UNIVERSITY OF
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Preamble

This document examines the application of an unsteady aerodynamic model to a sinusoidal variation in pitch angle. It is based on the concept of the Wagner or Kussner lift variations. Both of these models produce approximations consisting of a combination of unity and two exponential decays. To simplify the introductory analysis, only one exponential decay term is retained.

Nomenclature

Variable	Definition
U	Forward Speed
ρ	Air Density
λ	Exponential Decay Factor
$\Delta\alpha$	Incremental Pitch Angle Change
ϕ	Wagner Function
τ	Integrating time variable
ΔL	Incremental Lift Force
ΔC_L	Incremental Lift Coefficient
$\Delta C_{L\ QS}$	Incremental Lift Coefficient *(Quasi-Steady)*
$\Delta C_{L\ US}$	Incremental Lift Coefficient *(Unsteady)*
d	Lift Decrement
c	Wing Chord
b	Wing Semi-Chord
α, β	Dummy Parameters

Basic Analysis

The Wagner lift variation is given by:

$$\Delta L = 2\pi\rho U^2 b \cdot \Delta\alpha \cdot \phi(s) \tag{1.}$$

The Wagner function, ϕ, provides the unsteady lift variation using the reduced frequency as the independent variable.

The reduced frequency is defined by:

$$s = \frac{Ut}{b} \tag{2.}$$

Here, the term b is the semi chord and the reduced frequency represents the wing movement in terms of semi-chord – or alternatively, the increase in the streamwise extension of the wake.

Equation (1) can be converted to a lift coefficient thus:

$$\Delta C_L = \frac{\Delta L}{\frac{1}{2}\rho U^2 c} = \frac{2\pi\rho U^2 b\alpha\phi(s)}{\frac{1}{2}\rho U^2 c} = 2\pi\alpha\phi(s) \tag{3.}$$

Whence the ratio of the *Unsteady* lift coefficient change to that of the *Quasi Steady* result is given by:

$$\frac{\Delta C_{L\,US}}{\Delta C_{L\,QS}} = \phi(s) \tag{4.}$$

The Wagner function can be considered to consist of the quasi-steady response of unity and a decrement function, d:

$$\phi(s) = 1 - d(s) \tag{5.}$$

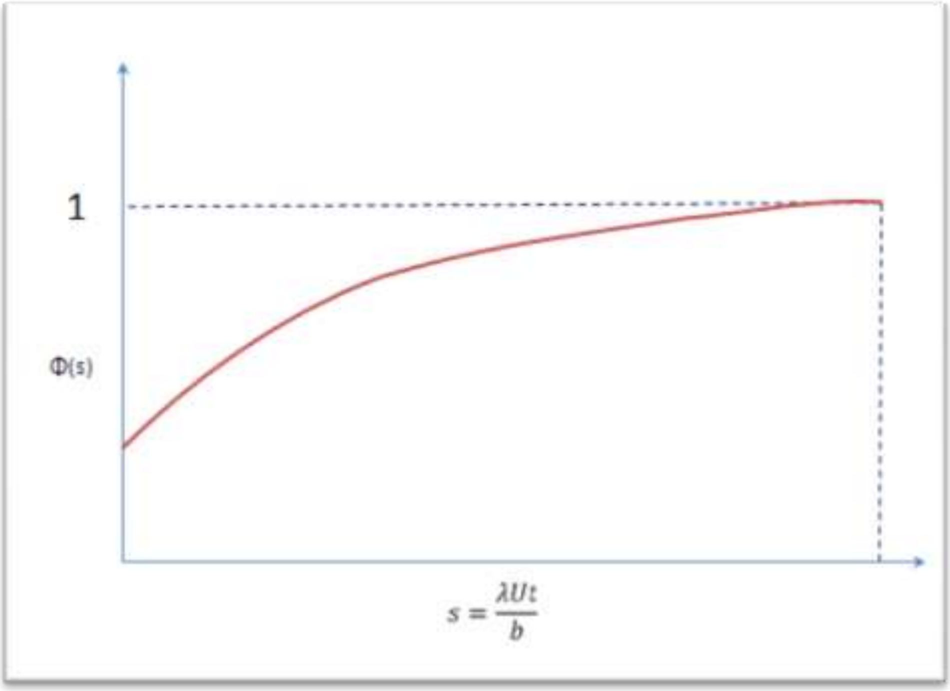

Figure 1

We use a simplified version of the conventional Wagner function, namely an single exponential decay term:

$$d(s) = Ae^{-\lambda s} \tag{6.}$$

The shape of this function is shown in Figure 1.

The response, R(t), to a forcing function f(t), with an indicial response g(t) is given by the Duhammel Superposition Integral:

$$R(t) = f(0) \cdot g(t) + \int_0^t \frac{df(\tau)}{d\tau} \cdot g(t - \tau) d\tau \tag{7.}$$

In our case, the function g is equal to ϕ, hence using (5) & (6), (7) becomes:

$$R = f(0)\left[1 - d\left(\frac{Ut}{b}\right)\right] + \int_0^t f'(\tau)\cdot\left[1 - d\left(\frac{U(t-\tau)}{b}\right)\right] d\tau \tag{8.}$$

Which can be simplified by integrating by parts:

$$R = f(t) - f(0)d\left(\frac{Ut}{b}\right) - \int_0^t f'(\tau)\cdot d\left(\frac{U(t-\tau)}{b}\right) d\tau \tag{9.}$$

In our case, the decrement function is given by:

$$d(t) = Ae^{-\frac{\lambda U}{b}t} \tag{10.}$$

Whence, (9) becomes:

$$R = f(t) - f(0)Ae^{-\frac{\lambda U}{b}t} - \int_0^t f'(\tau) \cdot Ae^{-\frac{\lambda U}{b}(t-\tau)} d\tau \qquad (11.)$$

We therefore have the lift coefficient ratio as:

$$\frac{C_L}{C_{L\ QS}} = f(t) - f(0)Ae^{-\frac{\lambda U}{b}t} - Ae^{-\frac{\lambda Ut}{b}} \int_0^t f'(\tau) \cdot e^{\frac{\lambda U\tau}{b}} d\tau \qquad (12.)$$

If the forcing function, f, satisfies:

$$f(0) = 0 \qquad (13.)$$

Equation (12) simplifies to:

$$\frac{C_L}{C_{L\ QS}} = f(t) - Ae^{-\frac{\lambda Ut}{b}} \int_0^t f'(\tau) \cdot e^{\frac{\lambda U\tau}{b}} d\tau \qquad (14.)$$

Particular Forcing Function – Sinusoidal Ramp

The function for this is piecewise and defined thus:

$$f(t) = \begin{cases} \theta \cdot sin^2\left(\frac{\pi t}{2T}\right) ; 0 \leq t \leq T \\ \theta \, ; \, t > T \end{cases} \tag{15.}$$

This satisfies the condition:

$$f(0) = 0 \tag{16.}$$

The profile of this function is shown in Figure 2:

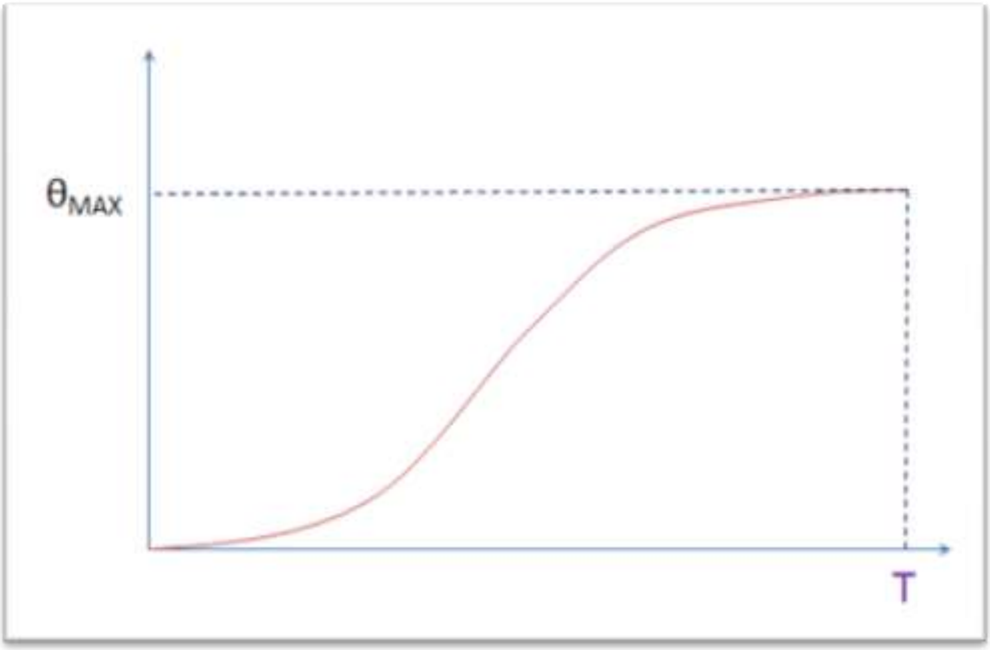

Figure 2

The derivative of this function with respect to t is given by:

$$f'(t) = \begin{cases} \frac{\pi\theta}{2T} \cdot \sin\left(\frac{\pi t}{T}\right) ; 0 \leq t \leq T \\ 0 \, ; \, t > T \end{cases} \tag{17.}$$

If we define two parameters thus:

$$\alpha = \frac{U\lambda}{b} \qquad \beta = \frac{\pi}{T} \tag{18.}$$

Then a typical integral is:

$$\int_0^t e^{\alpha\tau} \cdot \sin\beta\tau \cdot d\tau = \frac{\beta + e^{\alpha t}(\alpha\sin\beta t - \beta\cos\beta t)}{\alpha^2+\beta^2} \tag{19.}$$

In the case of the response after the forcing function has achieved its final value, the integral of (19) is the special case of:

$$\int_0^T e^{\alpha\tau} \cdot \sin\beta\tau \cdot d\tau = \frac{\beta + e^{\alpha T}(\alpha\sin\beta T - \beta\cos\beta T)}{\alpha^2+\beta^2} \tag{20.}$$

Noting that:

$$\beta T = \pi$$
$$\sin\beta T = 0 \tag{21.}$$
$$\cos\beta T = -1$$

Equation (20) simplifies to:

$$\int_0^T e^{\alpha\tau} \cdot \sin\beta\tau \cdot d\tau = \frac{\beta(1+e^{\alpha T})}{\alpha^2+\beta^2} \tag{22.}$$

From the above results, the final equation for the lift coefficient ratio is given by:

$$\frac{C_L}{C_{L\,QS}} = \begin{cases} \frac{\pi\theta}{2T} \cdot \sin\left(\frac{\pi t}{T}\right) - \frac{A\beta\theta}{2}\left[\frac{\beta e^{-\alpha t} + \alpha\sin\beta t - \beta\cos\beta t}{\alpha^2+\beta^2}\right] ; 0 \le t \le T \\ \frac{\pi\theta}{2T} \cdot \sin\left(\frac{\pi t}{T}\right) - \frac{A\beta^2\theta}{2}\left[\frac{e^{-\alpha t}(1+e^{\alpha T})}{\alpha^2+\beta^2}\right] ;\ t > T \end{cases} \tag{23.}$$

Results

The analysis was used to evaluate the lift variation for a sinusoidal-squared input.

The incidence variation is shown in Figure 3:

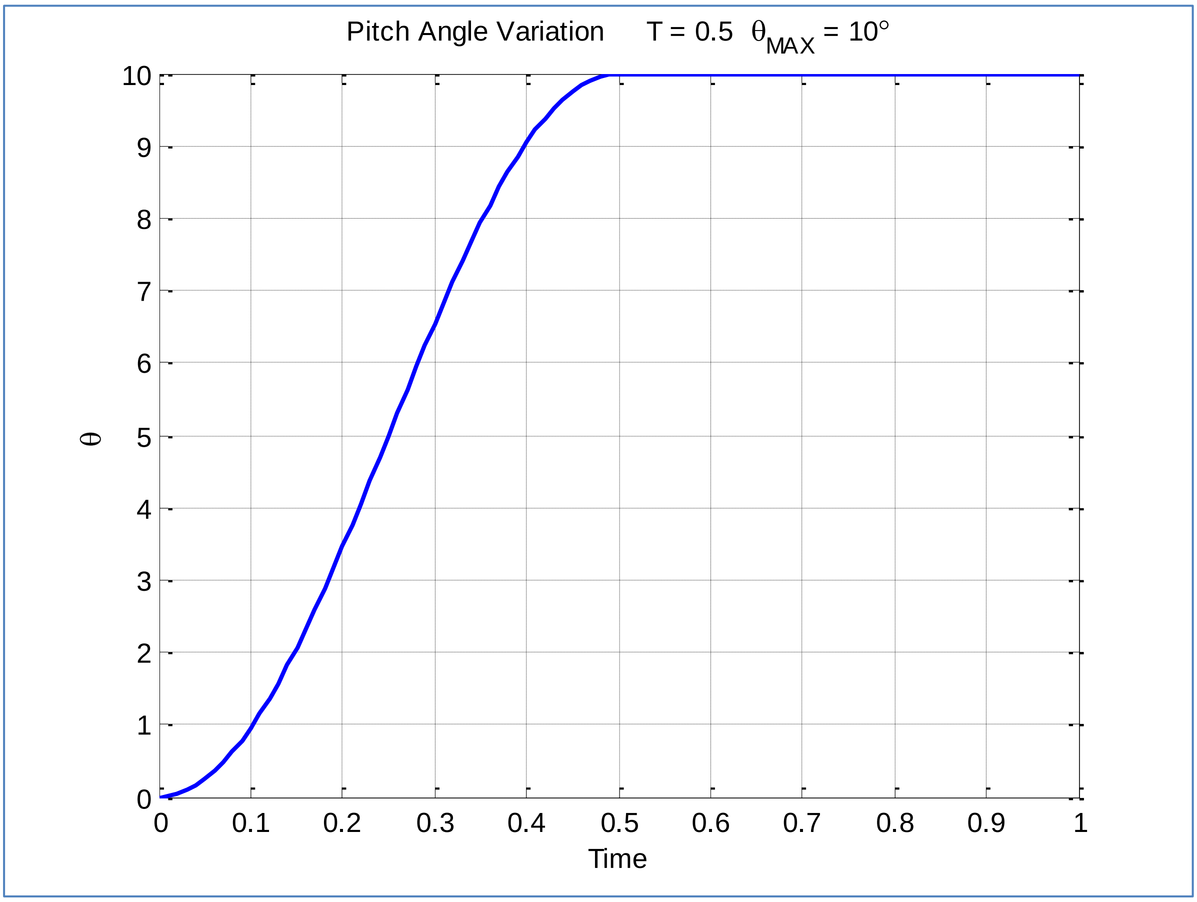

Figure 3 – Pitch Angle Variation

The Kussner function is shown in Figure 4:

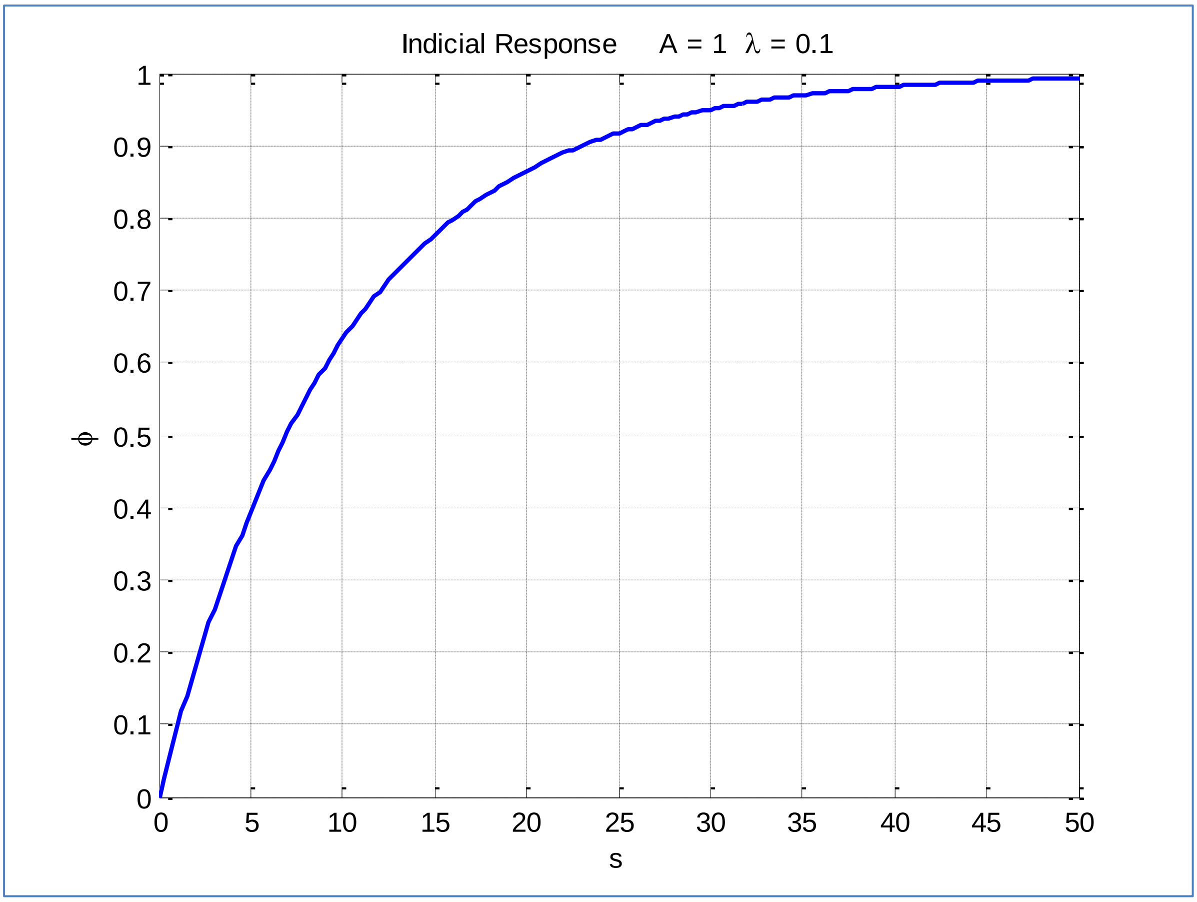

Figure 4 – Kussner Function

The resulting lift variation ratio is shown in Figure 5:

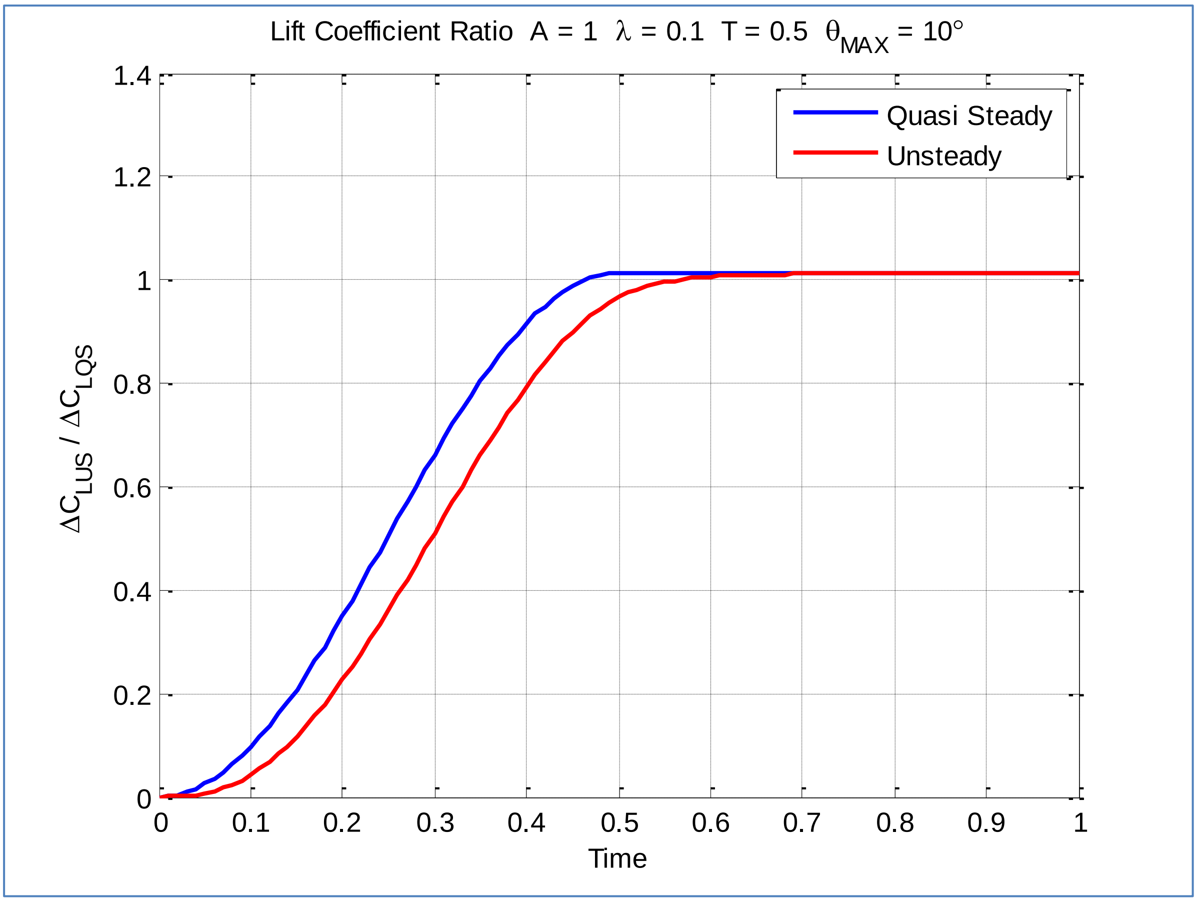

Figure 5 - Unsteady and Steady Lift Variation

Matlab File

```
%
%   Unsteady Aerodynamics -Exponential Lift Decrement
%
%   SJN 9/3/08
%
clear
colordef white
lambda=.1;
A=1;
U=100;
c=1;
dclda=5.8;

thetadeg=10;
T=.5;

tmax=1;
nt=101;

b=c/2;
thetamax=thetadeg*pi/180;
alf=U*lambda/b;
bet=pi/T;
den=alf^2+bet^2;

t=linspace(0,tmax,nt);
clqs=zeros(1,nt);
clus=clqs;
thetdeg=clqs;

for i=1:nt
    if t(i) < T
        f=thetamax*sin(bet*t(i)/2)^2;
        num=(bet*thetamax/2)*(alf*sin(bet*t(i))-bet*cos(bet*t(i))+bet*exp(-
alf*t(i)));
    else
        f=thetamax;
        num=(bet^2*thetamax/2)*(exp(-alf*(t(i)-T))+exp(-alf*t(i)));
    end
    clqs(i)=dclda*f;
    clus(i)=dclda*(f-A*num/den);
    thetdeg(i)=(180/pi)*f;
end
clf
plot(t,thetdeg,'b','LineWidth',2);
grid on
title(['Pitch Angle Variation     T = ',num2str(T),'  \theta_M_A_X =
',num2str(thetadeg),'\circ']);
xlabel('Time');
ylabel('\theta');

figure
clf
plot(t,clqs,'b','LineWidth',2);
hold on
```

```
plot(t,clus,'r','LineWidth',2);
grid on
title(['Lift Coefficient Ratio  A = ',num2str(A),'  \lambda =
',num2str(lambda),'  T = ',num2str(T),'  \theta_M_A_X =
',num2str(thetadeg),'\circ']);
legend('Quasi Steady','Unsteady');
xlabel('Time');
ylabel('\DeltaC_L_U_S / \DeltaC_L_Q_S');

figure
clf
s=linspace(0,50,201);
phi=1-A*exp(-lambda*s);
plot(s,phi,'b','LineWidth',2);
grid on
title(['Indicial Response     A = ',num2str(A),'  \lambda =
',num2str(lambda)]);
xlabel('s');
ylabel('\phi');
```