)$ is the duality group of $\mathcal{H}^+$, then it is also the smallest normal subgroup of G^+ such that the assignment $x \mapsto y$, $y \mapsto x$ induces an automorphism of $G^+/D(\mathcal{H}^+)$. We obtain this by adding extra relations in the presentation of G^+, substituting x for y and y for x. Then ,we will have $x^l = x^n = 1$ and $y^n = y^l = 1$, meaning that $x = 1$ and $y = 1$. That is, the group $G^+/D(\mathcal{H}^+)$ collapses to identity. Hence, $D(\mathcal{H}^+) = G^+$ and the hypermap $\mathcal{H}^+$ is totally dual. $\square$

Example 4.6.1. The torus map of type $\{4,4\}_{1,2}$ in Figure 4.4 is a self-dual oriented regular map (and also a chiral map).

Example 4.6.2. An example of a proper self-dual oriented hypermap (which is not a map) is $(\mathbb{Z}_3, 1, 1)$ of type $(3,3,3)$ (a hypermap on the sphere, in this case).

Example 4.6.3. The cube map on the sphere is a totally-dual hypermap since it has type $(3, 2, 4)$ and $gcd(3, 4) = 1$.

Definition 4.6.1. A hypermap has *intermediate duality index* if it is not self-dual or totally dual.

A few examples of hypermaps with intermediate duality index will be given in section 5.1.1.

Chapter 5

Duality on Oriented Regular Hypermaps

In this section, every hypermap will be oriented and regular unless it is mentioned the contrary. Therefore, we will deal with hypermap subgroups normal in Δ^+.

5.0.1 Duality index

Theorem 5.0.2. *For every $k \in \mathbb{N}$, there is a self-dual hypermap with order k.*

Proof: Let G be the cyclic group of order k generated by g. If we take $G = \langle g \rangle$ and $\mathcal{H} = (G, g, g)$, the hypermap with monodromy group G, then there is an automorphism of $\mathcal{H}$ that interchanges the two generators (they are both equal to g, in this case). Hence, the hypermap is self-dual. $\square$

Remark: We can get the same result using hypermaps $\mathcal{H} = (G, x, y)$, with G being the finite abelian group $C_{ab} \times C_a \cong \langle x, y | x^{ab} = y^{ab} = (xy^{-1})^a = x^{-1}y^{-1}xy = 1\rangle$ with a and b positive integers such that $a^2b = k$. When $b = 1$, $C_a \times C_a = \langle x, y | x^a = y^a = (xy^{-1})^a = x^{-1}y^{-1}xy = 1\rangle$ is the monodromy group of a self-dual hypermap of order a^2 and **every pair** of generators will give rise to the same hypermap.

This last theorem also means that for every $k \in \mathbb{N}$ there is a hypermap $\mathcal{H} = (G, a, b)$ such that $\frac{|G|}{d} = k$, with d being the duality index of $\mathcal{H}$. We just have to take G as the cyclic group of order k and the respective hypermap as in the previous proof. Because (G, g, g) is self-dual, $d = 1$ and we have $\frac{|G|}{d} = |G| = k$.

We will call $\frac{|G|}{d}$ the *duality coindex* of a hypermap of monodromy group G.

Can we do the same using only hypermaps that are not self-dual, for which $d \neq 1$? The proof we provide bellow will give the reader not only an affirmative answer but also the presentation of the monodromy groups of those hypermaps.

Theorem 5.0.3. *If $k \in \mathbb{N}$, there is a non self-dual hypermap $\mathcal{H} = (G, a, b)$ with duality coindex k.*

Proof:

Given $k \geq 3$, we can choose, by Dirichlet's Theorem, a prime $q \equiv 1 \bmod (k)$.

Let $G = \langle g, h | h^q = 1, g^k = 1, h^g = h^u \rangle \cong C_q \rtimes C_k$, where $u \in \mathbb{Z}_q$ has multiplicative order k, $C_q = \langle h \rangle$ and $C_k = \langle g \rangle$.

Then, if $h = ab$ and $g = a$, we have:

$$G = \langle a, b | (ab)^2 = a^k = 1, (ab)^a = (ab)^u \rangle$$

The duality group of this hypermap is the smallest normal subgroup N of G such that the assignment $a \mapsto b$, $b \mapsto a$ induces an automorphism of G/N. We obtain this quotient by adding extra relations, substituting a for b and b for a in the original ones.[1] In this case, we just have to add these relations: $b^k = 1$ and $(ba)^b = (ba)^u$.

Hence:

$$G/N = \langle (ab)^2 = a^k = b^k = 1, (ab)^a = (ab)^u, (ba)^b = (ba)^u \rangle$$

But $(ab)^a = ba$, so $ba = (ab)^u$, $ab = (ba)^u$. It follows that $ab = (ab)^{u^2}$ or, equivalently, $(ab)^{u^2-1} = 1$.

Because $k \geq 3$, we have $u \neq \pm 1 \bmod q \Rightarrow u^2 - 1 \neq 0 \bmod q \Rightarrow (u^2 - 1, q) = 1$. So,

$$(ab)^q = (ab)^{u^2-1} = 1 \Rightarrow ab = 1 \Rightarrow b = a^{-1}.$$

[1] This method will be used several times in the next sections and the group N in similar contexts will always mean the smallest normal subgroup N of the monodromy group G such that the interchange of generators induces an automorphism of G/N.

Thus $G/N = \langle a | a^k = 1 \rangle \cong C_k$. Therefore $|G/N| = k$ and, since G is not cyclic, the hypermap $\mathcal{H} = (G, a, b)$ is not self-dual.

If $k = 2$ we take $G = \langle x, y | x^6 = 1, x^4 = y \rangle$ (see Section 5.1). This works here because $|G| = 6$ and $d = 3$.

For $k = 1$, all we have to do is to choose any totally dual hypermap. $\square$

Now, another question can be asked: for each $d \in \mathbb{N}$, is it possible to find at least one hypermap with that duality index? And can we make some restrictions in the available hypermaps we are allowed to choose?

Theorem 5.0.4. *For every $d \in \mathbb{N}$, there is a hypermap with duality index equal to d.*

Proof: Let G be the cyclic group of order d generated by g. If we take $G = \langle g \rangle$ and $\mathcal{H} = (G, g, 1)$, the hypermap with monodromy group G, then its duality group must be equal to G, which means that the hypermap is totally dual and its duality index is $|G| = d$. $\square$

Remark: Obviously, $\mathcal{H} = (G, 1, g)$ also works here. Hence, for any duality index, we can always find, not just one, but two totally dual hypermaps with that index (which is not surprising since these two hypermaps are duals of each other).

It follows that we can get any duality index using hypermaps that are totally dual. Can we achieve the same result only with hypermaps that are not totally dual? Before we answer that question, we need to introduce some results and definitions about direct products of hypermaps.

5.0.2 Direct Products and Duality groups

If $\mathcal{H}$ and $\mathcal{K}$ are orientably regular hypermaps with hypermaps subgroups H, $K \leq \Delta^+$ then:

Definition 5.0.2. The *least common cover* $\mathcal{H} \vee \mathcal{K}$ and the *greatest common quotient* $\mathcal{H} \wedge \mathcal{K}$ are the orientably regular hypermaps with hypermap subgroups $H \cap K$ and $\langle H, K \rangle = HK$ respectively.

If $\mathcal{H} = (D_1, R_1, L_1)$ and $\mathcal{K} = (D_2, R_2, L_2)$ let $D = D_1 \times D_2$ and the permutations R and L be the ones that act on D induced by the actions $\rho \mapsto R_i$, $\lambda \mapsto L_i$ of Δ^+ on D_1 and D_2. If this action is transitive on D, we call $\mathcal{H} \times \mathcal{K} = (D, R, L)$, the *oriented direct product* of $\mathcal{H}$ and $\mathcal{K}$ with hypermap subgroup $H \cap K$.

Lemma 5.0.2. *If $\mathcal{H}$ and $\mathcal{K}$ are orientably regular hypermaps, then the following conditions are equivalent* [7]*:*

i) Δ^+ acts transitively on D;

ii) $\mathcal{H} \wedge \mathcal{K}$ is the orientable hypermap, with one dart;

iii) $HK = \Delta^+$.

If these conditions are satisfied we say that $\mathcal{H}$ and $\mathcal{K}$ are *orientably orthogonal* and we use the notation $\mathcal{H} \perp \mathcal{K}$. Then, $\mathcal{H} \times \mathcal{K}$ is well defined and isomorphic to $\mathcal{H} \vee \mathcal{K}$ with monodromy group $Mon(\mathcal{H} \times \mathcal{K}) = Mon(\mathcal{H}) \times Mon(\mathcal{K})$.

Having in mind that $\mathcal{H}$ is totally dual if and only if $HH^d = \Delta^+$ we have, as an important example, the following result:

Lemma 5.0.3. *$\mathcal{H}$ is totally dual $\Leftrightarrow \mathcal{H} \perp \mathcal{H}^d$.* $\square$

Once again we can adapt one of the theorems for chirality groups [7], writing it in this new context of duality:

Theorem 5.0.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be orientable regular hypermaps, with hypermap subgroups H and K, such that $\mathcal{K}$ is totally dual and covers H. Then the product $\mathcal{L} = \mathcal{K} \times \mathcal{H}^d$ is an orientably regular hypermap with duality group $D(\mathcal{L}) \cong H/K$.*

Proof: We are assuming that $\mathcal{K}$ is totally dual, so $KK^d = \Delta^+$. But $K \leq H$ which means that $KH^d \geq KK^d = \Delta^+$. Hence $KH^d = \Delta^+$. By the previous lemma, $\mathcal{K} \perp \mathcal{H}^d$. Therefore, $\mathcal{L} = \mathcal{K} \times \mathcal{H}^d$ exists and is a orientably regular hypermap with hypermap subgroup $L = K \cap H^d$. We know by Proposition 4.6.1 that

$$D(\mathcal{L}) \cong L^d / L_{\Delta^+}.$$

But

$$L_{\Delta^+} = K \cap L^d.$$

Hence

$$D(\mathcal{L}) \cong L^d / (K \cap L^d).$$

By the third isomorphism theorem, $L^d/(K \cap L^d) \cong KL^d/K$. Because $KL^d = H$ we have

$$D(\mathcal{L}) \cong H/K.$$

$\square$

Obviously, the monodromy group (Δ^+/H) of a totally dual o.r. hypermap is equal to its own duality group (since $HH^d = \Delta^+$).

It follows that for every $n \geq 1$ there is an orientably regular hypermap with duality group C_n (the monodromy group of the totally dual hypermap $\mathcal{H} = (C_n, g, 1)$ with $C_n = \langle g \rangle$). The same can be said, for instance, for $PSL(2, q)$ $(q \geq 4)$, A_n $(n \geq 5)$ and the generalized quaternion group (for n odd).

We can construct several other examples by choosing groups G_1 and G_2 such that $G_1 \perp G_2$ and both being monodromy groups of totally dual hypermaps.

All the hypermaps obtained in the previous examples are totally dual. However, Theorem 5.0.5 can also be useful to achieve some results about hypermaps that do not have that property and answer the question that we have raised at the end of the previous section.

Theorem 5.0.6. *For every $n \in \mathbb{N}$ there is a non totally dual hypermap with duality index n.*

Proof: Let K be a normal subgroup of Δ^+ such that $\Delta^+/K = C_{2n}$. Then $\mathcal{K} = (C_{2n}, g, 1)$, with $C_{2n} = \langle g \rangle$, is orientably regular and totally dual. If we take H such that $K \leq H$ and $|H : K| = n$ then $|\Delta^+ : H| = 2$, which means that $H \trianglelefteq \Delta^+$ and $\mathcal{H}$ is orientably regular.

Hence, the hypermap $\mathcal{L} = \mathcal{K} \times \mathcal{H}^d$ is orientably regular and

$$|Mon(\mathcal{L})| = |Mon(\mathcal{K})| \cdot |Mon(\mathcal{H}^d)| = 2n \times 2 = 4n$$

Then, by Theorem 5.0.5

$$\mathcal{D}(\mathcal{L}) = H/K$$

and $|H : K| = n$

$\mathcal{L}$ is not totally dual because $|Mon\mathcal{L}| = 4n > n$. $\square$

A group is called *strongly self dual* if for all its generating pairs there is an automorphism of G interchanging them. A good example of one of these groups is the quaternion group $Q = \langle x^4 = y^2, x^8 = 1, y^{-1}xy = x^{-1}\rangle$. In the next section, we will use a generalization of this quaternion group to find infinite families of non totally dual hypermaps.

5.0.3 Generalized Quaternion Groups

Definition 5.0.3. If $w = e^{i\pi n} \in \mathbb{C}$, the matrices:

$$x = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generate a subgroup Q_{2n} of order $4n$ in $GL(2, \mathbb{C})$ with presentation [28]:

$$\langle x, y | x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1}\rangle$$

which is called the *generalized quaternion group*.

As we have proved in Theorem 5.0.6, we can have a non totally dual hypermap of any duality index. However, that proof does not give us the presentation of the monodromy groups of any of those hypermaps. Explicit examples can be obtained using the generalized quaternion groups.

Theorem 5.0.7. *If d is odd there is a hypermap that is not totally dual and with duality index equal to d.*

Proof: Let $n = 2 + 4k,\ k = 0, 1, 2, \ldots$. If we take G to be the generalized quaternion group of order $4n$ then $|G| = 8 + 16k$ and has presentation:

$$G = \langle x, y | x^{2+4k} = y^2, x^{4+8k} = 1, y^{-1}xy = x^{-1}\rangle.$$

If we take N to be the smallest normal subgroup of G such that the assignment that interchanges the two generators induces an automorphism then G/N (which is obtain from G adding new relations) is the quaternion group and has order 8. But

$$|N| = \frac{|G|}{|G/N|}$$

Hence

$$|N| = \frac{8 + 16k}{8} = 2k + 1 \quad , \quad k = 0, 1, \ldots$$

From this we can conclude that for d odd there is a hypermap with monodromy group G that is not totally dual (since $|G/N| = 8 \neq 1$) and with duality index equal to $d = 2k + 1$. $\square$

Corollary 5.0.1. *Every cyclic group of odd order can be a duality group of a non totally dual hypermap.*

Proof: In the previous proof, $N = \langle x^4 \rangle \cong C_{1+2k}$. $\square$

In the proof of the Theorem 5.0.7, G/N is the quaternion group and any hypermap which has that group as monodromy group is self-dual. But all generating pairs are equivalent under automorphisms of the quaternion group. Then, there is only one (self-dual) hypermap, up to isomorphism, with monodromy group being the quaternion group.

Theorem 5.0.8. *If* $d \equiv 0 \pmod 4$ *it is possible to find a hypermap that is not totally dual and with duality index equal to d.*

Proof: Let $n = 4k$, $k = 1, 2, \ldots$ If we take G to be the generalized quaternion group of order $4n$ then $|G| = 16k$ and has presentation:

$$G = \langle x, y | x^{4k} = y^2, x^{8k} = 1, y^{-1}xy = x^{-1} \rangle$$

If we take N to be the smallest normal subgroup of G such that the assignment that interchanges the two generators induces an automorphism then

$$G/N = \langle x, y | x^{4k} = y^2, x^{8k} = 1, y^{-1}xy = x^{-1}, y^{4k} = x^2, y^{8k} = 1, x^{-1}yx = y^{-1} \rangle$$

Using the third and sixth relations, we have

$$(y^{-1}xy)yx = x^{-1}(xy^{-1}) = y^{-1}.$$

Therefore, applying the first relation $y^2 = x^{4k}$, we have:

$$y^{-1}xx^{4k}x = y^{-1} \Rightarrow x^{4k+2} = 1.$$

Then, using the second relation: $x^{4k+2} = x^{8k} \Rightarrow x^{4k-2} = 1$. Hence,

$$x^{4k+2} = x^{4k-2} = 1 \Rightarrow x^4 = 1.$$

From the first relation $x^{4k} = y^2$ we can now conclude that $y^2 = 1$ and, from the fourth one, that $x^2 = 1$. Therefore, the presentation of the group G reduces to $\langle x, y | x^2 = y^2 = (xy)^2 = 1 \rangle$, which defines a Klein 4-group. We have proved, this way, that G/N has order 4.

But

$$|N| = \frac{|G|}{|G/N|}.$$

Hence

$$|N| = \frac{16k}{4} = 4k \quad , \quad k = 1, 2, ...$$

This means that, for $d \equiv 0 \ (mod\ 4)$, there is a hypermap with monodromy group G that is not totally dual (since $|G/N| = 4 \neq 1$) and with duality index equal to $d = 4k$. □

Corollary 5.0.2. *Every cyclic group of order multiple of 4 can be a duality group of a non totally dual hypermap.*

Proof: In the previous proof, $N = \langle x^2 \rangle \cong C_{4k}$. □

Theorem 5.0.9. *Let n be odd. Then, the generalized quaternion group*

$$G = \langle x, y | x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle$$

of order $4n$ is the monodromy group of a totally dual hypermap.

Proof: If we take N to be the smallest normal subgroup of G such that the assignment that interchanges the two generators induces an automorphism then

$$G/N = \langle x, y | x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1}, y^n = x^2, y^{2n} = 1, x^{-1}yx = y^{-1} \rangle.$$

Hence, we have $x^{-1}yx = y^{-1}$ (last relation) but also $x^{-1} = y^{-1}xy$ (third relation). Therefore:

$$y^{-1}xyyx = y^{-1} \Rightarrow y^{-1}xy^2x = y^{-1}.$$

Using the first relation in this last equality we have:

$$y^{-1}xx^nx = y^{-1} \Rightarrow x^{n+2} = 1.$$

Let k be the order of x. Then, since $k|(n+2)$ and n is odd, k must also be odd. But from the second relation we also know that $x^{2n} = 1$ and, consequently, $k|2n$. Therefore, $k|n$. If odd k divides n and $n+2$, then $k = 1$ (and we have $x = 1$). Because $y^n = y^2$ and $y^2 = x^n$, we have $y^n = y^2 = 1$. Since n is odd, $y = 1$. Hence, $|G/N| = 1$, which means that the hypermap is totally dual. □

Corollary 5.0.3. *There are infinitely many totally dual hypermaps with generalized quaternion group as monodromy group.* □

Every hypermap having the generalized quaternion group (with the presentation given in our definition) as monodromy group has chirality index equal to 1. This can easily be checked because if we want to obtain a reflexible hypermap as a quotient of the original one, we just have to add the following relations to the ones that we already have for the generalized quaternion group: $x^{-n} = y^{-2}$, $x^{-2n} = 1$ and $yx^{-1}y^{-1} = x$. However, these relations do not change the presentation of the group. Hence, all the theorems above (where the generalized quaternion group appears in the proof) are, in fact, about reflexible hypermaps.

5.0.4 The symmetric and the alternating groups

In this section we will study some properties concerning duality and hypermaps with symmetric or alternating group as monodromy group.

Lemma 5.0.4. *For every $n \in \mathbb{N}$:*

a) there is a non-self-dual hypermap with monodromy group S_n.

b) there is a non-self-dual hypermap with monodromy group A_n.

Proof: a) If $n > 2$, we take $x = (12 \quad ... \quad n)$ and $y = (12)$. These permutations generate the group S_n [2] but, because they have different orders, there is no automorphism that interchanges those two generators. It follows that the hypermap $\mathcal{H} = (S_n, x, y)$ is not self-dual.

If $n = 2$ we use the proof of Theorem 5.0.4 with $S_2 \cong C_2$.

b) For $n > 3$, we just need to apply that same idea but using the permutations $x = (12...n)$ and $y = (123)$, if n is odd and greater than 3, or $x = (23..n)$ and $y = (123)$ if n is even. Then x and y generate A_n.

To prove this, we should notice, first, that any permutation of the alternating group A_n can be written as a product of 3-cycles. In [47, p. 98] it is proved, by induction, that A_n can be generated by just some of those 3-cycles:

$$\{(123), ..., (12i), ...(12n)\}$$

.

If n is odd greater that 3, let $x = (12...n)$ and $y = (123)$. Then, for t such that $3 \leq t < n$:

$$y^{-1}x(12t)x^{-1}y = (123)^{-1}(12...n)(12t)(12....n)^{-1}(123) = (12t+1)$$

If n is even, let $x = (23...n)$ and $y = (123)$. Then, for t such that $3 \leq t < n$:

$$yx(12t)^{-1}x^{-1}y^{-1} = (123)(23...n)(12t)^{-1}(12...n)^{-1}(123)^{-1} =$$

$$= (12t+1)$$

Hence, x and y generate A_n. Because they do not have the same order, the hypermap is not self-dual.

If $n = 3$ we use the proof of Theorem 5.0.4 with $A_n = A_3 \cong C_3$. $\square$

Example 5.0.4. If $G = \langle x = (12345), y = (123)\rangle = A_5$, what is the duality index of the hypermap (G, x, y)? Because A_5 is simple and the hypermap is not self-dual, the duality group (which is normal in A_5) must be A_5 itself. Hence, the duality index of the hypermap is $|A_5| = 5!/2$

More generally, if we take $(A_n, (12...n), (123))$, if n is odd, or $(A_n, (23...n), (123))$, if n is even, we get totally dual hypermaps with monodromy group A_n. Hence:

Theorem 5.0.10. *If $n \geq 5$ there is a totally dual hypermap with monodromy group A_n.*

Remark: If we take any simple group generated by two elements of different order, we can get a similar result. For instance, $PSL(2, q)$ for $q > 3$.

How can we find the duality group of a hypermap with monodromy group G? A naive algorithm would be:

- find all normal subgroups N_α of G.
- starting with the N_α of lower order, check if the hypermap with monodromy group G/N_α is self-dual
- Stop when finding one of these hypermaps

If $G = S_n$ the only possible duality groups are 1, A_n and S_n. Therefore, a hypermap with monodromy group S_n is self-dual, totally dual or has $n!/2$ as duality index. For $n \neq 6$, all automorphisms of S_n are inner and act by conjugation. It is then easy to check if there is an automorphism that transposes the two generators, knowing if the hypermap is self-dual or not. If the answer is affirmative then $\mathcal{D}(\mathcal{H}) = 1$. Otherwise, $\mathcal{D}(\mathcal{H}) = A_n$ or S_n and we need to see if the hypermap with monodromy group S_n/A_n, of order 2, is self-dual or not.

Theorem 5.0.11. *Every hypermap $\mathcal{H} = (S_n, x, y)$:*

i) is totally-dual if x or y is an even permutation;

ii) is self-dual or has duality index $n!/2$ if x and y are both odd permutations and $n \neq 4$.

iii) is self dual or has duality index 4 if x and y are both odd permutations and $n = 4$.

Proof: i) The only non-identity quotients S_n/N of S_n are $S_n/A_n \cong C_2$ and $S_4/V_4 \cong S_3$, when $n = 4$. In each case, because one of xN and yN is in the unique subgroup of index 2 of S_n/N and the other is not, there can be no automorphism of S_n/N transposing xN and yN. So, the only self-dual quotient is the trivial one and the hypermap is totally dual.

ii) S_n is not cyclic and, by definition $\langle x, y\rangle = S_n$. Hence, $x \neq y$. Suppose $\mathcal{H}$ is not self-dual. Because $\mathcal{H}/A_n = (S_n/A_n, xA_n, yA_n)$ and $|S_n/A_n| = 2$, the two generators xA_n and yA_n must be the same. It follows that the hypermap $\mathcal{H}/A_n$ is self-dual and $|D(\mathcal{H})| = n!/2$.

iii) Let x and y be two odd permutations generating S_4 and N the Klein group V_4 (a normal subgroup in S_4). That pair of generators may be formed by a transposition and a 4-cycle or by two 4-cycles. Say x is a transposition and y is a 4-cycle. Then, they map to distinct permutations in $S_4/N \cong S_3$. Because the third element in S_3 conjugates each to the other, the quotient hypermap is self-dual (whereas the hypermap itself is not), with duality group V_4. If x and y are both 4-cycles the hypermap is self-dual. $\square$

5.1 Counting Self-Dual and Totally Dual Regular Oriented Hypermaps

If $\mathcal{H}$ is a totally dual orientably regular hypermap with hypermap subgroup H, then $HH^d = \Delta^+$. Hence, $\mathcal{H} \perp \mathcal{H}^d$ and $\mathcal{L} = \mathcal{H} \times \mathcal{H}^d$ exists and is an orientably regular hypermap with hypermap subgroup $L = H \cap H^d$. This means that $\mathcal{L}$ is self-dual. Moreover,

$$|Mon(\mathcal{L})| = |\Delta^+/(H \cap H^d)| = |\Delta^+/H||H/H \cap H^d| = |D(\mathcal{H})|^2 = |Mon(\mathcal{H})|^2$$

Therefore, from any totally dual hypermap $\mathcal{H}$ we can build, this way, a self-dual hypermap (with monodromy group of order equal to $|D(\mathcal{H})|^2$).

If we take γ to be a mapping from the set of totally dual hypermaps to the set of self-dual hypermaps, sending $\mathcal{H}$ to $\mathcal{H} \times \mathcal{H}^d$, then γ is well defined. However, it might not be injective (see Figure 5.1).

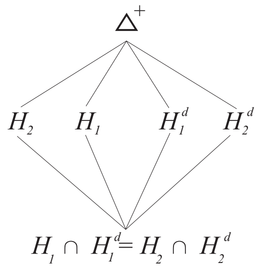


Figure 5.1: γ might not be injective.

Nevertheless, if we restrict γ to the set of hypermaps whose monodromy group is a non abelian simple group, different totally dual hypermaps will be sent to different self-dual hypermaps.

Example 5.1.1. If $\mathcal{H}_n = (A_n, (123), (12...n))$, for $n \geq 5$, then the hypermap is totally dual and, because A_n is simple and non abelian, different $\mathcal{H}_n$ will be sent to different self-dual hypermaps by γ.

Theorem 5.1.1. *If G has k generating pairs, is not cyclic and is the monodromy group of an orientably regular hypermap $\mathcal{H}$, then there are no more than $k/2$ self-dual hypermaps with G as monodromy group.*

Proof: If the hypermap $\mathcal{H} = (G, a, b)$ is self-dual, there must be an automorphism ϕ of G such that $\phi(a) = b$ and $\phi(b) = a$. This is the same that assuming that (a, b) and (b, a) belong to the same $Aut(G)$-class of generating pairs. The number of these orbits correspond to the number of orientably regular hypermaps. Because G is not cyclic, $b \neq a$. Hence, the result follows. $\square$

This allows us to build an upper bond for the number of self-dual hypermaps, with a particular (non-cyclic) monodromy group G, by calculating the number of generating pairs.

If $G = C_p$ with p prime then there are $p^2 - 1$ generating pairs of G and $Aut(C_p) = C_{p-1}$. Therefore, there are $\frac{p^2-1}{p-1} = p + 1$ $Aut(C_p)$-classes of generating pairs and each one of these corresponds to a different orientably regular hypermap. Representatives of the $Aut(C_p)$-classes are:

$$g_i = (1, i) \quad for \quad i = 0, 1, ..., p-1 \quad and$$

$$g_p = (0, 1)$$

It follows that there are 2 self-dual hypermaps, corresponding to the pairs $(1, 1)$ and $(1, p-1)$, and $p-1$ totally dual hypermaps corresponding to the other pairs.

5.1.1 Hypermaps with monodromy group of small order

Order	O.r. hypermaps	Totally Dual	Self Dual	Intermediate index
2	3	2	1	0
3	4	2	2	0
4	7	4	3	0
5	6	4	2	0
6	15	6	3	6
7	8	6	2	0

Table 5.1: Oriented regular hypermaps with monodromy groups of small order

In this section, we classify all hypermaps of small order, in terms of their duality. The results are summarized in Table 5.1. All the groups of prime order

p are isomorphic to C_p and it is then easy to fill the rows of that table (see previous section for details). For other groups, we have to be more careful and study each one of them separately.

We should also notice that the number of totally dual hypermaps with a specific monodromy group is always even because if $\mathcal{H}$ is totally dual, so is its dual $\mathcal{H}^d$. It follows that the third column of Table 5.1 can only contain even numbers.

Order 4: C_4 and $V = C_2 \times C_2$

► C_4

Number of generating pairs: 12

1.	$(0,1)$	7.	$(0,3)$
2.	$(1,0)$	8.	$(3,0)$
3.	$(1,1)$	9.	$(3,3)$
4.	$(1,2)$	10.	$(3,2)$
5.	$(2,1)$	11.	$(2,3)$
6.	$(1,3)$	12	$(3,1)$

For each one of this pairs we can associate a o.r. hypermap with monodromy group C_4. But

$$\mathcal{H}_1 = (C_4, 0, 1) \cong \mathcal{H}_7 = (C_4, 0, 3)$$

and also

$$\mathcal{H}_2 \cong \mathcal{H}_8$$

$$\mathcal{H}_3 \cong \mathcal{H}_9$$

$$\mathcal{H}_4 \cong \mathcal{H}_{10}$$

$$\mathcal{H}_5 \cong \mathcal{H}_{11}$$

$$\mathcal{H}_6 \cong \mathcal{H}_{12}$$

Since $AutC_4 \cong C_2$, these are the only isomorphisms between $\mathcal{H}_1$,..., $\mathcal{H}_{12}$. Hence, there are six non-isomorphic o.r. hypermaps with monodromy group C_4.

$\mathcal{H}_1$ and $\mathcal{H}_2$ are totally dual.

$\mathcal{H}_3$ and $\mathcal{H}_6$ are self-dual.

What about $\mathcal{H}_4$ and $\mathcal{H}_5$? They are clearly not self-dual but are they totally dual or have they intermediate duality index?

The only non trivial normal subgroup of C_4 is the group $\langle 2\rangle$. Therefore, if those hypermaps are not totally dual they must have $\langle 2\rangle$ as duality group. However,

$$\mathcal{H}_4/\langle 2\rangle = (C_4/\langle 2\rangle, 1+\langle 2\rangle, 2+\langle 2\rangle) = (C_4/\langle 2\rangle, 1+\langle 2\rangle, \langle 2\rangle)$$

is not self-dual. Hence, $\mathcal{H}_4$ is totally dual.

The same argument apply to $\mathcal{H}_5$ since

$$\mathcal{H}_5/\langle 2\rangle = (C_5/\langle 2\rangle, 2+\langle 2\rangle, 1+\langle 2\rangle) = (C_5/\langle 2\rangle, \langle 2\rangle, 1+\langle 2\rangle)$$

Therefore, H_5 is also totally dual. We can then conclude that there are four totally dual o.r. hypermaps and two self-dual o.r. hypermaps with monodromy group C_4.

► $V_4 = C_2 \times C_2$

The Klein four-group has six different generating pairs and, since the group of automorphisms of V is S_3 (of order 6), there is only one (self-dual) hypermap with monodromy group V_4.

Order 6: C_6 and S_3

► C_6

$C_6 = \{0, 1, 2, 3, 4, 5\} = \langle 1\rangle = \langle 5\rangle$

C_6 has two proper subgroups: $\langle 2\rangle = \langle 4\rangle$ and $\langle 3\rangle$, both normal since C_6 is abelian.

Number of generating pairs: 24

1.	$(1,0)$	13.	$(5,0)$
2.	$(1,1)$	14.	$(5,5)$
3.	$(1,2)$	15.	$(5,4)$
4.	$(1,3)$	16.	$(5,3)$
5.	$(1,4)$	17.	$(5,2)$
6.	$(1,5)$	18.	$(5,1)$
7.	$(0,1)$	19.	$(0,5)$
8.	$(2,1)$	20.	$(4,5)$
9.	$(3,1)$	21.	$(3,5)$
10.	$(4,1)$	22.	$(2,5)$
11.	$(2,3)$	23.	$(4,3)$
12.	$(3,2)$	24.	$(3,4)$

Number of automorphisms of C_6: 2

Hence, there are 12 different o.r. regular hypermaps with monodromy group C_6 (with $\mathcal{H}_i \cong \mathcal{H}_{12+i}$ for $i \in \{1,..,12\}$). The only two self-dual hypermaps are $\mathcal{H}_2$ and $\mathcal{H}_6$. On the other hand, $\mathcal{H}_1$ is clearly totally dual.

When C_6 is generated by the pair $(1,2)$ it has presentation:

$$C_6 = \langle x, y | x^6 = 1, x^2 = y\rangle.$$

Hence, considering N as the duality group,

$$|C_6/N| = |\langle x, y | x^6 = 1, x^2 = y, y^6 = 1, y^2 = x\rangle| = 3.$$

Therefore, $\mathcal{H}_3$ has duality index 2.

If we take C_6 generated by the pair $(1,4)$, we can write:

$$C_6 = \langle x, y | x^6 = 1, x^4 = y\rangle.$$

Hence, considering N as the duality group,

$$|C_6/N| = |\langle x, y | x^6 = 1, x^4 = y, y^6 = 1, y^4 = x\rangle| = 3.$$

Then, $\mathcal{H}_5$ has also duality index 2.

We can then conclude, without any further investigation, that the dual hypermaps of these ($\mathcal{H}_8$ and $\mathcal{H}_{10}$) have the same duality index (equal to 2).

On the other hand, when C_6 is generated by the pair $(1,3)$ it has presentation:

$$C_6 = \langle x, y | x^6 = 1, x^3 = y \rangle.$$

Therefore, considering N as the duality group,

$$|C_6/N| = |\langle x, y | x^6 = 1, x^3 = y, y^6 = 1, y^3 = x \rangle| = 2.$$

It follows that $\mathcal{H}_4$ and its dual, $\mathcal{H}_9$, have both duality index 3.

$\mathcal{H}_{11} = (C_6, 2, 3)$

$\mathcal{H}_{11}$ is clearly not self-dual. Hence, if it is not totally dual, the only possibilities for duality group are $\langle 2 \rangle$ or $\langle 3 \rangle$. However, in the first case, we would have:

$$(C_6/\langle 2 \rangle, 2 + \langle 2 \rangle, 3 + \langle 2 \rangle) = (C_6/\langle 2 \rangle, \langle 2 \rangle, 3 + \langle 2 \rangle)$$

which is not a self-dual hypermap because the orders of the generators are different. The same argument can be used in the second case:

$$(C_6/\langle 2 \rangle, 2 + \langle 3 \rangle, 3 + \langle 3 \rangle) = (C_6/\langle 2 \rangle, 2 + \langle 3 \rangle, \langle 3 \rangle).$$

Therefore, $\mathcal{H}_{11}$ and its dual $\mathcal{H}_{12}$ are totally dual.

We have then:

- two self-dual hypermaps ($\mathcal{H}_2$ and $\mathcal{H}_6$)
- three totally-dual hypermaps ($\mathcal{H}_1$, $\mathcal{H}_{11}$ and $\mathcal{H}_{12}$)
- four hypermaps with duality index 2 ($\mathcal{H}_3$, $\mathcal{H}_8$, $\mathcal{H}_5$ and $\mathcal{H}_{10}$)
- two hypermaps with duality index 3 ($\mathcal{H}_4$ and $\mathcal{H}_9$)

with monodromy group C_6

► S_3

$S_3 = \{(1), (12), (13), (23), (123), (132)\}$ has six elements and is not cyclic (and nonabelian). It has four proper subgroups, three of order 2 ($\langle (12) \rangle$, $\langle (13) \rangle$, $\langle (23) \rangle$) and one of order 3 (the one generated by (123)). Only this last subgroup is normal in S_3, which means that $\langle(123)\rangle$ is the only possible choice for duality group of a hypermap neither self-dual or totally dual.

The automorphism group of S_3 is isomorphic to S_3.

Number of generating pairs: 18

1.	$((12),(23))$	10.	$((23),(123))$
2.	$((13),(12))$	11.	$((13),(123))$
3.	$((12),(13))$	12.	$((13),(132))$
4.	$(((23),(12))$	13.	$((123),(12))$
5.	$((13),(23))$	14.	$((132),(12))$
6.	$((12),(13))$	15.	$((132),(23))$
7.	$((12),(123))$	16.	$((123),(23))$
8.	$((12),(132))$	17.	$((123),(13))$
9.	$((23),(132))$	18.	$((133),(13))$

But

$$\mathcal{H}_1 \cong \mathcal{H}_2 \cong \mathcal{H}_3 \cong \mathcal{H}_4 \cong \mathcal{H}_5 \cong \mathcal{H}_6$$

$$\mathcal{H}_7 \cong \mathcal{H}_8 \cong \mathcal{H}_9 \cong \mathcal{H}_{10} \cong \mathcal{H}_{11} \cong \mathcal{H}_{12}$$

$$\mathcal{H}_{13} \cong \mathcal{H}_{14} \cong \mathcal{H}_{15} \cong \mathcal{H}_{16} \cong \mathcal{H}_{17} \cong \mathcal{H}_{18}$$

Again, $Aut(S_3) \cong S_3$, of order 6, so these are the only isomorphisms. Hence, there are three non-isomorphic o.r. hypermaps with monodromy group S_3.

A presentation for S_3, using the first pair of generators, is:

$$\langle x, y | x^2 = y^2 = 1, xyx = yxy\rangle$$

meaning that $\mathcal{H}_1$ is self-dual. Because $\mathcal{H}_7$ is not self-dual (the generators have different orders) it must be totally dual (Theorem 5.0.11). The same applies to the remaining distinct hypermap. Therefore, there are one self-dual and two totally dual o.r. hypermaps with monodromy group S_3.

Looking at Table 5.1, we can now conclude that the smallest order of a monodromy group for which we can find an orientably regular hypermap with intermediate index is 6. For order 8 we can also find hypermaps with intermediate index. We just have to take D_8 as a monodromy group and choose the right generators in order to have a hypermap of duality index 2, as we will show in the proof of Theorem 5.2.1.

5.2 Duality and maps

In this section we will focus our attention on maps, which is the same as saying that each time we have $\mathcal{M} = (G, x, y)$, we will also have $(xy)^2 = 1$.

Some observations about maps and duality:

1) If G is a finite abelian group: $o(xy)$ divides $l.c.m.(o(x), o(y))$ and, as a consequence, at least one of the generators must be even. Therefore, there is no self-dual map of type $\{k, k\}$ for odd $k \in \mathbb{N}$ if the monodromy group is a finite abelian group.

2) In [5], it is proved that every nontrivial element in $PSL(2, q)$, $q \neq 9$, $q > 3$, is a member of a generating pair with one of the members having order 2. Hence, every $PSL(2, q)$ group is a monodromy group of an orientably regular map. Because, $PSL(2, q)$ is simple for $q > 3$, these hypermaps are self-dual or totally dual.

In the previous section we have proved that for any duality index there is a non totally dual hypermap of that index. In fact, that result can be stronger if we substitute the word *hypermap* for *map*, restricting our possible choices[2].

Theorem 5.2.1. *For every $d \in \mathbb{N}$ there is a (non totally dual) map with duality index equal to d.*

Proof: Let

$$D_{2m} = \langle x, y | x^m = y^2 = (xy)^2 = 1 \rangle$$

[2] Although this is, in a sense, a stronger result, the theorems we have proved using extended quaternion groups give us proper hypermaps, since $(xy)^2 \neq 1$, which is also an important restriction.

be the dihedral group of order $2m$. If we take $\mathcal{M} = (D_{4d}, x, y)$, then, considering N as before, we will have: $D_{4d}/N \cong D_4$. Therefore,

$$|N| = \frac{4d}{4} = d. \quad \square$$

5.3 Chiral Duality

The automorphism $\delta : r_i \mapsto r_{2-i}$ of Δ, corresponding to duality of hypermaps, restricts to an automorphism of Δ^+ that interchanges and invert the two generators $\rho = r_1 r_2$ and $\lambda = r_2 r_0$. This corresponds to the mirror image of the dual. As a consequence of that, **oriented** hypermaps have two duality operations to consider, one of them reversing the orientation. We have then two types of duality induced by the following automorphisms of Δ^+:

$$\alpha : x \mapsto y; \quad y \mapsto x,$$

$$\beta : x \mapsto y^{-1}; \quad y \mapsto x^{-1}.$$

Since the automorphisms of Δ^+ which induce them are conjugate in $Aut(\Delta^+)$, both dualities have the same general properties. Nevertheless, their effect on a *specific* hypermap might be distinct.

To make this observation clear to the reader, we will give some example of hypermaps such that:

a) $|\mathcal{D}_\alpha(\mathcal{H})| \neq |\mathcal{D}_\beta(\mathcal{H})|$

b) $\mathcal{D}_\alpha(\mathcal{H}) \cong \mathcal{D}_\beta(\mathcal{H})$

Examples

a) We can take $\mathcal{H} = (G, x, y)$ with

$$G = \langle x, y | x^4 = y^4 = 1, \quad xy = y^2 x^2 \rangle.$$

$$|G| = 20 \text{ (using GAP [23])}$$

$$G/N_\alpha = \langle x, y | x^4 = y^4 = 1, \quad xy = y^2 x^2, \quad yx = x^2 y^2 \rangle.$$

Using the two last relations, we have:

$$xyyx = y^2x^2x^2y \Leftrightarrow xy^2x = 1 \Leftrightarrow y^2 = x^2.$$

Therefore,

$$G/N = \langle x | x^4 = 1 \rangle$$
$$|G/N_\alpha| = 4.$$

However:

$$G/N_\beta = \langle x, y | x^4 = y^4 = 1, \quad xy = y^2x^2 \quad y^{-1}x^{-1} = x^{-2}y^{-2} \rangle = G.$$

Hence $|G/N_\beta| = 20$. It follows that G is β-self-dual but not α-self-dual.

b) If $\mathcal{H} = (G, x, y) = (A_5, (12345), (123))$ then $D_\alpha(\mathcal{H}) \cong A_5$ because the hypermap is α-totally dual. But $\mathcal{H}^\beta = (G, y^{-1}, x^{-1}) = (A_5, (132), (15432))$. Hence, we still have two permutations of different order. This means that the hypermap cannot be β-self-dual and, because A_5 is simple, we can conclude that it must be β-totally dual. Hence $D_\beta(\mathcal{H}) \cong D_\alpha(\mathcal{H}) \cong A_5$.

Another example, this time without using totally dual hypermaps, is the only proper chiral hypermap of genus 7 with monodromy group with presentation (see Conder's list [14]):

$$x^3, xy^{-1}x^{-1}y^{-2}$$

This hypermap has $|D_\alpha| = |D_\beta| = 3$ which means that $D_\alpha \cong D_\beta \cong \mathbb{Z}_3$.

Definition 5.3.1. Two distinct hypermaps $\mathcal{H}_1$ and $\mathcal{H}_2$ are *$\alpha\beta$-symmetric* if the order of their automorphism groups and their genus are the same and $|D_\alpha(\mathcal{H}_1)| = |D_\beta(\mathcal{H}_2)|$ and $|D_\alpha(\mathcal{H}_2)| = |D_\beta(\mathcal{H}_1)|$

One example of a pair ($\mathcal{H}_1$,$\mathcal{H}_2$) of $\alpha\beta$-symmetric hypermaps is the one formed by two proper chiral hypermaps of genus 19 whose automorphism groups have the following presentations (see Conder's list [14]):

$$x^4, y^4, xy^{-2}x^2y^{-1}x^{-1}y^{-1} \text{ and}$$
$$x^4, y^4, y^{-1}xy^{-1}x^2y^2x^{-1}$$

Both automorphism groups have order 144. Moreover, it is easy to check (using GAP [23] or making some calculations), that $|D_\alpha(\mathcal{H}_1)| = |D_\beta(\mathcal{H}_2)| = 18$ and $|D_\alpha(\mathcal{H}_2)| = |D_\beta(\mathcal{H}_1)| = 2$.

5.4 Orientably Regular Hypermaps - type and duality

The aim of this section is to know for what triples (l, m, n) we can find self-dual and totally dual hypermaps of that type. The relationship between type and self-dual orientably regular hypermaps is easier to study than the one concerning totally dual hypermaps. Therefore, we will deal with the *self-dual case* before.

If we want to find a self-dual hypermap of type (l, m, n), we obviously need to have $l = n$, but we can say more:

Theorem 5.4.1. *For every k, t there is a self-dual orientably regular hypermap of type (k, t, k)*

Proof: We just need to take the monodromy group

$$G = \Delta^+(k, t, k) = \langle x, y | x^k = y^k = (xy)^t = 1 \rangle.$$

Then

$$G/N = \langle x, y | x^k = y^k = (xy)^t = (yx)^t = 1 \rangle.$$

But $(yx)^t = y(xy)^{t-1}x = yy^{-1}x^{-1}x = 1$ (using the third relation). Therefore, we do not need the fourth relation $(yx)^t$ to present the group. Hence $G \cong G/N$, meaning that the hypermap $\mathcal{H} = (G, x, y)$ is self-dual. $\square$

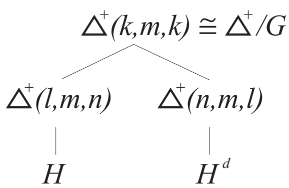


Figure 5.2: Hypermap of type (l, m, n).

If H is the hypermap subgroup of an oriented hypermap of type (l, m, n) then its dual has type (n, m, l) and we may look at both H and H^d as subgroups of $\Delta^+(k, m, k)$, with $k = lcm(l, n)$, which is the universal oriented hypermap of type (k, m, k) (and this is also, by the previous theorem, the monodromy group of a self-dual o.r. hypermap). It follows that every orientably regular hypermap subgroup is a subgroup of a monodromy group of a non trivial self-dual hypermap. In fact, it is a subgroup of an infinite number of monodromy groups of non trivial self-dual hypermaps if, instead of the *least* common multiple of l and m, we consider *any* common multiple.

As we have previously pointed, in a self-dual orientably regular hypermap, vertices and faces need, of course, to have the same valency but that might not be enough to assure that an orientably regular hypermap is self-dual. A hypermap of type (k, t, k) can be very far away from being self-dual. It can even be totally dual. For instance, the hypermap $\mathcal{H} = (C_p, 1, 2)$, with p prime, is totally dual (see section 5.1) and of type (p, p, p).

Can we have a totally dual orientably regular hypermap of each type (l, m, n)**?**

The answer is no. There are no totally dual hypermaps of type $(3, 2, 3)$, since this has to be the tetrahedron, which is self-dual. However, if we restrict ourselves to hyperbolic triples, we can enumerate orientably regular hypermaps of a given type (l, m, n) with automorphism groups isomorphic to $PSL(2, q)$ or $PGL(2, q)$ (this enumeration can be found in a joint work of Marston Conder, Primoz Potocnik, Josef Siran [17], based on the a paper by Sah [51]). Because these groups are simple or almost simple we can use them to try to find totally dual hypermaps or self-dual hypermaps. If $l \neq n$ then the hypermap cannot be self-dual. If $l = n$, we have to check if there is an automorphism of $PSL(2, q)$ or $PGL(2, q)$ that interchange the two generators.

We should also notice that for some triples (l, m, n) is very easy to find totally dual hypermaps of that type:

If $g.c.d.(l, n) = 1$ then

$$G_{l,m,n} = \langle a, b | a^l = b^n = (ab)^m = 1 \rangle$$

is obviously of type (l, m, n). Let N be the smallest normal subgroup of $G_{l,m,n}$ such that $G_{l,m,n}/N$ is reflexible. Then

$$G_{l,m,n}/N = \langle a, b | a^l = a^n = b^n = b^l = (ab)^m = (ba)^m = 1 \rangle.$$

Because l and n are co-prime the group $G_{l,m,n}/N$ collapse to the identity. Therefore $\mathcal{H} = (G_{l,m,n}, a, b)$ is a totally dual hypermap of type (l, m, n). This demonstrates that, if $g.c.d.(l, n) = 1$, for every triple (l, m, n) there is a totally dual hypermap of that type.

5.4.1 Duality-type in hypermaps with symmetric group or alternating group as monodromy groups

Definition 5.4.1. We say that an orientably regular hypermap has *duality-type* $\{l, n\}$ if l is the l.c.m. of the valency of the vertices and n is the l.c.m. of the valency of the faces (which is the same as saying that l and n are the orders of the generators interchanged by the duality operation).

If we restrict ourselves to the family of hypermaps whose monodromy group is the alternating or the symmetric group, can we have a totally dual hypermap of any duality-type $\{l, n\}$? In fact, we can not only prove that the answer is affirmative but also explicitly show how to construct those hypermaps. The reason why, here, we look for duality-type instead of type is because, in the first case, we just need to control the order of the two generators x and y, while in the second one we also need to pay attention to the order of xy, which is a harder problem to solve.

To complete that task we need a few concepts and theorems about generators of symmetric and alternating groups.

Let (G, Ω) be a permutation group. An equivalence relation $\sim$ is called *G-invariant* if whenever $\alpha, \beta \in \Omega$ satisfy $\alpha \sim \beta$ then $g(\alpha) \sim g(\beta)$ for all $\alpha, \beta \in \Omega$. Two obvious G-invariant equivalence relations are: (i) $\alpha \sim \beta$ if and only if $\alpha = \beta$ and (ii) $\alpha \sim \beta$ for all $\alpha, \beta \in \Omega$. We call (G, Ω) *imprimitive* if it admits some equivalence relation other than (i) or (ii). Otherwise, we call (G, Ω) *primitive.*

There are several examples of primitive groups.[3] For instance, any alternating group A_n is primitive, and so is $PGL(2, q)$, in its standard action, for any prime power q.

However, it is known - as underlined by Peter Cameron in the Encyclopedia of Design Theory [11]- that primitive groups are "rare (for almost all n, the

[3] Colva M. Roney-Dougal [49] has classified all the primitive permutation groups of degree less than 2500.

only primitive groups of degree n are the symmetric and alternating groups, see [12]); and small (of order at most $n^{c \cdot logn}$ with known exceptions), see [44] for the best possible result here". The fact that most of the primitive groups are alternating groups (which are simple) and symmetric groups (which have only one non trivial normal subgroup) makes *primitivity* a powerful concept to build totally dual hypermaps. Although the probability of a primitive group being an alternating or a symmetric group is very high, we need to be sure that we are not dealing with a different kind of group. The next definitions and theorems help us to avoid that situation.

Definition 5.4.2. Let G be a permutation group on Ω and k a natural number with $1 \leq k \leq n = |\Omega|$. G is called *k-fold transitive* if for every two ordered k-tuples $\alpha_1, ...\alpha_k$ and $\beta_1, ..., \beta_k$ of points of Ω (with $\alpha_i \neq \alpha_j$, $\beta_i \neq \beta_j$, for $i \neq j$) there exists $g \in G$ which takes α_t into β_t (for $t = 1, ..., k$).

Definition 5.4.3. The *minimal degree* of a permutation group is the minimum number of points moved by any non-identity element of the group.

Theorem 5.4.2 (Miller [46]). *The minimal degree of a primitive group which is neither alternating nor symmetric must exceed 4 whenever its degree exceeds 8.*

Theorem 5.4.3 (Marggraf [56], chapter II). *A primitive group of degree n, which contains a cycle of degree m with $1 < m < n$ is $n - m + 1$-fold transitive.*

And, as a consequence of the classification of simple groups [10]:

Theorem 5.4.4. *If a permutation group G is at least $6 - transitive$ then G is the alternating group or the symmetric group.*

With these tools we can then prove the following theorem:

Theorem 5.4.5. *For each pair n, $l \in \mathbb{N}$, with $n, l \geq 2$, we can find a totally dual orientably regular hypermap of duality-type $\{l, n\}$, with alternating or symmetric group as monodromy group .*

Proof:

a) If l, n are both odd and $l \neq n$ (we may assume, without loss of generality, that $l > n$) let $x = (1, 2, ..., l)$ and $y = (1, 2, ..., n)$ permutations of S_l.

The only equivalence relations preserved by x are the congruences $mod\ k_1$ (for some $k_1|l$). If that equivalence relations are also preserved by y then they have also to be the congruences $mod\ k_2$ (for some $k_2|n$), if the elements are less or equal than n (the other elements are fixed by y, so they can be in any equivalence class). Hence, the only equivalence relations preserved by x and y are the congruences $mod\ k$ (for some $k|l$ and $k|n$), if the elements are less or equal than n. Then,

$$l \equiv n(mod\ k) \Rightarrow l \sim n.$$

If z is the commutator $y^{-1}x^{-1}yx = (n, n-1, l)$ we have:

$$z(l) \sim z(n) \Leftrightarrow n \sim n-1$$

Therefore, because $(n-1, n) = 1$, the only equivalence relation preserved by $G =< x, y >$ is a trivial one.

Hence, G is primitive. But $z \in G$ and the minimal degree of a primitive group which is neither alternating or symmetric must exceed 4 whenever its degree exceeds 8 [46]. Therefore, if $l > 8$, G must be the alternating group and $\mathcal{H} = (G = A_l, x, y)$ is a totally dual hypermap of duality type $\{l, n\}$. [For $l < 8$ we can easily find two cycles of that given orders that generate A_l.]

b) If l is even and n is odd (or if l is odd and n is even) and $l \neq n$ a similar proof can be written using the cycles $x = (1, 2, ..., l)$ and $y = (1, 2, ..., n)$. In this case, however, $G =< x, y >= S_l$. Because not both of the generators are odd permutations, $\mathcal{H} = (G, x, y)$ is totally dual (see Theorem 5.0.11).

c) If l and n are both even and $l \neq n$ we take

$$x = (1, 2, ..., l)$$

$$y = (1, 2)(l, l+1, l+2, ..., l+n-1)$$

permutations of S_{l+n-1} and $G =< x, y >$. First, we need to show that G is primitive. Let $(l+1) \in B_1$, an equivalence class (or a group block).

i) If $l \in B_1$ (or any $k \in \{1, 2, ..., l\}$ also belongs to B_1) then, because $B_1 x = B_1$ ($l+1$ is fixed by x), all elements of the set $\{1, 2, .., l\}$ must belong to B_1.

   Then, $B_1 y = B_1$ (y fixes 3, for instance). Therefore $(l+2), (l+3), ..., (l+n-1)$ also belong to B_1. It follows that all elements of $\Omega = \{1, 2, ..., l+n-1\}$ belong to the same block and that the equivalence relation must be trivial.

ii) If l does not belong to B_1 we assume that $l \in B_2 \neq B_1$.

   Suppose $(l+2) \in B_2$. Then:

$$x(l+2) = l+2$$

$$x(l) = 1.$$

   Hence: $1 \in B_2$ and, consequently, $2 \in B_2$. But then:

$$y(1) = 2$$

$$y(l) = l+1.$$

   Therefore $(l+1) \in B_2$, which is a contradiction. Hence $(l+2)$ does not belong to B_2 and the same can be said to $(l+3), (l+4), ..., l+n-1$.

   On the other hand, if some $t \in \{3, ..., l-1\}$ belongs to B_2 we have

$$y(t) = t$$

$$y(l) = l+1.$$

   Then, $(l+2) \in B_2$ (contradiction).

   Finally, if $r \in \{1, 2\}$ belongs to B_2:

$$y^2(r) = r$$

$$y^2(l) = l+2.$$

   Hence, $(l+2) \in B_2$ (absurd).

Therefore $B_2 = \{l\}$. This means that all the blocks must have only one element too because if, for instance, $a, b \in B$ and w is the permutation that sends a to l, $w(a)$ and $w(b)$ must belong to the same block. Hence, in this case, $w(a) = l = w(b)$. That is not possible because $a \neq b$.

It follows that $G =< x, y >$ is a primitive group of degree $l + n - 1$. Because $l + n - 1 - l + 1 = n$ the group G is $n - transitive$ [56] and for $n > 5$, G must be the alternating or symmetric group. If $n = 4$ we have:

$$x = (1, 2, ..., l)$$

$$y = (1, 2)(l, l + 1, l + 2, l + 3).$$

Then, the degree of $y^2 = (l, l + 2)(l + 1, l + 3)$ is four and the group must be the alternating or the symmetric group [46]. Since x is an odd permutation, $G = S_{l+n-1}$. Then $\mathcal{H} = (S_{l+n-1}, x, y)$ is a totally dual hypermap because y is an even permutation (see Theorem 5.0.11).

d) If $n = l$ (and even) we can use the same generators as in c):

$$x = (1, 2, ..., l)$$

$$y = (1, 2)(l, l + 1, l + 2, ..., l + n - 1) = (1, 2)(l, l + 1, ..., 2l - 1).$$

$$ord(x) = l$$

$$ord(y) = [2, l] = l$$

Hence $G =< x, y >= S_{l+n-1} = S_{2l-1}$. Because x is an even permutation and y is odd, $\mathcal{H} = (S_{2l-1}, x, y)$ must be a totally dual hypermap (see Theorem 5.0.11).

e) Suppose now $n = l$ are odd.

Let the generators of the group be these two even permutations in S_{l+1}:

$$x = (1, 2, ..., l)$$

$$y = (2, 3, ..., l + 1).$$

The group generated by these two elements is primitive, since it is a 2-transitive group [50], and $x^{-1}y^{-1}xy = (1, l-1)(2, 3)$ has degree 4. Hence, by [46], $\langle x, y\rangle = A_l$ if $l + 1 \geq 8$, i.e. $l \geq 7$. Cases for low l are easy to solve.

□

In the 60's, Graham Highman conjectured that any Fuchsian group has among its homomorphic images all but finitely many of the alternating groups. He also proved that A_n is a factor group of $(2, 3, 7) = \langle a, b|a^2, b^3, (ab)^7\rangle$ for all large n. Because 2, 3 and 7 are prime numbers and $2 \neq 7$ we can conclude, from that result, that there is an **infinite** number of totally dual hypermpas of type $(2, 3, 7)$. That result, obtained by Higman, was later extended by others (see, for instance [16]) that proved that the same can be said for other families of triangle groups. The complete proof of the conjecture, however, was only published in 2007 by Brent Everitt [22]. In his paper, it is shown that we only need to consider the triangle groups (p, q, r), $3 \leq p < q < r$ to prove the main result (which is done making use of coset diagrams for those triangle groups). Hence, it is possible to say that if l, m, n are prime and $l \neq n$ we can always find **infinite** totally dual hypermaps of type (l, m, n), with alternating groups as monodromy groups.

If p, q, r are not all prime the alternating groups A_n, being factor groups of the triangle group (p, q, r), might correspond to hypermaps of type (p', q', r') with $p'|p$, $q'|q$ and $r'|r$ and not always type (p, q, r).

However, we can easily find some families of totally dual hypermaps with all elements having alternating groups as monodromy group:

Example 5.4.1. Totally dual hypermap of order $\frac{t!}{2}$ and

a) of type $(3, t, t)$, if t is odd and > 3;

b) of type $(3, t-2, t-1)$ if t is even and > 4.

For case a) we take: $A_t = \langle x, y|x = (123), y = (12...t)\rangle$. Then $xy = (134...t2)$ and $ord(xy) = t$. For case b): $A_t = \langle x, y|x = (123), y = (2...t)\rangle$. Then $xy = (12)(34...t)$ and $ord(xy) = t - 2$.

5.5 Genera, hypermaps and duality

If G is the monodromy group of an o.r. hypermap, then we can easily calculate its genus g using the formula:

$$-\mathcal{X} = 2g - 2 = |G| - V - E - F$$

where V, E and F are the number of hypervertices, hyperedges and hyperfaces, respectively and $\mathcal{X}$ is the Euler characteristic . But in an o.r. hypermap of type (l, m, n) we have $|G| = mE = lV = nF$; therefore, the previous formula can be written in the following way:

$$-\mathcal{X} = 2g - 2 = |G|(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}).$$

Hence:

$$g = |G|(\frac{1}{2} - \frac{1}{2l} - \frac{1}{2m} - \frac{1}{2n}) + 1$$

For maps, it can be slightly simplified:

$$g = |G|(\frac{1}{4} - \frac{1}{2l} - \frac{1}{2n}) + 1$$

Example 5.5.1. One of the most simple self-dual (hyper)maps of genus 2 is the one we can get by identifying the opposite sides of the octagon This map has one vertex of valency 8, one face of valency 8, four edges and its orientation-preserving automorphism group is C_8, the rotations around the vertex (Figure 5.3).

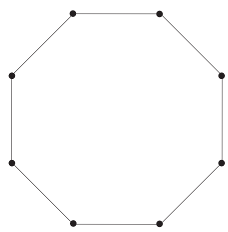


Figure 5.3: Self-dual hypermap of genus 2 (opposite sides of the octagon identified).

In fact, for each integer g there is a self dual o.r. (hyper)map with genus g. It is well known that we can find a map of genus g of type $\{4g, 4g\}$ for every

$g > 0$. Following a well known example (see, for instance, *Group Actions on Graphs, Maps and Surfaces*, a summary of a short course of lectures given by Marston Conder at the Group St Andrews conference in 2001, [14]), let G be the dihedral group of order $8g$ with generators u, v of order 2 and $4g$.

$$G =< u, v | u^2 = v^{4g} = (uv)^2 = 1 >$$

If $r_0 = u$, $r_1 = uv$ and $r_2 = uv^{2g}$ we have: $r_0 r_1 = v$ and $r_1 r_2 = v^{2g-1}$ of order $4g$ and $r_2 r_0 = v^{2g}$ of order 2. The map $\mathcal{M} = (G, r_0, r_1, r_2)$ is a g-sheeted covering of the torus map branched over its single vertex and its single face-center and is orientable since $\langle r_0 r_1, r_1 r_2 \rangle = \langle v \rangle$ has index 2 in $G = D_{8g}$. The hypermap is chiral and its genus is equal to:

$$|D_{8g}|(\frac{1}{8} - \frac{1}{16g} - \frac{1}{16g}) + 1 = 8g(\frac{1}{8} - \frac{2}{16g}) + 1 = g.$$

$\mathcal{M}$ is then a map of type $\{4g, 4g\}$ and of genus g. The same can be said for the associated oriented hypermap:

$$\overline{\mathcal{M}} = (\overline{G}, r_o r_1, r_1 r_2) = (\langle v \rangle, v, v^{2g-1}).$$

This o.r. map is self dual since $(4g, 2g-1) = 1$ and has also genus g.

The preceding example is simply the case $g = 2$ of this construction (Figure 5.3). If we identify the opposite edges of a $4g$-gon, as it is done for genus 2, we can get a self-dual hypermap on a surface of genus g .

If $g = 0$ (the sphere) the only o.r. hypermaps are the Platonic solids and the two infinite families of types $(2, 2, n)$, $(n, 1, n)$ and their duals (see, for instance, [8]). Some of these are self-dual and others totally dual or with intermediate duality index.

Example 5.5.2. The tetrahedron (with A_4 as monodromy group) is a self-dual map on the sphere (see Figure 5.4).

On the other hand, the cube and the octahedron (its dual) are examples of platonic solids that give rise to totally dual (hyper)maps of types $(3, 2, 4)$ and $(4, 2, 3)$, respectively. The same can be said about the icosahedron and dodecahedron, of types $(3, 2, 5)$ and $(5, 2, 3)$, respectively. These are just examples

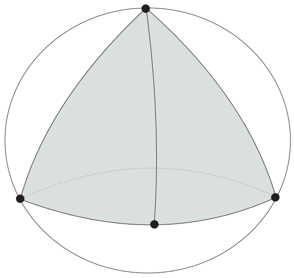


Figure 5.4: Tetrahedron on the sphere.

associated to the platonic solids but, on the sphere, it is possible to find an infinite number of totally dual hypermaps:

$$\mathcal{H} = (\mathbb{Z}_n = \langle t \rangle, t, 0)$$

These having a single hypervertex of valency n, n hyperfaces of valency 1 and one hyperedge of valency n. An infinite number of self-dual hypermaps on the sphere can also be obtained by using only dihedral groups as monodromy groups:

$$\mathcal{H} = (D_{2m}, x, y) \quad \text{with}$$

$$D_{2m} = \langle x, y | x^2, y^2, (xy)^m = 1 \rangle$$

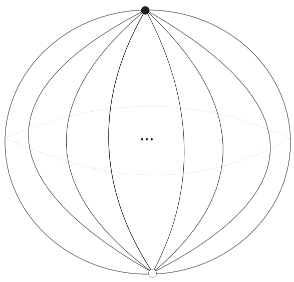


Figure 5.5: Totally dual hypermap on the sphere (Walsh representation)

The same can be said for hypermaps of intermediate index (again, we have an infinite number of these) but a stronger statement can be made:

Theorem 5.5.1. *In the sphere we can find a non totally dual (hyper)map of each duality index d.*

Proof: Let $\mathcal{H} = (D_{4d}, x, y)$. with

$$D_{4d} = \langle x, y | x^{2d} = y^2 = (xy)^2 = 1 \rangle.$$

Then $D_{4d}/N \cong V_4$ with $|N| = \frac{4d}{4} = d$ and the genus of $\mathcal{H}$ is

$$4d(-\frac{1}{4d} + \frac{1}{4} - \frac{1}{4}) + 1 = 0. \quad \square$$

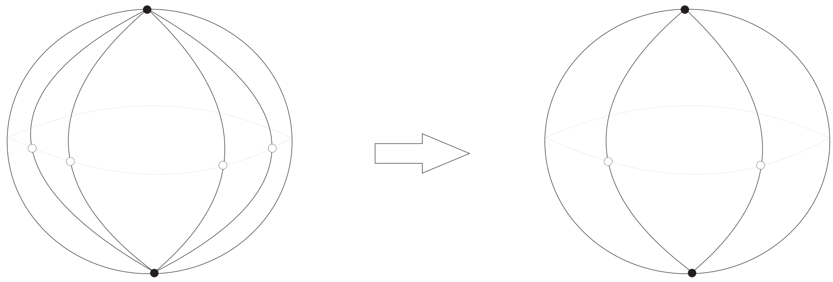


Figure 5.6: (D_8, x, y) of type $(4, 2, 2)$ collapsing to (V_4, x, y), a self-dual hypermap of type $(2, 2, 2)$.

A hypermap $\mathcal{H} = (G, x, y)$ of type (l, m, n) that has genus 1 (hypermap on the torus) must satisfy:

$$g = |G|(\frac{1}{2} - \frac{1}{2l} - \frac{1}{2m} - \frac{1}{2n}) + 1 = 1.$$

Which is the same as saying that:

$$\frac{1}{2} - \frac{1}{2l} - \frac{1}{2m} - \frac{1}{2n} = 0.$$

Therefore, on the torus, there are only three (infinite) families of o.r. hypermaps (and their duals), whose types are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

We have already shown one example of a self-dual hypermap on the torus: $(\mathbb{Z}_4, 1, 1)$ of type $(4, 2, 4)$. Another example would be the hypermap $(A_3, (123), (123))$ of type $(3, 3, 3)$. If

$$G = \Delta(3, 3, 3) = \langle x, y | x^3 = y^3 = (xy)^3 = 1 \rangle$$

then the finite quotients of this group, by torsion-free normal subgroups, will give hypermaps of type $(3, 3, 3)$ and of genus 1. In fact, an infinite number of self-dual hypermaps can be constructed on the torus (see Figure 5.7).

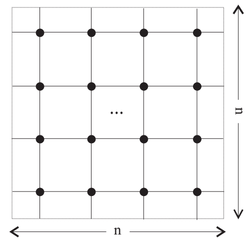


Figure 5.7: Self-dual maps of type $\{4, 4\}$ on the torus, with n^2 faces.

Totally dual hypermaps can also be found on the torus. For instance, $(\mathbb{Z}_4, 2, 1)$ is a totally dual hypermap of type $(2, 4, 4)$[4].Or we might take the group $G_n = \langle\alpha, \beta\rangle \leq S_{5+3n}$, for $n \in N$ with

$$\alpha = (1, 2)$$

$$\beta = (3, 4, 5)(6, 7, 8)...(3 + 3n, 4 + 3n, 5 + 3n)$$

Then $ord(\alpha) = 2$, $ord(\beta) = 3$ and $ord(\alpha\beta) = 6$. It follows that the hypermap $\mathcal{H} = (G_n, \alpha, \beta)$ is totally dual of type $(2, 6, 3)$. We can easily obtain other totally dual hypermaps on the torus by taking finite quotients, by torsion-free normal subgroups, of the universal hypermap of type $(2, 6, 3)$.

However, not all hypermaps on the torus are self-dual or totally dual. Examples of hypermaps with intermediate duality index can also be found: $(\mathbb{Z}_6, 1, 3)$ of type $(6, 2, 3)$ has duality index 3 (see section 5.1.1).

For genus 2, one example of a totally dual hypermap is the hypermap of type $(3, 3, 4)$, described in [6], with $SL_2(3)$ as monodromy group (and that is obviously totally dual since 3 and 4 are coprime). Another one is the map of type $\{3, 8\}$ listed by Conder in [14].

For every $g \in \mathbb{N}$ we can get an infinite number of totally dual hypermaps of genus greater than g by using, for instance, Hurwitz maps (hypermaps of type $(3, 2, 7)$). As we have already mentioned, Higman proved that almost all

[4]all the o.r. hypermaps with monodromy group $\mathbb{Z}_4$ are self-dual or totally dual (see section 5.1)

alternating groups are quotients of the triangle group $\Delta(2,3,7)$. Because the genus is given by

$$\frac{1}{84}|A_n| + 1 = \frac{n!}{168} + 1,$$

we can make it as big as we want. Some other examples of totally dual Hurwitz maps can be obtained using Luchini's result in [39] but in this case with $SL_n(q)$ as monodromy group. Since explicit generators are given in [39], we can not only obtain a sequence of hypermaps of growing genus but also the description of those hypermaps.

We have already proved that we can always find a self-dual hypermap for every genus. To finish this section we will prove that we can also find, for each integer $g > 1$, a non self-dual hypermap of that genus g. In fact, we can do this only using hypermaps of a certain type:

Theorem 5.5.2. *For each integer $g > 1$ there is a non self-dual (and non chiral) hypermap of genus g with vertices, edges and faces of the same order.*

Proof: We take $t = 2g + 1$ and the hypermap $\mathcal{H} = (\mathbb{Z}_t, 1, t-2)$ (of type (t,t,t) since all elements 1, $t-2$ and $t-1$ are coprime to the odd integer t). Then the genus of the hypermap is equal to:

$$t(\frac{1}{2} - \frac{3}{2t}) + 1 = \frac{t-3}{2} + 1 = \frac{t-1}{2} = g$$

If there was an automorphism interchanging the two generators 1 and $t-2$ then $(t-2)^2 \equiv 1 \pmod{t}$, which means that $t^2 - 4t + 4 \equiv 1 \pmod{t}$. Hence if $t \neq 3$ there is no automorphism that interchanges the generators.

The mirror image of $\mathcal{H}$ is $\overline{\mathcal{H}} = (\mathbb{Z}_t, t-1, 2) \cong \mathcal{H}$. Therefore, $\mathcal{H}$ is not chiral.
$\square$

Chapter 6

Duality on Non Orientable Regular Hypermaps

6.1 Duality Index

If instead of orientably regular hypermaps we look at regular hypermaps (regardless their orientability), we must work with monodromy groups generated by 3 elements of order 2. If $\mathcal{H} = (G, r_0, r_1, r_2)$ we define the *duality operation* to be the one that interchanges r_0 and r_2. The *duality group* $D(\mathcal{H})$ of a regular hypermap $\mathcal{H}$ is the smallest normal subgroup of $Mon(\mathcal{H})$ such that $\mathcal{H}/D(\mathcal{H})$ is self-dual. As a consequence of this, a *totally dual hypermap* would be a hypermap $\mathcal{H}$ such that $D(\mathcal{H}) = Mon(\mathcal{H})$.

If $\mathcal{H}$ is orientable, then its hypermap subgroup is not only a subgroup of Δ but also a subgroup of Δ^+ and the hypermap will never be totally dual by our definition (it will have at most index $|Mon(\mathcal{H})|/2$). It follows that all totally dual regular hypermaps that we will present in this chapter are necessarily imbeddings of hypergraphs on non orientable surfaces.

Example 6.1.1. $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is the monodromy group of the self-dual regular hypermap $\mathcal{H} = (G, (1,0,0), (0,1,0), (0,0,1))$. In fact, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is the largest abelian group that works as a monodromy group of a regular hypermap.

Every finite non-abelian simple group apart from $PSU(3,3)$ can be generated by three involutions [42]. As a consequence of this, we can say that every finite non-abelian simple group (apart from $PSU(3,3)$) is the monodromy group of some self-dual or totally dual hypermap.

But not only finite non-abelian simple groups can be taken as monodromy groups of a regular hypermaps. For instance, every symmetric group S_n, for $n \geq 4$ (or $n \geq 2$ if we allow $r_i = r_j$ for $i \neq j$), can also be generated by three involutions. Moreover, it is possible to choose those three involutions r_0, r_1 and r_2 in a way that two of them commute[1]. If we choose:

$$r_0 = (1,2)(n,3)(n-1,4)...$$

$$r_1 = (1)(n,2)(n-1,3)...$$

$$r_2 = (1,2)$$

then $r_1 r_0 = (1,2,...,n)$, so $S_n \geq \langle r_0, r_1, r_2 \rangle \geq \langle r_1 r_o, r_2 \rangle = S_n$ and $r_0 r_2 = r_2 r_0$.

The same can be said for certain classical groups of sufficiently high rank, such as $SL_n(q)$ for $n \geq 14$, all groups $SU_n(q)$ for $n \geq 40$, the groups $Sp_{2n}(q)$, $\Omega^+_{2n}(q)$ and $\Omega_{2n+1}(q)$ for $n \geq 20$ and q odd, and the groups $SL_n(\mathbb{Z})$ for $n \geq 14$. The constructive proof of these results can be found in [59], including explicit generators for each case (two of them commuting).

Theorem 6.1.1. *For every even t there is a self-dual regular hypermap with monodromy group of order t.*

Proof: We just need to take the hypermap $\mathcal{H} = (D_t, x, y, x)$ with

$$D_t = \langle x, y | x^2 = y^2 = (xy)^{t/2} = 1 \rangle$$

the dihedral group of order t, if $t > 4$ (the cyclic group of order 2, if $t = 2$, or the Klein 4-group, if $t = 4$) . $\square$

Corollary 6.1.1. *For every even k there is a regular hypermap $\mathcal{H} = (G, x, y, z)$ such that $\frac{G}{d} = k$, with d being the duality index of the hypermap.*

Proof: We just need to choose a convenient self-dual hypermap (with duality index 1) with monodromy group of order k. $\square$

[1] which means that, with such triple of generators, the hypermap $\mathcal{H} = (S_n, r_o, r_1, r_2)$ is, in fact, a map.

Obviously, Theorem 6.1.1 is not true for k prime because the only possible monodromy groups of that order are the cyclic groups which are not generated by three involutions. In fact, any group generated by three involutions cannot have odd order. Hence the result of last theorem is false for t odd. If we now

take the Coxeter group (for $n \geq 2$):

$$W_{2,n,2} = \langle x, y, z : x^2 = y^2 = z^2 = (xy)^2 = (yz)^n = (zx)^2 = 1 \rangle$$

(the direct product of the dihedral group of order $2n$ with $\mathbb{Z}_2$) then $|W_{2,n,2}| = 4n$. Considering N as before:

$$W_{2,n,2}/N = \langle x, y, z : x^2 = y^2 = z^2 = (xy)^2 = (yz)^n =$$

$$= (zx)^2 = (zy)^2 = (yx)^2 = (xz)^2 = 1 \rangle$$

Then, if n is even, $W_{2,n,2}/N = \{1, x, y, z, xy, zy, zx, xyz\}$ and $|W_{2,n,2}/N| = 8$ ($W_{2,n,2}/N \cong \mathbb{Z}_2^3$).

If n is odd, $W_{2,n,2}/N = \{1, x, y, xy\}$ and $|W_{2,n,2}/N| = 4$.

This means that $\mathcal{H} = (W_{2,n,2}, x, y, z)$ is not totally dual. Moreover, for every d it is possible to have a regular hypermap non totally dual and with d as duality index. We just need to take the group $W_{2,2d,2}$ of order $8d$ because

$$|W_{2,2d,2}/N| = 8$$

and, consequently, $|N| = d$. This works as a proof of the following theorem:

Theorem 6.1.2. *For every $d \in \mathbb{N}$ there is a (non totally dual) regular hypermap with duality index equal to d.*

When dealing with orientably regular hypermaps it was easy to find totally dual hypermaps of any duality index (using cyclic groups as monodromy groups) but it is a little more difficult to get the same result for non totally dual hypermaps. For regular hypermaps, the non totally dual case is not very difficult to deal with (see the previous theorem). The same can be said for the totally dual regular hypermaps: it is not possible to find totally dual regular hypermaps of odd index (in a totally dual hypermap, the monodromy group coincides with the duality group). However the question can be asked for even numbers: is it possible to find totally dual hypermaps of any even duality index? The answer is: no. If we want to find totally dual regular hypermaps of index 6 we must look at monodromy groups of order 6. But the only finite groups of order 6 are: $\mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$ and S_3. The first of these groups cannot be generated by three involutions and S_3 gives always rise to self-dual regular hypermaps or regular maps with intermediate index[2] [23].

When we are looking for totally dual hypermaps, we often choose hypermaps with simple monodromy groups because these must be either self-dual or totally dual (removing the possibility to deal with hypermaps of intermediate duality index). But if we only use finite simple groups we will never be able to build a totally dual hypermap for each even index d because then $|Mon(\mathcal{H})| = d$ and there is not a finite simple group of each even order. In fact, there are huge gaps if we list the orders of finite simple groups. However, we can at least give some examples of totally dual regular hypermaps since any nonabelian finite simple group can be generated by two elements of different orders and can be used as the monodromy group of a totally dual regular hypermap.

Remark: In the proof of the previous theorem, we have shown several examples of Coxeter groups that can work as monodromy groups of hypermaps with intermediate duality index and just one example ($W_{2,2,2}$) of a self-dual hypermap. However, we can also provide other examples of self-dual hypermaps with a Coxeter group as monodromy group. For instance, if we take $W_{3,3,2} = \langle x, y, z : x^2 = y^2 = z^2 = (xy)^3 = (yz)^3 = (zx)^2 = 1\rangle$, a Coxeter group on 3 generators and of order 24, then $\mathcal{H} = (W_{3,3,2}, x, y, z)$ is self-dual (as can be confirmed, using GAP [23]).

We have already seen that it is not possible to find totally dual regular hypermaps for every even duality index. Nevertheless, we can build families of that kind of hypermaps for some special even numbers, namely when those indexes are equal to $n!$ or $n!/2$.

[2]For instance, let $S_3 = \langle x, y, z\rangle$. If $x = y = (12)$ and $z = (23)$, then, in S_3/N, we have $x = y = z$, so $S_3/N \cong C_2$. Therefore, that hypermap with monodromy group $S_3 = \langle x, y, z\rangle$ will have intermediate duality index.

Theorem 6.1.3. *For every $n \in \mathbb{N}$ and ≥ 4 there is a totally dual regular (hyper)map $\mathcal{H}$ with duality index $|D(\mathcal{H})| = n!$.*

Proof: Suppose $\mathcal{H} = (S_n, r_0, r_1, r_2)$, with $n \geq 4$.

If $n = 2k+1$ we can use the same generators as in [13]:

$$r_0 = (12)$$

$$r_i = (34)(56)...(2k-1 \quad k)$$

$$r_j = (23)(45)...(2k \quad 2k+1)$$

here with $j = 2$, $i = 1$ when k is even and $j = 1$, $i = 2$ when k is odd.

Then, because r_0 and r_2 are permutations with different sign, the hypermap $\mathcal{H}$ is not self dual. The same can be said about:

$$\mathcal{H}/A_n = (S_n/A_n, r_0A_n, r_1A_n, r_2A_n) = (S_n/A_n, r_0A_n, r_1A_n, A_n).$$

Hence, the hypermap $\mathcal{H}$ is totally dual and $|D(\mathcal{H})| = |S_n| = n!$.

If n is even ($n = 2k$) the method is similar.

Because two of the permutations commute, $\mathcal{H}$ is, in fact, a map.□

Theorem 6.1.4. *For every $n \in \mathbb{N}$ and ≥ 9 there is a totally dual regular hypermap $\mathcal{H}$ with duality index $|D(\mathcal{H})| = \frac{n!}{2}$.*

Proof: Let $n \geq 9$. To build the required totally dual regular hypermap of that index n, we will use a result published by Nuzhin in 1992 [48]. In that theorem, it is demonstrated, by explicitly showing the generators, that the Alternating groups A_n (for $n \geq 9$) can be generated by three involutions. Nuzhin also proves that two of them commute but we do not need that in our construction (although this adds an extra information about the hypermaps). All we need to do is to take:

$$\mathcal{H} = (A_n, r_0, r_1, r_2)$$

with

$$\mathbf{r_0}:$$

$$(14)(23)(n-2 \; n-1), \text{ if } n = 4k+3 \geq 11$$

$$(12)(34) \text{ otherwise.}$$

$$\mathbf{r_1}:$$

$$(12)(34)...(n-4\ n-3)(n-2\ n-1) \text{ if } n = 4k+1 \geq 9$$

$$(34)(56)...(n-3\ n-2)(n-1\ n) \text{ if } n = 4k+2 \geq 10$$

$$(12)(34)...(n-6\ n-5)(n-4\ n-3) \text{ if } n = 4k+3 \geq 11$$

$$(12)(34)...(n-3\ n-2)(n-1\ n) \text{ if } n = 4k \geq 12$$

$$\mathbf{r_2}:$$

$$(23)(45)...(n-3\ n-2)(n-1\ n) \text{ if } n = 4k+1 \geq 9$$

$$(23)(45)...(n-4\ n-3)(n-2\ n-1) \text{ if } n = 4k+2 \geq 10$$

$$(45)(67)...(n-3\ n-2)(n-1\ n) \text{ if } n = 4k+3 \geq 11$$

$$(23)(45)...(n-2\ n-1)(1\ n) \text{ if } n = 4k \geq 12$$

Because r_0 and r_2 have always different cyclic structures, there is no automorphism interchanging them. The hypermap $\mathcal{H}$ is then totally dual and$|D(\mathcal{H})| = \frac{n!}{2}$.

Since two of the permutations commute, $\mathcal{H}$ is, in fact, a map. $\square$

In the same paper, Nuzhin also presents three involutions that generate A_5 (with two of them commuting):

$$(12)(34),\ (14)(23),\ (23)(45).$$

However, these generators will give rise to a self-dual hypermap since the generators have all the same cyclic structure. Using GAP [23] to compute all the generating triples of involutions we realize that all of them are triples of elements with two disjoint transpositions. Therefore, all possible regular hypermaps with monodromy group A_5 are self-dual [23].

Note: A_6, A_7 and A_8 are not generated by three involutions [using GAP [23] or Nuzhin's paper again [48]].

As a final observation, we would like to add that the well known fact that there are infinitely many nonorientable regular maps of type $\{3, 7\}$ (hypermaps of type $(3, 2, 7)$), allow us to conclude that we can build infinitely many totally dual maps, on nonorientable surfaces, just using Hurwitz maps.

Appendix A

The group $PSL_2(q)$ and its subgroups

A.1 Properties of the group $PSL_2(q)$

The group $SL_2(q)$ is the group of the 2×2 matrices with determinant equal to $+1$ and entries in the finite field $GF(q)$, with $q = p^n$ for some $n \in \mathbb{N}$ and p prime. The *projective special linear group* $PSL_2(q)$ is the group obtained from the group $SL_2(q)$ on factoring by the scalar matrices. $PSL_2(q) = SL_2(q)/\{\pm I\}$ has order equal to $\frac{q(q^2-1)}{d}$, where $d = (2, q-1)$, and is simple for $q > 3$. An element $A \in PSL_2(q)$ is hyperbolic, parabolic or elliptic if the number of fixed points of A in the projective line $\Omega = P^1(q)$ is 2, 1 or 0, respectively.

Then, the order t of A:

- divides $\frac{q-1}{d}$ if A is hyperbolic,
- is p if A is parabolic,
- divides $\frac{q+1}{d}$ if A is elliptic

and, conversely, $PSL_2(q)$ contains elements of each such order t.

A.2 Hypermaps and the subgroups of $PSL_2(q)$

To know for which $q = p^n$, $PSL_2(q)$ (or $L_2(q)$ in another notation) is the epimorphic image of a triangle group, can help us to build hypermaps (making use of the relationship between subgroups of the triangle group and compact orientable hypermaps without boundary). Instead of $PSL_2(q)$, other finite groups could be chosen but the projective special linear groups are *rich* enough to provide a large amount of different hypermaps and there are also available good results about their subgroups [21] and their generating triples [40], which can be extremely helpful.

The aim of Macbeath's paper [40] is to classify the pairs $(A, B) \in G = PSL_2(q)$ and be able to decide what kind of subgroups they generate. Because each pair (A, B) determines a unique C such that $ABC = 1$, the problem is equivalent to that of classifying triples (A, B, C) such that $ABC = 1$.

That classification is very useful to our study of hypermaps because if these elements A, B and C generate the whole group $PSL_2(q)$ then there is an epimorphism

$$\gamma : \Delta(l, m, n) \longrightarrow PSL_2(q)$$

(where l, m and n are the orders of the elements A, B and C) that sends $a \mapsto A$, $b \mapsto B$ and $c \mapsto C$.

It is proved by Dickson [21] that every subgroup of $PSL_2(q)$ is either exceptional, affine or projective (however, these families are not disjoint).

- The *projective subgroups* of $PSL_2(p^n)$ are isomorphic to $PSL_2(p^t)$, where $t|n$, or isomorphic to $PGL_2(p^{2t})$ where $2t|n$.
- The *affine subgroups* are the ones that fix a point in $\Omega = P^1(q)$ (hence their order divides $\frac{q(q-1)}{d}$) together with the cyclic groups of order dividing $\frac{q+1}{2}$, generated by an elliptic element.
- Finally, the *exceptional subgroups* are the finite triangle groups: dihedral groups or isomorphic to A_4, S_4 or A_5.

Let $G_0 = SL_2(q)$. In Macbeath's notation [40], a G_0-triple, is a triple (A, B, C) where A, B, $C \in G_0$ and $ABC = 1$. This gives rise to

- a k-triple (α, β, γ) of elements of the finite field, where $\alpha = trA$, $\beta = trB$ and $\gamma = trC$
- a N-triple $(l, m, n) \in \mathbb{N}^3$, where l, m, n are the orders of the corresponding elements in $PSL_2(q)$.[1]

A k-triple (α, β, γ) is called *singular* or *nonsingular* according as the quadratic form

$$Q_{\alpha\beta\gamma}(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 + \alpha\eta\zeta + \beta\zeta\eta + \gamma\xi\eta$$

is singular or nonsingular. A G_0-triple is called singular or nonsingular according as the associated k-triple (trace A, trace B, trace C) is singular or nonsingular.

We say that a G_0-triple is *exceptional* if the associated N-triple is one of the following:

$$(2, 2, n); (2, 3, 3); (3, 3, 3); (3, 4, 4); (2, 3, 4);$$

$$(2, 5, 5); (5, 5, 5); (3, 3, 5); (3, 5, 5); (2, 3, 5)$$

Macbeath [40] proves:

Theorem A.2.1. *A k-triple (α, β, γ) is singular if and only if there is a commutative G_0-triple associated with the k-triple (α, β, γ).*

Theorem A.2.2. *A triple that is neither singular nor exceptional generates a projective subgroup of $PSL_2(q)$.*

Therefore, to show that a triple generates the whole $PSL_2(q)$ group, the following method can be used:

1) show that the G_0-triple (α, β, γ) is not exceptional;
2) show that none of the G_0-triples (α, β, γ) is commutative and hence cannot generate an affine subgroup;
3) prove that the projective subgroup generated by the triple must be the whole group $PSL_2(q)$.

[1]If $A \in G_0$ then $det(A) = 1$. If $\alpha = trA$ the order of the corresponding element in $PSL_2(q)$ is uniquely determined except if $\alpha = \pm 2$ when the order can be 1 or p.

A.3 Examples of epimorphisms

Those results, proved by Macbeath, are very useful to demonstrate, adapting a method developed in [30], the following theorem:

Theorem A.3.1. *There is an epimorphism* $\Delta(3,3,7) \longrightarrow PSL_2(13)$.

Proof: Let $\alpha = \pm 1$ and β be the traces of the elements of G_0 whose images in G have orders 3 and $7 = \frac{13+1}{2}$, respectively. Therefore, there is a G_0-triple (A, B, C) with associated k-triple (α, α, β) and, hence, a N-triple $(3, 3, 7)$. Since $7 > 5$, the corresponding subgroup H generated by the three elements in G cannot be exceptional. Moreover, since $7 > 3$ there is no commutative G_0-triple with N-triple $(3, 3, 7)$, so H cannot be affine. No proper projective subgroup of $PSL_2(13)$ has an element of order $7 = \frac{13+1}{2}$, so $H = PSL_2(13)$ and $PSL_2(13)$ is an image of $\Delta(3, 3, 7)$. $\square$

This proves that there are regular hypermaps of those types $(3, 3, 7)$ for which $PSL_2(13)$ is the automorphism group. Moreover, the subgroups of $PSL_2(13)$ will give rise to non regular hypermaps of the same type. Similar results can be obtained for other groups $\Delta(l, m, n)$, using the same technique.

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