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Ground Track of a Satellite in a Circular Orbit around the Earth

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Introduction

This note describes the analysis of the ground track of a satellite with a given circular orbit inclination to the equator and given orbital period. The altitude of the orbit is calculated from that period.

Nomenclature

Variable Name	Description
X_G, Y_G, Z_G	Coordinates of satellite
θ	Inclination of Orbit Plane above the Equator
ϕ	Azimuth of Satellite in Orbit Plane
ψ	Azimuth Movement of Earth
r_{EARTH}	Mean Radius of Earth
r_{SAT}	Radius of Satellite Orbit
T_{SAT}, ω_{SAT}	Period (hr) and Angular Velocity of Satellite
τ_{EARTH}, ω_{EARTH}	Period (hr) and Angular Velocity of Earth
n	Centripetal Acceleration of Satellite expressed in g

Discussion

The basic geometry is shown in Figure 1, *(XOY is the equatorial plane and Z is from the Earth's centre to the North Pole)*:

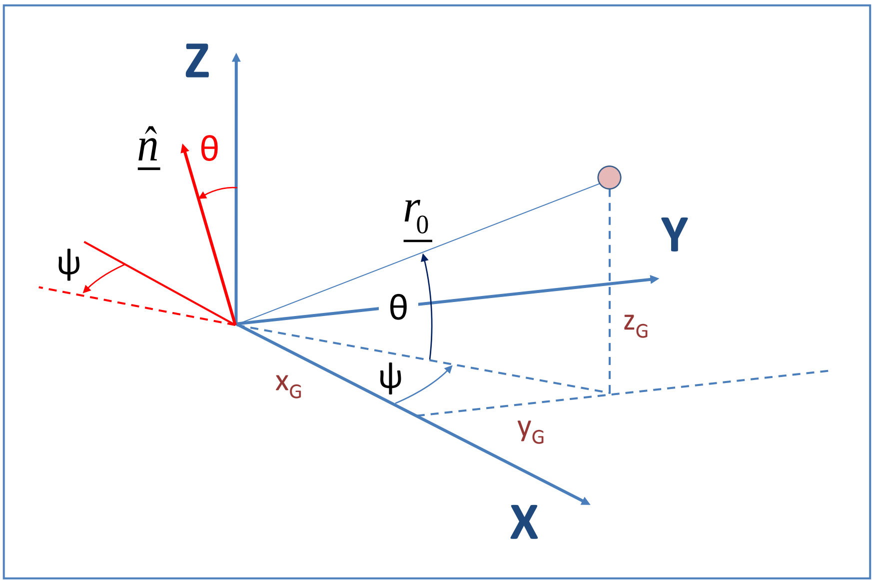

Figure 1 – Geometry of Satellite Orbit Plane Location

Referring to Figure 1, and using the results of the Appendix the satellite location (at t=0) – defining the orbit plane - is given by:

$$\underline{r} = r_{ORBIT}\left(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta\right) \tag{1.}$$

When the azimuthal rotation needs to be applied and to achieve this we need the unit normal to the orbit plane thus:

$$\underline{\hat{n}} = \left(-\sin\theta\cos\psi, -\sin\theta\sin\psi, \cos\theta\right) \tag{2.}$$

The azimuthal rotation is shown in Figure 2:

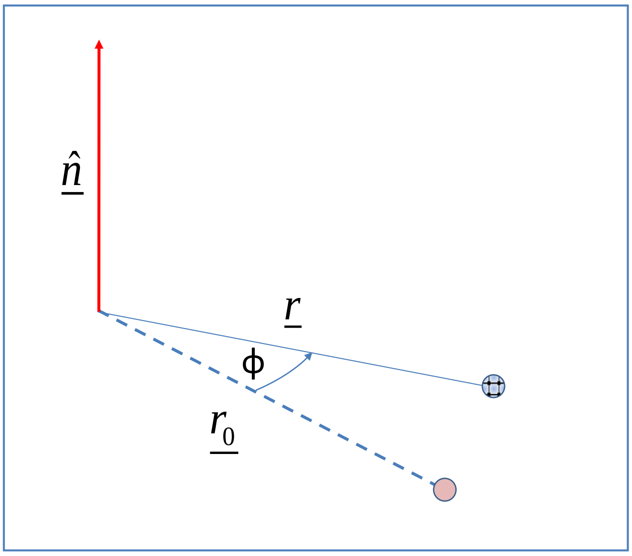

Figure 2 – Azimuthal Motion Definition of Satellite in Orbit Plane

We now assemble the terms needed and which are defined in the Appendix.

Therefore:

$$\begin{aligned}\underline{r}\cdot\underline{\hat{n}} &= -\sin\theta\cos\theta\cos^2\psi - \sin\theta\sin^2\psi\cos\theta + \sin\theta\cos\theta \\ &= 0\end{aligned} \tag{3.}$$

and

$$\begin{aligned}\underline{\hat{n}}\wedge\underline{r} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin\theta\cos\psi & -\sin\theta\sin\psi & \cos\theta \\ \cos\theta\cos\psi & \cos\theta\sin\psi & \sin\theta \end{vmatrix} \\ &= \underline{i}\left(-\sin^2\theta\sin\psi - \cos^2\theta\sin\psi\right) \\ &\quad + \underline{j}\left(\cos^2\theta\cos\psi + \sin^2\theta\cos\psi\right) \\ &\quad + \underline{k}\left(-\sin\theta\cos\theta\sin\psi\cos\psi + \sin\theta\cos\theta\sin\psi\cos\psi\right) \\ &= \left(-\sin\psi, \cos\psi, 0\right)\end{aligned} \tag{4.}$$

Giving finally:

$$
\begin{aligned}
\underline{r}' &= (\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)\cos\phi \\
&\quad + (-\sin\psi, \cos\psi, 0)\sin\phi \\
&= \begin{pmatrix} \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi, \\ \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi, \\ \sin\theta\cos\phi \end{pmatrix}
\end{aligned}
\tag{5.}
$$

The satellite location coordinates are therefore:

$$
\begin{aligned}
X_G &= \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi \\
Y_G &= \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi \\
Z_G &= \sin\theta\cos\phi
\end{aligned}
\tag{6.}
$$

From which the latitude and longitude can be obtained thus:

$$
\begin{aligned}
Latitude &= \tan^{-1}\left\{\frac{Z_G}{\sqrt{X_G^2 + Y_G^2}}\right\} \\
Longitude &= \tan^{-1}\left\{\frac{Y_G}{X_G}\right\}
\end{aligned}
\tag{7.}
$$

The two defining angles are:

$$
\begin{aligned}
\phi &= \omega_{SAT} t \\
\psi &= -\omega_{EARTH} t
\end{aligned}
\tag{8.}
$$

They are linked via the orbital periods thus:

$$
\psi = -\frac{\tau_{SAT}}{\tau_{EARTH}}\phi
\tag{9.}
$$

The inclusion of (9) represents the rotation of the Earth. Essentially to obtain the ground track we need to examine the location of the satellite relative to a *stationary* Earth.

Altitude v Orbit Period

The angular velocity of the Satellite and Earth are expressed by:

$$\omega_{SAT} = \frac{2\pi}{\tau_{SAT}} \qquad \omega_{EARTH} = \frac{2\pi}{\tau_{EARTH}} \tag{10.}$$

The inward acceleration for the satellite in orbit is:

$$n \cdot g = \omega_{SAT}^2 \cdot r_{SAT} \tag{11.}$$

From which we find:

$$r_{SAT} = \frac{n \cdot g}{\omega_{SAT}^2} \tag{12.}$$

Owing to the inverse square law of gravitation we can put:

$$n = \left(\frac{r_{EARTH}}{r_{SAT}} \right)^2 \tag{13.}$$

Combining (12) and (13) we obtain:

$$r_{SAT} = \sqrt[3]{\frac{r_{EARTH}^2 g}{\omega_{SAT}^2}} \qquad \frac{r_{SAT}}{r_{EARTH}} = \sqrt[3]{\frac{g}{r_{EARTH} \omega_{SAT}^2}} \tag{14.}$$

From which the altitude of the orbit is given by:

$$\begin{aligned} Altitude &= r_{SAT} - r_{EARTH} \\ &= \left(\frac{r_{SAT}}{r_{EARTH}} - 1 \right) \cdot r_{EARTH} \end{aligned} \tag{15.}$$

Example

An example of the calculation is shown below in Figures 3 and 4:

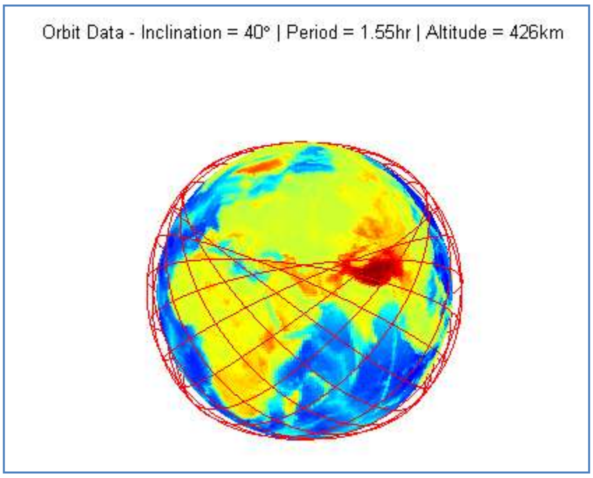

Figure 3

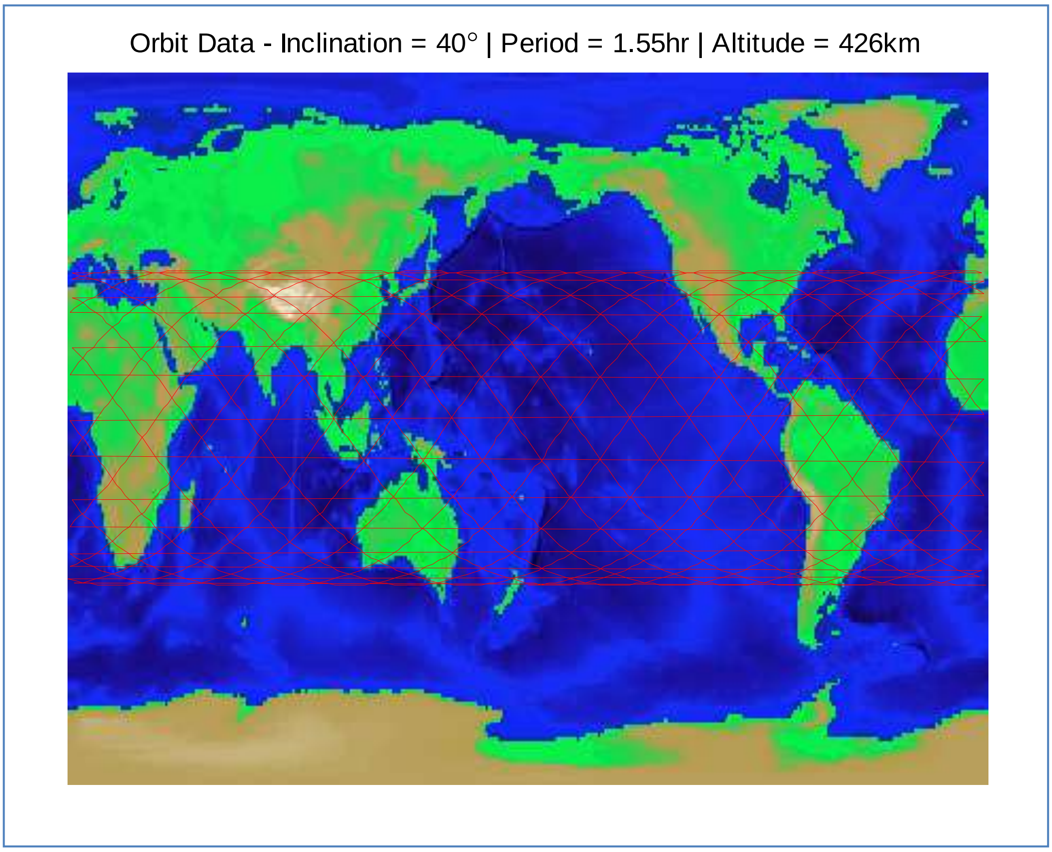

Figure 4

Appendix – Rotation of Vector about Axis

The problem is shown in Figure 1:

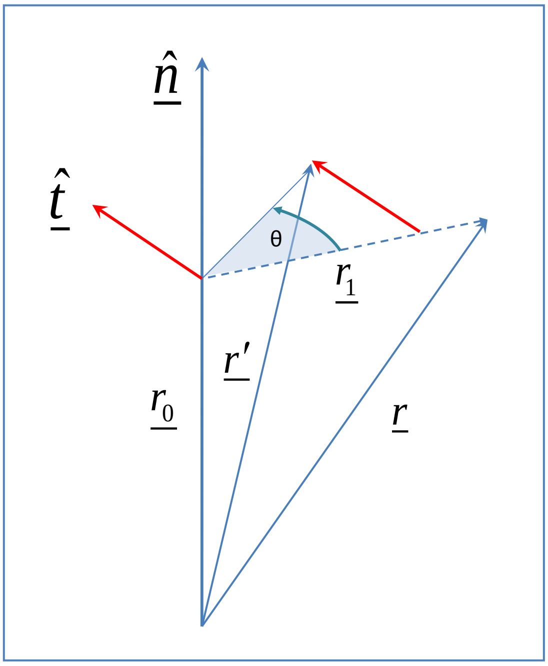

Figure 5 – Geometry of Rotation of Vector

Referring to Figure 1, the vector $\underline{r}$ is to be rotated about the axis defined by the unit vector $\hat{\underline{n}}$ by an angle θ. The analysis of the problem is as follows:

$$\underline{r}_0 = \left(\underline{r} \cdot \hat{\underline{n}}\right)\hat{\underline{n}} \tag{16.}$$

From which:

$$\underline{r}_1 = \underline{r} - \underline{r}_0 \tag{17.}$$

Using the vector product we find:

$$\hat{\underline{t}} = \frac{\left\{\hat{\underline{n}} \wedge \underline{r}_1\right\}}{\left|\underline{r}_1\right|} \tag{18.}$$

The new vector after rotation is given by:

$$\underline{r}' = \left|\underline{r}_1\right| \cos\theta \hat{\underline{r}}_1 + \left|\underline{r}_1\right| \sin\theta \hat{\underline{t}} + \underline{r}_0 \tag{19.}$$

Combining (17) & (18) we obtain:

$$\begin{aligned}\underline{\hat{t}} &= \frac{\underline{\hat{n}} \wedge \left(\underline{r} - \underline{r_0}\right)}{\left|\underline{r_1}\right|} \\ &= \frac{\underline{\hat{n}} \wedge \underline{r}}{\left|\underline{r_1}\right|} - \frac{\underline{\hat{n}} \wedge \underline{r_0}}{\left|\underline{r_1}\right|} \\ &= \frac{\underline{\hat{n}} \wedge \underline{r}}{\left|\underline{r_1}\right|}\end{aligned} \tag{20.}$$

since

$$\frac{\underline{\hat{n}} \wedge \underline{r_0}}{\left|\underline{r_1}\right|} = 0 \tag{21.}$$

Whence the new vector after rotation is given by:

$$\begin{aligned}\underline{r'} &= \underline{r_1} \cos\theta + \underline{\hat{n}} \wedge \underline{r} \sin\theta + \underline{r_0} \\ &= \left(\underline{r} - \underline{r_0}\right)\cos\theta + \underline{\hat{n}} \wedge \underline{r} \sin\theta \\ &= \underline{r}\cos\theta + \underline{r_0}\left(1 - \cos\theta\right) + \underline{\hat{n}} \wedge \underline{r}\sin\theta \\ &= \underline{r}\cos\theta + \left(\underline{r} \cdot \underline{\hat{n}}\right)\underline{\hat{n}}\left(1 - \cos\theta\right) + \underline{\hat{n}} \wedge \underline{r}\sin\theta\end{aligned} \tag{22.}$$

MATLAB File

```
%
%   Ground Track of Satellite
%
%   SJN 19/1/11
%
clear all
colordef black
%-----------------------------------------------------------------------------
% Input Data
thetaorbitdeg=40;
orbitperiodhr=1.55;
%-----------------------------------------------------------------------------

earthperiodhr=24;
omsathr=2*pi/orbitperiodhr;
omearthhr=2*pi/earthperiodhr;

rorearth=(9.81*(orbitperiodhr)^2*(3600^2)/(4*(pi^2)*6371000))^(1/3);
%-----------------------------------------------------------------------------
sataltkm=(rorearth-1)*6371;
%-----------------------------------------------------------------------------
% Time Specification
dtimehr=.01;
timemaxhr=24;
%-----------------------------------------------------------------------------
load('topo.mat','topo','topomap1');

timehr=0:dtimehr:timemaxhr;
phi=omsathr*timehr;
psi=-omearthhr*timehr;
theta=thetaorbitdeg*pi/180;

cphi=cos(phi);
sphi=sin(phi);
cpsi=cos(psi);
spsi=sin(psi);
ctheta=cos(theta);
stheta=sin(theta);

XG=rorearth*(ctheta*cpsi.*cphi-spsi.*sphi);
YG=rorearth*(ctheta*spsi.*cphi+cpsi.*sphi);
ZG=rorearth*(stheta*cphi);
%-----------------------------------------------------------------------------
% Plot Satellite Orbit in 3D
plot3(XG,YG,ZG,'Color','r','LineWidth',1);
hold on
%-----------------------------------------------------------------------------
% Plot Earth in 3D
[XEARTH,YEARTH,ZEARTH]=sphere(50);
props.FaceColor='texture';
props.Cdata =topo;
props.LineStyle='none';
surface(XEARTH,YEARTH,ZEARTH,props);
```

```
axis equal
axis off
title(['Orbit Data - Inclination = ',num2str(thetaorbitdeg),...
    '\circ | Period = ',num2str(orbitperiodhr),...
    'hr | Altitude = ',num2str(round(sataltkm)),'km']);
%--------------------------------------------------------------------------

figure

latitudedeg=asin(ZG/rorearth)*180/pi;
longitudedeg=atan2(YG,XG)*180/pi;
%--------------------------------------------------------------------------
% Plot Earth in 2D
image([-180 180],[-90 90],flipud(topo),'CDataMapping', 'scaled');
axis off
colormap(topomap1);
hold on
%--------------------------------------------------------------------------
% Plot Satellite Orbit in 2D
plot(longitudedeg,latitudedeg,'r');
title(['Orbit Data - Inclination = ',num2str(thetaorbitdeg),...
    '\circ | Period = ',num2str(orbitperiodhr),...
    'hr | Altitude = ',num2str(round(sataltkm)),'km']);
```