UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

School of Mathematics

Partial Translation Algebras for Certain Discrete Metric Spaces

by

Rosemary Johanna Putwain

Thesis submitted for the degree of Doctor of Philosophy

June 2010

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS

SCHOOL OF MATHEMATICS

Doctor of Philosophy

PARTIAL TRANSLATION ALGEBRAS FOR CERTAIN DISCRETE METRIC SPACES

by Rosemary Johanna Putwain

The notion of a *partial translation algebra* was introduced by Brodzki, Niblo and Wright in [11] to provide an analogue of the reduced group C^*-algebra for metric spaces. Such an algebra is constructed from a *partial translation structure*, a structure which any bounded geometry uniformly discrete metric space admits; we prove that these structures restrict to subspaces and are preserved by uniform bijections, leading to a new proof of an existing theorem. We examine a number of examples of partial translation structures and the algebras they give rise to in detail, in particular studying cases where two different algebras may be associated with the same metric space. We introduce the notion of a map between partial translation structures and use this to describe when a map of metric spaces gives rise to a homomorphism of related partial translation algebras. Using this homomorphism, we construct a C^*-algebra extension for subspaces of groups, which we employ to compute K-theory for the algebra arising from a particular subspace of the integers. We also examine a way to form a groupoid from a partial translation structure, and prove that in the case of a discrete group the associated C^*-algebra is the same as the reduced group C^*-algebra. In addition to this we present several subsidiary results relating to partial translations and cotranslations and the operators these give rise to.

Contents

Declaration of Authorship

I, Rosemary Putwain, declare that the thesis entitled *Partial Translation Algebras for Certain Discrete Metric Spaces* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission.

Signed: ....

Date: ....

Acknowledgements

Thanks must of course firstly go to my supervisor, Prof. Jacek Brodzki, for all his help and advice throughout the writing of this thesis, and also to my advisor, Prof. Graham Niblo, without neither of whom would the project have been possible. I am also greatly indebted to Dr. Nick Wright for a number of illuminating conversations, in particular relating to the C^*-algebra extension for subspaces of groups that we construct herein.

In addition to this, I would like to offer thanks to the many friends I made at the University of Southampton, in particular the cohabitants of the house at Harefield Road, who all helped to make my time there ever so much more enjoyable and spurred me on to complete my work. I must also mention my friends from home, including Rhi, Kate, Nick and many others, who always help me keep my feet on the ground and let my hair down when I need to most; I sincerely hope they will continue to do so for the rest of my days (and, perhaps more importantly, nights!). Of course I could not get away without also referring to my brilliant boyfriend Ollie, thanking him for his patience with me, for helping out where he could and for knowing me infuriatingly well and loving me all the same.

Last but most certainly not least I must express my gratitude to my wonderful mother, Trisha, for being eternally supportive and proud of me, even when she doesn't really know what it is I'm working on! I dedicate this document to her, as I truly believe I could never have reached this stage without her positive influence and guidance throughout my life so far. Thanks, Mum.

1 Introduction

It has long been recognised that certain C^*-algebras which may be associated with a group G can be used to characterise properties of the group itself. This is especially true of the reduced C^*-algebra of a discrete group; for example, Lance proved that a discrete group G is amenable if and only if its reduced C^*-algebra $C_r^*(G)$ is nuclear [23]. In the discrete case, the (right) reduced group C^*-algebra is contained within the uniform Roe algebra $C_u^*(G)$, but whilst such an algebra may be associated with any given metric space, the reduced C^*-algebra is only defined for groups. Hence it is useful for general metric spaces to consider an analogue of the reduced C^*-algebra; in [11], Brodzki, Niblo and Wright introduced an algebra to fill this role. Their construction works by mimicking the right action of a group on itself to abstract the notion of a translation structure for a space, then using the partial translations which contribute to this structure to generate a subalgebra of the uniform Roe algebra of the space. The main subject of this thesis is an exploration of various properties and examples of these partial translation structures and the algebras which arise from them.

The first chapter consists of motivational material and some results within the wider area of group C^*-algebras, before we restrict our focus to partial translation algebras. We begin with the general definition of a C^*-algebra, along with that of the reduced group C^*-algebra, the uniform Roe algebra and other related notions. A key example of a C^*-algebra is given by the space of complex-valued functions on a metric space X which vanish at infinity; an original proof of the known fact that the state space of this algebra is equivalent to the space of probability measures on X can be found in subsection 2.1.2.

In the second section of chapter 2 we include the definition of property A. This interesting metric property, reminiscent of the Følner condition for amenable groups, was introduced by Yu in [38]. Property A guarantees uniform embeddability into Hilbert space, which in turn gives the coarse Baum-Connes conjecture and therefore the Novikov conjecture. By the results of [3], [20], [18] and [25], for a discrete group G the property is equivalent to both the nuclearity of $C_u^*(G)$ and to the exactness of $C_r^*(G)$. In [11], the authors proved that the equivalence of property A and nuclearity of the uniform Roe algebra also holds for metric spaces which are sufficiently homogeneous in the sense that they admit a partial translation structure which satisfies certain properties (namely freeness and global control; see chapter 3 for definitions). They additionally proved that in this case property A is also equivalent to the exactness of the uniform Roe algebra, and thus property A provides a way to show that exactness and nuclearity of $C_u^*(X)$ are equivalent for a space X which has been deemed sufficiently group-like via the presence of a certain type of partial translation structure.

There are many different ways to define property A, so as well as stating Yu's original definition we consider two other accepted versions and prove their equivalence. One of these alternate definitions involves probability measures, so we are able to recast this in the language of states by way of our results from subsection 2.1.2.

The third section of chapter 2 takes a brief look at Hilbert space compression, a numerical invariant for a group, which is particularly interesting because of a result by Guentner and Kaminker which states that any finitely generated discrete group with Hilbert space compression greater than one half is exact. In other words, the reduced C^*-algebra of such a group is exact. It is known that exactness of a countable discrete group implies its uniform embeddability in a Hilbert space, but whether or not the converse is true is an open question; as Guentner and Kaminker also showed that any space with non-zero Hilbert space compression is uniformly embeddable in a Hilbert space, they in fact proved the converse for discrete groups with Hilbert space compression greater than one half. This section concludes with an alternative definition for compression which appears in [4].

One method for proving exactness of a discrete group, and hence showing that it has property A, is to show that the group acts amenably on a compact space [2]. It is well known that every free group acts on itself and has a tree for its Cayley graph; thus, in the final section of chapter 2 we provide a new proof of the known result that finitely generated free groups are exact by showing amenability of the action on the boundary of this tree. We describe two different methods for achieving this, the first using a variation of a construction by Brodzki, Campbell, Guentner, Niblo and Wright which was originally used to prove property A for finite dimensional CAT(0) cube complexes, and the second based on a construction of Germain related to discrete word hyperbolic groups.

The definition of a partial translation structure can be found in chapter

3, where we also prove a number of new results related to restricting partial translation structures to subspaces and mapping them via uniform bijections. Combining these results yields a new proof of a theorem by Brodzki, Niblo and Wright which states that any space admitting an injective uniform embedding into a discrete group admits a free and globally controlled partial translation structure.

Partial translation structures are made up of translations and cotranslations, which interact with each other in a way which mimics the interaction between left and right multiplication in a group. For the most part we focus on partial translations, as these define the generators for the partial translation algebra, however in the final section of chapter 3 we turn our attention to orbits of cotranslations. We show that any partial translation can be expressed as the orbit of a pair of elements under cotranslations, and that under certain conditions the cotranslation orbits alone form a partial translation structure. We also include a couple of useful remarks relating to group actions.

The following chapter defines the partial translation *algebra* arising from a partial translation structure, a subalgebra of the uniform Roe algebra, which in the canonical group case coincides with the reduced group C^*-algebra. Here we also prove a couple of results about the operators which arise from partial translations, as these are used to generate the algebra, in particular specifying when these will be unitary and explaining why they have finite propagation.

Next we consider some concrete examples of partial translation structures and algebras. In section 5.1, we take subspaces of certain groups and apply the theorems from chapter 3 to obtain two partial translation structures for each subspace, one via restricting the canonical partial translation structure of the group and one via the use of a uniform bijection, so that we may examine how these two methods can produce different results. These ideas are taken one step further for some of the examples in section 5.2, where we look at the partial translation algebras which arise, and thus prove that there are spaces for which these are not unique.

Maps of partial translation structures are an important theme for this thesis, and chapter 6 consists of the first stage in a project to ascertain when such maps give rise to C^*-homomorphisms between the associated partial translation algebras. Here we focus specifically on inclusion maps for subspaces of groups, and with the help of some examples decide this question first for the case where the group we consider is the integers and secondly for the more general case.

To identify homomorphisms between partial translation algebras in general it becomes necessary to formally define what it means for a map of metric spaces to be a morphism of partial translation structures (or *P.T.S. map*), hence we do so in section 7.1. We relate this definition to examples we have already considered in section 7.2, and then use it to evaluate some new examples involving maps from other groups to $\mathbb{Z}$. Finally, in section 7.5, we are able to expand upon the results of chapter 6 by proving that P.T.S. maps always give rise to homomorphisms of partial translation algebras, so long as there is a uniform bound on the number of partial translations being mapped to any single translation.

In chapter 8 we demonstrate the usefulness of such a homomorphism, by constructing an algebra extension where the middle term is the C^*-algebra arising from the restriction of a canonical group partial translation structure to a subspace. The surjective map in the sequence is the C^*-algebra homomorphism which features in the previous two chapters.

The algebra extension yields a six-term exact sequence in K-theory, which can be employed to compute K-theory for specific examples; generally speaking, this is no easy task! We show explicitly how one accomplishes such a computation with the help of our sequence in chapter 9, where an in-depth evaluation of a partial translation algebra is provided, culminating with the determination of the K-theory of that algebra. The algebra considered in this chapter arises from a restricted canonical partial translation structure defined over a set of integers, illustrating how a great deal of structure can be extrapolated even from the simplest of examples.

The final chapter of the thesis studies a link between partial translation structures and groupoids. After providing two equivalent definitions of a groupoid in section 10.1, in section 10.2 we introduce the "partial translation groupoid" which arises from any given partial translation structure. This groupoid is reminiscent of the beautiful construction of Skandalis, Tu and Yu appearing in [36], which allows one to encode the large scale structure of a (bounded geometry) space by means of a groupoid of partial translations, and has greatly clarified many foundational questions in coarse geometry. However, there are significant differences between the way in which the two groupoids are defined.

We show in section 10.2 that in cases where the translations and cotranslations satisfy certain conditions, which in particular hold for the canonical partial translation structure for a discrete group, our partial translation groupoid is itself a partial translation structure. It is also shown in this section that one can expand a free partial translation structure into a family of groupoids which is again a free partial translation structure, so long as every partial cotranslation is globally defined. Section 10.3 offers a definition of the reduced *groupoid* C^*-algebra (due to Roe), and goes on to prove that in the case of the groupoid associated to the canonical partial translation structure of a discrete group G, this object coincides with the reduced group C^*-algebra of G. In doing this we also employ an original proof of a lemma which gives an alternative description of the group ring of any countable discrete group.

2 C^*-algebras, Property A and Amenability

2.1 C^*-algebras and States

2.1.1 Definitions

A *Banach algebra* is a complex normed algebra A which is topologically complete and satisfies

$$\|ab\| \leq \|a\| \, \|b\| \text{ for all } a, b \in A.$$

A *Banach *-algebra* is a complex Banach algebra A with a conjugate linear involution *, the *adjoint*, which satisfies

$$\begin{aligned} (a+b)^* &= a^* + b^*, \\ (\lambda a)^* &= \overline{\lambda} a^*, \\ a^{**} &= a, \\ (ab)^* &= b^* a^*, \end{aligned}$$

for all $a, b \in A$, $\lambda \in \mathbb{C}$ [13].

A C^*- algebra A is a Banach *-algebra which also satisfies the C^*-identity

$$\|a^* a\| = \|a\|^2, \text{ for all } a \in A \text{ [13]}.$$

A *unital* C^*- (or Banach) algebra is one that contains the identity element 1 such that $1a = a1 = a$ for all $a \in A$ [12].

If A is a unital algebra we may define the *spectrum* of any element $a \in A$ by $Spec_A(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \text{ is not invertible}\}$ [13]. Say then that $a \in A$ is *positive* if $a^* = a$ (i.e. a is *self-adjoint* or *hermitian*) and if $Spec(a)$ is contained in the non-negative half-line $\mathbb{R}_+ = [0, \infty)$ [12].

Many interesting geometric properties of spaces and groups are captured by the structure of the C^*-algebras we may associate with them, particularly when we consider the *full* and *reduced* C^*-algebras of a discrete group G. The definitions of these are given below.

For any discrete group G, a *regular representation* of G is a linear representation afforded by the group action of G on itself. We may distinguish between the *left regular representation* λ, which is induced by the left multiplication action, and the *right regular representation* ρ, which comes from the multiplication on the right. The left and right multiplication actions of G on itself are implemented as closely as possible to obtain representations, so that for every $g \in G$ we define operators $\lambda_g, \rho_g : l^2(G) \to l^2(G)$ by

$$(\lambda_g(f))(h) = f(g^{-1}h) \text{ and } (\rho_g(f))(h) = f(hg) \text{ for all } g, h \in G, f \in l^2(G).$$

In particular, we have

$$(\lambda_g(\delta_x))(h) = \delta_x(g^{-1}h) = \delta_{gx}(h)$$

and

$$(\rho_g(\delta_x))(h) = \delta_x(hg) = \delta_{xg^{-1}}(h),$$

for all $g, h, x \in G$, where δ_x is the characteristic function of the element $x \in G$. For G countable, the functions δ_x form a basis for the Hilbert space $l^2(G)$ of square summable functions (in fact, sequences) on G.

Finally, for any element f of the group ring $\mathbb{C}G$, we may express f as a linear combination $f = \sum_{g \in G} f_g \delta_g$ with finitely many non-zero complex coefficients f_g, and then define

$$\lambda(f) = \sum_{g \in G} f_g \lambda_g \text{ and } \rho(f) = \sum_{g \in G} f_g \rho_g.$$

It can be seen that every λ_g is a unitary operator and hence that $\|\lambda(f)\| \leq$

$\sum_{g\in G} |f_g| = \|f\|_1$ (see [6], for example), and similarly for ρ, so that for G countable the regular representations extend to representations of the Banach algebra $l^1(G)$ on the Hilbert space $l^2(G)$.

DEFINITION 2.1 [Reduced Group C^*-algebra]

For a discrete group G, the *left reduced group C^*-algebra* $C^*_\lambda(G)$ is the closure of $\lambda(\mathbb{C}G)$ in the operator norm induced from $B(l^2(G))$ [8]. This coincides with the closure of $\lambda(l^1(G))$ in the same norm [6]. The closure of $\rho(\mathbb{C}G)$, or indeed $\rho(l^1(G))$, in this norm is called the *right reduced group C^*-algebra*, and is denoted by $C^*_\rho(G)$.

We have $C^*_\lambda(G) \cong C^*_\rho(G)$, for any discrete group G. The isomorphism is given by conjugating by the unitary operator $T : \delta_g \mapsto \delta_{g^{-1}}$ on $l^2(G)$ [11]. Indeed, we have

$$T\lambda_g T^{-1}(\delta_h) = T\lambda_g(\delta_{h^{-1}}) = T(\delta_{gh^{-1}}) = \delta_{hg^{-1}} = \rho_g(\delta_h),$$

for all $g, h \in G$. When the distinction between the left and right regular representations is not relevant we shall use the notation $C^*_r(G)$ for the reduced group C^*-algebra.

This algebra is of particular significance because, in the discrete case, we say that the group G is *exact* if $C^*_r(G)$ is exact; that is, if taking the minimal tensor product with $C^*_r(G)$ on each of the terms in a short exact sequence of C^*-algebras preserves the exactness of the sequence [19].

For a discrete group G equipped with a left invariant metric, the right reduced group C^*-algebra $C^*_\rho(G)$ is a subalgebra of the *uniform Roe algebra* $C^*_u(G)$ (if we equip G with a right invariant metric then the uniform Roe algebra contains $C^*_\lambda(G)$). We may in fact associate a uniform Roe algebra with any discrete metric space, and such an algebra can be used to encode analytically the coarse geometry of the space [29]; the algebra is defined as follows.

Recall that, for a metric space X, a kernel $u : X \times X \to \mathbb{C}$ has *finite propagation* if there exists $R \geq 0$ such that $u(x, y) = 0$ whenever $d(x, y) > R$. Recall also that a discrete metric space X is said to have *bounded geometry* if for all $R > 0$ there exists N such that the cardinality of $B_R(x)$ is at most N for all $x \in X$.

If X is a proper discrete metric space and $u : X \times X \to \mathbb{C}$ is a finite propagation kernel, then for each $x \in X$ there exist only finitely many

$y \in X$ satisfying $u(x,y) \neq 0$; this implies that u defines a linear operator from $l^2(X)$ to itself, given by $u * \xi(x) = \sum_{y \in X} u(x,y)\xi(y)$. We call operators defined using finite propagation kernels *finite propagation operators.* Note that if additionally X has bounded geometry then every finite propagation operator arising from a bounded kernel will be bounded itself.

DEFINITION 2.2 [The Uniform Roe Algebra]

Let X be a bounded geometry discrete metric space. The *uniform Roe algebra* $C_u^*(X)$ is the C^*-algebra completion of the algebra of bounded finite propagation operators on $l^2(X)$ [11].

In the case of a discrete group G, $C_u^*(G)$ is typically far larger than $C_r^*(G)$; in fact, it is non-separable unless G is finite [10]. For this reason it is useful to consider an analogue of the reduced group C^*-algebra for general metric spaces, and this is the role *partial translation algebras* were introduced to fill. These will be defined in chapter 4.

DEFINITION 2.3 [Full Group C^*-algebra]

The *full group C^*-algebra* $C^*(G)$ of a discrete group G is defined to be the C^*-enveloping algebra of $L^1(G)$, that is the completion of $C_c(G)$ with respect to the largest C^*-norm: $\|f\| = \sup_\pi\{\|\pi(f)\|\}$, where π ranges over all non-degenerate *-representations of $C_c(G)$ on Hilbert spaces [34].

Note that it follows from the triangle inequality that $\|\pi(f)\| \leq \|f\|_1$ for any such π, so the above norm is well-defined. It also follows from the definition that $C^*(G)$ has the universal property that any *-homomorphism from the group ring $\mathbb{C}G$ to some $B(H)$ (the C^*-algebra of bounded operators on some Hilbert space H) factors through the inclusion $\mathbb{C}G \hookrightarrow C^*(G)$ [34].

In general, a C^*-algebra A is said to be *nuclear* if for any C^*-algebra B there is a unique C^*-norm on $A \otimes B$ [23]. One useful property of the C^*-algebra $C^*(G)$ for a discrete group G is that it is isomorphic to $C_r^*(G)$ if and only if G is *amenable*, meaning that there exists a left translation invariant mean for G [13]. Amenability of G is also equivalent to nuclearity of $C_r^*(G)$ [23], and thus to nuclearity of $C^*(G)$.

2.1.2 *Prob*(X) is the State Space of $C_0(X)$

To define another interesting C^*-algebra we consider the algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space X which vanish at infinity. By the famous Gelfand-Naimark theorem [16], this example in fact covers all commutative C^*-algebras. It is also a known fact that the state space of this algebra is equivalent to the space of probability measures on X (for example, this is mentioned in [30]); we present a proof of this statement below.

Recall that for any given metric space X, a *measure* μ on X is a function mapping subsets of X to real values, such that if $E_1,\ E_2,\ldots,E_n \ \subset X$ are mutually disjoint subsets, i.e. $E_i \cap E_j = \emptyset$ for $i \neq j$, then $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$. A measure μ on a space X is called a *probability measure* if μ is a positive measure, i.e. it only maps to non-negative values, and if $\mu(X) = 1$. Denote the space of all probability measures on X by $Prob(X)$. A measure μ is called *Borel regular* if every Borel set $B \subseteq X$ is measurable, i.e. $\mu(A) = \mu(A \cap B) + \mu(A \backslash B)$ for all sets $A \subseteq X$, and if for all $A \subseteq X$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

The algebra $C_0(X)$ consists of continuous complex-valued functions on X which vanish at infinity. By definition, a continuous function $f \in C_0(X)$ if and only if for all $\epsilon > 0$ there exists a compact set $K \subseteq X$ such that $|f(x)| < \epsilon$ for all $x \in X \backslash K$. For any locally compact Hausdorff space X, $C_0(X)$ is a C^*-algebra. The significance of this example is illustrated by the situation where X is an abelian group, since in this case we have

$$C_0(\hat{X}) \cong C^*(X) \cong C_r^*(X),$$

where $\hat{X}$ denotes the group of *characters* of X, that is the group homomorphisms from X to S^1 [13].

A *state* on a C^*-algebra A is a linear map $\phi : A \to \mathbb{C}$ which has unit norm and which is *positive* in the sense that if f is a positive element of A then $\phi(f) \geq 0$ [13].

Note that a space X is called *Hausdorff* if any two distinct points of X can be separated by neighbourhoods, and that a Hausdorff space X is *locally compact* if every element of X has a compact neighbourhood.

THEOREM 2.4 (Riesz Representation Theorem) *If X is a locally compact Hausdorff space, then every bounded linear functional ϕ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that*

$$\phi(f) = \int_X f d\mu,$$

for every $f \in C_0(X)$ [32].

The following theorem considers a particular case of the Riesz Representation Theorem.

DEFINITION 2.5 A locally compact space X is said to be σ*-compact* if it is a countable union of finite subsets [5].

THEOREM 2.6 *Let X be a locally compact σ-compact Hausdorff space. There is a one-to-one correspondence between probability measures on X and real-valued states on the algebra $C_0(X)$.*

Proof.

Let X be a locally compact σ-compact Hausdorff space and suppose firstly that we have a measure $\mu \in Prob(X)$. Define a functional $\phi_\mu : C_0(X) \to \mathbb{R}$ by $\phi_\mu(f) := \int_X f d\mu$. A function $f \in C_0(X)$ is *positive* in the C^*-sense if and only if $f(x) \geq 0$ for all $x \in X$ (and so in particular $f(x)$ is real for all $x \in X$), so it is clear that the integral of such an element is a positive real number and hence ϕ_μ is a real-valued positive functional. We have:

$$\begin{aligned}
\|\phi_\mu\| &:= \sup_{\|f\|\leq 1} \{|\phi_\mu(f)|\} \\
&= \sup_{\|f\|\leq 1} \left\{\left|\int_X f d\mu\right|\right\} \\
&\leq \sup_{\|f\|\leq 1} \left\{\int_X |f|\, d\mu\right\} \\
&\leq \sup_{\|f\|\leq 1} \left\{\sup_{x\in X} |f(x)| \int_X d\mu\right\} \\
&= \sup_{\|f\|\leq 1} \{\|f\|\, \mu(X)\} \\
&= \sup_{\|f\|\leq 1} \{\|f\|\} \text{ (since we assume } \mu \text{ is a probability measure)} \\
&\leq 1.
\end{aligned}$$

To show that $\|\phi_\mu\| \geq 1$, firstly select an exhaustive family of compact sets $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots \subseteq X$ such that $K_n \subseteq K_{n+1}^\circ$ for all n; it is always possible to do this for X locally compact and σ-compact, by Proposition 15 on page 94 of [5]. Now choose a family of functions on X satisfying

$$f_n(x) = \begin{cases} 1 & \text{if } x \in K_n \\ 0 & \text{if } x \in (K_{n+1}^{\circ})^c \end{cases}$$

in such a way that for each n we have f_n continuous and $0 \leq f_n(x) \leq 1$ on $K_{n+1}^{\circ} \backslash K_n$. Note that it is possible to find such a family by the Tietze extension theorem [24]. Indeed, this theorem states that a continuous function from a closed subset of a topological space X to the interval $[-1, 1]$ may be extended to a continuous function on the whole of X, so long as the space X is normal. Every compact Hausdorff space is normal, and hence the set K_m is normal for all $m \in \mathbb{N}$, so the theorem tells us that each f_n can be defined so that it is continuous on, for example, the set K_{n+2}, and hence it is continuous on X, since it extends to the rest of the space by the zero function.

We have $X = \bigcup_n K_n$, and thus

$$\begin{aligned}
1 &= \mu(X) = \mu\left(\bigcup_n K_n\right) \\
&= \lim_{n \longrightarrow \infty} \mu(K_n) \\
&\leq \lim_{n \longrightarrow \infty} \int_{K_{n+1}} f_n \, d\mu \text{ (which exists since it is the limit of a bounded} \\
&\quad \text{sequence of monotonically increasing real numbers)} \\
&= \lim_{n \longrightarrow \infty} \int_X f_n \, d\mu \\
&= \lim_{n \longrightarrow \infty} \phi_\mu(f_n) = \lim_{n \longrightarrow \infty} |\phi_\mu(f_n)| \\
&\leq \lim_{n \longrightarrow \infty} \|\phi_\mu\| \, \|f_n\|_\infty \\
&= \|\phi_\mu\| .
\end{aligned}$$

Hence $\|\phi_\mu\| = 1$ and so ϕ_μ is a state.

Conversely, let us now suppose that we have a real-valued state ϕ on $C_0(X)$. By the Riesz representation theorem, there is a unique regular Borel measure μ_ϕ on X such that $\phi(f) = \int_X f d\mu_\phi$ for all $f \in C_0(X)$. The functional ϕ is positive, so $\int_X f d\mu_\phi \geq 0$ for all $f \in C_0(X)$ with $f \geq 0$, and μ_ϕ is a real positive measure.

Also, as ϕ is a state, we have

$$1 = \|\phi\| = \sup_{\|f\| \leq 1} \{|\phi(f)|\} \leq \sup_{\|f\| \leq 1} \{\|(f)\| \ \mu_\phi(X)\},$$

by the same argument as above. Thus there must be some $f \in C_0(X)$ with $\|f\| \leq 1$ and $\|f\| \, \mu_\phi(X) \geq 1$. Hence $\mu_\phi(X) \geq \frac{1}{\|f\|}$ for some $\|f\| \leq 1$, i.e. $\mu_\phi(X) \geq 1$.

We can also apply the same previous argument involving the family of compact subsets to conclude that $\mu_\phi(X) \leq \|\phi\|$; in other words, $\mu_\phi(X) \leq 1$ when ϕ is a state. Thus $\mu_\phi(X) = 1$ and so μ_ϕ is a probability measure.

□

REMARK 2.7 Some statements of the Riesz representation theorem also assert that when ϕ is a bounded linear functional on $C_0(X)$ and μ the unique regular complex Borel measure on X such that $\phi(f) = \int_X f d\mu$ for every $f \in C_0(X)$, then $\|\phi\|$ is equal to the total variation of μ, $|\mu|(X)$. This concurs with our theorem, since $|\mu|(X) = \mu(X)$ when μ is a positive measure.

2.2 Property A

In [38], Yu introduces a geometric property on discrete metric spaces, which he calls *property A*, that guarantees the existence of a uniform embedding into Hilbert space. It has not yet been determined exactly which classes of metric spaces or groups satisfy the property, although it is known that word hyperbolic groups, amenable groups and discrete subgroups of connected Lie groups are among those that do, and it has not even been established whether or not the existence of a uniform embedding into Hilbert space is actually equivalent to property A. We shall see more about this in the next section, as it has been shown by Guentner and Kaminker that equivalence certainly is the case when one imposes a condition on the space's *Hilbert space compression.*

Property A is also particularly interesting when related to C^*-algebras, as for a discrete group G it is equivalent both to the nuclearity of the uniform Roe algebra $C^*_u(G)$ and to the exactness of the reduced C^*-algebra $C^*_r(G)$, which means that we may say it is equivalent to the exactness of the group.

The original definition is as follows.

DEFINITION 2.8 [Property A]

A discrete metric space (X, d) has *property A* if for all $R, \epsilon > 0$ there exists a family of finite non-empty subsets A_x of $X \times \mathbb{N}$, indexed by $x \in X$, such that

- $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$ for all $x, y \in X$ with $d(x, y) < R$, and
- there exists S such that $d(x, y) \leq S$ for all $x \in X$ and $(y, n) \in A_x$ [11].

Many alternative characterisations of this definition have been formulated; for example, eight equivalent conditions are given in Theorem 3 of [11]. Another distinct statement of it is given by Roe in [30], which has particular relevance for us because it involves maps into $Prob(X)$, which we have shown to be equivalent to the state space of $C_0(X)$. The following terminology and remark will be useful in investigating how this connects with the characterisations given in [11].

DEFINITION 2.9 [Bounded Geometry]

A space X is said to have *bounded geometry* if for all $R > 0$ there exists N such that the cardinality of $B_R(x)$ is at most N for all $x \in X$.

DEFINITION 2.10 [(R, ϵ)-Variation]

A function from a metric space X to a Banach space, $x \mapsto \xi_x$, has *(R, ϵ)-variation* if $d(x, y) \leq R$ implies $\|\xi_x - \xi_y\| < \epsilon$ [11].

REMARK 2.11 As is noted in Definition 2.1 of [20], for X a discrete metric space, $Prob(X)$ may be regarded as the set of functions $f : X \to [0, 1]$ such that $\sum_{x \in X} f(x) = 1$. Thus, if $f \in Prob(X)$ then $f : X \to [0, 1] \subset \mathbb{C}$, and

$$\sum_{x \in X} |f(x)| = \sum_{x \in X} f(x) = 1 < \infty.$$

Hence $f \in l^1(X) = \{f : X \to \mathbb{C} \mid \sum_{x \in X} |f(x)| < \infty\}$, and so $Prob(X)$ is a subset of $l^1(X)$ when X is a discrete metric space. This could also be seen from Theorem 2.6, since $l^1(X)$ is known to be the *dual space* of $C_0(X)$, that is the space of all linear functionals on $C_0(X)$.

THEOREM 2.12 *Let X be a bounded geometry discrete metric space. Then the following are equivalent:*

(1) *X has property A;*

(2) *For all R, ϵ there exist vectors $\xi_x \in l^1(X)$ such that, for all $x \in X$, $\|\xi_x\|_1 = 1$, (ξ_x) has (R, ϵ)-variation, and there exists S such that ξ_x is supported in the S-ball about x;*

(3) *There exists a sequence of weak-$*$ continuous maps $f_n : X \to Prob(X)$ such that*

(i) *for each n there is an r such that, for each x, the measure $f_n(x)$ is supported within $B(x; r)$, and*

(ii)

$$\lim_{n\to\infty}\Big(\sup_{d(x,y)<s} \{\|f_n(x) - f_n(y)\|\}\Big) = 0$$

for all $s > 0$.

Proof.

The equivalence of **(1)** and **(2)** is given in [11]. Also, **(3)** is stated as the definition of property A in [30]. We shall check this by showing equivalence of **(2)** and **(3)**.

(3)⇒(2): Suppose we have a sequence of weak-$*$ continuous maps $f_n : X \to Prob(X)$ which satisfy conditions **(i)** and **(ii)**. So for each $x \in X$ we have a family of probability measures $(f_n(x))$. Now, by Remark 2.11, each $f_n(x)$ may be considered as a vector ξ_x^n in $l^1(X)$, and since each $f_n(x)$ is a probability measure, we know that the sum of the entries of this vector will be 1, i.e. $\|\xi_x^n\|_1 = 1$.

By **(i)**, for every n we have an r so that $f_n(x)$ is supported in $B(x; r)$, for all $x \in X$, so we may rename this r by S to obtain the third condition required for the vectors.

It remains to show (R, ϵ)-variation. By **(ii)**, for all $s > 0$ we have

$$\sup_{d(x,y)<s} \{\|f_n(x) - f_n(y)\|\} \overset{n\to\infty}{\longrightarrow} 0,$$

so clearly, for any $s > 0$,

$$d(x, y) < s \Rightarrow \|f_n(x) - f_n(y)\| < \epsilon$$

for some ϵ, regardless of which n we choose. As the supremum is tending to 0 as n tends to infinity, this will be true no matter how small an ϵ we select, if we simply choose a large enough value of n. So we may say that for each

n, (ξ_x^n) has (s, ϵ)-variation; thus for every $x \in X$ we may select for our ξ_x some (ξ_x^n) with n large enough so that the vector has (R, ϵ)-variation.

(2)⇒(3): Conversely, suppose we have vectors $\xi_x \in l^1(X)$ which satisfy the conditions of **(2)**. As these vectors have unit norm, we could consider each ξ_x as a function $f_x : X \to \mathbb{R}$ with $\sum_{y \in X} |f_x(y)| = 1$. Hence every $|f_x|$ is a function taking elements of X to non-negative real values, where the sum of all such values it may assume is 1; thus each of these real values must be less than or equal to 1. We therefore have $|f_x| : X \to [0, 1]$ and $\sum_{y \in X} |f_x(y)| = 1$, and so $|f_x| \in Prob(X)$.

Rename each $|f_x|$ by $f(x)$. These functions exist for all values of R and ϵ as well as for all $x \in X$, so we actually have a sequence of maps $f_n : X \to Prob(X)$, as required.

Clearly, condition **(i)** is satisfied, as this is stated as a condition in **(2)** (if ξ_x is supported in $B(x; S)$ then so must $|\xi_x|$ be). Also, (R, ϵ)-variation for all possible R and ϵ gives **(ii)**; simply arrange the sequence (f_n) in such a way that $n \to \infty$ as $\epsilon \to 0$.

Hence **(2)** and **(3)** are equivalent characterisations of property A.

□

As Roe's definition of property A involves probability measures defined on a metric space X, we may reformulate it using Theorem 2.6 to obtain an alternative statement phrased in the language of states on a C^*-algebra of functions on X. The only difficulty in doing this is finding an appropriate method for defining the support of a state ϕ on $C_0(X)$ as a subset of the space X itself, which coincides with the support of the probability measure corresponding to ϕ. To achieve this we first need to recall the following definitions.

DEFINITION 2.13 A topological space X is said to be *separable* iff there exists a countable subset which is dense in X [22].

DEFINITION 2.14 [Support of a Measure]

Let X be a separable metric space and let μ be a measure on X. Then we may define the *support* of μ to be the set of all points $x \in X$ having the property that $\mu(N_x) > 0$ for every open neighbourhood N_x of x [27]. Denote this set by $supp(\mu)$.

DEFINITION 2.15 Let X be a topological space. The collection of all neighbourhoods of a point $x \in X$ is called the neighbourhood system at x and is denoted by $\mathcal{U}_x$. An arbitrary subcollection $\mathcal{B}_x \subseteq \mathcal{U}_x$ is called a *fundamental system of neighbourhoods of* x if every neighbourhood $U \in \mathcal{U}_x$ is a superset to at least one $B \in \mathcal{B}_x$ [14].

REMARK 2.16 Note that within a metric space, to say a set X is separable is equivalent to saying that it is *Lindelöf* [35]; that is, that each open cover of X has a countable subcover [22]. Every σ-compact space is Lindelöf [22], and thus we may omit the condition of separability from the following lemma. Additionally, if X is both Lindelöf and locally compact then it is σ-compact [33], so we could alternatively replace "σ-compact" with "Lindelöf".

Note also that all metric spaces are Hausdorff topological spaces.

LEMMA 2.17 *Let X be a locally compact σ-compact metric space, let μ be a probability measure on X and let ϕ_μ be the unique state on $C_0(X)$ defined by $\phi_\mu(f) = \int_X f d\mu$ for all $f \in C_0(X)$. If we let $supp^*(\phi_\mu)$ denote the set of all points $x \in X$ with the property that whenever N_x is an open neighbourhood of x there exists some $f \in C_0(X)$ such that $supp(f) \subseteq N_x$ and $\phi_\mu(f) \neq 0$, then $supp^*(\phi_\mu) = supp(\mu)$.*

Similarly, if ϕ is a state on $C_0(X)$ and μ_ϕ is the unique probability measure on X defined by $\phi(f) = \int_X f d\mu_\phi$ for all $f \in C_0(X)$, then $supp^*(\phi) = supp(\mu_\phi)$.

Proof.

Suppose firstly that we have a state ϕ on $C_0(X)$, and let $x \in supp^*(\phi)$, where this set is defined as above. Then for every open neighbourhood N_x of x there exists some $f \in C_0(X)$ such that $supp(f) \subseteq N_x$ and $\phi(f) = \int_X f d\mu_\phi \neq 0$. Thus clearly $\mu_\phi(N_x) \neq 0$. Since ϕ is a state, μ_ϕ must be a probability measure by Theorem 2.6 and so takes only non-negative values; we therefore have $\mu_\phi(N_x) > 0$ for every open neighbourhood N_x, and so $x \in supp(\mu_\phi)$. Hence $supp^*(\phi) \subseteq supp(\mu_\phi)$. One can identically show that $supp^*(\phi_\mu) \subseteq supp(\mu)$, when beginning with a probability measure μ on X and using it to define a state ϕ_μ on $C_0(X)$.

To show the converse, suppose that μ is a probability measure on X. As we assume X to be locally compact, we know that every point $x \in$

X has a fundamental system of compact neighbourhoods, by the corollary appearing on page 90 of [5]. This means that, for any $x \in X$, for every open neighbourhood N_x of x there exists a compact set K_x containing x such that $K_x \subset N_x$, by Definition 2.15. Consider such N_x and K_x for an element $x \in supp(\mu)$. By definition of a compact neighbourhood, K_x contains an open set U containing x, and so $\mu(K_x) > 0$, by application of Definition 2.14 to the set U.

As K_x is a compact subset of a locally compact σ-compact space, it is also locally compact and σ-compact, and thus we may select an exhaustive family of compact sets $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots \subseteq K_x$ such that $K_n \subseteq K^\circ_{n+1}$ for all n, as in the proof of Theorem 2.6. As we know that there exists a subset $U \subset K_x$ with $\mu(U) > 0$, we must be able to find some m for which $\mu(K_m) > 0$. Now, in a similar way as we did in the proof of Theorem 2.6, we may define a function $f : X \to \mathbb{C}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in K_m \\ 0 & \text{if } x \in (K^\circ_{m+1})^c \end{cases}$$

in such a way that f is continuous and $0 \leq f(x) \leq 1$ on $K^\circ_{m+1} \backslash K_m$. As f vanishes outside a compact set it is clear that $f \in C_0(X)$. We have $supp(f) \subset N_x$ and $\phi_\mu(f) = \int_X f d\mu \neq 0$, and so we see that $x \in supp^*(\phi_\mu)$. This tells us that $supp(\mu) \subseteq supp^*(\phi_\mu)$, which means that these sets are equal. We could similarly show $supp(\mu_\phi) \subseteq supp^*(\phi)$ for any state ϕ on $C_0(X)$ to obtain $supp(\mu_\phi) = supp^*(\phi)$.

□

This result allows us to combine Roe's definition with Theorem 2.6 to obtain a new definition of property A for bounded geometry discrete metric spaces.

THEOREM 2.18 *Let X be a bounded geometry discrete metric space. Then X has property A if and only if there exists a sequence of weak-$*$ continuous maps f_n from X to the state space of $C_0(X)$ such that*

(i) *for each n there is an r such that, for each x, we have $supp^*(f_n(x)) \subseteq B(x; r)$, and*

(ii)

$$\lim_{n\to\infty} \sup_{d(x,y)<s} \|f_n(x) - f_n(y)\| = 0,$$

for all $s > 0$.

Proof.

Any discrete metric space is locally compact, and any bounded geometry metric space is σ-compact [30]; hence the result follows immediately by combining Theorem 2.6 and Lemma 2.17 with Theorem 2.12.

$\square$

2.3 Hilbert Space Compression

The Hilbert space compression of a finitely generated discrete group G is a numerical invariant of the group introduced by Guentner and Kaminker in [19] to parameterise the difference between G being uniformly embeddable in a Hilbert space and $C_r^*(G)$ being exact. It can be viewed as an approximate way to describe the geometry of a group by measuring the distortion up to which the group fails to embed into Hilbert space. Hilbert space compression is defined in the following manner, which is in fact formulated so as to make sense in the context of general metric spaces.

DEFINITION 2.19 [Large-scale Lipschitz]

A function $f : X \to Y$ between two metric spaces X and Y is called *large-scale Lipschitz* if there exist constants $C > 0$ and $D \geq 0$ such that

$$d_Y(f(x), f(y)) \leq C d_X(x, y) + D,$$

for all $x, y \in X$ [19].

Let $Lip^{ls}(X, Y)$ denote the set of large-scale Lipschitz maps from X to Y.

DEFINITION 2.20 [Compression]

The *compression* ρ_f of a map $f \in Lip^{ls}(X, Y)$ is defined by

$$\rho_f(r) = \inf_{d_X(x,y) \geq r} \{d_Y(f(x), f(y))\} \text{ [19].}$$

DEFINITION 2.21 [Hilbert Space Compression]

Let X be a metric space with unbounded metric.

1. The *asymptotic compression* R_f of a map $f \in Lip^{ls}(X,Y)$ is

$$R_f = \liminf_{r\to\infty} \left\{ \frac{\ln(\rho_f^*(r))}{\ln(r)} \right\},$$

   where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.

2. The *compression of* X *in* Y is

$$R(X,Y) = \sup\{R_f \mid f \in Lip^{ls}(X,Y)\}.$$

3. If Y is a Hilbert space then the *Hilbert space compression* of X is

$$R(X) = R(X,Y)$$

   [19].

Hilbert space compression is particularly interesting because of its links with exactness and property A; exactness of a countable discrete group implies its uniform embeddability in a Hilbert space, and Guentner and Kaminker proved that the converse is true when the Hilbert space compression of the group is strictly greater than 0.5, in other words they proved that these groups have property A. The proof of this appears in [19], and unfortunately it seems as though a similar proof could not be applied for other values of $R(X)$, even if we only relax the condition enough to allow the value of 0.5 itself.

The question of whether Guentner and Kaminker's result could be improved by imposing some sort of growth condition on the group has been investigated by Tessera [37]. Tessera proves that every bounded geometry proper metric measure space with subexponential growth has property A.

In [4], Arzhantseva, Druţu and Sapir employ a slightly different definition of compression. They begin with a 1-Lipschitz map $\phi : X \to Y$ and declare its compression to be

$$\sup\{\alpha \geq 0 \mid d_Y(\phi(u),\phi(v)) \geq d_X(u,v)^\alpha \ \forall\, u,v \text{ with large enough } d_X(u,v)\}.$$

If $\mathcal{E}$ is a class of metric spaces closed under re-scaling of the metric, the *$\mathcal{E}$-compression of X* is given by

$$\sup\{\text{compression of } \phi : X \to Y \mid \phi \text{ is 1-Lipschitz}, Y \in \mathcal{E}\} \text{ [4].}$$

The Hilbert space compression of X is then defined as the $\mathcal{E}$-compression where $\mathcal{E}$ is the class of Hilbert spaces. With this definition, the authors of [4] were able show that there exists a finitely generated group G_α of asymptotic dimension at most 3 with Hilbert space compression α for any α between 0 and 1, thus also constructing the first examples of groups that are uniformly embeddable into Hilbert spaces (and moreover exact) with Hilbert space compression 0. The compression defined here is always less than or equal to the one provided by Guentner and Kaminker.

2.4 Amenable Actions of Free Groups

Property A was introduced by Yu as a non-equivariant analogue of amenability which is defined for metric spaces. In particular, property A (and hence exactness) for a discrete group G is implied by the existence of a compact Hausdorff space X on which G acts amenably. Finitely generated free groups are a particularly nice class of examples of discrete groups, as their Cayley graphs are all (locally finite) *trees*, which are both hyperbolic spaces and CAT(0) cube complexes. These facts lead us to two new methods for proving the known result that finitely generated free groups are exact, and hence that they have property A.

For the first method we exploit a family of functions which were used by the authors of [7] in their proof that finite dimensional CAT(0) cube complexes have property A. Here we reconstruct the functions they concocted for the particular case where the CAT(0) cube complex under consideration is a tree, and use them to show that finitely generated free groups act amenably on their boundaries.

The second, more geometric, construction is based on that of Germain in Appendix B of [3], where he proved that discrete word hyperbolic groups act amenably on their own boundaries. Here we provide explicit descriptions of his functions for the simplified case of a group which has a tree as its Cayley graph, and employ an original proposition in our adaptation of his proof that such a group is exact.

2.4.1 Method One

To illustrate how the ideas introduced in previous sections can be combined, we explore new methods for proving the known result that finitely generated free groups are exact, and hence that they have property A. Firstly, let us fix notation and recall some basic information regarding free groups.

Let $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$ denote the free group with n generators. The *boundary* $\partial\mathbb{F}_n$ of $\mathbb{F}_n$ is the set of all infinite reduced words $\omega = b_1 b_2 \ldots b_n \ldots$, where $b_i \in \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$ for all i. Let $\overline{\mathbb{F}}_n = \mathbb{F}_n \cup \partial\mathbb{F}_n$. For any two points $x, y \in \overline{\mathbb{F}}_n$ there exists a unique geodesic path $[x, y]$ which connects x to y.

We may define a topology on $\overline{\mathbb{F}}_n$ by declaring that

$$U(x, F) = \{y \in \overline{\mathbb{F}}_n \mid [x, y] \cap F \subset \{x\}\}$$

is open for every $x \in \overline{\mathbb{F}}_n$ and every finite subset $F \subset \mathbb{F}_n$ [26]. Hence we may endow $\partial\mathbb{F}_n$ with the corresponding subspace topology.

LEMMA 2.22 *The free group $\mathbb{F}_n$ acts on its boundary continuously on the left by concatenation, $(\mathbb{F}_n \times \partial\mathbb{F}_n) \to \partial\mathbb{F}_n$, $(g, z) \mapsto g \cdot z$.*

Proof.

We have $g \cdot (h \cdot z) = (gh) \cdot z$ for all $g, h \in \mathbb{F}_n$, $z \in \partial\mathbb{F}_n$, and $e \cdot z = z$ for all $z \in \partial\mathbb{F}_n$, where e is the identity element in $\mathbb{F}_n$, so concatenation is indeed a left group action. We see that the action is continuous since for every open set $U(x, F) \cap \partial\mathbb{F}_n$ and for any $g \in \mathbb{F}_n$ we have

$$\begin{aligned} g^{-1} \cdot (U(x, F) \cap \partial\mathbb{F}_n) &= \{g^{-1}y \in \partial\mathbb{F}_n \mid [x, y] \cap F \subset \{x\}\} \\ &= \{g^{-1}y \in \partial\mathbb{F}_n \mid [g^{-1}x, g^{-1}y] \cap g^{-1}F \subset \{g^{-1}x\}\}, \end{aligned}$$

which is clearly open.

So $\partial\mathbb{F}_n$ is an $\mathbb{F}_n$-space.

□

It is clear from the following definition that $(\partial\mathbb{F}_n, \mathbb{F}_n)$ may thus be described as a *transformation group.*

DEFINITION 2.23 [Transformation Group]

A *transformation group* (X, G) consists of a left G-space X, where X is a locally compact space, G is a locally compact group and $(g, x) \mapsto g \cdot x$ is a continuous left action from $G \times X$ to X [2].

Given a discrete transformation group, amenability of the action is defined in the following way.

DEFINITION 2.24 [Amenable Action]

For G discrete, the transformation group (X, G) (or the G-action on X, or the G-space X) is said to be *amenable* if there exists a sequence $(m_i)_{i \in I}$ of weak *-continuous maps $x \mapsto m_i^x$ from X into the space $Prob(G)$ such that

$$\lim_{i \to \infty} (\sup_{x \in X} \{\|gm_i^x - m_i^{gx}\|_1\}) = 0$$

for all $g \in G$ [20].

We say that a group G is *exact* if there exists an amenable compact G-space.

There are a number of equivalent definitions for amenable actions, and we shall consider another when discussing our second method for proving the exactness of $\mathbb{F}_n$.

We base our proof of the below theorem on the construction of Brodzki, Campbell, Guentner, Niblo and Wright used to prove Theorem 4.2 in [7], which states that if G is a countable discrete group acting properly on a finite dimensional CAT(0) cube complex X, and z a vertex at infinity of X, then the stabiliser of z in G is amenable. The authors prove this by considering a sequence z_j of vertices converging to z and defining a family of functions $f_{n,x}^{z_j}$, indexed by $x \in X$ (a description of these functions as they appear for the particular case where X is a tree is provided below). They then take the limit of these functions as i tends to infinity, check that this limit is well-defined and denote it by $f_{n,x}$, for every natural number n and every vertex $x \in X$. To prove amenability of G, they show that the support of each $f_{n,x}$ is finite, that $f_{n,x}$ is almost equivariant, that, for any $R > 0$,

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \to 0$$

uniformly on the set $\{(x, x') \mid d(x, x') \leq R\}$ as $n \to \infty$, and that $f_{n,gx} = gf_{n,x}$ for all $g \in G$, $x \in X$ and $n \in \mathbb{N}$. To make use of this construction, we follow the framework for the creation of the family of functions $f_{n,x}$ in the particular case where the cube complex X is a tree. However, we shall normalise our functions so that we obtain probability measures. We then associate members of this family with elements of $\partial\mathbb{F}_n$, so that we obtain the type of maps we require, and finally show that these satisfy the condition stated in Definition 2.24.

A version of the following theorem which considers only the free group with two generators may be found in [2], where it is stated as an example without proof.

THEOREM 2.25 *The transformation group ($\partial\mathbb{F}_n, \mathbb{F}_n$) is amenable.*

Proof.

We prove amenability using Definition 2.24, so our goal is to create a sequence of continuous maps from $\partial\mathbb{F}_n$ into $Prob(\mathbb{F}_n)$ which satisfy the given condition.

As the Cayley graph of $\mathbb{F}_n$ is a tree, we begin by constructing the functions defined in [7] for this particular case. For $i \in \mathbb{N}$, x a fixed vertex of the tree, that is an element of $\mathbb{F}_n$, and $e \in \mathbb{F}_n$ the identity element, translating the construction into our case yields functions on the tree defined by:

$$\begin{aligned} g^e_{i,x}(y) &= \begin{cases} \begin{pmatrix} i - d(x,y) \\ 0 \end{pmatrix} & \text{if } y \neq e,\ y \in [e,x],\ d(x,y) \leq i \\ \begin{pmatrix} i - d(x,y) + 1 \\ 1 \end{pmatrix} & \text{if } y = e,\ d(x,y) \leq i \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \neq e,\ y \in [e,x],\ d(x,y) \leq i \\ i - d(x,y) + 1 & \text{if } y = e,\ d(x,y) \leq i \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then

$$\begin{aligned} \left\|g^e_{i,x}\right\|_1 &= \sum_{y \in \mathbb{F}_n} \left|g^e_{i,x}(y)\right| \\ &= \sum_{y \in \mathbb{F}_n} g^e_{i,x}(y) \text{ (as our functions take only non-negative values)} \\ &= \left.\begin{cases} d(x,e) + i - d(x,e) + 1 & \text{if } d(x,e) \leq i \\ (i+1) \cdot 1 & \text{if } d(x,e) > i \end{cases}\right\} = i + 1. \end{aligned}$$

So if we define functions

$$f^e_{i,x}(y) = \begin{cases} \frac{1}{i+1} & \text{if } y \neq e,\ y \in [e,x],\ d(x,y) \leq i \\ \frac{i-d(x,y)+1}{i+1} & \text{if } y = e,\ d(x,y) \leq i \\ 0 & \text{otherwise,} \end{cases}$$

then $\left\| f^e_{i,x} \right\|_1 = 1$, and $f^e_{i,x} : \mathbb{F}_n \to [0,1]$. Thus, for each $i \in \mathbb{N}$ and $x \in \mathbb{F}_n$, $f^e_{i,x}$ is a probability measure on $\mathbb{F}_n$.

In a similar way, probability measures $f^a_{i,x}$ may be defined for other elements $a \in \mathbb{F}_n$, and in particular for any point z_j in a sequence of elements of $\mathbb{F}_n$ which tend to a point z in the boundary $\partial\mathbb{F}_n$. These are defined in the following way:

$$f^{z_j}_{i,x}(y) = \begin{cases} \frac{1}{i+1} & \text{if } y \neq z_j,\ y \in [x,z_j],\ d(x,y) \leq i \\ \frac{i-d(x,y)+1}{i+1} & \text{if } y = z_j,\ d(x,y) \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Then let $f^z_{i,x}(y) = \lim_{j\to\infty} f^{z_j}_{i,x}(y)$.

As j tends to infinity, so does $d(x,z_j)$, and so as the value of j increases there will be fewer and fewer i's satisfying $d(x,z_j) \leq i$. At the limit, $d(x,z) = \infty$, so it is impossible to have $y = z$ and $d(x,y) \leq i$ for some finite i. Thus the middle term vanishes and we are left with

$$f^z_{i,x}(y) = \begin{cases} \frac{1}{i+1} & \text{if } y \in [x,z),\ d(x,y) \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if we fix a point $x \in \mathbb{F}_n$, we have continuous maps $f_{i,x} : z \mapsto f^z_{i,x}$ from $\partial\mathbb{F}_n$ into $Prob(\mathbb{F}_n)$. For simplicity let us take $x = e$, the identity element in $\mathbb{F}_n$.

It remains to show that $\lim_{i\to\infty} \left\| t f^z_{i,e} - f^{tz}_{i,e} \right\|_1 = 0$ for all $t \in \mathbb{F}_n$. We have

$$\begin{aligned}
tf_{i,e}^{z}(y) &:= f_{i,e}^{z}(t^{-1}y) \text{ for all } y \in \mathbb{F}_n \\
&= \begin{cases} \frac{1}{i+1} & \text{if } t^{-1}y \in [e,z),\ d(e,t^{-1}y) \leq i \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{i+1} & \text{if } y \in [t,tz),\ d(t,y) \leq i \\ 0 & \text{otherwise} \end{cases} \\
&= f_{i,t}^{tz}(y).
\end{aligned}$$

So we want to show that $\lim_{i\to\infty} \left\| f_{i,t}^{tz} - f_{i,e}^{tz} \right\|_1 = 0$. Now

$$\begin{aligned}
\left\| f_{i,t}^{tz} - f_{i,e}^{tz} \right\|_1 &= \left\| \frac{g_{i,t}^{tz}}{i+1} - \frac{g_{i,e}^{tz}}{i+1} \right\|_1 \\
&= \left\| \frac{g_{i,t}^{tz} - g_{i,e}^{tz}}{i+1} \right\|_1 \\
&= \frac{\left\| g_{i,t}^{tz} - g_{i,e}^{tz} \right\|_1}{i+1}.
\end{aligned}$$

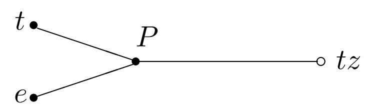
Note that

$$g_{i,t}^{tz}(y) = \begin{cases} 1 & \text{if } y \in [t,tz),\ d(t,y) \leq i \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_{i,e}^{tz}(y) = \begin{cases} 1 & \text{if } y \in [e,tz),\ d(e,y) \leq i \\ 0 & \text{otherwise.} \end{cases}$$

For $t \in \mathbb{F}_n$, let P denote the vertex where the paths $[t,tz)$ and $[e,tz)$ intersect, so that we may visualize the situation by way of the following diagram.



Note that P may be equal to either e or t.

Now $\left|(g_{i,t}^{tz} - g_{i,e}^{tz})(y)\right|$ is only non-zero if either $y \in [t,P]$ with $d(t,y) \leq i$, $y \in [e,P]$ with $d(e,y) \leq i$, or $y \in [P,tz)$ such that one of $d(t,y)$, $d(e,y)$ is less than or equal to i and the other is greater than i. Now both $d(t,P)$ and $d(e,P)$ can be at most $d(e,t)$, and the interval of $[P,tz)$ that can contain y for the latter case to occur must also be of length at most either $d(t,P)$

or $d(e, P)$. Thus in any case there are at most $2d(e, t)$ vertices y for which $\left|(g_{i,t}^{tz} - g_{i,e}^{tz})(y)\right|$ is non-zero, and at each of these points it takes the value 1.

So we have $\left\|g_{i,t}^{tz} - g_{i,e}^{tz}\right\|_1 \leq 2d(e, t)$, and therefore

$$\left\|f_{i,t}^{tz} - f_{i,e}^{tz}\right\|_1 \leq \frac{2d(e,t)}{i+1},$$

which tends to 0 as $i \to \infty$.

□

Recall that an action of a group G on a space X is called *transitive* if for any two $x, y \in X$ there exists $g \in G$ such that $gx = y$, and *free* if $gx \neq gy$ for all distinct $x, y \in X$ and for all $g \in G$. Finite free groups can also be characterised as those groups which act transitively and freely on a tree. These are the only conditions assumed when we show $tf_{i,e}^{z}(y) = f_{i,t}^{tz}(y)$, and they also ensure that there is an isomorphism between the group and the vertices of the tree, so that we really do construct probability measures on the group.

More generally, the following has been proved by Guentner.

THEOREM 2.26 *Let G be a countable discrete group acting on a tree without inversion. Then G is C^*-exact if and only if the vertex stabilisers of the action are C^*-exact [17].*

2.4.2 Method Two

As aforementioned, any finitely generated free group can be easily identified with a rooted tree; every tree is a 0-hyperbolic space. In [1], Adams proves that discrete hyperbolic groups act amenably on their own boundaries, and hence are exact. A supposedly simpler proof is given by E. Germain in Appendix B of [3]; we base our second proof of the exactness of finitely generated free groups on a construction of the functions used there for the case of a tree.

In this subsection we shall use the following characterisation of amenability of a group action. For this definition, recall that a function $f : X \to Y$ between Borel spaces is called a *Borel function* if and only if $f^{-1}(B)$ is a Borel set in X for any Borel set B in Y [15].

DEFINITION 2.27 [Amenable Action]

Let X be a locally compact space on which a discrete group G acts continuously on the left. The action of G on X is said to be *amenable* if there exists a sequence of positive compactly supported Borel functions f_i on $G \times X$ such that

$$\int_G f_i(g,x)dg > 0, \text{ for all } x \in X,$$

and

$$\lim_{i\to\infty}\left(\sup_{x\in X}\left\{\frac{\int_G |f_i(g,x) - h\cdot f_i(g,x)|dg}{\int_G f_i(g,x)dg}\right\}\right) = 0, \text{ for all } h \in G,$$

where $(h \cdot f)(g,x) = f(h^{-1}g, h^{-1}x)$ is the action induced by the diagonal action of G in $G \times X$ [3].

Since the propositions and lemma which follow hold for trees in general, and not just those which correspond to groups, let us begin with some basic terminology relating to trees. Recall that a tree is a locally finite connected graph with no circuits or loops, and that we write $a \sim a'$ when two vertices a and a' of a tree are connected by an edge. A *path* in a tree is a finite or infinite sequence of vertices $[a_0, a_1, \ldots]$ such that $a_i \sim a_{i+1}$ and $a_{i-1} \neq a_{i+1}$ for all i. If a and a' are two vertices then $[a, a']$ denotes the unique path joining them. A vertex is said to be *terminal* if it has only one neighbour.

We may endow a tree T with a metric d by assigning $d(a, a')$ to be the number of edges in the unique path $[a, a']$ for any vertices $a, a' \in T$. If we fix a vertex $e \in T$ as a root of the tree, we may define the *length* of any vertex a as $l(a) = d(e, a)$.

The *boundary* ∂T is the union of the set of terminal vertices and the set of equivalence classes of infinite paths under the relation $\simeq$, defined by $[a_0, a_1, \ldots] \simeq [a_1, a_2, \ldots]$; when the tree under consideration is the Cayley graph of a free group this coincides with the definition of the boundary given earlier. For any vertex $a \in T$, let $[a, \omega)$ denote the (unique) path starting at a in the class $\omega \in \partial T$.

PROPOSITION 2.28 *Let T be a locally finite tree and choose any vertex $a \in T$. Let a_k denote the unique point on the path $[a, \omega)$ such that $d(a, a_k) = k$ (the path $[a, \omega)$ is unique, so we may always find such a vertex a_k, assuming $[a, \omega)$ contains more than k vertices). Then a_k lies on the path $[a', \omega)$ for all*

$a' \in T$ *with* $d(a,a') \leq k$.

Proof.

Suppose for contradiction that there exists some $a' \in T$ with $d(a,a') \leq k$ such that a_k does not lie on the path $[a',\omega)$. Since $d(a,a') \leq k = d(a,a_k)$, we also know that a' does not lie on the (unique) path $[a_k,\omega) = [a_k, a_{k+1}, a_{k+2}, \ldots, \omega)$. Indeed, if it did then it would either equal a_k itself, contradicting the assumption that a_k does not lie on the path $[a',\omega)$, or it would be at a distance greater than k from a.

If we consider ω as a vertex in T then it must be terminal and so only has one neighbour, say ω', which must lie on both $[a',\omega)$ and $[a_k,\omega)$. As a' and a_k are distinct vertices, a_k does not lie on the path $[a',\omega)$ and a' does not lie on the path $[a_k,\omega)$, this provides us with a loop in T passing through a', a_k and ω', which contradicts the fact that T is a tree. Thus no such point exists and so a_k lies on $[a',\omega)$ for all $a' \in T$ with $d(a,a') \leq k$.

$\square$

Let T be a tree, $a \in T$ and $\omega \in \partial T$. For every positive integer k, define

$$I(a,\omega,k) := \left\{ \left[a',\omega\right) \;\middle|\; d(a,a') \leq k \right\}$$

to be the set of all paths in the class ω starting not too far from a. Choosing a length $l > 0$, define

$$F(a,\omega,k,l) := \text{the characteristic function of } \bigcup_{[a',\omega) \in I(a,\omega,k)} [a'_l, a'_{2l}]$$

to be the union of large segments of these paths, far enough from our reference point, where again a'_l denotes the vertex on the path $[a',\omega)$ with $d(a',a'_l) = l$. Finally, set

$$H(a,\omega,l) := \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} F(a,\omega,k,l)$$

to be our ad hoc average.

For F a compactly supported function on T, let $\|F\|$ denote its norm in $l^1(T)$.

The above functions were originally constructed by Germain in a more general sense so that they may be applied to hyperbolic groups rather than trees, however it is clear that a group structure is not required to make sense of their definitions. Germain went on to prove that for a discrete hyperbolic group Γ we have

$$\|H(a, \omega, l)\| \geq l,$$

for all $a \in \Gamma$, $\omega \in \partial\Gamma$, and

$$\sup_{\omega \in \partial\Gamma} \|H(ga, \omega, l) - H(a, \omega, l)\| = o(l),$$

for any fixed vertices $g, a \in \Gamma$. In the tree case we can be a little more explicit. Indeed, we have the following.

LEMMA 2.29 *For any vertex a in an infinite tree T, we have*

$$\|H(a, \omega, l)\| = l + 2k.$$

Proof.

Since $H(a, \omega, l) = \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} F(a, \omega, k, l)$, we wish to consider which values $\|F(a, \omega, k, l)\|$ may take for $k < \sqrt{l}$. Firstly, note that if $k < \sqrt{l}$ where k is a nonnegative integer and l is a positive integer, then in particular we have $2k \leq l$.

By Proposition 2.28, all paths in $I(a, \omega, k)$ contain the vertex a_k, and thus are equivalent to the path $[a_k, \omega)$. Clearly, if $d(a, a_k) = k$ and $d(a, a') \leq k$, then $d(a', a_k) \leq 2k$, and in the case of an infinite tree we will always be able to find at least one vertex $\tilde{a}$ such that $d(a, \tilde{a}) = k$ and $d(\tilde{a}, a_k) = 2k$.

Since we have that $2k \leq l$, that $d(a', a_k) \leq 2k$ for all $a' \in T$ with $d(a, a') \leq k$, and that $[a', \omega)$ coincides with $[a_k, \omega)$ from the point a_k onwards for all these vertices, it is clear that $F(a, \omega, k, l)$ will be the characteristic function of a segment of the path $[a_k, \omega)$. The starting vertex for this segment will be at a distance l from $\tilde{a}$, and the end vertex will be at a distance $2l$ from a_k, hence $F(a, \omega, k, l)$ will be the characteristic function of a path of length $(2l + d(\tilde{a}, a_k)) - l = 2l + 2k - l = l + 2k$. Thus $\|F(a, \omega, k, l)\| = l + 2k$, regardless of the choice of a.

Therefore, we have

$$\begin{aligned}\|H(a,\omega,l)\| &= \left\| \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} F(a,\omega,k,l) \right\| = \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} \|F(a,\omega,k,l)\| \\ &= \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} (l+2k) = \frac{\sqrt{l}(l+2k)}{\sqrt{l}} = l+2k.\end{aligned}$$

□

This can be used to show, as in Germain's proof, that we have

$$\sup_{\omega\in\partial\Gamma} \|H(a,\omega,l) - H(b,\omega,l)\| = o(l),$$

for any fixed vertices $a, b \in T$.

For his functions related to discrete hyperbolic groups, Germain additionally showed that $(\omega, t) \mapsto H(a,\omega,l)(t)$ is upper continuous for a and l fixed. He achieved this by proving that for k, l and a fixed, $(\omega, t) \mapsto F(a,\omega,k,l)(t)$ has a local maximum everywhere, via a short lemma, and his proof should follow exactly for the tree case as well.

Hence the functions we have constructed satisfy the requirements imposed by Germain. We may now allow our tree T to be the Cayley graph of some finitely generated free group $\mathbb{F}_n$, and follow his proof that discrete hyperbolic groups act amenably on their boundaries to obtain another method for showing that finitely generated free groups are exact.

THEOREM 2.30 *The action of $\mathbb{F}_n$ on its boundary $\partial\mathbb{F}_n$ is amenable.*

Proof.

We now prove amenability of the action using Definition 2.27. Let $f_i(g,\omega) = H(e,\omega,i)(g)$, for all $g \in \mathbb{F}_n$, $\omega \in \partial\mathbb{F}_n$. Then, as with Germain's version (see page 134 of [3]), f_i is positive, Borel and compactly supported (the support is contained within $B(e, 3l) \times \partial\mathbb{F}_n$). As $\|H(a,\omega,i)\| = i+2k > i$ for all $\omega \in \partial\mathbb{F}_n$, by Lemma 2.29, we have $\int_{\mathbb{F}_n} f_i(g,\omega)dg > 0$, for all $\omega \in \partial\mathbb{F}_n$ and $i > 0$.

Finally, $(h \cdot f_i)(g, \omega) = H(e, h^{-1}\omega, i)(h^{-1}g) = H(h, \omega, i)(g)$, and so

$$\begin{aligned}
&\lim_{i\to\infty}\left(\sup_{\omega\in\partial\mathbb{F}_n}\left\{\frac{\int_{\mathbb{F}_n}|f_i(g,\omega) - h\cdot f_i(g,\omega)|dg}{\int_{\mathbb{F}_n} f_i(g,x)dg}\right\}\right)\\
&= \lim_{i\to\infty}\left(\sup_{\omega\in\partial\mathbb{F}_n}\left\{\frac{\|H(e,\omega,i) - H(h,\omega,i)\|}{\|H(e,\omega,i)\|}\right\}\right)\\
&= \lim_{i\to\infty}\left(\frac{o(i)}{i+2k}\right) = 0,
\end{aligned}$$

for all $h \in \mathbb{F}_n$.

$\square$

Unfortunately, due to the nature of the functions H which we employ here, it does not appear that this method could be adapted to prove amenability of the action of a group on a tree where there is no existing group structure on the tree itself.

3 Partial Translation Structures

The notion of a partial translation structure was introduced in [11] as a means to finding a good analogue of the reduced C^*-algebra of a group, as well as capturing geometrically the interplay between the left and right action of a group on itself. When defined on discrete metric spaces, these structures in some sense blur the boundary between the world of such spaces and the world of groups, by mimicking the interaction between the left and the right multiplication on a discrete group. Here we recall the definitions required to understand partial translation structures and prove some new results which can be applied in the general case of metric spaces.

3.1 Basic Definitions

The following definitions all appear in [11]. The basic notion of a partial translation of a space was first introduced in [29].

Let X be a discrete metric space.

DEFINITION 3.1 [Partial Bijection]

A *partial bijection* from X to X is a subset s of $X \times X$ such that the coordinate projections of s onto X are injective.

A partial bijection can be viewed as a partially defined injection from X into X, and we will write $x = s(y)$ if $(x, y) \in s$.

DEFINITION 3.2 [Partial Translation]

A *partial translation* of X is a partial bijection t such that $d(x, y)$ is bounded for all $(x, y) \in t$. The *identity* translation, denoted 1, is the diagonal of $X \times X$. The *inverse* of a partial translation t is $t^{-1} = \{(y, x) \mid (x, y) \in t\}$.

DEFINITION 3.3 [Partial Cotranslation]

Let $\mathcal{T}$ be a collection of disjoint partial translations of X. A partial bijection σ of X is a *partial cotranslation for* $\mathcal{T}$ if $(\sigma x, \sigma y) \in t$ for all $t \in \mathcal{T}$ and for all $(x, y) \in t$ such that σ is defined on both x and y.

DEFINITION 3.4 [Partial Translation Structure]

A *partial translation structure* on X is a collection $\mathcal{T}$ of partial translations of X, such that for all $R > 0$ there is a finite subset $\mathcal{T}_R$ of disjoint partial translations in $\mathcal{T}$, and a collection Σ_R of partial cotranslations for $\mathcal{T}_R$, satisfying the following axioms.

1. The union of all the partial translations t in $\mathcal{T}_R$ contains the R-neighbourhood of the diagonal, that is the set of all $(x, y) \in X \times X$ such that $d(x, y) < R$.

2. There exists k such that for each x, x' in X there are at most k elements σ in Σ_R such that $\sigma x = x'$.

3. For each t in $\mathcal{T}_R$ and for all $(x, y), (x', y')$ in t, there exists σ in Σ_R such that $\sigma x = x'$ and $\sigma y = y'$.

Note that in the above we implicitly assume that the identity translation is an element of each cotranslation collection Σ_R.

Recall that a metric space X is said to be *uniformly discrete* if there exists some $r > 0$ such that either $x = y$ or $d(x, y) > r$ for all $x, y \in X$. It is also shown in [11] that every bounded geometry uniformly discrete metric space admits a partial translation structure.

For $\mathcal{T}$ a partial translation structure on X, and for $R > 0$, let $k_{\mathcal{T}}(R)$ denote the smallest k such that for x, x' in X there are at most k elements σ in Σ_R with $\sigma x = x'$. The smallest value any $k_{\mathcal{T}}(R)$ can take is 1, since we assume that every Σ_R contains at least the identity translation.

DEFINITION 3.5 [The Translation Invariant]

For (X, d) a bounded geometry uniformly discrete metric space, define the *translation invariant* of X to be the function

$$\kappa_X(R) = \inf\{k_{\mathcal{T}}(R) \mid \mathcal{T} \text{ a partial translation structure on } X\}.$$

DEFINITION 3.6 [Free Partial Translation Structure]

A partial translation structure $\mathcal{T}$ is called *free* if $k_{\mathcal{T}}(R) = 1$ for all $R > 0$.

DEFINITION 3.7 [Globally Controlled Partial Translation Structure]

We say a partial translation structure is *globally controlled* if the partial cotranslation orbit

$$\{(x', y') \mid \text{ there exists } \sigma \in \Sigma_R \text{ such that } \sigma x = x', \sigma y = y'\}$$

is a partial translation for all $R > 0$ and $x, y \in X$.

3.2 Restrictions and Images of Partial Translation Structures

In this section we prove that partial translation structures restrict to subspaces and are preserved by uniform bijections, and that these actions also preserve freeness and global control of the structures. This allows us to extend a result of Brodzki, Niblo and Wright, as well as to provide a shorter proof in the case covered by their theorem.

THEOREM 3.8 *Let X be a metric space endowed with partial translation structure $\mathcal{T}$. Then $\mathcal{T}$ can be restricted to a partial translation structure $\mathcal{T}_Y$ for any subspace Y of X.*

Proof.

Considering $\mathcal{T}$ as a collection of partial translations, we restrict to Y by setting

$$\mathcal{T}_Y = \mathcal{T} \cap (Y \times Y).$$

In other words, when viewing each partial translation $t \in \mathcal{T}$ as a subset of $X \times X$, we intersect it with $Y \times Y$ to obtain a partial translation t_Y of Y (we could also view it as a map and restrict its domain and codomain). As t_Y is a subset of t, the coordinate projections on t_Y are injective. Moreover, the distance $d(y, y')$ is bounded for all $(y, y') \in t_Y$, so t_Y will indeed be a partial translation of Y. It is also the case that $\mathcal{T}_Y$ is not an empty set so long as Y is not, since we know by the first condition of Definition 3.4 for $\mathcal{T}$ that at the very least $\cup_{t \in \mathcal{T}} t$ contains the diagonal of $X \times X$, so $\cup_{t_Y \in \mathcal{T}_Y} t_Y$ at the least contains the diagonal of $Y \times Y$.

It remains to check the three conditions of Definition 3.4 for $\mathcal{T}_Y$.

(1) We know by condition (1) for $\mathcal{T}$ that the union of partial translations in $\mathcal{T}_R$ contains the R-neighbourhood of the diagonal in $X \times X$, and thus it contains the R-neighbourhood of the diagonal in $Y \times Y$. Hence if we restrict every partial translation in $\mathcal{T}_R$ to $Y \times Y$, so that we let $\mathcal{T}_{YR} = \mathcal{T}_R \cap (Y \times Y)$, then the R-neighbourhood of the diagonal in $Y \times Y$ is still contained in this set. So condition (1) of Definition 3.4 holds for $\mathcal{T}_Y$.

(2) We can restrict the partial cotranslations for $\mathcal{T}$ in a similar way to the partial translations, so that we are left with partial cotranslations for $\mathcal{T}_Y$. Each cotranslation $\sigma \in \Sigma_R$ for some $R > 0$ must have its domain restricted to Y, but we must also restrict its codomain to ensure that this too is a subset of Y. So given a cotranslation $\sigma \in \Sigma_R$, let us define σ_Y to be the restriction of σ to the set $\{y \in Dom(\sigma) \cap Y \mid \sigma y \in Y\}$. This partial bijection will have image $\sigma(Y) \cap Y$, and it is clear from Definition 3.3 that it is indeed a partial cotranslation for $\mathcal{T}_{YR}$ if σ is one for $\mathcal{T}_R$.

If we then let $\Sigma_{YR} = \{\sigma_Y \mid \sigma \in \Sigma_R\}$, for all $R > 0$, then this set is a suitable collection of cotranslations in the sense of condition (2) of Definition 3.4. Indeed, if there exists k such that for each $x, x' \in X$ there are at most k elements $\sigma \in \Sigma_R$ such that $\sigma x = x'$, then for each $y, y' \in Y \subseteq X$ there certainly cannot be more than k elements $\sigma_Y \in \Sigma_{YR}$ such that $\sigma_Y y = y'$.

Note also that each Σ_{YR} is a non-empty set, since it at least contains the identity translation restricted to $Y \times Y$. Each $\mathcal{T}_{YR}$ is also non-empty as it at least contains the R-neighbourhood of the diagonal in $Y \times Y$.

(3) Let $t_Y \in \mathcal{T}_{YR}$, so that t_Y is the restriction to $Y \times Y$ of some $t \in \mathcal{T}_R$. Then every element of t_Y is also an element of t, and thus for all $(y_1, y_1'), (y_2, y_2') \in t_Y$ there exists $\sigma \in \Sigma_R$ such that $\sigma y_1 = y_2$ and $\sigma y_1' = y_2'$, by condition (3) of Definition 3.4 for $\mathcal{T}$. If we restrict σ to $\{y \in Dom(\sigma) \cap Y \mid \sigma y \in Y\}$, then by definition we obtain a partial cotranslation $\sigma_Y \in \Sigma_{YR}$, for which the same holds. Hence, for each $t_Y \in \mathcal{T}_{YR}$ there exists $\sigma_Y \in \Sigma_{YR}$ such that $\sigma_Y y_1 = y_2$ and $\sigma_Y y_1' = y_2'$ for all $(y_1, y_1'), (y_2, y_2') \in t_Y$. Thus condition (3) also holds for $\mathcal{T}_Y$, and so any partial translation structure

restricts to a subspace.

$\square$

COROLLARY 3.9 *Let X be a metric space with partial translation structure $\mathcal{T}$, and let $\mathcal{T}_Y$ be the restriction of $\mathcal{T}$ to a subspace Y of X. If $\mathcal{T}$ is free (resp. globally controlled) then $\mathcal{T}_Y$ is free (resp. globally controlled).*

Proof.

By definition, $k_{\mathcal{T}}(R)$ is the smallest k such that for all $x, x' \in X$ there exist at most k elements $\sigma \in \Sigma_R$ with $\sigma x = x'$, for all $R > 0$. For each $R > 0$ we have

$$k_{\mathcal{T}_Y}(R) \leq k_{\mathcal{T}}(R).$$

Indeed, let $y, y' \in Y \subseteq X$ and let $k_{\mathcal{T}}(R) = k$. Then there exist at most k elements $\sigma \in \Sigma_R$ with $\sigma y = y'$. Each such σ must have y in its domain, so as y and y' are both elements of Y we also have $\sigma_Y y = y'$ for the corresponding $\sigma_Y \in \Sigma_{YR}$, for each σ. As this is the only way elements of Σ_{YR} can be constructed, we have $k_{\mathcal{T}_Y}(R) \leq k = k_{\mathcal{T}}(R)$.

Thus if we assume that $\mathcal{T}$ is free, i.e. that $k_{\mathcal{T}}(R) = 1$ for all $R > 0$, then we have $k_{\mathcal{T}_Y}(R) \leq 1$ for all $R > 0$. Since 1 is the smallest value $k_{\mathcal{T}_Y}(R)$ can take, unless Σ_{YR} is an empty set, which is not possible, we in fact have $k_{\mathcal{T}_Y}(R) = 1$ for all $R > 0$, in other words $\mathcal{T}_Y$ is also a free partial translation structure.

Now suppose $\mathcal{T}$ is globally controlled. This means that the partial cotranslation orbit

$$\{(x'_1, x'_2) \mid \text{ there exists } \sigma \in \Sigma_R \text{ such that } \sigma x_1 = x'_1, \sigma x_2 = x'_2\}$$

is a partial translation for all $R > 0$ and for all $x_1, x_2 \in X$. Let us consider a partial cotranslation orbit for $\mathcal{T}_Y$, that is a set

$$\{(y'_1, y'_2) \mid \text{ there exists } \sigma_Y \in \Sigma_{YR} \text{ such that } \sigma_Y y_1 = y'_1, \sigma_Y y_2 = y'_2\}.$$

We wish to show that this is a partial translation for all $R > 0$ and $y_1, y_2 \in Y$.

Fix $R > 0$ and choose some $y_1, y_2 \in Y$. Since we are assuming that all y_1, y'_1, y_2 and y'_2 considered here are elements of Y, to state that $\sigma_Y y_1 = y'_1$ and $\sigma_Y y_2 = y'_2$ is equivalent to saying that $\sigma y_1 = y'_1$ and $\sigma y_2 = y'_2$, where $\sigma \in \Sigma_R$ is the partial cotranslation for $\mathcal{T}$ from which σ_Y was formed by restriction. Thus, since $Y \times Y$ is contained in $X \times X$, the partial cotranslation orbit for $\mathcal{T}_Y$ that we are considering is a subset of the corresponding partial cotranslation orbit for $\mathcal{T}$ (i.e. the one formed using the same $R > 0$ and points $y_1, y_2 \in Y$). Hence, as we assume this to be a partial translation, we know that the coordinate projections of our set onto Y are injective and that $d(y_1', y_2')$ is bounded for all (y_1', y_2') in the partial cotranslation orbit for $\mathcal{T}_Y$. Thus this is indeed also a partial translation.

In fact, the partial translation we obtain is the restriction to $Y \times Y$ of the corresponding partial cotranslation orbit for $\mathcal{T}$. Hence if $\mathcal{T}$ is globally controlled and every partial cotranslation orbit is a partial translation in $\mathcal{T}$, then every partial cotranslation orbit for $\mathcal{T}_Y$ is a partial translation in $\mathcal{T}_Y$.

□

We now wish to show that partial translation structures can be transported by uniform bijections. Let us begin by recalling the definition of uniformity and explaining how maps between metric spaces can be applied to partial translations.

DEFINITION 3.10 [Controlled Map]

A map ϕ between two metric spaces X and Y is called *controlled* if and only if, for every $R > 0$, there exists an $S > 0$ such that

$$d(x, y) \leq R \Rightarrow d(\phi(x), \phi(y)) \leq S,$$

for all $x, y \in X$.

DEFINITION 3.11 [Uniform Map]

A map of metric spaces $\phi : X \to Y$ is called *uniform* if and only if ϕ is controlled and for every $S > 0$ there exists an $R > 0$ such that

$$d(x, y) \geq R \Rightarrow d(\phi(x), \phi(y)) \geq S,$$

for all $x, y \in X$ [19].

Now suppose that X and Y are discrete metric spaces for which there exists an injective map $\phi : X \to Y$. For t a partial bijection on X, define a partial bijection on Y by

$$\phi(t) = \{(\phi(t(x)), \phi(x)) \mid (t(x), x) \in t\}.$$

In other words, if t represents a bijection between the set $S \subseteq X$ and $t(S)$, then $\phi(t)$ is a bijection between $\phi(S)$ and $\phi(t(S))$, mapping $y \in \phi(S)$ to $\phi(t(\phi^{-1}(y))) \in \phi(t(S))$. Thus, as a map defined on $\phi(X) \subseteq Y$, we have

$$\phi(t) = \phi \circ t \circ \phi^{-1}.$$

The coordinate projections of $\phi(t)$ are clearly injective if t is a partial translation and ϕ an injective map. By Definition 3.2, there exists some $R > 0$ such that $d(t(x), x) \leq R$ for all $x \in Dom(t)$. Hence if we assume ϕ to be controlled then there exists an $S > 0$ such that $d(\phi(t(x)), \phi(x)) \leq S$ for all $\phi(x) \in Dom(\phi(t)) = \phi(Dom(t))$. Therefore if ϕ is also controlled then it sends partial translations of X to partial translations of Y, by the method detailed above.

For $\mathcal{T}$ a partial translation structure on X, let $\phi(\mathcal{T})$ denote the disjoint collection of partial translations $\phi(t)$ for t a partial translation in $\mathcal{T}$.

LEMMA 3.12 *Let $\phi : X \to Y$ be an injective controlled map, and let $\mathcal{T}$ be a partial translation structure on X. Then ϕ establishes a bijection between partial cotranslations for $\mathcal{T}$ and partial cotranslations for $\phi(\mathcal{T})$.*

Proof.

We know from the above that ϕ maps partial bijections of X to partial bijections of Y. For every $R > 0$, let $\mathcal{T}_R$ be the subset of partial translations in $\mathcal{T}$ described in Definition 3.4, and Σ_R the set of partial cotranslations for $\mathcal{T}_R$. To show that $\phi(\sigma)$ is a partial cotranslation for $\phi(\mathcal{T}_R)$ when $\sigma \in \Sigma_R$, consider some partial translation t in $\mathcal{T}_R$. Let $(x, y) \in t$, so that $(\phi(x), \phi(y)) \in \phi(t) \in \phi(\mathcal{T}_R)$, and suppose that $\phi(\sigma)$ is defined on both $\phi(x)$ and $\phi(y)$. We wish to show that $(\phi(\sigma)\phi(x), \phi(\sigma)\phi(y)) \in \phi(t)$, i.e. that

$$\phi(t)(\phi(\sigma)\phi(y)) = \phi(\sigma)\phi(x).$$

We have

$$\begin{aligned}
\phi(t)(\phi(\sigma)\phi(y)) &= \phi \circ t \circ \phi^{-1}(\phi \circ \sigma \circ \phi^{-1}(\phi(y))) \\
&= \phi \circ t(\sigma y) \\
&= \phi \circ \sigma x, \text{ as } \sigma \text{ is a partial cotranslation for } \mathcal{T}_R, \\
&= \phi \circ \sigma \circ \phi^{-1}(\phi(x)) \\
&= \phi(\sigma)(\phi(x)), \text{ as required.}
\end{aligned}$$

So $\phi(\sigma)$ is indeed a partial cotranslation for $\phi(\mathcal{T}_R)$. It remains to check that $\sigma \mapsto \phi(\sigma)$ is a bijection.

Firstly, suppose that $\phi(\sigma) = \phi(\sigma')$ for two partial cotranslations $\sigma, \sigma' \in \Sigma_R$. We have that $\phi(\sigma) : \phi(Dom(\sigma)) \to \phi(Ran(\sigma))$ and $\phi(\sigma') : \phi(Dom(\sigma')) \to \phi(Ran(\sigma'))$, thus $\phi(Dom(\sigma)) = \phi(Dom(\sigma'))$ and $\phi(Ran(\sigma)) = \phi(Ran(\sigma'))$. As ϕ is injective, this means that $Dom(\sigma) = Dom(\sigma')$ and $Ran(\sigma) = Ran(\sigma')$. Let $S \subseteq X$ denote the set $Dom(\sigma) = Dom(\sigma')$.

We have $\phi(\sigma)\phi(s) = \phi(\sigma')\phi(s)$, for all $\phi(s) \in \phi(S)$. Hence $\phi \circ \sigma \circ \phi^{-1}(\phi(s)) = \phi \circ \sigma' \circ \phi^{-1}(\phi(s))$, that is $\phi \circ \sigma s = \phi \circ \sigma' s$, for all $s \in X$ such that $\phi(s) \in \phi(S)$. Therefore, since ϕ is injective, we have $\sigma s = \sigma' s$ for all $s \in S$ such that $\phi(s) \in \phi(S)$, i.e. for all $s \in S$. Hence we have injectivity.

We now need only show that, for every $R > 0$, every partial cotranslation for $\phi(\mathcal{T}_R)$ is of the form $\phi(\sigma)$ for some $\sigma \in \Sigma_R$. So let us consider an arbitrary partial cotranslation θ for $\phi(\mathcal{T}_R)$, which must be a partial bijection of $\phi(X)$. For each $x \in X$, if $\phi(x)$ lies in the domain of θ then $\theta\phi(x) = \phi(z)$ for some $z \in X$, which is uniquely determined, by injectivity of ϕ. Hence we may define a function $\sigma : X \to X$ by

$$\sigma x = z = \phi^{-1}(\theta\phi(x)),$$

for all x in $\{x \in X \mid \phi(x) \in Dom(\theta)\}$. Now θ is of the form $\phi(\sigma) = \phi \circ \sigma \circ \phi^{-1}$ for the partially defined map $\sigma : X \to X$. This function must be injective, since both θ and ϕ are.

As θ is a partial cotranslation for $\phi(\mathcal{T}_R)$, we have $\phi(t)(\theta\phi(y)) = \theta\phi(x)$ for all partial translations $\phi(t)$ in $\phi(\mathcal{T}_R)$ and for all $(\phi(x), \phi(y)) \in \phi(t)$ where θ is defined on both $\phi(x)$ and $\phi(y)$.

Thus $\phi(t)(\phi(\sigma)\phi(y)) = \phi(\sigma)\phi(x)$, that is $\phi \circ t \circ \phi^{-1}(\phi \circ \sigma \circ \phi^{-1}(\phi(y))) = \phi \circ \sigma \circ \phi^{-1}(\phi(x))$, and so $\phi \circ t(\sigma y) = \phi \circ \sigma x$, for all t, x and y satisfying the above. Hence $t(\sigma y) = \sigma x$, by injectivity of ϕ. Therefore $(\sigma x, \sigma y) \in t$.

This holds for all $\phi(t) \in \phi(\mathcal{T}_R)$ and for all $(\phi(x), \phi(y)) \in \phi(t)$, so it holds for all partial translations t in $\mathcal{T}_R$ and for all $(x, y) \in t$, since ϕ is a bijection onto its image. Thus σ is a partial cotranslation for $\mathcal{T}_R$ and θ is of the form $\phi(\sigma)$. The argument of this proof applies for any $R > 0$, so ϕ is surjective on, and hence establishes a bijection between, all partial cotranslations.

□

THEOREM 3.13 *Let X be a metric space equipped with a partial translation structure $\mathcal{T}$, and let $\phi : X \to Y$ be a uniform bijection. Then $\phi(\mathcal{T})$ is a partial translation structure for Y.*

Proof.

If we assume $\phi : X \to Y$ to be a uniform bijection then it is both controlled and injective, and thus sends partial translations in X to partial translations in Y. It remains to check the conditions of Definition 3.4 for $\phi(\mathcal{T})$.

(1) Firstly, note that we certainly require ϕ to be surjective for this condition to hold. Indeed, if it was not then there would exist some $y \in Y$ such that $y \neq \phi(x)$ for any $x \in X$. Then (y, y) could not possibly be an element of $\phi(t) = \{(\phi(t(x)), \phi(x)) \mid (t(x), x) \in t\}$ for any $t \in \mathcal{T}$, so it would not lie in the union of partial translations in $\phi(\mathcal{T}_R)$ for any $R > 0$, and thus none of these sets would even contain the whole of the diagonal in $Y \times Y$, let alone its R-neighbourhood.

Recall that to say ϕ is uniform means that it is controlled, and also that for all $S > 0$ there exists $R > 0$ such that

$$d(x, x') \geq R \Rightarrow d(\phi(x), \phi(x')) \geq S.$$

That is, for all $S > 0$ there exists $R > 0$ such that

$$d(\phi(x), \phi(x')) < S \Rightarrow d(x, x') < R.$$

So consider a point $(y, y') \in Y \times Y$ which lies in the S-neighbourhood of the diagonal for some $S > 0$. Since ϕ is surjective, we know that $(y, y') = (\phi(x), \phi(x'))$ for some $(x, x') \in X \times X$, and by uniformity of ϕ we know that (x, x') lies in the R-neighbourhood of the diagonal in $X \times X$ for some

$R > 0$. Thus (x, x') is an element of some partial translation t in $\mathcal{T}_R$ (viewed as a subset of $X \times X$), and therefore (y, y') lies in the partial translation $\phi(t)$, which is an element of $\phi(\mathcal{T}_R)$. So we have condition (1) for $\phi(\mathcal{T})$ if we define the collection $\phi(\mathcal{T})_S$ to be $\phi(\mathcal{T}_R)$. Note that the partial translations contained in this collection are indeed disjoint since the partial translations in each $\mathcal{T}_R$ are disjoint by assumption and ϕ is a bijection.

(2) By Lemma 3.12, we see that ϕ maps partial cotranslations for $\mathcal{T}_R$ bijectively to partial cotranslations for $\phi(\mathcal{T}_R)$, for every $R > 0$. Hence $\phi(\Sigma_R)$ is a collection of partial cotranslations for $\phi(\mathcal{T})_S$, and it is clear by bijectivity that condition (2) of Definition 3.4 holds for $\phi(\mathcal{T})$.

(3) Let $t \in \mathcal{T}_R$, so that $\phi(t) \in \phi(\mathcal{T})_S$, for some $S > 0$. Then by condition (3) of Definition 3.4 for $\mathcal{T}$, for every $(x, y), (x', y')$ in t, there exists σ in Σ_R such that $\sigma x = x'$ and $\sigma y = y'$. Hence for every $(\phi(x), \phi(y)), (\phi(x'), \phi(y'))$ in $\phi(t)$, there exists $\phi(\sigma)$ in $\phi(\Sigma_R)$ such that $\phi(\sigma)\phi(x) = \phi \circ \sigma \circ \phi^{-1}(\phi(x)) = \phi(\sigma x) = \phi(x')$ and, similarly, $\phi(\sigma)\phi(y) = \phi(y')$. Therefore condition (3) of Definition 3.4 also holds for $\phi(\mathcal{T})$, and so $\phi(\mathcal{T})$ is a partial translation structure for Y.

$\square$

COROLLARY 3.14 *Let X be a metric space equipped with a partial translation structure $\mathcal{T}$, and let $\phi : X \to Y$ be a uniform bijection. If $\mathcal{T}$ is free (resp. globally controlled) then $\phi(\mathcal{T})$ is free (resp. globally controlled).*

Proof.

For every $S > 0$ we consider the partial cotranslation collection $\phi(\Sigma)_S$ to be $\phi(\Sigma_R)$, where R is the value related to S by the uniformity condition for ϕ, as in the proof of Theorem 3.13. Thus to show that $k_{\phi(\mathcal{T})}(S) = 1$ for all $S > 0$ it suffices to show that, for any $R > 0$, for all $y, y' \in Y$ there can be no more than one $\phi(\sigma) \in \phi(\Sigma_R)$, such that

$$\phi(\sigma)(y) = y'.$$

As ϕ is a bijection, each $y \in Y$ can be written as $y = \phi(x)$ for some $x \in X$, and therefore the above is only the case when

$$\phi \circ \sigma \circ \phi^{-1}(\phi(x)) = \phi(x')$$

for some pair of elements $x, x' \in X$. That is, when

$$\sigma x = x',$$

since ϕ is injective. Hence, if we assume that $k_{\mathcal{T}}(R) = k$, then we know that there at most k elements $\sigma \in \Sigma_R$ satisfying this for any $x, x' \in X$, and thus there are at most k elements $\phi(\sigma) \in \phi(\Sigma_R) = \phi(\Sigma)_S$ which satisfy $\phi(\sigma)(y) = y'$, for all $y, y' \in Y$. So we have $k_{\phi(\mathcal{T})}(S) \leq k = k_{\mathcal{T}}(R)$. Therefore if $\mathcal{T}$ is free, in other words if $k_{\mathcal{T}}(R) = 1$ for all $R > 0$, then $\phi(\mathcal{T})$ is also free, as the minimum value $k_{\phi(\mathcal{T})}(S)$ can take is 1.

Moreover, as $\sigma \mapsto \phi(\sigma)$ is a bijection, by Lemma 3.12, and the implication can be reversed at each stage of the above argument, we in fact have

$$k_{\mathcal{T}}(R) = k_{\phi(\mathcal{T})}(S)$$

for all $R, S > 0$ which correspond in the sense of Definition 3.11.

Now suppose that $\mathcal{T}$ is a globally controlled partial translation structure. That is, each partial cotranslation orbit

$$\{(x_1', x_2') \mid \text{there exists } \sigma \in \Sigma_R \text{ such that } \sigma x_1 = x_1', \sigma x_2 = x_2'\}$$

is a partial translation, for $x_1, x_2 \in X$ and for all $R > 0$. Let us consider a general partial cotranslation orbit for $\phi(\mathcal{T})$:

$$\{(y_1', y_2') \mid \text{there exists } \phi(\sigma) \in \phi(\Sigma_R) \text{ such that } \phi(\sigma)(y_1) = y_1', \phi(\sigma)(y_2) = y_2'\},$$

for some $y_1, y_2 \in Y$ and some $R > 0$. Since ϕ establishes a bijection both between elements of X and elements of Y and between partial cotranslations for $\mathcal{T}$ and partial cotranslations for $\phi(\mathcal{T})$, the above set can be rewritten as

$$\begin{aligned} \{(\phi(x_1'), \phi(x_2')) \mid \text{there exists } \sigma \in \Sigma_R \text{ such that } \phi \circ \sigma \circ \phi^{-1}(\phi(x_1)) = \phi(x_1') \\ \text{and } \phi \circ \sigma \circ \phi^{-1}(\phi(x_2)) = \phi(x_2')\} \\ = \{(\phi(x_1'), \phi(x_2')) \mid \text{there exists } \sigma \in \Sigma_R \text{ such that } \sigma x_1 = x_1' \text{and } \sigma x_2 = x_2'\}, \end{aligned}$$

for some $x_1, x_2 \in X$. But this is just $\phi(t)$ for some partial cotranslation orbit

$$t = \{(x_1', x_2') \mid \text{there exists } \sigma \in \Sigma_R \text{ such that } \sigma x_1 = x_1', \sigma x_2 = x_2'\}$$

for $\mathcal{T}$, and hence it must be a partial translation of Y since we assume t to be a partial translation of X (if $\mathcal{T}$ is globally controlled) and we know that ϕ maps partial translations of X to partial translations of Y, as ϕ is controlled and injective. This will hold for any partial cotranslation for $\phi(\mathcal{T})$; thus $\phi(\mathcal{T})$ is globally controlled.

□

REMARK 3.15 Note that we could restate the above theorem and corollary to say that $\phi(\mathcal{T})$ is a (free, globally controlled) partial translation structure for $\phi(X)$ whenever $\phi : X \to Y$ is an injective uniform map. Since every map is surjective onto its own image, the proof would follow identically.

These theorems and corollaries allow us to construct a new and more concise proof of the following, which appears as Theorem 19 in [11].

COROLLARY 3.16 *Let X be a space admitting an injective uniform embedding into some discrete group G. Then X admits a free and globally controlled partial translation structure.*

Proof.

Let $\phi : X \to G$ denote the injective uniform embedding of X into G. Then ϕ is a uniform bijection onto its image $\phi(X) =: Y$, a subspace of G.

It is shown in [11] that any countable discrete group admits a canonical partial translation structure which is free and globally controlled; let us denote the canonical partial translation structure for G by $\mathcal{T}$. It can be seen from the proof of Proposition 15 in [11] that this structure consists of partial translations of the form

$$t_g = \{(x, xg) \mid x \in G\},$$

and partial cotranslations defined by $\sigma_h(x) = hx$, one of each for each element of the group. For every $R > 0$ we take $\mathcal{T}_R = \{t_g \mid d(e, g) < R\}$ and $\Sigma_R = \{\sigma_h \mid h \in G\}$.

Theorem 3.8, together with Corollary 3.9, now tells us that $\mathcal{T}$ restricts to a free and globally controlled partial translation structure $\mathcal{T}_Y$ on Y. If $\phi : X \to Y$ is a uniform bijection then so is its inverse $\phi^{-1} : Y \to X$, and hence $\phi^{-1}(\mathcal{T}_Y)$ is a free and globally controlled partial translation structure on X, by Theorem 3.13 and Corollary 3.14.

$\square$

REMARK 3.17 In particular, if X is a space admitting an injective uniform embedding ϕ into a discrete group G, then we have $\kappa_X(S) = 1$ for all $S > 0$. Indeed, by the proof of Proposition 15 in [11] we have that $k_{\mathcal{T}}(R) = 1$ for all $R > 0$, and so $\kappa_G(R)$ is also 1 for all R. Thus by the proof of Corollary 3.9 we have

$$k_{\mathcal{T}_Y}(R) \leq k_{\mathcal{T}}(R) = 1,$$

so that $k_{\mathcal{T}_Y}(R) = 1$ for all $R > 0$; and by the proof of Corollary 3.14 we have

$$k_{\phi^{-1}(\mathcal{T}_Y)}(S) = k_{\mathcal{T}_Y}(R) = 1,$$

for all $S > 0$, where R is the value associated with S by the uniformity condition for ϕ^{-1}. Therefore $\kappa_X(S)$ is also 1 for all $S > 0$.

3.3 Group Actions and Cotranslation Orbits

In order to gain a better understanding of how partial translation structures may arise, in this section we shall study cotranslation orbits and provide some details relating to group actions.

In the case of the canonical partial translation structure on a discrete group, the partial translations and cotranslations correspond with respectively the right and left multiplication actions of the group elements. Thus if such a group G acts on some space X on the left, it makes sense to study orbits of the canonical cotranslations within that space. We show that under certain conditions one may use these cotranslation orbits as the partial translations in a partial translation structure on X, with the same set of cotranslations as in the canonical partial translation structure on G.

Recall that the action of a group G on a space X is called *regular* if it is both transitive and free.

THEOREM 3.18 *Let G be a discrete group acting regularly on a bounded geometry metric space X in such a way that for all $R > 0$ there exists $S > 0$*

such that

$$d(x,y) \leq R \Rightarrow d(gx,gy) \leq S,$$

for all $g \in G$, $x, y \in X$. Then the cotranslation orbits form a partial translation structure on X.

Proof.

Let us begin by fixing a basepoint $x_0 \in X$. Now, by *cotranslation orbits*, we mean orbits of the action of G on points $(x_0, x) \in X \times X$. Thus our partial translations here are subsets of $X \times X$ of the form

$$t_x = \{(gx_0, gx) \mid g \in G\},$$

for some $x \in X$. We assume the action of G on X to be free, that is

$$gx \neq g'x$$

for all distinct $g, g' \in G$ and for all $x \in X$, so the coordinate projections of every such t_x are injective. To see that the distance function is bounded on each set t_x, let us first define the $\mathcal{T}_R$ sets for our partial translation structure. For every $R > 0$, let

$$\mathcal{T}_R = \{t_x \mid x \in B_S(x_0)\},$$

where each S corresponds to the R we are considering in the way stated in the assumptions of the theorem; we define our $\mathcal{T}_R$ sets in this manner to enable us to prove that condition (1) of Definition 3.4 holds. We only consider partial translations which are contained in some such $\mathcal{T}_R$, and so for every t_x there exists some $S > 0$ such that $d(x, x_0) < S$, and thus by reapplying the hypothesis of the theorem there exists some $S' > 0$ such that $d(gx, gx_0) < S'$ for all $g \in G$; hence each t_x is indeed a partial translation. Note also that every $\mathcal{T}_R$ is a finite set since we assume X to have bounded geometry. In addition to this, if the elements of some $\mathcal{T}_R$ were not disjoint, then we would have $(gx_0, gx) = (g'x_0, g'x')$ for some $g, g' \in G$, $x, x' \in X$, so that $g = g'$, by the freeness of the action. Then $gx = gx'$, so $g^{-1}gx = g^{-1}gx'$, and thus $x = x'$. Therefore every $\mathcal{T}_R$ is indeed a finite set of disjoint partial translations.

We define cotranslations for our entire set of partial translations by

$$\sigma_h : X \to X, \ \sigma_h(x) := hx,$$

so that we have one for each element $h \in G$, and $\Sigma_R = \{\sigma_h \mid h \in G\}$ for every $R > 0$. Then

$$(\sigma_h(gx_0), \sigma_h(gx)) = (hgx_0, hgx) = ((hg)x_0, (hg)x) \in t_x,$$

for all $\sigma_h \in \Sigma_R, t_x \in \mathcal{T}_R$ and $(gx_0, gx) \in t_x$.

It remains to check the three conditions of Definition 3.4.

(1) Let (x, y) be a point in the R-neighbourhood of the diagonal in $X \times X$, so that $x, y \in X$ and $d(x, y) \leq R$. We wish to find some $t_z \in \mathcal{T}_R$ such that $(x, y) \in t_z$. That is, we want to show that $(x, y) = (gx_0, gz)$ for some $g \in G$, $z \in B_S(x_0)$, where $d(hx, hy) \leq S$ for all $h \in G$ (such an S exists by assumption). By transitivity, there exists some $g \in G$ such that $gx_0 = x$. So now $(x, y) = (gx_0, g(g^{-1}y))$, and $d(x_0, g^{-1}y) = d(g^{-1}x, g^{-1}y) \leq S$. Hence if we set $z = g^{-1}y$ then $(x, y) \in t_z \in \mathcal{T}_R$, as required.

(2) Let $x, y \in X$. Transitivity of the action of G tells us that there exists some $h \in G$ such that $hx = y$. If there were also some element $h \neq h' \in G$ such that $h'x = y$ then we would have $hx = h'x$, which would contradict the freeness of the action. Hence freeness tells us that there is at most one cotranslation $\sigma_h \in \Sigma_R$ such that $\sigma_h(x) = y$ for all $x, y \in X$. In fact, there will be exactly one, by transitivity.

(3) Let $x \in B_S(x_0)$, so that $t_x \in \mathcal{T}_R$ for some $R > 0$, and let (gx_0, gx), $(g'x_0, g'x) \in t_x$. We wish to find some $h \in G$ such that $hgx_0 = g'x_0$ and $hgx = g'x$. We may simply take $h = g'g^{-1}$.

Therefore the cotranslation orbits do form a partial translation structure under the given assumptions.

□

When the space X on which a discrete group G acts is a subspace of G, we may also define a partial translation structure on it by restricting the group's canonical partial translation structure, by Theorem 3.8. We prove that in the case where the group action is on the left, these restricted partial translations are invariant under this action.

THEOREM 3.19 *Let X be a subset of a discrete group G on which G acts on the left, and equip X with the inherited subspace partial translation structure $\mathcal{T}_X$ arising from the canonical partial translation structure $\mathcal{T}$ on G. Then partial translations in $\mathcal{T}_X$ are invariant under the action of G.*

Proof.

Recall that a partial translation in $\mathcal{T}$ is of the form

$$t_g = \{(h, hg) \mid h \in G\}$$

and so partial translations in $\mathcal{T}_X$ are of the form

$$t_g|_X = \{(x, xg) \mid x, xg \in X\},$$

so that in each case we have one for every $g \in G$. We wish to show that $(x, xg) \in t_g|_X$ if and only if $(hx, hxg) \in t_g|_X$, for all $x \in X, g, h \in G$.

Firstly, suppose $(x, xg) \in t_g|_X$. Then we know that both x and xg must be elements of $X \subseteq G$. Hence we must also have $hx, hxg \in X$ for all $h \in G$, since G acts on X on the left. Therefore $(hx, hxg) = ((hx), (hx)g) \in t_g|_X$.

Conversely, suppose that $(hx, hxg) \in t_g|_X$, for some $x \in X, g, h \in G$. Then we must have $hx, hxg \in X$, and thus both $h^{-1}(hx) = x$ and $h^{-1}(hxg) = xg$ must also lie in X, since $h^{-1} \in G$ and G acts on X on the left. Hence $(x, xg) \in t_g|_X$, as required.

$\square$

The following lemma is simply a useful remark relating to certain group actions.

LEMMA 3.20 *Let X be a metric space and G a group which acts on X on both the left and the right. Suppose X is equipped with a metric d which is left invariant under the action of G, and suppose also that there exists some $x_0 \in X$ such that for every $x \in X$ there exists $g \in G$ such that $gx_0 = x$. Then each element of G acting on the right moves every $x \in X$ by a fixed distance.*

Proof.

For every $x \in X$ and $r \in G$ we have

$$\begin{aligned} d(xr, x) &= d(gx_0r, gx_0) \text{ for some } g \in G, \text{ by assumption,} \\ &= d(x_0r, x_0), \text{ by left invariance.} \end{aligned}$$

$\square$

REMARK 3.21 The assumptions of the above theorem could be replaced by "G acts transitively and by isometries on X on the left" (with no assumption on the right action except that there is one). Indeed, left invariance of d is equivalent to a left action by isometries, and the second assumption implies that for all $x, x' \in X$ there exists $g, g' \in G$ such that $gx_0 = x$ and $g'x_0 = x'$, so we have $g'g^{-1} \in G$ with $g'g^{-1}x = x'$, which implies transitivity, and it is clear that transitivity implies the given assumption.

The final result of this section can be applied to any metric space that is equipped with a partial translation structure, and requires no additional assumptions. Here we consider partial translations from a slightly different perspective and explore their relationship with cotranslation orbits in a little more detail.

PROPOSITION 3.22 *Let $\mathcal{T}$ be a partial translation structure on a space X, and let t be a partial translation contained in $\mathcal{T}_R$ for some $R > 0$. Then t can be expressed as the orbit of (x, y) under Σ_R for any $(x, y) \in t$.*

Proof.

Denote cotranslation orbits in the following way:

$$\begin{aligned} t_{xyR} &:= \{(x', y') \mid \text{there exists } \sigma \in \Sigma_R \text{ such that } (x', y') = (\sigma x, \sigma y)\} \\ &= \{(\sigma x, \sigma y) \mid \sigma \in \Sigma_R\}. \end{aligned}$$

Now assume $t \in \mathcal{T}_R$ for some $R > 0$, and $(x, y) \in t$. Then by definition of cotranslations we know that $(\sigma x, \sigma y) \in t$ for all $\sigma \in \Sigma_R$ (where defined), and thus $t_{xyR} \subseteq t$. However, by the third axiom of the partial translation structure definition, we also know that for all other $(x', y') \in t$ there exists some $\sigma \in \Sigma_R$ such that $\sigma x = x'$ and $\sigma y = y'$. So in other words $(x', y') = (\sigma x, \sigma y)$ for some $\sigma \in \Sigma_R$, for all $(x', y') \in t$; that is, $(x', y') \in t_{xyR}$ for all $(x', y') \in t$. Hence $t \subseteq t_{xyR}$ also, and therefore $t = t_{xyR}$ for all $(x, y) \in t$, where $t \in \mathcal{T}_R$.

$\square$

4 Partial Translation Algebras

In this chapter we formally introduce the analogue of the reduced group C^*-algebra for metric spaces that was mentioned at the start of the previous chapter. Such an algebra is obtained from a partial translation structure on a metric space X by viewing the partial translations as operators acting on $l^2(X)$. We explain this process below and discuss some properties of the operators involved.

DEFINITION 4.1 [Partial Translation Algebra]

Let X be a discrete metric space. For a partial translation structure $\mathcal{T}$ on X, the *partial translation algebra* $C^*(\mathcal{T})$ is the C^*-subalgebra of $C_u^*(X)$ generated by the partial translations (viewed as partial isometries) [11].

Let $\mathcal{T}$ be a partial translation structure on a discrete space X, and let $t \in \mathcal{T}$ be a partial translation. Then t defines an operator $T_t \in C^*(\mathcal{T})$ which acts on $l^2(X)$ in the following way:

$$T_t : \delta_x \mapsto \begin{cases} \delta_{t(x)} & \text{if } x \in Dom(t) \\ 0 & \text{otherwise,} \end{cases}$$

where δ_x denotes the delta function of the element $x \in X$. In the case where X is a space which may be specifically enumerated so that $l^2(X)$ can be identified with a space of sequences, this translates to T_t moving the term indexed by the element $x \in X$ in any sequence in $l^2(X)$ to the "$t(x)$"th position, whenever $x \in Dom(t)$. In particular, we use this method to understand operators arising from partial translations on a subset of $\mathbb{Z}$, several examples of which we will consider in the next chapter.

For every such T_t, it is not difficult to check that the adjoint operator T_t^* arises in a similar way from the partial translation t^{-1}, so that

$$T_t^* : \delta_x \mapsto \begin{cases} \delta_{t^{-1}(x)} & \text{if } x \in Dom(t^{-1}) = Ran(t) \\ 0 & \text{otherwise.} \end{cases}$$

For any pair of partial translations $t_1, t_2 \in \mathcal{T}$, we have

$$T_{t_1} T_{t_2} : \delta_x \mapsto \begin{cases} \delta_{t_1(t_2(x))} & \text{if } x \in Dom(t_2), t_2(x) \in Dom(t_1) \\ 0 & \text{otherwise,} \end{cases}$$

by definition. Now $Dom(t_1 t_2) = \{x \in Dom(t_2) \mid t_2(x) \in Dom(t_1)\}$, and hence we have

$$T_{t_1} T_{t_2} = T_{t_1 t_2},$$

where $T_{t_1 t_2}$ is the operator arising from $t_1 t_2$ (which is not necessarily an element of $\mathcal{T}$), defined in the usual way by

$$T_{t_1 t_2} : \delta_x \mapsto \begin{cases} \delta_{t_1 t_2(x)} & \text{if } x \in Dom(t_1 t_2) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $T_{t_1} T_{t_2} = T_{t_1 t_2}$ is the zero operator on $l^2(X)$ if t_1 and t_2 are not composable, that is if $Dom(t_1 t_2) = \emptyset$.

PROPOSITION 4.2 *Let t be a globally defined partial translation on a space X which also has a globally defined inverse t^{-1}. Then the corresponding operator T_t on $l^2(X)$ is unitary.*

Proof.

To say that t is globally defined means that $Dom(t) = X$. Thus the operator T_t simply acts on $l^2(X)$ by

$$T_t : \delta_x \mapsto \delta_{t(x)}$$

for all $x \in X$. Similarly, we will have

$$T_t^* : \delta_x \mapsto \delta_{t^{-1}(x)}$$

for all $x \in X$. Thus it is clear that $T_t T_t^* = Id = T_t^* T_t$, i.e. that T_t is unitary.

□

In general, the operator $T_t T_t^*$ acts as the identity on all basis elements δ_x such that $x \in Dom(t^{-1})$, and zero for all δ_x such that $x \notin Dom(t^{-1})$, whilst $T_t^* T_t$ acts as the identity for all δ_x such that $x \in Dom(t)$ and is zero elsewhere. Hence it is in fact the case that the only unitary operators arising from partial translations are those which arise from globally defined partial translations with globally defined inverses.

If we attempt to show the converse and to obtain a partial translation on X from a general unitary operator on $l^2(X)$, then we would have to assume that that operator acts by permuting basis elements, and also that it does not move them "too far" so that the corresponding partial bijection would satisfy the distance condition for partial translations. Thus, we may say that if T is a unitary operator on $l^2(X)$ which permutes the basis for $l^2(X)$ and does so in such a way that $d(x, y)$ is bounded whenever $T(\delta_x) = \delta_y$, then

T defines a globally defined partial translation on X which has a globally defined inverse.

COROLLARY 4.3 *If a partial translation t on a space X gives rise to a unitary operator on $l^2(X)$ and if $\phi : X \to Y$ is a uniform bijection, then the partial translation $\phi(t)$ gives rise to a unitary operator on $l^2(Y)$.*

Proof.

This is a consequence of the previous proposition together with the fact that if t is a globally defined partial translation on X then $\phi(t) = \phi \circ t \circ \phi^{-1}$ is a globally defined partial translation on Y.

$\square$

So, for example, if $\mathcal{T}_X$ is a partial translation structure on X, $\phi : X \to Y$ is a uniform bijection and $C^*(\mathcal{T}_X)$ is generated by a single unitary operator T_t, then $\phi(\mathcal{T}_X)$ is a partial translation structure on Y and the algebra $C^*(\phi(\mathcal{T}_X))$ will be generated by a single unitary operator $T_{\phi(t)}$.

REMARK 4.4 If we wished to consider a partial translation t which was globally defined but whose inverse was not, then the operator T_t arising from it would be an isometry, since we would have $T_t^* T_t = Id$, while $T_t T_t^*$ would only be the identity on basis elements δ_x for which $x \in Dom(t^{-1})$. In fact, the operator $T_t T_t^*$ would be a projection which could be expressed by

$$T_t T_t^* = 1 - P_{X \setminus Dom(t^{-1})}.$$

Thus if t^{-1} is defined for all but a finite number of elements of X (i.e. is *cofinite*), then $T_t T_t^*$ differs from the identity by an operator of finite rank.

If t^{-1} was globally defined but t was not then we would have $T_t T_t^* = Id$ and similar relations for $T_t^* T_t$ as for $T_t T_t^*$ in the above, so T_t would be a *coisometry.*

Almost everything we have considered so far in this chapter could be applied to operators arising from any partial bijections of the space X, but for a partial bijection t to be a *partial translation* of X, we also require that $d_X(x, t(x))$ is bounded for every $x \in Dom(t)$, so let us now turn our attention to the effect this distance condition has on operators arising from partial translations.

If T_t is an operator on $l^2(X)$ which arises from a partial translation t, then T_t maps any basis element δ_x to $\delta_{t(x)}$ in such a way that x is not infinitely far from $t(x)$ in X; in fact $d_X(x, t(x))$ is bounded by the specific bound given in the distance condition for t. As T_t acts on basis elements by permuting them and/or mapping to zero, if we express T_t in matrix form it will be a $\{0, 1\}$-matrix with at most one non-zero entry in each row and column; this matrix represents the characteristic function of the subset $t \subset X \times X$. The distance condition tells us that the non-zero entries of this matrix will be contained within a strip of finite width about the diagonal. Thus operators arising from partial translations have *finite propagation.*

5 Examples of Partial Translation Structures and Algebras

As previously mentioned, it is proved in [11] that every countable discrete group G admits a canonical partial translation structure which is free and globally controlled, consisting of partial translations of the form

$$t_g = \{(x, xg) \mid x \in G\},$$

and partial cotranslations defined by $\sigma_h(x) = hx$, one of each for each element of the group. For every $R > 0$ we take $\mathcal{T}_R = \{t_g \mid d(e, g) < R\}$ and $\Sigma_R = \{\sigma_h \mid h \in G\}$. We proved in chapter 3 that every partial translation structure may be restricted to a subspace, as well as that uniform bijections take partial translation structures to partial translation structures; in the section below we shall compare these two methods for obtaining a partial translation structure on a subspace of a countable discrete group, given that any such group comes equipped with a partial translation structure defined by the above. These ideas will then be extended to partial translation algebras.

5.1 Partial Translations Structures for Certain Subspaces of Groups

We aim to address the question of transport of partial translation structures; to this end, we begin by studying a number of examples involving a subspace X of a discrete group G for which there exists a uniform bijection $\phi : G \to$

X. We compare the partial translation structure induced by ϕ with the restriction of the canonical partial translation structure on G to X.

5.1.1 $\mathbb{Z}\backslash\{0\}$

Define the canonical partial translation structure $\mathcal{T}$ on $\mathbb{Z}$ to consist of partial translations of the form $t_n = \{(m+n, m) \mid m \in \mathbb{Z}\}$, and cotranslations of the form $\sigma_n : \mathbb{Z} \to \mathbb{Z}$, $\sigma_n(m) = m - n$, for each $n \in \mathbb{Z}$. Here we have

$$\mathcal{T}_R = \{t_n \mid \ |n| < R\} \text{ and } \Sigma_R = \{\sigma_n \mid n \in \mathbb{Z}\},$$

for all $R > 0$.

Note that if we were to follow the notation of [11] to the letter, then the partial translation taking any integer m to $m+n$ would be denoted by t_{-n} and the partial cotranslation taking m to $m - n = -n + m$ would be denoted by σ_{-n}. However, we use the above definitions for ease of notation throughout this chapter, with no loss of generality.

Firstly, let us consider the restriction of this partial translation structure to $X := \mathbb{Z}\backslash\{0\}$; that is, let $\mathcal{T}_X = \mathcal{T} \cap (X \times X)$. Then partial translations in $\mathcal{T}_X$ are of the form

$$t_n|_X = \{(m+n, m) \mid m \in X, m \neq -n\}.$$

Partial cotranslations must also be restricted, so each σ_n will now only be defined on $X\backslash\{n\}$. In addition to this, we now have

$$\mathcal{T}_{XR} = \mathcal{T}_R \cap (X \times X) = \{t_n \in \mathcal{T}_X \mid \ |n| < R\} \text{ and } \Sigma_{XR} = \{\sigma_n|_{X\backslash\{n\}} \mid n \in \mathbb{Z}\},$$

for all $R > 0$. Note that, with this definition, $\mathcal{T}_X$ will indeed be a partial translation structure, by Theorem 3.8.

Now consider the map $\phi : \mathbb{Z} \to X$, $n \mapsto \begin{cases} n & \text{if } n \geq 1 \\ n-1 & \text{if } n \leq 0 \end{cases}$. Since ϕ is a uniform bijection, $\phi(\mathcal{T})$ is a partial translation structure on X, by Theorem 3.13. By the proof of this theorem, it can be seen that partial translations in $\phi(\mathcal{T})$ are of the form $\phi(t_n) = \phi \circ t_n \circ \phi^{-1}$, for t_n a partial translation in

$\mathcal{T}$. We can describe these maps explicitly. So let $m \in X$; then

$$\begin{aligned}
\phi(t_n)(m) &= \phi(t_n(\phi^{-1}(m))) \\
&= \begin{cases} \phi(t_n(m)) & \text{if } m \geq 1 \\ \phi(t_n(m+1)) & \text{if } m \leq -1 \end{cases} \\
&= \begin{cases} \phi(m+n) & \text{if } m \geq 1 \\ \phi(m+n+1) & \text{if } m \leq -1 \end{cases} \\
&= \begin{cases} m+n & \text{if } m \geq 1,\ m+n \geq 1 \\ m+n-1 & \text{if } m \geq 1,\ m+n \leq 0 \\ m+n+1 & \text{if } m \leq -1,\ m+n+1 \geq 1 \\ m+n & \text{if } m \leq -1,\ m+n+1 \leq 0 \end{cases} \\
&= \begin{cases} m+n & \text{if } m > \max\{0,-n\} \\ & \text{or } m < \min\{0,-n\} \\ m+n-1 & \text{if } 0 < m \leq -n \\ m+n+1 & \text{if } -n \leq m < 0. \end{cases}
\end{aligned}$$

Thus it is clear that $\phi(t_n)(m)$ is an element of X for every $m \in X$, yet this is not the same as applying t_n restricted to X. Partial cotranslations could be defined in a similar way. Note that $d(n,m) \geq R$ implies that $d(\phi(n),\phi(m)) \geq R$, for any $n,m \in \mathbb{Z}$, $R > 0$, and hence we have

$$\phi(\mathcal{T})_R = \phi(\mathcal{T}_R) = \{\phi(t_n) \mid t_n \in \mathcal{T}_R\} \text{ and } \phi(\Sigma)_R = \phi(\Sigma_R) = \{\phi(\sigma_n) \mid \sigma_n \in \Sigma_R\},$$

for all $R > 0$. Nevertheless, mapping the canonical partial translation structure on $\mathbb{Z}$ via ϕ yields a very different structure from the one obtained by restricting to X.

5.1.2 $\mathbb{Z}\backslash\{n_1,\ldots,n_k\}$

Now that we have tackled the example of "$\mathbb{Z}$ with a hole" (specifically a hole at zero), we can extend these ideas to the case of $\mathbb{Z}$ with any finite number of holes. Let $n_1,\ldots,n_k \in \mathbb{Z}$ with $n_1 < \ldots < n_k$ be finitely many distinct integers, and define $X := \mathbb{Z}\backslash\{n_1,\ldots,n_k\}$. Equip $\mathbb{Z}$ with the same canonical partial translation structure $\mathcal{T}$.

Again, we consider first the restriction of $\mathcal{T}$ to X, so we let $\mathcal{T}_X = \mathcal{T}|_X =$

$\mathcal{T} \cap (X \times X)$. Then partial translations in $\mathcal{T}_X$ are of the form

$$\begin{aligned} t_n|_X &= \{(m+n, m) \mid m \in X,\ m+n \in X\} \\ &= \{(m+n, m) \mid m \in \mathbb{Z},\ m \neq n_1, \ldots, n_k, n_1 - n, \ldots, n_k - n\}. \end{aligned}$$

Every partial cotranslation σ_n has its domain restricted to

$$\mathbb{Z} \backslash \{n_1, \ldots, n_k, n_1 + n, \ldots, n_k + n\} = X \backslash \{n_1 + n, \ldots, n_k + n\}.$$

Now let $\phi : \mathbb{Z} \to X$ be the uniform bijection defined by

$$\phi : n \mapsto \begin{cases} n - k & \text{if } n < n_1 + k \\ n - k + i & \text{if } n_i + k - i < n < n_{i+1} + k - i, \\ & \text{for all } 1 \leq i \leq k-1 \\ n & \text{if } n_k < n. \end{cases}$$

and consider $\phi(\mathcal{T})$, another partial translation structure which may be defined for the space X. Note that $\phi^{-1} : X \to \mathbb{Z}$ is defined by

$$\phi^{-1} : n \mapsto \begin{cases} n + k & \text{if } n < n_1 \\ n + k - 1 & \text{if } n_1 < n < n_2 \\ \cdots & \\ n + k - i & \text{if } n_i < n < n_{i+1} \\ \cdots & \\ n + k - (k-1) & \text{if } n_{k-1} < n < n_k \\ n & \text{if } n_k < n. \end{cases}$$

Now, for every $m \in X$, we have

$\phi(t_n)(m)$

$$
\begin{aligned}
&= \phi(t_n(\phi^{-1}(m))) \\
&= \begin{cases} \phi(t_n(m+k)) & \text{if } m < n_1 \\ \phi(t_n(m+k-i)) & \text{if } n_i < m < n_{i+1},\ \forall\ 1 \le i < k \\ \phi(t_n(m)) & \text{if } n_k < m \end{cases} \\
&= \begin{cases} \phi(m+n+k) & \text{if } m < n_1 \\ \phi(m+n+k-i) & \text{if } n_i < m < n_{i+1},\ \forall\ 1 \le i < k \\ \phi(m+n) & \text{if } n_k < m \end{cases} \\
&= \begin{cases} \begin{cases} m+n & \text{if } m+n+k < n_1+k \\ m+n+j & \text{if } n_j+k-j < m+n+k < n_{j+1}+k-j, \\ & \text{for all } 1 \le j < k \\ m+n+k & \text{if } n_k < m+n+k \end{cases} \\ \text{if } m < n_1 \\ \begin{cases} m+n-i & \text{if } m+n+k-i < n_1+k \\ m+n-i+j & \text{if } n_j+k-j < m+n+k-i < n_{j+1}+k-j, \\ & \text{for all } 1 \le j < k \\ m+n+k-i & \text{if } n_k < m+n+k-i \end{cases} \\ \text{if } n_i < m < n_{i+1},\ \text{for all } 1 \le i < k \\ \begin{cases} m+n-k & \text{if } m+n < n_1+k \\ m+n-k+j & \text{if } n_j+k-j < m+n < n_{j+1}+k-j, \\ & \text{for all } 1 \le j < k \\ m+n & \text{if } n_k < m+n \end{cases} \\ \text{if } n_k < m. \end{cases}
\end{aligned}
$$

Again, it can be seen from this that the partial translation structure obtained in this manner will be very different from the restriction of $\mathcal{T}$ to X.

5.1.3 $2\mathbb{Z}$

The final subspace of $\mathbb{Z}$ we shall consider in this manner is $X = 2\mathbb{Z}$. The restriction $\mathcal{T}_X = \mathcal{T} \cap (X \times X)$ of our canonical partial translation structure to this set consists of partial translations

$$
\begin{aligned}
t_n|_X &= \{(m+n, m) \mid m \in X = 2\mathbb{Z}, m+n \in X = 2\mathbb{Z}\} \\
&= \{(m+n, m) \mid m, n \in X = 2\mathbb{Z}\},
\end{aligned}
$$

so that the set of partial translations in $\mathcal{T}_X$ is $\{t_n|_X \mid n \in X\} = \{t_{2n}|_X \mid n \in \mathbb{Z}\}$, where each t_n is a partial translation in $\mathcal{T}$. Partial cotranslations must be restricted in a similar way, so we restrict the domain of each one to X and then also require that $\sigma_n(m) = m - n \in X$ for all $m \in X$, in other words that $n \in X = 2\mathbb{Z}$ as well. Thus each partial cotranslation in $\mathcal{T}_X$ is of the form $\sigma_{2n}|_X$, for some $n \in \mathbb{Z}$. Here we have

$$\mathcal{T}_{XR} = \mathcal{T}_R \cap (X \times X) = \{t_{2n}|_X \mid |2n| < R\} \text{ and } \Sigma_{XR} = \{\sigma_{2n}|_X \mid n \in \mathbb{Z}\},$$

for all $R > 0$.

In addition to this, there exists a uniform bijection between $\mathbb{Z}$ and X given by

$$\phi : \mathbb{Z} \to X,\ n \mapsto 2n.$$

Again, partial translations in the partial translation structure $\phi(\mathcal{T})$ are of the form $\phi(t_n) = \phi \circ t_n \circ \phi^{-1}$, for some $t_n \in \mathcal{T}$. If we let $m \in X$ then

$$\begin{aligned} \phi(t_n)(m) &= \phi(t_n(\phi^{-1}(m))) \\ &= \phi(t_n(\frac{m}{2})) \\ &= \phi(\frac{m}{2} + n) \\ &= m + 2n, \end{aligned}$$

for every $n \in \mathbb{Z}$, and thus in this case we have the same set of partial translations as for $\mathcal{T}|_X$. Also, partial cotranslations in $\phi(\mathcal{T})$ will similarly be of the form $\sigma_{2n}|_X = \phi(\sigma_n)$, for some partial cotranslation σ_n in $\mathcal{T}$. Note that $d(n, m) \geq R$ implies that $d(\phi(n), \phi(m)) \geq 2R$, for any $n, m \in \mathbb{Z}$, $R > 0$, and so finally we have

$$\phi(\mathcal{T})_R = \phi(\mathcal{T}_{R/2}) = \{\phi(t_n) \mid t_n \in \mathcal{T}_{R/2}\} = \{t_{2n}|_X \mid |2n| < R\} \text{ and}$$

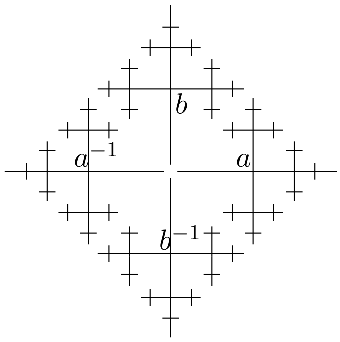
$$\phi(\Sigma)_R = \phi(\Sigma_{R/2}) = \{\phi(\sigma_n) \mid \sigma_n \in \Sigma_{R/2}\} = \{\sigma_{2n}|_X \mid n \in \mathbb{Z}\},$$

for all $R > 0$. So this method yields the same partial translation structure as the restriction of $\mathcal{T}$ to X.

5.1.4 $\mathbb{F}_2 \backslash \{e\}$

We shall now use similar methods to study a slightly different example, the case of "the free group with a hole". Removing the identity vertex from the Cayley graph of the free group on two generators leaves us with four disjoint isomorphic trees.

$X = \mathbb{F}_2 \backslash \{e\}$



Partial translations in the canonical partial translation structure on $\mathbb{F}_2$, if viewed as bijections from the group to itself, are defined as the actions of group elements by right translation; we continue the notation of the previous sections to allow the partial translation t_g to denote right translation by the element $g \in \mathbb{F}_2$ (rather than by g^{-1}). If we restrict this partial translation structure to X in the usual way, then we obtain partial translations including, for example:

$$
\begin{aligned}
t_a &: X \backslash \{a^{-1}\} \to X \backslash \{a\}, && x_1 \dots x_k \mapsto x_1 \dots x_k a, \\
t_b &: X \backslash \{b^{-1}\} \to X \backslash \{b\}, && x_1 \dots x_k \mapsto x_1 \dots x_k b, \\
t_{a^{-1}} &: X \backslash \{a\} \to X \backslash \{a^{-1}\}, && x_1 \dots x_k \mapsto x_1 \dots x_k a^{-1}, \\
t_{b^{-1}} &: X \backslash \{b\} \to X \backslash \{b^{-1}\}, && x_1 \dots x_k \mapsto x_1 \dots x_k b^{-1},
\end{aligned}
$$

where $x_i \in \{a, a^{-1}, b, b^{-1}\}$ for all $1 \leq i \leq k$. A general partial translation is given by right multiplication by a reduced word $y_1 \dots y_l$ with $y_i \in \{a, a^{-1}, b, b^{-1}\}$. Explicitly, we have

$$t_{y_1 \dots y_l} : X \backslash \{y_l^{-1} \dots y_1^{-1}\} \to X \backslash \{y_1 \dots y_l\}, \quad x_1 \dots x_k \mapsto x_1 \dots x_k y_1 \dots y_l.$$

Again, we can also define a partial translation structure on X by use of a uniform bijection between X and $\mathbb{F}_2$. Define

$$\phi : \mathbb{F}_2 \to X,\ x_1 \dots x_k \mapsto \begin{cases} x_1 \dots x_k a & \text{if } x_i = a \text{ for all } i \text{ or } x_1 \dots x_k = e \\ x_1 \dots x_k & \text{otherwise.} \end{cases}$$

Then we have the inverse map

$$\phi^{-1} : X \to \mathbb{F}_2,\ x_1 \dots x_k \mapsto \begin{cases} x_1 \dots x_{k-1} & \text{if } x_i = a \text{ for all } 1 \le i \le k \\ x_1 \dots x_k & \text{otherwise,} \end{cases}$$

and thus ϕ is a bijection. Moreover, the greatest distance ϕ can move an element by is 1, and hence (as ϕ is also injective) ϕ is uniform. Therefore, if $\mathcal{T}$ is a partial translation structure on $\mathbb{F}_2$ then $\phi(\mathcal{T})$ will be a partial translation structure on X, by Theorem 3.13. Considering again the canonical partial translation structure for $\mathbb{F}_2$, we obtain in this manner partial translations on X including

$$\begin{aligned}
\phi(t_a) : x_1 \dots x_k \mapsto\ & \phi \circ t_a \circ \phi^{-1}(x_1 \dots x_k) \\
= & \begin{cases} \phi \circ t_a(x_1 \dots x_{k-1}) & \text{if } x_i = a\ \forall\ 1 \le i \le k \\ \phi \circ t_a(x_1 \dots x_k) & \text{otherwise} \end{cases} \\
= & \begin{cases} \phi(x_1 \dots x_{k-1} a) & \text{if } x_i = a\ \forall\ 1 \le i \le k \\ \phi(x_1 \dots x_k a) & \text{otherwise} \end{cases} \\
= & \begin{cases} x_1 \dots x_{k-1} a a & \text{if } x_i = a\ \forall\ 1 \le i \le k \\ x_1 \dots x_k a a & \text{if } x_1 \dots x_k a = e, \\ & \text{i.e. if } x_1 \dots x_k = a^{-1} \\ x_1 \dots x_k a & \text{otherwise} \end{cases} \\
= & \begin{cases} a & \text{if } x_1 \dots x_k = a^{-1} \\ x_1 \dots x_k a & \text{otherwise,} \end{cases}
\end{aligned}$$

$$\phi(t_{a^{-1}}) : x_1 \dots x_k \mapsto \phi \circ t_{a^{-1}} \circ \phi^{-1}(x_1 \dots x_k)$$

$$\begin{aligned}
&= \begin{cases} \phi \circ t_{a^{-1}}(x_1 \dots x_{k-1}) & \text{if } x_i = a \ \forall\ 1 \leq i \leq k \\ \phi \circ t_{a^{-1}}(x_1 \dots x_k) & \text{otherwise} \end{cases} \\
&= \begin{cases} \phi(x_1 \dots x_{k-1} a^{-1}) & \text{if } x_i = a \ \forall\ 1 \leq i \leq k \\ \phi(x_1 \dots x_k a^{-1}) & \text{otherwise} \end{cases} \\
&= \begin{cases} \phi(a^{-1}) & \text{if } k = 1,\ x_k = a \\ \phi(x_1 \dots x_{k-2}) & \text{if } x_i = a \ \forall\ 1 \leq i \leq k \ (k > 1) \\ \phi(x_1 \dots x_k a^{-1}) & \text{otherwise} \end{cases} \\
&= \begin{cases} a^{-1} & \text{if } k = 1,\ x_k = a \\ x_1 \dots x_{k-1} & \text{if } x_i = a \ \forall\ 1 \leq i \leq k \ (k > 1) \\ x_1 \dots x_k a^{-1} & \text{otherwise} \end{cases} \\
&= \begin{cases} a^{-1} & \text{if } x_1 \dots x_k = a \\ x_1 \dots x_k a^{-1} & \text{otherwise} \end{cases} \\
&= (\phi(t_a))^{-1}(x_1 \dots x_k),
\end{aligned}$$

$$\phi(t_b) : x_1 \dots x_k \mapsto \phi \circ t_b \circ \phi^{-1}(x_1 \dots x_k)$$

$$\begin{aligned}
&= \begin{cases} \phi \circ t_b(x_1 \dots x_{k-1}) & \text{if } x_i = a \ \forall\ 1 \leq i \leq k \\ \phi \circ t_b(x_1 \dots x_k) & \text{otherwise} \end{cases} \\
&= \begin{cases} \phi(x_1 \dots x_{k-1} b) & \text{if } x_i = a \ \forall\ 1 \leq i \leq k \\ \phi(x_1 \dots x_k b) & \text{otherwise} \end{cases} \\
&= \begin{cases} x_1 \dots x_{k-1} b & \text{if } x_i = a \ \forall\ 1 \leq i \leq k \\ x_1 \dots x_{k-1} a & \text{if } k = 1,\ x_k = b^{-1}, \\ & \text{or } x_k = b^{-1},\ x_i = a \ \forall\ 1 \leq i \leq k-1 \\ x_1 \dots x_k b & \text{otherwise,} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\phi(t_{b^{-1}}) : x_1 \dots x_k \mapsto \phi \circ t_{b^{-1}} \circ \phi^{-1}(x_1 \dots x_k) \\
&= \begin{cases} \phi \circ t_{b^{-1}}(x_1 \dots x_{k-1}) & \text{if } x_i = a \ \forall\ 1 \le i \le k \\ \phi \circ t_{b^{-1}}(x_1 \dots x_k) & \text{otherwise} \end{cases} \\
&= \begin{cases} \phi(x_1 \dots x_{k-1} b^{-1}) & \text{if } x_i = a \ \forall\ 1 \le i \le k \\ \phi(x_1 \dots x_k b^{-1}) & \text{otherwise} \end{cases} \\
&= \begin{cases} x_1 \dots x_{k-1} b^{-1} & \text{if } x_i = a \ \forall\ 1 \le i \le k \\ x_1 \dots x_{k-1} a & \text{if } k = 1,\ x_k = b, \\ & \text{or } x_k = b,\ x_i = a \ \forall\ 1 \le i \le k-1 \\ x_1 \dots x_k b^{-1} & \text{otherwise} \end{cases} \\
&= (\phi(t_b))^{-1}(x_1 \dots x_k).
\end{aligned}$$

Notice that these are all globally defined partial translations with globally defined inverses. General partial translations could be formed in a similar way on an individual basis, and it is clear that we will end up with a very different set of partial translations from those obtained by restricting the canonical partial translation structure on $\mathbb{F}_2$.

5.2 Partial Translation Algebras

Now we shall expand upon some of the examples discussed in the previous section by considering the partial translation algebras that our partial translation structures generate. Recall that for $\mathcal{T}$ a partial translation structure, the *partial translation algebra* $C^*(\mathcal{T})$ is the C^*-subalgebra of $C^*_u(X)$ generated by the partial translations (viewed as partial isometries).

5.2.1 $\mathbb{Z}$

As our first three examples were subsets of $\mathbb{Z}$, let us begin by examining the algebra arising from the canonical partial translation structure on $\mathbb{Z}$ itself. We consider the set of partial translations in this case to be

$$\{t_n = \{(m+n, m) \mid m \in \mathbb{Z}\} \mid n \in \mathbb{Z}\},$$

which is generated by the partial translation t_1 and its inverse $t_1^{-1} = t_{-1}$. Viewing these as elements of $B(l^2(\mathbb{Z}))$, we obtain partial isometries $T_1 : \delta_n \mapsto \delta_{n+1}$ and $T_1^* : \delta_n \mapsto \delta_{n-1}$ of $l^2(\mathbb{Z})$. Thus the algebra arising in this case is generated by a single element T_1 such that $T_1^* T_1 = 1 = T_1 T_1^*$, i.e.

by a single *unitary* element, the bilateral shift. Hence we have $C^*(\mathcal{T}) = C^*_r(\mathbb{Z}) = C(S^1)$.

5.2.2 $\mathbb{Z}\backslash\{0\}$

Let $X = \mathbb{Z}\backslash\{0\}$, and again begin by considering $\mathcal{T}_X$, the restriction of the canonical partial translation structure on $\mathbb{Z}$ to X. The set of partial translations in $\mathcal{T}_X$ is given by

$$\{t_n = \{(m+n, m) \mid m \in X\backslash\{-n\}\} \mid n \in \mathbb{Z}\},$$

for example $t_1 = \{(m+1, m) \mid m \in X\backslash\{-1\}\}$. Viewing this element as a partial isometry of $l^2(X)$, we obtain the operator

$$T_1 : (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) \mapsto (\ldots, x_{-3}, x_{-2}, 0, x_1, \ldots),$$

which has an adjoint

$$T_1^* : (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) \mapsto (\ldots, x_{-1}, 0, x_2, x_3, \ldots).$$

Thus we have

$$T_1^* T_1 : (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) \mapsto (\ldots, x_{-2}, 0, x_1, x_2, \ldots)$$

and

$$T_1 T_1^* : (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) \mapsto (\ldots, x_{-2}, x_{-1}, 0, x_2, \ldots),$$

leading to the identities

$$\begin{aligned} 1 - T_1^* T_1 &= P_{-1}, \\ 1 - T_1 T_1^* &= P_1. \end{aligned}$$

Similarly, a general partial translation $t_n \in \mathcal{T}_X$ will give rise to a partial isometry of the form

$$\begin{aligned} T_n : \quad & (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots) \\ & \mapsto (\ldots, x_{-2-n}, x_{-1-n}, x_{1-n}, x_{2-n}, \ldots, x_{-1}, 0, x_1, \ldots), \end{aligned}$$

i.e. the x_{-n} term vanishes, a zero takes the x_n position, and all other elements are shifted right by n spaces. The following is proved in [10], as part of the proof of Theorem 2.

PROPOSITION 5.1 *The partial translations $t_0 = 1$, t_1 and t_2 are sufficient to generate $C^*(\mathcal{T}_X)$, the algebra arising from the restriction of the canonical partial translation structure on $\mathbb{Z}$ to $X = \mathbb{Z}\backslash\{0\}$.*

On the other hand, we also have the partial translation structure $\phi(\mathcal{T})$ on X, as defined in section 5.1.1. Recall that all partial translations in $\phi(\mathcal{T})$ are of the form

$$\phi(t_n)(m) = \begin{cases} m+n & \text{if } m > \max\{0,-n\} \\ & \text{or } m < \min\{0,-n\} \\ m+n-1 & \text{if } 0 < m \leq -n \\ m+n+1 & \text{if } -n \leq m < 0, \end{cases}$$

for some $n \in \mathbb{Z}$. For example, we have

$$\phi(t_1)(m) = \begin{cases} m+1 & \text{if } m > 0 \text{ or } m < -1 \\ m+2 = 1 & \text{if } m = -1, \end{cases}$$

for all $m \in X$. Now let T_n denote the partial isometry arising from the partial translation $\phi(t_n)$, that is

$$T_n : \delta_m \mapsto \delta_{\phi(t_n)(m)}.$$

So

$$T_1 : \delta_m \mapsto \begin{cases} \delta_1 & \text{if } m = -1 \\ \delta_{m+1} & \text{otherwise.} \end{cases}$$

Thus T_1 is the operator on $l^2(X)$ given by

$$T_1 : (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) \mapsto (\ldots, x_{-3}, x_{-2}, x_{-1}, x_1, \ldots).$$

The adjoint of this operator is obtained from the inverse of $\phi(t_1)$, which is defined by

$$(\phi(t_1))^{-1}(m) = \begin{cases} m-1 & \text{if } m > 1 \text{ or } m < 0 \\ m-2 = -1 & \text{if } m = 1 \end{cases}$$

(note that this is not the same as $\phi^{-1}(t_1)(m)$, but is equal to $\phi(t_{-1})(m)$). Hence

$$T_1^* : (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) \mapsto (\ldots, x_{-1}, x_1, x_2, x_3, \ldots).$$

So T_1 acts as the right shift by 1 and T_1^* acts as the left shift by 1, thus $T_1^* T_1 = 1 = T_1 T_1^*$, i.e. T_1 is unitary.

Now consider a general operator T_n in the generating set for $C^*(\phi(\mathcal{T}))$. Firstly, suppose $n \geq 1$. Then

$$T_n : \delta_m \mapsto \begin{cases} \delta_{m+n} & \text{if } m < -n \text{ or } m > 0 \\ \delta_{m+n+1} & \text{if } -n \leq m < 0. \end{cases}$$

So

$$\begin{aligned} &T_n : (..., x_{-n-1}, x_{-n}, x_{-n+1}, .., x_{-2}, x_{-1}, x_1, x_2, .., x_{n-1}, x_n, x_{n+1}, ...) \\ &\mapsto (..., x_{-2n-1}, x_{-2n}, x_{-2n+1}, .., x_{-n-2}, x_{-n-1}, x_{-n}, x_{-n+1}, .., x_{-1}, x_1, x_2, ...), \end{aligned}$$

i.e. T_n acts as a right shift by n spaces. For $n < 0$,

$$T_n : \delta_m \mapsto \begin{cases} \delta_{m+n} & \text{if } m > -n \text{ or } m < 0 \\ \delta_{m+n-1} & \text{if } 0 < m \leq -n. \end{cases}$$

So

$$\begin{aligned} &T_n : (..., x_{n-1}, x_n, x_{n+1}, .., x_{-2}, x_{-1}, x_1, x_2, .., x_{-n-1}, x_{-n}, x_{-n+1}, ...) \\ &\mapsto (..., x_{-1}, x_1, x_2, .., x_{-n-1}, x_{-n}, x_{-n+1}, x_{-n+2}, .., x_{-2n-1}, x_{-2n}, x_{-2n+1}, ...), \end{aligned}$$

i.e. T_n acts as a left shift by $|n| = -n$ spaces, that is again a right shift by n spaces. Thus it is clear that the partial translation algebra is generated by the single unitary operator T_1, and hence $C^*(\phi(\mathcal{T})) \cong C^*(\mathcal{T}) \cong C(S^1)$.

To summarise:

PROPOSITION 5.2 *Let $X = \mathbb{Z}\backslash\{0\} \subseteq \mathbb{Z}$, and let $\phi : \mathbb{Z} \to X$ be the uniform bijection defined by*

$$\phi : n \mapsto \begin{cases} n & \text{if } n \geq 1 \\ n-1 & \text{if } n \leq 0. \end{cases}$$

Then if $\mathcal{T}$ denotes the canonical partial translation structure on $\mathbb{Z}$, we have

$$C^*(\phi(\mathcal{T})) \cong C^*(\mathcal{T}) \cong C(S^1).$$

In particular, this algebra differs from the partial translation algebra which arises from the restriction of $\mathcal{T}$ to X.

5.2.3 $\mathbb{F}_2 \backslash \{e\}$

Let us now return to the example of "the free group with a hole" which was first discussed in section 5.1.4. Specifically, consider the partial translation structure $\mathcal{T}_X$, the restriction of the canonical partial translation structure on $\mathbb{F}_2$ to our space $X = \mathbb{F}_2 \backslash \{e\}$. A priori, there seems to be no reason why we would not require all of the partial translations to generate the algebra arising from $\mathcal{T}_X$. Compositions of the partial isometries arising from right multiplication by the four generating elements for $\mathbb{F}_2$ will have their domains restricted too much to form all the generating elements of $C^*(\mathcal{T}_X)$. For example, as a composition of partial translations, $t_a \circ t_a : x \mapsto xaa$ is defined on $X \backslash \{a^{-1}, a^{-1}a^{-1}\}$, whereas the partial translation t_{aa} acts on X in the same way but is defined on $X \backslash \{a^{-1}a^{-1}\}$, so the operators in $C^*(\mathcal{T}_X)$ which correspond to these two elements will not be the same. However, we prove the following.

THEOREM 5.3 *The partial translation algebra $C^*(\mathcal{T}_X)$ is generated by the operators arising from the partial translations t_e, t_a, t_b, t_{aa}, t_{bb}, t_{ab}, t_{ba} and their adjoints.*

Proof.

For every $x \in \mathbb{F}_2$, denote the operator arising from the partial translation t_x by T_x. We know that operators of this form generate the algebra $C^*(\mathcal{T}_X)$, so we shall take a general such T_x and attempt to express it in terms of the operators corresponding to elements of length one or two (i.e. those listed in the theorem). Firstly, note that for every $x \in \mathbb{F}_2$ the projection p_x onto the x-coordinate in $l^2(\mathbb{F}_2)$ can be written as

$$p_x = 1 - T_x T_x^*.$$

So, in particular, from our given set of operators we can generate p_x for $x \in \{a, b, a^{-1}, b^{-1}\}$. Now let $x = x_1 \dots x_k \in \mathbb{F}_2$, so that each x_i is an element of the set $\{a, b, a^{-1}, b^{-1}\}$. We have

$$T_{x_2 \dots x_k} T_{x_1} : \delta_y \mapsto \delta_{yx} \text{ for all } y \in X \backslash \{x^{-1}, x_1^{-1}\}.$$

So this operator agrees with T_x on all basis elements of $l^2(X)$ except for $\delta_{x_1^{-1}}$. As the operator acts as the zero map on this element, we can recover T_x from $T_{x_2 \dots x_k} T_{x_1}$ by simply adding on an operator which acts in the correct way on $\delta_{x_1^{-1}}$ and is zero everywhere else. Such an operator could be expressed by $T_{x_3 \dots x_k} p_{x_2} T_{x_1 x_2}$, which takes $\delta_{x_1^{-1}}$ to $\delta_{x_2 \dots x_k} = \delta_{x_1^{-1} x}$ and otherwise acts as zero. Hence

$$T_x = T_{x_2 \dots x_k} T_{x_1} + T_{x_3 \dots x_k} p_{x_2} T_{x_1 x_2}.$$

Since we may obtain all projections p_x for $x \in \{a, b, a^{-1}, b^{-1}\}$ from the set of operators $\{T_e = 1, T_a, T_b, T_{aa}, T_{bb}, T_{ab}, T_{ba}\}$, then by induction on the word length of an element $x \in \mathbb{F}_2$ which gives rise to an operator T_x and by considering the above equation, we may thus generate the entire generating set for $C^*(\mathcal{T}_X)$, and therefore the algebra itself.

$\square$

5.2.4 $\mathbb{F}_2 \backslash \{\mathbb{Z}\}$

We can adapt the previous theorem to the case where we remove a subgroup isomorphic to $\mathbb{Z}$ from $\mathbb{F}_2$, rather than just the identity element. Here we obtain the following.

THEOREM 5.4 *Let $\mathbb{F}_2 = \langle a, b \rangle$ and let $H = \{a^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$ be a subgroup of $\mathbb{F}_2$. Let $X = \mathbb{F}_2 \backslash H$, so that*

$$X = \{x \in \mathbb{F}_2 \mid x \neq a^n \text{ for any } n \in \mathbb{Z}\}.$$

The partial translation algebra $C^*(\mathcal{T}_X)$ is generated by the operators arising from the partial translations t_e, t_a, t_b, t_{bb} and their adjoints.

Proof.

For every $x \in \mathbb{F}_2$, denote the operator arising from the partial translation t_x by T_x. We may follow the method of the proof of Theorem 5.3 to show that any generator T_x of $C^*(\mathcal{T}_X)$ can be expressed by

$$T_x = T_{x_2 \dots x_k} T_{x_1} + T_{x_3 \dots x_k} (1 - T_{x_2} T_{x_2}^*) T_{x_1 x_2},$$

where $x = x_1 \dots x_k$, and hence that the operators $T_e = 1$, T_a, T_b, T_{aa}, T_{bb}, T_{ab}, T_{ba} and their adjoints are sufficient to generate the algebra $C^*(\mathcal{T}_X)$.

However, in this case we have $T_{aa} = T_a T_a$, since $Xa = X$, and hence T_{aa} may instantly be deleted from the above list.

In fact, we also have the following:

$$T_b T_a : \delta_x \mapsto \begin{cases} 0 & \text{if } xa = a^n b^{-1} \text{ for some } n \in \mathbb{Z} \\ \delta_{xab} & \text{otherwise} \end{cases}$$

and

$$T_{ab} : \delta_x \mapsto \begin{cases} 0 & \text{if } x = a^n b^{-1} a^{-1} \text{ for some } n \in \mathbb{Z} \\ \delta_{xab} & \text{otherwise,} \end{cases}$$

so that $T_b T_a = T_{ab}$. Additionally, both $T_a T_b$ and T_{ba} map those δ_x for which x does not equal $a^n b^{-1}$ for any $n \in \mathbb{Z}$ to δ_{xba}, and are zero otherwise, so that $T_a T_b = T_{ba}$.

Thus $C^*(\mathcal{T}_X)$ is generated by the operators $T_e = 1$, T_a, T_b, T_{bb} and their adjoints.

□

Note that the operator T_{bb} is defined by

$$T_{bb} : \delta_x \mapsto \begin{cases} 0 & \text{if } x = a^n b^{-1} b^{-1} \text{ for some } n \in \mathbb{Z} \\ \delta_{xbb} & \text{otherwise,} \end{cases}$$

whereas

$$T_b : \delta_x \mapsto \begin{cases} 0 & \text{if } x = a^n b^{-1} \text{ for some } n \in \mathbb{Z} \\ \delta_{xb} & \text{otherwise,} \end{cases}$$

so that $(T_b)^m(\delta_x)$, for $m > 0$, is always zero when $x = a^n b^{-1}$ for some $n \in \mathbb{Z}$. Thus it is clear that we do require both of these operators to generate the partial translation algebra mentioned in the above theorem. However, these two (and their adjoints) are sufficient to construct operators T_{b^m} for other values of $m \in \mathbb{Z}$. For example, we have

$$T_{bbb} = T_{bb} T_b + T_b T_{bb} - (T_b)^2 T_{b^{-1}} T_{bb}.$$

6 Transport of Partial Translation Structures for Subspaces of Groups

Our aim is to describe the conditions under which a map of metric spaces gives rise to a homomorphism between the corresponding partial translation algebras. As a first step toward this goal, we wish to understand when the inclusion map from a subspace $X \subseteq G$ into a group G induces a C^*-algebra homomorphism $\varphi : C^*(\mathcal{T}_X) \to C^*(\mathcal{T}) = C_r^*(G)$ from the partial translation algebra arising from the restriction of the group's canonical partial translation structure to X, to the reduced C^*-algebra of G. It can be seen that if the canonical partial translation structure restricts to the subspace X in such a way that there exists a nilpotent operator T in the generating set for $C^*(\mathcal{T}_X)$, arising from a restricted partial translation τ, then we will not obtain a homomorphism in this manner. Indeed, we have the following:

Let G be a discrete group endowed with a canonical partial translation structure $\mathcal{T}$, and let X be a subset of G. Restrict $\mathcal{T}$ to X and denote this restriction by $\mathcal{T}_X$; note that we know $\mathcal{T}_X$ to be a valid partial translation structure by Theorem 3.8. Recall that for every $g \in G$ we have a partial translation in $\mathcal{T}$ defined in the following way:

$$t_g = \{(x, xg) \mid x \in G\}.$$

Denote the restriction of this partial translation to X by

$$\tau_g = \{(x, xg) \mid x, xg \in X\}.$$

For every $g \in G$, we denote by T_g the partial isometry associated to t_g and by $\tilde{T}_g$ the partial isometry associated to τ_g. There is a natural linear map $\varphi : C^*(\mathcal{T}_X) \to C^*(\mathcal{T}) = C_r^*(G)$ that can be defined by setting

$$\varphi(\tilde{T}_g) = T_g$$

and extending by linearity.

LEMMA 6.1 *If there exists some $g \in G$ for which $\tilde{T}_g$ is a nilpotent operator in the C^*-algebra $C^*(\mathcal{T})$ then the map φ is not a C^*-algebra homomorphism.*

Proof.

Suppose for contradiction that $\varphi : C^*(\mathcal{T}_X) \to C_r^*(G)$ is a homomorphism. To say that $\tilde{T}_g$ is a nilpotent operator means that there exists some positive integer n such that $\tilde{T}_g^n = 0$. Thus we must have

$$\varphi(\tilde{T}_g^n) = \varphi(0) = 0.$$

However, the homomorphism properties of φ dictate that

$$\varphi(\tilde{T}_g^n) = (\varphi(\tilde{T}_g))^n = T_g^n = T_{g^n}.$$

Hence we obtain

$$T_{g^n} = 0,$$

for some $g \in G$, $n > 0$. By definition of the operator T_{g^n}, this means that the partial translation t_{g^n} that it arises from satisfies $Dom(t_{g^n}) = \emptyset$. However, g^n is an element of the group G, and we know that every partial translation of this form is globally defined. Thus we obtain a contradiction.

□

It seems reasonable to assume that this is the only possible obstruction which might be encountered when defining our C^*-algebra homomorphism. However, the following example reveals that there are more potential problems than nilpotency alone.

EXAMPLE 6.2 Let Y be the subspace of $\mathbb{Z}$ given by:

$$Y = \bigcup_{n \in \mathbb{N} \setminus \{0\}} Y_n,$$

where the components Y_n of Y are defined recursively in the following way:

$$\begin{aligned}
Y_1 &= \{0, 2\}, \\
Y_{n+1} &= \{M_n + 2, M_n + 4, \ldots, M_n + 2(n+1), \\
&\quad M_n + 2(n+1) + 3, M_n + 2(n+1) + 6, \ldots, M_n + 2(n+1) + 3n, \\
&\quad \ldots, \\
&\quad M_n + 2(n+1) + \ldots + (n+1) \cdot 1, M_n + 2(n+1) + \ldots + (n+1) \cdot 2, \\
&\quad M_n + 2(n+1) + 3n + 4(n-1) + \ldots + (n+1) \cdot 2 + (n+2) \cdot 1\},
\end{aligned}$$

where M_n denotes the maximal element of the component Y_n. Hence we have

$$\begin{aligned}
Y_1 &= \{0, 2\}, \\
Y_2 &= \{4, 6, 9\}, \\
Y_3 &= \{11, 13, 15, 18, 21, 25\}, \\
Y_4 &= \{27, 29, 31, 33, 36, 39, 42, 46, 50, 55\}, \\
&\vdots \quad ,
\end{aligned}$$

so that if we order the set Y in the natural way, the distances between consecutive elements of Y form the following sequence:

$$(2, 2, 2, 3, 2, 2, 2, 3, 3, 4, 2, 2, 2, 2, 3, 3, 3, 4, 4, 5, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 5, 5, 6, ..).$$

Every canonical partial translation on $\mathbb{Z}$ restricts to Y, except for t_1 and t_{-1}, and every restricted partial translation gives rise to an operator which is non-nilpotent, since for any $n \in \mathbb{Z}\backslash\{-1, 1\}$ we can find a string of arbitrarily many elements of Y separated by gaps of length $|n|$. However, if we attempt to define a map $\varphi : C^*(\mathcal{T}_Y) \to C^*(\mathcal{T}) = C^*_r(\mathbb{Z})$ by

$$\varphi : \tilde{T}_g \mapsto T_g,$$

then we have

$$\varphi(\tilde{T}_3\tilde{T}_{-2}) = \varphi(0) = 0,$$

whereas

$$\varphi(\tilde{T}_3)\varphi(\tilde{T}_{-2}) = T_3T_{-2} = T_1.$$

Hence the map φ fails to be a homomorphism.

It would appear that the failure of the above map to be a C^*-algebra homomorphism is due to a lack of semigroup structure on the set of restricted partial isometries. We propose that this is the only other factor which needs to be considered when attempting to define the related homomorphism of C^*-algebras. In the particular case where the group under consideration is the integers, this translates to the following theorem, which was suggested in discussion with Dr. N. Wright.

THEOREM 6.3 *Let X be a subset of the integers and use the usual notation to describe the restricted canonical partial translation structure on X. If the set $H = \{n \in \mathbb{Z} \mid \tilde{T}_n \neq 0\}$ is a group and if $\tilde{T}_n$ is non-nilpotent for all $n \in H$, then the canonical map $\varphi : C^*(\mathcal{T}_X) \to C_r^*(H) \cong C_r^*(\mathbb{Z})$ is a C^*-algebra homomorphism.*

Proof.

We let $\varphi : C^*(\mathcal{T}_X) \to C_r^*(H)$ denote the canonical map, so we let $\varphi(\tilde{T}_n) = T_n$ for all $n \in H$, $\varphi(0) = 0$, and extend to the rest of the algebra by linearity. This map is a homomorphism if and only if its kernel is the ideal

$$I = \left\langle \tilde{T}_{n+m} - \tilde{T}_n \tilde{T}_m \mid n, m \in H \right\rangle,$$

that is the closure of the linear span of elements of the form $\tilde{T}_{n+m} - \tilde{T}_n \tilde{T}_m$. Therefore to prove this we must show that $\tilde{T}_n \notin I$ for all $n \in H$.

Assume for contradiction that there does exist some $n \in H$ such that $\tilde{T}_n \in I$. Since I is an ideal in $C^*(\mathcal{T}_X)$, we then also have $\tilde{T}_{-n}\tilde{T}_n \in I$. As we assume H to be a group, $-n \in H$ also, and thus $\tilde{T}_{-n+n} - \tilde{T}_{-n}\tilde{T}_n \in I$, by definition of I. So now $\tilde{T}_{-n}\tilde{T}_n \in I$ and $\tilde{T}_0 - \tilde{T}_{-n}\tilde{T}_n \in I$, and therefore $\tilde{T}_{-n}\tilde{T}_n + (\tilde{T}_0 - \tilde{T}_{-n}\tilde{T}_n) = \tilde{T}_0 \in I$. Note that since I is an ideal in $C^*(\mathcal{T}_X)$ and $\tilde{T}_0$ is the identity operator this means that $C^*(\mathcal{T}_X) = I$, but this fact is not particularly helpful as we still need to prove that such a situation cannot arise.

If we multiply some generating element $\tilde{T}_{n+m} - \tilde{T}_n\tilde{T}_m \in I$ on the left by some generator $\tilde{T}_p$ of $C^*(\mathcal{T}_X)$, we obtain an element of the form $\tilde{T}_p\tilde{T}_{n+m} - \tilde{T}_p\tilde{T}_n\tilde{T}_m$, which can be expressed as:

$$\tilde{T}_p\tilde{T}_{n+m} - \tilde{T}_p\tilde{T}_n\tilde{T}_m = (\tilde{T}_{p+n+m} - \tilde{T}_p\tilde{T}_n\tilde{T}_m) - (\tilde{T}_{p+n+m} - \tilde{T}_p\tilde{T}_{n+m}),$$

where $n, m, p \in H$. Hence, more generally, we see that the ideal I is equal to the closed linear span of the set

$$\{\tilde{T}_{n_1} \ldots \tilde{T}_{n_k} - \tilde{T}_{m_1} \ldots \tilde{T}_{m_l} \mid n_1 + \ldots + n_k = m_1 + \ldots + m_l,\ n_i, m_i \in H \text{ for all } i\}.$$

So if $\tilde{T}_0$ is indeed an element of this ideal then it must lie within the closed span of such elements where $n_1 + \ldots + n_k = m_1 + \ldots + m_l = 0$. However, if this holds then we can assume that for each element both $\tilde{T}_{n_1} \ldots \tilde{T}_{n_k}$ and $\tilde{T}_{m_1} \ldots \tilde{T}_{m_l}$ are projections on $l^2(X)$, whilst $\tilde{T}_0$ is the identity operator on this space, and so it is equivalent to say that

$$\tilde{T}_0 \in \overline{\text{span}}\{\tilde{T}_0 - \tilde{T}_{m_1} \dots \tilde{T}_{m_l} \mid m_1 + \dots + m_l = 0,\ m_i \in H,\ \tilde{T}_{m_1} \dots \tilde{T}_{m_l} \neq 0\}.$$

Thus $\tilde{T}_0$ may be written as a linear combination of elements of the form $\tilde{T}_0 - \tilde{T}_{m_1} \dots \tilde{T}_{m_l}$ which satisfy the above conditions. Assume that we can find this expression and that it consists of finitely many elements, and then choose the positive integer m satisfying:

$$m = \max\{\sum_{1 \leq i \leq l} |m_i| \mid \tilde{T}_0 - \tilde{T}_{m_1} \dots \tilde{T}_{m_l} \text{ appears in the linear combination for } \tilde{T}_0\}.$$

Since H is a subgroup of $\mathbb{Z}$, we know that it is of the form $h\mathbb{Z}$ for some $h \in \mathbb{N}$. Then by definition the partial translation τ_h must give rise to a non-nilpotent operator, and therefore for any positive integer k we must be able to find some $x \in X$ such that the length of the orbit $\{h^n x \mid n \in \mathbb{N}\}$ in X is greater than k. So consider a segment of length $m+1$ of such an orbit. At the centre point y of this string of elements of X, every composition of partial translations $\tau_{m_1} \dots \tau_{m_l}$, for which the operator $\tilde{T}_{m_1} \dots \tilde{T}_{m_l}$ is used in our expression of $\tilde{T}_0$, is defined. Hence $\tilde{T}_0 - \tilde{T}_{m_1} \dots \tilde{T}_{m_l} = 0$ on $\mathbb{C}\delta_y$, whenever $m_i \in H$, $\sum |m_i| \leq m$ and $\sum m_i = 0$. Thus we obtain a subspace of $l^2(X)$ on which the identity operator is zero, and so the linear combination we constructed must not be valid. Since the orbits $\{h^n x \mid n \in \mathbb{N}\}$ can grow arbitrarily long, this problem cannot be avoided by taking limits, and so we cannot express $\tilde{T}_0$ as an infinite linear combination of elements in the required form either. Thus $\tilde{T}_0 \notin I$, and hence $\tilde{T}_n \notin I$ for all $n \in H$. Therefore the map φ is indeed a homomorphism with kernel I.

□

Unfortunately, the conditions of Theorem 6.3 are not sufficient to cover cases where the group in question is not isomorphic to $\mathbb{Z}$, as is illustrated by the following example.

EXAMPLE 6.4 Let $G = \mathbb{Z}^2$. For every $g \in G$ and $n \in \mathbb{N}$, construct the following subset of G:

$$X_{g,n} = \{e, g, 2g, \dots, ng\}.$$

For example, $X_{(1,1),5} = \{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5)\}$.

Since G is countable, we can enumerate these sets and choose some sequence of integers $(a_{g,n})$, such that $|a_{g,n}|$ tends to infinity very fast. If constructed correctly, we can use this sequence to separate our sets in such a way that no point within any given $X_{g,n}$ can be reached from some other set $X_{h,m}$ using a partial translation related to an element of that set. We then consider the subspace obtained in this manner, in other words we let

$$X = \bigcup_{g,n} a_{g,n} X_{g,n},$$

where $a_{g,n}X_{g,n} = \{a_{g,n}x \mid x \in X_{g,n}\}$. Now every canonical partial translation on $G = \mathbb{Z}^2$ restricts to this space and gives rise to a non-nilpotent operator, so that $H = G$, if we define H as in Theorem 6.3. However, the theorem fails for this example, since if we compose any two partial translations which do not correspond to multiples of the same group element then this composition gives rise to the zero operator.

The problem with this example is that whereas in $\mathbb{Z}$ we require there to be arbitrarily long orbits or *strings* of elements separated by gaps of the same length as the generator of H, which are given to us by the non-nilpotency of our operators, in a 2-dimensional group such as $\mathbb{Z}^2$ we need there to be arbitrarily large *balls* of elements separated by small enough gaps. To ensure that this happens in general we need to avoid not only nilpotent operators but in fact all *zero divisors*.

Recall that a non-zero element a of a ring R is called a *zero divisor* if there exists a non-zero $b \in R$ such that $ab = 0 = ba$.

Note that insisting that none of the operators arising from elements of our set H in the general case are zero divisors in the monoid $\{\tilde{T}_{h_1} \dots \tilde{T}_{h_l} \mid h_i \in H\}$ is enough to imply that H is a group. Indeed, we have the following lemma.

LEMMA 6.5 *Let G be a countable group and let X be a subset of G. Use the usual notation to describe the restricted canonical partial translation structure on X, and let $H = \{g \in G \mid \tilde{T}_g \neq 0\}$. If $\tilde{T}_g$ is not a zero divisor in the monoid $\{\tilde{T}_{h_1} \dots \tilde{T}_{h_l} \mid h_i \in H\}$ for all $g \in H$, then H is a subgroup of G.*

Proof.

Note that H consists solely of elements of the group G, and we know that the identity element $e \in G$ is contained in H, since $\tilde{T}_e$ is the identity operator on $l^2(X)$ and hence must be non-zero. Also, if $g \in H$ then $\tilde{T}_g \neq 0$, which is equivalent to saying that τ_g has non-empty domain, so there exists $x \in X$ such that $xg^{-1} \in X$. However, this then means that there exists $y = xg^{-1} \in X$ such that $yg \in X$, so $\tau_{g^{-1}}$ also has non-empty domain and therefore $\tilde{T}_{g^{-1}} \neq 0$, i.e. $g^{-1} \in H$. Thus the only way that H could fail to be a subgroup of G is if there exist some $g, h \in H$ such that $gh \notin H$. However, if this were the case then we would have $\tilde{T}_g, \tilde{T}_h \neq 0$ and $\tilde{T}_{gh} = 0$ for some $g, h \in H$. But we know that the operator $\tilde{T}_g \tilde{T}_h$ always behaves in the same way as $\tilde{T}_{gh}$ whenever $\tilde{T}_g \tilde{T}_h$ is non-zero, and so we must also have $\tilde{T}_g \tilde{T}_h = 0$, implying that $\tilde{T}_g$ and $\tilde{T}_h$ are zero divisors in $\{\tilde{T}_{h_1} \dots \tilde{T}_{h_l} \mid h_i \in H\}$.

□

Hence we may drop the requirement that H is a group from our statement of the theorem for the general case, since this will be given automatically. This leaves us with the following:

THEOREM 6.6 *Let G be a countable group and let X be a subset of G. Use the usual notation to describe the restricted canonical partial translation structure on X, and let $H = \{g \in G \mid \tilde{T}_g \neq 0\}$. If $\tilde{T}_g$ is not a zero divisor in the monoid $\{\tilde{T}_{h_1} \dots \tilde{T}_{h_l} \mid h_i \in H\}$ for all $g \in H$, then the canonical map $\varphi : C^*(\mathcal{T}_X) \to C^*_r(H) \subseteq C^*_r(G)$ is a C^*-algebra homomorphism.*

Proof.

Note that for the partial isometry $\tilde{T}_g$ to be non-zero, the corresponding partial translation τ_g must have non-empty domain, and so the way in which H is defined here ensures that any two non-empty restricted partial translations are composable. In particular, any partial translation on X can be composed with any word consisting of partial translations on X.

This proof will extend the method of Theorem 6.3, so we wish to show that for every $g \in H$ the operator $\tilde{T}_g$ is not contained within the ideal

$$I = \left\langle \tilde{T}_{gh} - \tilde{T}_g \tilde{T}_h \mid g, h \in H \right\rangle.$$

We can follow the previous proof almost exactly (simply by replacing $\tilde{T}_n$ with $\tilde{T}_g$ and $\tilde{T}_{-n}$ with $\tilde{T}_{g^{-1}}$) to see that if this were the case then we would have $\tilde{T}_e \in I$, where e denotes the identity element in G, and that $\tilde{T}_e$ could then be represented as a linear combination of elements of the form

$$\tilde{T}_e - \tilde{T}_{h_1} \dots \tilde{T}_{h_l},$$

where $h_1 \dots h_l = e$ and $h_i \in H$ for all $1 \leq i \leq l$. However, as we are no longer dealing with the integer case, we can no longer assume that H is a cyclic group, and so from this point on we need to apply a slightly different method.

Denote the linear combination of elements that is meant to represent $\tilde{T}_e$ by

$$\Theta = \sum_{i=1}^{k} a_i(\tilde{T}_e - \tilde{T}_{h_1^i} \dots \tilde{T}_{h_{l_i}^i}),$$

where $a_i \in \mathbb{C}$ and $h_j^i \in H$ for all i and j. To obtain a contradiction we wish to find some $x \in X$ such that $\Theta\delta_x = 0$. By our assumption that elements of H do not give rise to zero divisor operators, we have that there exists some $x \in X$ which lies in the domain of the composition of partial translations

$$(\tau_{h_1^1}\tau_{h_2^1} \dots \tau_{h_{l_1}^1})(\tau_{h_1^2}\tau_{h_2^2} \dots \tau_{h_{l_2}^2}) \dots (\tau_{h_1^k}\tau_{h_2^k} \dots \tau_{h_{l_k}^k}).$$

However, we assume that $h_1^i \dots h_{l_i}^i = e$ for all $1 \leq i \leq k$, and so each composition $\tau_{h_1^i}\tau_{h_2^i} \dots \tau_{h_{l_i}^i}$ acts as the identity translation wherever it is defined. Thus we see that the point x lies in the domain of every such composition, and so the corresponding operators act as the identity on the subspace spanned by δ_x. In particular,

$$(\tilde{T}_e - \tilde{T}_{h_1^i} \dots \tilde{T}_{h_{l_i}^i})\delta_x = 0, \text{ for all } 1 \leq i \leq k,$$

and so $\Theta\delta_x = 0$, as required. Specifically, since Θ differs from the identity operator $\tilde{T}_e$ on at least one basis vector we see that $\|\Theta - \tilde{T}_e\| \geq 1$, and since this holds for all finite linear combinations Θ of the desired form, we see that $\tilde{T}_e$ cannot lie within the closure of the span of such elements either. Thus $\tilde{T}_e$ is not an element of the ideal I.

Hence $\tilde{T}_g \notin I$ for all $g \in H$ and so φ is indeed a C^*-algebra homomorphism with kernel I.

□

Note that in some cases it is possible to define a homomorphism between partial translation algebras in the presence of nilpotent operators (and hence zero divisors), if we define it in such a way that it maps all nilpotent operators to zero. For example, let X be the subset of $\mathbb{Z}$ defined by $X = 2\mathbb{Z} \cup \{1\}$. Every canonical partial translation for $\mathbb{Z}$ restricts to X non-trivially, however each translation by an odd number is defined on only two points in X and hence gives rise to a nilpotent operator, whereas $H' = \{n \in \mathbb{Z} \mid \tilde{T}_n \neq 0 \text{ is non-nilpotent}\} = 2\mathbb{Z}$, and hence is a group. Now if we define a map φ from the canonical (i.e. restricted) partial translation algebra for X to the canonical partial translation algebra for $\mathbb{Z}$ as follows:

$$\varphi : \tilde{T}_n \mapsto \begin{cases} T_n & \text{if } n \in H' \\ 0 & \text{otherwise,} \end{cases}$$

and extend by linearity, then it is clear that this map is a C^*-algebra homomorphism.

One might suppose that this would be the case for any subset of $\mathbb{Z}$ for which we have $H' = \{n \in \mathbb{Z} \mid \tilde{T}_n \neq 0 \text{ is non-nilpotent}\}$ a non-trivial subgroup of $\mathbb{Z}$, and more generally that the map $\varphi : C^*(\mathcal{T}_X) \to C^*_r(H') \subseteq C^*_r(G)$ given by

$$\varphi : \tilde{T}_g \mapsto \begin{cases} T_g & \text{if } g \in H' \\ 0 & \text{otherwise} \end{cases}$$

would be a C^*-algebra homomorphism whenever X is a subset of a countable group G for which the set $H' = \{g \in G \mid \tilde{T}_g \neq 0 \text{ not a zero divisor in } C^*(\mathcal{T}_X)\}$ is a non-trivial subgroup of G. However, the following example shows that even the integer version of this is not true.

EXAMPLE 6.7 Let $X = \{n \in \mathbb{Z} \mid n \equiv 0 \bmod 4 \text{ or } n \equiv 1 \bmod 4\}$, so that

$$X = \{\ldots, -12, -11, -8, -7, -4, -3, 0, 1, 4, 5, 8, 9, 12, 13, \ldots\}.$$

The only even translations which restrict non-trivially to this space are those corresponding to multiples of four, and these all give rise to non-nilpotent operators, whereas every odd translation restricts but cannot be applied more than once at any point and hence gives rise to a nilpotent operator. Thus $H' = 4\mathbb{Z}$, which is certainly a non-trivial subgroup of $\mathbb{Z}$. However, we have

$$\tilde{T}_4 = \tilde{T}_1\tilde{T}_3 + \tilde{T}_3\tilde{T}_1,$$

and therefore

$$\tilde{T}_{4k} = (\tilde{T}_4)^k = (\tilde{T}_1\tilde{T}_3 + \tilde{T}_3\tilde{T}_1)^k,$$

so, as both $\tilde{T}_1$ and $\tilde{T}_3$ must be mapped to zero for a map between the partial translation algebras to be a homomorphism, we see that the only possible map from $C^*(\mathcal{T}_X)$ to $C_r^*(H')$ (and indeed $C_r^*(\mathbb{Z})$) is the zero map.

7 Maps of Partial Translation Structures

In this chapter we construct a definition of a morphism of partial translation structures, basing the definition on how the induced maps will behave with respect to partial translation algebras. This ultimately allows us to expand on the results of the previous chapter to identify when any given map between metric spaces gives rise to a homomorphism of the corresponding partial translation algebras. However, we first test our definition by revisiting a number of examples of maps already discussed to see which ones satisfy it, before considering some new examples of maps from different groups into $\mathbb{Z}$. In particular, we show that the only partial translation structure maps between the infinite dihedral group and $\mathbb{Z}$ are constant functions, and we compare two maps from $\mathbb{Z}\times\mathbb{Z}_2$ to $\mathbb{Z}$, where one is a map of partial translation structures and the other is not.

7.1 Definition of a Map of Partial Translation Structures

The main theorem of the previous chapter can be rephrased and generalised further if we formally define what it means for a map between metric spaces to act as a morphism between partial translation structures. To this end, let us firstly describe the conditions required for the presence of such a map.

DEFINITION 7.1 [Mappable Partial Translation Structure]

Let X be a metric space endowed with a partial translation structure $\mathcal{T}$. We say that $\mathcal{T}$ is *mappable* if $\mathcal{T}$ consists of a countable family of *pairwise disjoint* partial translations which is closed with respect to taking inverses, together with a single set of cotranslations, and if additionally $\mathcal{T}$ is free. These stipulations together with the original definition of a partial translation structure amount to the following conditions on $\mathcal{T}$:

1. For every $R > 0$ there exists a finite collection of partial translations

$\mathcal{T}_R$ contained in $\mathcal{T}$, such that the R-neighbourhood of the diagonal in $X \times X$ is contained within $\cup_{t \in \mathcal{T}_R} t$;

2. We have a collection of partial cotranslations Σ for partial translations in $\mathcal{T}$, such that for every $x, y \in X$ there exists at most one σ in Σ such that $\sigma x = y$;

3. For every partial translation t in $\mathcal{T}$ and for all $(x, y), (x', y')$ in t, there exists σ in Σ such that $\sigma x = x'$ and $\sigma y = y'$;

4. The inverse of any partial translation in $\mathcal{T}$ is also a partial translation in $\mathcal{T}$.

Note that the canonical partial translation structure on a group satisfies the above definition, as do restrictions of such partial translation structures to subspaces. Now let X and Y be two metric space endowed with mappable partial translation structures $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively, let $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, and recall that for a partial translation t in $\mathcal{T}_X$ we take $\phi(t)$ to denote the following subset of $Y \times Y$:

$$\phi(t) = \{(\phi(x), \phi(y)) \mid (x, y) \in t\}.$$

We now define morphisms between partial translation structures as follows.

DEFINITION 7.2 [Partial Translation Structure Map]

Let $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map of metric spaces, where $\mathcal{T}_X$ and $\mathcal{T}_Y$ are mappable partial translation structures. Let $\mathcal{T}(X)$ (respectively $\mathcal{T}(Y)$) denote the closure of $\mathcal{T}_X$ (respectively $\mathcal{T}_Y$) with respect to composition. We say that ϕ is a *partial translation structure map*, or *P.T.S. map*, if the following hold:

1. For every partial translation t in $\mathcal{T}_X$ there exists a partial translation $\phi_*(t)$ in $\mathcal{T}_Y$ such that $\phi(t) \subseteq \phi_*(t)$, so that we define a map ϕ_* from $\mathcal{T}_X$ to $\mathcal{T}_Y$;

2. Similarly, for every partial cotranslation σ in Σ_X there exists a partial cotranslation $\phi_*(\sigma)$ in Σ_Y such that $\phi_*(\sigma)\phi(x) = \phi(\sigma x)$, for all $x \in Dom(\sigma)$;

3. The map ϕ_* extends to a homomorphism from $\mathcal{T}(X)$ to $\mathcal{T}(Y)$.

The map $\phi_* : \mathcal{T}_X \to \mathcal{T}_Y$ is well-defined if we assume that partial translations in $\mathcal{T}_X$ and $\mathcal{T}_Y$ are pairwise disjoint, that both partial translation structures are free, and if we set ϕ_* applied to the empty translation to be the empty translation.

REMARK 7.3 A priori, there does not seem to be any justification for imposing additional conditions on how a P.T.S. map $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ behaves with respect to partial cotranslations, since these do not feature in the formation of partial translation algebras. Note that the condition we do have ensures that ϕ_* maps cotranslations for $\mathcal{T}_X$ to cotranslations for $\mathcal{T}_Y$, and this requires a certain degree of interaction with the partial translations. Indeed, we assume that, given some partial translation t in $\mathcal{T}_X$, for all $(t(x), x), (t(x'), x') \in t$ there exists a partial cotranslation $\sigma \in \Sigma_X$ such that $\sigma t(x) = t(x')$ and $\sigma x = x'$. Then $(\phi(t(x)), \phi(x)), (\phi(t(x')), \phi(x')) \in \phi(t)$ and we have a partial translation $\phi_*(t)$ in $\mathcal{T}_Y$ such that $\phi(t) \subseteq \phi_*(t)$. So by our assumptions for $\mathcal{T}_Y$ there exists a partial cotranslation $\sigma' \in \Sigma_Y$ such that $\sigma'\phi(t(x)) = \phi(t(x'))$ and $\sigma'\phi(x) = \phi(x')$, i.e.

$$\sigma'\phi(t(x)) = \phi(\sigma t(x)) \text{ and } \sigma'\phi(x) = \phi(\sigma x).$$

Thus $\sigma' = \phi_*(\sigma)$, by definition and by uniqueness of $\phi_*(\sigma)$.

Note additionally that in the case where X is a subspace of an abelian discrete group G (for example, $G = \mathbb{Z}$) with the inherited canonical partial translation structure, then all partial cotranslations are also partial translations and so do not require separate consideration. Indeed, we have

$$\sigma_g(x) := gx = xg =: t_{g^{-1}}(x),$$

for all $g \in G, x \in X$. Hence we may disregard condition 2 of Definition 7.2 in this case.

We shall prove that any group homomorphism is a P.T.S. map between the canonical group partial translation structures. We firstly note the following lemma.

LEMMA 7.4 *Let G be a discrete group with canonical partial translation structure $\mathcal{T}$, and let H be a subgroup of G. The canonical partial translation structure on H is the same as restriction of $\mathcal{T}$ to H.*

Proof.

Let $t_g = \{(x, xg) \mid x \in G\}$ be a partial translation in $\mathcal{T}$. To say that t_g restricts non-trivially to H means that there exists some $h \in H$ such that $hg \in H$. Since H is a group, we have $h^{-1} \in H$, and hence $h^{-1}hg = g \in H$. Conversely, if g is an element of H then $hg \in H$ for all $h \in H$, and thus every partial translation in $\mathcal{T}$ which corresponds to an element of H restricts non-trivially to H. Hence the restriction of $\mathcal{T}$ to H consists of partial translations $\{\tau_h \mid h \in H\}$, where $\tau_h = \{(x, xh) \mid h \in H\}$, the restriction of t_h to H; this is the same set of partial translations as appear in the canonical partial translation structure on H. By similar reasoning, both partial translation structures have the same set of cotranslations. Finally, for every $R > 0$, the restriction of $\mathcal{T}_R$ to H is equal to $\{\tau_h \mid d(e, h) < R\}$, which is the similarly required partial translation set for the canonical partial translation structure on H.

□

THEOREM 7.5 *Let G and H be discrete groups equipped with canonical partial translation structures $\mathcal{T}_G$ and $\mathcal{T}_H$ respectively. If $\phi : H \to G$ is a group homomorphism then ϕ is a P.T.S. map between $\mathcal{T}_G$ and $\mathcal{T}_H$.*

Proof.

Suppose that we have a group homomorphism $\phi : H \to G$. We know that canonical partial translation structures for groups are mappable, so it only remains to check the conditions of Definition 7.2 for ϕ.

(1) Let τ_h be a canonical partial translation for H, so that $h \in H$. We have

$$\begin{aligned} \phi(\tau_h) &= \{(\phi(x), \phi(xh)) \mid x \in H\} \\ &= \{(\phi(x), \phi(x)\phi(h)) \mid x \in H\} \\ &\subseteq t_{\phi(h)}, \end{aligned}$$

where $t_{\phi(h)} = \{(g, g\phi(h)) \mid g \in G\}$ is a canonical partial translation for G. Hence we may set $\phi_*(\tau_h) = t_{\phi(h)}$ for all $h \in H$, to obtain a map between the canonical partial translation structures.

(2) Let s_h be a partial cotranslation for $\mathcal{T}_H$, the canonical partial translation structure for H (recall that we have a single set of cotranslations in the group case). We have $s_h x = hx$ for all $x \in H$, and so $\phi(s_h x) = \phi(hx) =$

$\phi(h)\phi(x)$, since ϕ is a homomorphism. For every $h \in H$, let $\phi_*(s_h) = \sigma_{\phi(h)}$ be the partial cotranslation for $\mathcal{T}_G$, the canonical partial translation structure on G, corresponding to left multiplication by $\phi(h)$. We then have $\phi_*(s_h)\phi(x) = \sigma_{\phi(h)}\phi(x) = \phi(h)\phi(x) = \phi(s_h x)$, for every $x \in H = Dom(s_h)$, as required.

(3) Since H is a group, all canonical partial translations are composable, and for every $h_1, h_2 \in H$ we have $\tau_{h_1}\tau_{h_2} = \tau_{h_1 h_2}$. Hence $\phi_*(\tau_{h_1}\tau_{h_2}) = \phi_*(\tau_{h_1 h_2}) = t_{\phi(h_1 h_2)} = t_{\phi(h_1)\phi(h_2)} = t_{\phi(h_1)}t_{\phi(h_2)} = \phi_*(\tau_{h_1})\phi_*(\tau_{h_2})$. Therefore ϕ_* acts as a homomorphism on partial translations, as required.

□

By Lemma 7.4, if H is a subgroup of G then we may replace the canonical partial translation structure on H in the above by the restriction of $\mathcal{T}_G$ to H.

7.2 Examples of Maps of Partial Translation Structures Involving Subspaces of $\mathbb{Z}$

7.2.1 Examples

- If we consider a set of integers to which every canonical partial translation for $\mathbb{Z}$ restricts and for which any composition of restricted partial translations is defined, then $\mathcal{T}_X = \mathcal{T}(X)$ and there are no empty translations to consider, so it is clear that the inclusion map from this subspace into $\mathbb{Z}$ will be a P.T.S. map. This case includes examples such as $\mathbb{N}$, $\mathbb{Z}\backslash\{0\}$, and the space we discuss in chapter 9.

- Consider the subspace $2\mathbb{Z} \subset \mathbb{Z}$. The canonical partial translations t_n which restrict to this space are those for which $n \in 2\mathbb{Z}$. Since $2\mathbb{Z}$ is a group and every restricted partial translation is globally defined on $2\mathbb{Z}$, we have that $\tau_{n_1} \dots \tau_{n_k} = \tau_{n_1 + \dots + n_k}$ for all $n_1, \dots, n_k \in 2\mathbb{Z}$, where τ_{n_i} denotes the restriction of the partial translation t_{n_i} to $2\mathbb{Z}$. Thus $\mathcal{T}_{2\mathbb{Z}} = \mathcal{T}(2\mathbb{Z})$, the inclusion map $\phi : 2\mathbb{Z} \to \mathbb{Z}$ is the same as ϕ_*, and it can be seen that this map is again a P.T.S. map.

  For this subspace there is another function which we know to map between the two partial translation structures, namely

$$\phi' : 2\mathbb{Z} \to \mathbb{Z},\ n \mapsto \frac{n}{2}.$$

We have

$$\tau_n = \{(x, x+n) \mid x, x+n \in 2\mathbb{Z}\},$$

and thus if we assume that $n \in 2\mathbb{Z}$ then we have

$$\tau_n = \{(x, x+n) \mid x \in 2\mathbb{Z}\},$$

and every restricted partial translation on $2\mathbb{Z}$ is of this form. Now

$$\begin{aligned}\phi'(\tau_n) &= \{(\phi'(x), \phi'(x+n)) \mid x \in 2\mathbb{Z}\} \\ &= \{(\frac{x}{2}, \frac{x+n}{2}) \mid x \in 2\mathbb{Z}\} \\ &= \{(\frac{x}{2}, \frac{x}{2} + \frac{n}{2}) \mid \frac{x}{2} \in \mathbb{Z}\} \\ &= t_{\frac{n}{2}},\end{aligned}$$

where $\frac{n}{2} \in \mathbb{Z}$. Hence $\phi'_*(\tau_n) = \phi'(\tau_n) = t_{\frac{n}{2}}$ for all $n \in 2\mathbb{Z}$ and so ϕ' satisfies condition 1 of Definition 7.2. We do not need to check condition 2, by Remark 7.3. Since $\mathcal{T}(2\mathbb{Z}) = \mathcal{T}_{2\mathbb{Z}}$, ϕ'_* is already defined for compositions of partial translations, and so finally we have

$$\begin{aligned}\phi'_*(\tau_{n_1} \dots \tau_{n_k}) &= \phi'_*(\tau_{n_1+\ldots+n_k}) \\ &= t_{\frac{n_1+\ldots+n_k}{2}} \\ &= t_{\frac{n_1}{2}+\ldots+\frac{n_k}{2}} \\ &= t_{\frac{n_1}{2}} \dots t_{\frac{n_k}{2}} \\ &= \phi'_*(\tau_{n_1}) \dots \phi'_*(\tau_{n_k}),\end{aligned}$$

for all $n_1, \ldots, n_k \in 2\mathbb{Z}$. Thus ϕ'_* is itself a homomorphism on $\mathcal{T}(2\mathbb{Z}) = \mathcal{T}_{2\mathbb{Z}}$ and therefore ϕ' is also a P.T.S. map.

This can also be seen from the fact that ϕ' is an isomorphism of groups.

7.2.2 Non-examples

- Consider the subset Y of $\mathbb{Z}$ defined in Example 6.2, so that

  $$Y = \{0, 2, 4, 6, 9, 11, 13, 15, 18, 21, 25, 27, 29, 31, 33, 36, 39, 42, 46, 50, 55, ..\}.$$

  We again let τ_n denote the restriction of the canonical partial translation t_n and let ϕ denote the inclusion map of Y into $\mathbb{Z}$, so that

$\phi_*(\tau_n) = t_n$ for all $n \in \mathbb{Z}\backslash\{-1, 1\}$. As t_1 and t_{-1} do not restrict to Y, the image of the restriction of either of these under ϕ_* is defined to be the empty translation on $\mathbb{Z}$. Now if we attempt to extend ϕ_* to a homomorphism on $\mathcal{T}(Y)$ then we set, for example, $\phi_*(\tau_3)\phi_*(\tau_{-2}) = \phi_*(\tau_3\tau_{-2})$. However,

$$\begin{aligned}\tau_3\tau_{-2} &= \{(x, x+3-2) \mid x, x+3, x+3-2 \in Y\} \\ &= \{(x, x+1) \mid x, x+1, x+3 \in Y\}\end{aligned}$$

has empty domain, whereas

$$\phi_*(\tau_3)\phi_*(\tau_{-2}) = t_3t_{-2} = t_1,$$

which is non-empty. Hence our map would take the empty translation to a non-empty translation and so would not be well-defined. Thus we do not have a P.T.S. map in this case.

- Let $W = 4\mathbb{Z} \cup \{4n+1 \mid n \in \mathbb{Z}\}$, as in Example 6.7. We again take ϕ to be the inclusion map of the subspace into $\mathbb{Z}$, so that $\phi_*(\tau_n) = t_n$ for every non-empty restricted partial translation τ_n on W. In this case, the only partial translations which restrict are those corresponding to odd numbers or multiples of four. Now the composition

$$\tau_1\tau_1 = \{(x, x+2) \mid x, x+1, x+2 \in W\}$$

has empty domain and thus should be mapped to the empty translation under an extension of ϕ_*. However, we have

$$\phi_*(\tau_1)\phi_*(\tau_1) = t_1t_1 = t_2,$$

which is non-empty. Hence ϕ again fails to be a P.T.S. map.

- Finally, if we let $Z = 2\mathbb{Z} \cup \{1\}$, then every canonical partial translation restricts to Z, although restrictions of odd translations cannot be composed with more than once. The inclusion map is not a P.T.S. map in this case for similar reasons to the previous two examples, for example $\tau_1\tau_1\tau_1$ is the empty translation whereas $\phi_*(\tau_1)\phi_*(\tau_1)\phi_*(\tau_1) =$

$t_1t_1t_1 = t_3$. However, note that here we do have

$$\phi_*(\tau_n\tau_m) = \phi_*(\tau_n)\phi_*(\tau_m)$$

when ϕ_* is extended to compositions of *pairs* of restricted partial translations. This indicates that a degree of caution should be applied when checking the homomorphism condition for potential P.T.S. maps.

7.3 Maps of Partial Translation Structures and the Infinite Dihedral Group

Using our definition of maps of partial translation structures, it is now possible to classify all such maps in certain cases, which should then tell us something about the possibilities for homomorphisms between the associated C^*-algebras. In this section we consider maps from the infinite dihedral group to the integers.

THEOREM 7.6 *The only partial translation structure maps between the dihedral group $D_\infty = \langle r, s \mid s^2 = 1,\ srs = r^{-1} \rangle$ and $\mathbb{Z}$, with respect to the canonical partial translation structures on both groups, are constant functions. Any such map takes every partial translation to a restriction of the identity translation t_0.*

Proof.

We will use the following presentation of D_∞:

$$D_\infty = \{r^n, r^n s \mid n \in \mathbb{Z}\} = \{r^n s^i \mid n \in \mathbb{Z}, i \in \{0, 1\}\}.$$

Since D_∞ is a group, it is equipped with a canonical partial translation structure with the set of partial translations $\{t_d \mid d \in D_\infty\}$, where

$$t_d = \{(r^n s^i, r^n s^i d) \mid n \in \mathbb{Z}, i \in \{0, 1\}\}.$$

In particular, the partial translation corresponding to right multiplication by s is given by

$$t_s = \{(r^n, r^n s), (r^n s, r^n) \mid n \in \mathbb{Z}\},$$

so that it translates an element of D_∞ in one of two ways. Similarly, we have

$$t_r = \{(r^n, r^{n+1}), (r^n s, r^{n-1} s) \mid n \in \mathbb{Z}\}.$$

Now suppose we have a map $\phi : D_\infty \to \mathbb{Z}$ which is a map of the canonical partial translation structures. By the first part of Definition 7.2, this implies that for all $d \in D_\infty$ there exists $m_d \in \mathbb{Z}$ such that $\phi(t_d) \subseteq t_{m_d}$, where t_{m_d} is the partial translation $t_{m_d} = \{(x, x+m_d) \mid x \in \mathbb{Z}\}$, i.e. for all $d \in D_\infty$ there exists $m_d \in \mathbb{Z}$ such that

$$\{(\phi(r^n s^i), \phi(r^n s^i d)) \mid n \in \mathbb{Z}, i \in \{0,1\}\} \subseteq \{(x, x+m_d) \mid x \in \mathbb{Z}\}.$$

In other words, for all $d \in D_\infty$ there exists $m_d \in \mathbb{Z}$ such that

$$\phi(r^n s^i d) - \phi(r^n s^i) = m_d, \qquad \text{for all } n \in \mathbb{Z},\ i \in \{0,1\}.$$

So let us examine what can be intuited from this if we substitute various values for d.

Case 1: $d = s$.
Then there exists $m_s \in \mathbb{Z}$ such that

$$\phi(r^n s^i s) - \phi(r^n s^i) = m_s, \qquad \text{for all } n \in \mathbb{Z},\ i \in \{0,1\}.$$

In particular,

$$\phi(r^n s) - \phi(r^n) = m_s = \phi(r^n) - \phi(r^n s), \qquad \text{for all } n \in \mathbb{Z},$$

(with the left-hand side corresponding to $i = 0$ and the right to $i = 1$). Hence

$$2\phi(r^n s) = 2\phi(r^n), \qquad \text{for all } n \in \mathbb{Z},$$

i.e.

$$\phi(r^n s) = \phi(r^n), \qquad \text{for all } n \in \mathbb{Z}. \tag{1}$$

So it seems that our map ϕ must be somehow dependent on the powers of the generator r.

Case 2: $d = r$.
Then there exists $m_r \in \mathbb{Z}$ such that

$$\phi(r^n s^i r) - \phi(r^n s^i) = m_r, \qquad \text{for all } n \in \mathbb{Z},\ i \in \{0,1\}.$$

In particular,

$$\phi(r^{n+1}) - \phi(r^n) = m_r = \phi(r^n sr) - \phi(r^n s), \qquad \text{for all } n \in \mathbb{Z},$$

(with the left-hand side corresponding to $i = 0$ and the right to $i = 1$). Combining this with equation (1), we obtain

$$\phi(r^{n+1}) = \phi(r^n sr) = \phi(r^{n-1}s) = \phi(r^{n-1}), \qquad \text{for all } n \in \mathbb{Z}.$$

Thus

$$\phi(r^{n-1}) = \phi(r^{n+1}) \text{ and } \phi(r^n) = \phi(r^n s), \qquad \text{for all } n \in \mathbb{Z}. \tag{2}$$

Case 3: $d = rs$.
We have

$$\phi(r^n s^i rs) - \phi(r^n s^i) = m_{rs}, \text{ for all } n \in \mathbb{Z},\ i \in \{0, 1\}, \text{ and some } m_{rs} \in \mathbb{Z}.$$

In particular,

$$\phi(r^2 s) - \phi(r) = m_{rs} = \phi(srs) - \phi(s),$$

and therefore

$$\phi(r^2 s) + \phi(s) = \phi(r^{-1}) + \phi(r). \tag{3}$$

However, by equation (2), we have $\phi(r^2 s) = \phi(r^2) = \phi(r^0) = \phi(r^0 s) = \phi(s)$ and $\phi(r^{-1}) = \phi(r)$, and so equation (3) becomes:

$$2\phi(s) = 2\phi(r).$$

Hence, altogether, we have:

$$\phi(r^{2n}) = \phi(r^{2n}s) = \phi(s) = \phi(r) = \phi(r^{2n+1}) = \phi(r^{2n+1}s), \text{ for all } n \in \mathbb{Z}.$$

So now ϕ can only take one value. Say $\phi(d) = x \in \mathbb{Z}$ for all $d \in D_\infty$, then we have $\phi(D_\infty) = x$ and

$$\phi(t_d) = \{(x, x)\} \subset \mathbb{Z} \times \mathbb{Z}, \qquad \text{for all } d \in D_\infty.$$

This singleton set corresponds to the restriction of the partial translation $t_0 = \{(n, n) | n \in \mathbb{Z}\}$ to $\phi(D_\infty) = x$. Hence we have $\phi_*(t_d) = t_0$ for all

$d \in D_\infty$, where ϕ_* is as defined in Definition 7.2, which extends to a homomorphism on compositions of partial translations as required for ϕ to be a P.T.S. map.

$\square$

We will later see that this theorem suggests that the only C^*-algebra homomorphism from $C_r^*(D_\infty)$ to $C_r^*((Z)$ is the map which takes every operator to the identity.

7.4 Maps From $\mathbb{Z} \times \mathbb{Z}_2$ To $\mathbb{Z}$

We now consider examples of partial translation structures arising from maps between $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}_2$. Let $\phi,\ \psi : \mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}$ be defined by

$$\begin{aligned} \phi : (x,i) &\mapsto \begin{cases} 2x & \text{if } i = 0 \\ 2x+1 & \text{if } i = 1, \end{cases} \\ \psi : (x,i) &\mapsto x. \end{aligned}$$

We have a canonical partial translation structure $\mathcal{T}$ on $\mathbb{Z} \times \mathbb{Z}_2$ containing partial translations of the form

$$t_{(n,0)} = \{((x+n,i),(x,i)) \mid (x,i) \in \mathbb{Z} \times \mathbb{Z}_2\}$$

and

$$t_{(n,1)} = \{((x+n, i+1 \bmod 2),(x,i)) \mid (x,i) \in \mathbb{Z} \times \mathbb{Z}_2\},$$

two for every $n \in \mathbb{Z}$. In other words, all partial translations are of the form

$$t_{(n,k)} = \{((x+n, i+k \bmod 2),(x,i)) \mid (x,i) \in \mathbb{Z} \times \mathbb{Z}_2\}.$$

If we apply the uniform bijection ϕ to such a partial translation, we obtain a partial translation which acts on $\mathbb{Z}$ in the following way:

$$\begin{aligned}(\phi(t_{(n,k)}))(m) &= \phi \circ t_{(n,k)} \circ \phi^{-1}(m) \\ &= \begin{cases} \phi(t_{(n,k)}(\frac{m}{2}, 0)) & \text{if } m \text{ even} \\ \phi(t_{(n,k)}(\frac{m-1}{2}, 1)) & \text{if } m \text{ odd} \end{cases} \\ &= \begin{cases} \phi(\frac{m}{2} + n, k) & \text{if } m \text{ even} \\ \phi(\frac{m-1}{2} + n, k+1 \bmod 2) & \text{if } m \text{ odd} \end{cases} \\ &= \begin{cases} m + 2n & \text{if } m \text{ even, } k = 0 \\ m + 2n + 1 & \text{if } m \text{ even, } k = 1 \\ m + 2n - 1 & \text{if } m \text{ odd, } k = 1 \\ m + 2n & \text{if } m \text{ odd, } k = 0, \end{cases}\end{aligned}$$

for all $m \in \mathbb{Z}$. So

$$\phi(t_{(n,o)}) : m \mapsto m + 2n$$

and

$$\phi(t_{(n,1)}) : m \mapsto \begin{cases} m + 2n + 1 & \text{if } m \text{ even} \\ m + 2n - 1 & \text{if } m \text{ odd,} \end{cases}$$

for every $m, n \in \mathbb{Z}$. Hence $\phi(\mathcal{T})$ is a new partial translation structure on $\mathbb{Z}$ which is not comparable with the canonical one, and therefore ϕ is not a map of partial translation structures (since no partial translation of the form $\phi(t_{(n,1)})$ is contained in a partial translation from the canonical partial translation structure on $\mathbb{Z}$).

Compositions in $\phi(\mathcal{T})$ (where everything is globally defined), and thus in $C^*(\phi(\mathcal{T}))$, work in the following way (we omit "ϕ" from now on for ease of notation):

$$\begin{aligned}
t_{(n,0)} \circ t_{(n',0)}(m) &= t_{(n,0)}(m+2n') \\
&= m+2n'+2n \\
&= m+2(n'+n) \\
&= t_{(n'+n,0)}(m); \\
t_{(n,1)} \circ t_{(n',1)}(m) &= \begin{cases} t_{(n,1)}(m+2n'+1) & \text{if } m \text{ even} \\ t_{(n,1)}(m+2n'-1) & \text{if } m \text{ odd} \end{cases} \\
&= \begin{cases} (m+2n'+1)+2n-1 & \text{if } m \text{ even} \\ (m+2n'-1)+2n+1 & \text{if } m \text{ odd} \end{cases} \\
&= \begin{cases} m+2n'+2n & \text{if } m \text{ even} \\ m+2n'+2n & \text{if } m \text{ odd} \end{cases} \\
&= m+2(n'+n) \\
&= t_{(n'+n,0)}(m); \\
t_{(n,0)} \circ t_{(n',1)}(m) &= \begin{cases} t_{(n,0)}(m+2n'+1) & \text{if } m \text{ even} \\ t_{(n,0)}(m+2n'-1) & \text{if } m \text{ odd} \end{cases} \\
&= \begin{cases} m+2n'+1+2n & \text{if } m \text{ even} \\ m+2n'-1+2n & \text{if } m \text{ odd} \end{cases} \\
&= \begin{cases} m+2(n'+n)+1 & \text{if } m \text{ even} \\ m+2(n'+n)-1 & \text{if } m \text{ odd} \end{cases} \\
&= t_{(n'+n,1)}(m); \\
t_{(n,1)} \circ t_{(n',0)}(m) &= t_{(n,1)}(m+2n') \\
&= \begin{cases} m+2n'+2n+1 & \text{if } m \text{ even} \\ m+2n'+2n-1 & \text{if } m \text{ odd} \end{cases} \\
&= \begin{cases} m+2(n'+n)+1 & \text{if } m \text{ even} \\ m+2(n'+n)-1 & \text{if } m \text{ odd} \end{cases} \\
&= t_{(n'+n,1)}(m).
\end{aligned}$$

Hence

$$t_{(n,k)} \circ t_{(n',k')} = t_{(n'+n,k'+k \bmod 2)},$$

for all $n, n' \in \mathbb{Z},\ k, k' \in \mathbb{Z}_2$, and therefore it can be seen that the two unitary operators arising from the partial translations $t_{(0,1)}$ and $t_{(1,0)}$ generate the algebra $C^*(\phi(\mathcal{T}))$.

We have

$$t_{(0,1)}^2 = t_{(0,0)},$$

which gives rise to the identity operator. Thus $t_{(0,1)}$ gives rise to a self-adjoint generator of $C^*(\phi(\mathcal{T}))$. Also,

$$t_{(0,1)}t_{(1,0)} = t_{(1,1)} = t_{(1,0)}t_{(0,1)},$$

and

$$t_{(1,0)}^n = t_{(n,0)} \neq t_{(0,0)}, \text{ for all } n \neq 0.$$

We can generalise the above scenario by considering instead any map of the form $\phi_j : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}_j$, for $j \geq 2$, defined by:

$$\phi_j : (x,i) \mapsto \begin{cases} jx & \text{if } i = 0 \\ jx+1 & \text{if } i = 1 \\ \vdots & \vdots \\ jx+j-1 & \text{if } i = j-1, \end{cases}$$

i.e.

$$\phi_j : (x,i) \mapsto jx+i.$$

We then have

$(\phi_j(t_{(n,k)}))(m)$

$$
\begin{aligned}
&= \phi_j \circ t_{(n,k)} \circ \phi_j^{-1}(m) \\
&= \begin{cases} \phi_j(t_{(n,k)}(\frac{m}{j},0)) & \text{if } m \equiv 0 \bmod j \\ \phi_j(t_{(n,k)}(\frac{m-1}{j},1)) & \text{if } m \equiv 1 \bmod j \\ \vdots & \vdots \\ \phi_j(t_{(n,k)}(\frac{m-(j-1)}{j},j-1)) & \text{if } m \equiv j-1 \bmod j \end{cases} \\
&= \begin{cases} \phi_j(\frac{m}{j}+n, k \bmod j) & \text{if } m \equiv 0 \bmod j \\ \phi_j(\frac{m-1}{j}+n, k+1 \bmod j) & \text{if } m \equiv 1 \bmod j \\ \vdots & \vdots \\ \phi_j(\frac{m-(j-1)}{j}+n, k+j-1 \bmod j) & \text{if } m \equiv j-1 \bmod j \end{cases} \\
&= \begin{cases} m+jn+k \bmod j & \text{if } m \equiv 0 \bmod j \\ m-1+jn+(k+1) \bmod j & \text{if } m \equiv 1 \bmod j \\ \vdots & \vdots \\ m-(j-1)+jn+(k+j-1) \bmod j & \text{if } m \equiv j-1 \bmod j, \end{cases}
\end{aligned}
$$

for all $m \in \mathbb{Z}$. Thus we obtain partial translations which act on $\mathbb{Z}$ in the following way:

$$\phi_j(t_{(n,k)}) : m \mapsto m + jn + (k+m) \bmod j - m \bmod j,$$

and hence a partial translation structure containing j such partial translations for every $n \in \mathbb{Z}$. In each case taking compositions works in the same way as for our $\mathbb{Z} \times \mathbb{Z}_2$ example, and thus we have the following.

PROPOSITION 7.7 *For any integer $j \geq 2$, the partial translation algebra $C^*(\phi_j(\mathcal{T}))$ is generated by two commuting unitaries a and b (in fact the operators arising from $\phi_j(t_{(0,1)})$ and $\phi_j(t_{(1,0)})$ respectively), such that $a^j = 1$ and b has infinite order.*

Let us now turn our attention to the map $\psi : \mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}$, $(x,i) \mapsto x$. This is a surjective map, and it is also clearly uniform if we endow $\mathbb{Z} \times \mathbb{Z}_2$ with the product metric. However, it is non-injective, so we can not apply Theorem 3.13 to ascertain that it maps partial translations to partial translations. In spite of this, let us consider ψ applied to a generic partial translation from $\mathcal{T}$:

$$\begin{aligned}\psi(t_{(n,k)}) &= \{(\psi(x+n, i+k \bmod 2), \psi(x,i)) \mid (x,i) \in \mathbb{Z} \times \mathbb{Z}_2\} \\ &= \{(x+n, x) \mid x \in \mathbb{Z}\}.\end{aligned}$$

What we obtain is the partial translation t_n from the canonical partial translation structure on $\mathbb{Z}$, so the map ψ does in fact map partial translations to partial translations! This is due to the fact that, while ψ is not injective in general, it does satisfy:

$$\psi(x,i) = \psi(y,j) \Leftrightarrow \psi(t_{(n,k)}(x,i)) = \psi(t_{(n,k)}(y,j)),$$

for all $(x,i),(y,j) \in Dom(t_{(n,k)}) = \mathbb{Z} \times \mathbb{Z}_2$, for all partial translations $t_{(n,k)} \in \mathcal{T}$. The partial translations themselves are mapped non-injectively, but as the map is surjective we do obtain the entire set of canonical partial translations of $\mathbb{Z}$ (and nothing else), and so we can still consider ψ to be a map of partial translation structures. Since ψ is a group homomorphism, we know by Theorem 7.5 that it satisfies the conditions of Definition 7.2, and in fact the above condition is also a consequence of this.

This map gives rise to a map of partial translation algebras $\psi : C^*(\mathcal{T}) \to C_r^*(\mathbb{Z}) \cong C(S^1)$. We know that $C^*(\mathcal{T})$ is generated by two commuting unitaries, say T_0 and T_1, arising from $t_{(0,1)}$ and $t_{(1,0)}$ respectively, with $T_0^2 = 1$ and T_1 of infinite order, while $C_r^*(\mathbb{Z})$ is the algebra generated by a single unitary T arising from the partial translation t_1. We have

$$\psi(t_{(0,1)}) = t_0 = Id \quad \text{and} \quad \psi(t_{(1,0)}) = t_1.$$

Hence we have the following.

PROPOSITION 7.8 *The map $\psi : C^*(\mathcal{T}) \to C_r^*(\mathbb{Z}) \cong C(S^1)$ is a map of partial translation algebras which takes T_0 to the identity operator and T_1 to T.*

7.5 Maps of Partial Translation Algebras

We use our definition of a P.T.S. map to reformulate and ultimately extend our results from chapter 6. Let us begin with the case already discussed, where the map in question is the inclusion of a subspace into a countable group; here we can restate the definition of a P.T.S. map using the following.

LEMMA 7.9 *Let $(G, \mathcal{T}_G)$ be a countable discrete group with canonical partial translation structure, let X be a subspace of G and let $\mathcal{T}_X$ denote the restriction of $\mathcal{T}_G$ to X. Let $\phi : X \to G$ denote the inclusion map. Then ϕ is a P.T.S. map if and only if any composition $\tau_{g_1} \ldots \tau_{g_n}$ of non-empty partial translations $\tau_{g_1}, \ldots, \tau_{g_n}$ in $\mathcal{T}_X$ is not equal to the empty translation.*

Proof.

Let $\emptyset$ denote the empty translation, since it could be viewed as an empty subset of $X \times X$; to say that a partial translation is equal to the empty translation means that it has empty domain. Let us begin by supposing that $\tau_{g_1} \ldots \tau_{g_n} \neq \emptyset$ for all non-empty partial translations $\tau_{g_1}, \ldots, \tau_{g_n}$ in $\mathcal{T}_X$. We wish to check that $\phi : X \hookrightarrow G$ is a P.T.S. map.

Recall that we have one partial translation $t_g = \{(x, xg) \mid x \in G\}$ in $\mathcal{T}_G$ for every $g \in G$, and that every partial translation in $\mathcal{T}_X$ is a restriction to X of some such t_g. We assume G to be a countable group, and so as the partial translations in $\mathcal{T}_G$ are in one-to-one correspondence with the elements of G they must form a countable set. It is also clear that they are pairwise disjoint; indeed, if (x, xg) is an element of both t_g and $t_{g'}$ then $xg = xg'$ and thus $g = g'$, so that $t_g = t_{g'}$. The same conditions must hold for the partial translations in $\mathcal{T}_X$ as these are all restrictions of partial translations in $\mathcal{T}_G$. It is clear from the fact that G is a group that the inverse of every partial translation in $\mathcal{T}_G$ is also a partial translation in $\mathcal{T}_G$, and certainly if any such partial translation restricts to X then so does its inverse. It has also been shown previously that both $\mathcal{T}_G$ and $\mathcal{T}_X$ are free partial translation structures with a single set of cotranslations, and so all of the initial conditions are satisfied.

We define $\phi : X \to G$ to be the inclusion map, and so ϕ acts as the identity on elements of X. Thus, for every partial translation τ_g in $\mathcal{T}_X$, we have

$$\phi(\tau_g) = \{(\phi(x), \phi(xg)) \mid x, xg \in X\} = \{(x, xg) \mid x, xg \in X\} = \tau_g.$$

If τ_g is non-empty then it is a subset of the partial translation $t_g \in \mathcal{T}_G$, and if τ_g is empty then we consider it to only be a subset of the empty translation on G. Hence the map ϕ_* is well-defined and condition 1 of Definition 7.2 is satisfied. Similarly condition 2 will hold since partial cotranslations for $\mathcal{T}_X$ are formed by restricting partial cotranslations for $\mathcal{T}_G$.

Now, for any $g_1, \ldots, g_n \in G$, we have

$$\tau_{g_1} \ldots \tau_{g_n} = \{(x, xg_1 \ldots g_n) \mid x, xg_1, \ldots, xg_1 \ldots g_n \in X\},$$

so if this is not the empty translation then it must be a subset of the partial translation

$$t_{g_1} \ldots t_{g_n} = \{(x, xg_1 \ldots g_n) \mid x \in G\}.$$

Hence $\phi(\tau_{g_1} \ldots \tau_{g_n}) = \tau_{g_1} \ldots \tau_{g_n} \subseteq t_{g_1} \ldots t_{g_n}$, where $\phi(\tau_{g_i}) \subseteq t_{g_i}$ for all $1 \leq i \leq n$, whenever $\tau_{g_1}, \ldots, \tau_{g_n}$ are non-empty partial translations in $\mathcal{T}_X$. Therefore ϕ_* may clearly be extended to a homomorphism between $\mathcal{T}(X)$ and $\mathcal{T}(Y)$ and so ϕ is indeed a P.T.S. map.

On the other hand, if there exist non-empty partial translations $\tau_{g_1}, \ldots, \tau_{g_n}$ in $\mathcal{T}_X$ such that $\tau_{g_1} \ldots \tau_{g_n} = \emptyset$, then $\phi_*(\tau_{g_1}) \ldots \phi_*(\tau_{g_n}) = t_{g_1} \ldots t_{g_n}$ whilst $\phi_*(\tau_{g_1} \ldots \tau_{g_n}) = \phi_*(\emptyset) = \emptyset$, so ϕ_* does not extend to a homomorphism and thus ϕ fails to be a P.T.S. map.

$\square$

The above lemma allows us to prove the following strengthening of Theorem 6.6.

THEOREM 7.10 *Let $(G, \mathcal{T}_G)$ be a countable discrete group with canonical partial translation structure, let X be a subspace of G and let $\mathcal{T}_X$ be the restriction of $\mathcal{T}_G$ to X. For every $g \in G$, let T_g denote the operator on $l^2(G)$ arising from the partial translation $t_g = \{(x, xg) \mid x \in G\}$, and let $\tilde{T}_g$ denote the operator on $l^2(X)$ arising from the restriction of t_g to X. Set $H = \{g \in G \mid \tilde{T}_g \neq 0\}$. Then the canonical map $\varphi : C^*(\mathcal{T}_X) \to C^*_r(H) \subseteq C^*_r(G)$ is a C^*-algebra homomorphism if and only if the inclusion map $\phi : X \hookrightarrow G$ is a P.T.S. map.*

Proof.

To show that $\varphi : C^*(\mathcal{T}_X) \to C^*_r(H) \subseteq C^*_r(G)$ is a C^*-algebra homomorphism it is enough to show that $\tilde{T}_g$ is not a zero divisor in the monoid $\{\tilde{T}_{h_1} \ldots \tilde{T}_{h_l} \mid h_i \in H\}$ for all $g \in H$, by Theorem 6.6. If we assume firstly that $\phi : X \hookrightarrow G$ is a P.T.S. map, then by the above lemma we know that any composition $\tau_{g_1} \ldots \tau_{g_n}$ of non-empty partial translations $\tau_{g_1}, \ldots, \tau_{g_n}$ in

$\mathcal{T}_X$ is non-empty. This means precisely that any composition $\tilde{T}_{g_1}\dots\tilde{T}_{g_n}$ of non-zero generators $\tilde{T}_{g_1},\dots,\tilde{T}_{g_n}\in C^*(\mathcal{T}_X)$, i.e. operators for which $g_1,\dots,g_n\in H$, is non-zero. Hence no non-zero $\tilde{T}_g$ can be a zero divisor in the monoid $\{\tilde{T}_{h_1}\dots\tilde{T}_{h_l}\mid h_i\in H\}$, as required.

Conversely, if we assume that the canonical map $\varphi: C^*(\mathcal{T}_X)\to C^*_r(H)$ is a C^*-algebra homomorphism, then again by Lemma 7.9 it is only necessary to check that any composition $\tau_{g_1}\dots\tau_{g_n}$ of non-empty partial translations $\tau_{g_1},\dots,\tau_{g_n}$ in $\mathcal{T}_X$ is non-empty, to show that $\phi: X\hookrightarrow G$ is a P.T.S. map. But if there is an empty composition $\tau_{g_1}\dots\tau_{g_n}$ made up of non-empty partial translations, then this means there exist non-zero generators $\tilde{T}_{g_1},\dots,\tilde{T}_{g_n}$ of $C^*(\mathcal{T}_X)$ for which $\tilde{T}_{g_1}\dots\tilde{T}_{g_n}=0$. So then $\varphi(\tilde{T}_{g_1}\dots\tilde{T}_{g_n})=\varphi(0)=0$, whilst $\varphi(\tilde{T}_{g_1})\dots\varphi(\tilde{T}_{g_n})=T_{g_1}\dots T_{g_n}\neq 0$, and so φ cannot be a C^*-algebra homomorphism, which gives us a contradiction.

$\square$

With the P.T.S. map definition, Theorem 7.10 can now be considered as a special case of a far more general theorem (stated below). However, it is still helpful to have considered the inclusion of a subspace into a group as a separate case, since the zero divisors condition given in the original statement of the theorem is likely to provide a useful method for identifying P.T.S. maps in practice.

THEOREM 7.11 *Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be two metric spaces equipped with mappable partial translation structures, and suppose that $\phi:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is a P.T.S. map. Let $\mathcal{T}'_Y=\{t'\in\mathcal{T}_Y\mid t'=\phi_*(t)$ for some partial translation $t\in\mathcal{T}_X\}$, and suppose that there is a uniform bound on the number of partial translations $t\in\mathcal{T}_X$ for which $t'=\phi_*(t)$ for any given $t'\in\mathcal{T}'_Y$. Then the map $\hat{\phi}: C^*(\mathcal{T}_X)\to C^*(\mathcal{T}'_Y)\subseteq C^*(\mathcal{T}_Y)$, defined on the generating set of $C^*(\mathcal{T}_X)$ by*

$$\hat{\phi}(T_t)=T_{\phi_*(t)},$$

and extended by linearity, is a C^*-algebra homomorphism.

Proof.

Here we denote by T_t the partial isometry associated to the partial translation t. By definition, $\hat{\phi}$ is a linear map, so we only need check that

$$\hat{\phi}(T_t^*)=(\hat{\phi}(T_t))^* \text{ and } \hat{\phi}(T_{t_1}T_{t_2})=\hat{\phi}(T_{t_1})\hat{\phi}(T_{t_2})$$

hold for all partial translations t, t_1 and t_2 in $\mathcal{T}_X$. The uniform bound on the number of partial translations of X sent to any one partial translation of Y guarantees passage to completion; this works in the same way as the extension of a group homomorphism to a homomorphism between reduced group C^*-algebras.

So let us verify the two homomorphism conditions. From the definition of partial translation algebras, we have $T_t^* = T_{t^{-1}}$ for any partial isometry T_t arising from a partial translation t. Also, note that for any map $\phi : X \to Y$ we have

$$\phi(t^{-1}) = \{(\phi(x), \phi(y)) \mid (x, y) \in t^{-1}\} = \{(\phi(x), \phi(y)) \mid (y, x) \in t\} = (\phi(t))^{-1},$$

for all partial translations t in $\mathcal{T}_X$. Hence if $\phi(t) \subseteq \phi_*(t)$, i.e. $(\phi(t))^{-1} \subseteq (\phi_*(t))^{-1}$, then $\phi(t^{-1}) \subseteq (\phi_*(t))^{-1}$, and so $\phi_*(t^{-1}) = (\phi_*(t))^{-1}$, by disjointness of the partial translations in $\mathcal{T}_Y$. Thus we have

$$\hat{\phi}(T_t^*) = \hat{\phi}(T_{t^{-1}}) = T_{\phi_*(t^{-1})} = T_{(\phi_*(t))^{-1}} = T_{\phi_*(t)}^* = (\hat{\phi}(T_t))^*,$$

as required.

For the second condition, note that $T_{t_1 t_2} = T_{t_1} T_{t_2}$ for all partial isometries T_{t_1}, T_{t_2} associated to partial translations t_1, t_2. Hence we have

$$\begin{aligned}
\hat{\phi}(T_{t_1} T_{t_2}) &= \hat{\phi}(T_{t_1 t_2}) \\
&= T_{\phi_*(t_1 t_2)} \\
&= T_{\phi_*(t_1)\phi_*(t_2)} \text{ (since } \phi \text{ is a P.T.S. map)} \\
&= T_{\phi_*(t_1)} T_{\phi_*(t_2)} \\
&= \hat{\phi}(T_{t_1}) \hat{\phi}(T_{t_2}).
\end{aligned}$$

□

8 A C^*-Algebra Extension for Subspaces of Groups

Now that we have established when such a map arises, we can use the canonical C^*-algebra homomorphism from the partial translation algebra associated with the restriction of a group partial translation structure to a subspace to the reduced C^*-algebra of the group to construct an algebra extension under certain conditions. This short exact sequence of C^*-algebras was introduced in our paper [9], as joint work with Brodzki, Niblo and Wright. The sequence gives rise to a six-term exact sequence in K-theory, which can be used to compute the K-theory of some metric spaces, as we will see in the next chapter. We recall the definition of K-theory in the second part of this chapter.

8.1 The Extension

In this section we use the C^*-algebra homomorphism we have identified arising from inclusion into a group to construct a short exact sequence involving partial translation algebras associated with subspaces of groups.

The following proposition provides some ways in which to restate Theorem 6.6, the main theorem of chapter 6.

PROPOSITION 8.1 *Let G be a countable discrete group, let X be a subspace of G and let $\mathcal{T}_X$ be the restriction of the canonical partial translation structure on G to X. For every $g \in G$, let $\tilde{T}_g$ denote the operator on $l^2(X)$ arising from the restriction of the canonical partial translation t_g to X. Then the following are equivalent:*

1. *The subset $H = \{g \in G \mid \tilde{T}_g \neq 0\}$ is equal to G, i.e. every canonical partial translation restricts to X, and each restricted partial isometry $\tilde{T}_g$ is not a zero divisor in the monoid $\{\tilde{T}_{g_1} \ldots \tilde{T}_{g_l} \mid g_i \in G\}$;*
2. *X^c is not coarsely dense in G (with respect to the canonical coarse structure);*
3. *For all $r > 0$ there exists $x \in X$ such that $B_G(x, r) \subset X$.*

Proof.

(1)$\Leftrightarrow$(3): Condition (1) is equivalent to saying that $\tau_{g_1} \ldots \tau_{g_n} \neq \emptyset$, for all $g_1, \ldots, g_n \in G$, where τ_{g_i} denotes the restriction of the canonical partial translation t_{g_i} on G to X. In other words, it means that for all $g_1, \ldots, g_n \in G$ there exists some $x \in X$ such that $xg_1, xg_1g_2, \ldots, xg_1 \ldots g_n \in X$. It is clear that this will hold in the presence of condition (3); hence (3)$\Rightarrow$(1).

Now assume that (1) holds. For arbitrary $r > 0$, consider all elements of G of word length $\leq r$ with respect to a bounded geometry left invariant metric, recalling that we are always able to associate such a metric to G when

G is countable. These group elements are precisely those contained within the ball of radius r about the identity element in G, and hence by bounded geometry there are finitely many of them, so we may enumerate them as $g_1, \dots, g_N$. We may now form the word $g_1 g_1^{-1} g_2 g_2^{-1} \dots g_N g_N^{-1}$ in G, and by condition (1) there exists $x \in X$ such that $xg_1, xg_1g_1^{-1}, xg_1g_1^{-1}g_2, \dots,$ $xg_1g_1^{-1} \dots g_N g_N^{-1} \in X$. But this means that $xg_1, xg_2, \dots, xg_N \in X$, i.e. that x composed with any element of G of length $\leq r$ is an element of X. Therefore there exists $x \in X$ such that $B_G(x, r) \subset X$, as required.

(2)$\Leftrightarrow$(3): Recall that a subset Z of a metric space Y is *coarsely dense* if there exists $R > 0$ such that every point of Y lies within distance R of a point in Z [29]. So if we assume that (3) holds and suppose for contradiction that (2) does not, then we have that there exists some $R > 0$ such that for all $g \in G$ there exists $z \in X^c$ such that $z \in B_G(g, R)$. But by (3) we may choose $g \in X \subseteq G$ such that $B_G(g, R) \subset X$, which gives us a contradiction. Thus (3)$\Rightarrow$(2).

Now suppose that (2) does hold, so that for all $r > 0$ there exists $g \in G$ such that $z \notin B_G(g, r)$ for all $z \in X^c$. Fix an arbitrary $r > 0$ and consider the corresponding such $g \in G$, which must clearly be an element of X. If $z \notin B_G(g, r)$ for all $z \in X^c$, then $B_G(g, r) \subset X$. Hence (3) holds.

□

Note that the requirement that $H = G$ in condition (1) of the above excludes cases such as $X = 2\mathbb{Z} \subset \mathbb{Z}$, where we do have a homomorphism of partial translation algebras but it is clear that condition (3) does not hold.

Using this proposition we obtain a C^*-algebra extension for subspaces of groups involving partial translation algebras. Let us begin by describing the terms involved.

Let G be a countable group, let X be a subset of G and let $\mathcal{T}_X$ denote the restriction of the canonical partial translation structure for G to X. Let $P : l^2(G) \to l^2(G \backslash X)$ be the projection onto $l^2(G \backslash X)$, let $\mathcal{A} = C^*(C_r^*(G), P) \subset B(l^2(G))$ and let $I(P)$ denote the ideal in $\mathcal{A}$ generated by P, so that

$$I(P) = \overline{span}\{T_{g_0} P T_{g_1} P \dots T_{g_{k-1}} P T_{g_k} \mid k \geq 1,\ g_i \in G \text{ for all } 0 \leq i \leq k\}.$$

Before we state the main theorem, let us firstly note the following, which will be made use of later when we explore a specific example.

PROPOSITION 8.2

$$C^*(\mathcal{T}_X) \cap I(P) = (1-P)I(P)(1-P).$$

Proof.

Suppose $T \in C^*(\mathcal{T}_X) \cap I(P)$. Then T is an operator on $l^2(X)$ and so $T = (1-P)T(1-P)$. However, we also have $T \in I(P)$. Hence $T \in (1-P)I(P)(1-P)$.

Conversely, suppose that $T \in (1-P)I(P)(1-P)$. Then $T = (1-P)T'(1-P)$ for some $T' \in I(P)$. Since $(1-P)$ is the projection onto $l^2(X)$ it is then clear by the above definition of $I(P)$ that $T \in C^*(\mathcal{T}_X)$, as well as certainly being an element of the ideal generated by P. Hence the equality holds.

□

THEOREM 8.3 *The sequence*

$$0 \longrightarrow C^*(\mathcal{T}_X) \cap I(P) \xrightarrow{\iota} C^*(\mathcal{T}_X) \xrightarrow{\tilde{T}_g \mapsto T_g} C_r^*(G) \longrightarrow 0$$

is exact if and only if X^c is not coarsely dense in G.

Proof.

Firstly, suppose that we have the above with X^c not coarsely dense in G. The map ι in our sequence denotes an inclusion of an ideal into an algebra, so this is automatically injective. By Lemma 8.1, the fact that X^c is not coarsely dense means that $\{\tilde{T}_{g_1} \dots \tilde{T}_{g_l} \mid g_i \in G\}$ contains no zero divisors and that every partial translation restricts, which together ensure that the map from $C^*(\mathcal{T}_X)$ to $C_r^*(G)$ is surjective. Thus it remains to show that the image of the map ι, i.e. $C^*(\mathcal{T}_X) \cap I(P)$, is equal to the kernel of the map from $C^*(\mathcal{T}_X)$ to $C_r^*(G)$, which is a homomorphism by Theorem 6.6 together with Lemma 8.1. Therefore, by the first isomorphism theorem, we wish to prove that

$$C^*(\mathcal{T}_X)/C^*(\mathcal{T}_X) \cap I(P) \cong C_r^*(G).$$

Firstly, note that

$$\mathcal{A} = C_r^*(G) + I(P). \tag{4}$$

Indeed, since $\mathcal{A}$ is the C^*-algebra generated by $C_r^*(G)$ and P, it is clear that $C_r^*(G) + I(P) \subseteq \mathcal{A}$. On the other hand, $\mathcal{A}$ is by definition the smallest

C^*-algebra containing $C_r^*(G)$ and P, and $C_r^*(G) + I(P)$ also contains both $C_r^*(G)$ and P, so $\mathcal{A} \subseteq C_r^*(G) + I(P)$.

We claim secondly that

$$\mathcal{A} = C^*(\mathcal{T}_X) + I(P). \tag{5}$$

It is clear that $C^*(\mathcal{T}_X) + I(P) \subseteq \mathcal{A}$, as $I(P)$ is an ideal in $\mathcal{A}$ and $\mathcal{T}_X$ is generated by elements of the form $(1-P)T_g(1-P)$, where $T_g \in C_r^*(G)$. To show that $\mathcal{A} \subseteq C^*(\mathcal{T}_X)+I(P)$ it is enough to show that $T_g \in C^*(\mathcal{T}_X)+I(P)$ for all $g \in G$, by (4). But for any $g \in G$ we have

$$T_g = (1-P)T_g(1-P) + (1-P)T_gP + PT_g,$$

where $(1-P)T_g(1-P) \in C^*(\mathcal{T}_X)$ and $(1-P)T_gP + PT_g \in I(P)$ (note that $I(P)$ is a two-sided ideal), so this holds.

Combining (4) and (5), we obtain

$$C_r^*(G) + I(P) = C^*(\mathcal{T}_X) + I(P). \tag{6}$$

In addition to this, by the second isomorphism theorem, we have

$$(C_r^*(G) + I(P))/I(P) \cong C_r^*(G)/(C_r^*(G) \cap I(P)) \tag{7}$$

and

$$(C^*(\mathcal{T}_X) + I(P))/I(P) \cong C^*(\mathcal{T}_X)/(C^*(\mathcal{T}_X) \cap I(P)). \tag{8}$$

Now if we assume that X^c is not coarsely dense in G then we see from the definition of $I(P)$ that no operator in this ideal has support exhausting G and so $C_r^*(G) \cap I(P) = \{0\}$. So (7) can now be read as

$$(C_r^*(G) + I(P))/I(P) \cong C_r^*(G). \tag{9}$$

Combining this with (6) and (8), we obtain

$$C^*(\mathcal{T}_X)/C^*(\mathcal{T}_X) \cap I(P) \cong C_r^*(G),$$

as required.

Conversely, suppose that

$$0 \longrightarrow C^*(\mathcal{T}_X) \cap I(P) \stackrel{\iota}{\longrightarrow} C^*(\mathcal{T}_X) \stackrel{\tilde{T}_g \mapsto T_g}{\longrightarrow} C^*_r(G) \longrightarrow 0$$

is a short exact sequence. This means that the canonical map $\tilde{T}_g \mapsto T_g$ is a C^*-algebra homomorphism, which by Lemma 7.9 and Theorem 7.10 implies that any composition $\tau_{g_1} \ldots \tau_{g_n}$ of non-empty partial translations $\tau_{g_1}, \ldots, \tau_{g_n}$ in $\mathcal{T}_X$ is not equal to the empty translation. This is clearly equivalent to saying that no restricted partial isometry $\tilde{T}_g$ is a zero divisor in the monoid $\{\tilde{T}_{g_1} \ldots \tilde{T}_{g_l} \mid g_i \in G\}$. Exactness of the sequence also implies that the canonical map is surjective, which means that every partial translation in the canonical partial translation structure for G restricts non-trivially to the space X. Thus X^c is not coarsely dense in G, by Proposition 8.1.

□

Note in particular that in this sequence we employ a map of C^*-algebras arising from the inclusion map from a subspace into a group, which is not in general a group homomorphism!

We see from the following proposition that Theorem 8.3 tells us that we do not obtain the given short exact sequence of C^*-algebras in the case where our subset is in fact a subgroup of a countable group.

PROPOSITION 8.4 *Let G be a bounded geometry discrete group, let $H \neq G$ be a subgroup of G, and let X be the complement of H in G. Then X is coarsely dense in G.*

Proof.

Suppose for contradiction that $X = H^c$ is not coarsely dense in G, so that for every $r > 0$ there exists some $g \in G$ such that $g \notin B_r(x)$ for all $x \in X$. In particular, any such $g \in G$ must be an element of H, so we have that for every $r > 0$ there exists some $h \in H$ such that $B_r(h) \subset H$. But this means that for any $r > 0$ we can find some $h \in H$ such that $hg \in H$ for all $g \in G$ such that $l(g) \leq r$; in other words, $h^{-1}hg = g \in H$ for all $g \in G$ such that $l(g) \leq r$. From this we see that all group elements of finite length are contained within the subgroup H, and hence $H = G$, which is a contradiction to the assumption $H \neq G$.

□

On the other hand, it is clear from Theorem 8.3 that we do obtain such a C^*-algebra extension in the case where we form a subspace X by removing finitely many edges from the Cayley graph of the free group on two generators. The following theorem provides a little more information about the structure of such a space.

THEOREM 8.5 *Let $\mathbb{F}_2$ denote the free group on two generators, a and b, and let X be a space formed by removing finitely many edges from the Cayley graph of $\mathbb{F}_2$. Then every unbounded connected component of X is of the form*

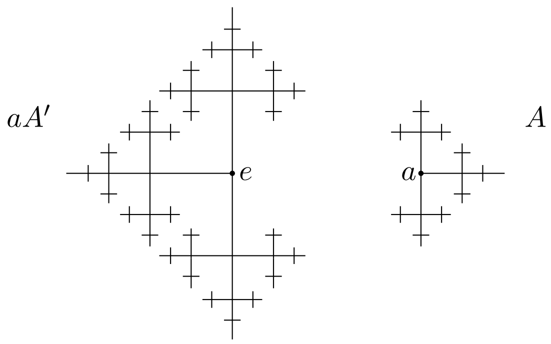
$$T = F \,\dot{\cup}\, T_1 \,\dot{\cup}\, T_2 \,\dot{\cup} \cdots \dot{\cup}\, T_m,$$

where F is a finite set of edges and each T_i is a left translation of one of A, B, A' or B', where these denote the sets of words in $\mathbb{F}_2$ beginning with a, b, a^{-1} and b^{-1} respectively.

Proof.

We proceed by induction on n, the number of edges removed from the Cayley graph of $\mathbb{F}_2$ to construct the space X.

Firstly, suppose that $n = 1$, so that X is formed by removing a single edge from the Cayley graph of $\mathbb{F}_2$. Since the graph is vertex transitive, we may assume without loss of generality that we remove one of the edges incident to the vertex representing the identity element e. However, it is clear that in this case the proposition holds. For example, if we remove the edge connecting the identity vertex to the vertex representing the element a then we obtain the components A and aA', a left translation of A'.



Similarly, if we removed the edge connecting e with b we would obtain the components B and bB', if we removed the edge connecting e with a^{-1} we would obtain the components A' and $a^{-1}A$, and if we removed the edge connecting e with b^{-1} we would obtain the components B' and $b^{-1}B$. Since we may translate on the left to move the vertex representing e to any other vertex in the Cayley graph for $\mathbb{F}_2$, we hence see that the proposition holds for $n = 1$.

Now suppose for our induction hypothesis that we can remove any k edges from the Cayley graph of $\mathbb{F}_2$ and that every unbounded component of the resulting space X will be of the form

$$T = F \dot\cup T_1 \dot\cup T_2 \dot\cup \cdots \dot\cup T_m$$

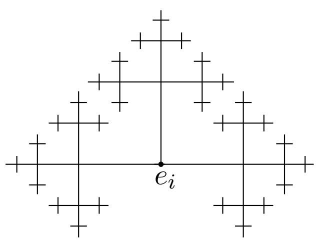
as required. We wish to prove that the result still holds if we remove one more edge from the space X.

If we remove an edge from some bounded component of X then it is clear that this is the case, so suppose that we remove an edge from some unbounded component $T = F \dot\cup T_1 \dot\cup T_2 \dot\cup \cdots \dot\cup T_m$. Again, if we remove one of the finite number of edges connecting the components T_i, that is an edge in F, then it is clear that the result will hold. So let us remove an edge from some T_i, where $1 \leq i \leq m$. We wish to show that the resulting (unbounded) components are each of the form

$$F_i \dot\cup T_{i_1} \dot\cup T_{i_2} \dot\cup \cdots \dot\cup T_{i_l},$$

where F_i is a finite set of edges and each T_{i_j} is a left translation of either A, B, A' or B'.

The component T_i is itself a left translation of either A, B, A' or B', which are all isomorphic, so let us assume without loss of generality that T_i is a left translation of B, so that it may be represented by the following diagram:



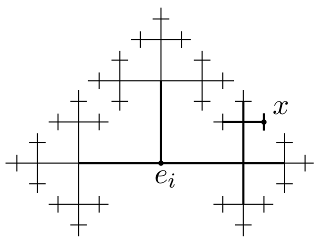
where e_i is some element of $\mathbb{F}_2$, and therefore a left translation of e. If we remove one of the edges incident to e_i we obtain a left translation of either A, B or A', plus left translations of the remaining two connected by two edges, and hence both components are of the required form. So suppose finally that we remove some other edge from T_i. Doing this splits the tree T_i into two unbounded components, one which contains the vertex e_i and one which does not. It is clear by observation that the component not containing e_i is a left translation of either A, B, A' or B', so it only remains to check that the other component is also of the required form. If we let x denote the vertex incident to the deleted edge which is closest to e_i, then taking

$$F_i = \{\text{all edges incident to a vertex lying on the path } [e_i, x]\}$$

accomplishes this.

To illustrate how this works we present the following example:

T_i



Here the thicker edges represent the set F_i.

□

It is evident from the above proof that any unbounded connected component T of such a space X contains at least one component T_i which is a left translation of either A, A', B or B', and is connected to the rest of the space only via the vertex representing the corresponding left translation of the identity element. Thus we see that for any $r > 0$ we could follow a path of length r away from this vertex, so that we stay within T_i, to reach a vertex x for which $B_{\mathbb{F}_2}(x, r) \subset T$. Therefore T^c is not coarsely dense in $\mathbb{F}_2$, by Lemma 8.1. Hence we see that given any space X formed by removing finitely many edges from the Cayley graph of $\mathbb{F}_2$ we may also apply Theorem 8.3 to any unbounded connected component T of X to construct a related short exact sequence of C^*-algebras.

8.2 The Related Sequence in K-theory

The short exact sequence described in Theorem 8.3 gives rise to the following six-term exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(C^*(\mathcal{T}_X)\cap I(P)) & \rightarrow & K_0(C^*(\mathcal{T}_X)) & \rightarrow & K_0(C_r^*(G)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(G)) & \leftarrow & K_1(C^*(\mathcal{T}_X)) & \leftarrow & K_1(C^*(\mathcal{T}_X)\cap I(P)). \end{array}$$

As we will use this sequence in the next chapter to compute the K-theory of a particular partial translation algebra, we recall here the basic definitions required to understand K-theory for unital C^*-algebras, as they appear in [6].

Let $\mathcal{A}$ be a unital C^*-algebra. A *projection* in $\mathcal{A}$ is an element $p \in \mathcal{A}$ such that $p = p^2 = p^*$, so a projection is a self-adjoint idempotent. If p and q are projections in $\mathcal{A}$, then write $p \sim q$ if they are Murray-von Neumann equivalent, that is if $p = v^*v$ and $q = vv^*$ for some partial isometry v in $\mathcal{A}$.

Denote by $P(\mathcal{A})$ the semigroup of projections in $M_\infty(\mathcal{A}) = \cup_{n\geq 1} M_n(\mathcal{A})$, the union of all finite-dimensional matrix algebras over $\mathcal{A}$, with respect to the direct sum, where

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{n+m}(\mathcal{A}),$$

for all $p \in M_n(\mathcal{A})$ and $q \in M_m(\mathcal{A})$. A projection $p \in M_n(\mathcal{A})$ is said to be *equivalent* to a projection $q \in M_m(\mathcal{A})$, with $n \leq m$, if and only if we have $p \oplus 0_{m-n} \sim q$ in $M_m(\mathcal{A})$. We say that two projections p and q are *stably equivalent* if and only if there exists a projection $r \in P(A)$ such that $p \oplus r$ is equivalent to $q \oplus r$.

Denote by $[P(\mathcal{A})]$ the semigroup of all stable equivalence classes of projections in $P(\mathcal{A})$, with addition induced from $P(\mathcal{A})$. We say that two *pairs* of elements of $[P(\mathcal{A})]$, $([p_1], [p_2])$ and $([q_1], [q_2])$, are equivalent if and only if $[p_1] \oplus [q_2] = [p_2] \oplus [q_1]$.

DEFINITION 8.6 $[K_0]$

We denote by $K_0(\mathcal{A})$ the abelian group consisting of all equivalence classes of pairs $([p_1], [p_2])$, with the componentwise addition [21].

Proposition 5.8 of [6] states that if $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a unital *-homomorphism of C^*-algebras, then there is an induced map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ which sends the class $[p] \in [P(\mathcal{A})]$ to $[\phi(p)] \in [P(B)]$; thus the homomorphism we construct from $C^*(\mathcal{T}_X)$ to $C_r^*(G)$ gives rise to a map of K-theory for these algebras.

DEFINITION 8.7 [Cones]

The *cone* of a C^*-algebra $\mathcal{A}$ is the algebra $C\mathcal{A}$ of continuous functions from $[0,1]$ to $\mathcal{A}$ which vanish at 0 [6].

DEFINITION 8.8 [Suspensions]

The *suspension* $S\mathcal{A}$ of a C^*-algebra $\mathcal{A}$ is the subalgebra of $C\mathcal{A}$ consisting of all functions which vanish at 1. So elements of $S\mathcal{A}$ are continuous functions $f : [0,1] \to \mathcal{A}$ such that $f(0) = f(1) = 0$ [21].

DEFINITION 8.9 [Higher K-theory]

We let $K_1(\mathcal{A}) = K_0(S\mathcal{A}) = K_0(C_0(\mathbb{R}) \otimes \mathcal{A})$. In general, we write

$$K_p(\mathcal{A}) = K_0(S^p\mathcal{A}) = K_0(C_0(\mathbb{R}^p) \otimes \mathcal{A}) \text{ [21]},$$

where $S^2\mathcal{A}$ denotes the suspension of $S\mathcal{A}$, and so on.

9 An Interesting Example of a Partial Translation Algebra Associated With a Subset of the Integers

The following example was originally formulated as a potential counter-example to one of the theorems discussed in chapter 6; fortunately it failed to fulfil this role, but it is none-the-less a rather intriguing case, and one for which we are able to make use of our C^*-algebra extension in the computation of its K-theory.

The Example

Let X be the subspace of $\mathbb{Z}$ defined by:

$$X = \bigcup_{i \in \mathbb{N} \setminus \{0\}} X_i,$$

where the components of X are given by

$$X_i = \{(i^2 - i), \ldots, (i^2 - 1)\}.$$

For example,

$$\begin{aligned} X_1 &= \{0\}, \\ X_2 &= \{2, 3\}, \\ X_3 &= \{6, 7, 8\}, \end{aligned}$$

and so on. Then

$$X = \{0, 2, 3, 6, 7, 8, 12, 13, 14, 15, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 35, ..\},$$

so that the subspace X is made up of increasingly large sets of consecutive numbers separated by increasingly large gaps.

We restrict the canonical partial translation structure on $\mathbb{Z}$ to X, following the same notational convention as laid out in section 5.1. For each $n \in \mathbb{Z}$ we denote the restriction of t_n to X by τ_n, so that

$$\tau_n = \{(x + n, x) \mid x, x + n \in X\}.$$

We consider the partial translation structure $\mathcal{T}_X$ on X with set of partial translations $\{\tau_n \mid n \in \mathbb{Z}\}$. Note that, as a subset of $X \times X$, every τ_n is non-empty, since we can always find two points in X which are distance n apart, for example the smallest and largest elements of the component X_{n+1}. Thus every canonical partial translation for $\mathbb{Z}$ restricts to a partial translation for X.

Note also that for any given $k, n \in \mathbb{N}$, τ_n^k will definitely be defined on $(kn+1)^2 - (kn+1) = kn(kn+1)$, the smallest element of X_{kn+1}, whereas if n is negative, τ_n^k will be defined on $(kn+1)^2 - 1 = kn(kn+2)$, the largest element of X_{kn+1}. The translation $\tau_0^k = \tau_0$ is defined on the whole of X for any $k \in \mathbb{N}$. Hence, if for any $n \in \mathbb{Z}$ we let $\tilde{T}_n$ denote the operator arising from τ_n, then we have that for every $k \in \mathbb{N}$ there exists $\delta_x \in l^2(X)$ such that $\tilde{T}_n^k(\delta_x) \neq 0$. In particular, for n positive we may take $x = kn(kn+1)$ (with $\tilde{T}_n^k(\delta_x) = \delta_{kn(kn+2)}$) and for n negative we may take $x = kn(kn+2)$ (with $\tilde{T}_n^k(\delta_x) = \delta_{kn(kn+1)}$). Thus every operator arising from a restricted partial translation on X is non-nilpotent, and so the conditions of Theorem 6.3 are satisfied for X. Therefore there exists a map of partial translation algebras from $C^*(\mathcal{T}_X)$ to $C^*_r(\mathbb{Z})$, defined in the obvious way, by sending each $\tilde{T}_n$ to the right shift by n, that is the operator arising from the canonical partial translation t_n.

However, what makes this a particularly interesting example is that despite the fact that every partial translation for $\mathbb{Z}$ restricts to our subspace and there are no nilpotent operators, there are also no unbounded orbits in X.

DEFINITION 9.1 For every $n, m \in \mathbb{Z}$, let

$$\Theta(n,m) = \tilde{T}_{m+n} - \tilde{T}_n\tilde{T}_m,$$

so that

$$\Theta(n,m) : \delta_x \mapsto \begin{cases} \delta_{x+m+n} & \text{if } x+m+n \in X, x+m \notin X \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$. So we could view $\Theta(n,m)$ as the operator arising from a partial translation of the form

$$\theta(n,m) = \{(x+m+n, x) \mid x, x+m+n \in X, x+m \notin X\},$$

which as a subset of $X \times X$ is equal to $\tau_{m+n} \backslash \tau_n\tau_m$, where

$$\tau_n\tau_m = \{(x+m+n, x) \mid x, x+m, x+m+n \in X\}.$$

Note that $\Theta(n,m) \neq \Theta(m,n)$ for $n \neq m$, but it can be easily checked that these operators satisfy the cocycle identity:

$$\Theta(l, m+n) + \tilde{T}_l\Theta(m,n) = \Theta(l+m, n) + \Theta(l,m)\tilde{T}_n,$$

for all $l, m, n \in \mathbb{Z}$.

Since $\tilde{T}_{m+n} = \tilde{T}_n\tilde{T}_m + \Theta(n,m)$ for all $n, m \in \mathbb{Z}$, our canonical C^*-algebra homomorphism $\varphi : C^*(\mathcal{T}_X) \to C^*_r(\mathbb{Z})$, must satisfy $\varphi(\Theta(n,m)) = 0$ for all $n, m \in \mathbb{Z}$.

We will see that the operators $\Theta(n,m)$ are in fact nilpotent whenever $n \neq -m$.

REMARK 9.2 For $n \in \mathbb{Z}$ positive, the domain of τ_n consists of finitely many points within the subset $X_1 \cup \ldots \cup X_n$, together with the infinite set

$$\{(n+k-1)(n+k), \ldots, (n+k-1)(n+k)+k-1 \mid k \in \mathbb{N}\},$$

which is made up of the first k elements of each set $X_{n+k} \subseteq X$, where $k \in \mathbb{N}$.

For $n \in \mathbb{Z}$ negative, the domain of τ_n consists of finitely many points within the subset $X_1 \cup \ldots \cup X_{|n|}$, together with the infinite set

$$\{(n-k)^2 - k, \ldots, (n-k)^2 - 1 \mid k \in \mathbb{N}\},$$

which is made up of the last k elements of each set $X_{|n|+k} \subseteq X$, where $k \in \mathbb{N}$.

In all cases, we have

$$Dom(\tau_n \tau_m) = \{x \in X \mid x+m+n, x+m \in X\} = Dom(\tau_{m+n}) \cap Dom(\tau_m).$$

Also recall that, where defined, $\tau_n \tau_m$ takes the same values as τ_{m+n}.

PROPOSITION 9.3 *For every $n, m \in \mathbb{Z}$ where $n \neq -m$, $\Theta(n,m) = \tilde{T}_{m+n} - \tilde{T}_n \tilde{T}_m$ is a nilpotent operator.*

Proof.

By definition, $\Theta(n,m)$ is nilpotent if there exists some $k \in \mathbb{N}$ such that $(\Theta(n,m))^k = 0$, i.e. such that $(\Theta(n,m))^k(\delta_x) = 0$ for all $x \in X$. This situation corresponds to the partial translation $(\theta(n,m))^k$ having empty domain, for some $k \in \mathbb{N}$, meaning that $(\tau_n \tau_m)^k$ and τ_{m+n}^k are defined on the same elements of X. We prove that this is always the case by comparing the domains of $\tau_n \tau_m$ and τ_{m+n}.

Case 1: n, m both positive.

In this case, in the parts of X where the gaps between the components are at least $m+n$, $Dom(\tau_n \tau_m) = Dom(\tau_{m+n}) \cap Dom(\tau_m)$ will be the same as $Dom(\tau_{m+n})$. So $\tau_n \tau_m$ and τ_{m+n} will definitely agree on $X_{m+n}, X_{m+n+1}, \ldots$, and hence $\theta(n,m)$ will be undefined here. In other words, $\theta(n,m)$ is defined on only finitely many elements of X, each contained within $X_1 \cup \ldots \cup X_{m+n-1}$, and so gives rise to a finite rank operator. In particular, if we let $k = m+n$, then it is certain that $(\theta(n,m))^k$ is undefined. Therefore $\Theta(n,m)$ is a finite rank nilpotent operator.

Case 2: n, m both negative.

To ensure that $\tau_n\tau_m$ and τ_{m+n} agree in the negative case we need to limit ourselves to components which occur *after* a gap of at least size $|m+n|$. We can certainly say that they behave in the same way on $X_{|m+n|+1}, X_{|m+n|+2}, \ldots$, which means that $\theta(n,m)$ is undefined here, and thus $\Theta(n,m)$ is again finite rank and nilpotent by the same argument as above.

This leaves us with cases where we compose negative and positive translations together.

Case 3: $|m| > |n|$ for either m negative and n positive or n negative and m positive.

Here m and $m+n$ are either both positive or both negative, and in either case $Dom(\tau_n\tau_m) = Dom(\tau_{m+n}) \cap Dom(\tau_m)$ coincides with $Dom(\tau_m)$ on the set $X_{|m|+1} \cup X_{|m|+2} \cup \ldots$, since $|m| > |m+n|$. Thus $\theta(n,m)$ is defined on some finite number of points within $X_1 \cup \ldots \cup X_{|m|}$, and the remainder of its domain coincides with $Dom(\tau_{m+n}) \backslash Dom(\tau_m)$, thus consisting of sets of $|n|$ consecutive points from each X_i for $i > |m|$. As the gaps between these sets of elements are of length at least $|m| > |m+n|$, we cannot apply $\theta(n,m)$ more than once at any point in this section of its domain. As the only other place where it is defined is a finite bounded set of points, $\theta(n,m)$ gives rise to a (infinite rank) nilpotent operator.

Case 4: $|n| > |m|$ for n positive and m negative.

In this case, to find the domain of $\tau_n\tau_m$ we intersect the domain of a positive/right translation, τ_{m+n}, with the domain of a negative/left translation, τ_m. It can be seen, using Remark 9.2, that here we obtain some finite number of points contained in $X_1 \cup \ldots \cup X_n$, together with the infinite set

$$\{(n+k-1)(n+k) + |m|, \ldots, (n+k-1)(n+k) + |m| + k - 1 \mid k \in \mathbb{N}\},$$

which is made up of the $(|m|+1), \ldots, (|m|+k)$th elements of each set $X_{n+k} \subseteq X$, where $k \in \mathbb{N}$. Note that we can see that $\tau_n\tau_m$ gives rise to a non-nilpotent operator from this, since the sets of consecutive numbers in the domain can grow arbitrarily large. However, the translation $\theta(n,m)$ will, after a finite fixed number of points, only be defined on sets of $|m|$ consecutive numbers with gaps of length at least $n > m+n$ between them, and hence will give rise to a nilpotent operator, as before.

Case 5: $|n| > |m|$ for m positive and n negative.

Finally, we are in a similar situation to above in that we intersect a left and right translation to form $Dom(\tau_n \tau_m)$, but now the overall action we wish to study is that of a negative translation. Here the domain consists of some finite number of points contained in $X_1 \cup \ldots \cup X_{|n|}$, together with the infinite set

$$\{(n-k)^2 - (m+k), \ldots, (n-k)^2 - (m+1) \mid k \in \mathbb{N}\},$$

which is made up of the $(|n|-m+1), \ldots, (|n|-m+k)$th elements of each set $X_{|n|+k} \subseteq X$, where $k \in \mathbb{N}$. So, as above, $\theta(n,m)$ is defined on some finite bounded set of points, together with sets of m consecutive numbers separated by gaps of length at least $|n| > |m+n|$, and thus $\Theta(n,m)$ is again infinite rank nilpotent.

$\square$

REMARK 9.4 Note that $\tilde{T}_0$ is the identity operator and $\tilde{T}_{-n} = \tilde{T}_n^*$ for all $n \in \mathbb{Z}$, so in the case not covered by the previous proposition, that is where $n = -m$, we obtain operators of the form

$$\Theta(-n,n) = 1 - \tilde{T}_n^* \tilde{T}_n \text{ and } \Theta(n,-n) = 1 - \tilde{T}_n \tilde{T}_n^*.$$

For $n \in \mathbb{Z}$ positive, we have

$$\tilde{T}_n^* \tilde{T}_n : \delta_x \mapsto \begin{cases} \delta_x & \text{if } x \in Dom(\tau_n) \\ 0 & \text{otherwise} \end{cases}, \text{ for all } x \in X,$$

which is a projection onto the space spanned by the vectors δ_x where x is not one of the last n elements of every X_{n+k}, for $k \in \mathbb{N}$, plus some finite rank projection onto the space spanned by finitely many vectors δ_y, where each y is a particular element of $X_1 \cup \ldots \cup X_n$. Thus

$$1 - \tilde{T}_n^* \tilde{T}_n : \delta_x \mapsto \begin{cases} \delta_x & \text{if } x \notin Dom(\tau_n) \\ 0 & \text{otherwise} \end{cases}, \text{ for all } x \in X,$$

is a projection onto the space spanned by the vectors δ_x where x is one of the last n elements of some X_{n+k}, plus a finite rank projection.

Similarly, $1 - \tilde{T}_n \tilde{T}_n^*$ is a projection onto the space spanned by the vectors δ_x corresponding to the first n elements of some X_{n+k}, plus a finite rank projection.

In any case, we can see that for every $n \in \mathbb{Z}\backslash\{0\}$, $\tilde{T}_n\tilde{T}_n^*$, $\tilde{T}_n^*\tilde{T}_n$, $1-\tilde{T}_n\tilde{T}_n^*$ and $1-\tilde{T}_n^*\tilde{T}_n$ are all infinite rank projections.

LEMMA 9.5 *The partial translation algebra $C^*(\mathcal{T}_X)$ contains all compact operators.*

Proof.

Consider the operator

$$\Theta(1,1) = \tilde{T}_2 - \tilde{T}_1\tilde{T}_1,$$

which we know to be an element of our algebra. This acts on $l^2(X)$ in the following way:

$$\Theta(1,1) : \delta_x \mapsto \begin{cases} \delta_2 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}, \text{ for all } x \in X.$$

Thus if we compose it with its adjoint we obtain a rank one projection p_0, the projection onto $\mathbb{C}\delta_0$. Then for any other $x \in X$ the composition $\tilde{T}_x p_0 \tilde{T}_x^*$ yields the rank one projection onto $\mathbb{C}\delta_x$; thus we may obtain all matrix units in the algebra and therefore generate all finite rank operators. Hence $C^*(\mathcal{T})$ contains all norm limits of finite rank operators and thus all compact operators.

□

Note that the compact operators can also be generated using only Θ operators, since for every $x \in X\backslash\{0\}$ the composition

$$\Theta(x-1,1)\Theta(-1,-1)\Theta(1,1)\Theta(-1,1-x) = \Theta(x-1,1)p_0\Theta(-1,1-x)$$

also yields the rank one projection onto $\mathbb{C}\delta_x$ (using the fact that $1 \notin X$ to ensure that our composition has non-trivial domain). Hence the kernel of our canonical map $\varphi : C^*(\mathcal{T}_X) \to C_r^*(\mathbb{Z})$ contains all compact operators. However, since every $\Theta(n,-n)$ is an infinite rank projection and these are also mapped to zero under φ, the kernel does not consist solely of compacts.

LEMMA 9.6 *For every $l, m, n \in \mathbb{Z}$,*

$$\tilde{T}_l\Theta(m,n) = \Theta(l+m,-m)\Theta(m,n).$$

Proof.

We have the following:

$$\tilde{T}_l : \delta_x \mapsto \begin{cases} \delta_{x+l} & \text{if } x+l \in X, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Theta(m,n) : \delta_x \mapsto \begin{cases} \delta_{x+n+m} & \text{if } x+n+m \in X, x+n \notin X, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\tilde{T}_l\Theta(m,n) : \delta_x \mapsto \begin{cases} \delta_{x+n+m+l} & \text{if } x+n+m, x+m+n+l \in X, x+n \notin X, \\ 0 & \text{otherwise.} \end{cases}$$

Also,

$$\Theta(l+m,-m) : \delta_x \mapsto \begin{cases} \delta_{x-m+l+m} = \delta_{x+l} & \text{if } x+l \in X, x-m \notin X, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\Theta(l+m,-m)\Theta(m,n) : \delta_x \mapsto \begin{cases} \delta_{x+n+m+l} & \text{if } x+n+m \in X, x+n \notin X, \\ & x+n+m+l \in X, \\ & x+n+m-m = x+n \notin X, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\Theta(l+m,-m)\Theta(m,n) = \tilde{T}_l\Theta(m,n),$$

as required.

□

PROPOSITION 9.7 *For every integer $n > 1$, we have*

$$\tilde{T}_n = \tilde{T}_1^n + K_n,$$

where K_n is a finite sum of compositions of elements of the form $\Theta(m,p)$, *where $m, p \in \mathbb{Z}$. For $n < -1$, we have*

$$\tilde{T}_n = (\tilde{T}_1^*)^{-n} + K_n',$$

for another such operator K_n'.

Proof.

Let us begin by concentrating on the first case. We proceed by induction.

For $n = 2$, we have

$$\tilde{T}_2 = \tilde{T}_1 \tilde{T}_1 + \Theta(1,1),$$

by definition of $\Theta(1,1)$. To see what happens in a slightly more interesting case, consider $n = 3$. Then

$$\begin{aligned}
\tilde{T}_n = \tilde{T}_3 &= \tilde{T}_1 \tilde{T}_2 + \Theta(1,2) \\
&= \tilde{T}_1(\tilde{T}_1^2 + \Theta(1,1)) + \Theta(1,2) \\
&= \tilde{T}_1^3 + \tilde{T}_1 \Theta(1,1) + \Theta(1,2) \\
&= \tilde{T}_1^3 + \Theta(2,-1)\Theta(1,1) + \Theta(1,2), \text{ by Lemma 9.6,}
\end{aligned}$$

so $K_3 = \Theta(2,-1)\Theta(1,1) + \Theta(1,2)$.

Assume the proposition holds for some $n > 3$, so that

$$\tilde{T}_n = \tilde{T}_1^n + K_n,$$

where K_n is an operator formed by adding together compositions of "Θ" elements.

We need to show that the statement now holds true for $n + 1$. By definition of $\Theta(1, n)$, we have

$$\begin{aligned}
\tilde{T}_{n+1} &= \tilde{T}_1 \tilde{T}_n + \Theta(1,n) \\
&= \tilde{T}_1(\tilde{T}_1^n + K_n) + \Theta(1,n), \text{ by our induction hypothesis.}
\end{aligned}$$

Thus

$$\tilde{T}_{n+1} = \tilde{T}_1 \tilde{T}_1^n + \tilde{T}_1 K_n + \Theta(1,n).$$

However, we know by Lemma 9.6 that we will also be able to express $\tilde{T}_1 K_n$ purely in terms of Θ elements, and so this is an operator of the required form. Thus we may denote $\tilde{T}_1 K_n + \Theta(1,n)$ by K_{n+1} to obtain

$$\tilde{T}_{n+1} = \tilde{T}_1^{n+1} + K_{n+1},$$

as required.

Recalling that $\tilde{T}_1^* = \tilde{T}_{-1}$, an almost identical proof follows for the negative case.

□

REMARK 9.8 Since every summand in one of the K_n operators as defined above involves composition with an operator $\Theta(l,m)$ for $l,m>0$ (or $l,m<0$ in the negative case), these operators are in fact finite rank. So every partial isometry $\tilde{T}_n$ is a *finite rank* perturbation of $\tilde{T}_1^n$ (or $(\tilde{T}_1^*)^{|n|}$). This can also be deduced purely by observation of the set X, since, for every $n \in \mathbb{Z}$, τ_n behaves in the same way as either τ_1^n or $\tau_{-1}^{|n|}$ outside of the finite set $X_1 \cup \ldots \cup X_{|n|}$.

Thus we have the following:

THEOREM 9.9

$$C^*(\mathcal{T}_X) = C^*(\tilde{T}_1, 1) + \mathfrak{K}.$$

This theorem demonstrates that, modulo compacts, the partial translation algebra is generated by the restriction of the bilateral shift.

The canonical homomorphism from this algebra to the reduced C^*-algebra of $\mathbb{Z}$ takes $\tilde{T}_1$ to the bilateral shift and compact operators to zero.

Another avenue worthy of investigation is the computation of the K-theory of our algebra. As a first attempt at understanding this, we define a trace on $C^*(\mathcal{T}_X)$.

DEFINITION 9.10 A *trace* on an algebra A is a linear function T from A to some vector space V, which satisfies the *trace condition*:

$$T(ab) = T(ba),$$

for all $a, b \in A$. Some sources also require that T be a *positive* map, i.e. that $T(a) \geq 0$ for all positive elements $a \in A$.

A trace T on a C^*-algebra A is said to be *faithful* if it satisfies:

$$T(a^*a) = 0 \implies a = 0,$$

for all $a \in A$.

DEFINITION 9.11 Let $\Pi\mathbb{C}$ denote the vector space of infinite sequences of complex numbers, and let J denote the subspace of $\Pi\mathbb{C}$ consisting of all finitely supported sequences whose sum is zero. Define a map

$$Tr : C^*(\mathcal{T}_X) \to \Pi\mathbb{C}/J,$$

by

$$T \mapsto (Tr_i(T)) \text{ mod finite support},$$

where $Tr_i(T)$ is the trace of T on the ith component of X, that is the trace of P_iTP_i, where $P_i : l^2(X) \to l^2(X_i)$ is the projection onto $l^2(X_i)$.

Quotienting the target space by J ensures that $Tr(\tilde{T}_n^*\tilde{T}_n) = Tr(\tilde{T}_n\tilde{T}_n^*)$ for all $n \in \mathbb{Z}$, while at the same time allowing traces of finite rank projections to be non-zero.

PROPOSITION 9.12 *The function Tr is a faithful (generalised) trace on $C^*(\mathcal{T}_X)$.*

Proof.

We say the trace is *generalised* because it maps into $\Pi\mathbb{C}/J$ rather than $\mathbb{C}$, as is typically the case.

We have the following:

(a) It is clear that the map Tr is linear because each Tr_i is linear by the properties of the usual matrix trace.

(b) Let $T, S \in C^*(\mathcal{T}_X)$. Since all elements of $C^*(\mathcal{T}_X)$ must have finite propagation, we know that the non-zero entries of the matrices of T and S must be contained within some strip of finite width about the diagonal. In other words, there exist non-negative integers R_T and R_S for which the (x, y)th entry of the matrix T is zero whenever $d(x, y) > R_T$, and the (x, y)th entry of the matrix S is zero whenever $d(x, y) > R_S$, for all $x, y \in X$. Thus the (x, y)th entries of the matrices representing TS and ST are zero for all $x, y \in X$ with $d(x,y) > R$, where $R = R_T + R_S$. In terms of the action on $l^2(X)$, this means that both TS and ST can only permute entries corresponding to elements of any component X_i with $i > R$ with other entries corresponding to elements of that component. Hence TS and ST act as block diagonal matrices on each of these components, and thus we have $Tr_i(TS) = Tr_i(ST)$ for all $i > R$.

On the other hand, we can restrict the operators T and S to the first R components of X to obtain finite matrices T_R and S_R respectively. The matrices $T_R S_R$ and $S_R T_R$ have the same trace, by definition. Thus the restrictions of TS and ST to the first R components have the same traces, which tells us that

$$Tr_1(TS) + \ldots + Tr_R(TS) = Tr_1(ST) + \ldots + Tr_R(ST).$$

As we only compute our trace modulo finitely supported sequences whose sum is zero, we thus obtain $Tr(TS) = Tr(ST)$.

(c) The positive elements of our algebra are of the form T^*T. For every such operator we have

$$\langle \delta_x, T^*T\delta_x \rangle = \|T\delta_x\|^2 \geq 0,$$

that is all of the diagonal entries are non-negative. Thus the trace of the operator is non-negative. Therefore Tr is a positive map.

Hence Tr is indeed a generalised trace.

(d) Suppose we have an operator of the form T^*T such that $Tr(T^*T) = 0$. This means that $Tr(T^*T) \in J$, and hence is a sequence consisting of finitely many non-zero entries whose sum is zero. However, we know that each of these entries is non-negative, and so they must all be equal to zero. Thus all diagonal entries of our operator are zero, and therefore we have $T^*T = 0$. Hence Tr is a faithful trace.

□

However, it turns out that the trace we have constructed is not surjective, and so is not as useful as we would have hoped when it comes to calculating K-theory. Fortunately though, we have another tool at our disposal, courtesy of Theorem 8.3.

By this theorem, we have $Ker(\varphi) = C^*(\mathcal{T}_X) \cap I(P)$, where φ is the canonical map from $C^*(\mathcal{T}_X)$ to $C_r^*(\mathbb{Z})$ and P denotes the projection onto $l^2(\mathbb{Z}\backslash X)$. Recall that $I(P)$ is the (two-sided) ideal in $\mathcal{A} = C^*(C_r^*(G), P)$ (in this case $\mathcal{A} = C^*(C_r^*(\mathbb{Z}), P)$) generated by P, in other words

$$I(P) = \overline{span}\{T_{g_0}PT_{g_1}P\ldots T_{g_{k-1}}PT_{g_k} \mid k \geq 1,\ g_j \in G \text{ for all } 0 \leq j \leq k\},$$

where in our case we are considering $G = \mathbb{Z}$. We have also seen, by Proposition 8.2, that we have

$$C^*(\mathcal{T}_X) \cap I(P) = (1-P)I(P)(1-P).$$

Combining the above statements, we now have

$$Ker(\varphi) = \overline{span}\{(1-P)T_{n_0}PT_{n_1}\ldots PT_{n_k}(1-P) \mid k \geq 1,\ n_j \in \mathbb{Z}\,\forall\, 0 \leq j \leq k\}.$$

This is the direct limit over R of the spaces

$$I(P,X)_R = \overline{span}\{(1-P)T_{n_0}PT_{n_1}\ldots PT_{n_k}(1-P) \mid n_j \in \mathbb{Z}\ \forall\ 0 \leq j \leq k, \\ k \geq 1,\ |n_0|, |n_k| \leq R\}.$$

Notice that if we view an operator of the form $PT_{n_1}P\ldots T_{n_{k-1}}P$, for $k \geq 1$, $n_j \in \mathbb{Z}$, as a partial translation on $\mathbb{Z}$, then this will only translate elements within $\mathbb{Z}\backslash X$. If we compose such an operator on the left with $(1-P)T_{n_0}$ and on the right with $T_{n_k}(1-P)$, where both n_0 and n_k are integers of norm not greater than R for some R, then the resulting associated partial translation could be viewed as a subset of $B_R(\mathbb{Z}\backslash X) \times B_R(\mathbb{Z}\backslash X)$ (within $X \times X$). In other words, on components X_i for $i \geq max\{n_j \mid 0 \leq j \leq k\}$, the translation will only be defined on some of either the first or last R elements of X_i, and it will shift these elements by a distance smaller than R. The behaviour of the translation on $X_1 \cup \ldots \cup X_{i-1}$ may differ from this pattern, but as this is a finite set the impact it has on the operator arising from the partial translation will only be a finite rank perturbation.

From the above we see in particular that the space $I(P,X)_1$ is generated by the compact operators (arising from rank one projections), together with the projections onto the first and last elements of each component, which we recognise as the operators $\Theta(1,-1)$ and $\Theta(-1,1)$. Note that with the current notation we have

$$\Theta(1,-1) = (1-P)T_{-1}PT_1(1-P) \text{ and } \Theta(-1,1) = (1-P)T_1PT_{-1}(1-P).$$

This tells us that

$$I(P,X)_1/\mathfrak{K} \cong \mathbb{C} \oplus \mathbb{C},$$

which implies the following short exact sequence:

$$0 \to \mathfrak{K} \to I(P,X)_1 \to \mathbb{C} \oplus \mathbb{C} \to 0.$$

This in turn gives rise to the following exact sequence in K-theory:

$$\begin{array}{ccccc} \mathbb{Z} & \to & K_0(I(P,X)_1) & \to & \mathbb{Z} \oplus \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \leftarrow & K_1(I(P,X)_1) & \leftarrow & 0, \end{array}$$

which indicates that $K_0(I(P,X)_1) \cong \mathbb{Z}^3$ and $K_1(I(P,X)_1) = 0$.

Since similar reasoning tells us that

$$I(P,X)_R/\mathfrak{K} \cong M_R(\mathbb{C}) \oplus M_R(\mathbb{C}),$$

which has the same K-theory as $\mathbb{C} \oplus \mathbb{C}$, we may replace $I(P,X)_1$ by $I(P,X)_R$, for any R, in both of the above exact sequences. As K-theory commutes with direct limits [31] and $C^*(\mathcal{T}_X) \cap I(P) = Ker(\varphi) \overset{lim}{=\!\!\rightarrow} I(P,X)_R$, we thus have

$$K_0(C^*(\mathcal{T}_X) \cap I(P)) \cong \mathbb{Z}^3 \quad \text{and} \quad K_1(C^*(\mathcal{T}_X) \cap I(P)) = 0. \tag{10}$$

Now, by chapter 8, we have the following six term exact sequence:

$$\begin{array}{ccccc} K_0(C^*(\mathcal{T}_X) \cap I(P)) & \to & K_0(C^*(\mathcal{T}_X)) & \to & K_0(C^*_r(\mathbb{Z})) \\ \uparrow & & & & \downarrow \\ K_1(C^*_r(\mathbb{Z})) & \leftarrow & K_1(C^*(\mathcal{T}_X)) & \leftarrow & K_1(C^*(\mathcal{T}_X) \cap I(P)). \end{array}$$

Hence by combining this with (10), together with the fact that we can identify $C^*_r(\mathbb{Z})$ with $C(S^1)$, we obtain the exact sequence

$$\begin{array}{ccccc} \mathbb{Z}^3 & \rightarrow & K_0(C^*(\mathcal{T}_X)) & \rightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \leftarrow & K_1(C^*(\mathcal{T}_X)) & \leftarrow & 0. \end{array}$$

We now wish to use the maps involved in this sequence to determine the remaining terms. For ease of reference, let us label them as follows.

$$\begin{array}{ccccc} \mathbb{Z}^3 & \stackrel{\alpha}{\rightarrow} & K_0(C^*(\mathcal{T}_X)) & \stackrel{\beta}{\rightarrow} & \mathbb{Z} \\ \zeta\uparrow & & & & \downarrow\gamma \\ \mathbb{Z} & \stackrel{\epsilon}{\leftarrow} & K_1(C^*(\mathcal{T}_X)) & \stackrel{\delta}{\leftarrow} & 0. \end{array}$$

Now since δ is the zero map and $Im(\delta) = Ker(\epsilon)$, we have $K_1(C^*(\mathcal{T}_X)) \cong Im(\epsilon)$, by the first isomorphism theorem. Since ϵ is a homomorphism and as the rest of the sequence must also be exact, we find by observation that either $K_1(C^*(\mathcal{T}_X)) = 0$ or $K_1(C^*(\mathcal{T}_X)) \cong \mathbb{Z}$.

If $K_1(C^*(\mathcal{T}_X)) \cong \mathbb{Z}$ then $Ker(\zeta) = \mathbb{Z}$ and so ζ is the zero map. Then $Ker(\alpha) = Im(\zeta) = 0$, so α is injective, and thus $Im(\alpha) \cong \mathbb{Z}^3/\{0\} = \mathbb{Z}^3$, by the first isomorphism theorem. Now we also have $Im(\beta) = Ker(\gamma) = \mathbb{Z}$, so β is surjective, and $Ker(\beta) = Im(\alpha) = \mathbb{Z}^3$. Hence $K_0(C^*(\mathcal{T}_X))/\mathbb{Z}^3 \cong \mathbb{Z}$, by the first isomorphism theorem for β, and so $K_0(C^*(\mathcal{T}_X)) \cong \mathbb{Z}^4$.

In the second case, if $K_1(C^*(\mathcal{T}_X)) = 0$, then $Ker(\zeta) = 0$ and so ζ represents an inclusion of $\mathbb{Z}$ into $\mathbb{Z}^3$. Then $Ker(\alpha) \cong \mathbb{Z}$ and so $Im(\alpha) = Ker(\beta) \cong \mathbb{Z}^2$. Hence $K_0(C^*(\mathcal{T}_X))/\mathbb{Z}^2 \cong \mathbb{Z}$, and so $K_0(C^*(\mathcal{T}_X)) \cong \mathbb{Z}^3$.

To find out which of these two situations occur, we need to look more carefully at what is happening to generators.

Recall that $K_0(C^*(\mathcal{T}_X))$ is the set of equivalence classes of pairs $([p], [q])$, where $[p]$ and $[q]$ are elements of $[P(C^*(\mathcal{T}_X))]$, that is the semigroup of stable equivalence classes of projections in $P(C^*(\mathcal{T}_X))$, the set of all projections in $M_\infty(C^*(\mathcal{T}_X)) = \cup_{n\geq 1} M_n(C^*(\mathcal{T}_X))$. Recall also that two projections p and q are called *stably equivalent* if and only if there exists a projection $r \in P(C^*(\mathcal{T}_X))$ such that $p \oplus r$ is equivalent to $q \oplus r$.

We know from our previous calculations that the generators of the $\mathbb{Z}^3$ term in our sequence correspond to stable equivalence classes associated with the projections $\Theta(1, -1)$ and $\Theta(-1, 1)$ and the rank one projection. We show that the first two of these represent the same stable equivalence class when considered as elements of $C^*(\mathcal{T}_X)$.

LEMMA 9.13 *The projections* $\Theta(1,-1)$ *and* $\Theta(-1,1)$ *are stably equivalent in* $C^*(\mathcal{T}_X)$.

Proof.

We consider the projection r defined by

$$\begin{aligned} r &= 1 - (\Theta(1,-1) + \Theta(-1,1)) \\ &= 1 - ((1 - \tilde{T}_1\tilde{T}_1^*) + (1 - \tilde{T}_1^*\tilde{T}_1)) \\ &= \tilde{T}_1\tilde{T}_1^* + \tilde{T}_1^*\tilde{T}_1 - 1, \end{aligned}$$

which is clearly an element of our algebra. Then

$$\begin{aligned} \Theta(1,-1) + r &= 1 - \tilde{T}_1\tilde{T}_1^* + \tilde{T}_1\tilde{T}_1^* + \tilde{T}_1^*\tilde{T}_1 - 1 \\ &= \tilde{T}_1^*\tilde{T}_1, \end{aligned}$$

whilst

$$\begin{aligned} \Theta(-1,1) + r &= 1 - \tilde{T}_1^*\tilde{T}_1 + \tilde{T}_1\tilde{T}_1^* + \tilde{T}_1^*\tilde{T}_1 - 1 \\ &= \tilde{T}_1\tilde{T}_1^*. \end{aligned}$$

It is clear that $\tilde{T}_1^*\tilde{T}_1$ and $\tilde{T}_1\tilde{T}_1^*$ are Murray-von Neumann equivalent, by definition, and hence $\Theta(1,-1)$ and $\Theta(-1,1)$ are stably equivalent.

□

This equivalence tells us that $Im(\alpha)$ has at the most two generators in $K_0(C^*(\mathcal{T}_X))$, and since the two possibilities for $Im(\alpha)$ were $\mathbb{Z}^3$ and $\mathbb{Z}^2$, we thus have $Im(\alpha) \cong \mathbb{Z}^2$, which concurs with our second case. Hence, finally, we obtain the following.

THEOREM 9.14

$$K_0(C^*(\mathcal{T}_X)) \cong \mathbb{Z}^3 \text{ and } K_1(C^*(\mathcal{T}_X)) = 0.$$

REMARK 9.15 Recall that where $\mathcal{T}$ is the canonical partial translation structure for $\mathbb{Z}$, we have $C^*(\mathcal{T}) = C_r^*(\mathbb{Z}) \cong C(S^1)$. It is known that $K_0(C(S^1)) \cong K_1(C(S^1)) \cong \mathbb{Z}$, and hence we see that the partial translation algebra arising from the restriction of $\mathcal{T}$ to a subset of $\mathbb{Z}$ can yield very different K-theory to that of $C^*(\mathcal{T})$ itself.

10 Partial Translation Groupoids

In this chapter we examine a groupoid that can be formed out of any partial translation structure. When considering the case of the canonical partial translation structure on a discrete group, this leads to another method for constructing the reduced group C^*-algebra.

10.1 Groupoids

A *groupoid* is a set with a partially defined associative composition, identity, and inverses. Formally, a groupoid consists of two sets, G and B, and three maps

$$s : G \to B, \; t : G \to B \text{ and } m : G \times_B G \to G,$$

called the *source*, *target* and *multiplication* maps respectively, where

$$G \times_B G = \{(g_1, g_2) \in G \times G \mid s(g_1) = t(g_2)\},$$

the set of *composable pairs*. We usually denote $m(g_1, g_2)$ by g_1g_2. These maps must satisfy the following:

(i) When g_1 and g_2 are composable, $s(g_1g_2) = s(g_2)$ and $t(g_1g_2) = t(g_1)$.

(ii) The map m, where defined, is associative; that is, $(g_1g_2)g_3 = g_1(g_2g_3)$ whenever $s(g_1) = t(g_2)$ and $s(g_2) = t(g_3)$.

(iii) For each $x \in B$ there exists a unique $e_x \in G$ with $s(e_x) = t(e_x) = x$, such that $e_xg = g$ whenever $t(g) = x$, and $he_x = h$ whenever $s(h) = x$.

(iv) For each $g \in G$ there exists a unique element $g^{-1} \in G$ with $s(g^{-1}) = t(g)$ and $t(g^{-1}) = s(g)$, such that $g^{-1}g = e_{s(g)}$ and $gg^{-1} = e_{t(g)}$.

The space B is called the *space of objects* of the groupoid and is sometimes denoted by $G^{(0)}$ [29].

Our aim is to construct a groupoid from any given partial translation structure.

The following equivalent definition of a groupoid appears in [28].

DEFINITION 10.1 A *groupoid* is a set G with a product map $G^2 \to G$, $(x, y) \mapsto xy$, where $G^2 \subseteq G \times G$ is the set of composable pairs, and an inverse map $G \to G$, $x \mapsto x^{-1}$, such that the following conditions are satisfied:

(i) $(x^{-1})^{-1} = x$, for all $x \in G$.

(ii) If (x, y), $(y, z) \in G^2$ then (xy, z), $(x, yz) \in G^2$ and $(xy)z = x(yz)$.

(iii) $(x^{-1}, x) \in G^2$ for all $x \in G$, and if $(x, y) \in G^2$ then $x^{-1}(xy) = y$.

(iv) $(x, x^{-1}) \in G^2$ for all $x \in G$, and if $(z, x) \in G^2$ then $(zx)x^{-1} = z$.

If $x \in G$, call $d(x) = x^{-1}x$ and $r(x) = xx^{-1}$ the *domain* and *range* of x respectively. The set $G^0 = d(G) = r(G)$ is the *unit space* of G, in the sense that $xd(x) = r(x)x = x$ for all $x \in G$.

This definition is more useful for our purposes, as it will allow us to construct a groupoid consisting of partial translations without explicitly stating the object set when we begin, letting it instead emerge naturally from the definition.

10.2 The Groupoid of a Partial Translation Structure

We may construct a *partial translation structure groupoid* using the method outlined in the following proposition.

PROPOSITION 10.2 *Let $\mathcal{T}$ be a partial translation structure on a space X, let G be the closure of the set of partial translations in $\mathcal{T}$ under inverses and finite compositions, and define inverse and product maps for G as follows:*

$$\begin{aligned} t = \{(t(x), x) \mid x \in Dom(t)\} &\mapsto t^{-1} = \{(x, t(x)) \mid x \in Dom(t)\} \\ \text{and } (t, t') &\mapsto tt' = \{(tt'(x), x) \mid x \in Dom(t')\}, \end{aligned}$$

respectively. Here G^2 is the set of all pairs of partial translations which can be composed, i.e.

$$(t, t') \in G^2 \text{ if and only if } Dom(t) = Ran(t').$$

Then G is a groupoid with respect to these maps.

Note that we really do require that $Dom(t) = Ran(t')$ for t and t' to be composable; if we attempt to relax the condition to $Ran(t') \subseteq Dom(t)$, then we would encounter problems with the inverse of the composition tt'. In particular, condition (iv) of Definition 10.1 would not be satisfied. Including all compositions and inverses in G ensures that both the inverse and product maps map into G.

Proof.

Let us check that the four axioms hold for G:

(i) Note that $t^{-1} = \{(t^{-1}(y), y) \mid y \in Ran(t)\}$. Thus

$$(t^{-1})^{-1} = \{(y, t^{-1}(y)) \mid y \in Ran(t)\} = \{(t(x), x) \mid x \in Dom(t)\} = t.$$

(ii) Suppose that (t_1, t_2), $(t_2, t_3) \in G^2$. This means that $Dom(t_1) = Ran(t_2)$ and $Dom(t_2) = Ran(t_3)$. Thus $Dom(t_1t_2) = Dom(t_2) = Ran(t_3)$ and so $(t_1t_2, t_3) \in G^2$, and $Ran(t_2t_3) = Ran(t_2) = Dom(t_1)$, so $(t_1, t_2t_3) \in G^2$.

Also,

$$\begin{aligned}(t_1t_2)t_3 &= \{((t_1t_2)t_3(x), x) \mid x \in Dom(t_3)\} \\ &= \{(t_1(t_2t_3)(x), x) \mid x \in Dom(t_2t_3)\} \\ &= t_1(t_2t_3).\end{aligned}$$

(iii) We have $t^{-1} = \{(t^{-1}(y), y) \mid y \in Ran(t)\}$, so $Dom(t^{-1}) = Ran(t)$ by definition, and hence $(t^{-1}, t) \in G^2$, for all $t \in G$. If (t, t') also lies in G^2, this tells us that $Dom(t) = Ran(t')$ and $tt' = \{(tt'(x), x) \mid x \in Dom(t')\}$ is defined. Then

$$\begin{aligned}t^{-1}(tt') &= \{(t^{-1}tt'(x), x) \mid x \in Dom(tt') = Dom(t')\} \\ &= \{(t'(x), x) \mid x \in Dom(t')\} \\ &= t'.\end{aligned}$$

(iv) We also have $t^{-1} = \{(x, t(x)) \mid x \in Dom(t)\}$, so it is clear that the range of this is equal to the domain of t, and thus (t, t^{-1}) also lies in G^2, for all $t \in G$. Let $(t', t) \in G^2$, so that we have $t't = \{(t't(x), x) \mid x \in$

$Dom(t)\}$ and $Dom(t') = Ran(t)$. Then

$$\begin{aligned}(t't)(t^{-1}) &= \{(t'tt^{-1}(x), x) \mid x \in Dom(t^{-1}) = Ran(t)\} \\ &= \{(t'(x), x) \mid x \in Dom(t')\} \\ &= t',\end{aligned}$$

recalling that $Dom(t^{-1}) = Ran(t)$ by axiom **(iii)**.

Thus G is indeed a groupoid.

□

For each $t \in G$ we have

$$d(t) = t^{-1}t = \{t^{-1}t(x), x) \mid x \in Dom(t)\} = Id_{Dom(t)}$$

and

$$r(t) = tt^{-1} = \{(tt^{-1}(x), x) \mid x \in Dom(t^{-1}) = Ran(t)\} = Id_{Ran(t)}.$$

Then $td(t) = t \circ Id_{Dom(t)} = t$ and $r(t)t = Id_{Ran(t)} \circ t = t$, for each $t \in G$.

It is implied from this (by identifying the map d with $s : G \to B$ and the map r with $t : G \to B$) that if we wished to use Roe's definition of a groupoid then our object set B would consist of all subsets of X which are either the domain or the range of one or more partial translations in G. Hence our groupoid differs from similar constructions appearing in, for example, [29] and [36], where the object set considered is the space X itself. Skandalis, Tu and Yu, the authors of [36], also define G^2 and the multiplication map in a different manner, by stating all partial translations to be composable and allowing for the empty translation.

PROPOSITION 10.3 *Let $\mathcal{T}$ be a partial translation structure for a space X, and let G be the closure of the set of partial translations in $\mathcal{T}$ with respect to inverses and finite compositions. Then G consists solely of partial translations on X (although G need not be a partial translation* structure*).*

Proof.

For each $g \in G$ we have that either g is already a partial translation in $\mathcal{T}$, in which case we are done, or g was formed by taking inverses and/or compositions of partial translations in $\mathcal{T}$. If an element $g \in G$ is of the form

$g = t^{-1} = \{(y,x) \mid (x,y) \in t\}$ for a partial translation $t \in \mathcal{T}$, then g must be well-defined as a partial bijection, because t is a partial bijection, and it is clear that the distance condition (part 1 of Definition 3.4) will also be satisfied, since $d(x,y) = d(y,x)$ for all $x, y \in X$. So suppose now that we have a finite composition of partial translations

$$g = t_n \cdots t_1 = \{(t_n \cdots t_1(x), x) \mid x \in Dom(t_1)\},$$

where $Ran(t_1) = Dom(t_2),\ Ran(t_2) = Dom(t_3), \ldots, Ran(t_{n-1}) = Dom(t_n)$. As each t_i is a partial translation, we know that the coordinate projections of t_i, viewed as a subset of $X \times X$, onto X are injective for all i, and thus the coordinate projections of our composition must also be injective. In addition to this, we know that $d(x_i, y_i)$ is bounded for all $(x_i, y_i) \in t_i$ for all i; say $d(x_i, y_i)$ is bounded by b_i. Then $d(x,y)$ is bounded by $\Sigma_i b_i$ for all $(x,y) \in g$, and hence g is also a partial translation on X. From this we can deduce that every element of G must be a partial translation on X.

□

Note that, unfortunately, cotranslations for $\mathcal{T}$ will not in general interact in the same way with G. This is illustrated by the following example.

EXAMPLE 10.4 Let $X = \mathbb{Z}\backslash\{0\}$, and let $\mathcal{T}$ denote the partial translation structure X inherits from the canonical partial translation structure on $\mathbb{Z}$, as in section 5.1.1. That is, partial translations in $\mathcal{T}$ are of the form

$$\tau_n = \{(m+n, m) \mid m \in X\backslash\{-n\} = \mathbb{Z}\backslash\{0, -n\}\},$$

for some $n \in \mathbb{Z}$, with partial cotranslations given by $\sigma_n(m) = m - n$, defined only for $m \in X\backslash\{n\} = \mathbb{Z}\backslash\{0, n\}$, for each $n \in \mathbb{Z}$.

Let G be the groupoid arising from $\mathcal{T}$. Then for each partial translation $\tau_n \in \mathcal{T}$, both τ_n and

$$\tau_n^{-1} = \{(m, m+n) \mid m \in X\backslash\{-n\}\} = \{(p-n, p) \mid p \in X\backslash\{n\}\}$$

will be elements of G (in fact τ_n^{-1} would already have been a partial translation in $\mathcal{T}$ in this example). Hence the composition

$$\tau_n^{-1}\tau_n = \{(\tau_n^{-1}\tau_n(m), m) \mid m \in Dom(\tau_n)\} = \{(m,m) \mid m \in X\backslash\{-n\}\}$$

will also be an element of G, for each $n \in \mathbb{Z}$.

We have that σ_{2n} is a partial cotranslation for $\mathcal{T}$ for each $n \in \mathbb{Z}$, i.e. $(\sigma_{2n}(x), \sigma_{2n}(y)) \in \tau_m$ for all $m \in \mathbb{Z}$ and for all $(x, y) \in \tau_m$ where σ_{2n} is defined on both x and y. However, each σ_{2n} is defined on n, and $(n, n) \in \tau_n^{-1}\tau_n$ for all $n \in \mathbb{Z}$, yet $(\sigma_{2n}(n), \sigma_{2n}(n)) = (-n, -n) \notin \tau_n^{-1}\tau_n$. Thus σ_{2n} is not a partial cotranslation for G for any $n \in \mathbb{Z}$.

Problems such as that occurring in the above are eliminated in the particular case where every partial translation in $\mathcal{T}$ is globally defined and has a globally defined inverse, however to combine our definitions of a partial translation structure groupoid and a partial translation structure we also require there to be a single set of globally defined cotranslations:

PROPOSITION 10.5 *Let $\mathcal{T}$ be a partial translation structure for a space X for which we may find a set of globally defined partial cotranslations Σ such that the conditions of Definition 3.4 are satisfied with $\Sigma = \Sigma_R$ for every $R > 0$. If $\mathcal{T}$ contains only globally defined partial translations, and if the inverse of every partial translation in $\mathcal{T}$ is also globally defined, then the groupoid arising from $\mathcal{T}$ is also a partial translation structure on X with set of cotranslations Σ.*

Proof.

Let G be the groupoid arising from $\mathcal{T}$, so that G is the closure of the set of partial translations in $\mathcal{T}$ with respect to compositions and inverses. We know by Proposition 10.3 that every element of G is indeed a partial translation for X; thus it only remains to check the conditions of the partial translation structure definition.

Firstly, note that every partial cotranslation for $\mathcal{T}$, i.e. every element of Σ, is now also a partial cotranslation for G. To see this we need to check that each $\sigma \in \Sigma$ also commutes with compositions and inverses of partial translations in $\mathcal{T}$. So let t and t' be partial translations in $\mathcal{T}$ which may be composed, so that $Dom(t) = Ran(t')$; we assume that every partial translation in $\mathcal{T}$ is globally defined, so in fact this is the case for any $t, t' \in \mathcal{T}$ (or indeed G, since the inverses are all globally defined as well). Now let $(x, y) \in tt'$, so $x = tt'(y)$, and suppose that $\sigma \in \Sigma$, which must therefore be defined for both x and y, since we assume all partial cotranslations to be globally defined. We know $(t'(y), y) \in t'$ and $(x, t'(y)) \in t$, thus $(\sigma t'(y), \sigma y) \in$

t' and $(\sigma x, \sigma t'(y)) \in t$, since σ is a partial cotranslation for $\mathcal{T}$ (and σ is also defined on $t'(y)$, as it is globally defined on X). Therefore

$$\sigma t'(y) = t'\sigma y \text{ and } \sigma x = \sigma t t'(y) = t\sigma t'(y).$$

Hence

$$\sigma x = \sigma t t'(y) = t\sigma t'(y) = t t' \sigma y,$$

and so $(\sigma x, \sigma y) \in tt'$, as required (recalling that we do not need to worry about domains here because everything is globally defined).

Now let t be some partial translation in $\mathcal{T}$ and consider its inverse $t^{-1} = \{(y,x) \mid (x,y) \in t\}$. If $\sigma \in \Sigma$, then by definition $(\sigma x, \sigma y) \in t$, and thus $(\sigma y, \sigma x) \in t^{-1}$, as required. So any partial cotranslation for $\mathcal{T}$ is also a partial cotranslation for G.

Note also that the partial translations of $\mathcal{T}$ are all contained in G. Hence, as we have all of the original partial translations as well as the same set of cotranslations, we may take the same $\mathcal{T}_R$ and Σ_R sets as would have been used for $\mathcal{T}$. Then the conditions for G to be a partial translation structure, which depend only on these sets, will be satisfied, since they must be satisfied for $\mathcal{T}$.

□

REMARK 10.6 The conditions of the previous proposition are satisfied in the case of the canonical partial translation structure for any countable discrete group. In fact, the groupoid formed in this case would just be the original set of partial translations, since all inverses and compositions will already be included due to the group structure.

If we are faced with a partial translation structure which does not have a single set of cotranslations, we could take a different approach to forming a new partial translation structure using groupoids, by creating a family of groupoids using the $\mathcal{T}_R$ sets:

THEOREM 10.7 *Let $(X, \mathcal{T})$ be a metric space with a free partial translation structure such that σ is globally defined for all $\sigma \in \Sigma_R$ and for all $R > 0$. Then if we expand each $\mathcal{T}_R$ set to form a groupoid, by including all compositions and inverses, we will still have a free partial translation structure for X.*

Proof.

For every $R > 0$, let G_R be the groupoid arising from $\mathcal{T}_R$, so that G_R is the closure of $\mathcal{T}_R$ with respect to compositions and inverses. Let G denote the structure we obtain in place of $\mathcal{T}$ by expanding every subset of partial translations $\mathcal{T}_R$ to G_R in this manner. The method of the proof of Proposition 10.3 can be applied to show that every element of each G_R is indeed a partial translation for X; thus it only remains to check the conditions of the partial translation structure definition for G.

Firstly, note that every partial cotranslation for $\mathcal{T}$ is now also a partial cotranslation for G. To see this we check that for any $R > 0$ each partial cotranslation in Σ_R also commutes with compositions and inverses of partial translations in $\mathcal{T}_R$. The proof follows identically as in the proof of the previous proposition.

For G to be a partial translation structure, we also require each G_R to be a collection of *disjoint* partial translations. For any $R > 0$, we know that $\mathcal{T}_R$ is a set of disjoint partial translations, as we are assuming $\mathcal{T}$ to be a partial translation structure. Firstly, let $t_1, t_2 \in \mathcal{T}_R$ be two composable partial translations; we wish to show that t_1t_2 is distinct from any other $t_3 \in \mathcal{T}_R$. So for contradiction suppose that $t_1t_2(x) = t_3(x)$ for some $x \in X$, so that $(t_1t_2(x), x) \in t_3$. Then we have $(\sigma t_1t_2(x), \sigma x) \in t_3$ for all $\sigma \in \Sigma_R$, i.e. $t_3(\sigma x) = \sigma t_3(x) = \sigma t_1t_2(x)$ for all $\sigma \in \Sigma_R$. By the above, $(\sigma t_1t_2(x), \sigma x) \in t_1t_2$ as well, so $\sigma t_1t_2(x) = t_1t_2(\sigma x)$, and therefore $t_3(\sigma x) = t_1t_2(\sigma x)$, for some $x \in X$ and for every $\sigma \in \Sigma_R$. We also know by condition (3) for $\mathcal{T}$ that for every $y \in Dom(t_3)$ there exists some $\sigma \in \Sigma_R$ such that $\sigma x = y$. Hence $t_3(y) = t_1t_2(y)$ for all $y \in Dom(t_3)$, and so $t_3 \subseteq t_1t_2$. Similarly, for all $y \in Dom(t_2) = Dom(t_1t_2)$ there exists some $\sigma \in \Sigma_R$ such that $\sigma x = y$. Thus $t_3(y) = t_1t_2(y)$ for all $y \in Dom(t_1t_2) \cup Dom(t_3)$; therefore $t_3 = t_1t_2$.

Hence composing two partial translations in $\mathcal{T}_R$ either produces another partial translation already contained in $\mathcal{T}_R$ or a new disjoint partial translation. This argument could be iterated for longer compositions, and could also be applied for inverses, to show that if $(x, t(x)) \in t'$ for some $t, t' \in \mathcal{T}_R$ then $t' = t^{-1}$ (Note that inverses of elements of any $\mathcal{T}_R$ are already disjoint from one another since the original elements are). Thus each G_R will be a set of disjoint partial translations of X.

It remains to check conditions (1) to (3) from Definition 3.4.

1. Each G_R is formed by expanding a $\mathcal{T}_R$ set, so if the R-neighbourhood

of the diagonal is contained in the union of partial translations in $\mathcal{T}_R$ then it is also included within the union of elements of the respective G_R set.

2. Each Σ_R set is unchanged, so this condition would also be unaffected.

3. We require, for every $R > 0$, that for all $t \in G_R$ there exists some $\sigma \in \Sigma_R$ such that $\sigma x = y$ and $\sigma t(x) = t(y)$ for all $(t(x), x), (t(y), y) \in t$. We assume this condition already holds for all elements of $\mathcal{T}_R$, so we only need check it for their compositions and inverses. Note that, for any $t \in \mathcal{T}_R$, $t^{-1} = \{(y, x) \mid (x, y) \in t\}$, so for inverses the condition is clear. So let $t, t' \in \mathcal{T}_R$ be composable, and consider two elements $(tt'(x), x), (tt'(y), y)$ contained in their composition. We wish to find some $\sigma \in \Sigma_R$ such that $\sigma x = y$ and $\sigma tt'(x) = tt'(y)$. By condition (3) for $\mathcal{T}$ applied to t and t' respectively, we have $\sigma_1 \in \Sigma_R$ such that $\sigma_1 t'(x) = t'(y)$ and $\sigma_1 tt'(x) = tt'(y)$, and $\sigma_2 \in \Sigma_R$ such that $\sigma_2 x = y$ and $\sigma_2 t'(x) = t'(y)$. However, since these cotranslations agree on $t'(x)$ and we assume $\mathcal{T}$ to be free, it must be the case that $\sigma_1 = \sigma_2$, and then this map satisfies the required condition.

$\square$

REMARK 10.8 Each G_R formed in the above manner acts *freely* on the space X, due to the disjointness of the partial translations.

10.3 The Reduced Groupoid C*-Algebra

As for any group G which is discrete we may define the reduced group C^*-algebra $C^*_r(G)$, it is similarly possible to associate a particular C^*-algebra with any groupoid which is *topological* and *étale*. Let us begin by explaining these required characteristics.

DEFINITION 10.9 A *topological groupoid* is a groupoid G which has a topology compatible with the groupoid structure, so that both the inverse map $G \to G$, $g \mapsto g^{-1}$, and the product map $G^2 \to G$, $(g, h) \mapsto gh$, are continuous, where G^2 has the induced subspace topology from $G \times G$ (endowed with the product topology). It is usually also assumed that the topology is Hausdorff and locally compact [28].

As consequences of this definition, the domain and range maps of any topological groupoid will also be continuous, and the inverse map will be a homeomorphism.

DEFINITION 10.10 A topological groupoid G is called *étale* if the range map $r : G \to G^0$ (equivalently, all the structure maps) is a local homeomorphism [29].

In general, the étale condition is an appropriate substitute for discreteness in the world of groupoids.

We are now ready to define the groupoid C^*-algebra. Let G be an étale topological groupoid. We call $C_c(G)$, the space of continuous and compactly supported functions from G to $\mathbb{C}$, the *groupoid algebra* of G. Multiplication here is given by the convolution $f_1 * f_2 = f$, where

$$f(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2), \qquad \text{for all } f_1, f_2 \in C_c(G),$$

the sum being taken over all composable pairs (g_1, g_2) with $g_1 g_2 = g$. The étale property for G ensures that this is a finite sum and that f is also a compactly supported continuous function on G, and thus $C_c(G)$ is indeed an algebra with $*$ as multiplication [29]. We may also define an involution on this algebra, by

$$f^*(g) = \bar{f}(g^{-1}).$$

For each $S \in G^0$, define a Hilbert space $\mathcal{H}_S = l^2(GS)$, where $GS = \{gh \mid d(g) = S\}$. Then left convolution defines a unitary representation π_S of $C_c(G)$ on $\mathcal{H}_S$, i.e. $\pi_S : C_c(G) \to \mathcal{B}(\mathcal{H}_S)$, $f \mapsto \pi_S(f)$, where $\pi_S(f)(\xi) = f * \xi$ for all $\xi \in \mathcal{H}_S$.

DEFINITION 10.11 The completion of $C_c(G)$ in the norm

$$\|f\| = \sup_{S \in G^0} \|\pi_S(f)\|$$

is the *reduced groupoid C^*-algebra $C_r^*(G)$* [29].

Recall that in the case of a partial translation groupoid acting on a space X, G^0 is the set of all identity maps on subsets of X which are equal to either the domain or the range of some element of G, and thus G^0 corresponds to the set of all such subsets themselves. Hence each G-orbit GS is of the form $\{tt' \mid Dom(t) = S = Ran(t')\}$, and the norm on $C_c(G)$ is defined as above with the supremum taken over all $S \subseteq X$ such that $S = Dom(t)$ or $S = Ran(t)$ for some $t \in G$.

When G is a discrete group, the reduced groupoid C^*-algebra coincides with the (reduced) group C^*-algebra [29]. We will show that this is also the case when we consider G to be the canonical partial translation structure on a discrete group, rather than the group itself. To achieve this we require the following lemma.

LEMMA 10.12 *Let G be a countable discrete group and let $\mathbb{C}G$ denote the group ring of G, that is the ring of all finitely supported functions on G. Then $\mathbb{C}G$ is isomorphic to the ring V of kernels $u : G \times G \to \mathbb{C}$ such that $u(rx, ry) = u(x, y)$ for all $r, x, y \in G$ and such that there exists $R > 0$ such that $u(x, y) = 0$ for all $x, y \in G$ with $l(x^{-1}y) > R$.*

Proof. Let us begin by verifying that V is a ring. We define the product of kernels in the obvious matrix way:

$$(u * u')(x, y) = \sum_{z \in G} u(x, z)u'(z, y).$$

We also have addition of kernels, given by

$$(u + u')(x, y) = u(x, y) + u'(x, y),$$

which is associative and commutative by the properties of $\mathbb{C}$. It is also clear that the zero kernel

$$u_0 : G \times G \to \mathbb{C},\ u_0(x, y) = 0,$$

is an element of V, along with $-u$ for every $u \in V$.

Let $u, u' \in V$. Then

$$(u + u')(rx, ry) = u(rx, ry) + u'(rx, ry) = u(x, y) + u'(x, y) = (u + u')(x, y)$$

for all $r, x, y \in G$, and if we let $R = \max\{R_1, R_2\}$, where $u(x, y) = 0$ for all $x, y \in G$ with $l(x^{-1}y) > R_1$ and $u'(x, y) = 0$ for all $x, y \in G$ with

$l(x^{-1}y) > R_2$, then

$$(u + u')(x, y) = u(x, y) + u'(x, y) = 0 + 0 = 0$$

for all $x, y \in G$ with $l(x^{-1}y) > R$. Thus $u + u' \in V$ for all $u, u' \in V$.

In addition to this, we have

$$\begin{aligned}
(u * u')(rx, ry) &= \sum_{z \in G} u(rx, z)u'(z, ry) \\
&= \sum_{z \in G} u(x, r^{-1}z)u'(r^{-1}z, y) \\
&= \sum_{w \in G} u(x, w)u'(w, y) \\
&= (u * u')(x, y),
\end{aligned}$$

for all $r, x, y \in G$. If we take R_1 and R_2 as above, let $R' = R_1 + R_2$ and suppose $l(x^{-1}y) > R'$, then for all $z \in G$ we have

$$l(x^{-1}z) + l(z^{-1}y) > R' = R_1 + R_2.$$

Thus $l(x^{-1}z) > R_1$ whenever $l(z^{-1}y) \leq R_2$ and $l(z^{-1}y) > R_2$ whenever $l(x^{-1}z) \leq R_1$, i.e. $u(x, z) = 0$ whenever $u'(z, y) \neq 0$ and $u'(z, y) = 0$ whenever $u(x, z) \neq 0$. Therefore

$$u * u' = \sum_{z \in G} u(x, z)u'(z, y) = 0$$

whenever $l(x^{-1}y) > R'$, and hence $u * u' \in V$ for all $u, u' \in V$.

Associativity and distributivity over addition of the product follow from associativity and distributivity of ordinary matrix multiplication. Thus V is indeed a ring.

Now, let us define a function ϕ on $\mathbb{C}G$ by $\phi(f) = u_f$, where

$$u_f(x, y) = f(x^{-1}y),$$

for all $x, y \in G$. Then $u_f : G \times G \to \mathbb{C}$, since $f : G \to \mathbb{C}$, and

$$u_f(rx, ry) = f((rx)^{-1}ry) = f(x^{-1}r^{-1}ry) = f(x^{-1}y) = u_f(x, y).$$

We also know that f is non-zero for only finitely many elements of G, thus we can choose $g \in G$ to be the element of greatest length in $supp(f)$. Then $f(x^{-1}y) = 0$ for all $x, y \in G$ with $l(x^{-1}y) > l(g)$, and hence we may set $R = l(g)$ so that we have $u_f(x, y) = 0$ for all $x, y \in G$ with $l(x^{-1}y) > R$. Therefore $\phi : \mathbb{C}G \to V$.

For all $u \in V$ we have $\phi^{-1}(u) = f_u$, where

$$f_u(g) = u(e, g),$$

for all $g \in G$, noting that as G is a countable discrete group it can be equipped with a metric that has bounded geometry, and so as $u(e, g)$ is only non-zero for $l(e^{-1}g) = l(g) \leq R$ for some finite R, f_u will have finite support. Indeed,

$$\phi(\phi^{-1}(u))(x, y) = \phi(f_u)(x, y) = f_u(x^{-1}y) = u(e, x^{-1}y) = u(x, y)$$

and

$$\phi^{-1}(\phi(f))(g) = \phi^{-1}(u_f)(g) = u_f(e, g) = f(e^{-1}g) = f(g),$$

for all $u \in V$, $f \in \mathbb{C}G$, $x, y, g \in G$. Hence ϕ is a bijection.

We have

$$\begin{aligned} \phi(f + f')(x, y) = u_{f+f'}(x, y) &:= u_f(x, y) + u_{f'}(x, y) \\ &= \phi(f)(x, y) + \phi(f')(x, y) \\ &= (\phi(f) + \phi(f'))(x, y) \end{aligned}$$

for all $f, f' \in \mathbb{C}G$, $x, y \in G$, and

$$\begin{aligned} \phi^{-1}(u + u')(g) = f_{u+u'}(g) &:= f_u(g) + f_{u'}(g) \\ &= \phi^{-1}(u)(g) + \phi^{-1}(u')(g) \\ &= (\phi^{-1}(u) + \phi^{-1}(u'))(g) \end{aligned}$$

for all $u, u' \in V$, $g \in G$.

Finally, we have

$$\begin{aligned}
\phi(f * f')(x,y) &= u_{f*f'}(x,y) \\
&= (f * f')(x^{-1}y) \\
&= \sum_{r\in G} f(x^{-1}yr^{-1})f'(r) \\
&= \sum_{r\in G} u_f(x, yr^{-1})u_{f'}(r^{-1}, e) \\
&= \sum_{r\in G} u_f(x, yr^{-1})u_{f'}(yr^{-1}, y) \\
&= \sum_{s\in G} u_f(x, s)u_{f'}(s, y), \text{ where } s = yr^{-1}, \\
&= (u_f * u_{f'})(x,y) \\
&= (\phi(f) * \phi(f'))(x,y)
\end{aligned}$$

for all $f, f' \in \mathbb{C}G$, $x, y \in G$, and

$$\begin{aligned}
\phi^{-1}(u * u')(g) &= f_{u*u'}(g) \\
&= (u * u')(e, g) \\
&= (u * u')(g^{-1}, e) \\
&= \sum_{r\in G} u(g^{-1}, r)u'(r, e) \\
&= \sum_{r\in G} u(e, gr)u'(e, r^{-1}) \\
&= \sum_{r\in G} f_u(gr)f_{u'}(r^{-1}) \\
&= \sum_{s\in G} f_u(gs^{-1})f_{u'}(s), \text{ where } s = r^{-1}, \\
&= (f_u * f_{u'})(g) \\
&= (\phi^{-1}(u) * \phi^{-1}(u'))(g)
\end{aligned}$$

for all $u, u' \in V$, $g \in G$. Therefore ϕ is a homomorphism, and thus an isomorphism.

$\square$

Recall that any element f of $\mathbb{C}G$ can be written as the sum

$$f = \sum_{g \in G} f_g \delta_g,$$

where $f_g = f(g)$ and δ_g is the 'Dirac delta function', that is the characteristic function of the singleton set $\{g\}$. In a similar vein, we may describe a basis for V by defining a kernel $u_g : G \times G \to \mathbb{C}$,

$$u_g(x, y) = \begin{cases} 1 & \text{if } x^{-1}y = g \\ 0 & \text{otherwise,} \end{cases}$$

for each $g \in G$. Then

$$\begin{aligned} u_g(rx, ry) &= \begin{cases} 1 & \text{if } (rx)^{-1}ry = x^{-1}r^{-1}ry = g \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } x^{-1}y = g \\ 0 & \text{otherwise,} \end{cases} \\ &= u_g(x, y), \end{aligned}$$

for all $x, y \in G$, and if we let $R = l(g)$ then $u_g(x, y) = 0$ for all $x, y \in G$ with $l(x^{-1}y) > R$. Thus u_g is an element of V for each $g \in G$.

Moreover, any $u \in V$ can be represented by the sum

$$\sum_{g \in G} u(e, g) u_g.$$

Indeed, let $u \in V$; then

$$\begin{aligned} u(x, y) &= u(e, x^{-1}y) \\ &= u(e, x^{-1}y) u_{x^{-1}y}(x, y) \\ &= \sum_{g \in G} u(e, g) u_g(x, y), \end{aligned}$$

for all $x, y \in G$.

Also note that we could equivalently define each u_g by

$$u_g(x,y) = \begin{cases} 1 & \text{if } (x,y) \in t_g \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } y = xg \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } x^{-1}y = g \\ 0 & \text{otherwise,} \end{cases}$$

where t_g is a partial translation in the canonical partial translation structure on G, that is $t_g = \{(x, xg) \mid x \in G\}$, for each $g \in G$. Thus each $u_g \in V$ corresponds to a partial translation t_g.

Let $\phi : \mathbb{C}G \to V$ be the isomorphism from the proof of Lemma 10.12. Then it is clear from the definition of ϕ that $\phi(\delta_g) = u_g$ for every $g \in G$. Hence overall we have that any finitely supported function on a countable discrete group G may be written as a linear combination of partial translations from the canonical partial translation structure for G, simply by replacing each δ_g by the corresponding t_g.

Now let $\mathcal{T}$ denote the set of partial translations in the canonical partial translation structure on G. Then as previously noted (see Remark 10.6), $\mathcal{T}$ is itself a groupoid. Hence we may define the reduced groupoid C^*-algebra $C^*_r(\mathcal{T})$ as the completion of $C_c(\mathcal{T})$ in the norm

$$\|f\| = \sup_{S \in G^0} \|\pi_S(f)\|.$$

Note that, due to the discreteness of G, the set of continuous compactly supported functions on $\mathcal{T}$ is in fact the set of finitely supported functions on $\mathcal{T}$, that is the set of all finite linear combinations of elements of $\mathcal{T}$, which we have just shown to be equivalent to $\mathbb{C}G$. Also, every partial translation in $\mathcal{T}$ is globally defined and so the only element of G^0 is G itself. Thus $C^*_r(\mathcal{T})$ is the completion of $\mathbb{C}G$ in the norm

$$\|f\| = \|\pi(f)\|,$$

where π is the unitary representation of $\mathbb{C}G$ on $\mathcal{H}_G = l^2(GG) = l^2(G)$, defined using left convolution. Thus we obtain the following.

THEOREM 10.13 *If G is a countable discrete group and $\mathcal{T}$ the set of partial translations in the canonical partial translation structure for G, then*

$$C_r^*(\mathcal{T}) \cong C_r^*(G),$$

where $C_r^*(\mathcal{T})$ denotes the reduced groupoid C^*-algebra of the groupoid $\mathcal{T}$ and $C_r^*(G)$ is the reduced group C^*-algebra of G.

References

[1] Adams, S., "Boundary Amenability For Word Hyperbolic Groups and An Application To Smooth Dynamics Of Simple Groups", *Topology*, **33**, no.4, 765-783, 1994.

[2] Anantharaman-Delaroche, C., "Amenability and Exactness for Dynamical Systems and Their C^*-Algebras", *Trans. Amer. Math. Soc.*, **354**, no.10, 4153-4178, 2002.

[3] Anantharaman-Delaroche, C., and Renault, J., "Amenable Groupoids", *Monographie de L'Enseignement Mathématique* **36**, Genève, 2000.

[4] Arzhantseva, G., Druţu, C., and Sapir, M., "Compression Functions of Uniform Embeddings of Groups into Hilbert and Banach Spaces", *Journal für die Reine und Angewandte Mathematik*, **633**, 213-235, 2009.

[5] Bourbaki, N., "General Topology, Chapters 1-4", *Elements of Mathematics*, Springer-Verlag Berlin Heidelberg New York, 1989.

[6] Brodzki, J., "Topological Invariants of C^*-Algebras", Lecture notes from the LMS/EPSRC graduate school at Southampton, September 2004.

[7] Brodzki, J., Campbell, S., Guentner, E., Niblo, G. A., and Wright, N. J., "Property A and $CAT(0)$ Cube Complexes", *J. Funct. Anal.*, **256**, Issue 5, 1408-1431, 2009.

[8] Brodzki, J., and Niblo, G. A., "Approximation Properties for Discrete Groups", in, Bojarski, B., Mishchenko, A., Troitsky, E., and Weber, A. (eds.) "C*-algebras and Elliptic Theory", Basel, Switzerland, Birkhäuser, 23-35 (Trends in Mathematics), 2006.

[9] Brodzki, J., Niblo, G. A., Putwain, R. J., and Wright, N. J., "A C^*-Algebra Extension for Subspaces of Groups", Unpublished.

[10] Brodzki, J., Niblo, G. A., and Wright, N. J., "Partial Translation Algebras for Trees", *Journal of Noncommutative Geometry*, **3**, no.1, 83-98, 2009.

[11] Brodzki, J., Niblo, G. A., and Wright, N., "Property A, Partial Translation Structures, and Uniform Embeddings in Groups", *J. London Math. Soc.*, **76**, no.2, 479-497, 2007.

[12] Conway, J. B., "A Course in Operator Theory", Graduate Studies in Mathematics **21**, American Mathematical Society, 2000.

[13] Davidson, K. R., "C^*-Algebras by Example", Fields Institute Monographs **6**, 1996.

[14] Dshalalow, J. H., "Real Analysis: An Introduction to the Theory of Real Functions and Integration", CRC Press, 2000.

[15] Edgar, G., "Integral, Probability, and Fractal Measures", Springer-Verlag New York, 1998.

[16] Gelfand, I. M., and Naimark, M. A., "On the Embedding of Normed Rings into the Ring of Operators in Hilbert Space", *Rec. Math. [Mat. Sbornik]*, **12**, no.54, 197-213, 1943.

[17] Guentner, E., "Exactness of the One Relator Groups", *Proc. Amer. Math. Soc.* **130**, no.4, 1087-1093 (electronic), 2002.

[18] Guentner, E., and Kaminker, J., "Exactness and the Novikov Conjecture", *Topology*, **41**, no.2, 411-418, 2002.

[19] Guentner, E., and Kaminker, J., "Exactness and Uniform Embeddability of Discrete Groups", *J. London Math. Soc.*, **70**, no.3, 703-718, 2004.

[20] Higson, N., and Roe, J., "Amenable Group Actions and the Novikov Conjecture", *J. Reine Agnew. Math.*, **519**, 143-153, 2000.

[21] Higson, N., and Roe, J., "Analytic K-homology", Oxford Mathematical Monographs, OUP, 2000.

[22] Kelley, J. L., “General Topology”, Springer-Verlag New York Berlin Heidelberg, Graduate Texts in Mathematics **27**, 1955.

[23] Lance, C., “On Nuclear C^*-algebras”, *J. Functional Analysis*, **12**, 157-176, 1973.

[24] Mukherjea, A., and Pothoven, K., “Real and Functional Analysis”, Plenum Press, 1978.

[25] Ozawa, N., “Amenable Actions and Exactness for Discrete Groups”, *C. R. Acad. Sci. Paris Sér. I Math*, **330**, no.8, 691-695, 2000.

[26] Ozawa, N., “Crude Notes for a Mini-course at NCGOA 2005”, http://www.math.vanderbilt.edu/ bisch/ncgoa05/talks/ozawa.pdf, 2005.

[27] Parthasarathy, K. R., “Probability Measures on Metric Spaces”, *Probability and Mathematical Statistics* **3**, Academic Press Inc., New York, 1967.

[28] Renault, J., “A Groupoid Approach to C^*-Algebras”, *Lecture Notes in Mathematics* **793**, Springer-Verlag Berlin Heidelberg New York, 1980.

[29] Roe, J., “Lectures on Coarse Geometry”, University Lecture Series **31**, American Mathematical Society, 2003.

[30] Roe, J., “Warped Cones and Property A”, *Geom. Topol.*, **9**, 163-178, 2005.

[31] Rosenberg, J., “Algebraic K-theory and Its Applications” Second Edition, Graduate Texts in Mathematics **147**, Springer, 1994.

[32] Rudin, W., “Real and Complex Analysis”, McGraw-Hill, 1986.

[33] Seebach, J. A., and Steen, L. A., “Counterexamples in Topology”, Holt, Rinehart and Winston, Inc., 1970.

[34] Shtern, A. I., “Group Algebra of a Locally Compact Group”, in Hazewinkel, Michiel, ‘Encyclopedia of Mathematics’, Kluwer Academic Publishers, 2001.

[35] Sierpiński, W., “General Topology”, Translated by C. C. Krieger, *Mathematical Expositions* **7**, University of Toronto Press, 1952.

[36] Skandalis, G., Tu, J. L., and Yu, G., "The Coarse Baum-Connes Conjecture and Groupoids", *Topology*, **41**, 807-834, 2002.

[37] Tessera, R., "Hilbert Compression of Metric Measure Spaces with Subexponential Growth", April 2006.

[38] Yu, G., "The Coarse Baum-Connes Conjecture for Spaces Which Admit a Uniform Embedding into Hilbert Space", *Invent. Math.*, **139**, 201-240, 2000.