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INDEPENDENCE AND HETEROGENEITY IN GAMES OF INCOMPLETE INFORMATION

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Abstract

This paper provides a sufficient condition for existence and uniqueness of equilibrium, which is in monotone pure strategies, in games of incomplete information. First, we show that if each player's incremental *ex post* payoff is uniformly increasing in its own action and type, and its type is sufficiently uninformative of the types of its opponents (independence), then its expected payoff satisfies a strict single crossing property in its own action and type, for any strategy profile played by its opponents. This ensures that a player's best response to any strategy profile is a monotone pure strategy. Secondly, we show that if, in addition, there is sufficient heterogeneity of the conditional density of types, then the best response correspondence is a contraction mapping. This ensures equilibrium existence and uniqueness. In contrast to existing results, our uniqueness result does not rely on strategic complementarities; this allows for a wider range of applications.

Keywords: Incomplete Information, Heterogeneity, Existence, Unique pure strategy equilibrium.

JEL classification: C72; D82.

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1 Introduction

This paper studies existence and uniqueness of equilibrium which is in monotone pure strategies in games of incomplete information. Several papers have established existence of pure strategy equilibria in incomplete information games under a variety of assumptions. For example, Milgrom and Weber (1985) show existence in games with a finite number of actions and (conditionally) independent types. Milgrom and Roberts (1990) and Vives (1990) work with supermodular games. The pure strategy equilibria in these games need not be monotone. A few papers have studied existence of monotone pure strategy equilibria: in particular, Athey (2001) and McAdams (forthcoming) (discussed further below). Finally, a strand of the literature has established that in a particular class of supermodular games (called global games), a unique equilibrium exists which is in monotone pure strategies; see Carlsson and van Damme (1993) and Frankel et al. (2003).

This paper provides a sufficient condition for existence and uniqueness of equilibrium in monotone pure strategies in a broad class of games of incomplete information. The class of games we consider includes most supermodular and all global games, for example; but is broader, since it does not require that players' actions are strategic complements. Our argument has two steps. First, we show that a player's incremental expected payoff satisfies a strict single crossing property in its own action and type, for any strategy profile of its opponents, if its incremental *ex post* payoff is uniformly increasing in its own action and type, and its type is sufficiently uninformative about the types of its opponents. Since the strict single crossing property holds for any strategy profile adopted by the opponents, each player's best response is a monotone pure strategy. Secondly, we show that if, in addition, there is sufficient heterogeneity of the conditional density of types, then the best response correspondence is a contraction mapping. This ensures that equilibrium exists and that it is unique.

The argument is easiest to see in an example with independent, private values and binary actions. Suppose that a player's payoff difference between the two actions is separable in two terms: the first is strictly increasing in the type of the player; the second depends on the actions of other players. Since the players are assumed to be independent, player i's type tells it nothing about the types of its opponents. Hence player i's expectation of the payoff difference is strictly increasing in its type, irrespective of its opponents' strategies; therefore in equilibrium, players use monotone pure strategies. To show that there exists a unique equilibrium in monotone pure strategies, we must show that there is a unique threshold type who is indifferent between the two actions. Consider a threshold player; and consider its estimate of the payoff effect of its opponents' actions. If this estimate is sufficiently insensitive to the threshold player's type, then the threshold player's expected payoff difference is strictly increasing in its type. Hence there can be only one solution to the indifference condition: if there were multiple solutions, the function would have to be decreasing at (at least) one of the solutions. If the conditional density of types is sufficiently flat, then threshold player's estimate of its opponents's action will not vary very much with its type.

These arguments can be extended to more general payoffs and for more general distributions that allow types to be dependent. For more general payoffs, we require that the incremental *ex post* payoffs are increasing in own action and type. For more general type distributions, we require that the likelihood of other players' types is not too sensitive to the type of an individual player. In the case when the conditional density is differentiable, this condition requires that the *Fisher information* is bounded above. Finally, a conditional density with a small upper bound ensures that the equilibrium correspondence is unique.

Athey (2001) establishes existence of monotone pure strategy equilibria. She shows that a single crossing condition—that each player's expected payoff satisfies monotone incremental returns in its own type given any *non-decreasing* strategy profile played by its opponents—ensures existence of equilibrium. She shows further that games in which *ex post* payoffs are supermodular in all players' actions and types, and in which types are affiliated, satisfy the single crossing condition. We also derive a single crossing condition; we show that a condition slightly stronger than supermodularity with respect *own* action and type, and uninformative types, ensures this single crossing condition. If these assumptions are supplemented by heterogeneity, then we can establish that the only equilibrium that exists is in monotone pure strategies.

The technical details of our argument are quite different from those of Athey (and extended by McAdams (forthcoming) to the case of multidimensional actions and types). The key step for both Athey and McAdams is to establish convexity of the best-response correspondence, in order to apply a fixed point theorem. In contrast, we use a contraction mapping argument. Our approach has a number of advantages. First, it gives both existence and uniqueness of equilibrium. Secondly, it seems to be a very flexible analytical approach. For example, we are able to accommodate the extension to multidimensional actions and types relatively easily; in contrast, as McAdams shows, multidimensional actions (in particular) present a challenge when establishing convexity of the best response correspondence. Thirdly, the contraction approach leads to parameteric restrictions that have a clear economic interpretation. In particular, the information requirements, in terms of independence and heterogeneity, are intuitive.

Our analysis helps to clarify the mechanism at work in a number of previous papers that have found, in a variety of situations, that heterogeneity can ensure uniqueness of equilibrium. For example, in a canonical two-by-two public good model in Fudenberg and Tirole (1991, pp. 211–213), there are two pure strategy equilibria in the common knowledge game. If the distribution of types satisfies certain conditions, there is only one equilibrium in the incomplete information game. One such condition is that the maximum value of the density is sufficiently small; following Grandmont (1992), this can be interpreted as requiring a sufficient degree of heterogeneity between the players. Burdzy et al. (2001) demonstrate that there can be a unique equilibrium in a model in which players face exogenous shocks, can change their action only occasionally, and are heterogeneous in the frequency with which they can change their action. Herrendorf et al. (2000) show how heterogeneity in the manufacturing productivity (rather than the information) of agents in a two-sector, increasing returns-to-scale model can remove indeterminacy and multiplicity of equilibrium. Glaeser and Scheinkman (2002) show that if there is not too much heterogeneity among players, then there can be multiple equilibria in social interaction games. In all of these papers, heterogeneity lays some part in ensuring the uniqueness of equilibrium. Our analysis shows exactly what form of heterogeneity is needed, and exactly what mechanism is at work when heterogeneity yields uniqueness.

An important alternative approach to establishing equilibrium uniqueness in incomplete information games concerns the class of games known as 'global games'. Global games are games of incomplete information whose type space is determined by the players each observing a noisy signal of an underlying state; see Carlsson and van Damme (1993), Morris and Shin (1998), and Morris and Shin (2002). If players' actions are strict strategic complements, there are 'dominance regions' (i.e., types for which there is a strictly dominant action), and players' signals are sufficiently informative about the true underlying state, then global games have a unique, dominance solvable equilibrium. (Existence of equilibrium is assured by the results of Milgrom and Roberts (1990) on supermodular games.)

An attractive feature of the global game approach is that a very small (informational) perturbation of a complete information model with multiple equilibria can yield a unique equilibrium. In contrast, our approach typically requires sufficiently large perturbations from the complete information case. The major advantage of our approach, relative to global games, is that we do not require strategic complementarities or dominance regions.

This allows our results to be used in a wider range of applications.[1]

The rest of the paper is structured as follows. In section 2, we analyze a simple model, based on a particular payoff function and the normal distribution, to make the basic points of the paper. We extend the analysis in section 3 to show how the conclusions can be generalized to other payoffs and distributions. In the initial version of the model, we follow the set-up of Athey; in particular, we assume that the action sets are finite and one-dimensional, and type sets bounded and one-dimensional. In section 3.2, we show how the analysis can be extended to relax these assumptions. Section 4 concludes. Longer proofs are in the appendix.

2 A Simple Model

Suppose that there is a continuum of players, of measure 1. There are two possible actions. The payoff to any player from action 0 is zero. The payoff to player i from action 1 is $t_i + g(n)$. t_i is player i's type, which is private information observed only by player i. It is drawn from a normal distribution with mean y and variance σ^2. Players' types are correlated—the degree of correlation between the types of player i and $j \neq i$ is $\rho \in [0, 1)$ (note that perfect correlation is ruled out). Hence when player i has a private type of t_i, its posterior of the type t_{-i} of any other player $-i$ is normally distributed

[1] Global games have been used to analyze currency attacks (see Morris and Shin (1998)) and the pricing of debt (see Morris and Shin (forthcoming)), to name only two examples. But there are many applications in which the assumption of strategic complementarity is inappropriate. For example, in industrial organization, it is reasonable that positive network effects might hold in a new market when a small number of firms have entered; but that the network effects become negative once too many firms enter and the market becomes crowded. In the Internet, each new web site, or the addition of information to an existing site, increases the value of the Internet to every existing user. However, as usage of the Internet grows, so does congestion. Goldstein and Pauzner (2002) study a model of bank runs based on Diamond and Dybvig (1983). In their model, an agent's incentive for early withdrawal of funds from a bank is non-monotonic in the number of agents withdrawing. The incentive is highest when the number of agents demanding withdrawal reaches the level at which the bank goes bankrupt; after that point, the incentive decreases. (Despite this lack of complete strategic complementarity, Goldstein and Pauzner are able to establish uniqueness of equilibrium.)

with mean $\rho t_i + (1-\rho)y$ and variance $\sigma^2(1-\rho^2)$. y, σ^2 and ρ are common knowledge. Finally, $n \in [0,1]$ is the proportion of players choosing action 1. $g : [0,1] \to \mathbb{R}$ is an interaction function, describing how a player's utility is affected by the actions of other players. We assume that it is continuous and bounded i.e., there exists a finite k such that $\sup_{n\in[0,1]} |g(n)| \leq k/2$.[2]

Consider any strategy profile played by all players other than i. This profile induces a distribution $s(t) : \mathbb{R} \to [0,1]$ that gives the proportion of players choosing action 1 for a given value of t. The expected utility gain for player i of choosing action 1, conditional on being type t_i, is then

$$\Delta U(t_i, s) \equiv t_i + \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} g(s(t)) \exp\left[-\frac{1}{2}\left(\frac{t-\rho t_i-(1-\rho)y}{\sigma\sqrt{1-\rho^2}}\right)^2\right] dt. \quad (1)$$

So player i's expected utility has two components: the expected stand-alone utility (the first term of the expression), and the expected interaction utility (the second term).

2.1 The Independent Case

Consider first the case of independent types: $\rho = 0$. Clearly in this case, the expected interaction utility does not depend on player i's type. It is then immediate that $\Delta U(t_i, s)$ is a strictly increasing function of t_i for any $s(\cdot)$. This means that the best response to any distribution $s(\cdot)$ induced by any strategy profile is a monotone pure strategy.

Proposition 1 *In the independent case, $\rho = 0$, the best response $BR(s)$ to any distribu-*

[2] The assumptions that types are unbounded and the interaction term is bounded means that there are *dominance regions* i.e., for sufficiently low (high) values of t_i, it is strictly dominant to choose action 0 (1) for any player i. Our argument does not rely on this feature; see section 3.2 for further discussion.

tion $s(\cdot)$ induced by any strategy profile is a monotone pure strategy, taking the form

$$BR(s) = \begin{cases} 0 & t < \tilde{t}, \\ 1 & t \geq \tilde{t} \end{cases}$$

for some $\tilde{t} \in (\underline{t}, \bar{t})$.

Hence, any equilibrium must be in monotone pure strategies. Given the threshold point $\tilde{t}$ in a symmetric monotone pure strategy equilibrium, the expected utility of a player of type $\tilde{t}$ is

$$\Delta U(\tilde{t}) \equiv \tilde{t} + \frac{1}{\sqrt{2\pi}\sigma}\left(\int_{-\infty}^{\tilde{t}} g(0) \exp\left[-\frac{1}{2}\left(\frac{t-y}{\sigma}\right)^2\right] dt \right.$$
$$\left. + \int_{\tilde{t}}^{\infty} g(1) \exp\left[-\frac{1}{2}\left(\frac{t-y}{\sigma}\right)^2\right] dt\right). \quad (2)$$

The equilibrium threshold point satisfies the equation

$$\Delta U(\tilde{t}) \triangleq 0. \quad (3)$$

MS show in the case of strict strategic complements (i.e., $g(\cdot)$ strictly increasing) that a necessary and sufficient condition for there to be a unique solution to equation (3) is that σ is sufficiently large i.e., that there is enough heterogeneity. A similar argument is given in HVW, who give a sufficient, but not necessary condition based on heterogeneity. The next proposition shows that the assumption of strategic complementarity is not needed for this result.

Proposition 2 *For any continuous and bounded interaction function $g(\cdot)$, in the independent case, there exists a $\sigma^* \geq 0$ such that if $\sigma > \sigma^*$, then there is a unique equilibrium.*

Proof. There is a unique rationalizable action for (almost) all types iff $d\Delta U(\tilde{t})/d\tilde{t} > 0$ for any $\tilde{t}$ at which $\Delta U(\tilde{t}) = 0$. Differentiation of equation (2) gives

$$\frac{d\Delta U(\tilde{t})}{d\tilde{t}} = 1 + \left(\frac{g(0) - g(1))}{\sqrt{2\pi}\sigma}\right) \exp\left[-\left(\frac{\tilde{t} - y}{\sigma}\right)^2\right].$$

Since $|g(0) - g(1)| \leq k$, a sufficient condition for $d\Delta U(\tilde{t})/d\tilde{t} > 0$ is

$$1 > \frac{k}{\sqrt{2\pi}\sigma}$$

which completes the proof. □

2.2 Positive Correlation

Now suppose that there is a degree of correlation: $\rho \in (0, 1)$. In this section, we derive joint conditions on heterogeneity σ, correlation ρ, and the interaction function bound k such that the best response of player i to any strategy profile played by all other players is a monotone pure strategy. Once this fact is established, sufficient heterogeneity again ensures uniqueness of equilibrium. Hence the basic mechanism that generates uniqueness in the case of independence extends to positive, but limited correlation.

Proposition 3 *If*

$$\sqrt{\frac{1-\rho^2}{\rho^2}} > \frac{k}{\sqrt{2\pi}\sigma}, \tag{4}$$

then the best response to any strategy profile is a monotone pure strategy.

Proof. See Appendix A.

In order to establish uniqueness of equilibrium in the correlated case, we now derive a condition for there to be a unique monotone pure strategy equilibrium, assuming that such an equilibrium exists. This result is stated in proposition 4; as in proposition 2, it basically requires sufficiently large heterogeneity (for any given values of ρ and k). We then combine the results of propositions 3 and 4 to give a sufficient condition for equilibrium uniqueness.

Proposition 4 *If*

$$\sqrt{\frac{1+\rho}{1-\rho}} > \frac{k}{\sqrt{2\pi}\sigma}, \tag{5}$$

and a monotone pure strategy equilibrium exists, then there is a unique monotone pure strategy equilibrium.

Proof. As in the proof of proposition 2, there is a unique threshold for (almost) all types iff $d\Delta U(\tilde{t})/d\tilde{t} > 0$ for any $\tilde{t}$ at which $\Delta U(\tilde{t}) = 0$, where

$$\Delta U(\tilde{t}) \equiv \tilde{t} + \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \left(\int_{-\infty}^{\tilde{t}} g(0) \exp\left[-\frac{1}{2} \left(\frac{t - \rho\tilde{t} - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \right] dt \right.$$
$$\left. + \int_{\tilde{t}}^{\infty} g(1) \exp\left[-\frac{1}{2} \left(\frac{t - \rho\tilde{t} - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \right] dt \right).$$

Differentiation shows that a sufficient condition for $d\Delta U(\tilde{t})/d\tilde{t} > 0$ is

$$1 > \frac{k}{\sqrt{2\pi}\sigma} \left(\frac{1-\rho}{\sqrt{1-\rho^2}} \right).$$

This completes the proof. □

Proposition 5 *If*

$$1 > \min\left[\sqrt{\frac{1-\rho^2}{\rho^2}}, \sqrt{\frac{1+\rho}{1-\rho}} \right] > \frac{k}{\sqrt{2\pi}\sigma} \tag{6}$$

then there is a unique equilibrium which is in monotone pure strategies.

Proof. To have a unique equilibrium in monotone pure strategies, equations (4) and (5) must both hold. Also observe that

$$\sqrt{\frac{1-\rho^2}{\rho^2}} \geq \sqrt{\frac{1+\rho}{1-\rho}} \quad \text{for } \rho \in [0, \tfrac{1}{2}]$$
$$\sqrt{\frac{1-\rho^2}{\rho^2}} \leq \sqrt{\frac{1+\rho}{1-\rho}} \quad \text{for } \rho \in [\tfrac{1}{2}, 1).$$

So condition (5) implies (4) for $\rho \in (0, \frac{1}{2}]$ while the converse holds for $\rho \in [\frac{1}{2}, 1)$. The result follows. □

Proposition 5 gives a joint condition on the model parameters ρ, σ and k that is sufficient for equilibrium uniqueness. The proposition is illustrated in figure 1, which gives an intuitive interpretation of the result.

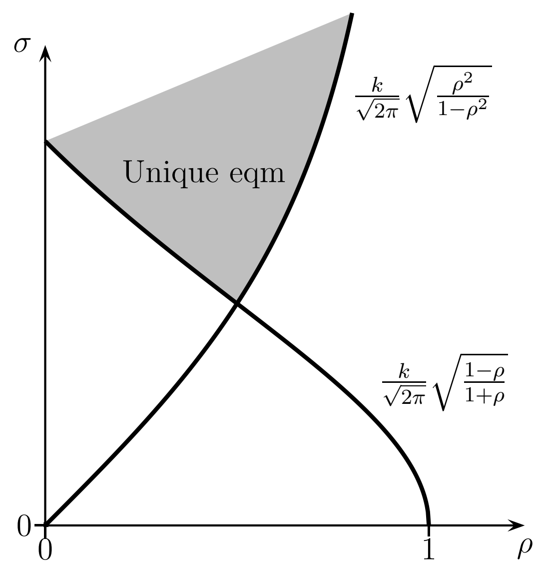

Figure 1: Proposition 5

Two facts stand out from the figure. First, the figure demonstrates the statements made in the introduction of the paper—that there is a unique equilibrium (in monotone pure strategies) if and only if there is sufficient heterogeneity of types. In figure 1, the sufficient condition requires the correlation between players' types to be sufficiently low and/or the variance of the prior distribution sufficiently high. (For certain parameter values, there is also a lower bound on the value of ρ.)

Secondly, our sufficient condition for uniqueness of equilibrium is stricter than that of MS. In the figure, the MS result gives a unique equilibrium for all parameter values lying in the area under the downward-sloping curve. We require in addition that parameter values lie in the area beneath the upward-sloping line. But, in contrast to MS, we do not require that players' actions are strategic complements—proposition 5 holds for any bounded interactions between the players. So, while our sufficient condition is indeed stricter than MS's when actions are strategic complements, it is less strict in the sense that it applies to a larger class of games.

These observations highlight the mechanism at work here: the conditions ensure that a monotone pure strategy is a best response to all other strategies; and that there is a unique monotone pure strategy equilibrium.

3 The General Model

The simple model establishes the role that independence, and hence small correlation, plays in ensuring equilibrium uniqueness. There is a possibility, however, that the conclusions depend on the simplifying assumptions of the model. In this section, we extend the model in a few directions to show that this is not the case. In particular, we allow for a more general payoff structure and distribution of types.

3.1 Finite and Single-Dimensional Action Games

Consider a game of incomplete information between I players, $i \in I \equiv \{1, \ldots, I\}$, where each player first observes its own type, $t_i \in T_i \equiv [\underline{t}_i, \bar{t}_i] \subset \mathbb{R}$ and then takes an action a_i from an action set A_i that is a closed, finite subset of the unit interval that contains 0 and 1 i.e., $\{0, 1\} \subseteq A_i \subset [0, 1]$. (The restriction to the unit interval is simply a normalization.) Let $\boldsymbol{a}$ denote an action profile: $\boldsymbol{a} = (a_1, \ldots, a_I)$; and let $A \equiv \times A_i$ the space of action profiles. A type profile and the space of type profiles are similarly defined as $\boldsymbol{t}$ and $T \equiv \times T_i$. Finally, let $\boldsymbol{a}_{-i}$ denote the profile of actions of all other players, and A_{-i} the space of all such action profiles. A similar notation is adopted for type profiles, strategy profiles, marginals etc..

Player i's payoff function is $u_i : A \times T \to \mathbb{R}$. We assume that

U1. Bounded Payoffs. The payoff function $u_i : A \times T \to \mathbb{R}$ is bounded and measurable.

Let

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, \boldsymbol{t}) \equiv u_i(a_i, \boldsymbol{a}_{-i}, \boldsymbol{t}) - u_i(a_i', \boldsymbol{a}_{-i}, \boldsymbol{t}).$$

The joint distribution of players' types is given by the probability measure η on the (Borel) subsets of T. The marginal distribution on each T_i is denoted η_i. We make the following assumption:

D1. Conditional Densities. The types have conditional densities with respect to the Lebesgue measure. The conditional density of $\boldsymbol{t}_{-i}$ given t_i, is denoted $f(\boldsymbol{t}_{-i}|t_i)$ for $i \in I$ and is strictly positive.

Players use behavioural strategies. A behavioural strategy for player i is a measurable function $\mu_i : \mathcal{A}_i \times T_i \to [0, 1]$ where $\mathcal{A}_i$ is the collection of Borel subsets of A_i, with the following properties: (i) for every $B \in \mathcal{A}_i$, the function $\mu_i(B, \cdot) : T_i \to [0, 1]$ is measurable; (ii) for every $t_i \in T_i$, the function $\mu_i(\cdot, t_i) : \mathcal{A}_i \rightarrow [0,1]$ is a probability measure. Hence when player i observes its type t_i, it selects an action in A_i according to the measure $\mu_i(\cdot, t_i)$. A pure strategy in behavioural form is simply a function that returns a probability measure that is concentrated on the graph of a classical pure strategy.[3]

Let $\boldsymbol{\mu}_{-i}$ denote the vector of behavioural strategies played by the opponents of player i. Assumption 1 allows the *interim* expected payoff of player i (i.e., when it knows its type t_i and has chosen its action a_i) to be written as:

$$U_i(a_i, t_i; \boldsymbol{\mu}_{-i}) = \int_{T_{-i}} \int_{A_{-i}} u_i(\boldsymbol{a}, \boldsymbol{t}) \prod_{j \neq i} d\mu_j(\cdot, t_j) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}.$$

We make a further assumption on payoff functions:

U2. Uniformly Positive Sensitivity to Own Action and Type. There is a $\delta \in (0, \infty)$ such that for all $a_i \geq a_i'$, $t_i \geq t_i'$, $\boldsymbol{a}_{-i}, \boldsymbol{t}_{-i}$ and $i \in I$,

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \geq \delta(a_i - a_i')(t_i - t_i').$$

Assumption U2 essentially requires that a higher type makes a higher action more appealing to a player. It is similar to, but stronger than, an assumption that a player's payoff function $u_i(a_i, \boldsymbol{a}_{-i}, \boldsymbol{t})$ is supermodular in (a_i, t_i).[4] In our case, supermodularity of u_i in (a_i, t_i) implies that $\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) \geq \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i})$; clearly, therefore, the

[3] An alternative approach would use distributional strategies. A distributional strategy for player i is a probability measure μ_i on $A_i \times T_i$ such that the marginal distribution on T_i is η_i i.e., $\mu_i(A_i \times S) = \eta_i(S)$ for any Borel subset S of T_i; see Milgrom and Weber (1985). As Milgrom and Weber show, there is a many-to-one mapping from behavioural strategies to distributional strategies. In fact, there is little difference between the two approaches here, since we establish quickly (see theorem 1) that in equilibrium, only monotone pure strategies are used. It is slightly more convenient, however, to use behavioural strategies.

[4] Let X be a lattice i.e., a partially ordered set that includes both the meet $\wedge$ (the greatest lower bound) and join $\vee$ (the least upper bound) of any two elements in the set. A function $h : X \rightarrow \mathbb{R}$ is supermodular if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, $h(\boldsymbol{x} \vee \boldsymbol{y}) + h(\boldsymbol{x} \wedge \boldsymbol{y}) \geq h(\boldsymbol{x}) + h(\boldsymbol{y})$. In the case that h is twice

uniform boundedness assumption is stronger. Nevertheless, the assumption is satisfied in a large number of games, including most supermodular games (see Athey (2001) for a longer discussion of this class of games); we note in passing that global games belong to this class.

In addition, assumption U1 and the finiteness of action sets assumed in this section imply Lipschitz conditions, expressed in the following corollary (which is stated without proof, as the statements are immediate).

Corollary 1 *Assumption U1 and finite action sets imply that*

U3. Uniformly Bounded Sensitivity to Own Action. *For each* $\boldsymbol{a}_{-i}$ *and* $\boldsymbol{t}$*, there is an* $\omega \in (0, \infty)$ *such that for all* $a_i \geq a_i'$ *and* $i \in I$*,*

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, \boldsymbol{t}) \leq \omega(a_i - a_i').$$

U4. Uniformly Bounded Sensitivity to Opponents' Action. *There is a* $\kappa \in (0, \infty)$ *such that for all* $a_i \geq a_i', \boldsymbol{t}$ *and* $i \in I$*,*

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, \boldsymbol{t}) - \Delta u_i(a_i, a_i', \boldsymbol{a}'_{-i}, \boldsymbol{t}) \leq \kappa(a_i - a_i')\|\boldsymbol{a}_{-i} - \boldsymbol{a}'_{-i}\|$$

where $\|\boldsymbol{a}_{-i} - \boldsymbol{a}'_{-i}\| \equiv \max_{j \neq i} |a_j - a_j'|$.

In this section, conditions U3 and U4 are consequences of previous assumptions. In section 3.2 when we consider games with a continuum of actions, an additional (continuity) assumption must be made.

differentiable, h is supermodular if and only if

$$\frac{\partial^2}{\partial x_i \partial x_j} h(\boldsymbol{x}) \geq 0$$

for all i, j; see Topkis (1998).

We make the following assumptions about the conditional density:

D2. There is a $\iota \in (0,\infty)$ such that for any $t_i > t_i'$ and $i \in I$, $\sqrt{I(t_i,t_i')} \leq \iota(t_i - t_i')$, where

$$I(t_i,t_i') \equiv \mathrm{Var}_{T_{-i}}\left(\frac{f(\boldsymbol{t}_{-i}|t_i) - f(\boldsymbol{t}_{-i}|t_i')}{f(\boldsymbol{t}_{-i}|t_i)}\right).$$

D3. There is a $\nu \in [0,\infty)$ such that $f_j(t_j|t_i) \leq \nu$ for all $i,j \in I$ and $j \neq i$ where

$$f_j(t_j|t_i) = \int_{\times_{k\neq i,j} T_k} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}.$$

The function defined in assumption D2 is the expectation of the square of a likelihood ratio:

$$\mathbb{E}_{T_{-i}}\left[\left(\frac{f(\boldsymbol{t}_{-i}|t_i')}{f(\boldsymbol{t}_{-i}|t_i)}\right)^2\right],$$

and so is a measure of *differential information*. In the case that the conditional density $f(\boldsymbol{t}_{-i}|t_i)$ is differentiable in t_i, the function is related to the *Fisher information* of a player's type about the types of the opponents. To see this, consider the limit as $t_i' \to t_i$:

$$\lim_{t_i \to t_i'} \frac{I(t_i,t_i')}{t_i - t_i'} \to \mathcal{I}(t_i) \equiv \mathrm{Var}_{T_{-i}}\left(\frac{\partial \ln f(\boldsymbol{t}_{-i}|t_i)}{\partial t_i}\right).$$

$\mathcal{I}(t_i)$ is the variance of a score function and so is the Fisher information, measuring how sensitive the likelihood of other players' types is to the type of player i. Hence assumption D2 bounds the Fisher information in the model.

Assumption D3 introduces a particular type of heterogeneity, in terms of the upper bound ν on the conditional density. This condition is similar to the one used by Grandmont (1992): we, like him, require the density function to be sufficiently flat.

In the next lemma (the proof of which is in the appendix), we derive a sufficient condition that ensures that a player's interim expected payoff function satisfies the strict single crossing condition. We then use this property in theorem 1 to argue that all players use monotone pure strategies.

Lemma 1 *Given assumptions U1–U3 and D1–D2, if* $\iota < \delta/\omega$*, then player* i*'s (interim) expected payoff satisfies the strict single crossing property in* (a_i, t_i) *for any* $\boldsymbol{\mu}_{-i}$ *i.e.,* $U_i(a_i, t_i', \boldsymbol{\mu}_{-i}) \geq U_i(a_i', t_i', \boldsymbol{\mu}_{-i})$ *implies* $U_i(a_i, t_i, \boldsymbol{\mu}_{-i}) > U_i(a_i', t_i, \boldsymbol{\mu}_{-i})$ *for all* $a_i > a_i'$ *and* $t_i > t_i'$*.*

Proof. See Appendix B.

Theorem 1 *Given assumptions U1–U3 and D1–D2, if* $\iota < \delta/\omega$*, then the best response of player* i *to any profile of opponents' strategies is a monotone pure strategy.*

Proof. The action set A_i is totally ordered (because $\{0, 1\} \subseteq A_i \subset [0, 1]$), implying that $U_i(a_i, t_i, \boldsymbol{\mu}_{-i})$ is quasisupermodular in a_i. Moreover, A_i is independent of t_i, and $T_i \in \mathbb{R}$ is also totally ordered. Finally, $U_i(a_i, t_i, \boldsymbol{\mu}_{-i})$ satisfies the strict single crossing property when $\iota < \delta/\omega$, from lemma 1. Therefore by the Monotone Selection Theorem 4′ of Milgrom and Shannon (1990),

$$s_i^*(t_i, \boldsymbol{\mu}_{-i}) = \arg\max_{a_i \in A_i} U_i(a_i, t_i, \boldsymbol{\mu}_{-i})$$

is monotone non-decreasing in t_i. (The strict single crossing property implies that there is indifference only on sets of measure zero.) □

The sufficient condition in theorem 1 ensuring that each agent plays a monotone pure strategy is stronger than that found in the simple model of section 2 (see proposition 2). The Fisher information with the normal distribution is

$$I(t_i) = \frac{\rho^2}{\sigma^2(1 - \rho^2)};$$

in contrast, the sufficient condition in proposition 4 for the normal distribution bounds

$$\frac{\rho^2}{2\pi\sigma^2(1-\rho^2)}.$$

The factor of 2π that does not appear in the bound in this section means that the sufficient condition in theorem 1 is more demanding. Nevertheless, it is doing much the same work as the condition in proposition 4. Both require that a player's type tells it sufficiently little about the types of other players—in the case of proposition 4, by ensuring that heterogeneity is sufficiently large and/or correlation sufficiently small; in the case of theorem 1, by bounding the Fisher information.

The assumptions required for theorem 1—in particular, assumptions U2 and D2—can be contrasted to the conditions used by Athey (2001). In both papers, the first step is to establish that an expected payoff satisfy a single crossing property in incremental returns (SCP-IR).[5] Athey imposes such an assumption from the outset, when all other players use non-decreasing strategies; she shows that the assumption is satisfied in games where agents' *ex post* utility is supermodular in $\boldsymbol{a}$ and $(a_i, t_j),\ j \in I$ and types are affiliated (see Athey (2001, theorem 3)). In contrast, we assume that the *ex post* utility function u_i satisfies a condition slightly stronger than supermodularity in *own* action and type, (a_i, t_i), and that types are not too associated. We can then show that the expected payoff satisfies a SCP-IR for *any* strategy profiles of opponents.

The second step is to show that there is a unique equilibrium in monotone pure strategies. A sufficient condition for this is given in the next theorem.

Theorem 2 *Given assumptions U1–U4 and D1–D3 hold, if*

$$\iota + \frac{\kappa\nu}{\lambda\omega} \leq \frac{\delta}{\omega} \tag{7}$$

[5]A function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies single crossing of incremental returns in (x, θ) if, for all $x_H > x_L$ and $\theta_H > \theta_L$, $h(x_H, \theta_L) - h(x_L, \theta_L) \geq (>)0$ implies $h(x_H, \theta_H) - h(x_L, \theta_H) \geq (>)0$. See Milgrom and Shannon (1990).

where $\lambda < 1$, then there is a unique equilibrium, which is in monotone pure strategies.

Proof. See Appendix C.

Note that compared to theorem 1, which requires only that ι is less than δ/ω, the sufficient condition in theorem 2 is stricter.

What is condition (7) ensuring? It does the two things that were illustrated in the simple model in section 2. First, it ensures that a player's own type dominates interaction effects in payoff terms enough to make any best response a non-decreasing pure strategy. Roughly speaking, if condition (7) is satisfied, then each player places more weight on its own type than on the possible actions of its opponents when choosing its best action. Secondly, the condition ensures that there is a unique equilibrium in monotone pure strategies. It does so by using in the general case the mechanism that was used in the binary action case. In order for there to be multiple equilibria in non-decreasing strategies, it must be that there are multiple values of a player's type that leaves that player indifferent between the two actions. The direct effect of a player's type is monotonic: the utility difference between the actions increases with type, other things equal. So, in order for there to be multiple equilibria, the indirect effect, operating through the player's assessment of its opponents' actions, must dominate. Condition (7) ensures that the direct, own-type effect is sufficiently strong; or that the interaction effect is sufficiently weak; or that the player's type is sufficiently uninformative about the types (and hence likely action) of others. It therefore ensures that the direct effect dominates and multiplicity is not possible.

It is worth comparing condition (7) with condition (5) established in proposition 4. Recall that there, a contraction mapping was found for monotone pure strategies when

$$\frac{\delta}{\kappa} > \frac{1}{\sigma\sqrt{2\pi}}\sqrt{\frac{1-\rho}{1+\rho}}$$

in the private value case. (In fact, $\delta = 1$ in the simple model; it is written here as a general parameter for comparability.) Condition (7) requires that

$$\frac{\delta}{\kappa} \geq \frac{1}{\sigma} \frac{\rho}{\sqrt{1-\rho^2}} + \frac{2}{\lambda \sigma \sqrt{2\pi} \sqrt{1-\rho^2}}$$

where the expressions for the Fisher information and the maximum value of the density of the normal have been used. Condition (7) therefore implies condition (5) if

$$\frac{1}{\sigma} \frac{\rho}{\sqrt{1-\rho^2}} + \frac{2}{\lambda \sigma \sqrt{2\pi} \sqrt{1-\rho^2}} > \frac{1}{\sigma \sqrt{2\pi}} \left(\sqrt{\frac{1-\rho}{1+\rho}} \right)$$

i.e., $\rho(1+\sqrt{2\pi}) > 1 - 2/\lambda$, which certainly holds since $\lambda < 1$. In summary: the sufficient condition in theorem 2 is stricter than the sufficient condition in proposition 4.

Finally, we note that Athey (2001, p. 879) commented that "[t]here is not a global "contraction mapping" theorem". We agree with this observation: assumptions U2–U4 and D2–D3 are restrictive, but needed if a contraction is to be established. As we mentioned in the introduction, we make stronger assumptions than Athey and so obtain stronger results.

3.2 Extensions

In this section, we consider how the sufficient condition for equilibrium uniqueness established in theorem 2 stands up to various extensions of the model.

Consider first the extension to a continuum of actions for each player, so that $A_i = [0,1]$, $i \in I$. The argument of Athey (2001, theorem 2) can be used in a direct way to establish the uniqueness of equilibrium in this case. One extra assumption is required:

U5. Payoff Continuity. Each $u_i(\boldsymbol{a}, \boldsymbol{t})$ is continuous in $\boldsymbol{a}$.

Note that in this case, the Lipschitz conditions U3 and U4 are implied by assumptions

U1 and U5. With this assumption, the conditions in theorem 2 ensure that there is a unique equilibrium in monotone pure strategies.

We have assumed that the type sets T_i are bounded: $T_i \equiv [\underline{t}_i, \bar{t}_i] \subset \mathbb{R}$. If this assumption does not hold, then the metric used to establish the contraction (see the proof of theorem 2 in the appendix) is not well-defined. In this case, we need an additional assumption:

U6. Limit Dominance. There exist $\underline{t}_i, \bar{t}_i \in T_i$ such that

(a) $\Delta u_i(0, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) > 0$ for all $a_i' \neq 0$, $\boldsymbol{a}_{-i} \in A_{-i}$, $\boldsymbol{t}_{-i} \in T_{-i}$, and $t_i \leq \underline{t}_i$,

(b) $\Delta u_i(1, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) > 0$ for all $a_i' \neq 1$, $\boldsymbol{a}_{-i} \in A_{-i}$, $\boldsymbol{t}_{-i} \in T_{-i}$, and $t_i \geq \bar{t}_i$.

With this assumption, the previous arguments again apply.

Suppose now that the type and action sets are multi-dimensional (c.f., McAdams (forthcoming)). Let the common support of types be $T = [\underline{t}, \bar{t}]^h \subset \mathbb{R}^h$ for some finite h; and the common action set A of all players be a finite sublattice of k-dimensional Euclidean space with respect to the product order on $\mathbb{R}^k$, where we normalize so that $\{0,1\}^k \subseteq A_i \subset [0,1]^k$. A typical action for player i is $a_i \equiv (a_i^1, \ldots, a_i^k)$; a typical action profile is $\boldsymbol{a} \equiv (a_1, \ldots, a_I)$.

Some of the previous assumptions have to be restated in straightforward ways:

U2'. Uniformly Positive Sensitivity to Own Type. There is a $\delta \in (0, \infty)$ such that for all $a_i \geq a_i'$, $t_i \geq t_i'$, $\boldsymbol{a}_{-i}, \boldsymbol{t}_{-i}$ and $i \in I$,

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \geq \delta \max_l (a_i^l - a_i'^l) \max_p (t_i^p - t_i'^p).$$

U3'. Uniformly Bounded Sensitivity to Own Action. For each $\boldsymbol{a}_{-i}$ and $\boldsymbol{t}$, there is an $\omega \in (0, \infty)$ such that for all $a_i \geq a'_i$ and $i \in I$,

$$\Delta u_i(a_i, a'_i, \boldsymbol{a}_{-i}, \boldsymbol{t}) \leq \omega \max_l (a^l_i - a'^l_i).$$

U4'. Uniformly Bounded Sensitivity to Opponents' Action. There is a $\kappa \in (0, \infty)$ such that for all $a_i, a'_i, \boldsymbol{t}$ and $i \in I$,

$$\Delta u_i(a_i, a'_i, \boldsymbol{a}_{-i}, \boldsymbol{t}) - \Delta u_i(a_i, a'_i, \boldsymbol{a}'_{-i}, \boldsymbol{t}) \leq \kappa \max_l (a^l_i - a'^l_i) \|\boldsymbol{a}_{-i} - \boldsymbol{a}'_{-i}\|$$

where $\|\boldsymbol{a}_{-i} - \boldsymbol{a}'_{-i}\| \equiv \max_{j \neq i} \max_l |a^l_j - a'^l_j|$.

D2'. There is a $\iota \in (0, \infty)$ such that for any $t_i > t'_i$ and $i \in I$, $\sqrt{I(t_i, t'_i)} \leq \iota \max_p (t^p_i - t'^p_i)$, where

$$I(t_i, t'_i) \equiv \text{Var}_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i) - f(\boldsymbol{t}_{-i}|t'_i)}{f(\boldsymbol{t}_{-i}|t_i)} \right).$$

We make the additional assumption:

U7. Quasi-supermodularity. $u_i(a_i, \boldsymbol{a}_{-i}, \boldsymbol{t})$ is quasi-supermodular in $a_i \in A_i$ for all $\boldsymbol{a}_{-i} \in A_{-i}$, $\boldsymbol{t} \in T$ and $i \in I$.[6]

Quasi-supermodularity expresses a weak kind of complementarity between the choice variables.

We are then able to extend theorems 1 and 2 to the multi-dimensional case. In the one-dimensional case, the appropriate notion was a monotone pure strategy. In the multi-dimensional case, this generalizes to an *isotone* pure strategy: in an isotone pure

[6] A function $h : X \to \mathbb{R}$ on a lattice X is quasi-supermodular if (i) $h(x) \geq h(x \wedge y)$ implies $h(x \vee y) \geq h(y)$ and (ii) $h(x) > h(x \vee y) > h(y)$.

strategy, $t_i > t'_i$ implies $a_i(t_i) \geq a_i(t'_i)$ i.e., the action chosen by a type that is higher in all dimensions is no lower, in all dimensions.

Theorem 3 *Given assumptions U1, U2'–U3', U7, D1 and D2', if $\iota < \delta/\omega$, then the best response of player i to any profile of opponents' strategies is a isotone pure strategy.*

Theorem 4 *Given assumptions U1, U2'–U4', U7, D1, D2' and D3, if*

$$\iota + \frac{\kappa\nu}{\lambda\omega} \leq \frac{\delta}{\omega} \tag{8}$$

where $\lambda < 1$, then there is a unique equilibrium, which is in isotone pure strategies.

Both theorems are proved by noting that previous proofs are amended in a straightforward way to accommodate multi-dimensional types and actions; and by noting that, with the addition of assumption U7, all the conditions for the proof of theorem 1 are satisfied.

4 Conclusions

In this paper, we have provided a sufficient condition for there to be a unique equilibrium, which is in monotone pure strategies, in games of incomplete information. The condition relates payoff parameters to informational conditions (independence and heterogeneity) in a way that ensures that the equilibrium mapping is a contraction. Using the contraction approach allows us to accommodate a number of extensions relatively easily. This flexibility should prove useful in applications.

Appendix

A Proof of Proposition 3

A sufficient condition for player i's best response to any distribution $s(\cdot)$ induced by any strategy profile to be a monotone pure strategy is that the expected utility $\Delta U(t_i, s)$ (see equation (1)) is a strictly increasing function of t_i. This requires that

$$1 > \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \left| \frac{\partial}{\partial t_i} \left(\int_{-\infty}^{+\infty} f(s(t)) \exp\left[-\frac{1}{2} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \right] dt \right) \right|$$
$$= \frac{\rho}{\sqrt{2\pi}\sigma^2(1-\rho^2)} \left| \int_{-\infty}^{+\infty} f(s(t)) \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right) \exp\left[-\frac{1}{2} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \right] dt \right|.$$

Since the normal distribution is symmetric around the mean,

$$\frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \left| \int_{-\infty}^{+\infty} f(s(t)) \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right) \exp\left[-\frac{1}{2} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \right] dt \right|$$
$$\leq \frac{\kappa}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \int_{\rho t_i + (1-\rho)y}^{+\infty} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right) \exp\left[-\frac{1}{2} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \right] dx.$$

A change of variables

$$z \equiv \frac{1}{2} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \qquad dt \equiv \left(\frac{\sigma^2\sqrt{1-\rho^2}}{t - \rho t_i - (1-\rho)y} \right) dz$$

shows that

$$\frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \int_{\rho t_i + (1-\rho)y}^{+\infty} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right) \exp\left[-\frac{1}{2} \left(\frac{t - \rho t_i - (1-\rho)y}{\sigma\sqrt{1-\rho^2}} \right)^2 \right] dt$$
$$= \frac{\rho}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}.$$

Hence the sufficient condition is

$$1 > \frac{\kappa\rho}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}$$

which proves the claim.

B Proof of Lemma 1

Let

$$\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) \equiv U_i(a_i, t_i, \boldsymbol{\mu}_{-i}) - U_i(a_i', t_i, \boldsymbol{\mu}_{-i}).$$

For the strict single crossing property to hold, it is sufficient to show that if $\iota < \delta/\omega$, then

$$\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) > \Delta U_i(a_i, a_i', t_i', \boldsymbol{\mu}_{-i})$$

for any $a_i > a_i'$ and $t_i > t_i'$.

$$\begin{aligned}
&\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i', t_i', \boldsymbol{\mu}_{-i}) \\
&= \int_{T_{-i}} \int_{A_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) \prod_{j\neq i} d\mu_j(\cdot, t_j) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\quad - \int_{T_{-i}} \int_{A_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \prod_{j\neq i} d\mu_j(\cdot, t_j) f(\boldsymbol{t}_{-i}|t_i') d\boldsymbol{t}_{-i} \\
&= \int_{T_{-i}} \int_{A_{-i}} [\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i})] \prod_{j\neq i} d\mu_j(\cdot, t_j) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\quad - \int_{T_{-i}} \int_{A_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \prod_{j\neq i} d\mu_j(\cdot, t_j) \left[f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)\right] d\boldsymbol{t}_{-i}. \qquad \text{(B.9)}
\end{aligned}$$

From assumption U2, we obtain for the first term that

$$\int_{T_{-i}} \int_{A_{-i}} [\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i})] \prod_{j \neq i} d\mu_j(\cdot, t_j) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \geq \delta(a_i - a_i')(t_i - t_i'). \quad \text{(B.10)}$$

Now consider the second term in equation (B.9). The integral can be separated, so that

$$\begin{aligned}
&\int_{T_{-i}} \int_{A_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \prod_{j \neq i} d\mu_j(\cdot, t_j) \left[f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i) \right] d\boldsymbol{t}_{-i} \\
&= \int_{T_{-i}} \left[\int_{A_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \prod_{j \neq i} d\mu_j(\cdot, t_j) \right] \frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\leq \left(\int_{T_{-i}} \left[\int_{A_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \prod_{j \neq i} d\mu_j(\cdot, t_j) \right]^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \right)^{1/2} \\
&\qquad \times \left(\int_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)} \right)^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \right)^{1/2} \quad \text{(B.11)}
\end{aligned}$$

where in the last line, we use the Cauchy-Schwartz inequality.

Using assumption U3 and the fact $a_i > a_i'$ yields an upper bound on the first term of the product in equation (B.11),

$$\left(\int_{T_{-i}} \left[\int_{A_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \prod_{j \neq i} d\mu_j(\cdot, t_j) \right]^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \right)^{1/2} \leq \omega(a_i - a_i'). \quad \text{(B.12)}$$

For the second term of the product in equation (B.11),

$$\left(\int_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)} \right)^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \right)^{1/2} = \sqrt{\operatorname{Var}_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)} \right)}$$

because

$$\mathbb{E}_{T_{-i}}\left[\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)}\right] = \int_{T_{-i}} \frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}$$
$$= \int_{T_{-i}} (f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)) d\boldsymbol{t}_{-i} = 0$$

since $\int_{T_{-i}} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} = \int_{T_{-i}} f(\boldsymbol{t}_{-i}|t_i') d\boldsymbol{t}_{-i} = 1$. Therefore from assumption D2,

$$\left(\int_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)}\right)^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}\right)^{1/2} \leq \iota(t_i - t_i') \tag{B.13}$$

Combining equation (B.9) with equations (B.10)–(B.13) yields

$$\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i', t_i', \boldsymbol{\mu}_{-i}) \geq (\delta - \omega\iota)(a_i - a_i')(t_i - t_i') > 0 \tag{B.14}$$

which proves the lemma.

C Proof of Theorem 2

To show existence and uniqueness of equilibrium we first construct the best response correspondence, and then show that under the assumed parameter restrictions the equilibrium correspondence is a contraction. Then the result follows from the contraction mapping theorem.

We start by constructing the equilibrium correspondence. Let

$$a_i^+ = \min\{a \in A_i | a > a_i\}$$
$$a_i^- = \max\{a \in A_i | a < a_i\},$$

i.e., a_i^+ and a_i^- are the two actions adjacent to a_i. Since the action set is countable, both

a_i^+ and a_i^- are well defined. For any given vector of opponents' behavioural strategies, $\boldsymbol{\mu}_{-i}$, define $\tau_i(a_i; \boldsymbol{\mu}_{-i})$ by

$$\Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}_{-i}), \boldsymbol{\mu}_{-i}) \triangleq 0 \tag{C.15}$$

i.e., $\tau_i(a_i; \boldsymbol{\mu}_{-i})$ is the type at which player i is indifferent between actions a_i and a_i^-. From lemma 1, if $\delta/\omega > \iota$, then the function ΔU_i is strictly increasing in t_i and so $\tau_i(a_i; \boldsymbol{\mu}_{-i})$ is uniquely defined by equation (C.15). Furthermore, $\tau_i(a_i; \boldsymbol{\mu}_{-i}) > \tau_i(a_i^-; \boldsymbol{\mu}_{-i})$. We maintain the assumption that $\delta/\omega > \iota$ and show that the assumption is consistent with the sufficient condition derived in this proof.

Define

$$\chi_i(t_i; a_i, \boldsymbol{\mu}_{-i}) = \begin{cases} 0 & t_i \notin [\tau_i(a_i; \boldsymbol{\mu}_{-i}), \tau_i(a_i^+; \boldsymbol{\mu}_{-i})], \\ 1 & t_i \in [\tau_i(a_i; \boldsymbol{\mu}_{-i}), \tau_i(a_i^+; \boldsymbol{\mu}_{-i})] \end{cases}$$

for $a_i \in A_i$ and $t_i \in T_i$. Recall that the best response of player i to any vector of opponents' strategies $\boldsymbol{\mu}_{-i}$ is a monotonic pure strategy. Therefore player i's best response $\mu_i(a_i, t_i; \boldsymbol{\mu}_{-i})$ is an indicator function, so that $\mu_i(a_i, t_i; \boldsymbol{\mu}_{-i}) = \chi_i(t_i; a_i, \boldsymbol{\mu}_{-i})$ for all $a_i \in A_i$. An equilibrium is then defined by

$$\boldsymbol{\mu} = (\chi_i(t_i; a_i, \boldsymbol{\mu}_{-i}))_{i \in I} \equiv \boldsymbol{\phi}(\boldsymbol{\mu})(\boldsymbol{a}, \boldsymbol{t}).$$

Let $\mathcal{X}$ denote the set of indicator functions: $\mathcal{X} : \prod_{i \in I}(A_i \times T_i) \to \{0,1\}^I$. The mapping $\boldsymbol{\phi}(\boldsymbol{\mu})$ maps $\mathcal{X}$ into itself. So $\boldsymbol{\phi}(\boldsymbol{\mu})$ is the equilibrium correspondence.

Now we show that $\boldsymbol{\phi}(\boldsymbol{\mu})$ is a contraction. First we demonstrate that space $\mathcal{X}$ is complete under an appropriate metric. Consider any two vectors of behavioural strategies,

$\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$. Let $d(\boldsymbol{\mu}, \boldsymbol{\mu}')$ denote the metric

$$d(\boldsymbol{\mu}, \boldsymbol{\mu}') \equiv \max_{i \in I} \max_{a_i \in A_i} \int_{T_i} |\mu_i(a_i, t_i) - \mu_i'(a_i, t_i)| dt_i. \tag{C.16}$$

Notice that the assumption that the type spaces of all players are bounded means that the distances defined by the metric exist and are finite. Moreover, the metric is a variant of the L^1 metric, and so it is easy to show that it is indeed a metric. The space $(\mathcal{X}, d)$ is complete, since an indicator function is a function of bounded variation (i.e., can be expressed as the difference between monotonic functions), and so, by Helly's selection theorem (see Kolmogorov and Fomin (1970, p. 372)), has a convergent (sub)sequence.

So for existence and uniqueness of equilibrium, it is sufficient to show that $\boldsymbol{\phi}(\boldsymbol{\mu})$ is a contraction under the metric d i.e., that there is a $\lambda < 1$ such that $d(\boldsymbol{\phi}(\boldsymbol{\mu}), \boldsymbol{\phi}(\boldsymbol{\mu}')) \leq \lambda d(\boldsymbol{\mu}, \boldsymbol{\mu}')$. Consider

$$\begin{aligned} d(\boldsymbol{\phi}(\boldsymbol{\mu}), \boldsymbol{\phi}(\boldsymbol{\mu}')) &= \max_{i \in I} \max_{a \in A_i} \int_{T_i} \left| \chi_i(a_i, t_i; \boldsymbol{\mu}_{-i}) - \chi_i(a_i, t_i; \boldsymbol{\mu}'_{-i}) \right| dt_i \\ &= \max_{i \in I} \max_{a \in A_i} \left(|\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i^+; \boldsymbol{\mu}'_{-i})| - |\tau_i(a_i^+; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})| \right) \\ &\leq \max_{i \in I} \max_{a \in A_i} \left(|\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})| - |\tau_i(a_i^+; \boldsymbol{\mu}_{-i}) - \tau_i(a_i^+; \boldsymbol{\mu}'_{-i})| \right) \\ &\leq \max_{i \in I} \max_{a \in A_i} |\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})| \end{aligned}$$

where in the second line, we use the fact that χ_i is an indicator function. A sufficient condition for $\boldsymbol{\phi}(\boldsymbol{\mu})$ to be a contraction under the metric defined in equation (C.16) is therefore that there is a $\lambda \in (0, 1)$ such that

$$\max_{i \in I} \max_{a \in A_i} |\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})| \leq \lambda d(\boldsymbol{\mu}, \boldsymbol{\mu}') \tag{C.17}$$

First note that $\Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}_{-i}), \boldsymbol{\mu}_{-i}) = \Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}'_{-i}) = 0$. This implies that

$$\begin{aligned}&|\Delta U_i(a_i, a_i^-, \tau_i(a_i;\boldsymbol{\mu}_{-i}), \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}_{-i})| \\ &\qquad = |\Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}'_{-i}) - \Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}_{-i})|.\end{aligned} \tag{C.18}$$

Our aim is to bound the left-hand side from below with a bound proportional to $|\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})|$; and the right-hand side from above with a bound proportional to $d(\boldsymbol{\mu}, \boldsymbol{\mu}')$.

To bound the left-hand side from below, observe that lemma 1, assumption U2 and that the fact $a_i > a_i^-$ together imply that

$$\begin{aligned}&\Big|\Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}_{-i}), \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}_{-i})\Big| \\ &\qquad \geq (\delta - \iota\omega)(a_i - a_i^-)\Big|\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})\Big|.\end{aligned} \tag{C.19}$$

To bound the right-hand side from above, observe that the definition of ΔU_i implies

$$\begin{aligned}&\Big|\Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}'_{-i}) - \Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}_{-i})\Big| \\ &\leq \int_{T_{-i}} \int_{A_{-i}} \left|\Delta u_i(a_i, a_i^-, \boldsymbol{a}_{-i}, \tau_i(a_i; \boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i}) \left[\prod_{j\neq i} d\mu_j(\cdot, t_j) - \prod_{j\neq i} d\mu'_j(\cdot, t_j)\right]\right| f(\boldsymbol{t}_{-i}|\tau_i(a_i; \boldsymbol{\mu}'_{-i}))d\boldsymbol{t}_{-i}.\end{aligned}$$

Recall that under the maintained assumption $\delta/\omega > \iota$, players use pure strategies. So for any particular t_j, $\mu_j(a_j, t_j)$ is an indicator function i.e., for the behavioural strategy $\mu_j(\cdot, t_j)$, there exists almost surely a unique $a \in A_j$ such that $\mu_j(a_j, t_j) = 1$ for $a_j = a$ and $\mu_j(a, t_j) = 0$ for all $a_j \in A_j$ with $a_j \neq a$. Therefore

$$\begin{aligned}&\int_{A_{-i}} \left|\Delta u_i(a_i, a_i^-, \boldsymbol{a}_{-i}, \tau_i(a_i; \boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i}) \left[\prod_{j\neq i} d\mu_j(\cdot, t_j) - \prod_{j\neq i} d\mu'_j(\cdot, t_j)\right]\right| \\ &\qquad = \Big|\Delta u_i(a_i, a_i^-, \boldsymbol{a}^{\mu}_{-i}, \tau_i(a_i; \boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i^-, \boldsymbol{a}^{\mu'}_{-i}, \tau_i(a_i; \boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i})\Big|.\end{aligned} \tag{C.20}$$

where $\boldsymbol{a}^{\mu}_{-i}$ and $\boldsymbol{a}^{\mu'}_{-i}$ are the action profiles prescribed by the two strategy profiles $\boldsymbol{\mu}_{-i}$ and

$\boldsymbol{\mu}'_{-i}$.

Next observe that if $\boldsymbol{a}^{\mu}_{-i} = \boldsymbol{a}^{\mu'}_{-i}$, then $\max_{j\neq i} \max_{a_j \in A_j} |\mu_j(a_j,t_j) - \mu'_j(a_j,t_j)| = 0$, and the right hand side of equation (C.20) is also zero. Alternatively, if $\boldsymbol{a}^{\mu}_{-i} \neq \boldsymbol{a}^{\mu'}_{-i}$, then $\max_{j\neq i} \max_{a_j \in A_j} |\mu_j(a_j,t_j) - \mu'_j(a_j,t_j)| = 1$. Hence

$$\begin{aligned}\int_{A_{-i}} \left| \Delta u_i(a_i, a_i^-, \boldsymbol{a}_{-i}, \tau_i(a_i;\boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i}) \left[\prod_{j\neq i} d\mu_j(\cdot,t_j) - \prod_{j\neq i} d\mu'_j(\cdot,t_j) \right] \right| \\ = \left| \Delta u_i(a_i, a_i^-, \boldsymbol{a}^{\mu}_{-i}, \tau_i(a_i;\boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i^-, \boldsymbol{a}^{\mu'}_{-i}, \tau_i(a_i;\boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i}) \right| \\ \times \max_{j\neq i} \max_{a_j\in A_j} |\mu_j(a_j,t_j) - \mu'_j(a_j,t_j)|.\end{aligned} \tag{C.21}$$

Using assumption U4 and the fact that $\|\boldsymbol{a}^{\mu}_{-i} - \boldsymbol{a}^{\mu'}_{-i}\| \leq 1$, the right hand side of equation (C.21) can therefore be bounded above:

$$\begin{aligned}\int_{A_{-i}} \left| \Delta u_i(a_i, a_i^-, \boldsymbol{a}_{-i}, \tau_i(a_i;\boldsymbol{\mu}_{-i}), \boldsymbol{t}_{-i}) \left[\prod_{j\neq i} d\mu_j(\cdot,t_j) - \prod_{j\neq i} d\mu'_j(\cdot,t_j) \right] \right| \\ \leq \kappa(a_i - a_i^-) \max_{j\neq i} \max_{a_j\in A_j} |\mu_j(a_j,t_j) - \mu'_j(a_j,t_j)|.\end{aligned}$$

It follows from this that the right hand side of equation (C.18) is bounded above:

$$\begin{aligned}\Big| \Delta U_i(a_i, &a_i^-, \tau_i(a_i;\boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}'_{-i}) - \Delta U_i(a_i, a_i^-, \tau_i(a_i;\boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}_{-i}) \Big| \\ &\leq \int_{T_{-i}} \kappa(a_i - a_i^-) \max_{j\neq i} \max_{a_j\in A_j} |\mu_j(a_j,t_j) - \mu'_j(a_j,t_j)| f(\boldsymbol{t}_{-i}|\tau_i(a_i;\boldsymbol{\mu}'_{-i})) d\boldsymbol{t}_{-i} \\ &\leq \kappa(a_i - a_i^-) \max_{j\neq i} \max_{a_j\in A_j} \int_{T_{-i}} |\mu_j(a_j,t_j) - \mu'_j(a_j,t_j)| f(\boldsymbol{t}_{-i}|\tau_i(a_i;\boldsymbol{\mu}'_{-i})) d\boldsymbol{t}_{-i} \\ &= \kappa(a_i - a_i^-) \max_{j\neq i} \max_{a_j\in A_j} \int_{T_j} |\mu_j(a_j,t_j) - \mu'_j(a_j,t_j)| f_j(t_j|\tau_i(a_i;\boldsymbol{\mu}'_{-i})) dt_j \end{aligned} \tag{C.22}$$

Assumption D3 requires that $f(t_j|t_i) \leq \nu$; this leads to

$$
\begin{aligned}
\Big| \Delta U_i(a_i, a_i^-, &\tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}'_{-i}) - \Delta U_i(a_i, a_i^-, \tau_i(a_i; \boldsymbol{\mu}'_{-i}), \boldsymbol{\mu}_{-i}) \Big| \\
&\leq \kappa(a_i - a_i^-) \max_{j \neq i} \max_{a_j \in A_j} \int_{T_j} |\mu_j(a_j, t_j) - \mu'_j(a_j, t_j)| \nu dt_j \\
&\leq \kappa\nu(a_i - a_i^-) \max_{j \in I} \max_{a_j \in A_j} \int_{T_j} |\mu_j(a_j, t_j) - \mu'_j(a_j, t_j)| dt_j \\
&= \kappa\nu(a_i - a_i^-) d(\boldsymbol{\mu}, \boldsymbol{\mu}'). \qquad \text{(C.23)}
\end{aligned}
$$

Putting equation (C.18) with the inequalities (C.19) and (C.23) together yields

$$
|\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})| \leq \frac{\kappa\nu}{\delta - \iota\omega} d(\boldsymbol{\mu}, \boldsymbol{\mu}') \qquad \text{(C.24)}
$$

where the assumption that $\delta/\omega > \iota$ is still maintained. Since the above inequality holds for any $i \in I$ and any $a_i \in A_i$, we also have

$$
\max_{i \in I} \max_{a \in A_i} |\tau_i(a_i; \boldsymbol{\mu}_{-i}) - \tau_i(a_i; \boldsymbol{\mu}'_{-i})| \leq \frac{\kappa\nu}{\delta - \iota\omega} d(\boldsymbol{\mu}, \boldsymbol{\mu}')
$$

which implies from (C.17) that

$$
d(\boldsymbol{\phi}(\boldsymbol{\mu}_{-i}), \boldsymbol{\phi}(\boldsymbol{\mu}'_{-i})) \leq \frac{\kappa\nu}{\delta - \iota\omega} d(\boldsymbol{\mu}, \boldsymbol{\mu}').
$$

Hence $\boldsymbol{\phi}$ is a contraction under the metric $d(\cdot, \cdot)$ if for $\lambda < 1$,

$$
\frac{\kappa\nu}{\delta - \iota\omega} \leq \lambda. \qquad \text{(C.25)}
$$

Finally, note that if the condition in equation (C.25) is satisfied, then $\delta > \iota\omega$ and so the initial assumption is verified.

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