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Joukowski Aerofoil Modelling in MATLAB

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Preliminary Discussion

The 2D Potential Flow is built up using 3 basic components, namely:

- Free Stream
- Doublet
- Vortex

The assumption is made whereby, the flow velocity is given by:

$$\underline{q} = u + iv = \nabla \phi \tag{1.}$$

The Complex Potential is:

$$W = \phi + i\psi \tag{2.}$$

From which we have:

$$\frac{dW}{dZ} = u - iv \tag{3.}$$

And:

$$q = |\underline{q}| = \left|\frac{dW}{dZ}\right| = \sqrt{u^2 + v^2}$$
$$C_P = 1 - \left(\frac{q}{U}\right)^2 \tag{4.}$$

Free Stream

If the incident freestream flow is angled by α to the X axis of velocity U, the Complex Potential (W) is given by:

$$W = Uz \cdot e^{i\alpha} \tag{5.}$$

Doublet

A doublet placed at the origin is:

$$\begin{aligned} W &= \frac{\mu_{Doublet}}{z} \\ \frac{dW}{dz} &= -\frac{\mu_{Doublet}}{z^2} \end{aligned} \tag{6.}$$

However, the doublet will create the flow of a freestream passing a circular cylinder. If the cylinder radius is r_c, then the doublet strength specification can be modified to give:

$$\begin{aligned} W &= \frac{Ua^2}{z} e^{i\alpha} \\ \frac{dW}{dz} &= -\frac{Ua^2}{z^2} e^{i\alpha} \end{aligned} \tag{7.}$$

Vortex

A vortex placed at the origin is:

$$W = k_{Vortex} i \log z$$
$$\frac{dW}{dz} = \frac{i \cdot k_{Vortex}}{z} \qquad (8.)$$

Total

Hence, assembling the three components gives the overall result:

$$W = Uze^{-i\alpha} + \frac{Ua^2}{z} e^{i\alpha} + k_{Vortex} i \log z$$
$$\frac{dW}{dz} = Ue^{-i\alpha} - \frac{Ua^2}{z^2} e^{i\alpha} + \frac{i \cdot k_{Vortex}}{z} \qquad (9.)$$

Joukowski Transformation

The Joukowski transformation takes a circle and transforms it into an aerofoil type of shape using the following complex expression:

$$Z = U\zeta + \frac{Ua^2}{\zeta} \tag{10.}$$

Where a is the radius of the reference circle centred at the origin.

The circle which transforms to the aerofoil shape is defined as shown in Figure 1:

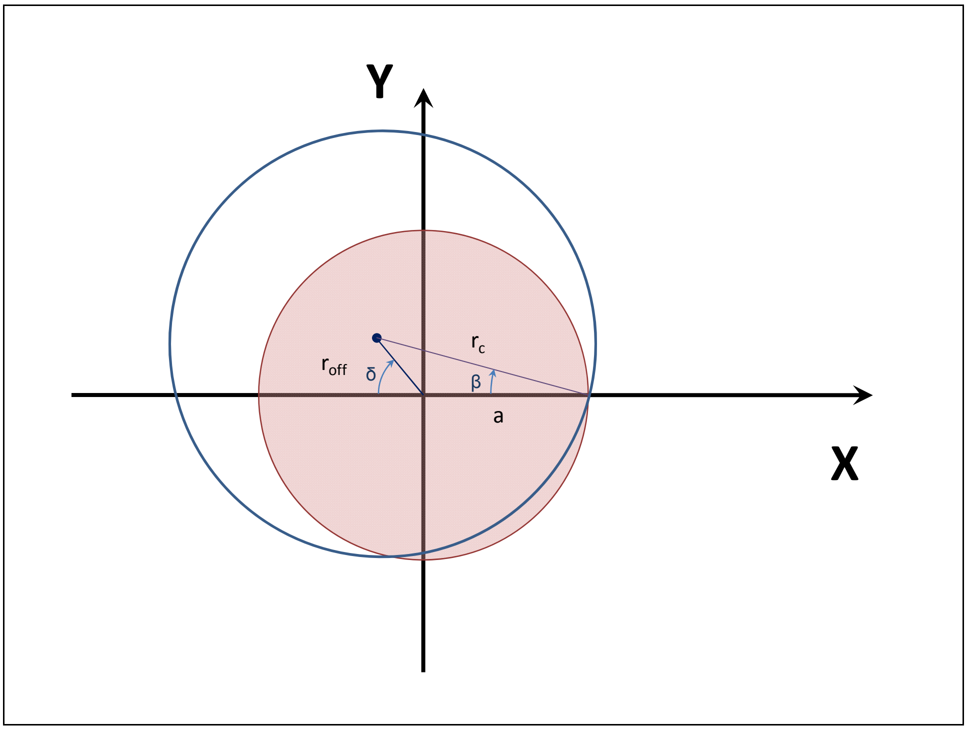


Figure 1

Using the above definitions, the circle is given by:

$$\zeta_{circ} = r_c e^{i\theta} + r_{off}(-\cos\delta + i\sin\delta) \tag{11.}$$

The reference angle, β, is given by:

$$\sin\beta = \frac{r_{off}\sin\delta}{r_c} \tag{12.}$$

An example of the transformation geometry is shown in Figure 2:

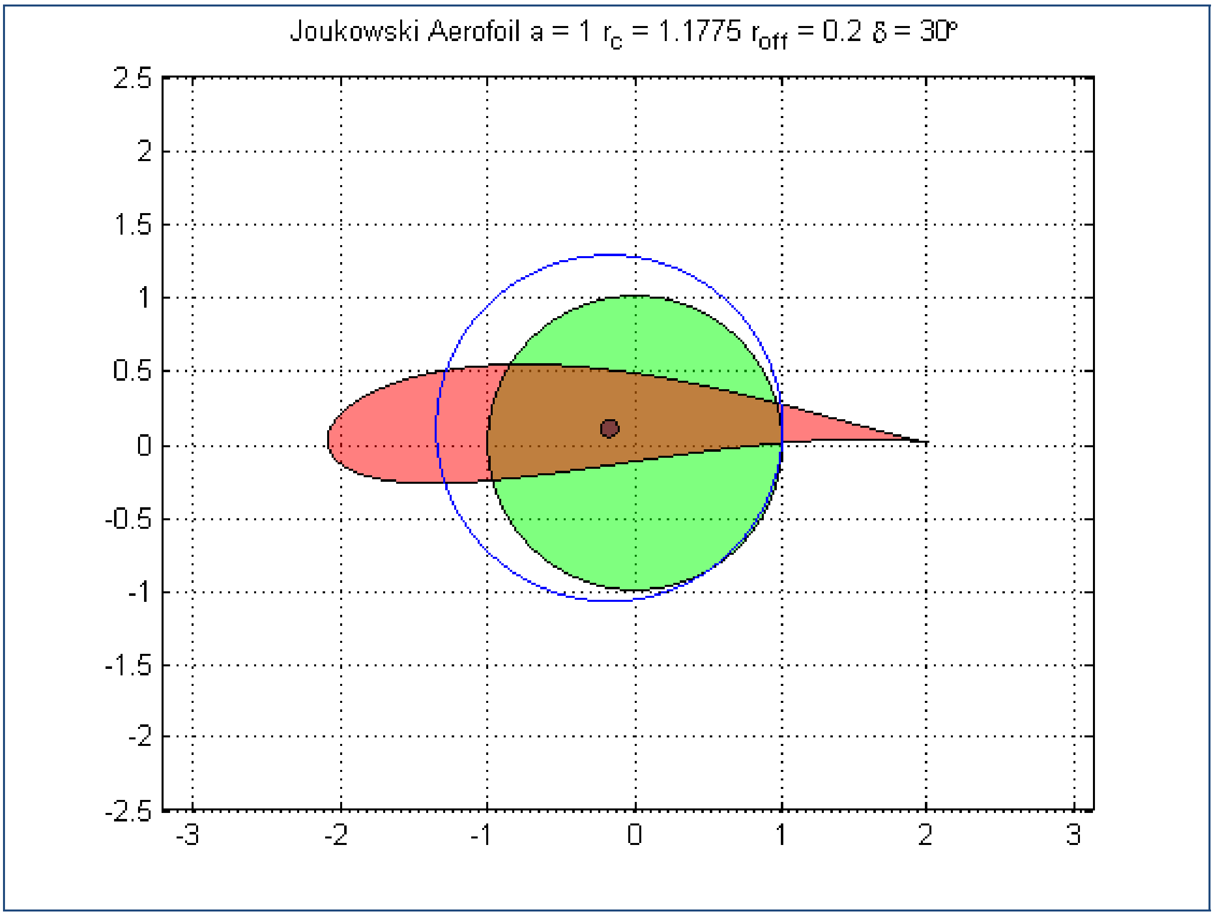


Figure 2 - Ref Circle (Green), Base Circle (Blue), Aerofoil (Red)

The inverse transformation from the Z to the ζ plane is given by:

$$\zeta = \frac{Z + \sqrt{Z^2 - 4a^2}}{2} \quad if \quad real(Z) \geq 0$$
$$\zeta = \frac{Z - \sqrt{Z^2 - 4a^2}}{2} \quad if \quad real(Z) < 0 \qquad (13.)$$

Note the use of both solutions of the inverse transformation quadratic equation.

Kutta Condition

The fully-developed flow over an aerofoil requires the flow to leave the trailing edge tangentially. This is known as the ***Kutta condition*** and is achieved by adjusting the vortex strength. The requirement is a result of the viscous nature of the fluid. Potential flow does not allow for viscosity so this is an analytical tweak to enable a necessary result of viscosity to be included in a potential flow model.

In order for the Kutta condition to be met the vortex strength must be of the correct magnitude, given by:

$$K = 2aU\sin\left(\alpha+\beta\right) \tag{14.}$$

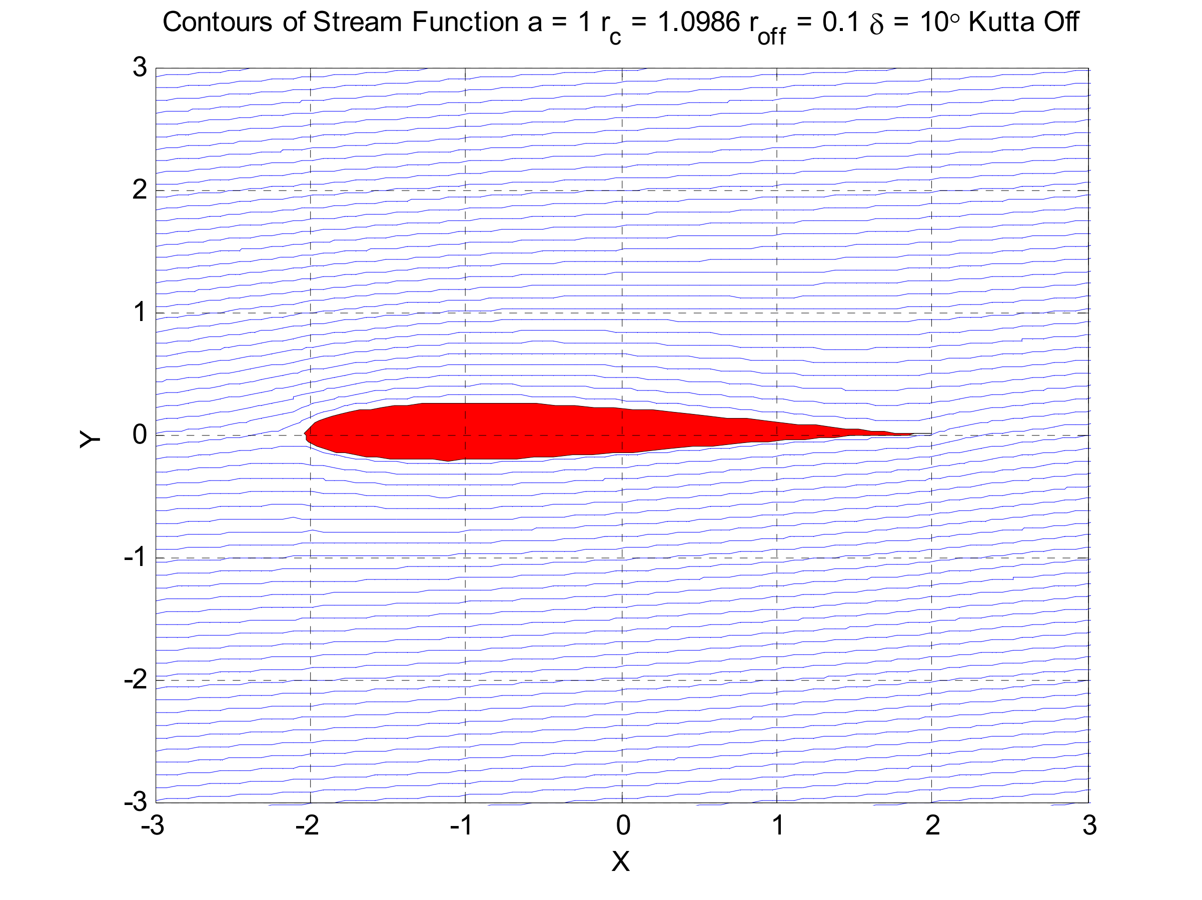
Whence the final flow Complex Potential is defined (in the ζ plane) by:

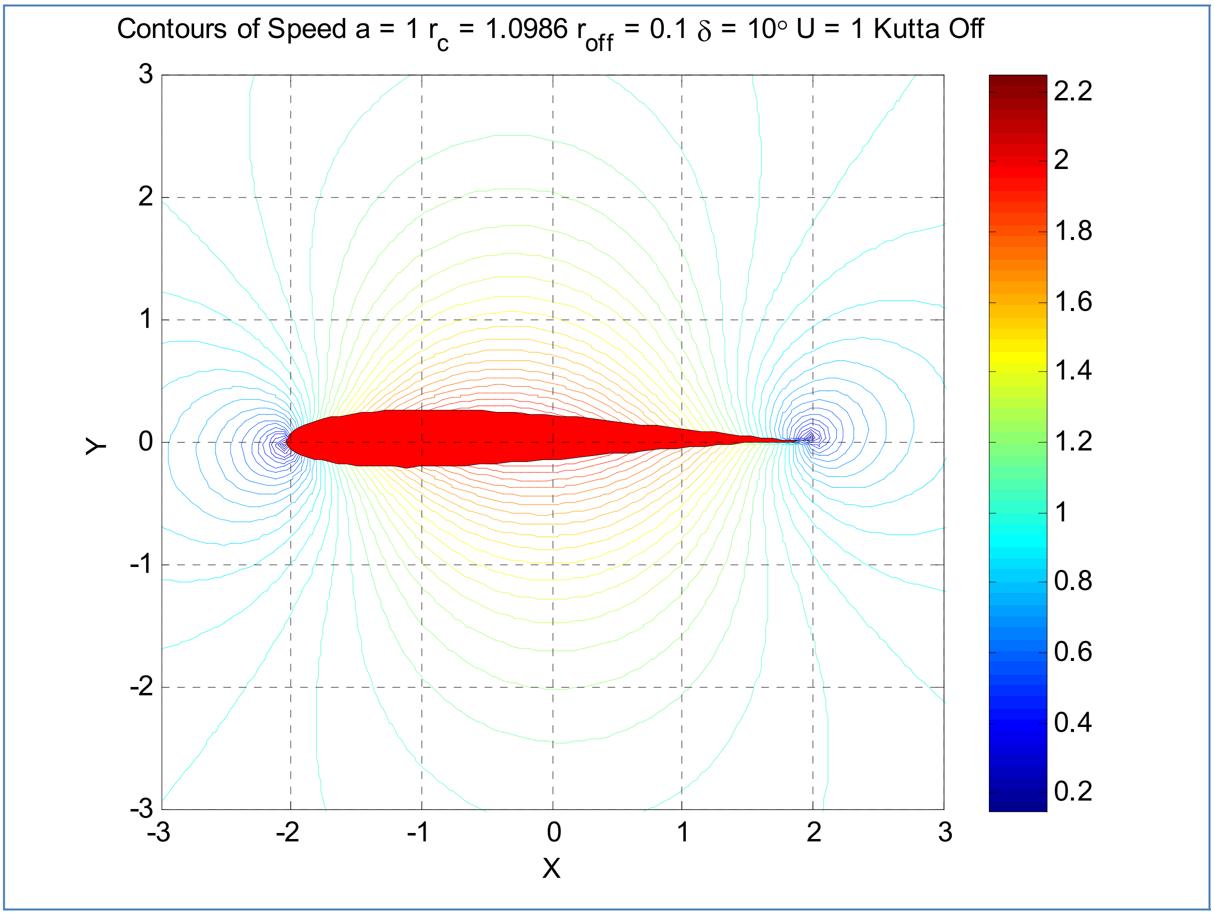
$$\begin{aligned} W &= U\zeta e^{-i\alpha} + \frac{Ua^2}{\zeta}e^{i\alpha} + 2aU\sin\left(\alpha+\beta\right)i\log\zeta \\ \frac{dW}{dz} &= Ue^{-i\alpha} - \frac{Ua^2}{\zeta^2}e^{i\alpha} + \frac{i\cdot 2aU\sin\left(\alpha+\beta\right)}{\zeta} \end{aligned} \tag{15.}$$

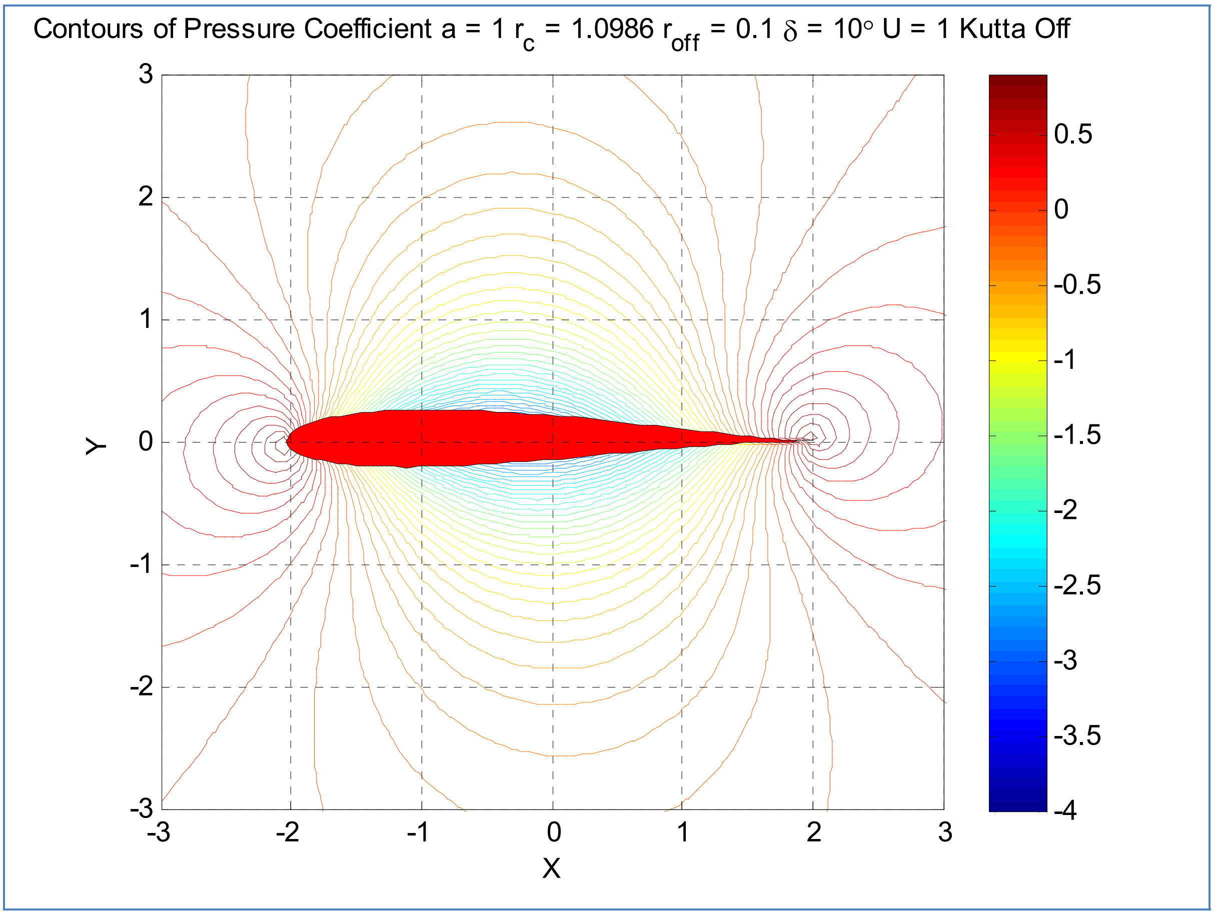
Examples

Using the above transformation, the following figures show contours of stream function, speed and pressure coefficient:

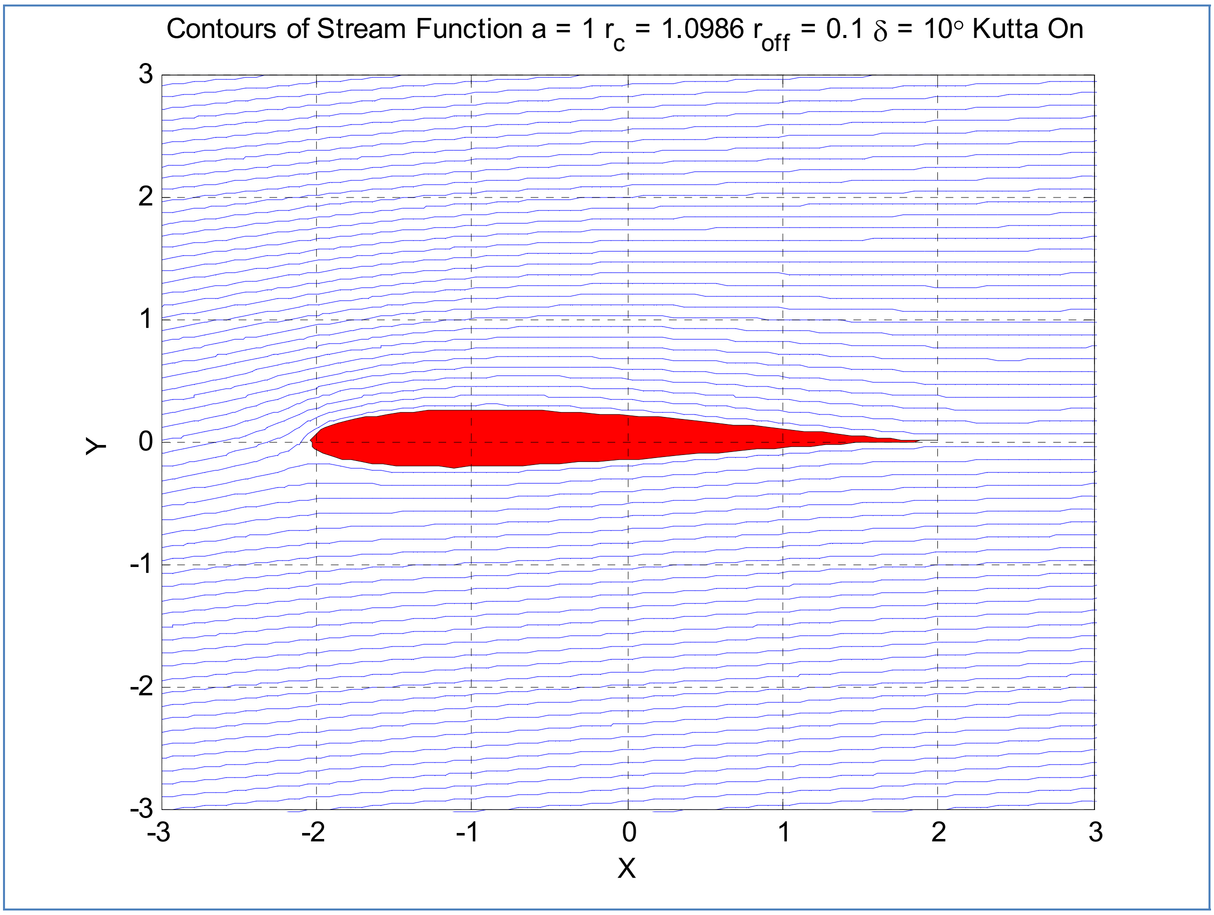
Kutta Condition Off

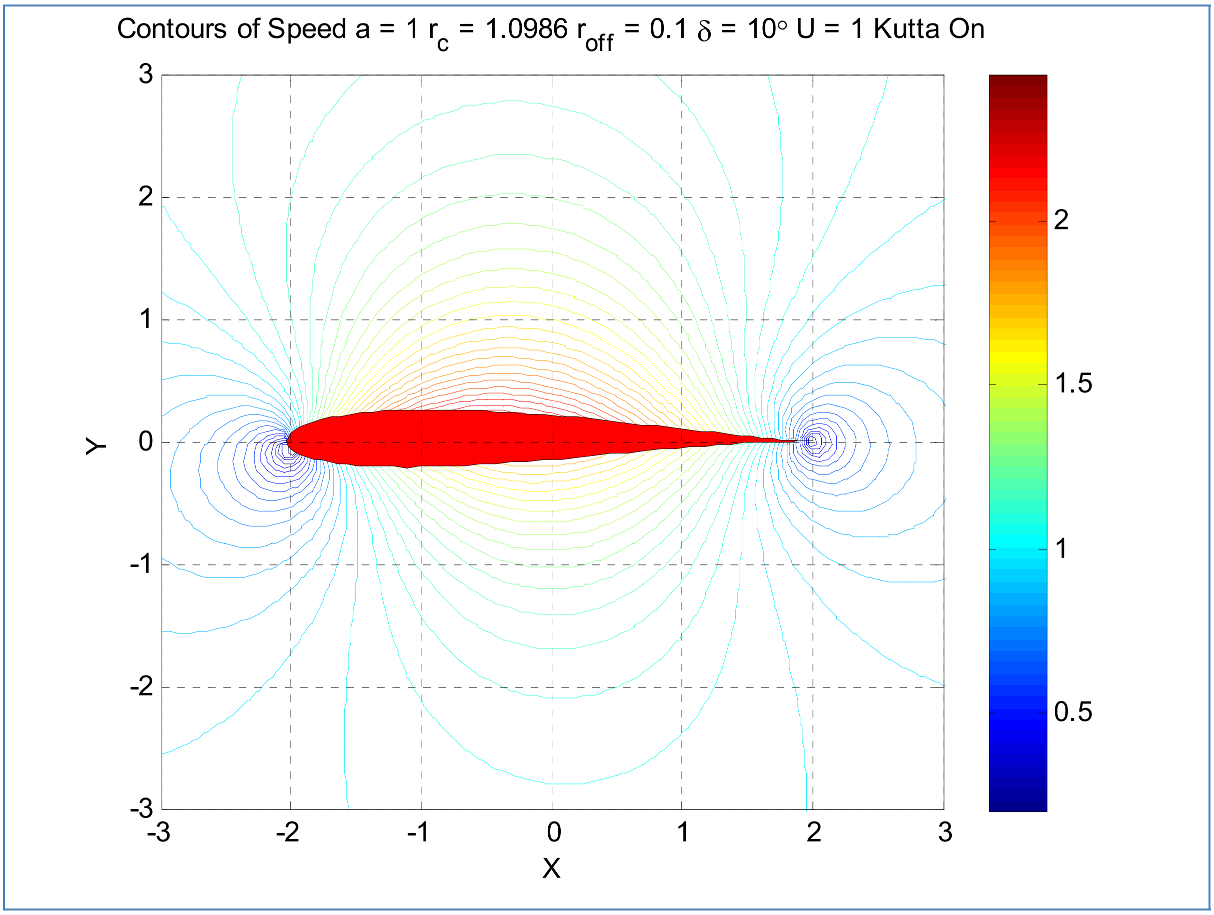



Contours of Speed a = 1 r_c = 1.0986 r_off = 0.1 δ = 10° U = 1 Kutta Off
Y
X
3
2
1
0
-1
-2
-3
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2

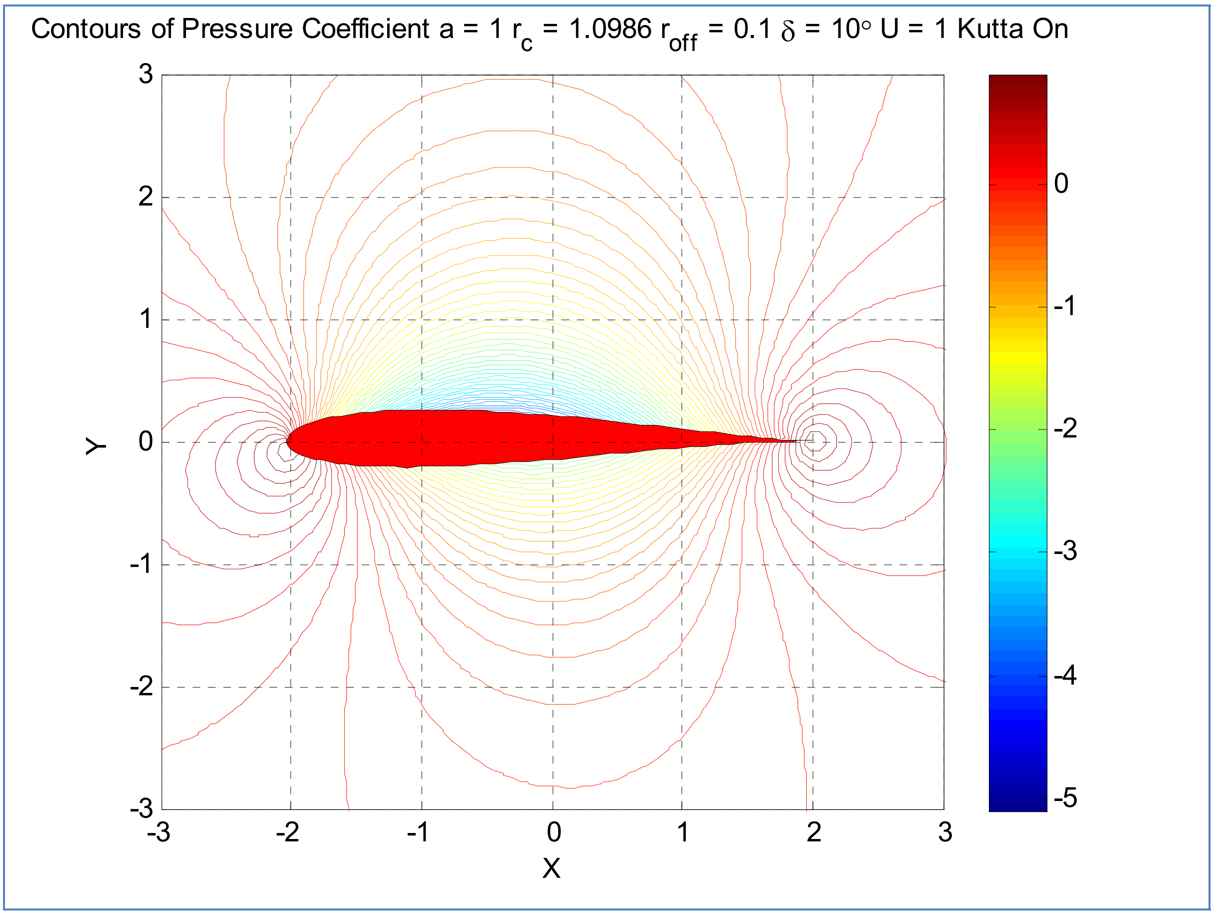


Kutta Condition On


Contours of Stream Function a = 1 r_c = 1.0986 r_off = 0.1 δ = 10° Kutta On
Y
X
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3



Contours of Speed a = 1 r_c = 1.0986 r_off = 0.1 δ = 10° U = 1 Kutta On
Y
X
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3
2
1.5
1
0.5



Matlab File

```
%
%   Joukowski Aerofoils
%
%   SJN 5/11/09
%
clear all
xmax=5;
nx=150;
a=1;
deltadeg=10;
roff=.1;

U=1;
alfdeg=5;

kutta=1;

%-----------------------------------------------------------------------------
delta=deltadeg*pi/180;
offset=roff*(-cos(delta)+1i*sin(delta));
circrad=abs(offset-a);
beta=asin(roff*sin(delta)/circrad);
%-----------------------------------------------------------------------------
```

```
alf=alfdeg*pi/180;
kuttagamma=2*circrad*U*sin(alf+beta);
if kutta~=0
    vortex=kuttagamma;
else
    vortex=0;
end
%------------------------------------------------------------------------
x=linspace(-xmax,xmax,nx);
y=x;
[X,Y]=meshgrid(x,y);
Z=X+1i*Y;
ZETA=.5*(Z-sqrt(Z.^2-4*a^2)).*(real(Z)<0)+...
    .5*(Z+sqrt(Z.^2-4*a^2)).*(real(Z)>=0);
%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
% CONTOURS OF STREAM FUNCTION
W=U*exp(1i*-alf)*ZETA...
    +U*exp(1i*alf)*circrad^2./(ZETA-offset)...
    +vortex*1i*log(ZETA-offset);
v1=-40:.1:40;
contour(X,Y,imag(W),v1,'b');
hold on
if kutta~=0
    title(['Contours of Stream Function a = ',num2str(a),...
        ' r_c = ',num2str(circrad),...
        ' r_o_f_f = ',num2str(roff),...
        ' \delta = ',num2str(deltadeg),'\circ',' Kutta On']);
else
    title(['Contours of Stream Function a = ',num2str(a),...
        ' r_c = ',num2str(circrad),...
        ' r_o_f_f = ',num2str(roff),...
        ' \delta = ',num2str(deltadeg),'\circ',' Kutta Off']);
end
xlabel('X');
ylabel('Y');
%------------------------------------------------------------------------
theta=linspace(0,2*pi,101);
ctheta=cos(theta);
stheta=sin(theta);
ZETACIRC=circrad*(ctheta+1i*stheta)+offset;
ZCIRC=ZETACIRC+a^2./ZETACIRC;
fill(real(ZCIRC),imag(ZCIRC),'r','FaceAlpha',1);
%axis(xmax*[-1 1 -1 1]);
axis(3*[-1 1 -1 1]);
%axis([1.5 2.5 -.5 .5]);
%axis equal
grid on
%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
figure % CONTOURS OF SPEED

dWdZ=U*exp(1i*-alf)...
    -U*exp(1i*alf)*circrad^2./(ZETA-offset).^2 ...
    +vortex*1i./(ZETA-offset);
    Q=abs(dWdZ);
v2=0:.05:3;
contour(X,Y,Q,v2);
colorbar
hold on
%------------------------------------------------------------------------
```

```
theta=linspace(0,2*pi,101);
ctheta=cos(theta);
stheta=sin(theta);
ZETACIRC=circrad*(ctheta+1i*stheta)+offset;
ZCIRC=ZETACIRC+a^2./ZETACIRC;
fill(real(ZCIRC),imag(ZCIRC),'r','FaceAlpha',1);
%axis(xmax*[-1 1 -1 1]);
axis(3*[-1 1 -1 1]);
%axis([1.5 2.5 -.5 .5]);
%axis equal
grid on
if kutta~=0
    title(['Contours of Speed a = ',num2str(a),...
        ' r_c = ',num2str(circrad),...
        ' r_o_f_f = ',num2str(roff),...
        ' \delta = ',num2str(deltadeg),'\circ',' U = ',num2str(U),' Kutta
On']);
else
    title(['Contours of Speed a = ',num2str(a),...
        ' r_c = ',num2str(circrad),...
        ' r_o_f_f = ',num2str(roff),...
        ' \delta = ',num2str(deltadeg),'\circ',' U = ',num2str(U),' Kutta
Off']);
end
xlabel('X');
ylabel('Y');

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
figure % CONTOURS OF PRESSURE COEFFICIENT

CP=1-Q.^2/U^2;
v3=-10:.1:1;
contour(X,Y,CP,v3);
colorbar
hold on
%--------------------------------------------------------------------------
theta=linspace(0,2*pi,101);
ctheta=cos(theta);
stheta=sin(theta);
ZETACIRC=circrad*(ctheta+1i*stheta)+offset;
ZCIRC=ZETACIRC+a^2./ZETACIRC;
fill(real(ZCIRC),imag(ZCIRC),'r','FaceAlpha',1);
%axis(xmax*[-1 1 -1 1]);
axis(3*[-1 1 -1 1]);
%axis([1.5 2.5 -.5 .5]);
%axis equal
grid on
if kutta~=0
    title(['Contours of Pressure Coefficient a = ',num2str(a),...
        ' r_c = ',num2str(circrad),...
        ' r_o_f_f = ',num2str(roff),...
        ' \delta = ',num2str(deltadeg),'\circ',' U = ',num2str(U),' Kutta
On']);
else
    title(['Contours of Pressure Coefficient a = ',num2str(a),...
        ' r_c = ',num2str(circrad),...
        ' r_o_f_f = ',num2str(roff),...
        ' \delta = ',num2str(deltadeg),'\circ',' U = ',num2str(U),' Kutta
Off']);
end
```

```
xlabel('X');
ylabel('Y');
```