

# **Aerodynamics & Flight Mechanics Research Group**

## **The Modes of Vibration of a Mass Spring System**

S. J. Newman

Technical Report AFM-11/22

January 2011

UNIVERSITY OF SOUTHAMPTON

SCHOOL OF ENGINEERING SCIENCES

AERODYNAMICS AND FLIGHT MECHANICS RESEARCH GROUP

**The Modes of Vibration of a Mass Spring System**

by

**S. J. Newman**

AFM Report No. AFM 11/22

January 2011

© School of Engineering Sciences, Aerodynamics and Flight Mechanics Research Group



## COPYRIGHT NOTICE

(c) SES University of Southampton All rights reserved.

SES authorises you to view and download this document for your personal, non-commercial use. This authorization is not a transfer of title in the document and copies of the document and is subject to the following restrictions: 1) you must retain, on all copies of the document downloaded, all copyright and other proprietary notices contained in the Materials; 2) you may not modify the document in any way or reproduce or publicly display, perform, or distribute or otherwise use it for any public or commercial purpose; and 3) you must not transfer the document to any other person unless you give them notice of, and they agree to accept, the obligations arising under these terms and conditions of use. This document, is protected by worldwide copyright laws and treaty provisions.



# Preamble

---

This document examines a system of masses and springs placed in line on a frictionless table. From these equations, the modal frequencies and shapes can be derived.

## *1 Degree of Freedom*

---

The system is shown in Figure 1

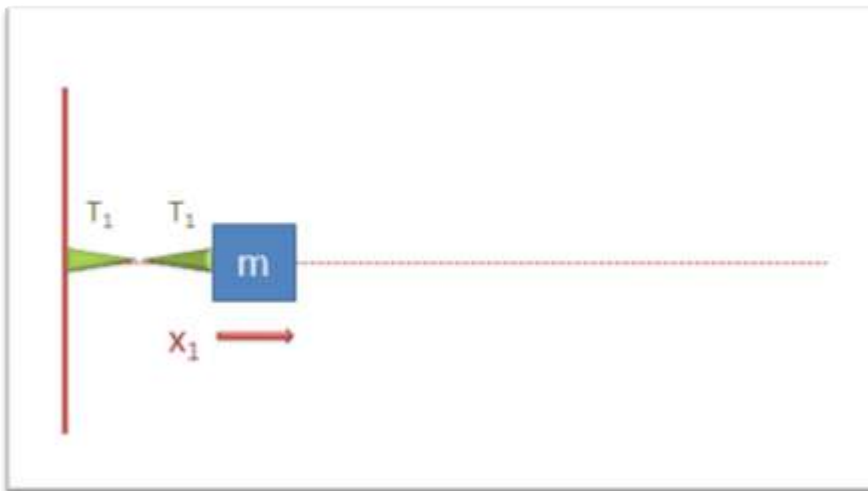


Figure 1

The spring tension is given by:

$$T_1 = kx_1 \quad (1.)$$

The equation of motion is then:

$$-T_1 = m\ddot{x}_1 \quad (2.)$$

From which we obtain:

$$m\ddot{x}_1 = -kx_1 \quad (3.)$$

whence:

$$m\ddot{x}_1 + kx_1 = 0 \quad (4.)$$



This standard result gives SHM with circular frequency given by:

$$\omega_0^2 = \frac{k}{m} \quad (5.)$$



2 Degree of Freedom

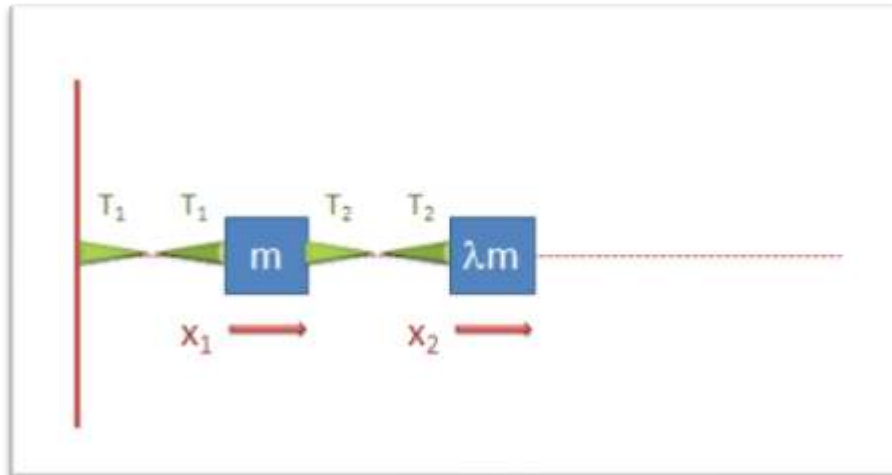


Figure 2

The spring tensions are given by:

$$\begin{aligned} T_1 &= kx_1 \\ T_2 &= k(x_2 - x_1) \end{aligned} \quad (6.)$$

The equations of motion is then:

$$\begin{aligned} T_2 - T_1 &= m\ddot{x}_1 \\ -T_2 &= \lambda m\ddot{x}_2 \end{aligned} \quad (7.)$$

From which we obtain:

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - 2x_1) \\ \lambda m\ddot{x}_2 &= -k(x_2 - x_1) \end{aligned} \quad (8.)$$

and thus:

$$\begin{aligned} m\ddot{x}_1 + k(2x_1 - x_2) &= 0 \\ \lambda m\ddot{x}_2 + k(-x_1 + x_2) &= 0 \end{aligned} \quad (9.)$$

and in matrix form:



$$m \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (10.)$$

---



3 Degree of Freedom

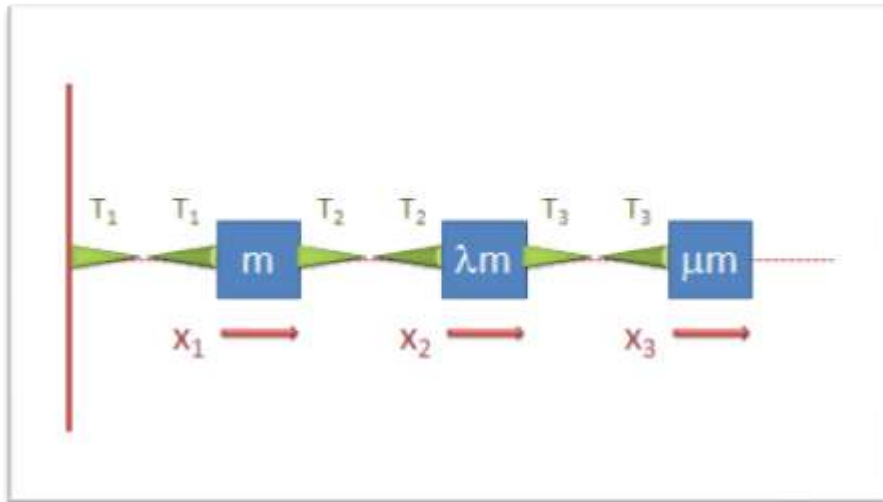


Figure 3

The spring tensions are given by:

$$\begin{aligned} T_1 &= kx_1 \\ T_2 &= k(x_2 - x_1) \\ T_3 &= k(x_3 - x_2) \end{aligned} \quad (11.)$$

The equations of motion are then:

$$\begin{aligned} T_2 - T_1 &= m\ddot{x}_1 = k(x_2 - 2x_1) \\ T_3 - T_2 &= \lambda m\ddot{x}_2 = k(x_3 - 2x_2 + x_1) \\ -T_3 &= \mu m\ddot{x}_3 = k(x_2 - x_3) \end{aligned} \quad (12.)$$

from which we obtain:

$$\begin{aligned} m \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ = 0 \end{aligned} \quad (13.)$$





4 Degree of Freedom

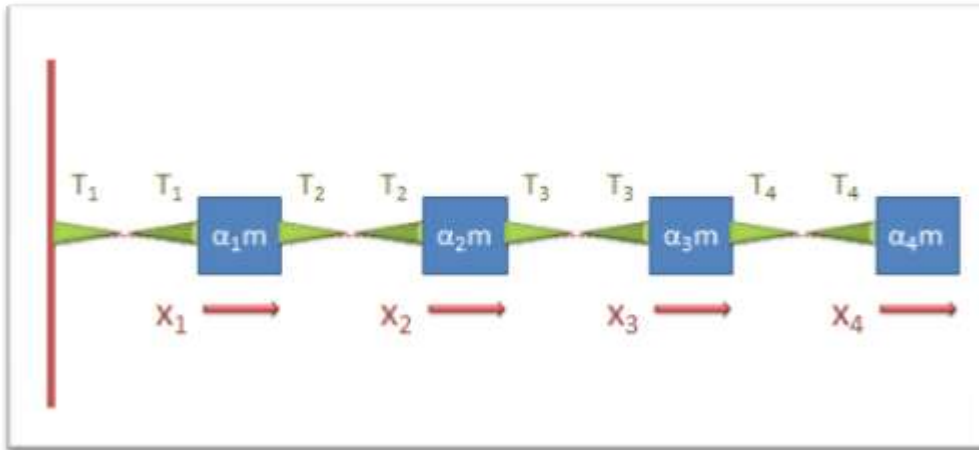


Figure 4

The spring tensions are given by:

$$\begin{aligned}
 T_1 &= kx_1 \\
 T_2 &= k(x_2 - x_1) \\
 T_3 &= k(x_3 - x_2) \\
 T_4 &= k(x_4 - x_3)
 \end{aligned}
 \tag{14.}$$

The equations of motion are then:

$$\begin{aligned}
 T_2 - T_1 &= \alpha_1 m \ddot{x}_1 = k(x_2 - 2x_1) \\
 T_3 - T_2 &= \alpha_2 m \ddot{x}_2 = k(x_3 - 2x_2 + x_1) \\
 T_4 - T_3 &= \alpha_3 m \ddot{x}_3 = k(x_4 - 2x_3 + x_2) \\
 -T_4 &= \alpha_4 m \ddot{x}_4 = k(x_3 - x_4)
 \end{aligned}
 \tag{15.}$$

From which we obtain:



$$m \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad (16.)$$

The above matrix equations are of the form:

$$M\ddot{X} + KX = 0 \quad (17.)$$

Here M is a mass matrix and K is a stiffness matrix.

The basic pattern can now be seen.

If we define the following matrices:

$$M_1 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (18.)$$

$$M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The mass and stiffness matrices can be constructed via:

$$\begin{aligned} K &= k\{2M_1 - M_3 - M_2 - M_2'\} \\ K &= k\bar{K} \end{aligned} \tag{19.}$$

$$\begin{aligned} M &= m \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \\ M &= m\bar{M} \end{aligned} \tag{20.}$$



# Modal Solution

---

In order to determine the modes, we use equation (17).

If the masses are executing SHM we have:

$$\ddot{X} = -\omega^2 X = 0 \quad (21.)$$

Substituting into (17) gives:

$$-\omega^2 MX + KX = 0 \quad (22.)$$

whence:

$$M^{-1}KX - \omega^2 X = 0 \quad (23.)$$

i.e. we have an eigenvalue problem where the modal frequencies are the eigenvalues and the mode shapes the corresponding eigenvectors.



# Non-Dimensionalisation

---

Using (19) & (20):

$$\frac{k}{m} (\bar{M}^{-1} \bar{K}) X - \omega^2 X = 0 \quad (24.)$$

Defining:

$$\bar{\omega} = \frac{\omega}{\omega_0} \quad (25.)$$

$$(\bar{M}^{-1} \bar{K}) X - \bar{\omega}^2 X = 0 \quad (26.)$$

This is the non-dimensional equivalent.



# MATLAB File - Animation

---

```
%  
%   N DOF Modes  
%  
%   SJN 23/02/08  
%  
colordef black  
clear  
nmode=4; % Define No of masses/springs  
%masses=ones(1,nmode); % Define masses - matrix leading diagonal  
masses=[1 1 1 1]; % Define masses - matrix leading diagonal  
%-----  
m1=eye(nmode);  
m2=zeros(nmode);  
for i=1:nmode-1  
    m2(i,i+1)=1;  
end  
m3=zeros(nmode);  
m3(nmode,nmode)=1;  
stiffmat=2*m1-m3-m2-m2'; % Assemble Stiffness Matrix  
massmat=diag(masses);  
modalmat=inv(massmat)*stiffmat;  
[modeshapes,modfreq]=eig(modalmat); % Calculate Modal Data via Eigenvalues  
for iif=1:nmode  
    freq(iif)=sqrt(modfreq(iif,iif));  
end  
%-----  
ntime=1000;  
tmax=100;  
%-----  
t=linspace(0,tmax,ntime);  
%-----  
xmax=5;  
ymax=1;  
earthwidth=.5;  
xmin=-xmax;  
ymin=-ymax;  
xwidth=2*xmax/(nmode+1);  
xplot=xwidth*(1:nmode)+xmin;  
ywidth=2*ymax/(nmode+1);  
yplot=ywidth*(1:nmode)+ymin;  
%-----  
%   Commence Time Loop  
%-----  
for it=1:ntime  
    clf  
    hold on  
    axis off  
    axis([xmin xmax ymin ymax]);  
    for im=1:nmode  
        moderesp=modeshapes(:,im)*sin(freq(im)*t(it))';  
        l1=[xmin moderesp(nmode)+xplot(nmode)];  
        l2=[yplot(im) yplot(im)];  
        plot(l1,l2,'y','LineWidth',3);  
        for imm=1:nmode
```



```

plot(xplot(imm)+moderesp(imm),yplot(im),'oy','LineWidth',2,'MarkerSize',10,
'MarkerFaceColor','r');
    end
    end
    xearth=[(xmin+earthwidth) xmin xmin (xmin+earthwidth)];
    yearth=[ymin ymin ymax ymax];
    fill(xearth,yearth,'y');
    title(['No of Modes = ',num2str(nmode),' <<<<<>>>> Masses
[' ,num2str(masses), ']]');
%-----
%   Input Figure into Display Matrix
m(i)=getframe;
%-----
end
%-----
%   Create AVI
%movie2avi(m,'fredn','compression','none');
%-----

```

