Mathematical Medley, June 2003
Volume 30, No. 1, pp. 30-37

A Generalization of Synthetic Division and A General Theorem of Division of Polynomials

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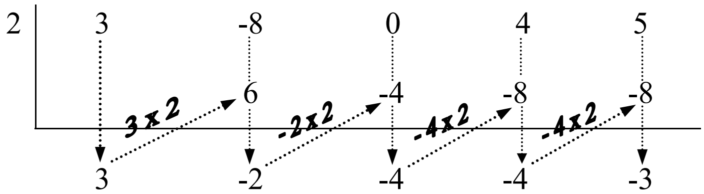
1. INTRODUCTION. Synthetic division has long been a standard topic in college algebra course. However, the college algebra textbooks usually introduce the method to divide a polynomial of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ by a binomial of $g(x) = x - c$, without mentioning if this classical method can be applied when the divisor is a polynomial of degree being higher than 1, and some further explicitly stated that it is not applicable to such a divisor. For example, Larson, Hostetler, and Edwards claimed, "synthetic division works only for divisors of the form $x - k$. You cannot use synthetic division to divide a polynomial by a quadratic such as $x^2 - 3$." [**1**, p. 270]. Similar statements are also found in many other texts; e.g., see [**2**, p. 237], [**3**, p. 569], [**4**, p. 227], [**5**, p. 45], [**6**, p. 284], [**7**, p. 25], and [**8**, p. 198].

In this article, I will present a generalization of the classical synthetic division that can be used to divide a polynomial by another polynomial of any degree, and then, following the generalized method, I will establish a general theorem about the division of any two polynomials.

Let me start with a quick review of the traditional synthetic method by looking at the following example:

$$(3x^4 - 8x^3 + 4x + 5) \div (x - 2).$$

Using the following array of numbers,



we get that the quotient is $q(x) = 3x^3 - 2x^2 - 4x - 4$, and the remainder is $r = -3$.

Note in the array, which I call "the synthetic array of numbers" below in this article, the dotted arrows are used to indicate how the numbers are related.

2. SYNTHETIC DIVISION WITH A DIVISOR OF DEGREE 2. Before I describe the generalization of the method for a divisor of any degree, let me first give a specific example where the divisor is of degree 2. Suppose $f(x) = 5x^5 + 13x^4 + 2x^2 - 7x - 10$,

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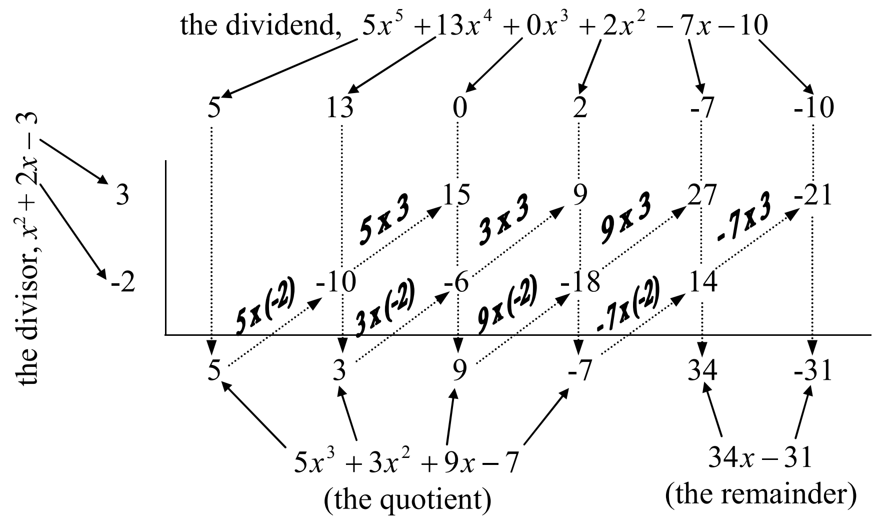
and $g(x) = x^2 + 2x - 3$. The following array of numbers shows how the extended version of synthetic division works. Note how the coefficients of $g(x)$, 2 and –3, are used and placed in the self-explanatory diagram.



As we know, the traditional synthetic division can be regarded as a shortcut method of doing long division of two polynomials when the divisor is a binomial. When we extend it to the situation where the divisor is a quadratic, the same idea still holds. The following long division explains why the above example works. Note in order to see clearly how the coefficients of the quotient $5x^3 + 3x^2 + 9x - 7$ are obtained, the quotient is better placed at the top of $5x^5 + 13x^4 + 0x^3 + 2x^2$ instead of $0x^3 + 2x^2 - 7x - 10$.

$$
\begin{array}{r|l}
 & 5x^3 + 3x^2 + 9x - 7 \\
\hline
x^2 + 2x - 3 & 5x^5 + 13x^4 + 0x^3 + 2x^2 - 7x - 10 \\
 & \underline{5x^5 + 10x^4 - 15x^3} \\
 & \quad 3x^4 + 15x^3 + 2x^2 \\
 & \quad \underline{3x^4 + 6x^3 - 9x^2} \\
 & \qquad 9x^3 + 11x^2 - 7x \\
 & \qquad \underline{9x^3 + 18x^2 - 27x} \\
 & \qquad\quad -7x^2 + 20x - 10 \\
 & \qquad\quad \underline{-7x^2 - 14x + 21} \\
 & \qquad\qquad\quad 34x - 31
\end{array}
$$

Following the above example, one can similarly extend the method without much difficulty for a divisor of degree 3 or 4 or higher.

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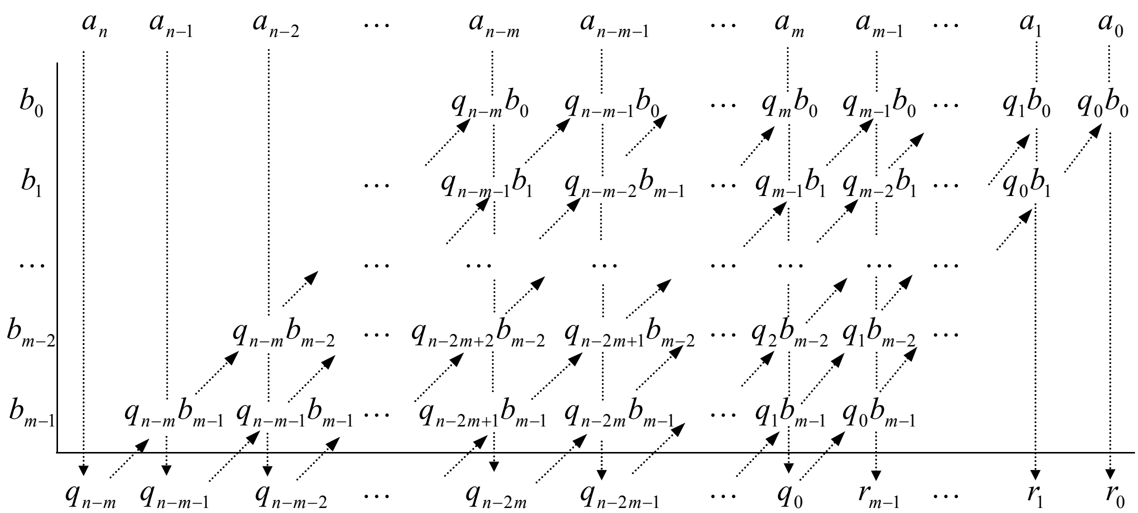
Now let me move to generalize the synthetic division method for a divisor being of any degree.

3. A GENERALIZATION OF SYNTHETIC DIVISION. Without sacrificing the generality, let us assume that the dividend is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and the divisor is $g(x) = b_m x^m - b_{m-1}x^{m-1} - \cdots - b_1 x - b_0$, where $a_n \neq 0, b_m \neq 0$ and $m \leq n$. For convenience, we first assume that $b_m = 1$.

Introduced below is the generalized synthetic method. To understand how it works, one can use the long division algorithm, similar to the case discussed in the previous section in which the divisor is a quadratic, though a little more tedious manipulation and careful observation are needed in the essentially straightforward process. I would say that a relatively hard part is to realize that one needs to consider two separate situations, that is, one for $m \leq \frac{n}{2}$ and the other for $m > \frac{n}{2}$, as the corresponding synthetic arrays of numbers, which can be seen using the long division, are different in format. Again, notice that in both arrays, the arrows are used to indicate how the numbers are connected.

(a) When $n - m \geq m$ or equivalently $m \leq \frac{n}{2}$, the synthetic array of numbers is shown below:



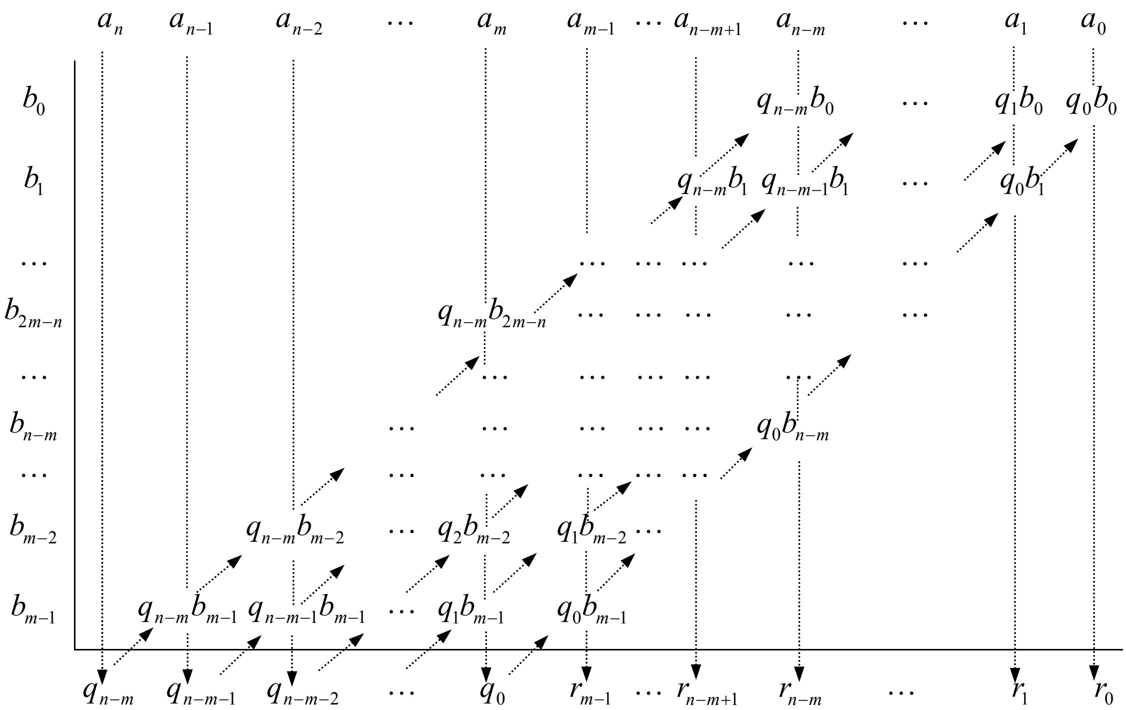
From this array, it is easy to obtain the following general expressions of the coefficients of the quotient $q(x) = q_{n-m}x^{n-m} + q_{n-m-1}x^{n-m-1} + \cdots + q_1 x + q_0$ and the remainder $r(x) = r_{m-1}x^{m-1} + r_{m-2}x^{m-2} + \cdots + r_1 x + r_0$, namely,

$$q_{n-m-i} = \begin{cases} a_n, & i = 0, \\ a_{n-i} + \sum_{j=1}^{i} q_{n-m-i+j} b_{m-j}, & i = 1, 2, \cdots, m, \\ a_{n-i} + \sum_{j=1}^{m} q_{n-m-i+j} b_{m-j}, & i = m+1, m+2, \cdots, n-m; \end{cases}$$

and

$$r_{m-k} = a_{m-k} + \sum_{i=0}^{m-k} q_i b_{m-k-i}, \ \ k = 1, 2, \cdots, m.$$

(**b**) When $n - m < m$ or equivalently $m > \frac{n}{2}$, the synthetic array of numbers is as follows:



Note that in the given array, for the convenience of presentation we assume that $2m - n < n - m$ or equivalently $m < \frac{2}{3}n$, therefore the row containing the coefficient b_{2m-n} of the divisor is shown above the row containing the coefficient b_{n-m} in the diagram. If $2m-n \geq n-m$ or equivalently $m \geq \frac{2}{3}n$, the positions of the relevant rows can be accordingly presented in the synthetic array.

In any case, it is not difficult to see that the coefficients of the quotient, q_{n-m-i}, and the remainder, r_{m-k}, can be generally expressed in the following way:

$$q_{n-m-i} = \begin{cases} a_n, & i = 0, \\ a_{n-i} + \sum_{j=1}^{i} q_{n-m-i+j} b_{m-j}, & i = 1, 2, \cdots, n-m; \end{cases}$$

and

$$r_{m-k} = \begin{cases} a_{m-k} + \sum_{i=0}^{n-m} q_i b_{m-k-i}, & k = 1, 2, \cdots, 2m-n, \\ a_{m-k} + \sum_{i=0}^{m-k} q_i b_{m-k-i}, & k = 2m-n+1, 2m-n+2, \cdots, m. \end{cases}$$

Notice that in the above generalized synthetic method, the sign before each of $b_{m-1}, b_{m-2}, \cdots, b_1, b_0$ is arranged to be negative, and $b_m = 1$. Obviously, this arrangement is only for convenience, but not necessary.

4. A GENERAL THEOREM OF DIVISION OF POLYNOMIALS. Now let us consider the general situation of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ where $a_n \neq 0, b_m \neq 0$, and $m \leq n$. Notice that b_m here is not necessarily 1.

To find $q(x)$ and $r(x)$ so that $f(x) = q(x)g(x) + r(x)$, we consider its equivalent equation $\frac{f(x)}{b_m} = q(x)\frac{g(x)}{b_m} + \frac{r(x)}{b_m}$, and let $F(x) = \frac{f(x)}{b_m}$ and $G(x) = \frac{g(x)}{b_m}$. The coefficients of $F(x)$ and $G(x)$ are $\frac{a_i}{b_m}$ $(i = 1, 2, ..., n)$ and $\frac{b_j}{b_m}$ $(j = 1, 2, ..., m)$, respectively, and the leading coefficient of $G(x)$ is 1. Therefore, we can apply the generalized synthetic method introduced in Section 3 to obtain the quotient $Q(x)$ and remainder $R(x)$ of the division of $F(x)$ by $G(x)$. From $F(x) = q(x)G(x) + \frac{r(x)}{b_m}$, we can get $q(x) = Q(x)$ and $r(x) = b_m R(x)$.

With the above idea, one can now easily deduce the following general theorem of the division of any two polynomials.

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Theorem. *For any two polynomials,* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *and* $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$*, where* $a_n \neq 0, b_m \neq 0$ *and* $m \leq n$*, there exist* $q(x) = q_{n-m}x^{n-m} + q_{n-m-1}x^{n-m-1} + \cdots + q_1 x + q_0$ *and* $r(x) = r_{m-1}x^{m-1} + r_{m-2}x^{m-2} + \cdots + r_1 x + r_0$ *so that*

$$f(x) = q(x)g(x) + r(x).$$

The coefficients of $q(x)$ *and* $r(x)$ *are obtained as follows:*

(**a**) When $m \leq \frac{n}{2}$,

$$q_{n-m-i} = \begin{cases} \dfrac{a_n}{b_m}, & i = 0, \\ \dfrac{a_{n-i}}{b_m} - \displaystyle\sum_{j=1}^{i} q_{n-m-i+j}\dfrac{b_{m-j}}{b_m}, & i = 1, 2, \cdots, m, \\ \dfrac{a_{n-i}}{b_m} - \displaystyle\sum_{j=1}^{m} q_{n-m-i+j}\dfrac{b_{m-j}}{b_m}, & i = m+1, m+2, \cdots, n-m; \end{cases}$$

and

$$r_{m-k} = a_{m-k} - \sum_{i=0}^{m-k} q_i b_{m-k-i}, \quad k = 1, 2, \cdots, m.$$

(**b**) When $m > \frac{n}{2}$,

$$q_{n-m-i} = \begin{cases} \dfrac{a_n}{b_m}, & i = 0, \\ \dfrac{a_{n-i}}{b_m} - \displaystyle\sum_{j=1}^{i} q_{n-m-i+j}\dfrac{b_{m-j}}{b_m}, & i = 1, 2, \cdots, n-m; \end{cases}$$

and

$$r_{m-k} = \begin{cases} a_{m-k} - \displaystyle\sum_{i=0}^{n-m} q_i b_{m-k-i}, & k = 1, 2, \cdots, 2m-n, \\ a_{m-k} - \displaystyle\sum_{i=0}^{m-k} q_i b_{m-k-i}, & k = 2m-n+1, 2m-n+2, \cdots, m. \end{cases}$$

Remarks. Two remarks are in order to end this article. First, I realized that the traditional synthetic division could be extended for a divisor being a quadratic when I observed the classroom teaching of a high school mathematics teacher in Illinois during my doctoral study, though I cannot explicitly identify the teacher and the school here because of the agreement for conducting the study. The teacher showed one concrete example, which was recorded in my doctoral dissertation [**9**, p. 142], to explain how it works for a divisor of degree 2. It provoked me to explore the possibility of generalizing the method for a divisor of any integral degree. Later, I also found examples somehow containing the same idea in [**10**] with the divisor being a quadratic and [**11**, pp. 58-59] with the divisor being of degree 3. But unfortunately, all of them did not go further after mentioning one or two concrete examples, failing to establish a general algebraic expression for the divisor being of any degree. To me, this fact appears to be a reason that they received little attention; as mentioned earlier, many college algebra texts still maintain that synthetic division can only be applied when the divisor is of the form $x-a$, which according to this article is not true and should be changed. Second, it seems to me clear that the general theorem of division of polynomials presented in this article has a good potential in application. For instance, the so-called "Division Theorem" of polynomials (e.g., see [**11**, p. 59]), which says that for two polynomials f and g with deg $g \geq 1$ there are polynomials q and r such that $f = qg + r$ and deg $r <$ deg g, is essentially a corollary of the general theorem. Nevertheless, to explore how this general theorem can be applied in relevant areas of mathematics is beyond the intention of this article.

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