UNIVERSITY OF Southampton
School of Engineering Sciences

Aerodynamics & Flight Mechanics Research Group

Coleman Transformation for N Rotor Blades

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Technical Report AFM-11/20

January 2011

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SCHOOL OF ENGINEERING SCIENCES

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Definition of the Transformation

The rotor consists of N blades, each of which is denoted by the suffix, k. Each individual blade then possesses a lag angle of ζ_k.

The Coleman transformation converts the N lag angles (ζ_k) to N variables, θ_j, via the following:

$$\theta_j = \sum_{k=0}^{N-1} \zeta_k e^{j \cdot \frac{2\pi i}{N} \cdot k} \tag{1}$$

For brevity, define:

$$\alpha = e^{\frac{2\pi i}{N}} \tag{2}$$

i.e. α is the primary Nth root of unity.

Thus;

$$\theta_j = \sum_{k=0}^{N-1} \zeta_k \alpha^{j \cdot k} \tag{3}$$

for j=1,2,……,N-1.

Therefore:

$$
\begin{aligned}
&\theta_0 = \zeta_0 + \zeta_1 + \zeta_2 + ... + \zeta_{N-1} \\
&\theta_1 = \zeta_0 + \zeta_1 \alpha + \zeta_2 \alpha^2 + ... + \zeta_{N-1} \alpha^{N-1} \\
&\theta_2 = \zeta_0 + \zeta_1 \alpha^2 + \zeta_2 \alpha^4 + ... + \zeta_{N-1} \alpha^{2(N-1)} \\
&\Downarrow \\
&\theta_{N-1} = \zeta_0 + \zeta_1 \alpha^{N-1} + \zeta_2 \alpha^{2(N-1)} + ... + \zeta_{N-1} \alpha^{(N-1)(N-1)}
\end{aligned}
\tag{4}
$$

The α terms represent the azimuthal spacing between the blades and are thus referenced to a rotating set of axes.

To cater for a non-rotating set of axes, the above equations are multiplied by:

$$
e^{i\omega t} \tag{5}
$$

also, denoting the blade azimuth via:

$$
\psi_k = \omega t + \frac{2\pi}{N} \cdot k \tag{6}
$$

from which:

$$
e^{i\omega t} \cdot \alpha^k = e^{i\psi_k} \tag{7}
$$

Since, α is an Nth root of unity we also have:

$$\alpha^{N-j} = \frac{\alpha^N}{\alpha^j} = \frac{1}{|\alpha|^{2j}} \cdot \left(\overline{\alpha}\right)^j = \overline{\alpha^j} \qquad (8)$$

Since:

$$|\alpha| = 1 \qquad (9)$$

Rewriting the equations and premultiplying by powers of $e^{i\omega t}$ we find:

$$\begin{aligned}
&\theta_0 = \zeta_0 + \zeta_1 + \zeta_2 + ... + \zeta_{N-1} \\
&e^{i\omega t} \cdot \theta_1 = e^{i\omega t} \cdot \zeta_0 + \zeta_1 e^{i\omega t} \cdot \alpha + \zeta_2 e^{i\omega t} \cdot \alpha^2 + ... + \zeta_{N-1} e^{i\omega t} \cdot \alpha^{N-1} \\
&e^{2i\omega t} \cdot \theta_2 = e^{2i\omega t} \cdot \zeta_0 + \zeta_1 e^{2i\omega t} \cdot \alpha^2 + \zeta_2 e^{2i\omega t} \cdot \alpha^4 + ... + \zeta_{N-1} e^{2i\omega t} \cdot \alpha^{2(N-1)} \\
&\Downarrow \\
&e^{-2i\omega t} \cdot \theta_{N-2} = e^{-2i\omega t} \zeta_0 + \zeta_1 e^{-2i\omega t} \cdot \alpha^{N-2} + \zeta_2 e^{-2i\omega t} \cdot \alpha^{2(N-2)} + ... + \zeta_{N-1} e^{-2i\omega t} \cdot \alpha^{(N-2)(N-1)} \\
&e^{-i\omega t} \cdot \theta_{N-1} = e^{-i\omega t} \cdot \zeta_0 + \zeta_1 e^{-i\omega t} \cdot \alpha^{N-1} + \zeta_2 e^{-i\omega t} \cdot \alpha^{2(N-1)} + ... + \zeta_{N-1} e^{-i\omega t} \cdot \alpha^{(N-1)(N-1)}
\end{aligned} \qquad (10)$$

Since α is an Nth root of unity the last two equations can be rewritten:

$$\begin{aligned}
&e^{-2i\omega t} \cdot \theta_{N-2} = e^{-2i\omega t} \zeta_0 + \zeta_1 e^{-2i\omega t} \cdot \alpha^{-2} + \zeta_2 e^{-2i\omega t} \cdot \alpha^{-4} + ... + \zeta_{N-1} e^{-2i\omega t} \cdot \alpha^{-2(N-1)} \\
&e^{-i\omega t} \cdot \theta_{N-1} = e^{-i\omega t} \cdot \zeta_0 + \zeta_1 e^{-i\omega t} \cdot \alpha^{-1} + \zeta_2 e^{-i\omega t} \cdot \alpha^{-2} + ... + \zeta_{N-1} e^{-i\omega t} \cdot \alpha^{-(N-1)}
\end{aligned} \qquad (11)$$

Construction of the Coleman Variables

The first equation

$$\theta_0 = \zeta_0 + \zeta_1 + \zeta_2 + ... + \zeta_{N-1} \tag{12}$$

is left untouched.

The equations are then paired:

$$\begin{array}{l} e^{i\omega t}\theta_1 \Leftrightarrow e^{-i\omega t}\theta_{N-1} \\ e^{2i\omega t}\theta_2 \Leftrightarrow e^{-2i\omega t}\theta_{N-2} \\ e^{3i\omega t}\theta_3 \Leftrightarrow e^{-3i\omega t}\theta_{N-3} \end{array} \tag{13}$$

these are then added and subtracted to give real expressions in cosine and sine.

For instance, let us examine the first pairing:

$$\frac{\left(e^{i\omega t} \cdot \theta_1 + e^{-i\omega t} \cdot \theta_{N-1}\right)}{2} = \frac{1}{2}\left(\begin{array}{l} e^{i\omega t} \cdot \zeta_0 + e^{-i\omega t} \cdot \zeta_0 \\ + \zeta_1 e^{i\omega t} \cdot \alpha + \zeta_1 e^{-i\omega t} \cdot \alpha^{N-1} \\ + \zeta_2 e^{i\omega t} \cdot \alpha^2 + \zeta_2 e^{-i\omega t} \cdot \alpha^{2(N-1)} .... \\ + \zeta_{N-1} e^{i\omega t} \cdot \alpha^{N-1} + \zeta_{N-1} e^{-i\omega t} \cdot \alpha^{(N-1)(N-1)} \end{array} \right) \tag{14}$$

and since:

$$\begin{array}{l} \alpha^N = 1 \\ \alpha^{N-1} = \alpha^{-1} \\ \alpha^{(N-1)(N-1)} = \alpha^{-(N-1)} \end{array} \tag{15}$$

we find:

$$\frac{\left(e^{i\omega t}\cdot\theta_1+e^{-i\omega t}\cdot\theta_{N-1}\right)}{2}=\frac{1}{2}\left(\begin{array}{l} e^{i\omega t}\cdot\zeta_0+e^{-i\omega t}\cdot\zeta_0 \\ +\zeta_1 e^{i\omega t}\cdot\alpha+\zeta_1 e^{-i\omega t}\cdot\alpha^{-1} \\ +\zeta_2 e^{i\omega t}\cdot\alpha^2+\zeta_2 e^{-i\omega t}\cdot\alpha^{-2}.... \\ +\zeta_{N-1} e^{i\omega t}\cdot\alpha^{N-1}+\zeta_{N-1} e^{-i\omega t}\cdot\alpha^{-(N-1)} \end{array}\right) \qquad (16)$$

Recalling (7), we have finally:

$$\frac{\left(e^{i\omega t}\cdot\theta_1+e^{-i\omega t}\cdot\theta_{N-1}\right)}{2}=\left(\begin{array}{l} \zeta_0\cdot\cos\psi_0 \\ +\zeta_1\cdot\cos\psi_1 \\ +\zeta_2\cdot\cos\psi_2 \\ .... \\ +\zeta_{N-1}\cdot\cos\psi_{N-1} \end{array}\right) \qquad (17)$$

and by a similar process:

$$\frac{\left(e^{i\omega t}\cdot\theta_1-e^{-i\omega t}\cdot\theta_{N-1}\right)}{2}=\left(\begin{array}{l} \zeta_0\cdot\sin\psi_0 \\ +\zeta_1\cdot\sin\psi_1 \\ +\zeta_2\cdot\sin\psi_2 \\ .... \\ +\zeta_{N-1}\cdot\sin\psi_{N-1} \end{array}\right) \qquad (18)$$

If the rotor has an odd number of blades (N) this process will complete the transformation. If the number of blades (N) is even then there will be one equation (θ_M) left. In this case the multiplication by $e^{Mi\omega t}$ does <u>not</u> take place and we have:

$$\theta_M=\zeta_0+\zeta_1\alpha^M+\zeta_2\alpha^{2M}+....+\zeta_{N-1}\alpha^{M(N-1)} \qquad (19)$$

here we have:

$$M = \frac{N}{2}$$

as can be seen the multiplying factors for the N lag angles, ζ_k, are powers of α^M.

Since:

$$\alpha = e^{\frac{2\pi i}{N}} \tag{20}$$

we find:

$$\begin{aligned}\alpha^M &= \left(e^{\frac{2\pi i}{N}}\right)^M \\ &= e^{\frac{2\pi M i}{N}} \\ &= e^{\pi i} \\ &= -1\end{aligned} \tag{21}$$

In other words the coefficients of the ζ_k terms are powers of -1, in other words of alternating sign.

Reference

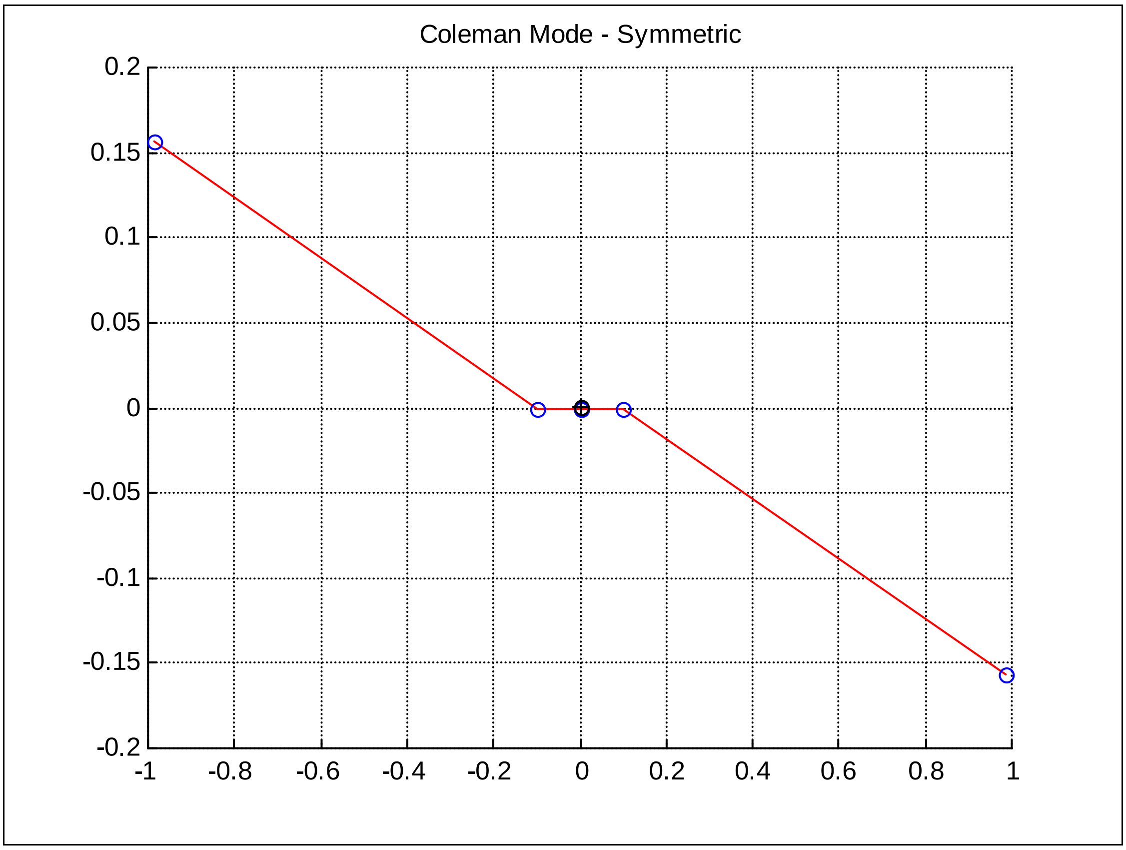
R.P. **Coleman**, A.M. **Feingold**, Theory of self-excited mechanical oscillations of helicopter rotors with hinged blades, NACA Report TN 1351, 1958

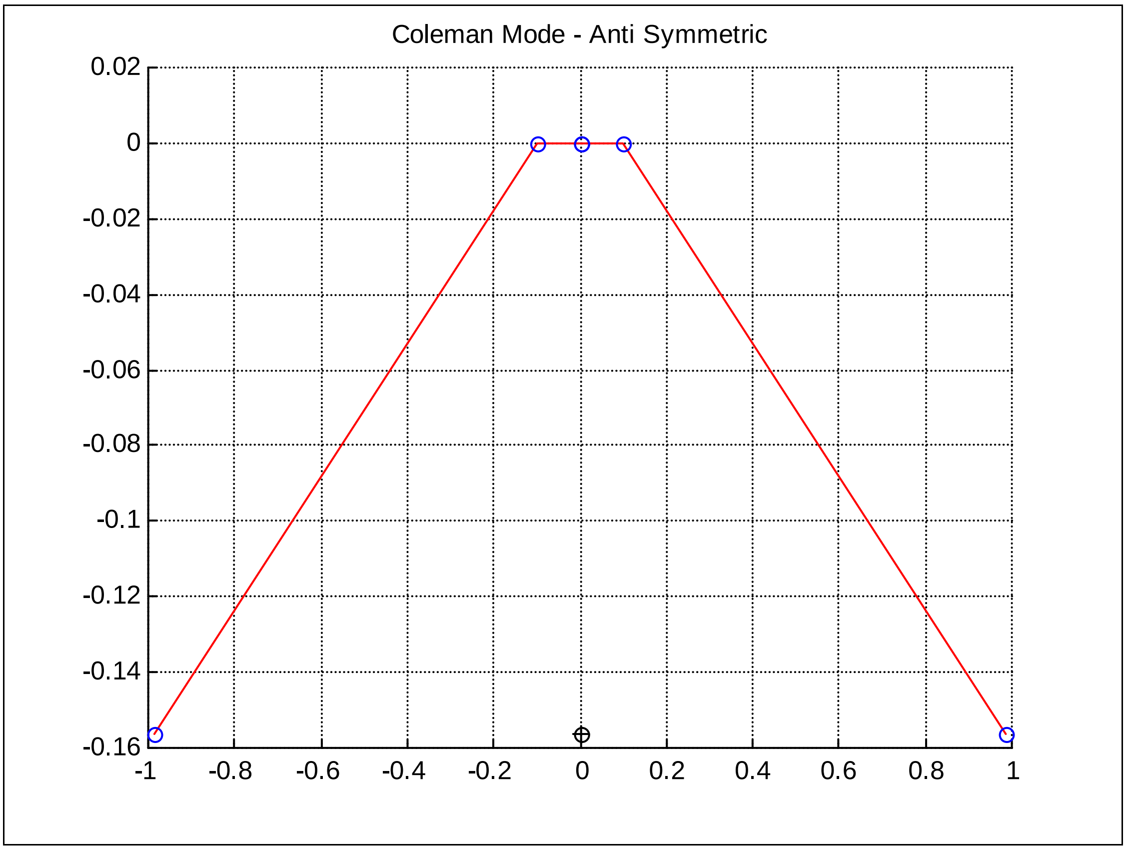
Examples

We therefore have the following examples of the transformation:

N=2

$$\Theta_0 = \zeta_0 + \zeta_1$$
$$\Theta_1 = \zeta_0 - \zeta_1$$





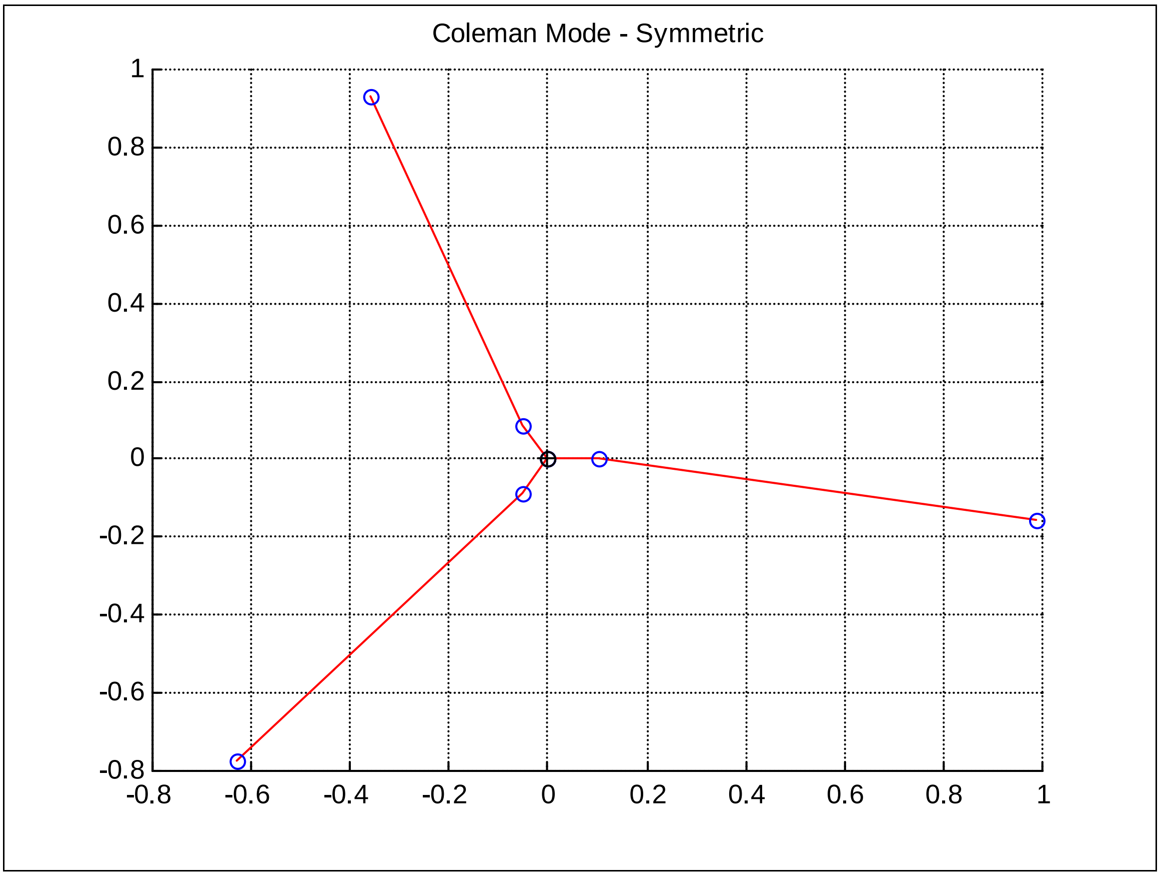
Coleman Mode - Anti Symmetric
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14
-0.16
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

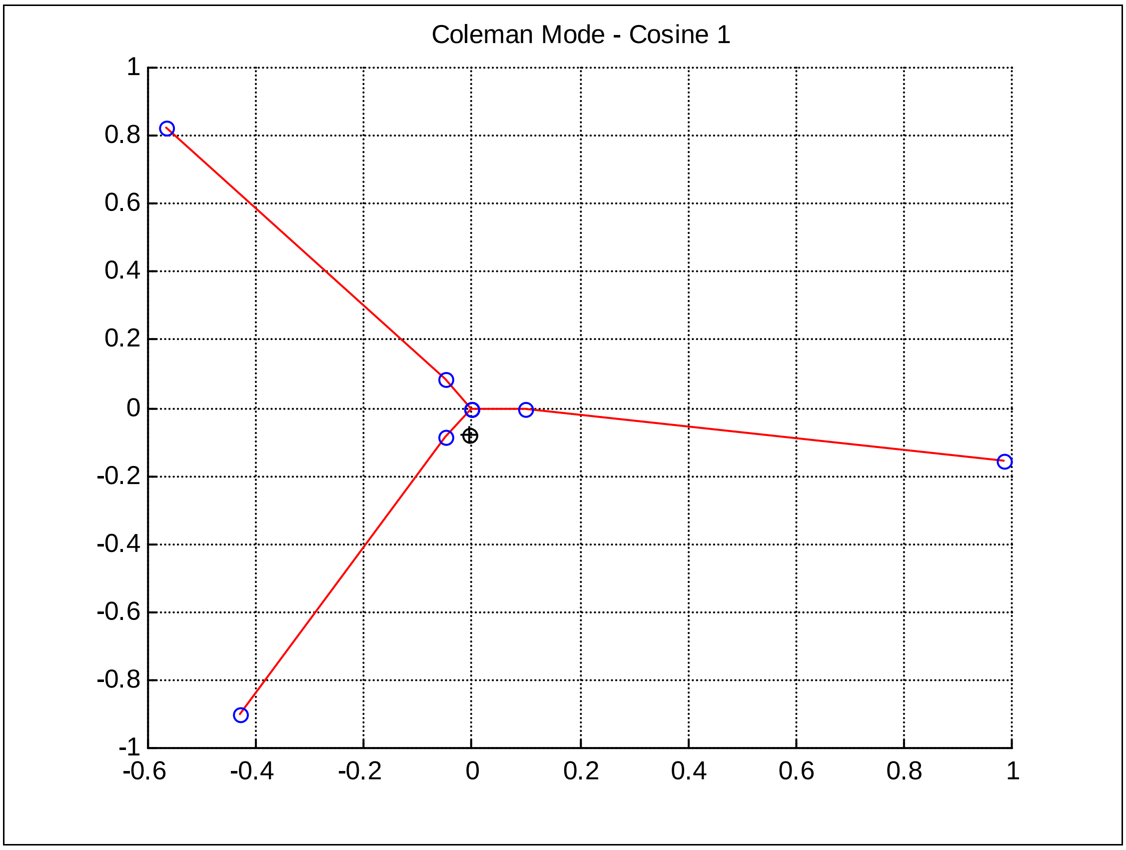
N=3

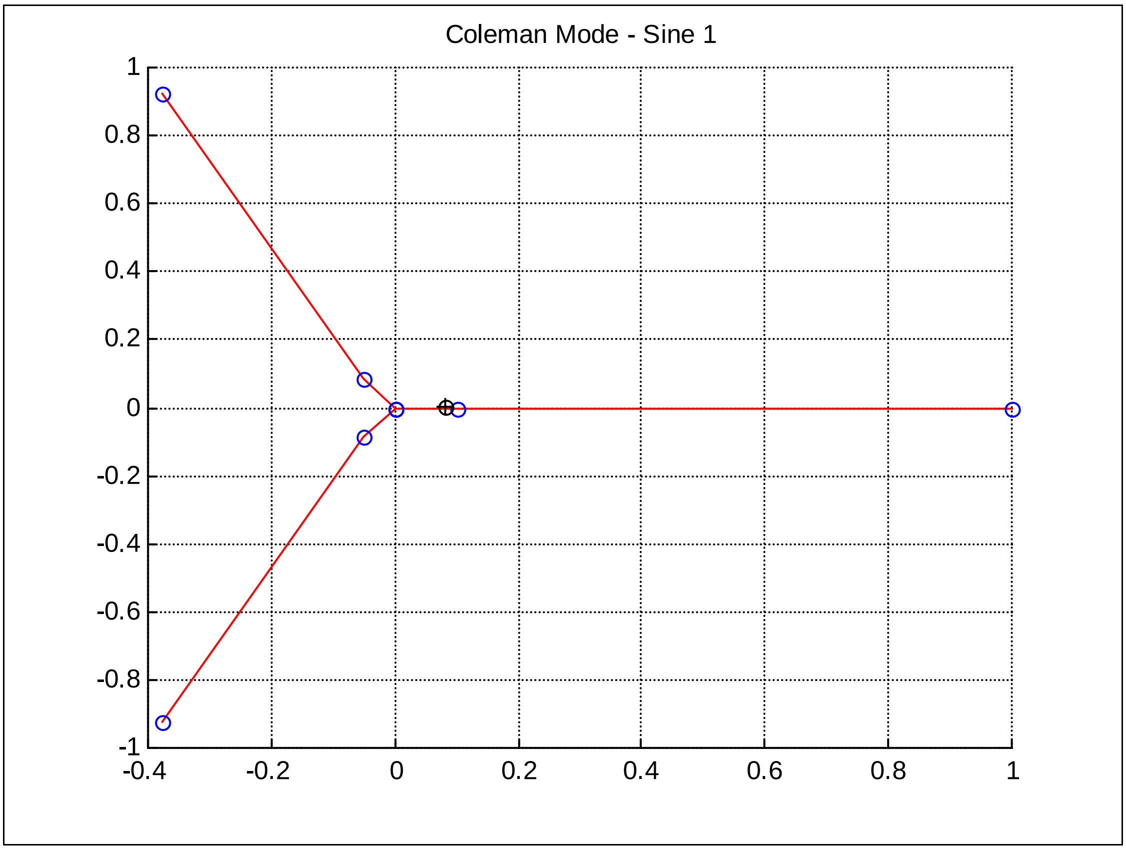
$$\Theta_0 = \zeta_0 + \zeta_1 + \zeta_2$$

$$\Theta_1 = \zeta_0 \cos\psi_0 + \zeta_1 \cos\psi_1 + \zeta_2 \cos\psi_2$$

$$\Theta_2 = \zeta_0 \sin\psi_0 + \zeta_1 \sin\psi_1 + \zeta_2 \sin\psi_2$$


Coleman Mode - Symmetric
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Coleman Mode - Cosine 1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Coleman Mode - Sine 1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

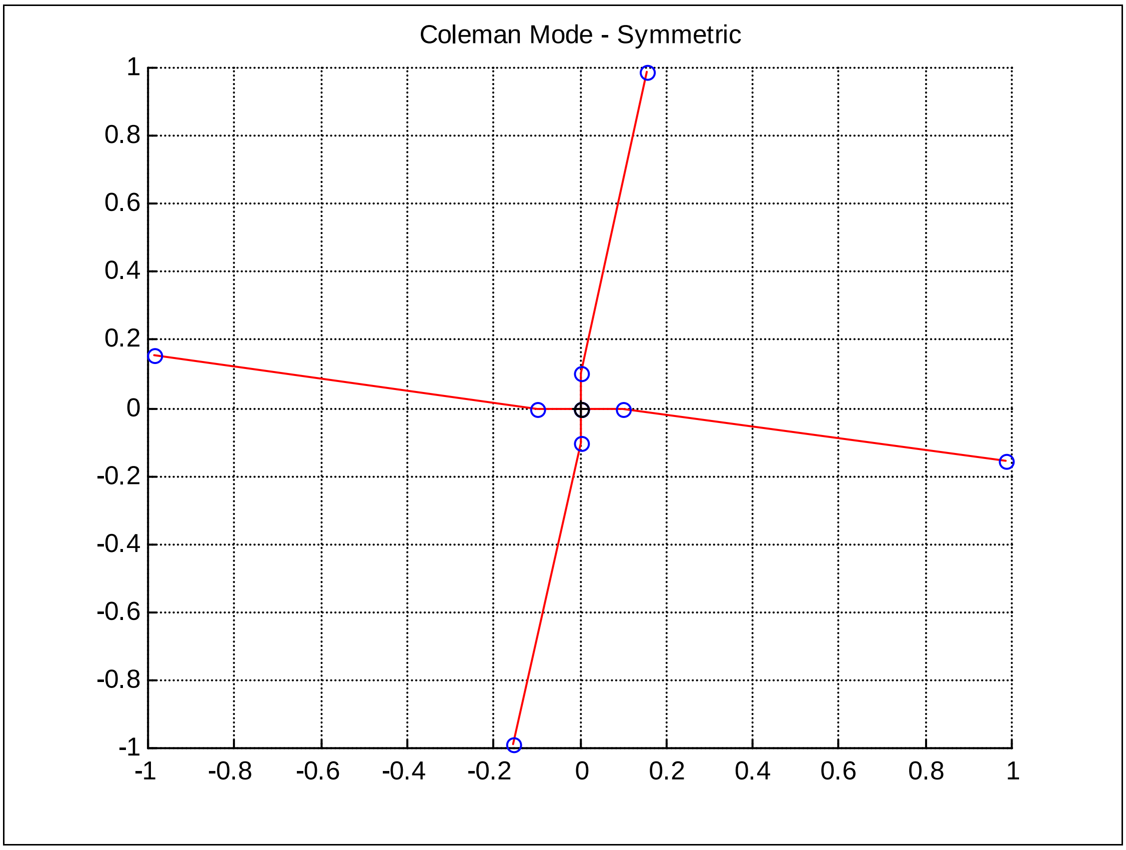
N=4

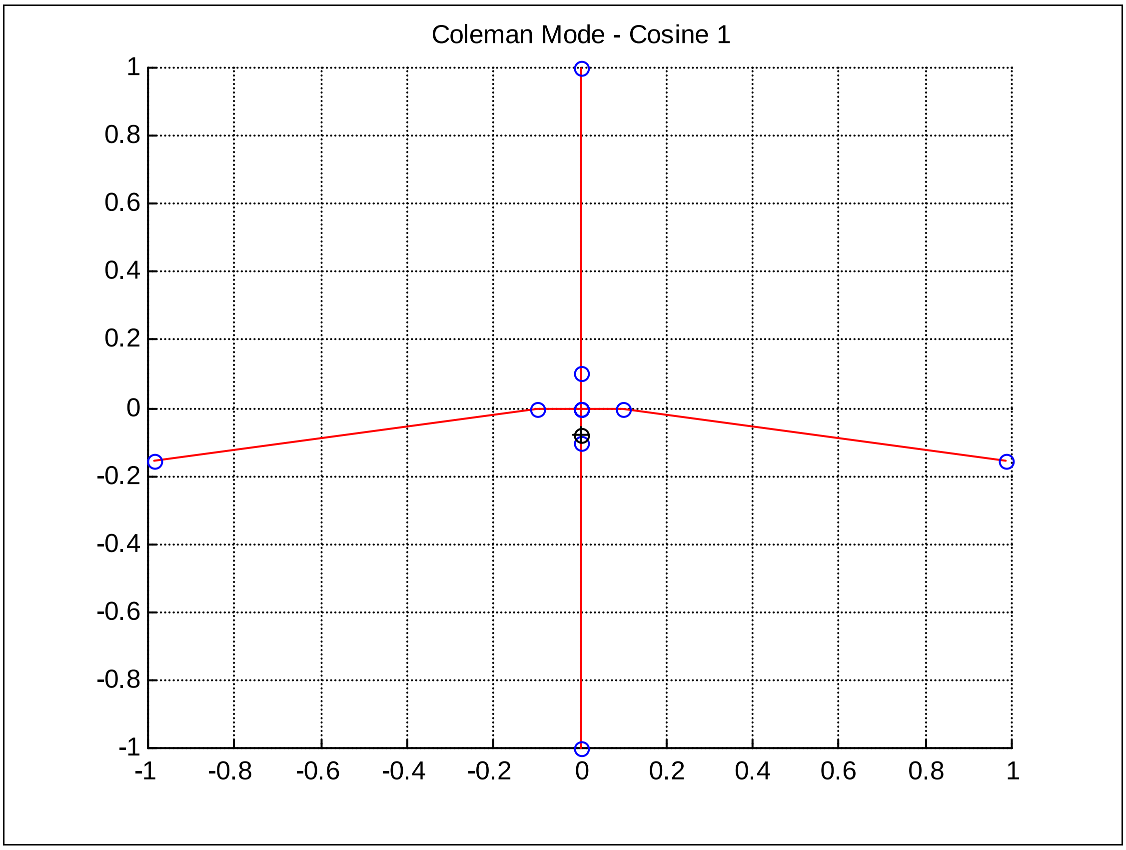
$$\Theta_0 = \zeta_0 + \zeta_1 + \zeta_2 + \zeta_3$$

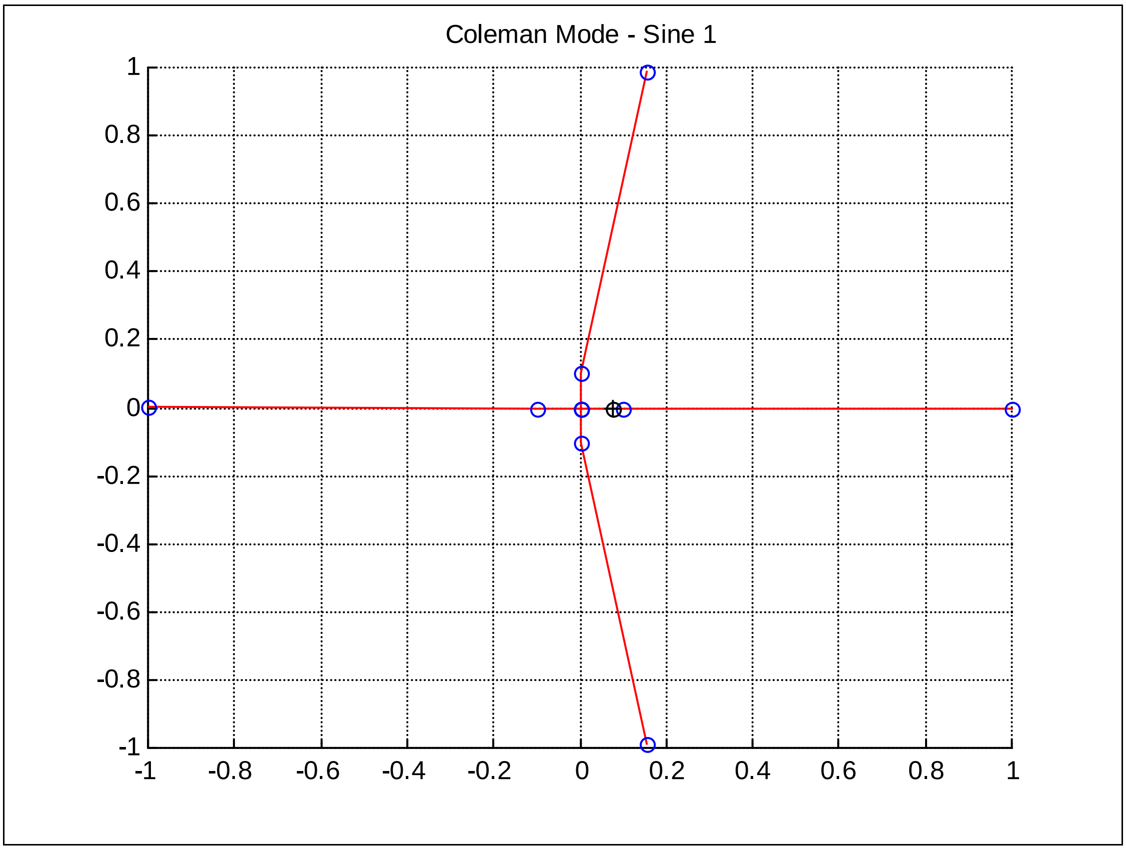
$$\Theta_1 = \sum_k \zeta_k \cos\psi_k$$

$$\Theta_2 = \sum_k \zeta_k \sin\psi_k$$

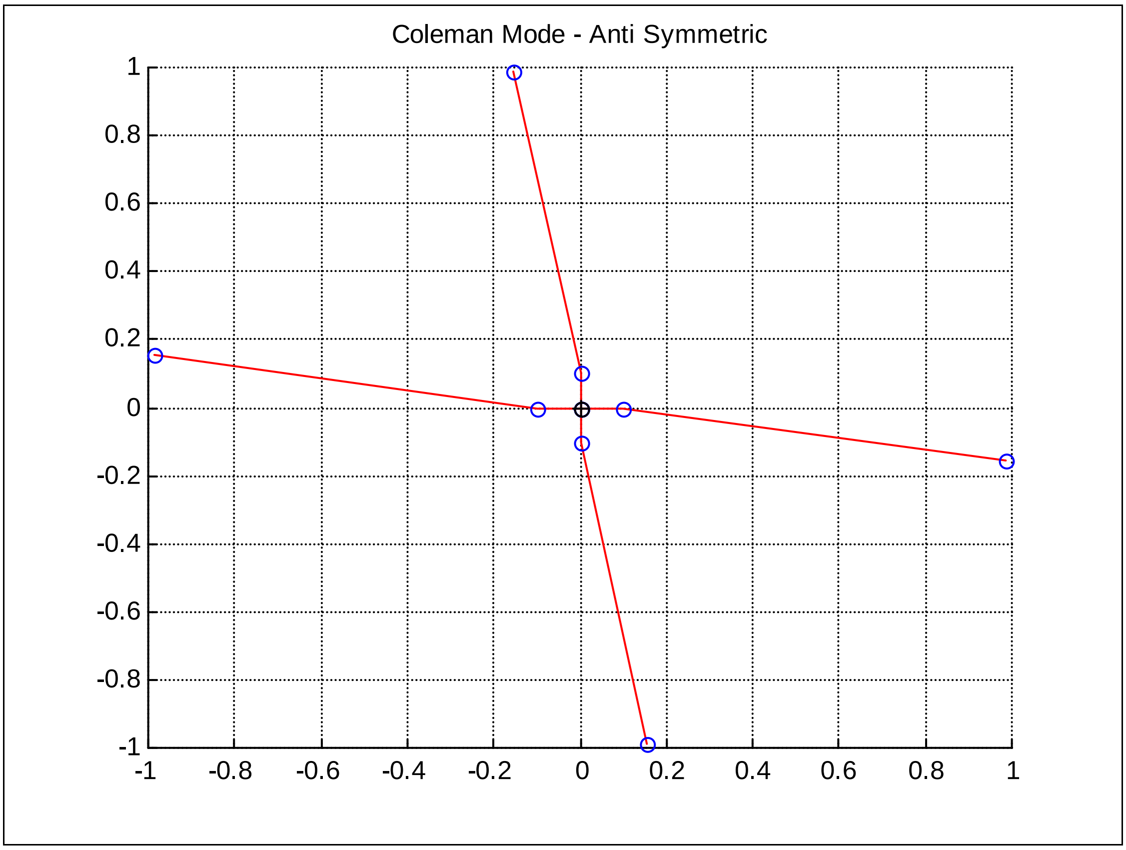
$$\Theta_3 = \zeta_0 - \zeta_1 + \zeta_2 - \zeta_3$$


Coleman Mode - Symmetric
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Coleman Mode - Cosine 1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



Coleman Mode - Sine 1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



Coleman Mode - Anti Symmetric
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

N=5

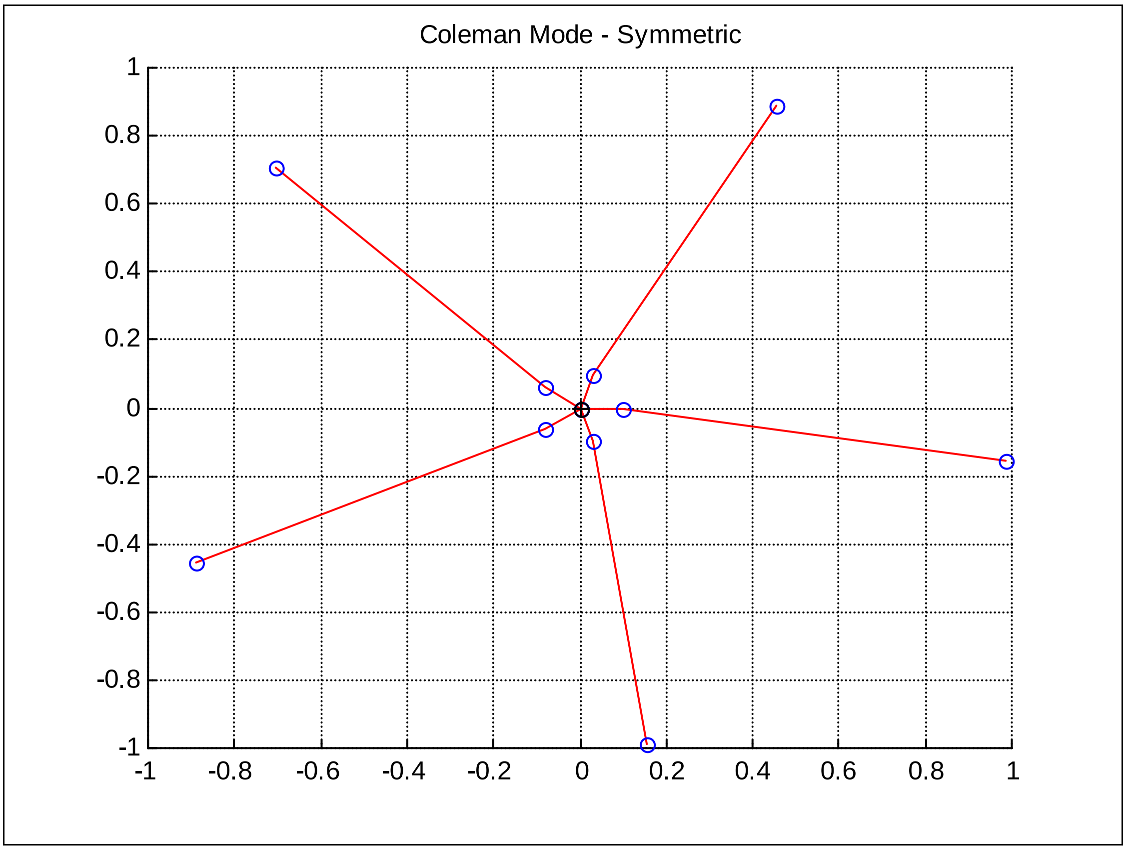
$$\Theta_0 = \zeta_0 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4$$

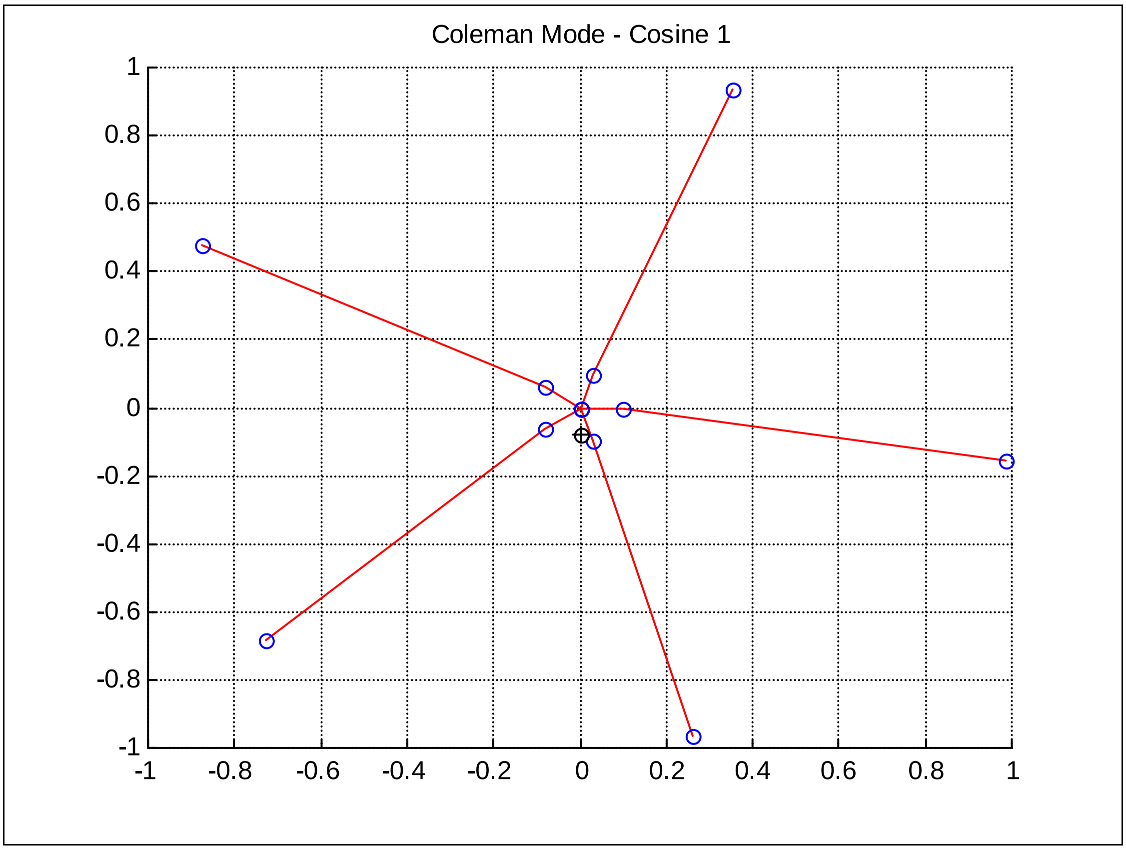
$$\Theta_1 = \sum_k \zeta_k \cos \psi_k$$

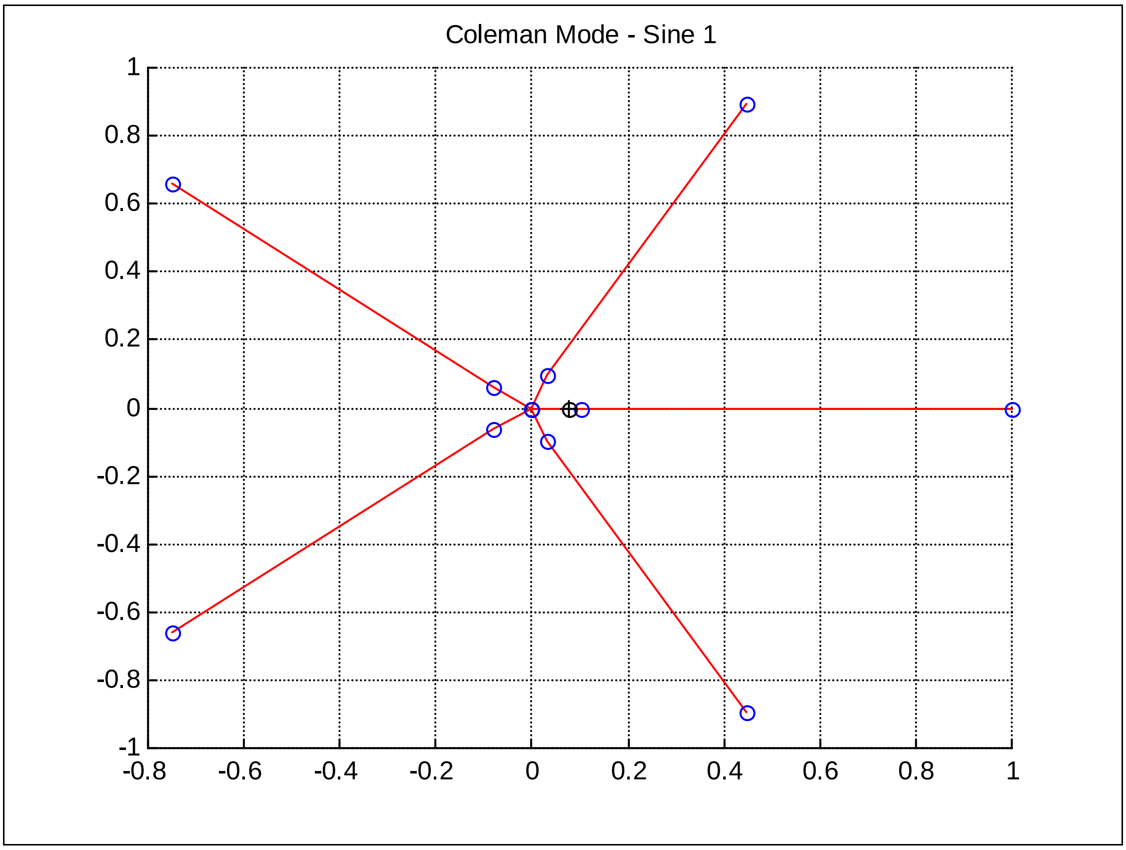
$$\Theta_2 = \sum_k \zeta_k \sin \psi_k$$

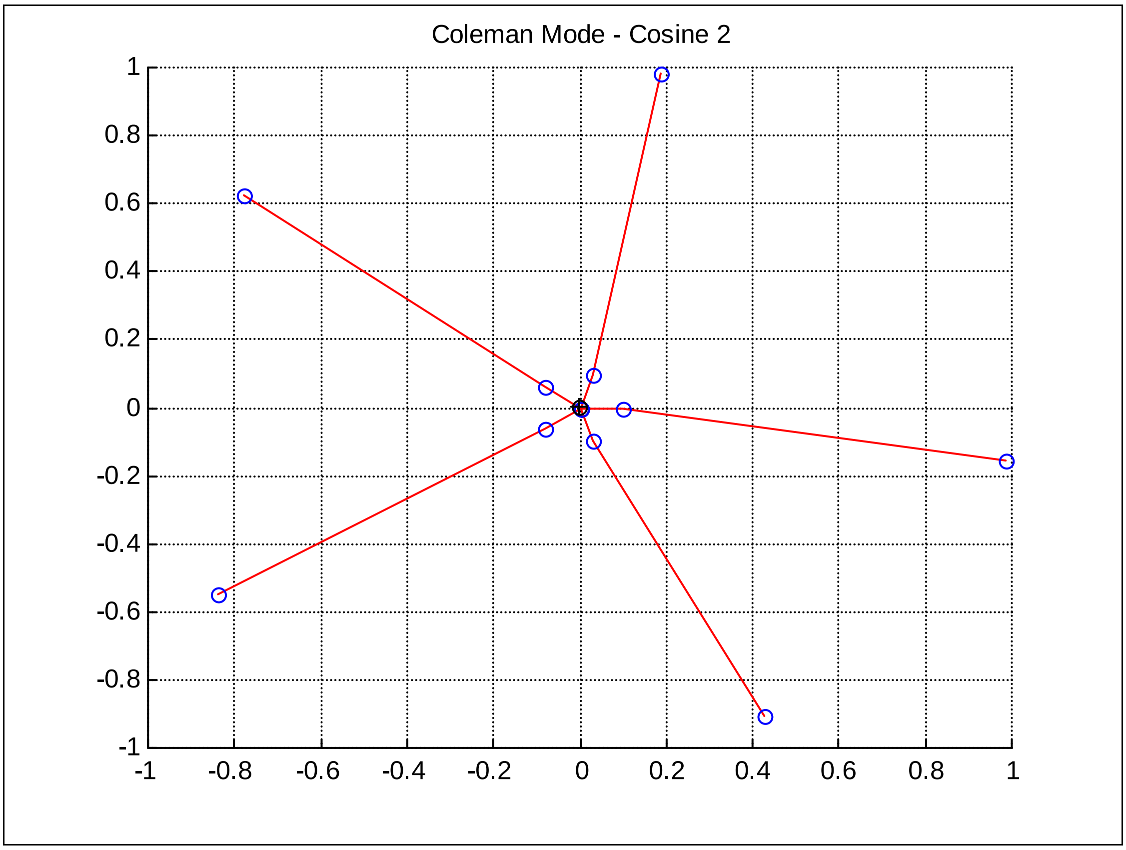
$$\Theta_3 = \sum_k \zeta_k \cos 2\psi_k$$

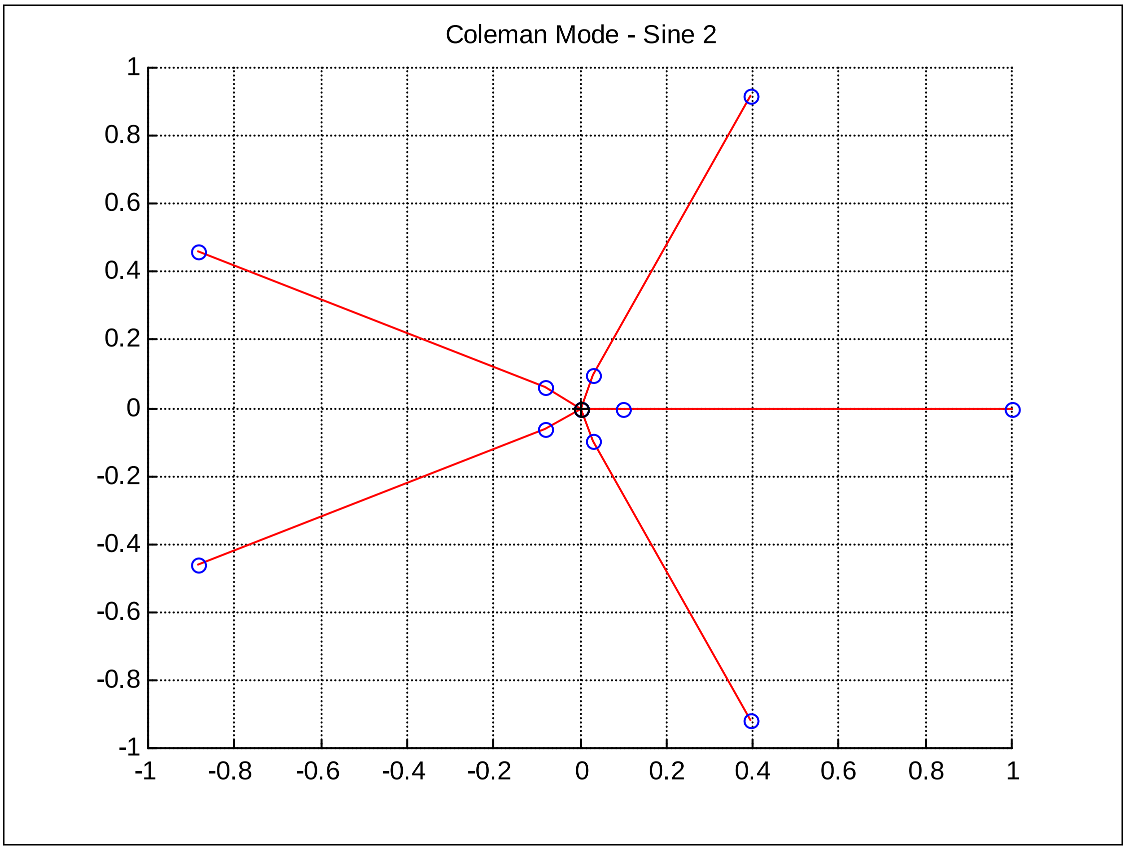
$$\Theta_4 = \sum_k \zeta_k \sin 2\psi_k$$




Coleman Mode - Cosine 1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Coleman Mode - Sine 1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Coleman Mode - Cosine 2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Coleman Mode - Sine 2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Or in tabular form:

N	Σ+	Σ+-	Cosψ	Cos 2ψ	Cos 3ψ	Cos 4ψ	Sinψ	Sin 2ψ	Sin 3ψ	Sin 4ψ
1	✓									
2	✓	✓								
3	✓		✓				✓			
4	✓	✓	✓				✓			
5	✓		✓	✓			✓	✓		
6	✓	✓	✓	✓			✓	✓		
7	✓		✓	✓	✓		✓	✓	✓	
8	✓	✓	✓	✓	✓		✓	✓	✓	
9	✓		✓	✓	✓	✓	✓	✓	✓	✓
10	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓