

Aerodynamics & Flight Mechanics Research Group

Fitting an Inclined Ellipse into a Rectangular Box

S. J. Newman

Technical Report AFM-11/26

January 2011

UNIVERSITY OF SOUTHAMPTON

SCHOOL OF ENGINEERING SCIENCES

AERODYNAMICS AND FLIGHT MECHANICS RESEARCH GROUP

Fitting an Inclined Ellipse into a Rectangular Box

by

S. J. Newman

AFM Report No. AFM 11/26

January 2011

© School of Engineering Sciences, Aerodynamics and Flight Mechanics Research Group



COPYRIGHT NOTICE

(c) SES University of Southampton All rights reserved.

SES authorises you to view and download this document for your personal, non-commercial use. This authorization is not a transfer of title in the document and copies of the document and is subject to the following restrictions: 1) you must retain, on all copies of the document downloaded, all copyright and other proprietary notices contained in the Materials; 2) you may not modify the document in any way or reproduce or publicly display, perform, or distribute or otherwise use it for any public or commercial purpose; and 3) you must not transfer the document to any other person unless you give them notice of, and they agree to accept, the obligations arising under these terms and conditions of use. This document, is protected by worldwide copyright laws and treaty provisions.



Introduction

This note describes the analysis of the requirement to fit an inclined ellipse into a rectangle, touching all four sides.

Discussion

The problem is shown in Figure 1:

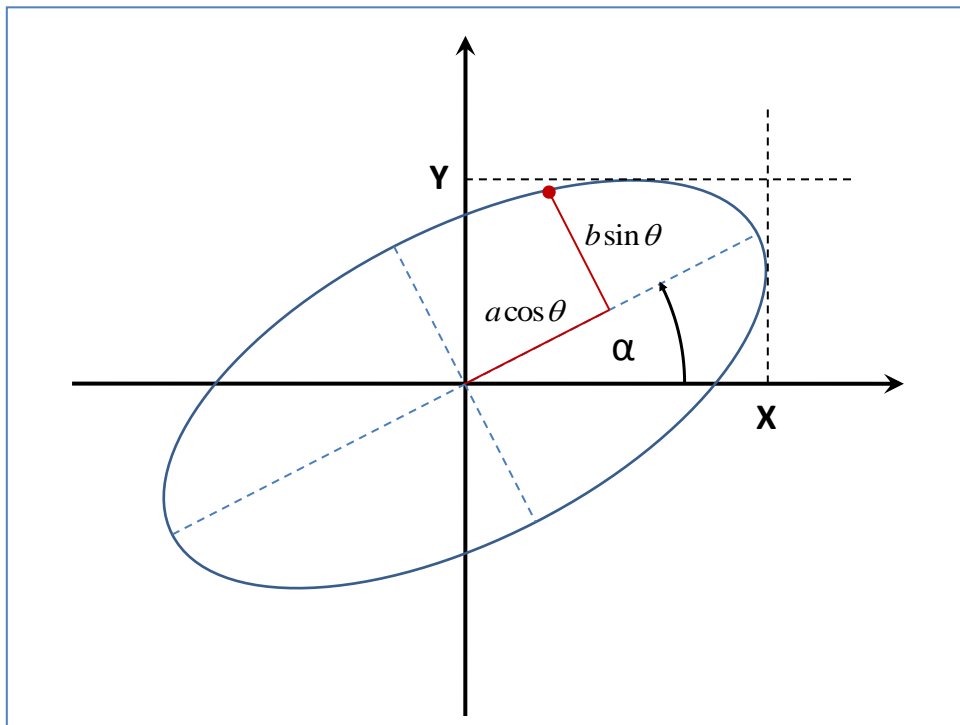


Figure 1 – Parametric Expression of Inclined Ellipse

Referring to Figure 1, the general point of the ellipse is given by:

$$\begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned} \quad (1.)$$

When rotated by an angle α , the general point of the ellipse becomes:

$$\begin{aligned} X &= a \cos \theta \cdot \cos \alpha - b \sin \theta \cdot \sin \alpha \\ Y &= a \cos \theta \cdot \sin \alpha + b \sin \theta \cdot \cos \alpha \end{aligned} \quad (2.)$$

In order to determine the conditions for a fit in the rectangle we need to establish the extrema.

To achieve this we determine the derivatives thus:



$$\begin{aligned}\frac{dX}{d\theta} &= -a \sin \theta \cdot \cos \alpha - b \cos \theta \cdot \sin \alpha \\ \frac{dY}{d\theta} &= -a \sin \theta \cdot \sin \alpha + b \cos \theta \cdot \cos \alpha\end{aligned}\quad (3.)$$

This requires the following criteria:

$$\frac{dX}{d\theta} = 0$$

$$\begin{aligned}a \sin \theta \cdot \cos \alpha &= -b \cos \theta \cdot \sin \alpha \\ \tan \theta &= \frac{-b \sin \alpha}{a \cos \alpha}\end{aligned}\quad (4.)$$

This condition can be viewed geometrically as in Figure 2:

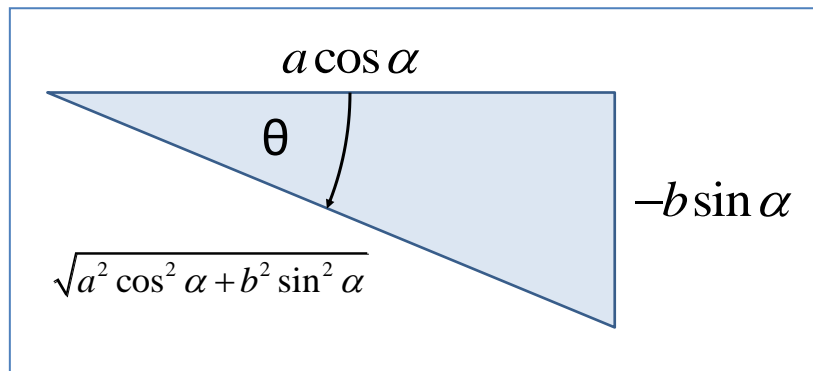


Figure 2

From this we obtain:

$$\begin{aligned}\sin \theta &= \frac{-b \sin \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}} \\ \cos \theta &= \frac{a \sin \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}}\end{aligned}\quad (5.)$$

Substituting these results into the upper equation of (2) we obtain:

$$\begin{aligned}X &= \frac{a \cos \alpha \cdot a \cos \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}} + \frac{b \sin \alpha \cdot b \sin \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}} \\ &= \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}\end{aligned}\quad (6.)$$



$$\frac{dY}{d\theta} = 0$$

$$a \sin \theta \cdot \sin \alpha = -b \cos \theta \cdot \cos \alpha$$

$$\tan \theta = \frac{b \cos \alpha}{a \sin \alpha} \quad (7.)$$

This condition can be viewed geometrically as in Figure 2:

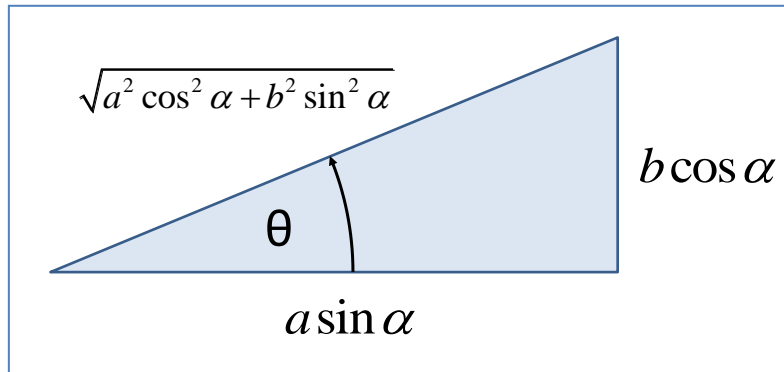


Figure 3

From this we obtain:

$$\sin \theta = \frac{b \cos \alpha}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}}$$

$$\cos \theta = \frac{a \sin \alpha}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \quad (8.)$$

Substituting these results into the lower equation of (2) we obtain:

$$Y = \frac{a \sin \alpha \cdot a \sin \alpha}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} + \frac{b \cos \alpha \cdot b \cos \alpha}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}}$$

$$= \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \quad (9.)$$

We therefore have the following conditions:

$$X^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

$$Y^2 = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha \quad (10.)$$

Or expressed in matrix form:

$$\begin{bmatrix} \cos^2 \alpha & \sin^2 \alpha \\ \sin^2 \alpha & \cos^2 \alpha \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \end{bmatrix} = \begin{bmatrix} X^2 \\ Y^2 \end{bmatrix} \quad (11.)$$



Now the determinant becomes:

$$\begin{aligned}
 \Delta &= \begin{vmatrix} \cos^2 \alpha & \sin^2 \alpha \\ \sin^2 \alpha & \cos^2 \alpha \end{vmatrix} \\
 &= \cos^4 \alpha - \sin^4 \alpha \\
 &= (\cos^2 \alpha - \sin^2 \alpha) \cdot (\cos^2 \alpha + \sin^2 \alpha) \\
 &= (\cos^2 \alpha - \sin^2 \alpha) \\
 &= \cos 2\alpha
 \end{aligned} \tag{12.}$$

The solution then becomes:

$$\begin{bmatrix} a^2 \\ b^2 \end{bmatrix} = \frac{\begin{bmatrix} \cos^2 \alpha & -\sin^2 \alpha \\ -\sin^2 \alpha & \cos^2 \alpha \end{bmatrix}}{\Delta} \begin{bmatrix} X^2 \\ Y^2 \end{bmatrix} \tag{13.}$$

Whence:

$$\begin{aligned}
 a &= \sqrt{\frac{X^2 \cdot \cos^2 \alpha - Y^2 \cdot \sin^2 \alpha}{\cos 2\alpha}} \\
 b &= \sqrt{\frac{-X^2 \cdot \sin^2 \alpha + Y^2 \cdot \cos^2 \alpha}{\cos 2\alpha}}
 \end{aligned} \tag{14.}$$

Observing (14) shows that in order to get a real solution the following criteria must hold:

$$\begin{aligned}
 \tan \alpha &< \frac{X}{Y} \\
 \tan \alpha &< \frac{Y}{X}
 \end{aligned} \tag{15.}$$

i.e.:

$$\tan \alpha < \min\left(\frac{X}{Y}, \frac{Y}{X}\right) \tag{16.}$$

In other words, provided the angle of the ellipse conforms to (16) then an ellipse can be drawn. At the limit however, one of the ellipse axes is reduced to zero and the ellipse becomes a straight line joining two opposite corners of the rectangle. The only other special case is when $\alpha=45^\circ$. In this case we have a solution only if $X=Y$ and the ellipse becomes a circle.



Example

An example of the procedure is shown below in Figure 4:

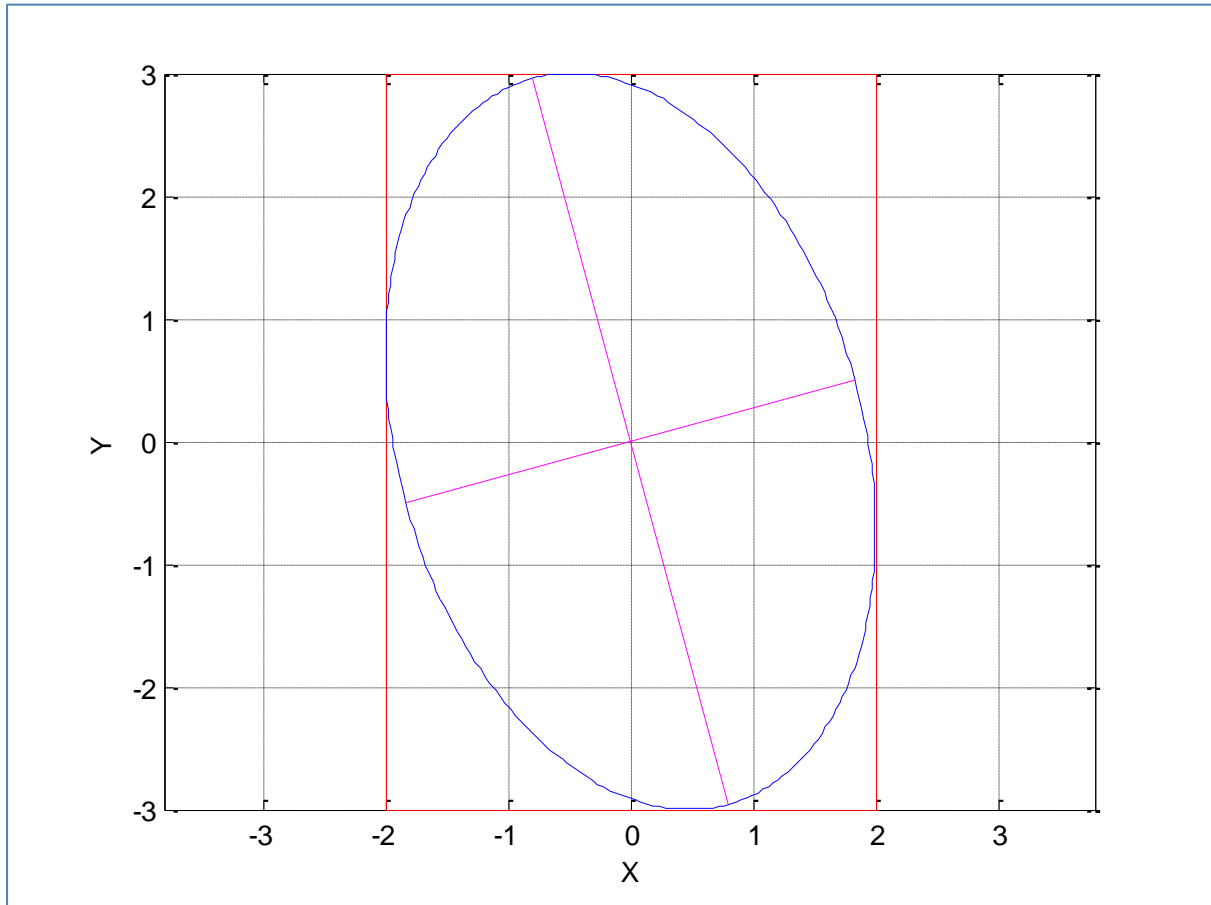


Figure 4



MATLAB File

```
%  
%   Inclined Ellipse  
%  
%   SJN 5/1/11  
%  
clear all  
colordef black  
X=2;  
Y=3;  
  
ntheta=251;  
  
alfdeg=15;  
alf=alfdeg*pi/180;  
calf=cos(alf);  
salf=sin(alf);  
a=sqrt( (X^2*calf^2-Y^2*salf^2) / cos(2*alf) );  
b=sqrt( (-X^2*salf^2+Y^2*calf^2) / cos(2*alf) );  
xbox=[-X X X -X -X];  
ybox=[-Y -Y Y Y -Y];  
plot(xbox,ybox,'r');  
hold on  
theta=linspace(0,2*pi,ntheta);  
ctheta=cos(theta);  
sthesin=sin(theta);  
xellip=a*ctheta*calf-b*sthesin*salf;  
yellip=a*ctheta*salf+b*sthesin*calf;  
plot(xellip,yellip,'c');  
xmajor=[-a*calf a*calf];  
ymajor=[-a*salf a*salf];  
plot(xmajor,ymajor,'y');  
xminor=[b*salf -b*salf];  
yminor=[-b*calf b*calf];  
plot(xminor,yminor,'y');  
grid on  
xlabel('X');  
ylabel('Y');  
axis equal
```

