

Aerodynamics & Flight Mechanics Research Group

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Technical Report AFM-11/03

January 2011

UNIVERSITY OF SOUTHAMPTON

SCHOOL OF ENGINEERING SCIENCES

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Introduction

This report derives the induced velocity of a vortex ring filament using the Biot Savart Law.

Biot Savart Law

The basic formula for evaluating the induced velocity of a vortex ring is the Biot-Savart law.

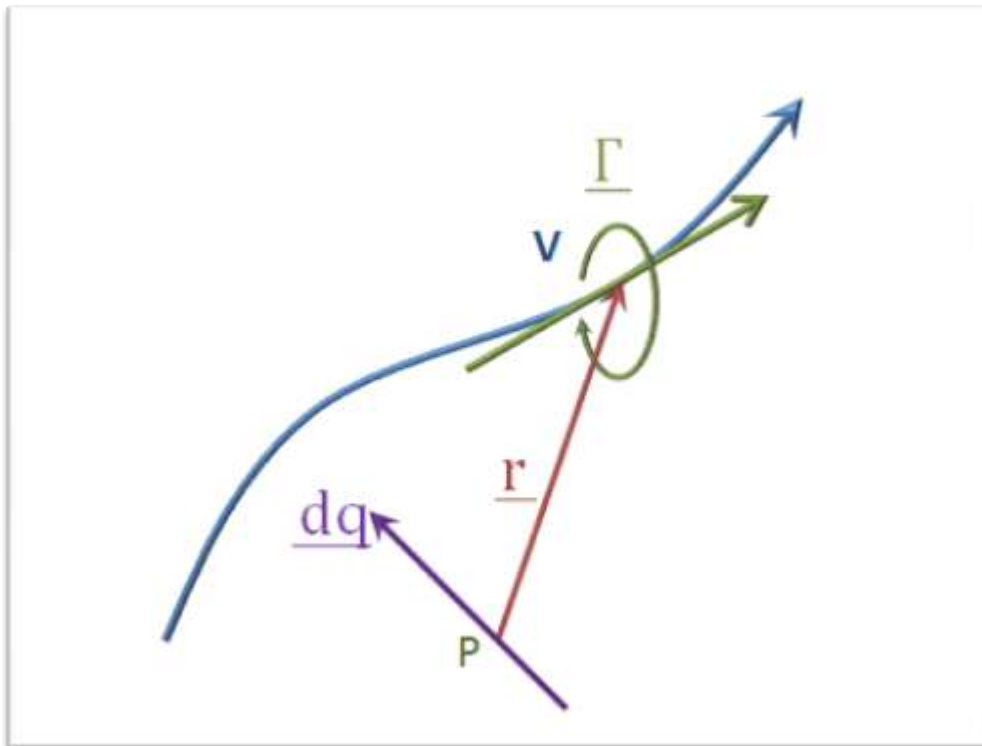


Figure 1

Figure 1 shows the geometry, the vortex element is positioned at V and the vortex strength is $\underline{\Gamma}$. The induced velocity at point P is \underline{dq} , which is given by:

$$\underline{q} = \frac{\underline{r} \wedge \underline{\Gamma}}{4\pi r^3} \quad (1.)$$



Induced Velocity of a Vortex Ring

The vortex ring has radius R , is placed in the XOY plane with the centre at the origin, O . The control point, P , without loss of generality, can be considered as lying in the XOZ plane with coordinates $(r,0,h)$. In order to evaluate the overall induced velocity, we need to consider a vortex element at azimuth, ψ .

This is shown in Figure 2.

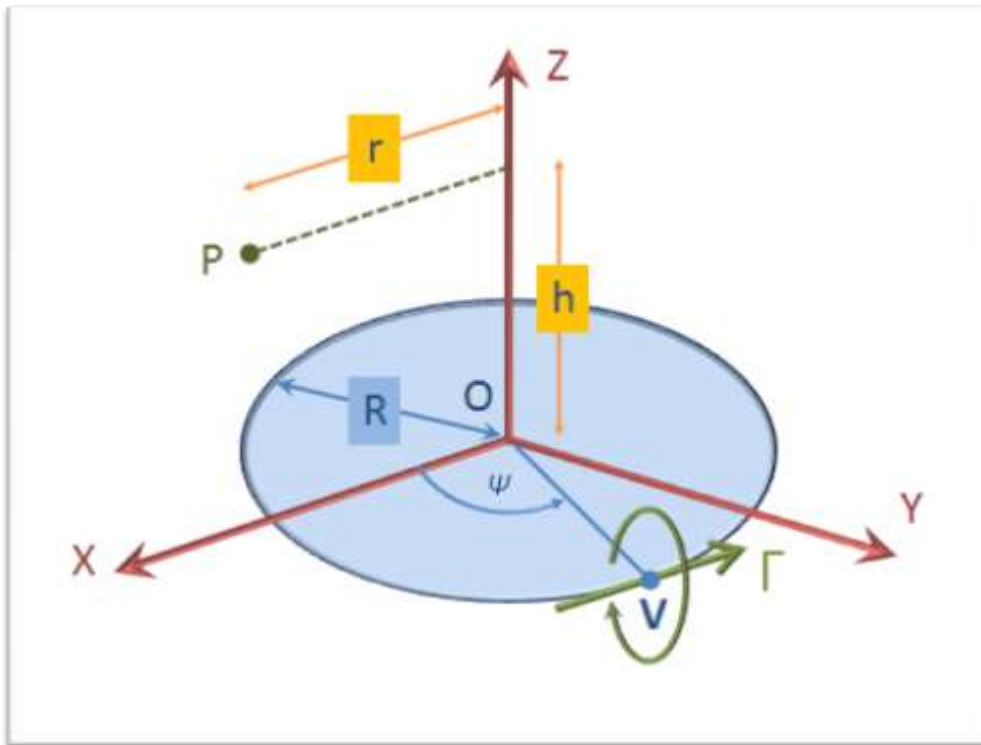


Figure 2

The vortex filament is given by:

$$\underline{\underline{\Gamma}} = \Gamma(-\sin \psi, \cos \psi, 0)Rd\psi \quad (2.)$$

The vector of the vortex filament location is:

$$\underline{\underline{V}} = R(\cos \psi, \sin \psi, 0) \quad (3.)$$



The vector of the control point location is:

$$\underline{P} = (r, 0, h) = R(x, 0, \bar{h}) \quad (4.)$$

The vector representing the vortex relative to the control point is given by:

$$\underline{r} = \underline{V} - \underline{P} = R(\cos \psi - x, \sin \psi, -\bar{h}) \quad (5.)$$

Invoking the Biot Savart law, the induced velocity at P due to the vortex filament is given by:

$$\underline{dq} = \frac{\Gamma R}{4\pi r^3} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \psi & \sin \psi & -\bar{h} \\ -\sin \psi & \cos \psi & 0 \end{vmatrix} R d\psi \quad (6.)$$

which becomes:

$$\underline{dq} = \frac{\Gamma R}{4\pi r^3} (\bar{h} \cos \psi, -\bar{h} \sin \psi, 1 - x \cos \psi) R d\psi \quad (7.)$$

The distance between the vortex filament and the control point is given by:

$$\begin{aligned} r^2 &= R^2 \{(\cos \psi - x)^2 + (\sin \psi)^2 + (-\bar{h})^2\} \\ &= R^2 \{1 + x^2 + \bar{h}^2 - 2x \cos \psi\} \end{aligned} \quad (8.)$$

Therefore, the overall induced velocity due to the complete ring is:

$$\underline{q} = \frac{\Gamma}{4\pi R} \int_{-\pi}^{\pi} \frac{(\bar{h} \cos \psi, -\bar{h} \sin \psi, 1 - x \cos \psi)}{\{1 + x^2 + \bar{h}^2 - 2x \cos \psi\}^{3/2}} d\psi \quad (9.)$$

Because the integrand consists of even functions, except the sine, the following conclusions can be made, namely:

$$q_y = 0 \quad (10.)$$



and

$$(q_x, q_z) = \frac{\Gamma}{4\pi R} \int_{-\pi}^{\pi} \frac{(\bar{h} \cos \psi, 1 - x \cos \psi)}{\{1 + x^2 + \bar{h}^2 - 2x \cos \psi\}^{3/2}} d\psi \quad (11.)$$

Which becomes:

$$(q_x, q_z) = \frac{\Gamma}{2\pi R} \int_0^{\pi} \frac{(\bar{h} \cos \psi, 1 - x \cos \psi)}{\{1 + x^2 + \bar{h}^2 - 2x \cos \psi\}^{3/2}} d\psi \quad (12.)$$

This integral can be evaluated using substitutions and standard results. The first substitution is:

$$\psi = 2\theta$$

$$d\psi = 2d\theta \quad (13.)$$

$$\cos \psi = 2 \cos^2 \theta - 1$$

Form which (12) becomes:

$$(q_x, q_z) = \frac{\Gamma}{\pi R} \int_0^{\frac{\pi}{2}} \frac{(\bar{h}(2 \cos^2 \theta - 1), 1 - 2x \cos^2 \theta + x)}{\{1 + x^2 + \bar{h}^2 + 2x - 4x \cos^2 \theta\}^{3/2}} d\theta \quad (14.)$$



The second substitution is given by:

$$\begin{aligned}\theta &= \frac{\pi}{2} - \phi \\ \cos \theta &= \sin \phi \\ d\theta &= -d\phi\end{aligned}\tag{15.}$$

From which we obtain:

$$(q_x, q_z) = \frac{\Gamma}{\pi R} \int_0^{\frac{\pi}{2}} \frac{(\bar{h}(2 \sin^2 \phi - 1), 1 - 2x \sin^2 \phi + x)}{\{1 + x^2 + \bar{h}^2 + 2x - 4x \sin^2 \phi\}^{3/2}} d\phi\tag{16.}$$

Which can be rearranged to:

$$(q_x, q_z) = \frac{\Gamma}{\pi R} \int_0^{\frac{\pi}{2}} \frac{(\bar{h}(2 \sin^2 \phi - 1), 1 + x - 2x \sin^2 \phi)}{\{(1 + x)^2 + \bar{h}^2 - 4x \sin^2 \phi\}^{3/2}} d\phi\tag{17.}$$

(17) can be simplified to:

$$\begin{aligned}(q_x, q_z) \\ &= \frac{\Gamma}{\pi R \{(1 + x)^2 + \bar{h}^2\}^{3/2}} \int_0^{\frac{\pi}{2}} \frac{(\bar{h}(2 \sin^2 \phi - 1), 1 + x - 2x \sin^2 \phi)}{\{1 - k^2 \sin^2 \phi\}^{3/2}} d\phi\end{aligned}\tag{18.}$$

Where:

$$k^2 = \frac{4x}{(1 + x)^2 + \bar{h}^2}\tag{19.}$$



This will become the modulus of elliptic integrals.

Elliptic Integrals

The standard elliptic integrals are given by:

$$\begin{aligned}
 K &= \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \\
 E &= \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \phi} \, d\phi \\
 D &= \int_0^{\frac{\pi}{2}} \frac{\sin^2 \phi \, d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{K - E}{k^2}
 \end{aligned}
 \tag{20.}$$

The complementary modulus is defined by:

$$k'^2 = 1 - k^2 \tag{21.}$$

The following results will be used in the solution:

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{d\phi}{\{1 - k^2 \sin^2 \phi\}^{\frac{3}{2}}} &= \frac{E}{k'^2} \\
 \int_0^{\frac{\pi}{2}} \frac{\sin^2 \phi \, d\phi}{\{1 - k^2 \sin^2 \phi\}^{\frac{3}{2}}} &= \frac{(K - D)}{k'^2}
 \end{aligned}
 \tag{22.}$$



Final Solution

The above results mean that the induced velocity can be expressed in terms of elliptic integrals, namely:

$$\begin{bmatrix} q_x \\ q_z \end{bmatrix} = \frac{\Gamma}{\pi R \cdot k'^2 \left\{ (1+x)^2 + \bar{h}^2 \right\}^{3/2}} \begin{bmatrix} \bar{h}(2(K-D) - E) \\ (1+x)E - 2x(K-D) \end{bmatrix} \quad (23.)$$

Which with some rearrangement becomes:

$$\begin{bmatrix} q_x \\ q_z \end{bmatrix} = \frac{\Gamma}{\pi R \cdot k'^2 \left\{ (1+x)^2 + \bar{h}^2 \right\}^{3/2}} \begin{bmatrix} \bar{h}(2K - E - 2D) \\ E - x(2K - E - 2D) \end{bmatrix} \quad (24.)$$

Generalisation

The above analysis can be extended to a general problem by noting that the control point and vortex ring centre will define the XOZ plane. The two velocity components, q_x & q_z , can then be obtained using the above results. These can then be expressed in terms of an arbitrary axes system by resolution.



Special Case – q_z at Ring Centre

To scale the circulation, we can use the velocity at the ring centre normal to the ring plane. If we denote this to q_{z0} , we have the following simplifications:

$$k = 0 \quad (25.)$$

From which the elliptic integrals become:

$$\begin{aligned} K &= \frac{\pi}{2} \\ E &= \frac{\pi}{2} \\ D &= \frac{\pi}{4} \end{aligned} \quad (26.)$$

Whence the centre velocity becomes:

$$q_{z0} = \frac{\Gamma}{\pi R} \cdot \frac{\pi}{2} = \frac{\Gamma}{2R} \quad (27.)$$

Which simplifies to:

$$q_{z0} = \frac{\Gamma}{2R} \quad (28.)$$

From which the induced velocity of the vortex ring in terms of the centre velocity is given by:

$$\begin{bmatrix} q_x \\ q_z \end{bmatrix} = q_{z0} \frac{2}{\pi \cdot k'^2 \left\{ (1+x)^2 + \bar{h}^{-2} \right\}^{3/2}} \begin{bmatrix} \bar{h}(2K - E - 2D) \\ E - x(2K - E - 2D) \end{bmatrix} \quad (29.)$$



