

Aerodynamics & Flight Mechanics Research Group

The Modes of Vibration of a Mass Spring System

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Preamble

This document examines a system of masses and springs placed in line on a frictionless table. From these equations, the modal frequencies and shapes can be derived.

1 Degree of Freedom

The system is shown in Figure 1

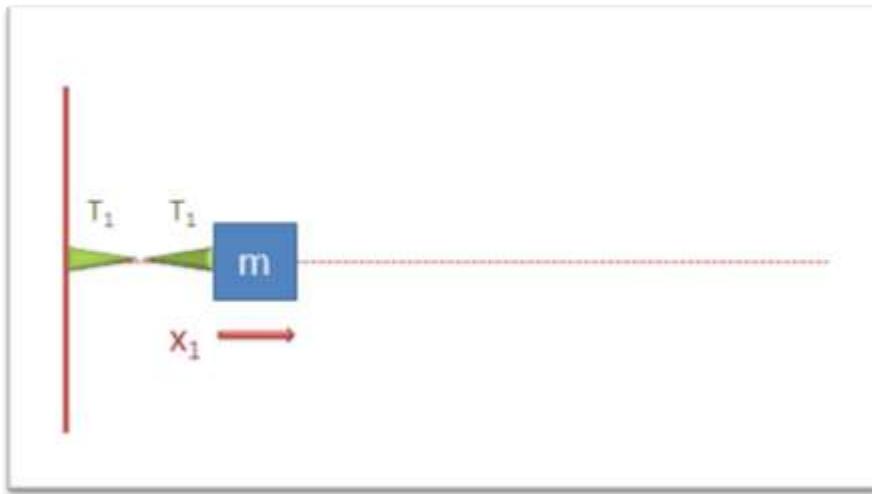


Figure 1

The spring tension is given by:

$$T_1 = kx_1 \quad (1.)$$

The equation of motion is then:

$$-T_1 = m\ddot{x}_1 \quad (2.)$$

From which we obtain:

$$m\ddot{x}_1 = -kx_1 \quad (3.)$$

whence:

$$m\ddot{x}_1 + kx_1 = 0 \quad (4.)$$



This standard result gives SHM with circular frequency given by:

$$\omega_0^2 = \frac{k}{m} \quad (5.)$$



2 Degree of Freedom

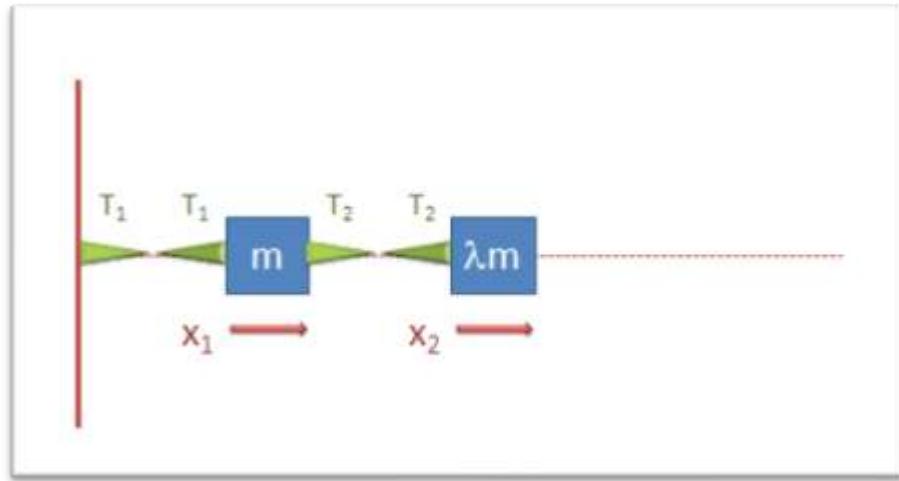


Figure 2

The spring tensions are given by:

$$\begin{aligned} T_1 &= kx_1 \\ T_2 &= k(x_2 - x_1) \end{aligned} \quad (6.)$$

The equations of motion is then:

$$\begin{aligned} T_2 - T_1 &= m\ddot{x}_1 \\ -T_2 &= \lambda m\ddot{x}_2 \end{aligned} \quad (7.)$$

From which we obtain:

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - 2x_1) \\ \lambda m\ddot{x}_2 &= -k(x_2 - x_1) \end{aligned} \quad (8.)$$

and thus:

$$\begin{aligned} m\ddot{x}_1 + k(2x_1 - x_2) &= 0 \\ \lambda m\ddot{x}_2 + k(-x_1 + x_2) &= 0 \end{aligned} \quad (9.)$$

and in matrix form:



$$m \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (10.)$$



3 Degree of Freedom

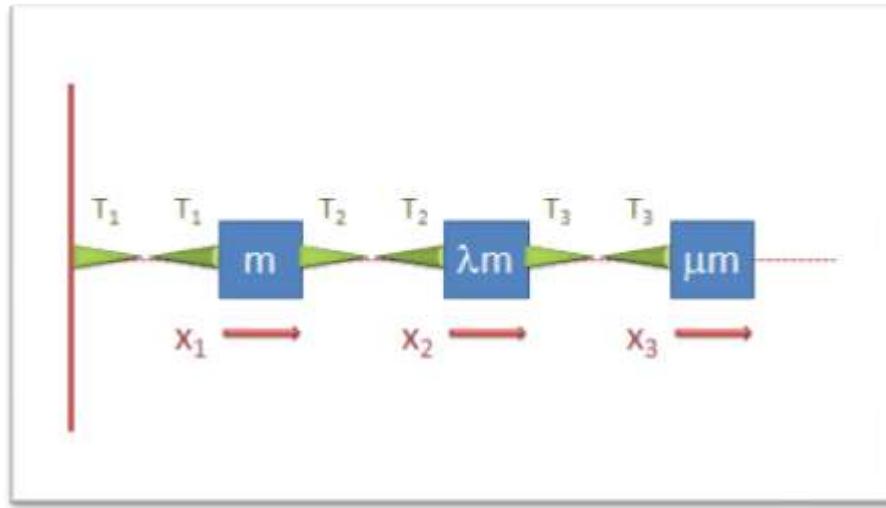


Figure 3

The spring tensions are given by:

$$\begin{aligned} T_1 &= kx_1 \\ T_2 &= k(x_2 - x_1) \\ T_3 &= k(x_3 - x_2) \end{aligned} \quad (11.)$$

The equations of motion are then:

$$\begin{aligned} T_2 - T_1 &= m\ddot{x}_1 = k(x_2 - 2x_1) \\ T_3 - T_2 &= \lambda m\ddot{x}_2 = k(x_3 - 2x_2 + x_1) \\ -T_3 &= \mu m\ddot{x}_3 = k(x_2 - x_3) \end{aligned} \quad (12.)$$

from which we obtain:

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (13.)$$



4 Degree of Freedom

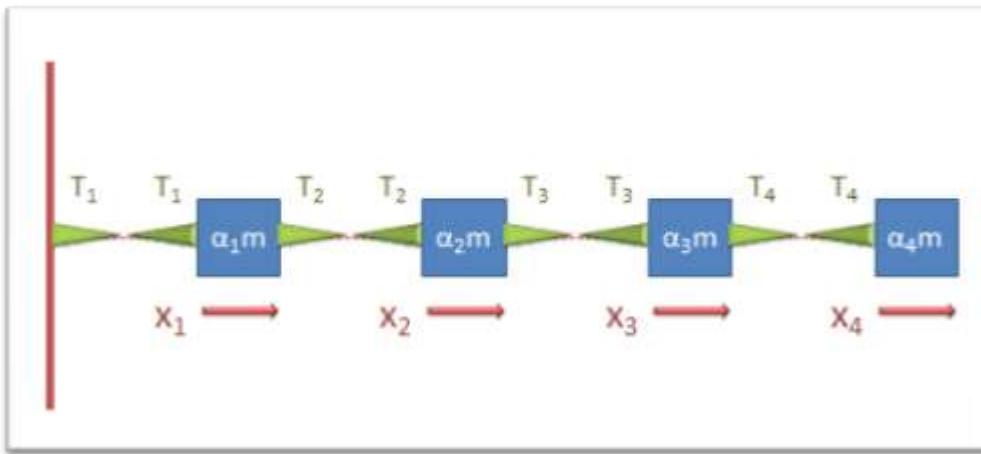


Figure 4

The spring tensions are given by:

$$\begin{aligned}
 T_1 &= kx_1 \\
 T_2 &= k(x_2 - x_1) \\
 T_3 &= k(x_3 - x_2) \\
 T_4 &= k(x_4 - x_3)
 \end{aligned} \tag{14.}$$

The equations of motion are then:

$$\begin{aligned}
 T_2 - T_1 &= \alpha_1 m \ddot{x}_1 = k(x_2 - 2x_1) \\
 T_3 - T_2 &= \alpha_2 m \ddot{x}_2 = k(x_3 - 2x_2 + x_1) \\
 T_4 - T_3 &= \alpha_3 m \ddot{x}_3 = k(x_4 - 2x_3 + x_2) \\
 -T_4 &= \alpha_4 m \ddot{x}_4 = k(x_3 - x_4)
 \end{aligned} \tag{15.}$$

From which we obtain:



$$\begin{aligned}
 & m \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} \\
 & + k \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0
 \end{aligned} \tag{16.}$$

The above matrix equations are of the form:

$$M\ddot{X} + KX = 0 \tag{17.}$$

Here M is a mass matrix and K is a stiffness matrix.

The basic pattern can now be seen.

If we define the following matrices:

$$\begin{aligned}
 M_1 = I &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 M_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned} \tag{18.}$$



$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The mass and stiffness matrices can be constructed via:

$$\begin{aligned} K &= k\{2M_1 - M_3 - M_2 - M_2'\} \\ K &= k\bar{K} \end{aligned} \tag{19.}$$

$$\begin{aligned} M &= m \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \\ M &= m\bar{M} \end{aligned} \tag{20.}$$



Modal Solution

In order to determine the modes, we use equation (17).

If the masses are executing SHM we have:

$$\ddot{X} = -\omega^2 X = 0 \quad (21.)$$

Substituting into (17) gives:

$$-\omega^2 M X + K X = 0 \quad (22.)$$

whence:

$$M^{-1} K X - \omega^2 X = 0 \quad (23.)$$

i.e. we have an eigenvalue problem where the modal frequencies are the eigenvalues and the mode shapes the corresponding eigenvectors.



Non-Dimensionalisation

Using (19) & (20):

$$\frac{k}{m} (\bar{M}^{-1} \bar{K}) X - \omega^2 X = 0 \quad (24.)$$

Defining:

$$\bar{\omega} = \frac{\omega}{\omega_0} \quad (25.)$$

$$(\bar{M}^{-1} \bar{K}) X - \bar{\omega}^2 X = 0 \quad (26.)$$

This is the non-dimensional equivalent.



MATLAB File - Animation

```

%
% N DOF Modes
%
% SJN 23/02/08
%
colordef black
clear
nmode=4; % Define No of masses/springs
%masses=ones(1,nmode); % Define masses - matrix leading diagonal
masses=[1 1 1 1]; % Define masses - matrix leading diagonal
%-----
m1=eye(nmode);
m2=zeros(nmode);
for i=1:nmode-1
    m2(i,i+1)=1;
end
m3=zeros(nmode);
m3(nmode,nmode)=1;
stiffmat=2*m1-m3-m2'; % Assemble Stiffness Matrix
massmat=diag(masses);
modalmat=inv(massmat)*stiffmat;
[modeshapes,modefreq]=eig(modalmat); % Calculate Modal Data via Eigenvalues
for iif=1:nmode
    freq(iif)=sqrt(modefreq(iif,iif));
end
%-----
ntime=1000;
tmax=100;
%-----
t=linspace(0,tmax,ntime);
%-----
xmax=5;
ymax=1;
earthwidth=.5;
xmin=-xmax;
ymin=-ymax;
xwidth=2*xmax/(nmode+1);
xplot=xwidth*(1:nmode)+xmin;
ywidth=2*ymax/(nmode+1);
yplot=ywidth*(1:nmode)+ymin;
%-----
% Commence Time Loop
%-----
for it=1:ntime
    clf
    hold on
    axis off
    axis([xmin xmax ymin ymax]);
    for im=1:nmode
        moderesp=modeshapes(:,im)*sin(freq(im)*t(it))';
        l1=[xmin moderesp(nmode)+xplot(nmode)];
        l2=[yplot(im) yplot(im)];
        plot(l1,l2,'y','LineWidth',3);
        for imm=1:nmode

```



```
plot(xplot(imm)+moderesp(imm),yplot(im),'oy','LineWidth',2,'MarkerSize',10,  
'MarkerFaceColor','r');  
end  
end  
xearth=[(xmin+earthwidth) xmin xmin (xmin+earthwidth)];  
yearth=[ymin ymin ymax ymax];  
fill(xearth,yearth,'y');  
title(['No of Modes = ',num2str(nmode),' <<<<>>>> Masses  
[',num2str(masses),']']);  
%-----  
% Input Figure into Display Matrix  
m(i)=getframe;  
%-----  
end  
%-----  
% Create AVI  
%movie2avi(m,'freadn','compression','none');  
%-----
```

