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The Modes of Vibration of a Mass Spring System

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Preamble

This document examines a system of masses and springs placed in line on a frictionless table. From these equations, the modal frequencies and shapes can be derived.

I Degree of Freedom

The system is shown in Figure 1

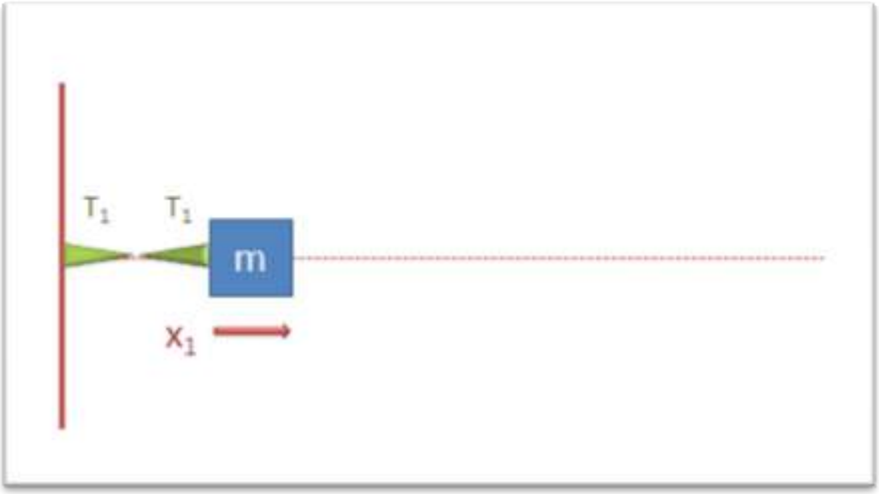

Figure 1

The spring tension is given by:

$$T_1 = kx_1 \tag{1.}$$

The equation of motion is then:

$$-T_1 = m\ddot{x}_1 \tag{2.}$$

From which we obtain:

$$m\ddot{x}_1 = -kx_1 \tag{3.}$$

whence:

$$m\ddot{x}_1 + kx_1 = 0 \tag{4.}$$

This standard result gives SHM with circular frequency given by:

$$\omega_0^2 = \frac{k}{m} \tag{5.}$$

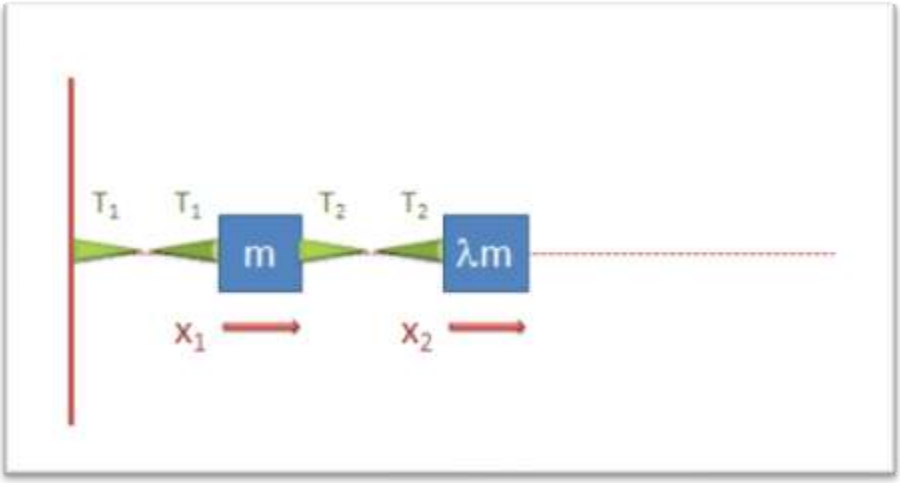

Figure 2

The spring tensions are given by:

$$\begin{aligned} T_1 &= kx_1 \\ T_2 &= k(x_2 - x_1) \end{aligned} \tag{6.}$$

The equations of motion is then:

$$\begin{aligned} T_2 - T_1 &= m\ddot{x}_1 \\ -T_2 &= \lambda m\ddot{x}_2 \end{aligned} \tag{7.}$$

From which we obtain:

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - 2x_1) \\ \lambda m\ddot{x}_2 &= -k(x_2 - x_1) \end{aligned} \tag{8.}$$

and thus:

$$\begin{aligned} m\ddot{x}_1 + k(2x_1 - x_2) = 0 \\ \lambda m\ddot{x}_2 + k(-x_1 + x_2) = 0 \end{aligned} \tag{9.}$$

and in matrix form:

$$m\begin{bmatrix}1 & 0\\ 0 & \mu\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{10.}$$

3 Degree of Freedom

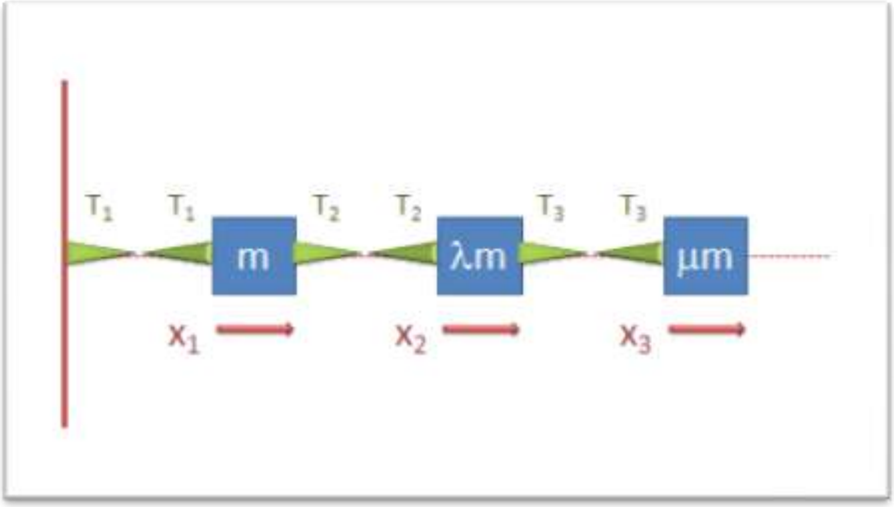

Figure 3

The spring tensions are given by:

$$\begin{aligned} T_1 &= kx_1 \\ T_2 &= k(x_2 - x_1) \\ T_3 &= k(x_3 - x_2) \end{aligned} \tag{11.}$$

The equations of motion are then:

$$\begin{aligned} T_2 - T_1 &= m\ddot{x}_1 = k(x_2 - 2x_1) \\ T_3 - T_2 &= \lambda m\ddot{x}_2 = k(x_3 - 2x_2 + x_1) \\ -T_3 &= \mu m\ddot{x}_3 = k(x_2 - x_3) \end{aligned} \tag{12.}$$

from which we obtain:

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{13.}$$

4 Degree of Freedom

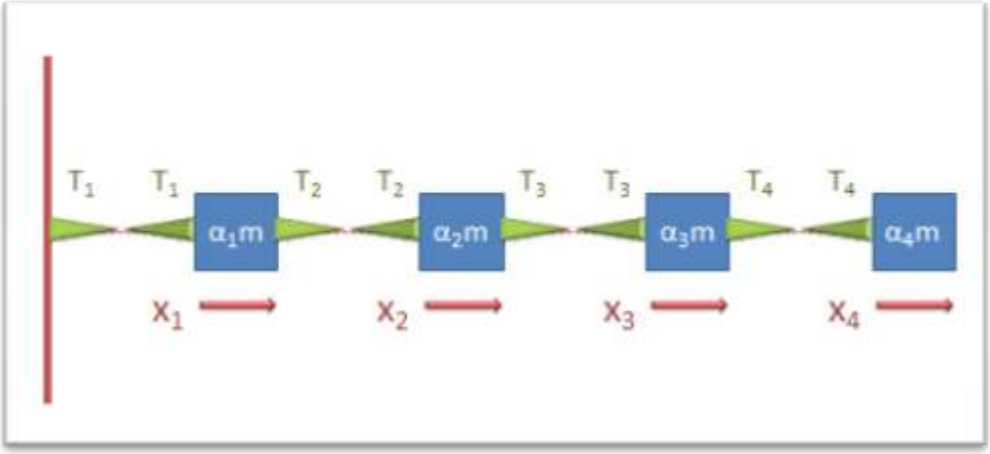

Figure 4

The spring tensions are given by:

$$\begin{aligned} T_1 &= kx_1 \\ T_2 &= k(x_2 - x_1) \\ T_3 &= k(x_3 - x_2) \\ T_4 &= k(x_4 - x_3) \end{aligned} \tag{14.}$$

The equations of motion are then:

$$\begin{aligned} T_2 - T_1 &= \alpha_1 m\ddot{x}_1 = k(x_2 - 2x_1) \\ T_3 - T_2 &= \alpha_2 m\ddot{x}_2 = k(x_3 - 2x_2 + x_1) \\ T_4 - T_3 &= \alpha_3 m\ddot{x}_3 = k(x_4 - 2x_3 + x_2) \\ -T_4 &= \alpha_4 m\ddot{x}_4 = k(x_3 - x_4) \end{aligned} \tag{15.}$$

From which we obtain:

$$m\begin{bmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \alpha_2 & 0 & 0\\ 0 & 0 & \alpha_3 & 0\\ 0 & 0 & 0 & \alpha_4\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{bmatrix} + k\begin{bmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = 0 \tag{16.}$$

The above matrix equations are of the form:

$$M\ddot{X} + KX = 0 \tag{17.}$$

Here M is a mass matrix and K is a stiffness matrix.

The basic pattern can now be seen.

If we define the following matrices:

$$M_1 = I = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$M_2 = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{18.}$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The mass and stiffness matrices can be constructed via:

$$K = k\{2M_1 - M_3 - M_2 - M_2{}'\}$$
$$K = k\bar{K} \tag{19.}$$

$$M = m\begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix}$$
$$M = m\bar{M} \tag{20.}$$

Modal Solution

In order to determine the modes, we use equation (17).

If the masses are executing SHM we have:

$$\ddot{X} = -\omega^2 X = 0 \qquad (21.)$$

Substituting into (17) gives:

$$-\omega^2 MX + KX = 0 \qquad (22.)$$

whence:

$$M^{-1}KX - \omega^2 X = 0 \qquad (23.)$$

i.e. we have an eigenvalue problem where the modal frequencies are the eigenvalues and the mode shapes the corresponding eigenvectors.

Non-Dimensionalisation

Using (19) & (20):

$$\frac{k}{m}(\bar{M}^{-1}\bar{K})X - \omega^2 X = 0 \tag{24.}$$

Defining:

$$\bar{\omega} = \frac{\omega}{\omega_0} \tag{25.}$$

$$(\bar{M}^{-1}\bar{K})X - \bar{\omega}^2 X = 0 \tag{26.}$$

This is the non-dimensional equivalent.

MATLAB File - Animation

```
%
%   N DOF Modes
%
%   SJN 23/02/08
%
colordef black
clear
nmode=4; % Define No of masses/springs
%masses=ones(1,nmode); % Define masses - matrix leading diagonal
masses=[1 1 1 1]; % Define masses - matrix leading diagonal
%--------------------------------------------------------------------------
m1=eye(nmode);
m2=zeros(nmode);
for i=1:nmode-1
    m2(i,i+1)=1;
end
m3=zeros(nmode);
m3(nmode,nmode)=1;
stiffmat=2*m1-m3-m2-m2'; % Asemble Stiffness Matrix
massmat=diag(masses);
modalmat=inv(massmat)*stiffmat;
[modeshapes,modefreq]=eig(modalmat); % Calculate Modal Data via Eigenvalues
for iif=1:nmode
    freq(iif)=sqrt(modefreq(iif,iif));
end
%--------------------------------------------------------------------------
ntime=1000;
tmax=100;
%--------------------------------------------------------------------------
t=linspace(0,tmax,ntime);
%--------------------------------------------------------------------------
xmax=5;
ymax=1;
earthwidth=.5;
xmin=-xmax;
ymin=-ymax;
xwidth=2*xmax/(nmode+1);
xplot=xwidth*(1:nmode)+xmin;
ywidth=2*ymax/(nmode+1);
yplot=ywidth*(1:nmode)+ymin;
%--------------------------------------------------------------------------
%   Commence Time Loop
%--------------------------------------------------------------------------
for it=1:ntime
    clf
    hold on
    axis off
    axis([xmin xmax ymin ymax]);
    for im=1:nmode
        moderesp=modeshapes(:,im)*sin(freq(im)*t(it))';
        l1=[xmin moderesp(nmode)+xplot(nmode)];
        l2=[yplot(im) yplot(im)];
        plot(l1,l2,'y','LineWidth',3);
        for imm=1:nmode
```

```
plot(xplot(imm)+moderesp(imm),yplot(im),'oy','LineWidth',2,'MarkerSize',10,
'MarkerFaceColor','r');
        end
    end
    xearth=[(xmin+earthwidth) xmin xmin (xmin+earthwidth)];
    yearth=[ymin ymin ymax ymax];
    fill(xearth,yearth,'y');
    title(['No of Modes = ',num2str(nmode),' <<<<<>>>>>  Masses
[',num2str(masses),']']);
%---------------------------------------------------------------------------
%   Input Figure into Display Matrix
    m(i)=getframe;
%---------------------------------------------------------------------------
end
%---------------------------------------------------------------------------
%   Create AVI
%movie2avi(m,'fredn','compression','none');
%---------------------------------------------------------------------------
```