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Fitting an Inclined Ellipse into a Rectangular Box

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Introduction

This note describes the analysis of the requirement to fit an inclined ellipse into a rectangle, touching all four sides.

Discussion

The problem is shown iun GFigure 1:

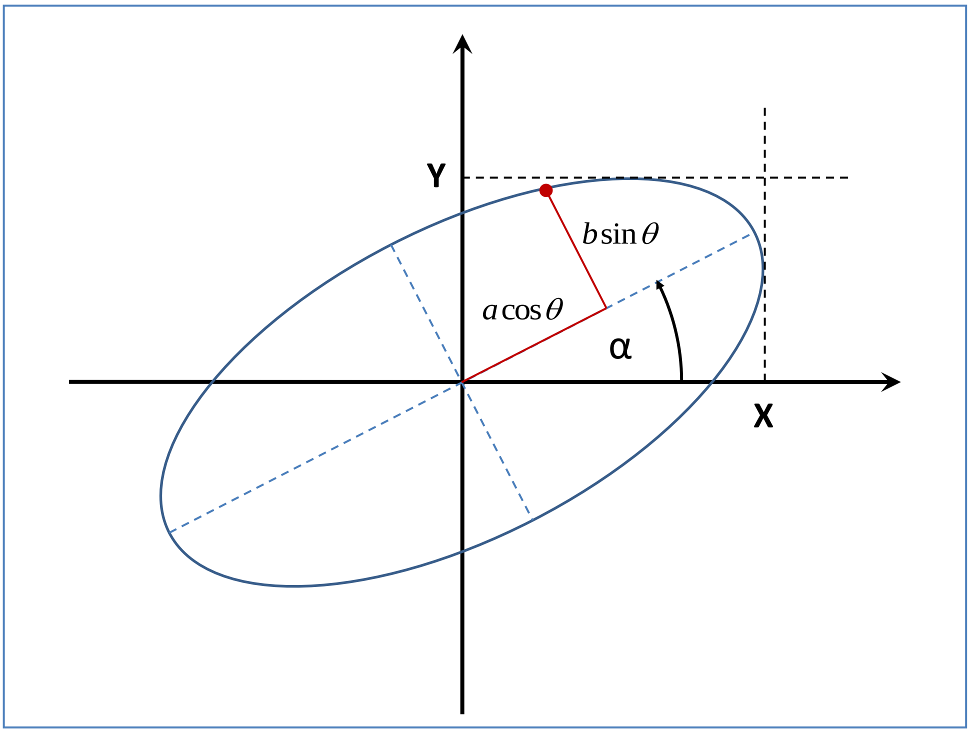

Figure 1 – Parametric Expression of Inclined Ellipse

Referring to Figure 1, the general point of the ellipse is given by:

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned} \tag{1.}$$

When rotated by an angle α, the general point of the ellipse becomes:

$$\begin{aligned} X &= a\cos\theta\cdot\cos\alpha - b\sin\theta\cdot\sin\alpha \\ Y &= a\cos\theta\cdot\sin\alpha + b\sin\theta\cdot\cos\alpha \end{aligned} \tag{2.}$$

In ordser to determine the conditions for a fit in the rectangle we need to establish the extrema.

To achieve this we determine the derivatives thus:

$$\frac{dX}{d\theta} = -a\sin\theta \cdot \cos\alpha - b\cos\theta \cdot \sin\alpha$$
$$\frac{dY}{d\theta} = -a\sin\theta \cdot \sin\alpha + b\cos\theta \cdot \cos\alpha \qquad (3.)$$

This requires the following criteria:

$$\frac{dX}{d\theta} = 0$$

$$a\sin\theta \cdot \cos\alpha = -b\cos\theta \cdot \sin\alpha$$
$$\tan\theta = \frac{-b\sin\alpha}{a\cos\alpha} \qquad (4.)$$

This condition can be viewed geometrically as in Figure 2:

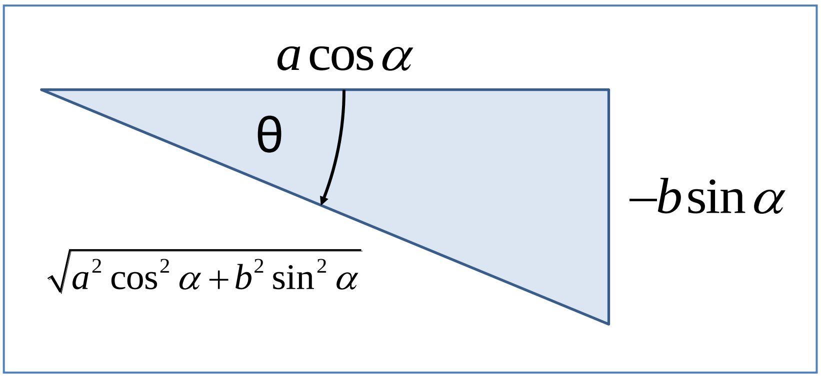

Figure 2

From this we obtain:

$$\sin\theta = \frac{-b\sin\alpha}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}}$$
$$\cos\theta = \frac{a\sin\alpha}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}} \qquad (5.)$$

Substituting these results into the upper equation of (2) we obtain:

$$X = \frac{a\cos\alpha \cdot a\cos\alpha}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}} + \frac{b\sin\alpha \cdot b\sin\alpha}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}}$$
$$= \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \qquad (6.)$$

$$\frac{dY}{d\theta} = 0$$

$$a\sin\theta \cdot \sin\alpha = -b\cos\theta \cdot \cos\alpha$$

$$\tan\theta = \frac{b\cos\alpha}{a\sin\alpha} \qquad (7.)$$

This condition can be viewed geometrically as in Figure 2:

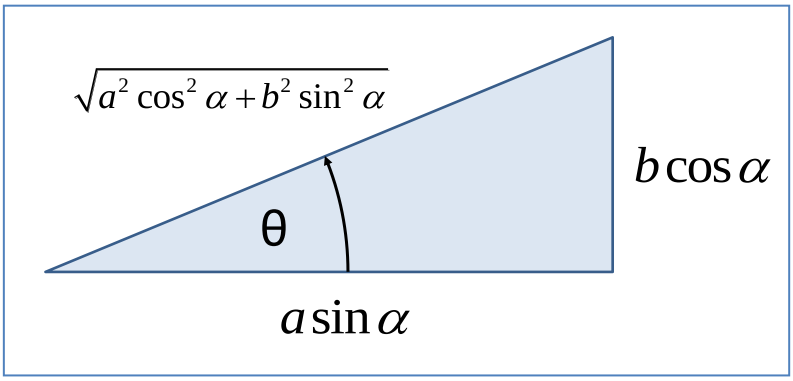

Figure 3

From this we obtain:

$$\sin\theta = \frac{b\cos\alpha}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$$

$$\cos\theta = \frac{a\sin\alpha}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}} \qquad (8.)$$

Substituting these results into the lower equation of (2) we obtain:

$$Y = \frac{a\sin\alpha \cdot a\sin\alpha}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}} + \frac{b\cos\alpha \cdot b\cos\alpha}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$$

$$= \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha} \qquad (9.)$$

We therefore have the following conditions:

$$X^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

$$Y^2 = a^2\sin^2\alpha + b^2\cos^2\alpha \qquad (10.)$$

Or expressed in matrix form:

$$\begin{bmatrix} \cos^2\alpha & \sin^2\alpha \\ \sin^2\alpha & \cos^2\alpha \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \end{bmatrix} = \begin{bmatrix} X^2 \\ Y^2 \end{bmatrix} \qquad (11.)$$

Now the determinant becomes:

$$\begin{aligned}\Delta &= \begin{vmatrix}\cos^2\alpha & \sin^2\alpha \\ \sin^2\alpha & \cos^2\alpha\end{vmatrix} \\ &= \cos^4\alpha - \sin^4\alpha \\ &= \left(\cos^2\alpha - \sin^2\alpha\right)\cdot\left(\cos^2\alpha + \sin^2\alpha\right) \\ &= \left(\cos^2\alpha - \sin^2\alpha\right) \\ &= \cos 2\alpha\end{aligned} \tag{12.}$$

The solution then becomes:

$$\begin{bmatrix}a^2 \\ b^2\end{bmatrix} = \frac{\begin{bmatrix}\cos^2\alpha & -\sin^2\alpha \\ -\sin^2\alpha & \cos^2\alpha\end{bmatrix}}{\Delta}\begin{bmatrix}X^2 \\ Y^2\end{bmatrix} \tag{13.}$$

Whence:

$$\begin{aligned}a &= \sqrt{\frac{X^2\cdot\cos^2\alpha - Y^2\cdot\sin^2\alpha}{\cos 2\alpha}} \\ b &= \sqrt{\frac{-X^2\cdot\sin^2\alpha + Y^2\cdot\cos^2\alpha}{\cos 2\alpha}}\end{aligned} \tag{14.}$$

Observing (14) shows that in order to get a real solution the following criteria must hold:

$$\begin{aligned}\tan\alpha &< \frac{X}{Y} \\ \tan\alpha &< \frac{Y}{X}\end{aligned} \tag{15.}$$

i.e.:

$$\tan\alpha < \min\left(\frac{X}{Y}, \frac{Y}{X}\right) \tag{16.}$$

In other words, provided the angle of the ellipse conforms to (16) then an ellipse can be drawn. At the limit however, one of the ellipse axes is reduced to zero and the ellipse becomes a straight line joining two opposite corners of the rectangle. The only other special case is when $\alpha=45^o$. In this case we have a solution only if X=Y and the ellipse becomes a circle.

Example

An example of the procedure is shown below in Figure 4:

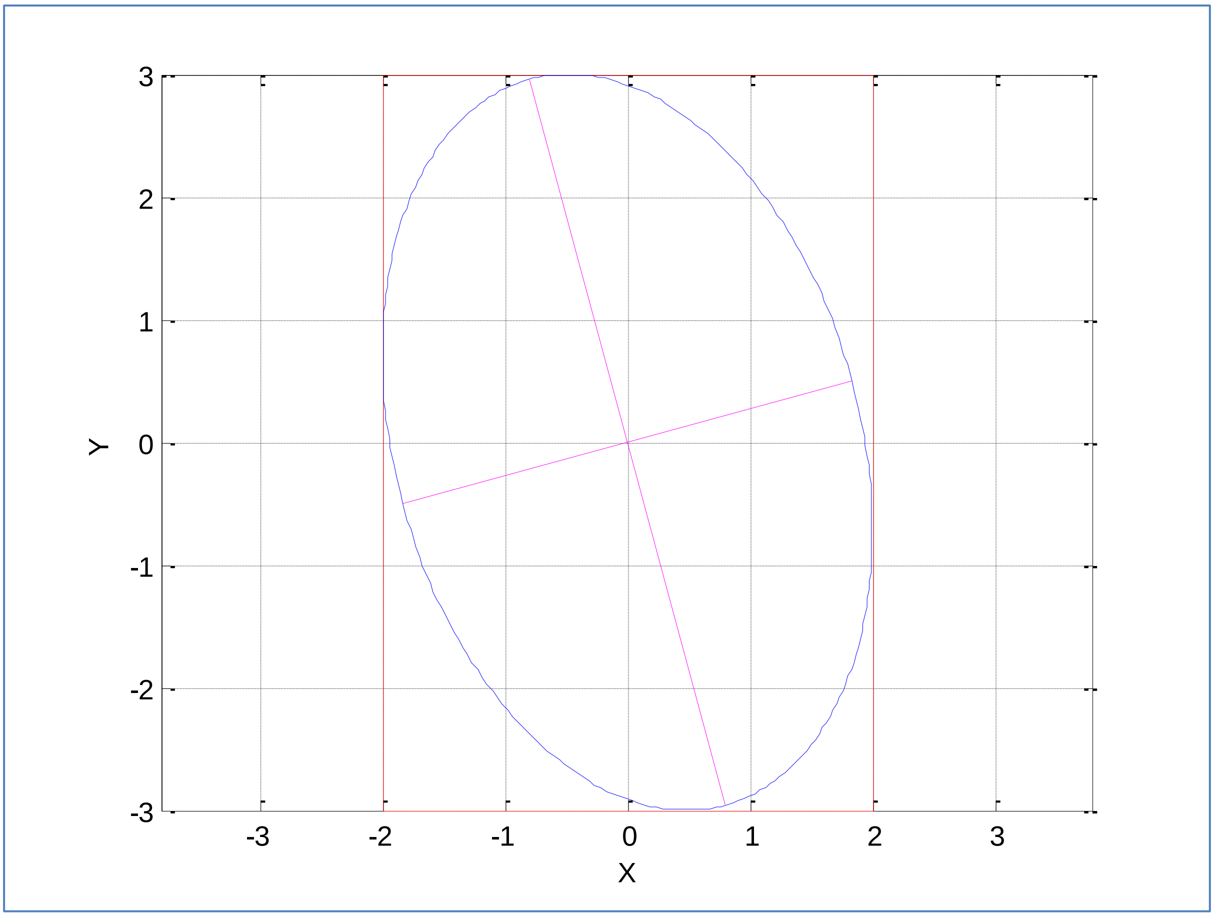

Figure 4

MATLAB File

```
%
%   Inclined Ellipse
%
%   SJN 5/1/11
%
clear all
colordef black
X=2;
Y=3;

ntheta=251;

alfdeg=15;
alf=alfdeg*pi/180;
calf=cos(alf);
salf=sin(alf);
a=sqrt(  (X^2*calf^2-Y^2*salf^2)  /  cos(2*alf)  );
b=sqrt(  (-X^2*salf^2+Y^2*calf^2)  /  cos(2*alf)  );
xbox=[-X X X -X -X];
ybox=[-Y -Y Y Y -Y];
plot(xbox,ybox,'r');
hold on
theta=linspace(0,2*pi,ntheta);
cthet=cos(theta);
sthet=sin(theta);
xellip=a*cthet*calf-b*sthet*salf;
yellip=a*cthet*salf+b*sthet*calf;
plot(xellip,yellip,'c');
xmajor=[-a*calf a*calf];
ymajor=[-a*salf a*salf];
plot(xmajor,ymajor,'y');
xminor=[b*salf -b*salf];
yminor=[-b*calf b*calf];
plot(xminor,yminor,'y');
grid on
xlabel('X');
ylabel('Y');
axis equal
```