Aerodynamics & Flight Mechanics
Research Group

Unsteady Aerodynamics with Exponential Lift Decrement

S. J. Newman

Technical Report AFM-11/08

January 2011
UNIVERSITY OF SOUTHAMPTON
SCHOOL OF ENGINEERING SCIENCES
AERODYNAMICS AND FLIGHT MECHANICS RESEARCH GROUP

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AFM Report No. AFM 11/08

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Preamble

This document examines the application of an unsteady aerodynamic model to a sinusoidal variation in pitch angle. It is based on the concept of the Wagner or Kussner lift variations. Both of these models produce approximations consisting of a combination of unity and two exponential decays. To simplify the introductory analysis, only one exponential decay term is retained.

Nomenclature

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<td>Incremental Lift Coefficient (Unsteady)</td>
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<td>Lift Decrement</td>
</tr>
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<td>Wing Chord</td>
</tr>
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<tr>
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Basic Analysis

The Wagner lift variation is given by:

\[ \Delta L = 2\pi \rho U^2 b \cdot \Delta \alpha \cdot \phi(s) \]  \hspace{1cm} (1.)

The Wagner function, \( \phi \), provides the unsteady lift variation using the reduced frequency as the independent variable.

The reduced frequency is defined by:

\[ s = \frac{Ut}{b} \]  \hspace{1cm} (2.)

Here, the term \( b \) is the semi chord and the reduced frequency represents the wing movement in terms of semi-chord – or alternatively, the increase in the streamwise extension of the wake.

Equation (1) can be converted to a lift coefficient thus:

\[ \Delta C_L = \frac{\Delta L}{\frac{1}{2} \rho U^2 c} = \frac{2\pi \rho U^2 b \alpha \phi(s)}{\frac{1}{2} \rho U^2 c} = 2\pi \alpha \phi(s) \]  \hspace{1cm} (3.)

Whence the ratio of the Unsteady lift coefficient change to that of the Quasi Steady result is given by:

\[ \frac{\Delta C_L_{US}}{\Delta C_L_{QS}} = \phi(s) \]  \hspace{1cm} (4.)

The Wagner function can be considered to consist of the quasi-steady response of unity and a decrement function, \( d \):
We use a simplified version of the conventional Wagner function, namely an single exponential decay term:

\[ d(s) = Ae^{-\lambda s} \]  

(6.)

The shape of this function is shown in Figure 1.

The response, \( R(t) \), to a forcing function \( f(t) \), with an indicial response \( g(t) \) is given by the Duhammel Superposition Integral:

\[ R(t) = f(0) \cdot g(t) + \int_{0}^{t} \frac{df(\tau)}{d\tau} \cdot g(t - \tau) d\tau \]  

(7.)

In our case, the function \( g \) is equal to \( \phi \), hence using (5) & (6), (7) becomes:
\[ R = f(0) \left[ 1 - d \left( \frac{Ut}{b} \right) \right] \]

\[ + \int_0^t f'(\tau) \cdot \left[ 1 - d \left( \frac{U(t - \tau)}{b} \right) \right] d\tau \]

Which can be simplified by integrating by parts:

\[ R = f(t) - f(0) d \left( \frac{Ut}{b} \right) \]

\[ - \int_0^t f'(\tau) \cdot d \left( \frac{U(t - \tau)}{b} \right) d\tau \]

In our case, the decrement function is given by:

\[ d(t) = Ae^{-\frac{\lambda U}{b}t} \]
Whence, (9) becomes:

\[ R = f(t) - f(0)Ae^{\frac{\lambda U}{b}t} - \int_{0}^{t} f'(\tau) \cdot Ae^{\frac{\lambda U}{b}(t-\tau)} d\tau \]  

(11.)

We therefore have the lift coefficient ratio as:

\[ \frac{C_L}{C_{L QS}} = f(t) - f(0)Ae^{\frac{\lambda U}{b}t} - Ae^{\frac{\lambda Ut}{b}} \int_{0}^{t} f'(\tau) \cdot e^{\frac{\lambda Ut}{b}} d\tau \]  

(12.)

If the forcing function, f, satisfies:

\[ f(0) = 0 \]  

(13.)

Equation (12) simplifies to:

\[ \frac{C_L}{C_{L QS}} = f(t) - Ae^{\frac{\lambda Ut}{b}} \int_{0}^{t} f'(\tau) \cdot e^{\frac{\lambda Ut}{b}} d\tau \]  

(14.)
Particular Forcing Function – Sinusoidal Ramp

The function for this is piecewise and defined thus:

\[ f(t) = \begin{cases} 
\theta \cdot \sin^2 \left( \frac{\pi t}{2T} \right) ; & 0 \leq t \leq T \\
\theta ; & t > T 
\end{cases} \]  \hspace{1cm} (15.)

This satisfies the condition:

\[ f(0) = 0 \]  \hspace{1cm} (16.)

The profile of this function is shown in Figure 2:

![Figure 2](image_url)

The derivative of this function with respect to \( t \) is given by:

\[ f'(t) = \begin{cases} 
\frac{\pi \theta}{2T} \cdot \sin \left( \frac{\pi t}{T} \right) ; & 0 \leq t \leq T \\
0 ; & t > T 
\end{cases} \]  \hspace{1cm} (17.)
If we define two parameters thus:

\[
\alpha = \frac{U\lambda}{b} \\
\beta = \frac{\pi}{T}
\]  \hspace{1cm} (18.)

Then a typical integral is:

\[
\int_{0}^{t} e^{\alpha \tau} \cdot \sin \beta \tau \cdot d\tau = \beta + e^{\alpha t} \frac{(\alpha \sin \beta t - \beta \cos \beta t)}{\alpha^2 + \beta^2} 
\]  \hspace{1cm} (19.)

In the case of the response after the forcing function has achieved its final value, the integral of (19) is the special case of:

\[
\int_{0}^{T} e^{\alpha \tau} \cdot \sin \beta \tau \cdot d\tau = \beta + e^{\alpha T} \frac{(\alpha \sin \beta T - \beta \cos \beta T)}{\alpha^2 + \beta^2} 
\]  \hspace{1cm} (20.)

Noting that:
\[ \beta T = \pi \]
\[ \sin \beta T = 0 \]
\[ \cos \beta T = -1 \]  

(21.)

Equation (20) simplifies to:

\[ \int_{0}^{T} e^{\alpha \tau} \cdot \sin \beta \tau \cdot d\tau = \frac{\beta (1 + e^{\alpha T})}{\alpha^2 + \beta^2} \]  

(22.)

From the above results, the final equation for the lift coefficient ratio is given by:

\[ \frac{C_L}{C_{L_{QS}}} = \begin{cases} \frac{\pi \theta}{2T} \cdot \sin \left(\frac{\pi t}{T}\right) - \frac{A \beta \theta}{2} \left[ \frac{\beta e^{-\alpha t} + \alpha \sin \beta t - \beta \cos \beta t}{\alpha^2 + \beta^2} \right] & ; 0 \leq t \leq T \\ \frac{\pi \theta}{2T} \cdot \sin \left(\frac{\pi t}{T}\right) - \frac{A \beta^2 \theta}{2} \left[ \frac{e^{-\alpha t} (1 + e^{\alpha T})}{\alpha^2 + \beta^2} \right] & ; t > T \end{cases} \]  

(23.)
Results

The analysis was used to evaluate the lift variation for a sinusoidal-squared input.

The incidence variation is shown in Figure 3:

![Pitch Angle Variation](image1)

**Figure 3 – Pitch Angle Variation**

The Kussner function is shown in Figure 4:
The resulting lift variation ratio is shown in Figure 5:
Figure 5 - Unsteady and Steady Lift Variation
% Unsteady Aerodynamics - Exponential Lift Decrement
% SJN 9/3/08

clear
colordef white

lambda = .1;
A = 1;
U = 100;
c = 1;
dclda = 5.8;

thetadeg = 10;
T = .5;
tmax = 1;
nt = 101;

b = c/2;
theta_max = thetadeg*pi/180;
alf = U*lambda/b;
bet = pi/T;
den = alf^2 + bet^2;

t = linspace(0, tmax, nt);
clqs = zeros(1, nt);
clus = clqs;

thetdeg = clqs;

for i = 1:nt
    if t(i) < T
        f = theta_max*sin(bet*t(i)/2)^2;
        num = (bet*theta_max/2)*(alf*sin(bet*t(i)) - bet*cos(bet*t(i)) + bet*exp(-alf*t(i)));
    else
        f = theta_max;
        num = (bet^2*theta_max/2)*(exp(-alf*(t(i)-T)) + exp(-alf*t(i)));
    end
    clqs(i) = dclda*f;
    clus(i) = dclda*(f - A*num/den);
    thetdeg(i) = (180/pi)*f;
end

clf
plot(t, thetdeg, 'b', 'LineWidth', 2);
grid on
title(['Pitch Angle Variation $T = ', num2str(T), '$ \theta_{M_A_X} = ', num2str(thetadeg), '$\circ$']);
xlabel('Time');
ylabel('\theta');

figure
clf
plot(t, clqs, 'b', 'LineWidth', 2);
hold on
plot(t,clus,'r','LineWidth',2);
grid on
title(['Lift Coefficient Ratio A = ',num2str(A),', \lambda = ',num2str(lambda),', T = ',num2str(T),', \theta_{M\_A\_X} = ',num2str(thetadeg),'\circ']);
legend('Quasi Steady','Unsteady');
xlabel('Time');
ylabel('\Delta C_{L\_U\_S} / \Delta C_{L\_Q\_S}');

figure
cif
s=linspace(0,50,201);
phi=1-A*exp(-lambda*s);
plot(s,phi,'b','LineWidth',2);
grid on

title(['Indicial Response A = ',num2str(A),', \lambda = ',num2str(lambda)]);
xlabel('s');
ylabel('\phi');