UNIVERSITY OF
Southampton
School of Engineering Sciences

Aerodynamics & Flight Mechanics Research Group

Unsteady Aerodynamic Lift Variation of an Aerofoil using Kussner's Function
(Example - the Passage of a Transverse Vortex)

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Preamble

This document examines the application of an unsteady aerodynamic model to a vertical gust variation. It is based on the concept of the Kussner lift variation.

The Kussner function, Ψ, provides the unsteady lift variation using a reduced frequency as the independent variable.

The passage of a transverse vortex is used as an example.

Nomenclature

Variable	Definition
U	Forward Speed
ρ	Air Density
k_1, k_2	Exponential Kussner Factors
$\Delta\alpha$	Incremental Pitch Angle Change
Ψ	Kussner Function
σ	Integrating variable
ΔL	Incremental Lift Force
ΔC_L	Incremental Lift Coefficient
$\Delta C_{L\ QS}$	Incremental Lift Coefficient *(Quasi-Steady)*
$\Delta C_{L\ US}$	Incremental Lift Coefficient *(Unsteady)*
V	Induced Velocity of Vortex
c	Wing Chord
b	Wing Semi-Chord
h	Height of Vortex below Aerofoil

Basic Analysis

The Kussner lift variation for a vertical gust, V_N, in a uniform stream, U, is given by:

$$\Delta L = 2\pi\rho U b \cdot \left\{ \begin{matrix} V_N(0)\cdot\psi(s) \\ +\int_0^s \frac{dV_N(\sigma)}{d\sigma}\cdot\psi(s-\sigma)d\sigma \end{matrix} \right\} \qquad (1.)$$

Here, the independent variable is the distance moved by the uniform stream expressed in semi-chords, b.

The second integral of the RHS can be transformed using integration by parts thus:

$$\int_0^s \frac{dV_N(\sigma)}{d\sigma}\cdot\psi(s-\sigma)d\sigma$$

$$= \left[V_N(\sigma)\cdot\psi(s-\sigma)\right]_0^S + \int_0^s V_N(\sigma)\cdot\psi'(s-\sigma)d\sigma \qquad (2.)$$

From which we obtain:

$$\int_0^s \frac{dV_N(\sigma)}{d\sigma}\cdot\psi(s-\sigma)d\sigma$$

$$= V_N(s)\cdot\psi(0) - V_N(0)\cdot\psi(s) + \int_0^s V_N(\sigma)\cdot\psi'(s-\sigma)d\sigma \qquad (3.)$$

Which simplifies to:

$$\int_0^s \frac{dV_N(\sigma)}{d\sigma} \cdot \psi(s-\sigma)d\sigma = -V_N(0) \cdot \psi(s) + \int_0^s V_N(\sigma) \cdot \psi'(s-\sigma)d\sigma \tag{4.}$$

Since

$$\psi(0) = 0 \tag{5.}$$

The lift variation is now given by:

$$\Delta L = 2\pi\rho U b \left\{ \int_0^s V_N(\sigma) \cdot \psi'(s-\sigma)d\sigma \right\} \tag{6.}$$

The standard form of the Kussner Function is:

$$\psi(s) = 1 - \frac{1}{2}e^{-k_1} - \frac{1}{2}e^{-k_2} \tag{7.}$$

Where:

$$\begin{gathered} k_1 = 0.31 \\ k_2 = 1 \end{gathered} \tag{8.}$$

The derivative of the Kussner Function is:

$$\psi^{'}(s) = \frac{1}{2}\{k_1 e^{-k_1} + k_2 e^{-k_2}\} \tag{9.}$$

Combining (6) & (10) gives the lift variation as:

$$\Delta L = 2\pi\rho U b \frac{1}{2}\left\{\int_0^s V_N(\sigma)\left[k_1 e^{-k_1(s-\sigma)} + k_2 e^{-k_2(s-\sigma)}\right]d\sigma\right\} \tag{10.}$$

The exponential terms in s are constant for the integrals so (11) can be simplified thus:

$$\Delta L = 2\pi\rho U b \frac{1}{2}\left\{e^{-k_1 s}\int_0^s V_N(\sigma)[k_1 e^{k_1\sigma}]d\sigma + e^{-k_2 s}\int_0^s V_N(\sigma)[k_2 e^{k_2\sigma}]d\sigma\right\} \tag{11.}$$

The RHS of (12) contains terms of the form:

$$I_k(s) = e^{-ks}\left\{\int_0^s V_N(\sigma)e^{k\sigma}d\sigma\right\} \tag{12.}$$

In general, the form of $V_N(\sigma)$ makes the RHS of (12) not integrable in closed form. In order to achieve this, a numerical procedure must be used. As the form of the Kussner Function consists of numerical constants and exponential functions only, the numerical scheme becomes very straightforward.

Using the first two terms of a Taylor Series applied to the I_k integral we have:

$$I_k(s+\delta s) = I_k(s) + \frac{dI_k(s)}{ds}\cdot\delta s \tag{13.}$$

Applying the scheme of (14) to (13) gives:

$$\frac{dI_k(s)}{ds} = -k \cdot e^{-ks} \left\{ \int_0^s V_N(\sigma) e^{k\sigma} d\sigma \right\} + e^{-ks} \cdot e^{ks} \cdot V_N(\sigma) \quad (14.)$$

From which we obtain:

$$\frac{dI_k(s)}{ds} = -k \cdot I_k(s) + V_N(\sigma) \quad (15.)$$

The scheme then becomes finally:

$$\begin{aligned} I_k(s + \delta s) &= I_k(s) + \big(V_N(s) - k \cdot I_k(s)\big) \cdot \delta s \\ &= I_k(s)(1 - k\delta s) + V_N(s) \cdot \delta s \end{aligned} \quad (16.)$$

If we now consider the lift in terms of strength of bound vorticity, Γ’:

$$\Delta L = \rho U \Gamma' \quad (17.)$$

Combining (12), (13) & (18) gives:

$$\Gamma'(s) = \pi b \left\{ I_{k_1}(s) + I_{k_2}(s) \right\} \quad (18.)$$

Example - Passage of a Transverse Vortex

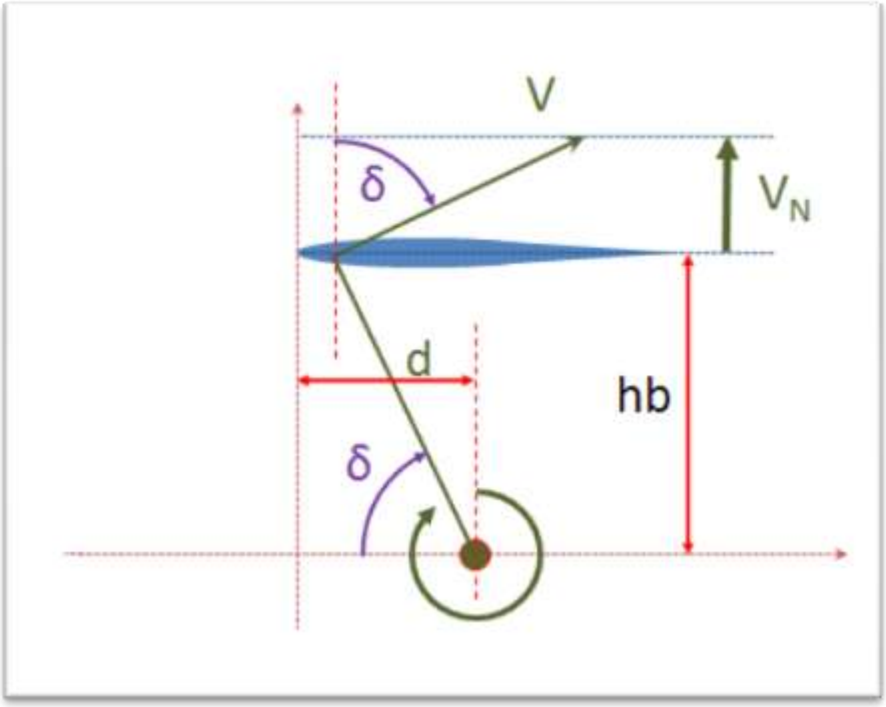

Figure 1

The axis origin is placed at the aerofoil leading edge with the X axis in a horizontal downstream direction and the Y axis in the lift direction.

The vortex commences its motion, at time zero, a distance x_0b *ahead* of the leading edge at a height of hb *below* the aerofoil centreline. At time t, the vortex position is:

$$\begin{gathered} x = Ut - x_0 b = b(s - x_0) \\ y = -hb \end{gathered} \tag{19.}$$

The induced velocity of the vortex, V, is given by:

$$V = \frac{\Gamma}{2\pi d} \tag{20.}$$

The vertical component is:

$$V_N = \frac{\Gamma}{2\pi d} \cdot \cos\delta \tag{21.}$$

$$\cos\delta = \frac{Ut - x_0 b}{d} = \frac{b(s - x_0)}{d} \tag{22.}$$

$$d^2 = (Ut - x_0 b)^2 + (hb)^2 = b^2 \cdot \{(s - x_0)^2 + h^2\} \tag{23.}$$

$$V_N = \frac{\Gamma}{2\pi b} \cdot \frac{(s - x_0)}{(s - x_0)^2 + h^2} \tag{24.}$$

If we define a normalised vertical velocity as:

$$V_N{}' = \frac{(s - x_0)}{(s - x_0)^2 + h^2} \tag{25.}$$

We have:

$$V_N = \frac{\Gamma}{2\pi b} \cdot V_N{}' \tag{26.}$$

Then, working with the normalised vertical velocity, we redefine normalised integrals as:

$$I'_k(s) = e^{-ks} \left\{ \int_0^s V'_N(\sigma) e^{k\sigma} d\sigma \right\} \tag{27.}$$

And the recursive scheme becomes:

$$\begin{aligned} I'_k(s+\delta s) &= I'_k(s) + \left(V'_N(s) - k\cdot I'_k(s)\right)\cdot \delta s \\ &= I'_k(s)(1-k\delta s) + V'_N(s)\cdot \delta s \end{aligned} \quad (28.)$$

Whence (18) becomes

$$\Gamma'(s) = \frac{\Gamma}{2\pi b}\cdot \pi b\left\{I'_{k_1}(s) + I'_{k_2}(s)\right\} \quad (29.)$$

And finally:

$$\frac{\Gamma'(s)}{\Gamma} = \frac{1}{2}\left\{I'_{k_1}(s) + I'_{k_2}(s)\right\} \quad (30.)$$

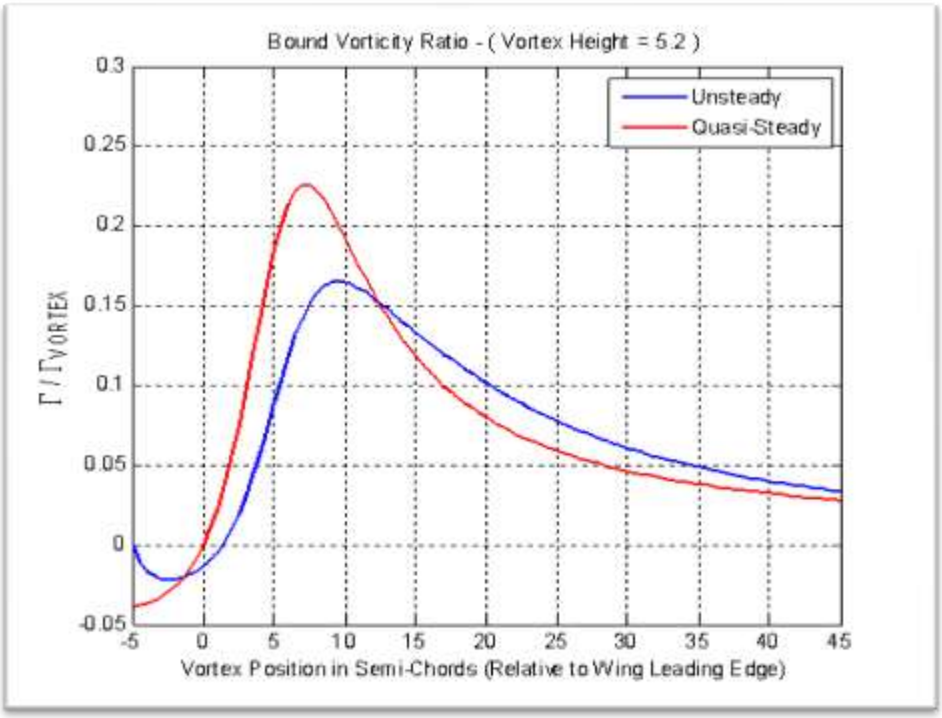

Figure 2

Constant Vertical Velocity

For comparison, if the vertical velocity, $V_{N,}$ is constant, then the following applies:

$$\Delta L = \frac{1}{2}\rho U^2 \cdot \Delta\alpha \cdot c \cdot 2\pi = \rho U\pi \cdot V_N.2b = \rho U \cdot \Gamma_0 \qquad (31.)$$

From which:

$$\Gamma_0 = 2\pi b \cdot V_N \qquad (32.)$$

Matlab File

```
%
%   Unsteady Aerodynamics - Transverse Vortex
%   Kussner Function
%
%   SJN 14/3/08
%
clear
colordef white
k1=.13;
k2=1;
srange=50;
ns=101;
x0=5;
h=5.2;

deltas=srange/(ns-1);
s=linspace(0,srange,ns);
vortx=s-x0;

vdash=(vortx)./((vortx-x0).^2+h^2);
%vdash=ones(1,ns);
ik1=zeros(1,ns);
ik2=ik1;

for is=1:ns-1
    ik1(is+1)=ik1(is)*(1-k1*deltas)+vdash(is)*deltas;
    ik2(is+1)=ik2(is)*(1-k2*deltas)+vdash(is)*deltas;
end
bndrat=(k1*ik1+k2*ik2)/2;
bnd0rat=vdash;
clf
plot(vortx,bndrat,'b','LineWidth',2);
hold on
plot(vortx,bnd0rat,'r','LineWidth',2);
grid on
title(['Bound Vorticity Ratio - (',' Vortex Height = ',num2str(h),' )']);
xlabel('Vortex Position in Semi-Chords (Relative to Wing Leading Edge)');
ylabel('\Gamma / \Gamma_V_O_R_T_E_X');
legend('Unsteady  ','Quasi-Steady');
```