UNIVERSITY OF Southampton
School of Engineering Sciences

Aerodynamics & Flight Mechanics Research Group

The Rotor in Axial Flight

S. J. Newman

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by

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Introduction

The following analysis is an appraisal of Actuator Disc Theory and Blade Element Theory applied to a rotor in axial flight. The solutions to the methods require scrutiny to choose which is correct for the particular flight condition. The report discusses the manner in which the solutions interact and proposes a scheme to assemble them together in a unified whole.

The analysis is then extended to Annulus Theory.

Nomenclature

T	Thrust
R	Rotor Radius
V_T	Tip Speed
V_C	Climb Velocity
V_D	Descent Velocity
V_i	Rotor Downwash Velocity
N	No of Blades
C	Blade Chord (Constant)
ρ	Air Density
a	Lift Curve Slope
s	Rotor Solidity
θ_0	Collective Pitch
κ	Overall Blade Twist (Linear)
x	Non- Dimensional Spanwise Variable
μ_z	Vertical Advance Ratio
λ_i	Non Dimensional Rotor Downwash
C_T	Thrust Coefficient
BET	Blade Element Theory
ADT	Actuator Disc Theory
AT	Annulus Theory

Diagrams

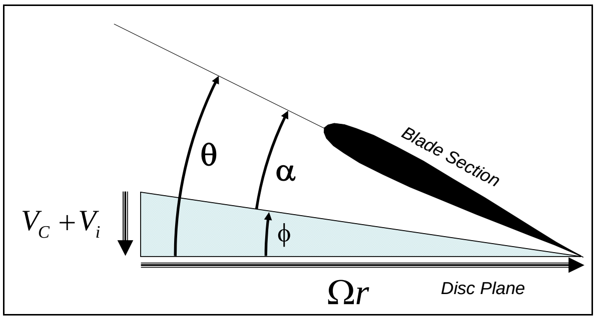

Diagram 1 - BET for Climb

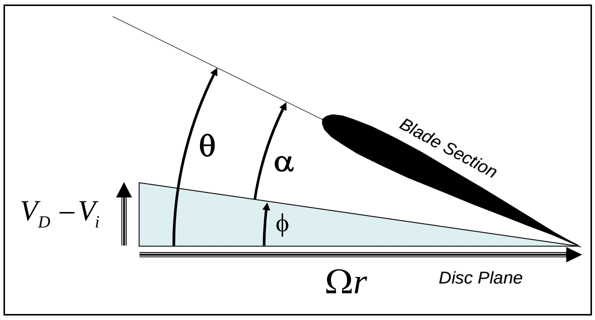

Diagram 2 - BET for Descent

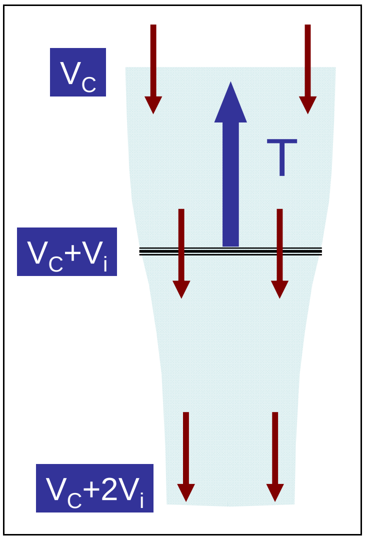

Diagram 3 - ADT for Climb

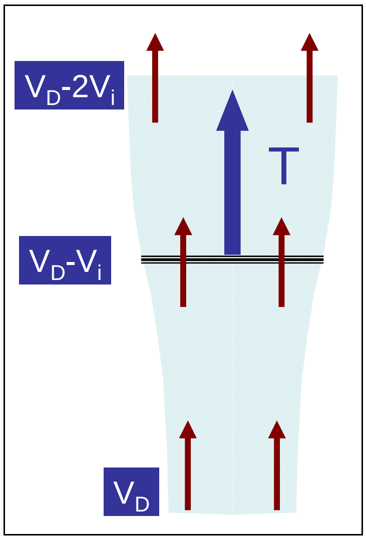

Diagram 4 - ADT for Descent

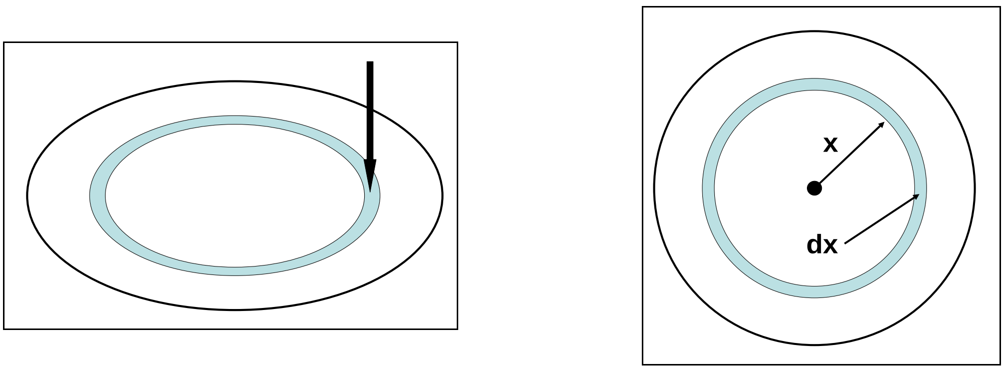

Diagram 5 - Annulus Theory

Blade Element Theory with Momentum Theory

Climb Root

BET gives:

$$T = \frac{1}{2}\rho V_T^2 \cdot NcR \cdot a\int_0^1 x^2\left(\theta_0 - \kappa x - \frac{\mu_z + \lambda_i}{x}\right) \cdot dx \qquad (1.)$$

which on normalisation becomes:

$$\begin{aligned} \frac{C_T}{sa} &= \frac{\theta_0}{3} - \frac{\kappa}{4} - \frac{\mu_z + \lambda_i}{2} \\ &= \frac{\theta_{75}}{3} - \frac{\mu_z + \lambda_i}{2} \end{aligned} \qquad (2.)$$

ADT gives the following expression for rotor thrust:

$$\begin{aligned} T &= \rho A(V_z + V_i) \cdot 2V_i \\ &= 2\rho A V_T^2 (\mu_z + \lambda_i)\lambda_i \end{aligned} \qquad (3.)$$

which on normalisation becomes:

$$C_T = 4(\mu_z + \lambda_i)\lambda_i$$
$$\frac{C_T}{4} = \mu_z \lambda_i + \lambda_i^2 \qquad (4.)$$

Combining equations (2) & (4) results in the quadratic equation:

$$\lambda_i^2 + \lambda_i \left\{ \mu_z + \frac{sa}{8} \right\} - \frac{sa}{4} \left\{ \frac{\theta_{75}}{3} - \frac{\mu_z}{2} \right\} = 0 \qquad (5.)$$

i.e.:

$$A\lambda_i^2 + \lambda_i B + C = 0 \qquad (6.)$$

where the coefficients are defined as:

$$A = 1$$
$$B = \mu_z + \frac{sa}{8} \qquad (7.)$$
$$C = -\frac{sa}{4} \left\{ \frac{\theta_{75}}{3} - \frac{\mu_z}{2} \right\}$$

if we introduce the product sa as a basic normalisation variable, the above equations reduce to:

$$
\begin{aligned}
\overline{\mu_z} &= \frac{\mu_z}{sa} \\
\overline{\lambda_i} &= \frac{\lambda_i}{sa} \\
\overline{\theta_{75}} &= \frac{\theta_{75}}{sa} \\
\overline{C_T} &= \frac{C_T}{(sa)^2} \\
&= \frac{\overline{\theta_{75}}}{3} - \frac{\overline{\mu_z} + \overline{\lambda_i}}{2}
\end{aligned}
\qquad (8.)
$$

$$
\overline{\lambda}_i^2 + \overline{\lambda}_i \left\{ \overline{\mu}_z + \frac{1}{8} \right\} - \frac{1}{4} \left\{ \frac{\overline{\theta}_{75}}{3} - \frac{\overline{\mu}_z}{2} \right\} = 0 \qquad (9.)
$$

$$
\overline{A}\overline{\lambda}_i^2 + \overline{\lambda}_i \overline{B} + \overline{C} = 0 \qquad (10.)
$$

$$
\begin{aligned}
\overline{A} &= 1 \\
\overline{B} &= \overline{\mu}_z + \frac{1}{8} \\
\overline{C} &= -\frac{1}{4} \left\{ \frac{\overline{\theta}_{75}}{3} - \frac{\overline{\mu}_z}{2} \right\}
\end{aligned}
\qquad (11.)
$$

It is now possible to solve this problem within a domain where rotor scale factors have been completely removed.

The solutions of (10) & (11) are:

$$\overline{\lambda}_i = \frac{-\left(\overline{\mu}_z + \frac{1}{8}\right) \pm \sqrt{\left(\overline{\mu}_z + \frac{1}{8}\right)^2 + \left\{\frac{\overline{\theta_{75}}}{3} - \frac{\overline{\mu_z}}{2}\right\}}}{2} = \frac{-\left(\overline{\mu}_z + \frac{1}{8}\right) \pm \sqrt{\left(\overline{\mu}_z - \frac{1}{8}\right)^2 + \frac{\overline{\theta_{75}}}{3}}}{2} \qquad (12.)$$

These are denoted **C+** & **C-**

In order for either solution to exist, the following condition must be met:

$$\overline{B}^2 - 4\overline{A}\overline{C} \geq 0 \qquad (13.)$$

i.e.:

$$\left(\overline{\mu}_z - \frac{1}{8}\right)^2 + \frac{\overline{\theta_{75}}}{3} \geq 0 \qquad (14.)$$

which is a parabolic region defined by:

$$\overline{\theta_{75}} = -3 \cdot \left(\overline{\mu}_z - \frac{1}{8}\right)^2 \qquad (15.)$$

Provided that this region is avoided, the remaining part of the domain will give rise to a positive or negative solution.

For a positive solution we have from inspection of (12):

C+

$$\left(\overline{\mu}_z + \frac{1}{8}\right) \leq 0$$
$$\overline{\mu}_z \leq -\frac{1}{8} \tag{16.}$$

OR

$$\left|\sqrt{\left(\overline{\mu}_z - \frac{1}{8}\right)^2 + \frac{\overline{\theta_{75}}}{3}}\right| \geq \left|\overline{\mu}_z + \frac{1}{8}\right| \tag{17.}$$

which on squaring out gives:

$$\overline{\theta_{75}} \geq \frac{3}{2}\overline{\mu}_z \tag{18.}$$

the domain now sub-divides as shown in Figure 1:

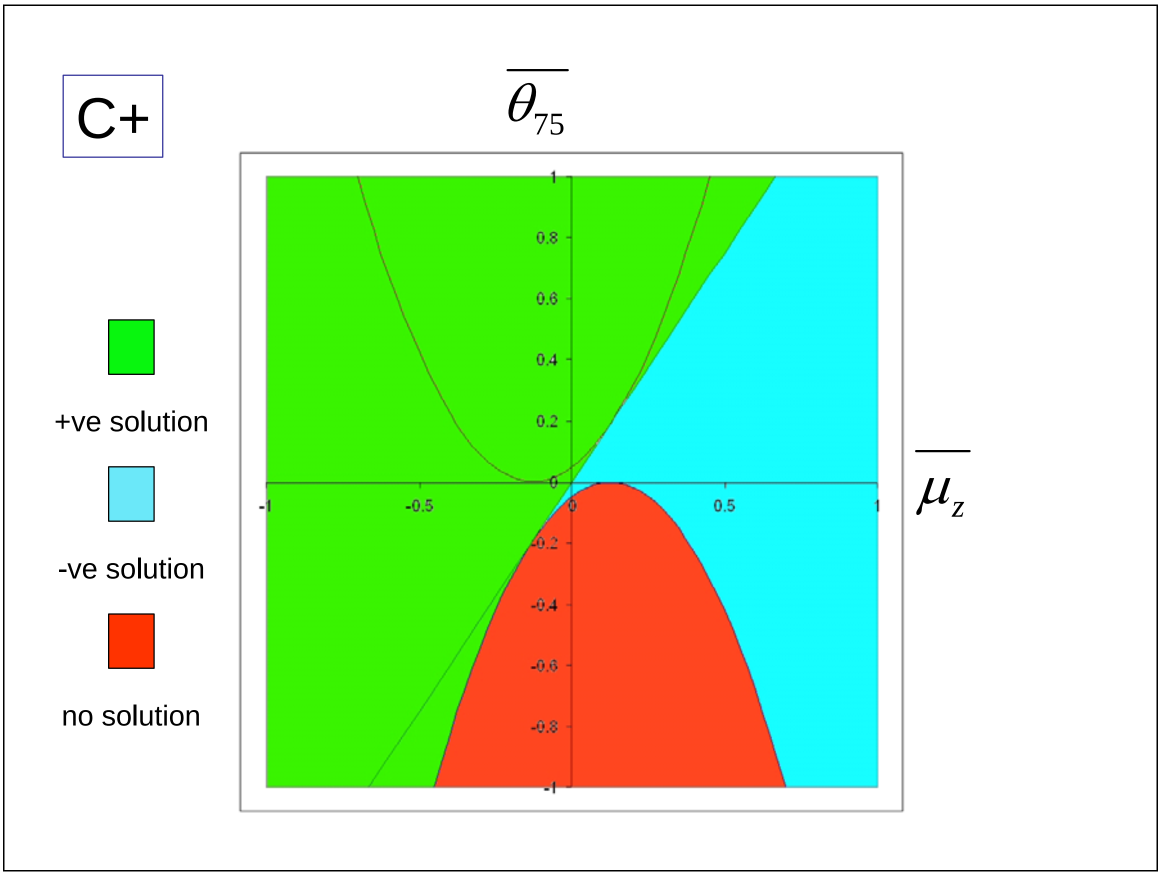

Figure 1 - C+ Solution Space

C-

a positive root will be obtained if:

$$\left(\overline{\mu}_z + \frac{1}{8}\right) \leq 0$$
$$\overline{\mu}_z \leq -\frac{1}{8} \tag{19.}$$

AND

$$\left|\sqrt{\left(\overline{\mu}_z - \frac{1}{8}\right)^2 + \frac{\overline{\theta_{75}}}{3}}\right| \leq \left|\overline{\mu}_z + \frac{1}{8}\right| \qquad (20.)$$

which on squaring out gives:

$$\overline{\theta_{75}} \leq \frac{3}{2}\overline{\mu}_z \qquad (21.)$$

the domain now sub-divides as shown in Figure 2:

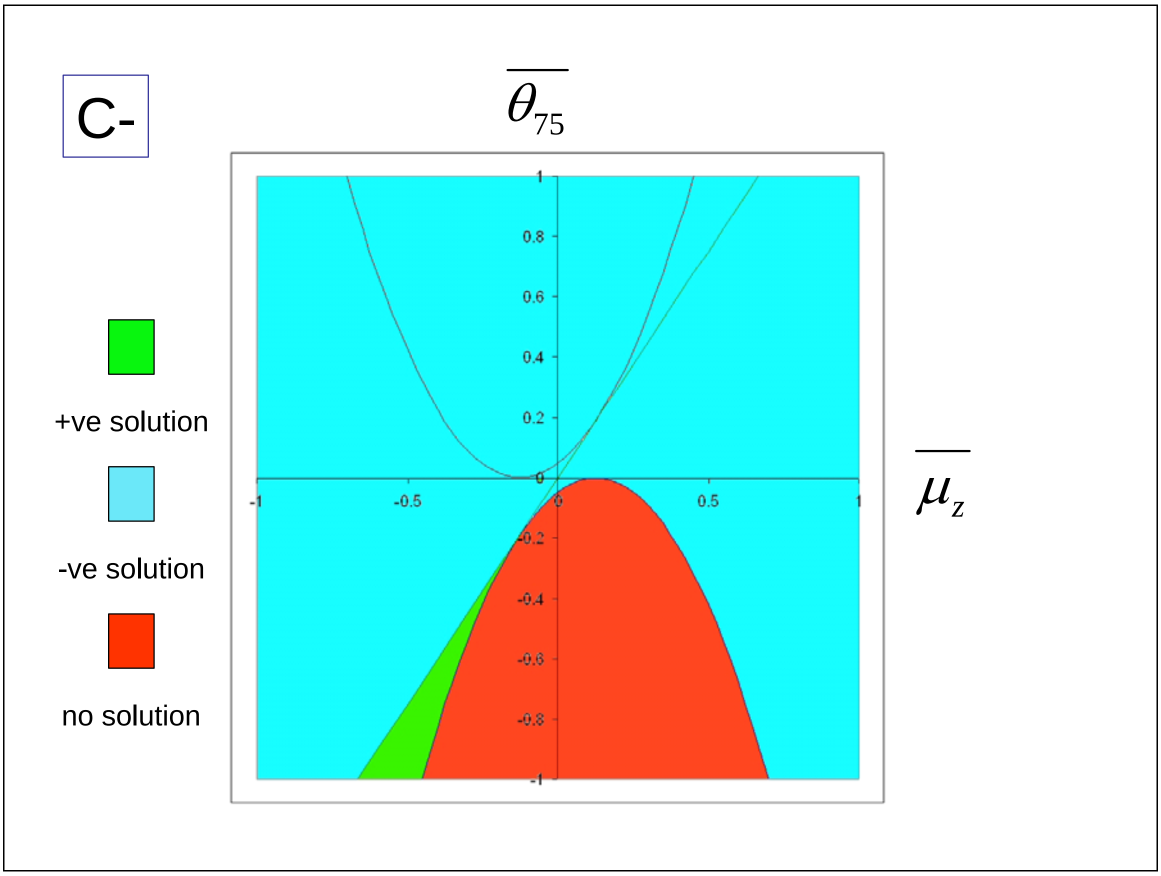

Figure 2 – C- Solution Space

Descent Root

BET gives:

$$T = \frac{1}{2}\rho V_T^2 \cdot NcR \cdot a\int_0^1 x^2\left(\theta_0 - \kappa x - \frac{\mu_z + \lambda_i}{x}\right)\cdot dx \tag{22.}$$

which on normalisation becomes:

$$\begin{aligned}\frac{C_T}{sa} &= \frac{\theta_0}{3} - \frac{\kappa}{4} - \frac{\mu_z + \lambda_i}{2} \\ &= \frac{\theta_{75}}{3} - \frac{\mu_z + \lambda_i}{2}\end{aligned} \tag{23.}$$

ADT gives the following expression for rotor thrust:

$$\begin{aligned}T &= \rho A(-V_z - V_i)\cdot 2V_i \\ &= 2\rho A V_T^2(-\mu_z - \lambda_i)\lambda_i\end{aligned} \tag{24.}$$

which on normalisation becomes:

$$\begin{aligned}C_T &= -4(\mu_z + \lambda_i)\lambda_i \\ \frac{C_T}{4} &= -\mu_z\lambda_i - \lambda_i^2\end{aligned} \tag{25.}$$

Equations (23) & (25) combine to give:

$$\lambda_i^2 + \lambda_i \left\{ \mu_z - \frac{sa}{8} \right\} + \frac{sa}{4} \left\{ \frac{\theta_{75}}{3} - \frac{\mu_z}{2} \right\} = 0 \qquad (26.)$$

i.e.

$$A\lambda_i^2 + \lambda_i B + C = 0 \qquad (27.)$$

where the coefficients are defined by:

$$\begin{aligned} A &= 1 \\ B &= \mu_z - \frac{sa}{8} \\ C &= \frac{sa}{4} \left\{ \frac{\theta_{75}}{3} - \frac{\mu_z}{2} \right\} \end{aligned} \qquad (28.)$$

normalisation using sa gives:

$$\overline{\lambda}_i^2 + \overline{\lambda}_i \left\{ \overline{\mu}_z - \frac{1}{8} \right\} + \frac{1}{4} \left\{ \frac{\overline{\theta}_{75}}{3} - \frac{\overline{\mu}_z}{2} \right\} = 0 \qquad (29.)$$

i.e.:

$$\overline{A}\overline{\lambda}_i^2 + \overline{\lambda}_i \overline{B} + \overline{C} = 0 \qquad (30.)$$

where:

$$
\begin{aligned}
&\overline{A} = 1 \\
&\overline{B} = \overline{\mu}_z - \frac{1}{8} \\
&\overline{C} = +\frac{1}{4}\left\{\frac{\overline{\theta_{75}}}{3} - \frac{\overline{\mu_z}}{2}\right\}
\end{aligned}
\qquad (31.)
$$

It is now possible to solve this problem within a domain where rotor scale factors have been completely removed.

The solutions of (30) & (31) are:

$$
\begin{aligned}
\overline{\lambda}_i &= \frac{-\left(\overline{\mu}_z - \frac{1}{8}\right) \pm \sqrt{\left(\overline{\mu}_z - \frac{1}{8}\right)^2 - \left\{\frac{\overline{\theta_{75}}}{3} - \frac{\overline{\mu_z}}{2}\right\}}}{2} \\
&= \frac{-\left(\overline{\mu}_z - \frac{1}{8}\right) \pm \sqrt{\left(\overline{\mu}_z + \frac{1}{8}\right)^2 - \frac{\overline{\theta_{75}}}{3}}}{2}
\end{aligned}
\qquad (32.)
$$

These are denoted **D+** & **D-**

In order for a solution, the following condition must be met:

$$
\overline{B}^2 - 4\overline{A}\overline{C} \geq 0 \qquad (33.)
$$

i.e.:

$$\left(\overline{\mu}_z - \frac{1}{8}\right)^2 - \frac{\overline{\theta_{75}}}{3} + \frac{\overline{\mu_z}}{2} \geq 0$$
$$\left(\overline{\mu}_z + \frac{1}{8}\right)^2 - \frac{\overline{\theta_{75}}}{3} \geq 0 \qquad (34.)$$

This is a parabolic region, but defined by:

$$\overline{\theta_{75}} = 3 \cdot \left(\overline{\mu}_z + \frac{1}{8}\right)^2 \qquad (35.)$$

Provided that this region is avoided, the remaining part of the domain will give rise to a positive or negative solution.

For a positive solution we have from inspection of (32):

D+

$$\left(\overline{\mu}_z - \frac{1}{8}\right) \leq 0$$
$$\overline{\mu}_z \leq \frac{1}{8} \qquad (36.)$$

OR

$$\left| \sqrt{\left(\overline{\mu}_z + \frac{1}{8} \right)^2 - \frac{\overline{\theta_{75}}}{3}} \right| \geq \left| \overline{\mu}_z - \frac{1}{8} \right| \qquad (37.)$$

which on squaring out gives:

$$\overline{\theta_{75}} \leq \frac{3}{2} \overline{\mu}_z \qquad (38.)$$

the domain now sub-divides as shown in Figure 3:

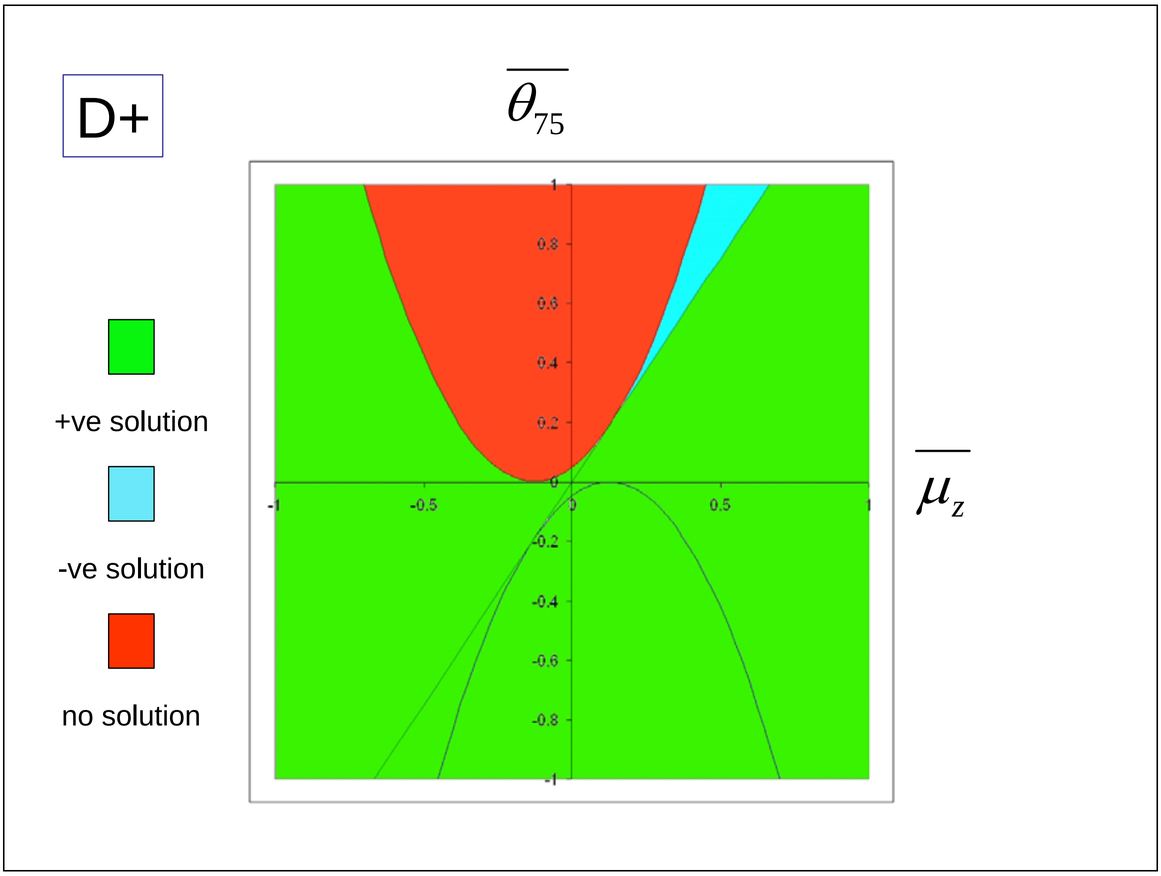

Figure 3 - D+ Solution Space

a positive root will be obtained if:

$$\left(\overline{\mu}_z - \frac{1}{8}\right) \leq 0$$
$$\overline{\mu}_z \leq \frac{1}{8} \tag{39.}$$

AND

$$\left|\sqrt{\left(\overline{\mu}_z + \frac{1}{8}\right)^2 - \frac{\overline{\theta_{75}}}{3}}\right| \leq \left|\overline{\mu}_z - \frac{1}{8}\right| \tag{40.}$$

which on squaring out gives:

$$\overline{\theta_{75}} \geq \frac{3}{2}\overline{\mu}_z \tag{41.}$$

the domain now sub-divides as shown in Figure 4:

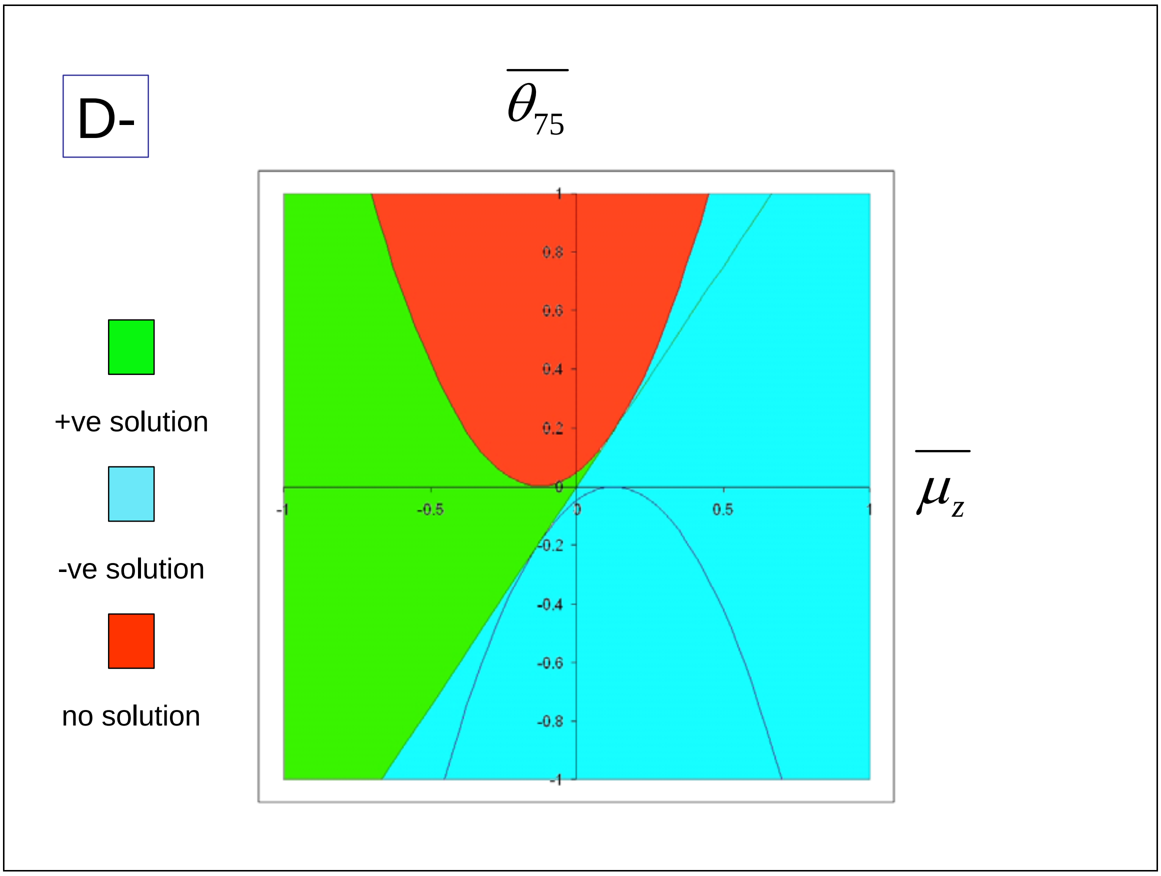

Figure 4 – D- Solution Space

Combined Solution

It is now possible to determine a calculation strategy permitting climb and descent with positive & negative pitch angles.

The following procedure is proposed:

The C+ & D- solutions form the basis. *(The C- & D+ solutions are not appropriate.)* These are shown together in Figure 5:

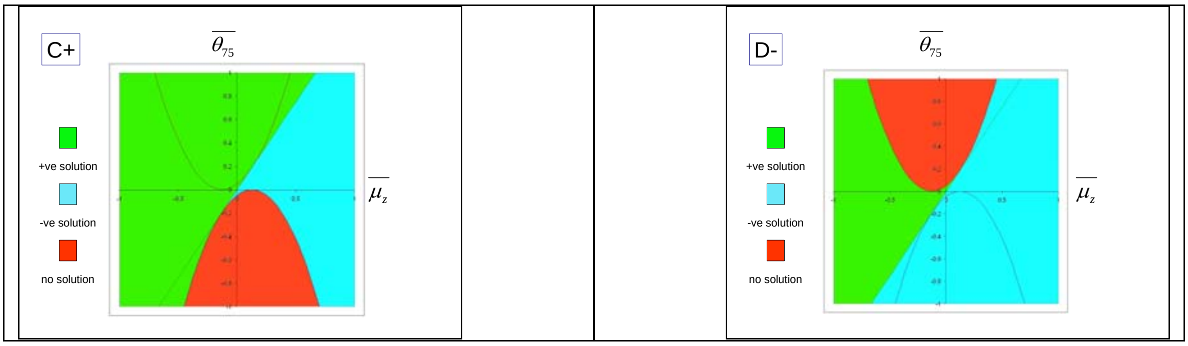

Figure 5 – Comparison of C+ & D- Solution Spaces

for any point in the domain of $\bar{\mu}_z$ & $\bar{\theta}_{75}$ $\bar{\theta}_{75}$ a choice of C+ or D- solution is required.

The tables show the choice order.

Select 1st Choice if it exists
Otherwise
Select 2nd Choice

The choices are as follows:

θ_{75}	μ_z	1st Choice	2nd Choice
=>0	=>0	C+	C+
	<0	D-	C+
<0	=>0	C+	D-
	<0	D-	D-

This can be simplified to:

μ_z	1st Choice	2nd Choice
=>0	C+	D-
<0	D-	C+

Extension to Annulus Theory

The thrust over an elemental annulus is given by BET:

$$dT = \frac{1}{2}\rho(\Omega r)^2 \cdot Ncdr \cdot a\left(\theta - \frac{\mu_z + \lambda_i}{x}\right) \tag{42.}$$

Momentum Theory - Climb

from momentum theory over the annulus:

$$dT = \rho\left(V_C + V_i\right)\cdot 2\pi r dr \cdot 2V_i \tag{43.}$$

Equating (42) & (43) gives:

$$\left(V_C + V_i\right)V_i = \frac{1}{8}\frac{Nc}{\pi}\cdot\left(\frac{{V_T}^2 x}{R}\right)\cdot a\left(\theta - \frac{\mu_z + \lambda_i}{x}\right) \tag{44.}$$

From which we have:

$$\left(\mu_z + \lambda_i\right)\lambda_i = \frac{sa}{8}\left(\theta x - \mu_z - \lambda_i\right) \tag{45.}$$

Hence:

$$\lambda_i^2 + \lambda_i\left(\mu_z + \frac{sa}{8}\right) - \frac{sa}{8}\left(\theta x - \mu_z\right) = 0 \tag{46.}$$

Normalizing on sa gives finally:

$$\overline{\lambda}_i^2 + \overline{\lambda}_i\left(\overline{\mu}_z + \frac{1}{8}\right) - \frac{1}{8}\left(\overline{\theta}x - \overline{\mu}_z\right) = 0 \quad (47.)$$

Momentum Theory - Descent

The thrust over an elemental annulus is given by BET:

$$dT = \rho\left(V_D - V_i\right) \cdot 2\pi r dr \cdot 2V_i \quad (48.)$$

Noting that:

$$V_D = -V_C \quad (49.)$$

from momentum theory over the annulus:

$$dT = -\rho\left(V_C + V_i\right) \cdot 2\pi r dr \cdot 2V_i \quad (50.)$$

Equating gives:

$$\left(V_C + V_i\right)V_i = -\frac{1}{8}\frac{Nc}{\pi} \cdot \left(\frac{V_T^2 x}{R}\right) \cdot a\left(\theta - \frac{\mu_z + \lambda_i}{x}\right) \quad (51.)$$

From which we have:

$$\left(\mu_z + \lambda_i\right)\lambda_i = -\frac{sa}{8}\left(\theta x - \mu_z - \lambda_i\right) \tag{52.}$$

Hence:

$$\lambda_i^2 + \lambda_i\left(\mu_z - \frac{sa}{8}\right) + \frac{sa}{8}\left(\theta x - \mu_z\right) = 0 \tag{53.}$$

Normalizing on sa gives finally:

$$\overline{\lambda}_i^2 + \overline{\lambda}_i\left(\overline{\mu}_z - \frac{1}{8}\right) + \frac{1}{8}\left(\overline{\theta} x - \overline{\mu}_z\right) = 0 \tag{54.}$$

Comparison of Actuator Disc Theory (ADT) & Annulus Theory (AT)

The quadratic equations for ADT & AT are presented below:

Climb

ADT

$$\overline{\lambda}_i^2 + \overline{\lambda}_i\left\{\overline{\mu}_z + \frac{1}{8}\right\} - \frac{1}{4}\left\{\frac{\overline{\theta}_{75}}{3} - \frac{\overline{\mu}_z}{2}\right\} = 0 \tag{55.}$$

AT

$$\bar{\lambda}_i^2 + \bar{\lambda}_i\left(\bar{\mu}_z + \frac{1}{8}\right) - \frac{1}{8}\left(\bar{\theta}x - \bar{\mu}_z\right) = 0 \qquad (56.)$$

Descent

ADT

$$\bar{\lambda}_i^2 + \bar{\lambda}_i\left\{\bar{\mu}_z - \frac{1}{8}\right\} + \frac{1}{4}\left\{\frac{\bar{\theta}_{75}}{3} - \frac{\bar{\mu}_z}{2}\right\} = 0 \qquad (57.)$$

AT

$$\bar{\lambda}_i^2 + \bar{\lambda}_i\left(\bar{\mu}_z - \frac{1}{8}\right) + \frac{1}{8}\left(\bar{\theta}x - \bar{\mu}_z\right) = 0 \qquad (58.)$$

Inspection of these equation pairs shows the following equivalence between the local pitch angle (AT) and the pitch at 75% radius (ADT):

$$\bar{\theta}x = \frac{2}{3}\cdot\bar{\theta}_{75} \qquad (59.)$$

In other words, the AT solution can be achieved using the same choice of solution developed for the ADT.

The thrust variation can then be evaluated by:

$$T = \frac{1}{2}\rho V_T^2 \cdot NcR \cdot a \int_0^1 x^2 \left(\theta - \frac{\mu_z + \lambda_i}{x} \right) \cdot dx \qquad (60.)$$

whence:

$$\frac{C_T}{sa} = \int_0^1 x^2 \left(\theta - \frac{\mu_z + \lambda_i}{x} \right) \cdot dx \qquad (61.)$$

where finally:

$$\begin{aligned} \overline{C}_T &= \frac{C_T}{(sa)^2} \\ &= \int_0^1 x^2 \left(\overline{\theta} - \frac{\overline{\mu}_z + \overline{\lambda}_i}{x} \right) \cdot dx \end{aligned} \qquad (62.)$$