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IN THE HYPERBOLIC THEORY OF SPECIAL RELATIVITY IS SPACE ALSO HYPERBOLIC?

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1. ABSTRACT

This paper discusses the hyperbolic (Bolyai-Lobachevsky) theory of relativity as given by the writer at the previous PIRT conference at Budapest and London [1]. The writer's viewpoint differs from that of Ungar [2] the basic concept in the hyperbolic theory being *rapidity* w given by $\tan^{-1}(v/c)$ which replaces ordinary velocity v. Velocity is always less than the velocity of light c but correspondingly rapidity may become infinite, as is also *hyperbolic velocity* V, a scaled version cw of rapidity. Corresponding to the velocity space $|\mathbf{v}| < c$ is the infinite hyperbolic space of hyperbolic velocities V.

An interesting axiomatic question was raised by Borel's assertion [3] that the principle of Special Relativity is equivalent to the assumption that kinematic space is a hyperbolic space with negative radius of curvature equal to the velocity of light. Such an assertion needs detailed justification which Borel did not attempt. Certainly it ensures validity of Einstein's principle of the constancy of the speed of light but leaves open the question of whether a hyperbolic velocity space could explain all phenomena of the special theory.

A further question concerns the nature of space itself which has importance for the philosophy of Special Relativity. It seems plausible that, to avoid known difficulties concerned with absolute motion, space should be assumed hyperbolic instead of Euclidean. This assumption, implicit in Varićak's work [4], has not been followed up in any detail by other writers.

Certainly at terrestrial distances any deviation from Euclidean will be beyond observation and so effectively space is Euclidean. But for philosophy and cosmology the question of whether it is actually Euclidean is important. If space were hyperbolic then the much-discussed problems of absolute motion and the nonexistence of the aether could just disappear. Equivalence of all observers is assured so that, possibly, the way could be open for reintroduction of the aether

The assumption that space, i.e. the universe, is hyperbolic leads naturally to Milne's theory [5] previously discussed in these conferences by Prokhovnik [6], Wegener [7] and the writer [8]. As pointed out by Walker [9], Milne's model is essentially the same as the asymptotic solution of the Friedmann model for the negative curvature case. A simplified form of the Milne model fits beautifully the hyperbolic kinematic representation when use is made of a hyperbolic restatement of the Hubble law for recession velocities previously described by the writer [10]. This law is equivalent to the exponential red-shift law discussed by Prokhovnik [6] and Hawkins [11] in connection with observations of Humason, Mayal & Sandage [12] on nebula recession. The resulting hyperbolic model of the universe has negative radius of curvature equal to the Hubble radius and from this fact an estimate may be made of the minute deviation from Euclidean space at terrestrial distances.

2. HYPERBOLIC GEOMETRY

Euclid's axiom of parallels **led to the study of the consequences of its negation and so to the discovery of hyperbolic geometry where this axiom is false by Gauss, Bolyai, Lobachevski and others.**

Asymptotic parallels **may be defined from a point P to a line l as limiting positions of line segments PQ, PQ', PQ", ...from P to l as the point Q of intersection moves off to infinity. In Euclidean geometry asymptotic parallels coincide; but in hyperbolic geometry they do not.**

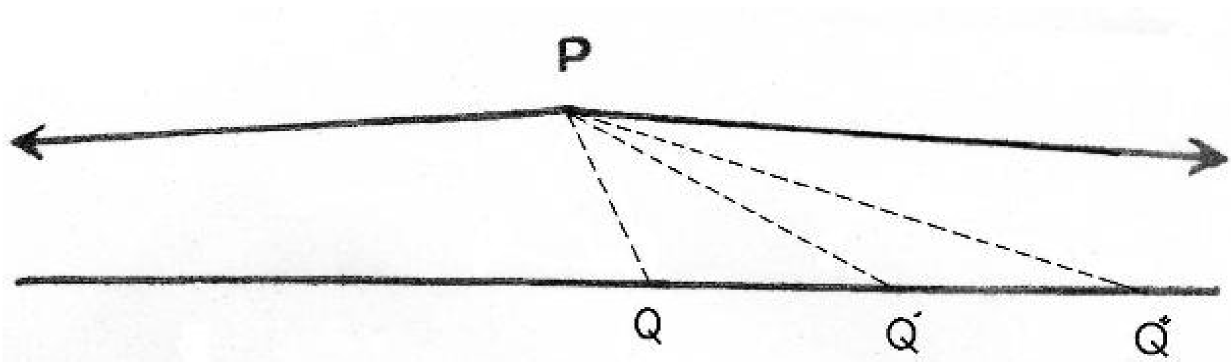

Fig 1 **Asymptotic parallels to a line from a point**

If space were hyperbolic the divergence from Euclidean geometry would be minute. Nevertheless there would be important consequences for the axiomatics of relativity, cosmology and the philosophy of physics.

The disbelief and controversy **which accompanied early studies of hyperbolic geometry hindered recognition of its importance Now, almost a hundred years after the first works on its application to relativity by Varićak and others, the basic importance of this geometry for physics still remains, in general, unrecognized.**

3. THE BELTRAMI REPRESENTATION

***(Beltrami-Klein* or *Klein disc model*)**

Straight line segments within a circle satisfy all the axioms of hyperbolic geometry if the circle is regarded as an infinite horizon so that line segments meeting on the circle are considered asymptotically parallel (figure).

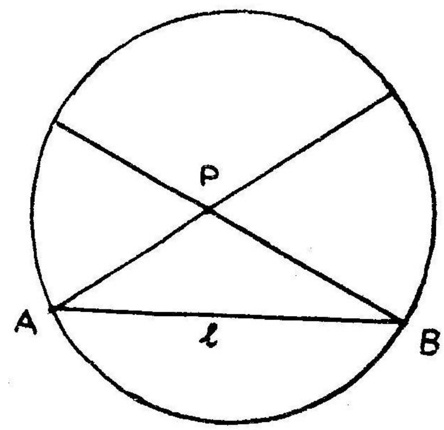

***Fig* 2 Asymptotic parallels from a point P**

***Dual nature of the representation*: An important aspect of this representation is its dual nature - it is simultaneously Euclidean and non-Euclidean (hyperbolic). This fact can be made clear by introducing a new radial coordinate ρ related to the radial distance r by**

$$\rho = R\,\mathrm{th}^{[-1]}(r/R) \qquad 0 < r < R;$$
$$r = R\,\mathrm{th}\,(\rho/R) \qquad 0 < \rho < \infty$$

R is here the radius of the circle. Then the points within the circle become mapped as

$$x = r\cos\theta = R\,\mathrm{th}\,(\rho/R)\cos\theta$$
$$y = r\sin\theta = R\,\mathrm{th}\,(\rho/R)\sin\theta$$

With the new radial coordinate ρ the space is infinite and is in fact a hyperbolic space. This can be seen by another viewpoint as follows.

4. GEOMETRY OF THE HYPERBOLOID SURFACE

Consider the surface

$$- x^2 - y^2 + z^2 = R^2$$

This is a hyperboloid in two sheets (fig). The upper half-sheet has parameterization

$$x = R \operatorname{sh} u \cos \theta$$
$$y = R \operatorname{sh} u \sin \theta$$
$$z = R \operatorname{ch} u$$

Geodesic distance ρ from the vertex is found as:

$$\rho = Ru$$

***Projection on to a circle*: The upper surface maps on to a circle by central projection, a point (x, y, z) mapping on to point (X, Y):**

$$X = x/z, \; Y = y/z$$

Substituting ρ for u gives:

$$X = R \operatorname{th}(\rho/R) \cos \theta$$
$$Y = R \operatorname{th}(\rho/R) \sin \theta$$

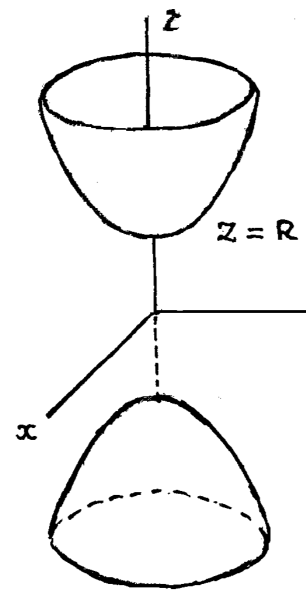

***Fig 3*   Hyperboloid in 2 sheets**

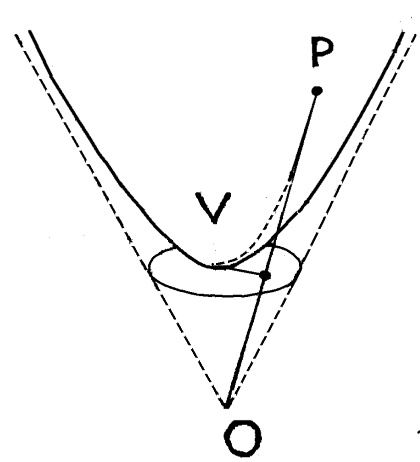

***Fig 4*   Projection on to a circle**

Central projection maps the hyperbolic sheet on to the circle giving the Beltrami representation. Vice-versa, the Beltrami representation can be projected upwards to the hyperbola. These ideas extend to the Minkowski space-time diagram.

5. RAPIDITY

In relativity *rapidity* replaces velocity. It was used in early papers on hyperbolic theory (Varićak 1910, etc). The name is due to Robb (1911)

Rapidity: **w is defined in terms of velocity v by**

$$v = c\,\mathrm{th}\,w \qquad\qquad w = \mathrm{th}^{-1}(v/c)$$

When $-c < v < c$ the value of w lies in the range $-\infty < w < \infty$.

**** Additivity:*** **to the composition law**

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}$$

there corresponds for rapidities

$$w = w_1 + w_2$$

**** Trigonometric identities*:**

$$\mathrm{ch}\,w = \frac{1}{\sqrt{(1 - v^2/c^2)}} \qquad\qquad \mathrm{sh}\,w = \frac{(v/c)}{\sqrt{(1 - v^2/c^2)}}$$

**** Lorentz transformation:***

$$ct = ct'\,\mathrm{ch}\,w + x'\,\mathrm{sh}\,w$$
$$x = ct'\,\mathrm{sh}\,w + x'\,\mathrm{ch}\,w$$

**** Hyperbolic velocity:***

$$V = cw = c\,\mathrm{th}^{-1}(v/c) \;(\approx v \text{ when } v << c)$$

6. DOPPLER FORMULAE

***The classical formula*: (Hubble 1936)**

$$z = v/c$$

where z is the redshift

$$z = (\lambda_1 - \lambda_0)/\lambda_0$$

***Einstein's formula*: (using wave-length)**

$$\lambda_1/\lambda_0 = \sqrt{(1 + v/c)}/\sqrt{(1 - v/c)}$$

Taking logs gives

$$\ln \lambda_1/\lambda_0 = \mathrm{th}^{[-1]}\, v/c = w$$

There follows either

(a) *exponential law:*

$$\lambda_1/\lambda_0 = \exp w$$

$$z = \exp w - 1$$

(b) *logarithmic redshift law* :

$$Z = \ln \lambda_1/\lambda_0 = \mathrm{th}^{[-1]}\, v/c = w = V/c$$

i.e.

$$Z = V/c$$

exactly analogous to the classical formula*. When Z and V are small the formula reduces to the classical formula.

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* Note: Exponential redshift was used by Varićak and later by Hawkins and Prokhovnik. Logarithmic redshift was discussed by the writer [10]

7. EINSTEIN ADDITION RULE IN HYPERBOLIC SPACE

A basic step in the construction of the hyperbolic theory of special relativity was the recognition that the Einstein addition rule can be reinterpreted geometrically in hyperbolic space (Robb 1911, Varićak 1910 etc, Borel 1913)

***Einstein addition rule*: Magnitude squared v^2 of composition of velocities v_1, v_2 inclined at an angle θ is**

$$v^2 = \frac{\{v_1^2 + v_2^2 + 2\, v_1 v_2 \cos\theta - (v_1 v_2/c \sin\theta)^2\}}{(1 + v_1.v_2/c^2 \cos\theta)^2}$$

Rearrange:

$$\frac{1}{\sqrt{(1- v^2/c^2)}} = \frac{1 + v_1/c.v_2/c^2 \cos\theta.}{\sqrt{(1 - v_1^2/c^2)}\sqrt{(1 - v_2^2/c^2)}}$$

Use identities relating rapidities w and velocities v

$$\mathrm{ch}\, w = \frac{1}{\sqrt{(1- v^2/c^2)}} \qquad \mathrm{sh}\, w = \frac{v}{\sqrt{(1- v^2/c^2)}}.$$

Cosine law for hyperbolic triangles

$$\mathrm{ch}\, w = \mathrm{ch}\, w_1\, \mathrm{ch}\, w_2 + \mathrm{sh}\, w_1\, \mathrm{sh}\, w_2 \cos(\pi - \theta)$$

or

$$\mathrm{ch}\, V/c = \mathrm{ch}\, V_1/c\, \mathrm{ch}\, V_2/c + \mathrm{sh}\, V_1/c\, \mathrm{sh}\, V_2/c \cos(\pi - \theta)$$

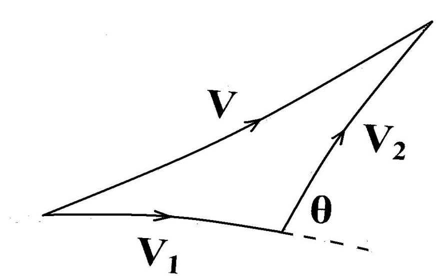

Fig 5 Hyperbolic velocity addition

8. BOREL'S ASSERTION

From the hyperbolic interpretation of velocity Borel (1913) asserted:

> ***'The principle of relativity corresponds to the hypothesis that the kinematic space is a space of constant negative curvature, the space of Bolyai and Lobachevski. The value of the radius of curvature is the speed of light.'***

***Advantages of using hyperbolic velocity*:**

(a) The characteristic limiting property of the velocity of light is very understandably explained.

(b) It leads to a highly satisfactory restatement of many known properties in special relativity in optics and dynamics.

Axiomatic questions:

(a) Do the hyperbolic properties of velocity equate with the principle of relativity?

(b) Can they explain all the phenomena of relativity? Are additional assumptions necessary; if so what are they?

What is the nature of space itself? **The assumption that kinematic space is hyperbolic leaves the nature of space itself unspecified.**

(a) *Euclidean space*: This is commonly assumed though it is not compatible with the Lorentz transformation. There arise well known questions concerning absolute and relative motions and existence of the aether resulting in rival theories (Einsteinian, Lorentzian)

(b) *Hyperbolic space*: Would such difficulties disappear if space were hyperbolic? What new difficulties would arise?

9. PHYSICAL SPACE AS HYPERBOLIC

A theory of special relativity based on the assumption that both physical space and kinematic space are hyperbolic is due to Varićak whose papers appeared in 1910, 1912, and his book in 1924. His work is difficult reading but it remains the only attempt to present a unified theory and is a valuable source of ideas.

***Weierstrass coordinates*: The connection between the geometry of hyperbolic space and the familiar methods and formulae of relativity comes about through the use of Weierstrass coordinates. These give a Cartesian representation (including time) in the Euclidean tangent space at a point in the hyperbolic space. These coordinates denoted x, y, z, t as usual, may be represented in parametric form similar to that for Minkowski space and with them to the invariant quadratic form $(ct)^2 - (x^2+y^2+z^2)$.**

***Lorentz transformation*: In hyperbolic space the Lorentz group is the natural automorphism group. Varićak showed directly that the familiar one-dimensional Lorentz transformation is equivalent to a translation along the x axis. Lorentz contraction is explained as purely geometrical.**

***Physical space close to Euclidean*: There are for physics few differences with standard theory. Differences exist on the cosmological scale so making necessary the combination of cosmology with the special theory.**

***Hyperbolic space and optics:* What is meant by a parallel beam of light must be reinterpreted*.**

***Hyperbolic space and dynamics*: There is difficulty in reconciling these completely. But, as indicated by Varićak, there is a possibility of treating also rotation.**

*** The hyperbolic divergence of light rays was discussed by Varićak (1910 etc.) and later by Prokhovnik (1967) and the writer (PIRT conf. 1994a.)**

10. BELTRAMI VELOCITY SPACE

The representation of relativistic velocities in special relativity by the Beltrami space originated with Fock (1955) though it was previously implicit in Milne's cosmological model (1935).

Cartesian form: **Velocities (v_x, v_y, v_z) satisfy the inequality**

$$v_x^2 + v_y^2 + v_z^2 < c^2$$

The velocity (kinematic) space is the interior of a sphere parametrized by spherical coordinates θ, φ as

$$v_x = v \sin \varphi \cos \theta$$
$$v_y = v \sin \varphi \sin \theta$$
$$v_z = v \cos \varphi$$

Hyperbolic form: **This spherical region is now taken as a Beltrami - Klein representation of a hyperbolic space with parametrization of radial distance by rapidity w (or hyperbolic velocity V) so that**

$$v_x = c \operatorname{th} w \sin \varphi \cos \theta = c \operatorname{th} (V/c) \sin \varphi \cos \theta$$
$$v_y = c \operatorname{th} w \sin \varphi \sin \theta = c \operatorname{th} (V/c) \sin \varphi \sin \theta$$
$$v_z = c \operatorname{th} w \cos \varphi = c \operatorname{th} (V/c) \cos \varphi$$

It now becomes an infinite hyperbolic space, the metric squared differential taking the standard form for a hyperbolic space of negative radius of curvature c*:

$$dV^2 + c^2 \operatorname{sh}^2 V/c \, [d\varphi^2 + \sin\varphi^2 \, d\theta^2]$$

As described above the Beltrami space can be obtained directly by central projection of the hyperboloid of Minkowski 4 vectors.

11. MILNE'S COSMOLOGICAL THEORY

Milne's theory is a cosmological version of special relativity with its own style of presentation. It appeared in the 1930s and was developed especially by the British school of cosmologists (Milne, McCrea, McVittie, Whitrow). Milne published two books *Relativity, Gravitation and World Structure* (1935) and *Kinematical Relativity* (1948) as well as numerous papers. A convenient summary is given in North: *The Measure of the Universe*.

Central to the theory is the determination of time and distance estimates by light signals using the method indicated by Einstein (1905). Milne also wrote at length on an associated gravitational theory. If special relativity is to include a theory of hyperbolic space, then a modified and simplified form of the Milne theory appears to be most suitable.

At the beginning of Milne (1935) is shown a diagram similar to that alongside. The sphere is the observable universe with a statistical distribution of points representing a stationary distribution of freely moving particles. At the boundary, particles move outward with the limiting velocity of light, the distribution becoming infinitely dense there.

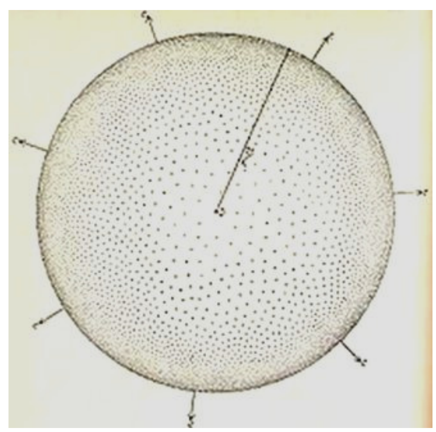

***Fig 5*: Milne's diagram**

The distribution is that in the Beltrami model for a uniform distribution in infinite hyperbolic space.

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Note: The diagram can be found with annotations in Wikipedia (article 'Milne')

12. THE HYPERBOLIC FRIEDMANN MODEL

The Milne model essentially coincides with an asymptotic form of the hyperbolic Friedmann model (Walker 1935 etc)

Robertson-Walker metric: **Cosmic time t, comoving radial coordinate r, spherical coordinates θ, φ:**

$$ds^2 = c^2\, dt^2 - R(t)^2 \{ \frac{dr^2}{1 + r^2} + r^2(d\varphi^2 + \sin\varphi^2\, d\theta^2)\}$$

Transform by replacing r by th u

$$ds^2 = c^2\, dt^2 - R(t)^2\{du^2 + sh^2u\, (d\varphi^2 + \sin\varphi^2\, d\theta^2)\}$$

Asymptotic form for R(t)**: General Relativity field equations have asymptotic solution for R(t):**

$$R(t) \sim ct$$

Radial light path**: (ds = 0, dθ = 0, dφ = 0) leads to**

$$c\, dt = R(t)\, du$$

Integration between t_1, t_0 :

$$u = \int_{t_1}^{t_0} c\, dt/R(t) \sim \int_{t_1}^{t_0} dt/t = \ln (t_0/t_1)$$

General Relativity Doppler shift**: change in wavelength is**

$$\lambda_0/\lambda_1 = R(t_0)/R(t_1) = t_0/t_1 = \exp u$$

Identifying u with rapidity w* the Doppler law is

$$Z = \ln(\lambda_0/\lambda_1) = w = V/c$$

*** See Prokhovnik 1967, 1985**

13. THE HUBBLE RADIUS

The Hubble law **relates nebula recession velocity v to radial distance r using the Hubble constant H as**

$$v = H r$$

It determines a radial velocity field for any observer.

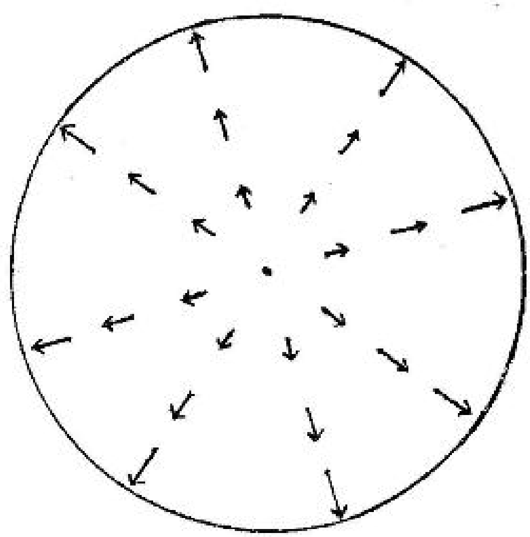

Fig 6 **Radial nebula field.**

The Hubble radius R_H is the value of r for which v is the speed of light: (3×10^5 km/sec)

$$R_H = c/H$$

H: known to be in the range 50 to 100 km sec^{-1}/megaparsec. A working value of 75 gives for R_H in megaparsecs

$$R_H = = 3 \times 10^5 /75 = 4 \times 10^3 \text{ (megaparsecs)}$$

A megaparsec is 3×10^{19} km or 3×10^{22} m. So that

$$R_H = 12 \times 10^{25} = 1.2 \times 10^{26} \text{ (metres)}$$

14. DUAL VIEWS FOR THE OBSERVABLE UNIVERSE

Euclidean view: **observable universe seen as Euclidean sphere with receding nebula field following Hubble law**

$$v = H r \qquad v < c \qquad (r < c/H = R_H)$$

Doppler red shift defined for emitted and observed wave-lengths λ_0, λ_1 by

$$z = (\lambda_1 - \lambda_0)/\lambda_0$$

Classical Doppler shift law gives velocity v observed from z

$$z = v/c \ (= r/R_H)$$

The observable universe is finite and *private* to each observer.

***Hyperbolic view*: The sphere of the observable universe is regarded as a Beltrami space with hyperbolic metric**

$$\rho = R_H \, th^{[-1]}(r/R_H) \qquad 0 < \rho < \infty$$

The hyperbolic Hubble law comes by substitution for v in:

$$V = c \, th^{[-1]}(v/c) = c \, th^{[-1]}(r/R_H) = H \rho$$

Redshift law uses logarithmic redshift Z

$$Z = V/c$$

The observable universe is infinite and *public* (identical for different observers). Both distance and velocity can be parametrized by nondimensional rapidity

$$w = th^{[-1]}(v/c) = th^{[-1]}(r/R_H)$$

15. ESTIMATION OF THE ANGLE OF LOBACHEVSKI

Diagram:

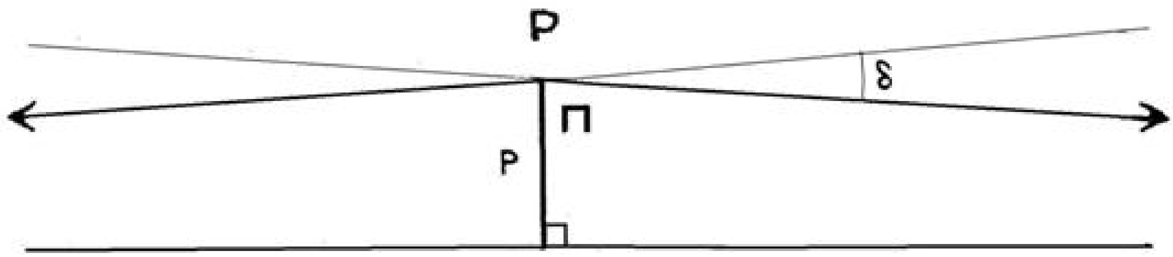

Fig 7 **The deviation from parallelism**

***Notation*:**

Π: angle between perpendicular and an asymptotic parallel
p: perpendicular distance of the point from the line
R: radius of negative curvature of the space.

The formula of Lobachevski

$$\tan(\Pi/2) = \exp(-p/R)$$

***Calculation*: Put Π as $\pi/2 - \delta/2$ where δ is the very small angle between asymptotic parallels. Then approximately**

$$\delta = 2p/R \text{ (radians)}$$

Set R to the Hubble radius R_H.

$$R_H = 1.2 \times 10^{26} \text{ (metres)}$$

For a displacement p of 1 metre the value of δ is

$$\delta = 2 \times 57/(1.2 \times 10^{26}) \approx 10^{-24} \text{ (degrees)}$$

16. REFERENCES

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