UNIVERSITY OF Southampton
School of Engineering Sciences

Aerodynamics & Flight Mechanics Research Group

Derivation of a Simple Ground Resonance Model

S. J. Newman

Technical Report AFM-11/19

January 2011

UNIVERSITY OF SOUTHAMPTON

SCHOOL OF ENGINEERING SCIENCES

AERODYNAMICS AND FLIGHT MECHANICS RESEARCH GROUP

Derivation of Simple Ground Resonance Model

by

S. J. Newman

AFM Report No. AFM 11/19

January 2011

© School of Engineering Sciences, Aerodynamics and Flight Mechanics Research Group

UNIVERSITY OF Southampton

COPYRIGHT NOTICE

(c) SES University of Southampton All rights reserved.

SES authorises you to view and download this document for your personal, non-commercial use. This authorization is not a transfer of title in the document and copies of the document and is subject to the following restrictions: 1) you must retain, on all copies of the document downloaded, all copyright and other proprietary notices contained in the Materials; 2) you may not modify the document in any way or reproduce or publicly display, perform, or distribute or otherwise use it for any public or commercial purpose; and 3) you must not transfer the document to any other person unless you give them notice of, and they agree to accept, the obligations arising under these terms and conditions of use. This document, is protected by worldwide copyright laws and treaty provisions.

Introduction

This method allows the offset rotor centre of gravity to be determined from the blade lagging motion and how the resulting frequencies can couple with those of the fuselage sitting on its undercarriage.

Each rotor blade is modelled by a concentrated mass connected by a light rod to the lagging hinge, as shown in Figure 1.

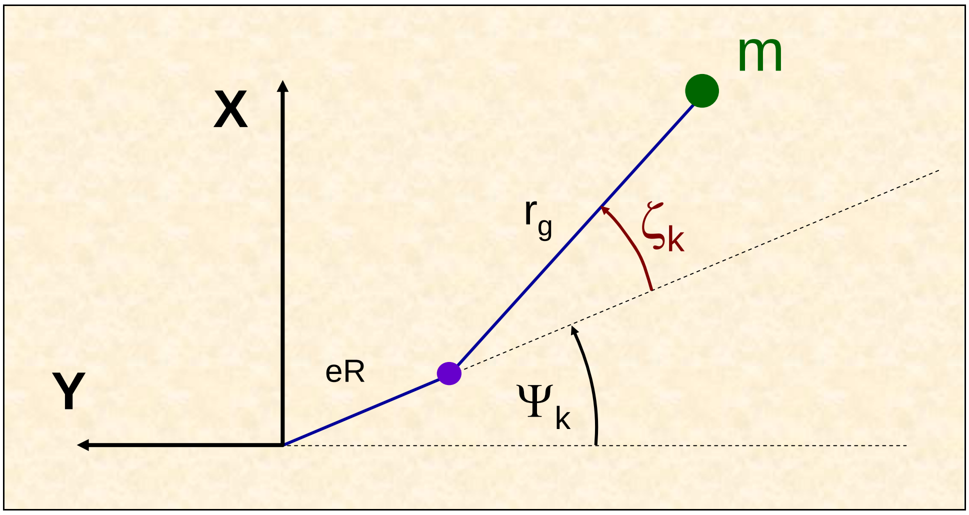

Figure 1 - Blade Coordinate System

The location of the mass is given by:-

$$\begin{aligned} X_k &= eR\sin\psi_k + r_g \sin(\psi_k + \zeta_k) \\ Y_k &= -eR\cos\psi_k - r_g\cos(\psi_k + \zeta_k) \end{aligned} \quad (1)$$

For small ζ_k we can use the approximations

$$\begin{aligned} \cos\zeta_k &\approx 1 \\ \sin\zeta_k &\approx \zeta_k \end{aligned} \quad (2)$$

from which:-

$$
\begin{aligned}
X_k &= eR\sin\psi_k + r_g\sin\psi_k + r_g\cos\psi_k\zeta_k \\
&= (eR + r_g)\sin\psi_k + r_g\zeta_k\cos\psi_k \\
Y_k &= -eRcos\psi_k - r_g\cos\psi_k + r_g\sin\psi_k\zeta_k \\
&= -(eR + r_g)\cos\psi_k + r_g\zeta_k\sin\psi_k
\end{aligned}
\tag{3}
$$

For a rotor with N such blades, if the rotor CG is located at (X_c,Y_c) then we have:-

$$
\begin{aligned}
X_c &= \frac{1}{mN}\sum_{k=1}^{N} m\,X_k \\
Y_c &= \frac{1}{mN}\sum_{k=1}^{N} m\,Y_k
\end{aligned}
\tag{4}
$$

A blade is selected to be at an azimuth of ψ and so the kth blade is at an azimuth angle of:-

$$
\psi_k = \psi + (k - 1)\frac{2\pi}{N} \tag{5}
$$

Where the inter-blade spacing is:

$$
\phi = \frac{2\pi}{N} \tag{6}
$$

Whence from equations (2,3) we find for the Y component of the rotor centre of gravity:-

$$
\begin{aligned}
Y_c &= \frac{1}{mN}\sum_k m\left[-(eR + r_g)\cos\psi_k + r_g\,\zeta_k \sin\psi_k\right] \\
&= \frac{1}{N}\left[-(eR + r_g)\sum_k \cos\psi_k + r_g\sum_k \zeta_k \sin\psi_k\right] \\
&= \frac{r_g}{N}\sum_k \zeta_k \sin\psi_k
\end{aligned}
\qquad (7)
$$

Where the following result has been used:-

$$\sum_k \cos\psi_k = 0 \qquad (8)$$

Similarly, for the X component of the rotor centre of gravity, we obtain:-

$$X_c = \frac{r_g}{N}\sum_k \zeta_k \cos\psi_k \qquad (9)$$

If we now assume the lagging behaviour of the blades to be SHM of amplitude ζ_0 and frequency $K\Omega$, we have for the lag angle of the blade at an azimuth angle of ψ_k :-

$$\zeta_k = \zeta_0 \cos(K\,\psi_k) \qquad (10)$$

Substituting this expression into equations (5) and (7) we find:-

$$
\begin{aligned}
X_c &= \frac{r_g \zeta_0}{N} \sum_k \cos K \psi_k \bullet \cos \psi_k \\
&= \frac{r_g \zeta_0}{2N} \sum_k \left[\cos(K+1)\psi_k + \cos(K-1)\psi_k \right] \\
Y_c &= \frac{r_g \zeta_0}{N} \sum_k \cos K \psi_k \bullet \sin \psi_k \\
&= \frac{r_g \zeta_0}{2N} \sum_k \left[\sin(K+1)\psi_k - \sin(K-1)\psi_k \right]
\end{aligned}
\qquad (11)
$$

We now require summations of the form:-

$$
\sum_{k=1}^{N} \cos n \psi_k \qquad \sum_{k=1}^{N} \sin n \psi_k \qquad (12)
$$

where n, in this instance, is not necessarily an integer.

To perform this sum we define the following summations:-

$$
\begin{aligned}
C &= \sum_{k=1}^{N} \cos n(\psi + [k-1]\phi) \\
S &= \sum_{k=1}^{N} \sin n(\psi + [k-1]\phi)
\end{aligned}
\qquad (13)
$$

These can be combined into a single complex quantity:-

$$
\begin{aligned}
C + i\,S &= \sum_{k=1}^{N} (\cos + i\sin)\{\, n(\psi + [k-1]\phi)\,\} \\
&= \sum_{k=1}^{N} e^{i\,n(\psi + [k-1]\phi)}
\end{aligned}
\qquad (14)
$$

This is a geometric progression of N terms, with the first term $e^{in\psi}$, and the multiplying factor $e^{in\phi}$.

This results in:-

$$
C + iS = e^{in\psi}\left(\frac{e^{iN\phi n} - 1}{e^{i\phi n} - 1}\right) \qquad (15)
$$

Noting that:-

$$
\begin{aligned}
e^{iN\phi n} - 1 &= e^{\frac{iN\phi n}{2}}\left(e^{\frac{iN\phi n}{2}} - e^{-\frac{iN\phi n}{2}}\right) \\
&= e^{\frac{iN\phi n}{2}}\, 2\,i\sin\left(\frac{N\phi n}{2}\right) \\
e^{i\phi n} - 1 &= e^{\frac{i\phi n}{2}}\left(e^{\frac{i\phi n}{2}} - e^{-\frac{i\phi n}{2}}\right) \\
&= e^{\frac{i\phi n}{2}}\, 2\,i\sin\left(\frac{\phi n}{2}\right)
\end{aligned}
\qquad (16)
$$

From which we obtain:-

$$C + i\,S = e^{in\left(\psi + \frac{N-1}{2}\phi\right)} \bullet \frac{2\,i\sin\left(\frac{nN\phi}{2}\right)}{2\,i\sin\left(\frac{n\phi}{2}\right)}$$
$$= \frac{\sin\left(\frac{nN\phi}{2}\right)}{\sin\left(\frac{n\phi}{2}\right)} \bullet (\cos + i\sin)\left[n\left(\psi + \frac{(N-1)\phi}{2}\right)\right] \tag{17}$$

Since the inter-blade spacing, ϕ, is given by:-

$$\phi = \frac{2\pi}{N} \tag{18}$$

whence on substitution in (15):-

$$C + i\,S = \frac{\sin(n\pi)}{\sin\left(\frac{n\pi}{N}\right)} \bullet (\cos + i\sin)\left[n\left(\psi + \frac{(N-1)\pi}{N}\right)\right] \tag{19}$$

From which we obtain on equating real and imaginary parts:-

$$C = \frac{\sin(n\pi)}{\sin\left(\frac{n\pi}{N}\right)} \bullet \cos\left[n\left(\psi + \frac{(N-1)\pi}{N}\right)\right]$$
$$S = \frac{\sin(n\pi)}{\sin\left(\frac{n\pi}{N}\right)} \bullet \sin\left[n\left(\psi + \frac{(N-1)\pi}{N}\right)\right] \tag{20}$$

For brevity, define:-

$$S_{K+1} = \frac{\sin\left(\overline{K+1}\pi\right)}{\sin\left(\overline{K+1}\frac{\pi}{N}\right)}$$
$$S_{K-1} = \frac{\sin\left(\overline{K-1}\pi\right)}{\sin\left(\overline{K-1}\frac{\pi}{N}\right)} \qquad (21)$$
$$\Phi = \frac{N-1}{N}\pi$$

whence:-

$$X_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1}\cos(K+1)(\psi+\Phi) + S_{K-1}\cos(K-1)(\psi+\Phi)\,\right]$$
$$Y_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1}\sin(K+1)(\psi+\Phi) - S_{K-1}\sin(K-1)(\psi+\Phi)\,\right] \qquad (22)$$

which can be re-expressed as:

$$X_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1}\cos(K+1)(\psi+\Phi) + S_{K-1}\cos(1-K)(\psi+\Phi)\,\right]$$
$$Y_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1}\sin(K+1)(\psi+\Phi) + S_{K-1}\sin(1-K)(\psi+\Phi)\,\right] \qquad (23)$$

This means that the centre of mass of the N blades can be represented by two masses rotating around the shaft at different radii and rotational speeds.

$$\begin{aligned} Mass &= mN \\ Radius &= \frac{r_g \zeta_0}{2N} \bullet S_{K\pm 1} \\ Rotational\ Speed &= (K \pm 1)\Omega \end{aligned} \tag{24}$$

A typical value for K is 0.3 from which K+1=1.3 and K-1=-0.7, the former result is known as the progressive mode and the latter the regressive mode. Both masses move in the direction of rotation of the rotor relative to an inertial frame, however, relative to the rotor, the progressive rotates faster than the rotor whilst the regressive mode rotates in opposition to the rotor.

The Prolate Epicycloid

The epicycloid is a curve traced out by a point on a circle whilst rolling around another circle. This can be extended to a point outside of the rolling circle forming the prolate variety.

With reference to Figure 2, the brown circle is rolling around the green circle. The point under scrutiny is P which lies outside of the brown circle placed a distance l from the centre. The reference position is with the centre of the rolling circle lying on the abscissa together with the point P.

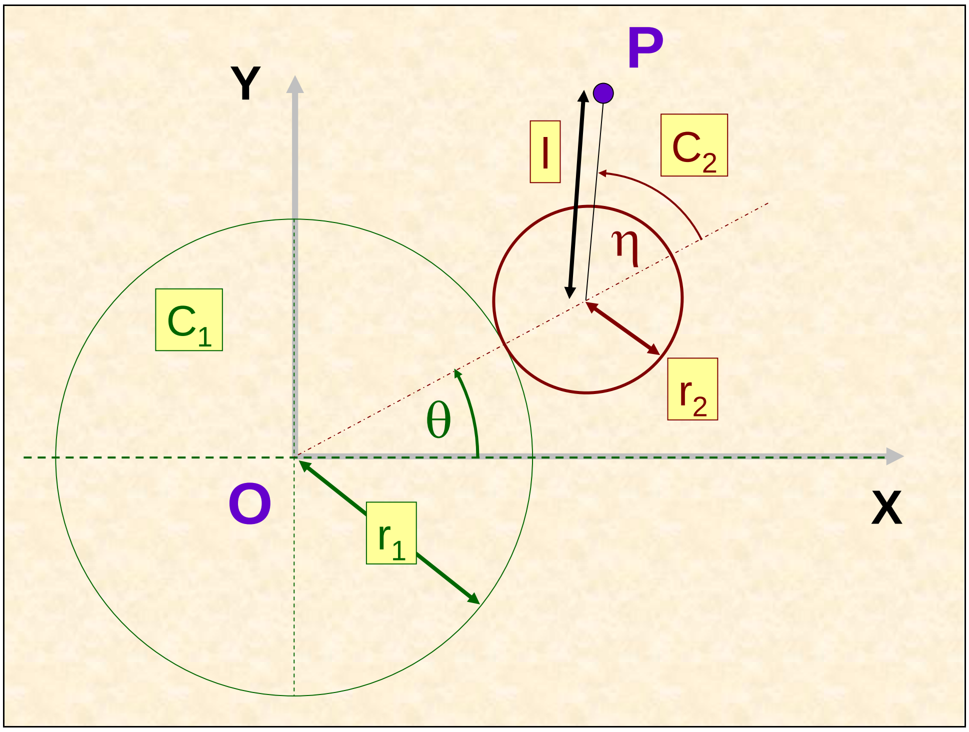

Figure 2 - Generation of Prolate Epicycloid

As the circles roll without slipping, the rotation angles are linked via the following relationship:

$$r_1\theta = r_2\eta$$

$$\eta = \frac{r_1}{r_2}\theta \tag{25}$$

The coordinates of point P are then given by:

$$\begin{aligned} X &= (r_1 + r_2)\cos\theta + l\cos(\theta+\eta) \\ Y &= (r_1 + r_2)\sin\theta + l\sin(\theta+\eta) \end{aligned} \tag{26}$$

Defining the following terms:

$$\mu = \frac{r_1}{r_2}$$
$$\lambda = \frac{l}{r_2} \qquad (27)$$

from which we find:

$$\frac{\lambda}{\mu} = \frac{l}{r_1}$$
$$\eta = \mu\theta \qquad (28)$$

Here, μ represents the ratio of the radii of the generating circles and λ the prolate ratio. The former will give the number of cusps or loops of the epicycloids and the latter the depth of each loop.

Combining (26-28) we find:

$$\begin{aligned}
X &= r_1\left\{\left(1+\frac{r_2}{r_1}\right)\cos\theta + \frac{l}{r_1}\cos(\theta+\eta)\right\} \\
&= r_1\left\{\left(1+\frac{1}{\mu}\right)\cos\theta + \frac{\lambda}{\mu}\cos(\theta+\mu\theta)\right\} \\
&= r_1\left\{\left(\frac{1+\mu}{\mu}\right)\cos\theta + \frac{\lambda}{\mu}\cos([1+\mu]\theta)\right\} \\
Y &= r_1\left\{\left(1+\frac{r_2}{r_1}\right)\sin\theta + \frac{l}{r_1}\sin(\theta+\eta)\right\} \\
&= r_1\left\{\left(1+\frac{1}{\mu}\right)\sin\theta + \frac{\lambda}{\mu}\sin(\theta+\mu\theta)\right\} \\
&= r_1\left\{\left(\frac{1+\mu}{\mu}\right)\sin\theta + \frac{\lambda}{\mu}\sin[1+\mu]\theta)\right\}
\end{aligned} \qquad (29)$$

Returning to the CG coordinates from (23) we define an angle which includes the phase angle Φ:

$$\alpha = \psi + \Phi \qquad (30)$$

which results in:

$$X_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1} \cos (K+1)\alpha + S_{K-1} \cos (1-K)\alpha \,\right]$$
$$Y_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1} \sin (K+1)\alpha + S_{K-1} \sin (1-K)\alpha \right] \tag{31}$$

In addition, we define the following angle:

$$\chi = (1-K)\alpha \tag{32}$$

From this we have:

$$(1+K)\alpha = \frac{1+K}{1-K}\chi \tag{33}$$

We define the term σ via:

$$(1+K)\alpha = \sigma\chi$$
$$\sigma = \frac{1+K}{1-K} \tag{34}$$

Collecting these substitutions together we have the CG coordinates as:

$$X_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1} \cos \sigma\chi + S_{K-1} \cos \chi \,\right]$$
$$Y_c = \frac{r_g \zeta_0}{2N}\left[\, S_{K+1} \sin \sigma\chi + S_{K-1} \sin \chi \right] \tag{35}$$

Recalling (29):

$$
\begin{aligned}
X &= r_1\left\{\left(\frac{1+\mu}{\mu}\right)\cos\theta + \frac{\lambda}{\mu}\cos([1+\mu]\theta)\right\} \\
Y &= r_1\left\{\left(\frac{1+\mu}{\mu}\right)\sin\theta + \frac{\lambda}{\mu}\sin[1+\mu]\theta)\right\}
\end{aligned}
\qquad (36)
$$

Comparing (35) & (36), these coordinates match if the following results are satisfied:

$$
\begin{aligned}
1+\mu &= \sigma \\
r_1\left(\frac{1+\mu}{\mu}\right) &= \frac{r_g\zeta_0}{2N}\cdot S_{K-1} \\
r_1\left(\frac{\lambda}{\mu}\right) &= \frac{r_g\zeta_0}{2N}\cdot S_{K+1}
\end{aligned}
\qquad (37)
$$

From this we find the radii ratio as:

$$
\begin{aligned}
\mu &= \sigma - 1 \\
&= \frac{1+K}{1-K} - 1 \\
&= \frac{2K}{1-K}
\end{aligned}
\qquad (38)
$$

Whence:

$$
\begin{aligned}
\frac{1+\mu}{\mu} &= \frac{1+K}{1-K}\Big/\frac{2K}{1-K} \\
&= \frac{1+K}{2K}
\end{aligned}
\qquad (39)
$$

The fixed (green) circle radius becomes:

$$r_1 = \frac{2K}{1+K} \cdot \frac{r_g \zeta_0}{2N} \cdot S_{K-1} \qquad (40)$$

and the prolate ratio is:

$$\lambda = \frac{S_{K+1}}{S_{K-1}} \cdot \frac{1+K}{1-K} \qquad (41)$$

The ratio of radii (μ) gives the number of cusps or loops in the epicycloid. & through (38) it is linked to the lag frequency (K). If μ is an integer, then the epicycloid is complete in one revolution.

Examples

Equation (38) can be reformed as:

$$K = \frac{\mu}{\mu+2} \qquad (42)$$

Figure 3 plots the lag frequencies which result in integer values of μ. These are listed in Table 1.

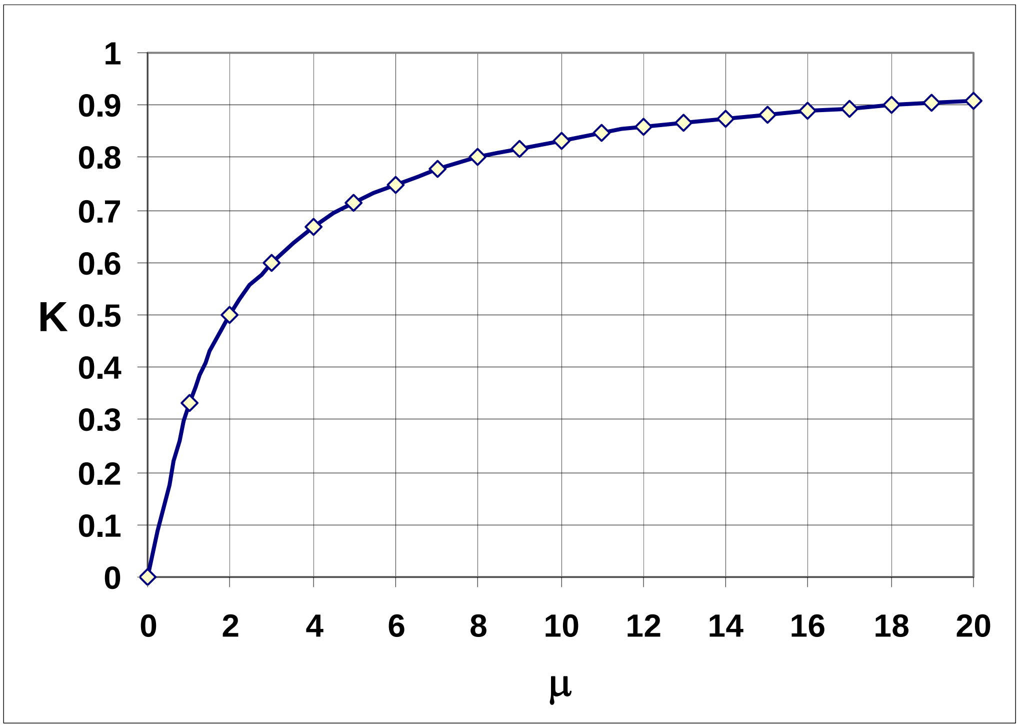

Figure 3 - Lag Frequencies

μ	K
0	0
1	0.333333
2	0.5
3	0.6
4	0.666667
5	0.714286
6	0.75
7	0.777778
8	0.8
9	0.818182
10	0.833333
11	0.846154
12	0.857143
13	0.866667
14	0.875
15	0.882353
16	0.888889
17	0.894737
18	0.9
19	0.904762
20	0.909091

Table 1

Using (21) & (41), the prolate ratio, which governs whether a smooth curve, a curve with cusps or loops results, is determined by both the lag frequency (K) and the number of blades (N).

Fig No.	K_{LAG}	N	μ	λ
4	0.6	4	3	-1.2997
5	0.3333	6	1	-1.0642
6	0.6	8	3	-1.0646
7	0.6	3	3	-1.6349
8	0.75	6	6	-1.1517

Table 2

Examples of these results – detailed in Table 2 - are shown in the following figures:

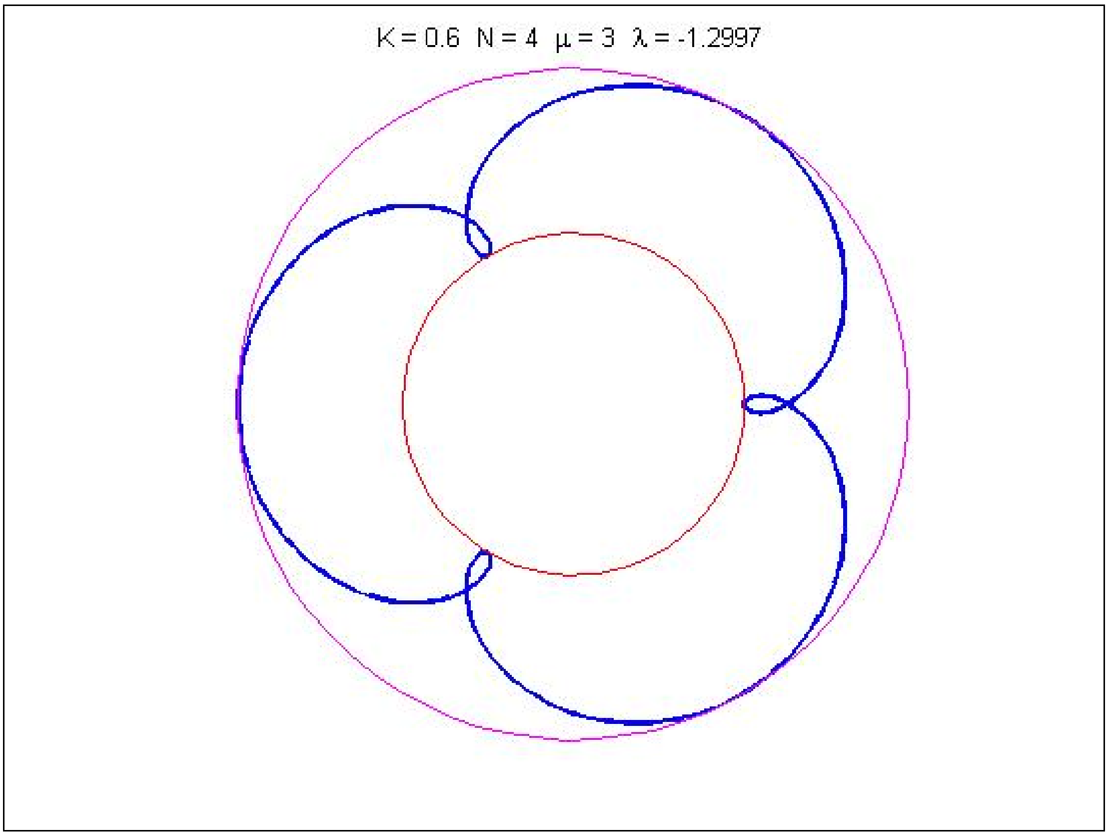

Figure 4

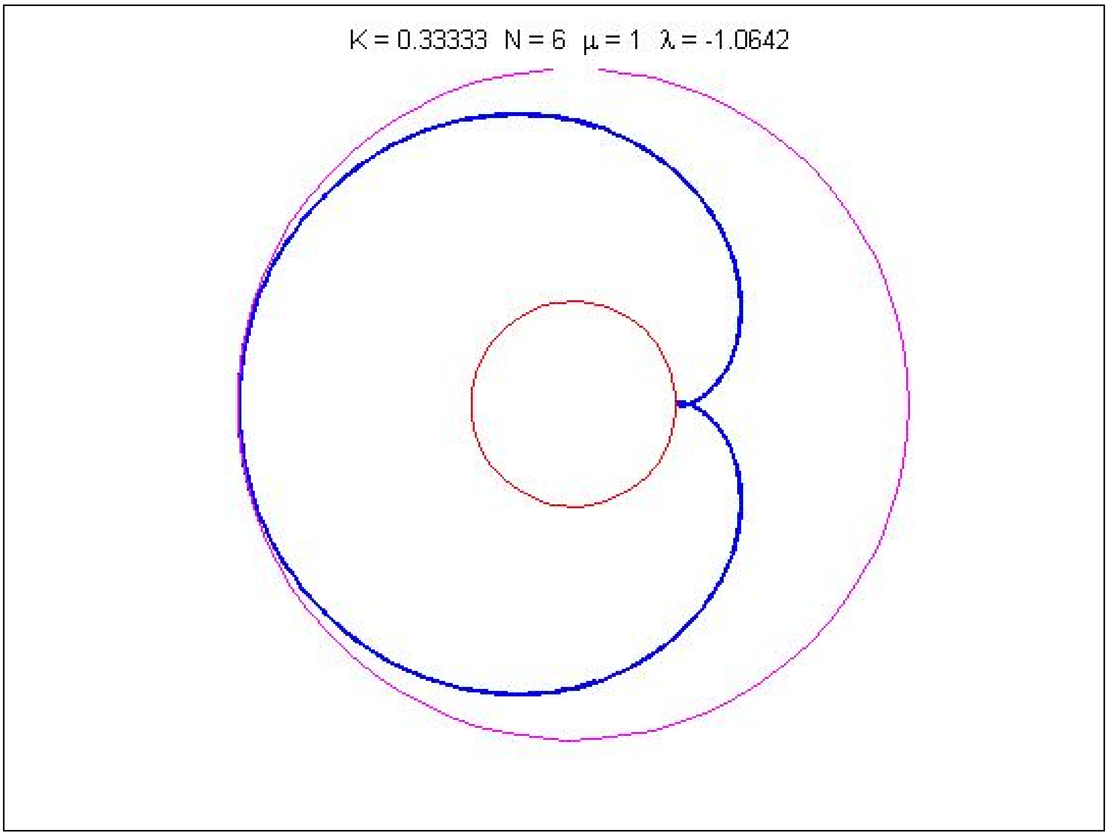

Figure 5

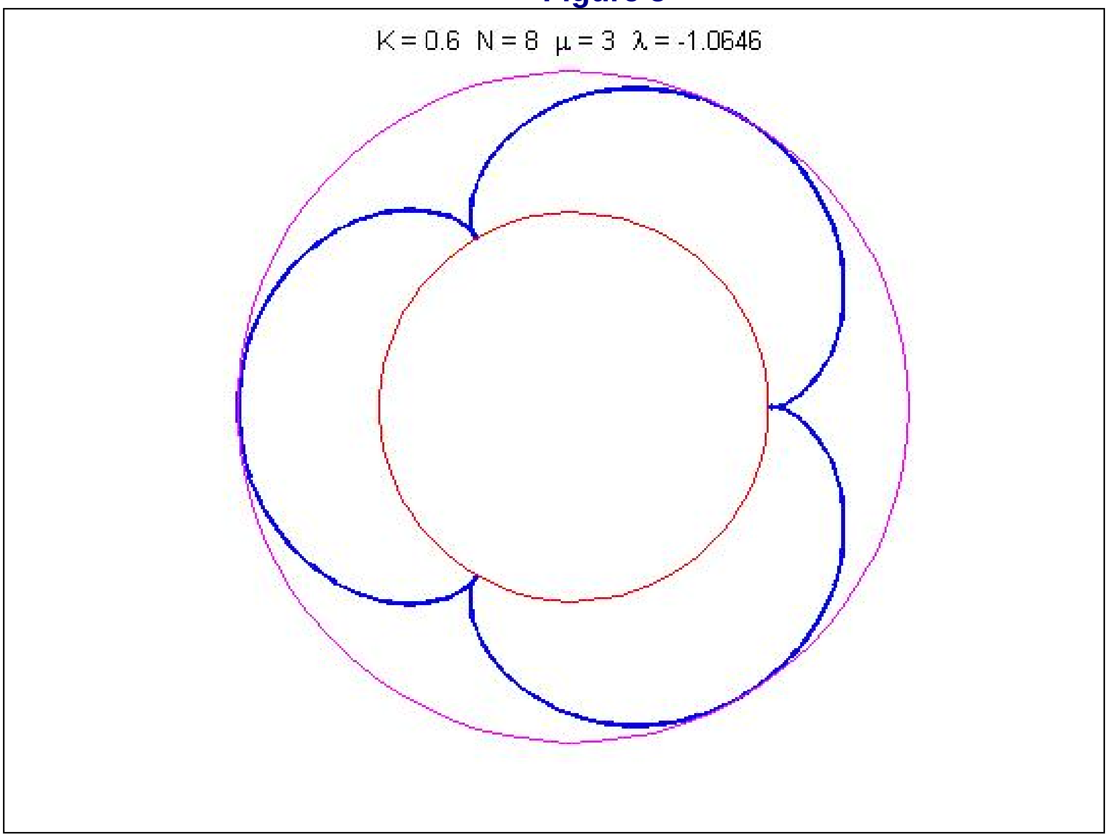

Figure 6

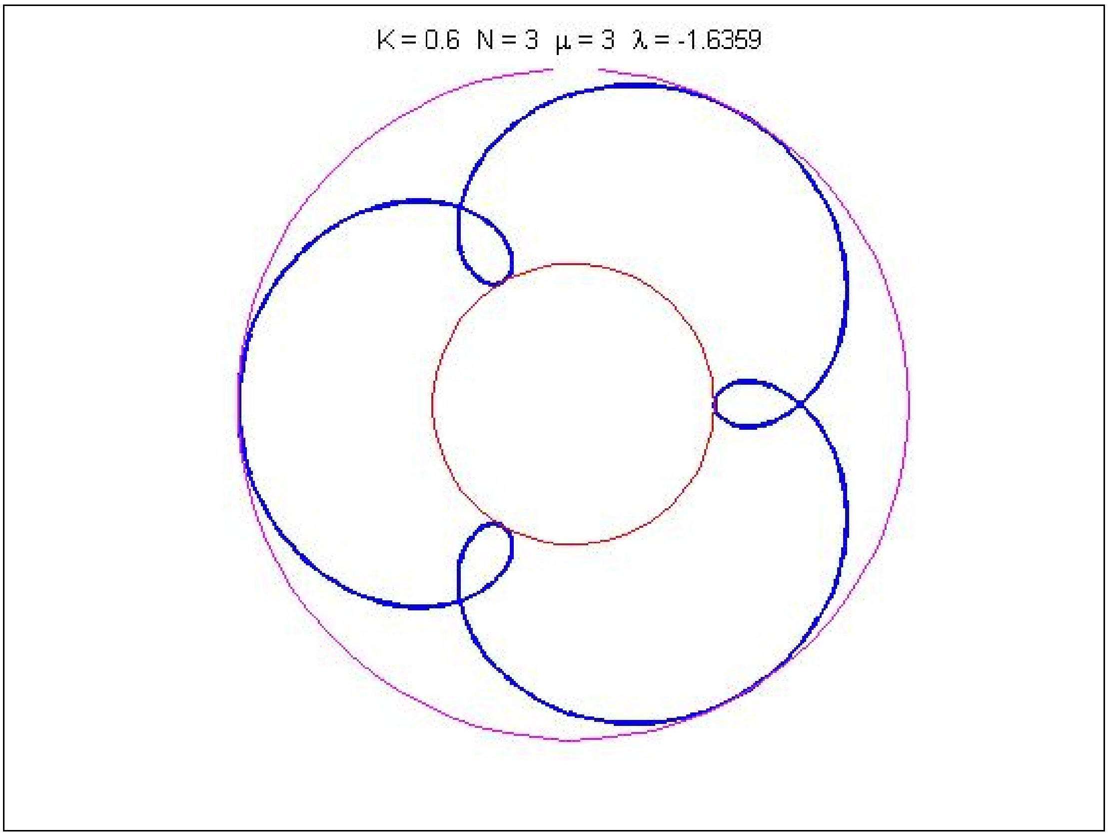

Figure 7

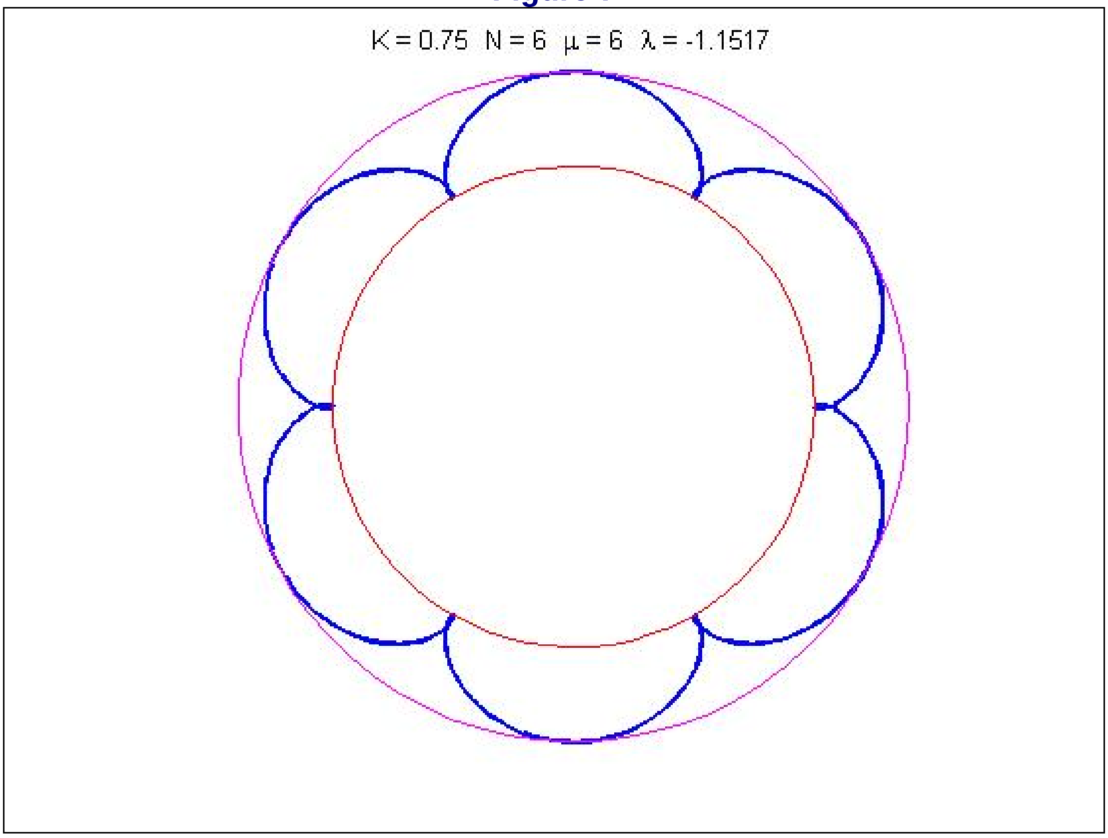

Figure 8

MATLAB File – Animation

```
%
%   Ground Resonance CG Plot
%
%   SJN 22/7/07
%
colordef black

%   Initial Data

plotsize=1.3;
rtipcirc=.05;

e=.1;
klag=.75;
zeta0=.01;
nblade=6;

npsi=72;
nrev=10;

cgradfact=150;

phi=2*pi/nblade; % Inter-Blade Angle

%   Establish Two Mass Location Parameters

bigphi=(nblade-1)*pi/nblade;
kp1=klag+1;
km1=klag-1;
skp1=sin(kp1*pi)/sin(kp1*pi/nblade);
skm1=sin(km1*pi)/sin(km1*pi/nblade);
rcg=2*cgradfact*(1-e)*zeta0/(2*nblade);

%   Initialise CG Locus

xcgloc=[];
ycgloc=[];

%   Set up Blade Azimuth Array

psib=zeros(1,nblade);
for ii=1:nblade
    psib(ii)=(ii-1)*phi;
end

psi=zeros(1,npsi*nrev);

for jj=1:npsi*nrev+1
    psi(jj)=(jj-1)*2*pi/npsi;
end
```

UNIVERSITY OF Southampton

```
%   Define Circle

tipang=linspace(0,2*pi,72);
xtipcirc=rtipcirc*cos(tipang);
ytipcirc=rtipcirc*sin(tipang);

%   Commence Azimuth Loop

for i=1:npsi*nrev+1 % Commence Azimuth Loop

    clf;

%   Clear Rotor CG Position

    xcg=0;
    ycg=0;

%   Commence Blade Loop

    for k=1:nblade % Commence Blade Loop

        zeta=zeta0*cos(klag*(psi(i)+psib(k)));
        xlag=e*cos(psi(i)+psib(k));
        ylag=e*sin(psi(i)+psib(k));
        xblade=(1-e)*cos(psi(i)+psib(k)+zeta);
        yblade=(1-e)*sin(psi(i)+psib(k)+zeta);
        xtip=xlag+xblade;
        ytip=ylag+yblade;

        xcg=xcg+xtip;
        ycg=ycg+ytip;

        xplt=[0,xlag,xtip];
        yplt=[0,ylag,ytip];

        plot(xplt,yplt,'r','LineWidth',1);
        hold on
        fill(.5*xtipcirc+xlag,.5*ytipcirc+ylag,'b');
        fill(xtipcirc+xtip,ytipcirc+ytip,'g');


        axis([-plotsize,plotsize,-plotsize,plotsize]);
        axis square
        axis off

    end % End Blade Loop

%   CG Location

    xcg=cgradfact*xcg/nblade;
    ycg=cgradfact*ycg/nblade;
    xcgloc=[xcgloc,xcg];
    ycgloc=[ycgloc,ycg];

    fill(xtipcirc+xcg,ytipcirc+ycg,'y');

 %   Two Mass CG Locations
```

```
    xcgp1=-rcg*skp1*sin(kp1*(psi(i)+bigphi));
    ycgp1=rcg*skp1*cos(kp1*(psi(i)+bigphi));
    fill(xtipcirc+xcgp1,ytipcirc+ycgp1,'m');

    xcgm1=rcg*skm1*sin(km1*(psi(i)+bigphi));
    ycgm1=rcg*skm1*cos(km1*(psi(i)+bigphi));
    fill(xtipcirc+xcgm1,ytipcirc+ycgm1,'c');

    m(i)=getframe(gcf);

end % End Azimuth Loop
figure(2)
plot(xcgloc,ycgloc,'w');
%movie2avi(m,'fred');
```