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The Induced Velocity of a Vortex Ring Filament

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Technical Report AFM-11/03

January 2011

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Introduction

This report derives the induced velocity of a vortex ring filament using the Biot Savart Law.

Biot Savart Law

The basic formula for evaluating the induced velocity of a vortex ring is the Biot-Savart law.

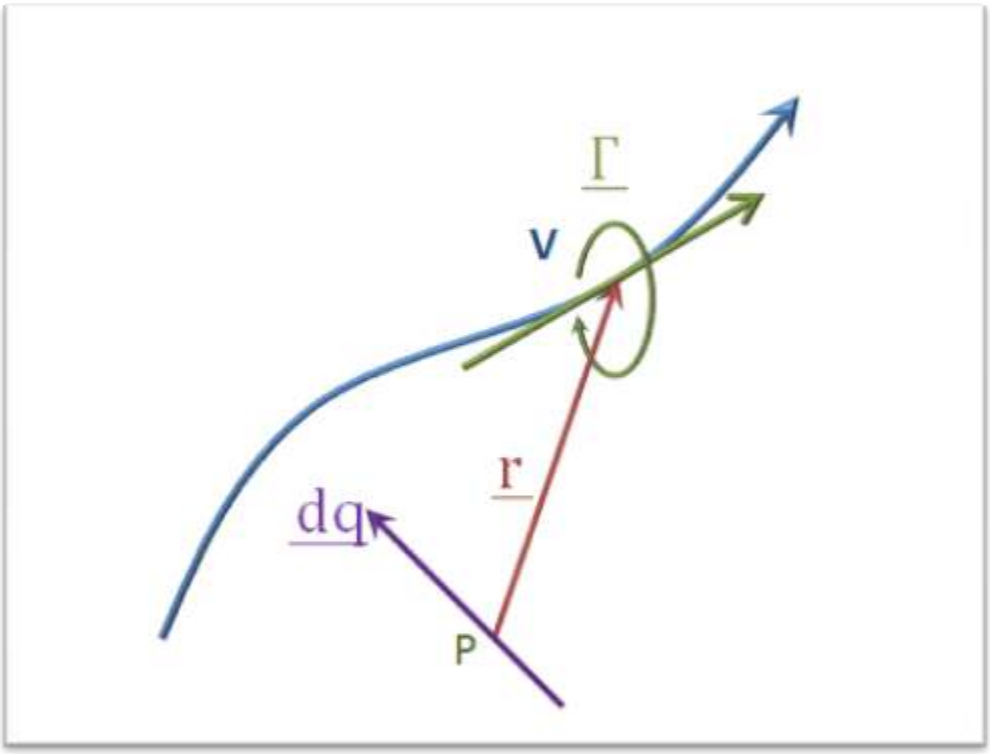

Figure 1

Figure 1 shows the geometry, the vortex element is positioned at V and the vortex strength is $\underline{\Gamma}$. The induced velocity at point P is dq, which is given by:

$$\underline{q} = \frac{\underline{r} \wedge \underline{\Gamma}}{4\pi r^3} \quad (1.)$$

Induced Velocity of a Vortex Ring

The vortex ring has radius R, is placed in the XOY plane with the centre at the origin, O. The control point, P, without loss of generality, can be considered as lying in the XOZ plane with coordinates (r,0,h). In order to evaluate the overall induced velocity, we need to consider a vortex element at azimuth, ψ.

This is shown in Figure 2.

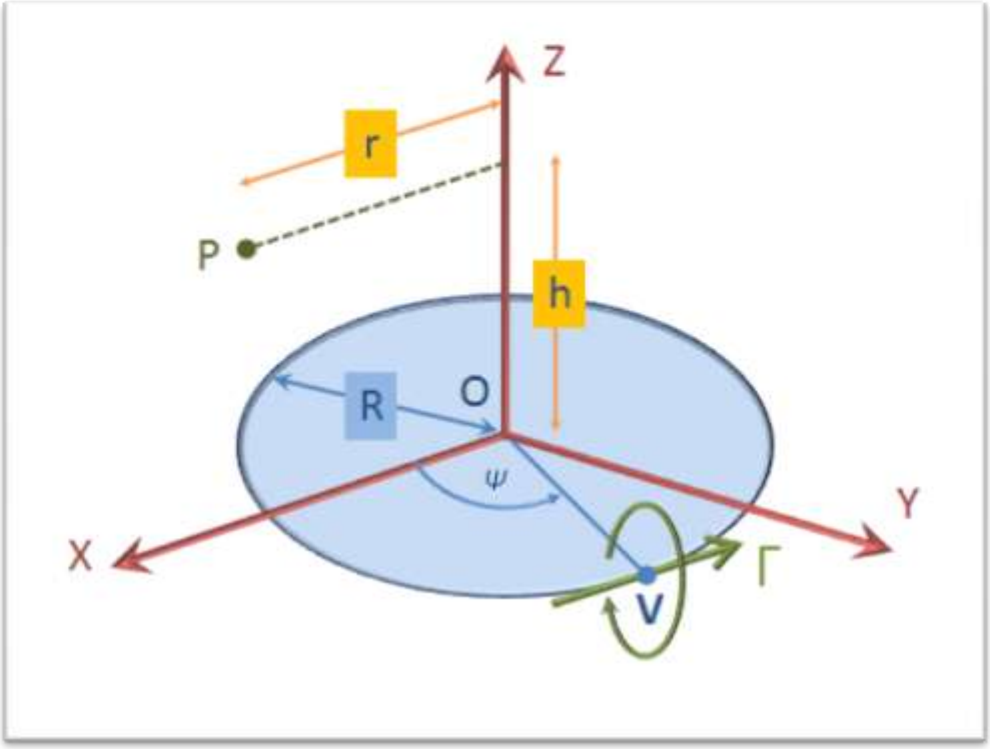

Figure 2

The vortex filament is given by:

$$\underline{\Gamma} = \Gamma(-\sin\psi, \cos\psi, 0)Rd\psi \tag{2.}$$

The vector of the vortex filament location is:

$$\underline{V} = R(\cos\psi, \sin\psi, 0) \tag{3.}$$

The vector of the control point location is:

$$\underline{P} = (r, 0, h) = R\left(x, 0, \bar{h}\right) \tag{4.}$$

The vector representing the vortex relative to the control point is given by:

$$\underline{r} = \underline{V} - \underline{P} = R\left(\cos\psi - x, \sin\psi, -\bar{h}\right) \tag{5.}$$

Invoking the Biot Savart law, the induced velocity at P due to the vortex filament is given by:

$$\underline{dq} = \frac{\Gamma R}{4\pi r^3}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\psi & \sin\psi & -\bar{h} \\ -\sin\psi & \cos\psi & 0 \end{vmatrix} R d\psi \tag{6.}$$

which becomes:

$$\underline{dq} = \frac{\Gamma R}{4\pi r^3}\left(\bar{h}\cos\psi, -\bar{h}\sin\psi, 1 - x\cos\psi\right) R d\psi \tag{7.}$$

The distance between the vortex filament and the control point is given by:

$$\begin{aligned} r^2 &= R^2\left\{(\cos\psi - x)^2 + (\sin\psi)^2 + \left(-\bar{h}\right)^2\right\} \\ &= R^2\left\{1 + x^2 + \bar{h}^2 - 2x\cos\psi\right\} \end{aligned} \tag{8.}$$

Therefore, the overall induced velocity due to the complete ring is:

$$\underline{q} = \frac{\Gamma}{4\pi R}\int_{-\pi}^{\pi} \frac{\left(\bar{h}\cos\psi, -\bar{h}\sin\psi, 1 - x\cos\psi\right)}{\left\{1 + x^2 + \bar{h}^2 - 2x\cos\psi\right\}^{3/2}} d\psi \tag{9.}$$

Because the integrand consists of even functions, except the sine, the following conclusions can be made, namely:

$$q_y = 0 \tag{10.}$$

and

$$(q_x, q_z) = \frac{\Gamma}{4\pi R} \int_{-\pi}^{\pi} \frac{\left(\bar{h}\cos\psi, 1 - x\cos\psi\right)}{\left\{1 + x^2 + \bar{h}^2 - 2x\cos\psi\right\}^{3/2}} d\psi \qquad (11.)$$

Which becomes:

$$(q_x, q_z) = \frac{\Gamma}{2\pi R} \int_{0}^{\pi} \frac{\left(\bar{h}\cos\psi, 1 - x\cos\psi\right)}{\left\{1 + x^2 + \bar{h}^2 - 2x\cos\psi\right\}^{3/2}} d\psi \qquad (12.)$$

This integral can be evaluated using substitutions and standard results. The first substitution is:

$$\psi = 2\theta$$
$$d\psi = 2d\theta \qquad (13.)$$
$$\cos\psi = 2\cos^2\theta - 1$$

Form which (12) becomes:

$$(q_x, q_z) = \frac{\Gamma}{\pi R} \int_{0}^{\frac{\pi}{2}} \frac{\left(\bar{h}(2\,cos^2\theta - 1), 1 - 2x\,cos^2\theta + x\right)}{\left\{1 + x^2 + \bar{h}^2 + 2x - 4x\cos^2\theta\right\}^{3/2}} d\theta \qquad (14.)$$

The second substitution is given by:

$$\theta = \frac{\pi}{2} - \phi$$
$$\cos\theta = \sin\phi \qquad (15.)$$
$$d\theta = -d\phi$$

From which we obtain:

$$(q_x, q_z) = \frac{\Gamma}{\pi R} \int_0^{\frac{\pi}{2}} \frac{\left(\bar{h}(2\sin^2\phi - 1), 1 - 2x\sin^2\phi + x\right)}{\left\{1 + x^2 + \bar{h}^2 + 2x - 4x\sin^2\phi\right\}^{3/2}} d\phi \qquad (16.)$$

Which can be rearranged to:

$$(q_x, q_z) = \frac{\Gamma}{\pi R} \int_0^{\frac{\pi}{2}} \frac{\left(\bar{h}(2\sin^2\phi - 1), 1 + x - 2x\sin^2\phi\right)}{\left\{(1 + x)^2 + \bar{h}^2 - 4x\sin^2\phi\right\}^{3/2}} d\phi \qquad (17.)$$

(17) can be simplified to:

$$(q_x, q_z) = \frac{\Gamma}{\pi R\left\{(1 + x)^2 + \bar{h}^2\right\}^{3/2}} \int_0^{\frac{\pi}{2}} \frac{\left(\bar{h}(2\sin^2\phi - 1), 1 + x - 2x\sin^2\phi\right)}{\{1 - k^2\sin^2\phi\}^{3/2}} d\phi \qquad (18.)$$

Where:

$$k^2 = \frac{4x}{(1 + x)^2 + \bar{h}^2} \qquad (19.)$$

This will become the modulus of elliptic integrals.

Elliptic Integrals

The standard elliptic integrals are given by:

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$
$$E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi \tag{20.}$$
$$D = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \phi\, d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{K - E}{k^2}$$

The complementary modulus is defined by:

$$k'^2 = 1 - k^2 \tag{21.}$$

The following results will be used in the solution:

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\{1 - k^2 \sin^2 \phi\}^{\frac{3}{2}}} = \frac{E}{k'^2}$$
$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 \phi\, d\phi}{\{1 - k^2 \sin^2 \phi\}^{\frac{3}{2}}} = \frac{(K - D)}{k'^2} \tag{22.}$$

Final Solution

The above results mean that the induced velocity can be expressed in terms of elliptic integrals, namely:

$$\begin{bmatrix} q_x \\ q_z \end{bmatrix} = \frac{\Gamma}{\pi R \cdot k'^2 \left\{(1+x)^2 + \bar{h}^2\right\}^{3/2}} \begin{bmatrix} \bar{h}(2(K-D)-E) \\ (1+x)E - 2x(K-D) \end{bmatrix} \tag{23.}$$

Which with some rearrangement becomes:

$$\begin{bmatrix} q_x \\ q_z \end{bmatrix} = \frac{\Gamma}{\pi R \cdot k'^2 \left\{(1+x)^2 + \bar{h}^2\right\}^{3/2}} \begin{bmatrix} \bar{h}(2K-E-2D) \\ E - x(2K-E-2D) \end{bmatrix} \tag{24.}$$

Generalisation

The above analysis can be extended to a general problem by noting that the control point and vortex ring centre will define the XOZ plane. The two velocity components, q_x & q_z, can then be obtained using the above results. These can then be expressed in terms of an arbitrary axes system by resolution.

Special Case – q_z at Ring Centre

To scale the circulation, we can use the velocity at the ring centre normal to the ring plane. If we denote this to q_{z0}, we have the following simplifications:

$$k = 0 \qquad (25.)$$

From which the elliptic integrals become:

$$\begin{aligned} K &= \frac{\pi}{2} \\ E &= \frac{\pi}{2} \\ D &= \frac{\pi}{4} \end{aligned} \qquad (26.)$$

Whence the centre velocity becomes:

$$q_{z0} = \frac{\Gamma}{\pi R} \cdot \frac{\pi}{2} = \frac{\Gamma}{2R} \qquad (27.)$$

Which simplifies to:

$$q_{z0} = \frac{\Gamma}{2R} \qquad (28.)$$

From which the induced velocity of the vortex ring in terms of the centre velocity is goven by:

$$\begin{bmatrix} q_x \\ q_z \end{bmatrix} = q_{z0} \frac{2}{\pi \cdot k'^2 \left\{(1+x)^2 + \bar{h}^2\right\}^{3/2}} \begin{bmatrix} \bar{h}(2K - E - 2D) \\ E - x(2K - E - 2D) \end{bmatrix} \qquad (29.)$$