COHOMOLOGICAL FINITENESS PROPERTIES OF THE BRIN-THOMPSON-HIGMAN GROUPS $2V$ AND $3V$

D. H. KOCHLOUKOVA, C. MARTÍNEZ-PÉREZ, AND B. E. A. NUCINKIS

Abstract. We show that Brin's generalisations $2V$ and $3V$ of the Thompson-Higman group V are of type FP_∞. Our methods also give a new proof that both groups are finitely presented.

1. Introduction

In this paper we study cohomological finiteness conditions of certain generalisations of Thompson's group V, which is a simple, finitely presented group of homeomorphisms of the Cantor-set C. The finiteness conditions we consider, are the homotopical finiteness property F_∞ for a group, which was first defined by C.T.C.Wall, and its homological version FP_∞, which was studied in detail in [3]. We say that a group G is of type F_∞ if it admits a $K(G,1)$ with finite k-skeleton in all dimensions k. A group is of type FP_∞ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a resolution with finitely generated projective $\mathbb{Z}G$-modules. A group is of type F_∞ if and only if it is of type FP_∞ and is finitely presented. There are, however, examples of groups of type FP_∞, which are not finitely presented [2].

In [8] K.S. Brown showed that Thompson's groups F, T and V as well as some generalisations such as Higman's groups $V_{n,r}$ (see [11]) are of type F_∞. The idea there is to express these groups as groups of algebra-automorphisms and let them act on a poset determined by the algebra. It is then shown that the geometric realisation of this poset yields the required finiteness properties.

In [6] M. Brin defined a group sV generalising V for every natural number $s \geq 2$. Analogously to V, these groups are defined as subgroups of the homeomorphism group of a finite Cartesian product of the Cantor-set. For each s, the group sV is simple, finitely presented and contains a copy of every finite group [7, 5]. It was also shown in [4] that for $s \neq t$, sV is not isomorphic to tV.

Our main result is the following:

Main Theorem. *Brin's groups $2V$ and $3V$ are of type F_∞.*

The proof of the Main Theorem is split in two parts : Theorems 4.17 and 5.6. We partially follow the proof of [8] that V has type F_∞. Our proof is more intricate, as the fact that some particular complex K_Y is t-connected

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if Y is sufficiently large requires more work than in Brown's proof. As in [8] we view sV as a group of algebra automorphisms and consider a poset $\mathfrak{A}$ on which sV acts. This action has the following properties:

(i) Vertex stabilisers are finite.
(ii) The complex $|\mathfrak{A}|$ is contractible.
(iii) There is a filtration $\{|\mathfrak{A}_n|\}_{n\geq 1}$ of sV-subcomplexes of $|\mathfrak{A}|$ such that each complex $|\mathfrak{A}_n|$ is finite modulo sV.
(iv) For $s = 2$ and $s = 3$ the connectivity of the pair of complexes $(|\mathfrak{A}_{n+1}|, |\mathfrak{A}_n|)$ tends to infinity as $n \to \infty$.

We then apply Brown's criterion [8, Cor. 3.3] to conclude that $2V$ and $3V$ are finitely presented and of type F_∞. The key result towards the proof of our main theorem for $s = 2$ is Theorem 4.6. Finally, in the last section, we prove Theorem 5.3 as a variation of Theorem 4.6 and show that the method above can be applied for $s = 3$.

2. Construction of the algebra and the group

In this section we shall define the generalised Higman Algebra, also called Cantor-Algebra, in a general setting. We then define sV as a group of automorphisms of this Algebra.

Consider a finite set $\{1, \dots, s\}$. We call its elements colours. Also consider a finite set of integers $\{n_1, \dots, n_s\}$, $n_i > 1$. We call each n_i the arity of the colour i. We begin by defining an Ω-algebra U. For details the reader is referred to [10]. We say U is an Ω-algebra, if, for each colour i, the following operations are defined in U:

i) One n_i-ary operation λ_i:

$$\lambda_i : U^{n_i} \to U.$$

We call these operations ascending operations, or contractions.

ii) n_i 1-ary operations $\alpha_i^1, \dots, \alpha_i^{n_i}$:

$$\alpha_i^j : U \to U.$$

We call these operations 1-ary descending operations.

Throughout this paper all operations act on the right. By definition, $\Omega = \{\lambda_i, \alpha_i^j\}_{i,j}$. In what follows it will be convenient to consider the following map, which we also call operation: For each colour i, and any $v \in U$, we denote

$$v\alpha_i := (v\alpha_i^1, v\alpha_i^2, \dots, v\alpha_i^{n_i}).$$

Therefore α_i is a map

$$\alpha_i : U \to U^{n_i}.$$

We call these maps descending operations, or expansions. Unless otherwise stated, whenever we use the term "descending operation", we refer to one of the α_i.

For any subset Y of U, a simple expansion of colour i of Y consists of substituting some element $y \in Y$ by the n_i elements of the tuple $y\alpha_i$. And a simple contraction of colour i of Y is the set obtained by substituting a certain collection of n_i distinct elements of Y, say $\{a_1, \dots, a_{n_i}\}$, by

$(a_1, \ldots, a_{n_i})\lambda_i$. We also use the word operation to refer to the effect of a simple expansion, respectively contraction on a set .

A morphism between Ω-algebras is a map commuting with all operations in Ω. Let $\mathfrak{B}_0$ be a category of Ω-algebras. An object $U_0(X) \in \mathfrak{B}_0$ is a free object in $\mathfrak{B}_0$ with X as a *free basis* if for any $S \in \mathfrak{B}_0$ any mapping

$$\theta : X \to S$$

can be extended in a unique way to a morphism

$$U_0(X) \to S.$$

We also say $U_0(X)$ is free on X in the category $\mathfrak{B}_0$. Following [10, III.2], we construct the free object on any set X in the category of all Ω-algebras as follows: take the set of finite sequences of elements of the disjoint union $\Omega \cup X$ with the Ω-algebra structure defined by juxtaposition. Then $U_0(X)$ is the sub Ω-algebra generated by X.

Definition 2.1. The free object constructed above is called the Ω-word algebra and denoted $W_\Omega(X)$. An *admissible* subset is any $Y \subset W_\Omega(X)$, which can be obtained from X by a finite number of operations α_i and λ_j, i.e. by a finite number of simple contractions or expansions.

Now we consider the variety of Ω-algebras satisfying a certain set of identities.

Definition 2.2. Let Σ_1 be the following set of laws in a countable (possibly finite) alphabet X.

i) For any $u \in W_\Omega(X)$ and any colour i,

$$u\alpha_i\lambda_i = u.$$

ii) For any colour i and any n_i-tuple $(u_1, \ldots, u_{n_i}) \in W_\Omega(X)^{n_i}$,

$$(u_1, \ldots, u_{n_i})\lambda_i\alpha_i = (u_1, \ldots, u_{n_i}).$$

The variety $\mathfrak{V}_1$ of Ω-algebras which satisfy the identities in Σ_1, obviously contains nontrivial algebras. Hence it is a nontrivial variety. Therefore by [10, IV 3.3] it contains free algebras on any set X. Let $U_1(X)$ be the free Ω-algebra on X in $\mathfrak{V}_1$. Moreover, by the proof of [10, IV 3.1]

$$U_1(X) = W_\Omega(X)/\mathfrak{q}_1,$$

where $\mathfrak{q}_1$ is the fully invariant congruence generated by Σ_1, i.e. the smallest equivalence set in $W_\Omega(X) \times W_\Omega(X)$ containing Σ_1, which admits any endomorphism of $W_\Omega(X)$ and is Ω-closed (see [10, IV Section 1]). In fact there is an epimorphism

$$\theta_1 : W_\Omega(X) \to U_1(X)$$

and $\mathfrak{q}_1$ corresponds precisely to $\mathrm{Ker}(\theta_1)$.

Definition 2.3. Let $U \in \mathfrak{V}_1$ and let Y be a subset of U. A set Z obtained from Y by a finite number of simple expansions is called a descendant of Y. In this case we denote

$$Y \leq Z.$$

Conversely, Y is called an ascendant of Z and can be obtained after a finite number of simple contractions.

In what follows we will consider Ω-algebras satisfying some additional identities as described below.

Definition 2.4. Let Σ be the set of identities

$$\Sigma = \Sigma_1 \cup \{r_{ij} \mid 1 \leq i < j \leq s\},$$

where r_{ij} consists of certain identifications between sets of simple expansions of $w\alpha_i$ and $w\alpha_j$ for any $w \in W_\Omega(X)$ which do not depend on w.
Let X be a set and $U(X) = U_1(X)/\mathfrak{q}$ where $\mathfrak{q}$ is the fully invariant congruence generated by Σ. There is an epimorphism

$$\theta_2 : U_1(X) \twoheadrightarrow U(X)$$
$$a_1 \mapsto \bar{a}_1.$$

Let $\theta : W_\Omega(X) \to U(X)$ be the composition of θ_1 with θ_2. We say that a subset Y of $U_1(X)$ or of $U(X)$ is *admissible* if it is the image by θ_1 or θ of an admissible subset of $W_\Omega(X)$. We call the set of identities Σ *valid* if the following condition holds: for any admissible set $Y \subseteq U_1(X)$ we have $|Y| = |\bar{Y}|$, i.e. θ_2 is injective on admissible subsets.
Let $\mathfrak{V}$ be the variety of all Ω-algebras which satisfy the identities in a valid Σ. Note that $\mathfrak{V}$ contains nontrivial Ω-algebras, so it has free objects on every set X. In fact, the algebra $U(X)$ above is a free object on X.

Lemma 2.5. *Any admissible subset is a free basis in $W = U(X)$.*

Proof. This can be proven using the same argument as in [11]: Let X be a free basis of W, let $i \in \{1, \dots, s\}$ be any colour of arity n_i and

$$Y = (X \smallsetminus \{x\}) \cup \{x\alpha_i^j \mid 1 \leq j \leq n_i\}.$$

We will show that Y is a free basis of W. Recall that $\mathfrak{V}$ is the variety of Ω-algebras satisfying the identities Σ. Then, given any $S \in \mathfrak{V}$ and any mapping $\theta : Y \to S$, there is a unique way to obtain a map $\theta^* : X \to S$ such that $\theta^*(\tilde{x}) = \theta(\tilde{x})$ for $\tilde{x} \in X \smallsetminus \{x\}$ and $\theta^*(x) = (\theta(x\alpha_i^1), \dots, \theta(x\alpha_i^{n_i}))\lambda_i$. As there is a unique $\hat{\theta} : W \to S$ extending θ^*, the same happens with the original θ.

Analogously, one proves that considering n_i distinct elements $x_1, \dots, x_{n_i}$ of X, the admissible subset

$$Y = (X \smallsetminus \{x_1, \dots, x_{n_i}\}) \cup \{(x_1, \dots, x_{n_i})\lambda_i\}$$

is a free basis of W. □

Definition 2.6. Consider the set of s colours $\{1, \dots, s\}$, all of which have arity 2, together with the relations:

$$\Sigma := \Sigma_1 \cup \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq i \neq j \leq s; l, t = 1, 2\}.$$

We call the Ω-algebra $W = U(\{x_0\})$, defined by the Σ above, the generalised Higman algebra on s colours.

Remark 2.7. (Geometric interpretation of the generalised Higman algebra). Consider the unit cube $\mathfrak{C}$ of $\mathbb{R}^s$. Fix a bijection between the set of colours $\{1, \dots, s\}$ and the set of hyperplanes which are parallel to the faces of $\mathfrak{C}$. To each operation α_i we associate a halving using a hyperplane parallel to the hyperplane corresponding to i. In this case we say we halve in direction i. Then, to each side of this halving we associate one of the components of α_i: α_i^1 and α_i^2. This association will stay fixed. For a sequence of 1-ary descending operations $u = \alpha_{i_1}^{r_1} \dots \alpha_{i_t}^{r_t}$ with $r_j \in \{1, 2\}$ we perform the following operations in $\mathfrak{C}$: First, halve it in direction i_1 and take the r_1-half. Repeat the process with operation $\alpha_{i_2}^{r_2}$ for this half. At the end, we get a subset (subparallelepiped) of $\mathfrak{C}$. For simplicity we call the subparallelepipeds s-subcubes or simply s-cubes. Note that at any stage, if $i \neq j$, the effect of $\alpha_i^{r_i} \alpha_j^{r_j}$ equals the effect of $\alpha_j^{r_j} \alpha_i^{r_i}$.

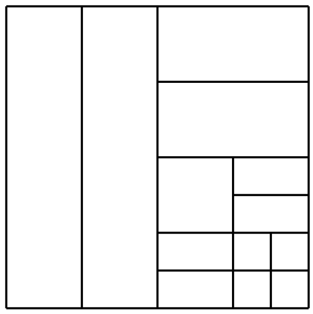


Figure 1

The family of s-subcubes of the s-cube $\mathfrak{C}$, which can be obtained in this way corresponds to *the set $x(D)$ of descendants of x* in the generalised Higman algebra $U(\{x_0\})$, where x is an element belonging to some admissible subset. Analogously, we may identify any admissible subset A with a collection of $|A|$ s-cubes. In particular, the set of descendants of A corresponds to the set of those subsets in the collection of $|A|$ s-cubes, which are obtained in the prescribed way.

Remark 2.8. In the following diagram we use two different types of carets to visualise the two colours in the generalised Higman algebra on 2 colours, each of arity 2.

Figure 2

The first type of caret corresponds to vertical cutting and the second one to horizontal. We view an admissible set that is a descendent of an element x as the set of leaves of a rooted tree with root x. The rooted tree is constructed by gluing one of the two types of carets when passing to descendants. The following two rooted trees represent the same admissible set:

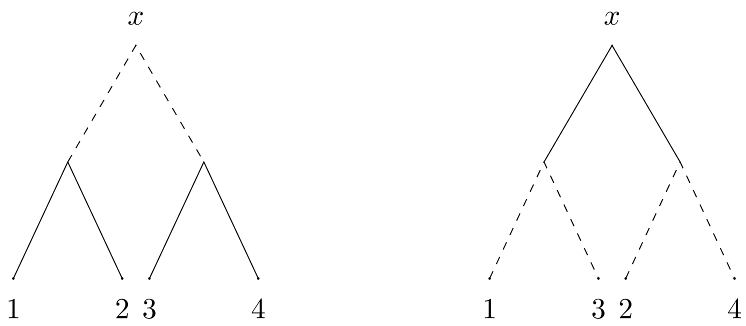


Figure 3

Considering the geometric interpretation of the generalised Higman algebra, both of the rooted trees above represent the following subdivision of the square:

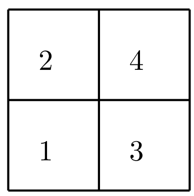


Figure 4

Lemma 2.9. *The generalised Higman algebra $W = U(\{x_0\})$ is valid.*

Proof. To begin we claim that for any pair of admissible subsets Y and $Z \subseteq U_1(\{x_0\})$, such that Z is obtained from Y after a simple expansion, we have $|\bar{Z}| = |\bar{Y}| + 1$. Recall that $\bar{Z}$ and $\bar{Y}$ are the images of Z and Y in $U(\{x_0\})$. Any admissible set in $U_1(\{x_0\})$ is a descendant of an admissible set with only one element, say y. So for $x = \bar{y}$ we have that $\bar{Z}, \bar{Y} \in x(D)$, where $x(D)$ is as defined in Remark 2.7. Using the geometric interpretation of $x(D)$ as a subdivision of an s-cube we get the claim.

Conversely, if Z is a simple contraction of Y then Y is a simple expansion of Z. Thus $|\bar{Y}| = |\bar{Z}| + 1$.

Finally, an induction on the number of simple contractions and expansions needed to obtain an admissible subset $\bar{Y} \subseteq U(\{x_0\})$ from $\{x_0\}$ yields the result. $\square$

Definition 2.10. The Brin-Thompson-Higman group on $W_0 = U(X)$, which we denote $G(W_0)$, is the group of algebra automorphisms of W_0 which are induced by a bijection $Z \to Y$ for any two admissible sets Z and Y of the same cardinality. If W is the generalised Higman algebra $U(\{x_0\})$, then $G(W)$ is the Brin group on s colours and is denoted sV.

The following diagram illustrates an element g of $2V$ sending each leaf to the leaf with the same label.

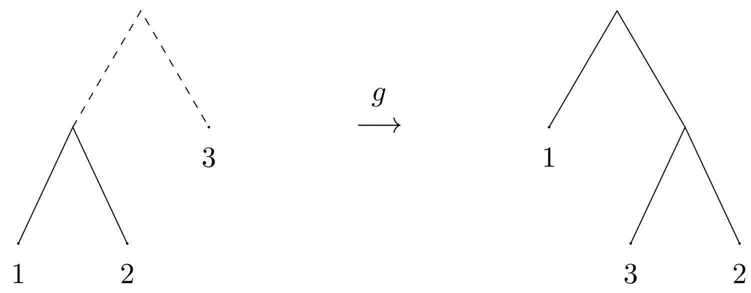


Figure 5

Remark 2.11. Looking at the geometric interpretation of the generalised Higman algebra, Section 2.3 of [6] implies that this is exactly the definition of Brin's generalisation $2V$ of V as a group of self-homeomorphisms of $C \times C$, where C denotes the Cantor-set. The element g in Figure 5 corresponds to the following picture:

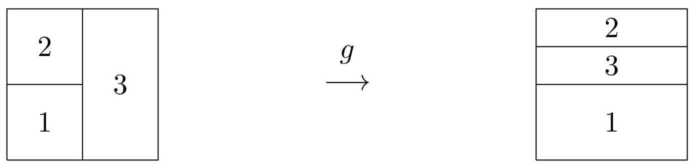


Figure 6

The equivalence of definitions for higher dimensional sV follows from Section 4.1 [6]. If there is only one colour, then V is exactly the Higman- Thompson group as defined in [8].

3. The poset of admissible subsets

In this section we consider the Brin-Higman algebra on s colours with basis $\{x\}$. We write U for $U(\{x\})$.

Definition 3.1. The set of admissible subsets is a poset with the order defined by $A < B$ if B is a descendant of A. We denote this poset by $\mathfrak{A}$ and by $|\mathfrak{A}|$ its geometric realization. Note that any descendant and any ascendant of an admissible subset is also admissible.

Given admissible subsets Y and Z of U, we say that they have a unique least upper bound T if $Y \leq T$ and $Z \leq T$, and whenever $Y \leq S$ and $Z \leq S$, then $T \leq S$. Analogously, we define the notion of greatest lower bound.

Lemma 3.2. *Let A, Y and Z be admissible subsets with $A \leq Y$ and $A \leq Z$. Then there is a unique least upper bound of Y and Z.*

Proof. Consider the geometric representation of the set of descendants of A as subdivisions of s-dimensional cubes (in fact s-dimensional parallelepipeds but we call them cubes for simplicity) labeled by the elements of A, see Remark 2.7. Then the result of performing both sets of subdivisions corresponding to Y and Z yields an upper bound T. Clearly, for any other upper bound S of Y and Z we have $T \leq S$. $\square$

Lemma 3.3. *Let Y, Y_1 and Z be admissible subsets with*
$$Y \geq Y_1 \leq Z.$$
Then there is some admissible subset Z_1 with
$$Y \leq Z_1 \geq Z.$$

Proof. Observe that Y and Z are both descendants of Y_1. Then by Lemma 3.2 there exists an upper bound Z_1 of Y and Z. So we have $Y \leq Z_1 \geq Z$. □

Proposition 3.4. *Any two admissible subsets have some upper bound.*

Proof. Let Y and Z be two admissible subsets. By definition we can obtain Z from Y by a finite number of expansions or contractions. Therefore we may put
$$Y \geq Y_1 \leq Y_2 \geq Y_3 \leq \ldots \geq Y_r \leq Z.$$
By Lemma 3.3 we get
$$Y \leq Z_1 \geq Y_2 \geq Y_3 \leq \ldots$$
and we may shorten the previous chain by omitting Y_2 to get a chain
$$Y \leq Z_1 \geq Y_3 \leq \ldots$$
Thus after finitely many steps we get
$$Y \leq T \geq Z \text{ or } Y \geq T \leq Z$$
for some T. In the second case we apply Lemma 3.3. □

Proposition 3.4 has the following consequence: for any admissible subset A, any element $g \in G(sV)$ can be represented by its action in the set of descendants of A, i.e. there is some $A \leq Z$ with $A \leq Zg$. To see this, choose Z to be some upper bound of A and Ag^{-1}. Then $A \leq Z$ and $Ag^{-1} \leq Z$, so $A \leq Zg$.

Lemma 3.5. *$|\mathfrak{A}|$ is contractible.*

Proof. It is a consequence of Proposition 3.4 as the poset $\mathfrak{A}$ is directed. □

Remark 3.6. Observe that as in the case of V considered in [8], the stabiliser of any admissible set Y in sV is finite, as it consists precisely of the permutations of the elements of Y.

We consider the filtration of $|\mathfrak{A}|$ given by
$$\mathfrak{A}_n := \{Y \in \mathfrak{A} \mid |Y| \leq n\}.$$

Lemma 3.7. *Each $|\mathfrak{A}_n|/sV$ is finite.*

Proof. For any Y and $Z \in \mathfrak{A}_n$ with $|Y| = |Z|$ we may consider the element $g \in sV$ given by $yg = y\sigma$, where $\sigma : Y \to Z$ is a fixed bijection. Thus sV acts transitively on the admissible sets of the same size. □

Contrary to what happens with upper bounds, it is not true in general that any two admissible subsets have some lower bound. But the existence of greatest lower bounds in some particular cases will be crucial in the subsequent sections. To overcome this problem, we assume that our contractions are descendants of the same A and consider greatest lower bounds above A. For simplicity we use the following notation.

Definition 3.8. Let Λ be a finite set of admissible sets, A_1 and A_2 be admissible sets. We write

$$A_1 \leq \Lambda \text{ if for every } B \in \Lambda \text{ we have } A_1 \leq B$$

and

$$\Lambda \leq A_2 \text{ if for every } B \in \Lambda \text{ we have } B \leq A_2.$$

Definition 3.9. Let A be an admissible set and $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets with $A \leq \Omega$. Assume there exists an admissible set M such that $A \leq M \leq \Omega$ and for any other admissible set B with $A \leq B \leq \Omega$, we have $B \leq M$. Then we call M a greatest lower bound of Ω above A and denote $M = \mathrm{glb}_A(\Omega)$.

Definition 3.10. Let $A \leq Y$ be admissible sets and let $r \geq 0$ be an integer. We say that A involves contractions of r elements of Y, or involves r elements of Y for short, if $|Y \smallsetminus A| = r$; we also say that $Y \smallsetminus A$ are the elements of Y contracted in A. Two contractions $A_1, A_2 \leq Y$ are said to be disjoint if the respective sets of elements of Y contracted in A_1 and A_2 are disjoint.

In the particular case of disjoint contractions of a certain admissible Y the existence of greatest lower bounds follows easily:

Lemma 3.11. *Let $\Omega = \{M_0, \dots, M_t\}$ be a set of pairwise disjoint contractions of Y. Then*

$$\varnothing \neq \bigcap_i \{L \mid L \leq M_i\}$$

has a maximal element M which we call a global greatest lower bound for Ω and denote by $gglb(\Omega)$. In particular for any $A \leq \Omega$, M is a $glb_A(\Omega)$. Moreover

$$|\textit{elements of } Y \textit{ involved in } M| = \sum_{0 \leq i \leq t} |\textit{elements of } Y \textit{ involved in } M_i|$$

Proof. We obtain M by successively performing the contractions M_i. □

Lemma 3.12. *Let A be an admissible set and $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets such that $A \leq \Omega$. Then for an admissible subset M we have $M = glb_A(\Omega)$ if and only if $A \leq M \leq \Omega$ and there is no expansion N with $M < N$ and $N \leq \Omega$.*

Proof. Assume first $M = \mathrm{glb}_A(\Omega)$. If $M < N \leq \Omega$, then $A \leq N \leq \Omega$ and therefore $N \leq M$ which is a contradiction.

Conversely, we prove that if there is no N as before, then M is a greatest lower bound above A. Assume there is some admissible set B such that $A \leq B \leq \Omega$. Recall that by Lemma 3.2 there exists a unique smallest upper bound C of B and M above A. Then

$$A \leq \{B, M\} \leq C \leq \Omega.$$

If $M < C$ we have a contradiction and therefore $M = C$, and thus $B \leq M$. □

Lemma 3.13. *Let A be an admissible set and $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets such that $A \leq \Omega$. Then there exists $M = glb_A(\Omega)$.*

Proof. Observe that the following set is finite and non-empty

$$\mathfrak{S} = \{N \text{ admissible} \mid A \leq N \leq \Omega\}.$$

This means that we may choose an element $M \in \mathfrak{S}$ maximal with respect to the ordering. By Lemma 3.12, $M = \mathrm{glb}_A(\Omega)$. □

For later use, we record now the following obvious consequence of the definition of greatest lower bounds and Lemma 3.12:

Lemma 3.14. *Let A be an admissible set and $\Omega = \{Y_0, \dots, Y_t\}$ be a finite set of admissible sets such that $A \leq \Omega$. Consider $A \leq B$ and a subset $\Lambda \subseteq \Omega$ such that $B \leq \Lambda$. Then*

$$glb_A \Omega \leq glb_A \Lambda = glb_B \Lambda.$$

4. Connectivity of $|K_Y|$ and proof of the main result for s=2

Let Y be any admissible subset of the Brin-Higman algebra on s colours. We put

$$K_Y := K_{<Y} = \{Z \mid Z \text{ is admissible with } Z < Y\}.$$

Note that K_Y is a poset. We also consider its geometric realisation which we denote $|K_Y|$.

Our next objective will be to prove that in the case of two colours and $|Y|$ big enough, this complex $|K_Y|$ is t-connected. To do this, we will argue as follows: firstly we will show that the considered complex can be "pushed down" in the sense that its t-connectedness is equivalent to the connectedness of a certain subcomplex Σ_{4t} defined in Section 4.1. Then we will use an argument similar to Brown's argument in [8] to prove that Σ_{4t} is t-connected for $|Y|$ big enough and to deduce, in the last subsection, that $2V$ is of type F_∞.

In the first subsection we shall begin with some general observations, valid for an arbitrary number s of colours.

4.1. Some general observations.

Definition 4.1. Denote by C_r the following subposet of K_Y:

$$C_r := \{A \in K_Y \mid A < Y \text{ and } A \text{ involves at most } r \text{ elements of } Y\},$$

and denote by Σ_r the following subcomplex of $|K_Y|$:

$$\Sigma_r := \{\sigma : A_t < A_{t-1} < \ldots < A_1 < A_0 \mid \sigma \in |K_Y|, A_t \in C_r\}.$$

We denote by Σ_r^t the t-skeleton of Σ_r.

To construct the pushing-procedure we will need to control the number of elements involved in the greatest lower bounds of certain sets of simple contractions of Y. To do that, we will use the notion of length which we define next.

Definition 4.2. Consider $A \in K_Y$. For any $i \in Y$, there is a unique $m \in A$ such that the s-cube labeled m contains the s-cube labeled i. Then i is obtained by a certain number of successive subdivisions of m. We call that number the length of i as descendant of A and denote it by $l(A, i)$. We say that two elements $i, j \in Y$ are glueable in A if there exists some simple contraction $Z < Y$ (of any color) contracting exactly i, j such that $A \leq Z$. Note that in that case $l(A, i) = l(A, j)$.

We also say that $i \in Y$ is locally maximal with respect to A if for any other $j \in Y$ obtained from the same $m \in A$ we have $l(A, i) \geq l(A, j)$. Clearly, in that case any other vertex which is glueable to i in A is also locally maximal.

For example, consider the following admissible subset A in the case of two colours and its descendant Y:

Figure 7

Here we have $l(A, 5) = 2$ and 6 and 5 are glueable. So are 1 and 2. Moreover, all the elements except of 4 are locally maximal with respect to A.

Lemma 4.3. *Let $A \leq B < Y$ be admissible subsets. If $i \in Y$ is locally maximal with respect to A then it is also locally maximal with respect to B.*

Proof. Let $m_A \in A$, $m_B \in B$ be the elements in the respective set from which i is obtained. It suffices to note that any $j \in Y$ obtained from m_B is also obtained from m_A. $\square$

If $A \leq Y$ and we use the geometric description of Y as partitions of s-cubes, then the length of $i \in Y$ is related to the size of the subcube labeled i. If two vertices i, j are glueable, then the cubes labeled i and j have exactly the same sizes and are neighbours. This implies that, for fixed i, there are at most $2s$ vertices which are glueable to i. The next result implies that this bound in fact is $2(s-1)$.

Lemma 4.4. *Let $A \leq \{Y_0, Y_1\} < Y$, where Y_1 and Y_2 are different, not disjoint, simple contractions of Y of colours a and b. Label $\{1, 2\}$ the vertices contracted in Y_0 and $\{2, 3\}$ those contracted in Y_1. Then the vertices labeled 1 and 3 are different and $a \neq b$.*

Proof. We use the geometric realisation of sV. Assume that $a = b$. As $Y_0 \neq Y_1$ this would mean that the s-cubes labelled 1 and 3 are situated at opposite sides of the s-cube labeled 2. This, however, is impossible since α_a^1 and α_a^2 do not commute. In particular, if one side of an s-cube can be deleted in a contraction, then the opposite side can not be deleted. Therefore $a \neq b$ and the s-cubes labeled 1 and 3 are on the sides of the s-cube labeled 2 corresponding to different directions. In particular the s-cubes labeled 1 and 3 are different. $\square$

In the following definition we consider a special graph Γ_A that will be quite useful in the next subsections.

Definition 4.5. Let $A \leq Y$ be a contraction and consider the coloured graph Γ_A whose vertices are the vertices of Y, and with an edge of colour a between vertices i, j if there is a simple contraction Z with $A \leq Z < Y$

which contracts i, j with colour a. Note that whenever $A \leq B \leq Y$ then $\Gamma_B \subseteq \Gamma_A$ and the graph Γ_Y consists of the vertices of Y with no edges. Also, any family of simple contractions $\Omega = \{Y_0, \ldots, Y_t\}$ of Y such that $A \leq \Omega$ yields a subgraph of Γ_A formed by the edges associated to the Y_i's. We say that the family is connected if this subgraph is connected. Observe that if Ω is connected, then all the contractions $Y_i \in \Omega$ have the same length in A. In particular, if the vertices involved in Y_i are locally maximal with respect to A then so are the vertices involved in any other Y_j.

4.2. **Construction of the Pushing-procedure.** From now on, we assume we have only two colours. Also recall that both are of arity 2. In this subsection we prove the following result:

Theorem 4.6. *There exists an order reversing poset map*

$$M : \{\textit{Poset of simplices of } |K_Y|\} \to K_Y$$

such that for any t-simplex $\sigma : A_t < A_{t-1} < \ldots < A_0$ we have

$$A_t \leq M(\sigma) \in C_{4t}.$$

In the next lemma we describe certain connected components of the graph Γ_A. Recall that for $M \in K_Y$ the vertices involved in contraction in M, or just involved in M for short are the elements of $Y \smallsetminus M$.

Lemma 4.7. *Let $A \leq \{Y_0, Y_1\} < Y$, where Y_0 and Y_1 are different, not disjoint, simple contractions of Y such that the vertices involved in them are locally maximal with respect to some B with $A \leq B \leq \{Y_0, Y_1\}$. Then the connected component of Γ_A containing them is a square and for $M = glb_A(\{Y_0, Y_1\})$, the vertices involved in M are precisely those in the square. In particular, $M \in C_4$.*

Proof. Label with $\{1, 2\}$ the vertices involved in Y_0 and with $\{2, 3\}$ those involved in Y_1. Note that $B \leq M \leq Y_0, Y_1$ so the vertices 1,2,3 are also locally maximal respect to M. Moreover 1,2,3 are obtained from the same element $m \in M$. We shall show that the only possibility occurring is the picture of Figure 4, where m is the square subdivided into 4 small squares.

Consider one of the possible chains of subdivisions of m yielding 1,2,3 and let α_b be the first subdivision of the chain. If $1, 2, 3$ were all in the same half, i.e., all descendants of the same $m\alpha_b^r$ for a fixed $r \in \{1, 2\}$ then a geometric argument proves that also $M_1 = \{m\alpha_b^1, m\alpha_b^2\} \cup (M \smallsetminus m) \leq Y_1, Y_2$, which is impossible by the definition of greatest lower bounds. Hence we may assume that 1,2 are partitions of $m\alpha_b^1$ and 3 is a partition of $m\alpha_b^2$. Moreover, by the commutativity relations, there are no more subdivisions corresponding to colour b in the path of subdivisions needed to obtain 1,2,3 from m. The fact that $M \leq Y_1$ implies that the first subdivision α_b can be inverted, i.e., it must be possible to perform the successive subdivisions in such a way that the second step consists of subdividing in direction a both halves $m\alpha_b^1$ and $m\alpha_b^2$. But again the commutativity relations imply that we may assume that this second subdivision using colour a (i.e. subdivision in direction a) yields precisely the line between the rectangles 1 and 2, and that the rectangles $1, 2, 3$ correspond precisely to three of the rectangles $m\alpha_b^i\alpha_a^j$ for $i, j = 1, 2$. It would be possible that the fourth rectangle were also subdivided, but the hypothesis that the length $l(M,1)$ is maximal implies that this is not the case. So the fourth is also a rectangle of the same size which we label 4 and therefore the rooted tree yielding $1,2,3$ from m is any of the trees of Figure 3. Clearly, the associated graph in Γ_A is a square. $\square$

Observe that the previous Lemma implies that for the contractions Z_0 of $\{3,4\}$ of colour a and Z_1 of $\{1,4\}$ of colour b we also have $A \leq M \leq \{Z_0, Z_1\}$. Moreover $M = \mathrm{glb}_A(Y_0, Y_1, Z_0) = \mathrm{glb}_A(Y_0, Y_1, Z_0, Z_1)$.

Example 4.8. If we have more than 2 colours, the obvious corresponding version of Lemma 4.7, that two non-disjoint simple contractions are contained in a square in Γ_A, will be false. Consider the following example: Suppose we have 3 colours a, b, c, let $A = \{m\}$ and $Y = \{1,2,3,4,5,6,7\}$ with

$$1 = m\alpha_b^2\alpha_a^2\alpha_c^1, \quad 2 = m\alpha_b^1\alpha_a^2\alpha_c^1, \quad 3 = m\alpha_b^1\alpha_a^1\alpha_c^1, \quad 4 = m\alpha_b^1\alpha_a^1\alpha_c^2,$$
$$5 = m\alpha_b^1\alpha_a^2\alpha_c^2, \quad 6 = m\alpha_b^2\alpha_a^2\alpha_c^2, \quad 7 = m\alpha_b^2\alpha_a^1$$

Consider the following tree-diagram, where dotted lines represent halving in direction a, dashed lines halving in direction b and normal lines halving in direction c.

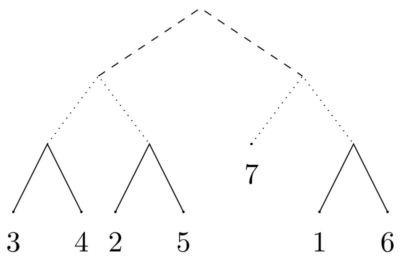


Figure 8

If we wanted all nodes of the same length, we would only have to subdivide 7 further, for example into $m\alpha_b^2\alpha_a^1\alpha_a^1$ and $m\alpha_b^2\alpha_a^1\alpha_a^2$. Let Y_0 be the simple contraction of Y of colour b involving $\{1,2\}$ and Y_1 the simple contraction of Y of colour a involving $\{2,3\}$. Note that $A \leq Y_0, Y_1$ and any contraction of both Y_0 and Y_1 has to involve contraction of either 7 elements in the first case or 8 elements in the second. One easily checks that (in both cases) there is no square in Γ_A containing Y_0 and Y_1. The maximal connected component of the graph Γ_A (in both cases) is what will be called an open book in section 5, where we consider the case of three colours in detail. We may also represent the elements of Y as subdivisions of a cube labelled m, the following picture ilustrates the case when Y has 7 elements.

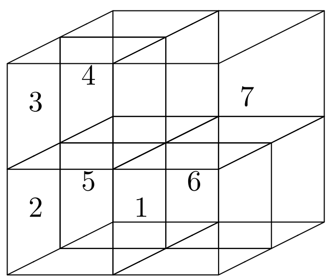


Figure 9

Moreover, if we enlarge in a suitable way we can easily build examples in which the common contraction of Y_0, Y_1 has to involve arbitrarily many elements of Y. For example, by looking at the associated tree-diagram, we could insert another subdivision in direction c as in the figure below to obtain a Y' with 13 vertices. As before let Y_0 and Y_1 simple contractions involving $\{1, 2\}$ (with colour b) and $\{2, 3\}$ (with colour a) respectively. Here, any contraction of both, Y_0 and Y_1, would involve 13 elements.

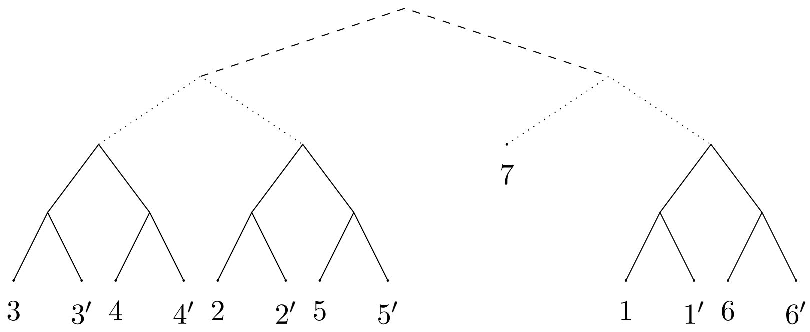


Figure 10

The effect of this in the representation of Figure 9 would be to halve each of the cubes 1, 2, 3, 4, 5 and 6 (with a plane parallel to the plane between 3 and 4) to yield the new cubes 1, 1', etc.

Proposition 4.9. *Let $A \leq \Omega = \{Y_0, \dots, Y_t\}$ where $t \geq 1$ and Y_i are simple contractions of Y. Assume further that there are admissible sets $A \leq A_t \leq A_{t-1} \leq \dots \leq A_0$ such that for each i we have $A_i \leq Y_i$ and the elements involved in Y_i are locally maximal with respect to A_i. Then for $M = glb_A(\Omega)$,*

$$M \in C_{4t}.$$

Proof. We may subdivide Ω into its connected components

$$\Omega = \bigcup_{i=1}^{r} \Omega_i.$$

For any $i \in \{1, \dots, r\}$ there is $j_i \in \{0, 1, \dots, t\}$ such that $A_{j_i} \leq Y_{l_i}$ for any $Y_{l_i} \in \Omega_i$ with the elements of Y contracted in Y_{l_i} locally maximal with respect to A_{j_i} (recall that Ω_i is connected). Put $M_i = \mathrm{glb}_A(\Omega_i)$.

If Ω_i contains at least two different contractions, Lemma 4.7 gives that its connected component in Γ_A is a square. In particular Ω_i is contained in the set of four contractions representing the four sides of the square. Moreover, by the observation after Lemma 4.7, $M_i \in C_4$.

On the other hand, if all the elements of Ω_i are equal to some Z, then $M_i = Z \in C_2$. Clearly, all M_i are pairwise disjoint so if we put $M = \mathrm{glb}_A(\{M_1, \dots, M_r\})$, then $M = \mathrm{glb}_A(\Omega)$ and Lemma 3.11 implies for $r \leq t$

$$|\text{vertices contracted in } M| \leq \sum_{i=1}^{r} |\text{vertices contracted in } M_i| \leq 4r \leq 4t.$$

If $r = t + 1$ then the elements of Ω are pairwise disjoint and by Lemma 3.11 $M \in C_{2t+2} \subseteq C_{4t}$.

□

Now we are ready to prove Theorem 4.6.

Proof. (of Theorem 4.6) Fix any map

$$M : K_Y \to \{\text{Simple contractions of } Y\}$$

such that for any $A \in K_Y$, if i is any of the elements contracted in $M(A)$, then i is locally maximal with respect to A. We extend the above map M to a map

$$M : \{\text{Poset of simplices of } K_Y\} \to K_Y$$

as follows: for any t-simplex $\sigma : A_t < A_{t-1} < \ldots < A_0$ we put

$$M(\sigma) := \mathrm{glb}_{A_t}(M(A_t), \ldots, M(A_1), M(A_0)).$$

Proposition 4.9 and Lemma 3.14 imply that M is a well defined order reversing poset map and that

$$A_t \leq M(\sigma) \in C_{4t}.$$

□

4.3. **Construction of the null-homotopy.**

Remark 4.10. Denote by X^t the t-skeleton of a simplicial complex X. A simplicial complex X is t-connected if it is 0-connected, i.e. path-connected, and its t-th homotopy group vanishes. As $\pi_t(X, x_0) = [S^t, s_0; X, x_0]$, this means that every continuous pointed map

$$\mu : (S^t, s_0) \xrightarrow{\nu} (X^t, x_0) \xrightarrow{i_t} (X, x_0)$$

is null-homotopic, i.e. homotopic to the constant map in (X, x_0). Note, if i_t is null-homotopic, then the composition $\mu = i_t \circ \nu$ will also be null-homotopic. We aim to show that i_t is null-homotopic for $|Y|$ big enough and $X = |K_Y|$.

Because of the following general result the poset map M constructed in Theorem 4.6 will be useful.

Lemma 4.11. *Let $\mathfrak{P}$ be a poset and consider an order reversing poset map*

$$M : \{\textit{Poset of simplices of } \mathfrak{P}\} \to \mathfrak{P},$$

such that for any $\sigma : A_t < \ldots < A_0$, $A_t \leq M(\sigma)$ in $\mathfrak{P}$. Then M induces a map

$$f_t : |\mathfrak{P}|^t \to |\mathfrak{P}|$$

which is homotopy equivalent in $|\mathfrak{P}|$ to the inclusion $i_t : |\mathfrak{P}|^t \to |\mathfrak{P}|$ and such that $f_t(\sigma)$ is contained in the realization of the subposet of those $B \in \mathfrak{P}$ such that $M(\sigma) \leq B$.

Proof. Consider the map

$$h : \{\text{Poset of simplices of } \mathfrak{P}\} \to \mathfrak{P}$$

such that $h(\sigma) = A_t$. Then as $h(\sigma) \leq M(\sigma)$ by a classical result in posets [1, 6.4.5] we have $M \simeq h$. This means that $|h| \simeq |M|$. Denote $j : \mathfrak{P} \to \{\text{Poset of simplices of } \mathfrak{P}\}$ the inclusion, then $h \circ j = 1_{\mathfrak{P}}$. Therefore $|1_{\mathfrak{P}}| \simeq |M \circ j|$. Considering the composition

$$f_t : |\mathfrak{P}|^t \stackrel{i_t}{\to} |\mathfrak{P}| \stackrel{|j|}{\to} |\{\text{Poset of simplices of } \mathfrak{P}\}| \stackrel{|M|}{\to} |\mathfrak{P}|$$

we deduce $f_t = |M| \circ |j| \circ i_t \simeq i_t$. Finally note that $|j|$ takes any simplex σ to the geometric realisation of the poset of those simplices δ such that $\delta \subseteq \sigma$. Thus $f_t(\sigma)$ is contained in the realization of the subposet of those $B \in \mathfrak{P}$ such that $M(\sigma) \leq B$. □

As a corollary of Definition 4.1, Theorem 4.6 and Lemma 4.11 we obtain the following result.

Proposition 4.12. *For any t there is a map*

$$f_t : |K_Y|^t \to |K_Y|$$

which is homotopy equivalent to the inclusion $i_t : |K_Y|^t \to |K_Y|$ and such that $f_t(\sigma) \subseteq \Sigma^t_{4t}$.

Lemma 4.13. *For any fixed r, t there exists a function $\nu_r(t)$ such that if $|Y| \geq \nu_r(t)$, the inclusion of Σ^t_r in $|K_Y|$ is null-homotopic.*

Proof. We adapt Brown's argument in [8, 4.20] to our context. For $|Y|$ big enough we will construct, by induction on t, a null-homotopy

$$F_t : \Sigma^t_r \times I \to |K_Y|$$

such that $F_t(-,0)$ is the identity map and $F_t(-,1)$ is the constant map sending everything to the point $a \in K_Y$. More precisely, we do the following: we show that there are functions $\nu_r(t)$, $\mu_r(t)$ such that for $|Y| \geq \nu_r(t)$ there is a homotopy F_t as before, such that for any t-simplex $\sigma \in \Sigma^t_r$, $F_t(\sigma \times I) \subseteq \widehat{\Sigma}_{\mu_r(t)}$, where $\widehat{\Sigma}_s$ is the set of subcomplexes T of Σ_s such that the union of all elements of Y that are contracted in the vertex of some simplex of T has at most s elements.

The case $t = 0$: We choose any simple contraction a of Y. Hence it involves 2 vertices, i.e. elements of Y. Let A be a point of Σ^0_r i.e. A is a contraction of Y involving at most r vertices. Now, if $|Y| \geq r + 4$, we may choose a set of 2 vertices disjoint to both those contracted in A and those contracted in a. Let b_0 be a simple contraction of any colour of Y corresponding to these two vertices. Then

$$A \geq gglb(A, b_0) \leq b_0 \geq gglb(b_0, a) \leq a$$

is a path linking A with a in $\widehat{\Sigma}^0_{r+4}$. Therefore we get the claim with

$$\nu_r(0) = r + 4,$$

$$\mu_r(0) = r + 4.$$

Induction step: We assume there is a null-homotopy $F_{t-1} : \Sigma^{t-1}_r \times I \to K_Y$. We want to extend F_{t-1} to F_t. Let $\sigma : A_t < A_{t-1} < \ldots < A_0$

be a t-simplex in Σ_r^t. For any face τ of σ of dimension $t-1$ we have $F_{t-1}(\tau \times I) \subseteq \widehat{\Sigma}_{\mu_r(t-1)}$. This means that if we denote $\delta\sigma = \cup_{i=1}^{t+1}\tau_i$, then

$$\Delta := F_{t-1}(\delta\sigma \times I) = \cup F_{t-1}(\tau_i \times I) \subseteq \widehat{\Sigma}_{(t+1)\mu_r(t-1)}.$$

Now, if $|Y| \geq 2 + (t+1)\mu_r(t-1)$ there are at least 2 vertices of Y not involved in any contraction in $F_{t-1}(\delta\sigma \times I)$. Let b be a simple contraction of any colour of Y contracting these 2 vertices.

We claim that the homotopy F_{t-1} can be extended to $F_t : \sigma \times I \to |K_Y|$ with

$$F_t(\sigma \times I) \subseteq \widehat{\Sigma}_{2+(t+1)\mu_r(t-1)}.$$

As b is a contraction of Y disjoint to all contractions A of Y such that $A \in F_{t-1}(\delta\sigma \times I)$, we may consider the global greatest lower bound of b and A which we denote $\mathrm{gglb}(A, b)$. Note that this is just the result of contracting in A those elements which are contracted in b. Analogously we denote by $\mathrm{gglb}(\Delta, b)$ the subcomplex given by $\mathrm{gglb}(A, b)$ for all $A \in \Delta$. The same notation is also used for simplices in Δ. Also note that for all $A \in \Delta$, $\mathrm{gglb}(A, b) \leq b$ and we can always form the cone with base $\mathrm{gglb}(\Delta, b)$ and vertex b.

We claim that the homotopy $F_t(\sigma \times I)$ can be built up by gluing:

i) the cylinder given by Δ and $\mathrm{gglb}(\Delta, b)$
ii) the cone formed by $\mathrm{gglb}(\Delta, b)$ and b.

Note that for any l-simplex $\tau : A_l < A_{l-1} < \ldots < A_0$ lying in Δ then the following $l+1$-simplices:

$$\mathrm{gglb}(A_l, b) < \mathrm{gglb}(A_{l-1}, b) < \ldots < \mathrm{gglb}(A_i, b) < A_i < A_{i-1} < \ldots < A_0$$

for $i = l, \ldots, 0$ fill up the cylinder formed by τ and $\mathrm{gglb}(\tau, b)$ (recall that $\mathrm{gglb}(\tau, b)$ is given by $\mathrm{gglb}(A_l, b) < \mathrm{gglb}(A_{l-1}, b) < \ldots < \mathrm{gglb}(A_0, b)$).

Furthermore, the cone formed by $\mathrm{gglb}(\tau, b)$ and b is also filled up via the $t+1$-simplex

$$\mathrm{gglb}(A_l, b) < \mathrm{gglb}(A_{l-1}, b) < \ldots < \mathrm{gglb}(A_0, b) < b.$$

We shall now explain how the above constructions yield the extension of the homotopy :

(1) Consider the cylinder with base the simplex σ and top the simplex $gglb(\sigma, b)$ and glue to the cylinder the cone with base $gglb(\sigma, b)$ and apex b.

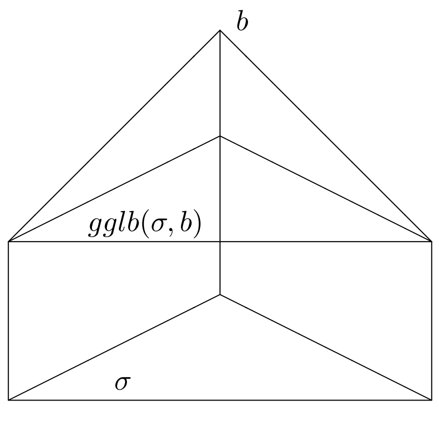


Figure 11

Let $\sigma \cup \widetilde{\Sigma}$ be the boundary of Figure 11. Then σ is homotopic to $\widetilde{\Sigma}$ via a homotopy, see Figure 11, fixing $\partial\sigma$.
(2) The following picture illustrates the homotopy F_{t-1} squeezing $\partial\sigma$ to the point a.

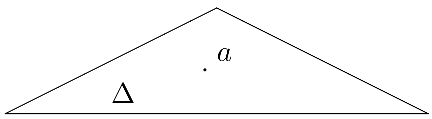


Figure 12

(3) Consider the cylinder with bottom Δ and top $gglb(\Delta, b)$ and glue to it the cone with bottom $gglb(\Delta, b)$ and vertex b.

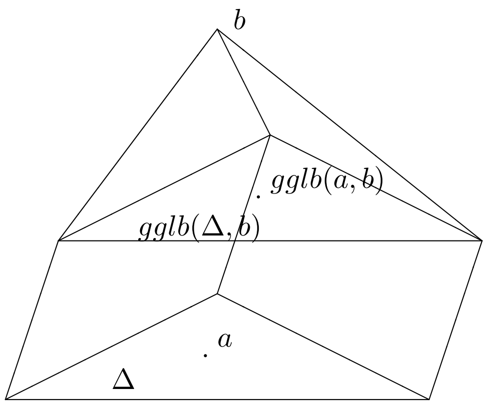


Figure 13

Note that $\Delta \cup \widetilde{\Sigma}$ is the boundary of Figure 13. Thus $\widetilde{\Sigma}$ and Δ are homotopy equivalent via a homotopy, see Figure 13, fixing $\partial\Delta = \partial\sigma$. Set $\mu_r(t) = 2 + (t+1)\mu_r(t-1)$. Then by (1) and (3) σ and Δ are homotopy equivalent via a homotopy, inside $\widehat{\Sigma}_{\mu_r(t)}$, which fixes $\partial\sigma$. This completes the proof of the fact that F_{t-1} is extendable to a homotopy F_t (inside $\widehat{\Sigma}_{\mu_r(t)}$) that contracts σ to the point a. Therefore the inductive step is proven for

$$\nu_r(t) = \mu_r(t) = 2 + (t+1)\mu_r(t-1).$$

□

Theorem 4.14. *There exists a function $\alpha(t)$ such that if $|Y| \geq \alpha(t)$, the inclusion of $|K_Y|^t$ in $|K_Y|$ is null-homotopic.*

Proof. Consider the homotopy equivalent maps $i_t, f_t : |K_Y|^t \to |K_Y|$ given by Proposition 4.12. Since the image of f_t is contained in Σ^t_{4t}, f_t factors through the inclusion of Σ^t_{4t} in K_Y. But we have just proven that this last inclusion is null-homotopic whenever $|Y| \geq \nu_{4t}(t)$ and therefore in that case f_t and i_t are also null-homotopic. Therefore it suffices to set $\alpha(t) := \nu_{4t}(t)$. $\square$

Corollary 4.15. *There exists a function $\alpha(t)$ such that if $|Y| \geq \alpha(t)$, K_Y is t-connected.*

4.4. Finiteness properties of $2V$.

Now we are ready to prove that the group $2V$ is of type FP_∞. To do that, we will verify the conditions of [8, Cor. 3.3] with respect to the complex $|\mathfrak{A}|$ defined in Definition 3.1. As before, consider the filtration of $|\mathfrak{A}|$ given by

$$\mathfrak{A}_n := \{Y \in \mathfrak{A} \mid |Y| \leq n\}.$$

Lemmas 3.5 and 3.7 and Remark 3.6 imply that all that remains is to prove the following theorem.

Theorem 4.16. *The connectivity of the pair of complexes $(|\mathfrak{A}_{n+1}|, |\mathfrak{A}_n|)$ tends to infinity as $n \to \infty$.*

Proof. We use the same argument as in [8, 4.17] i.e. note that $|\mathfrak{A}_{n+1}|$ is obtained from $|\mathfrak{A}_n|$ by gluing cones with base K_Y and top Y for every $Y \in \mathfrak{A}_{n+1} \smallsetminus \mathfrak{A}_n$. By Corollary 4.15, if $n+1 \geq \alpha(t)$ we have that K_Y is t-connected, hence $(|\mathfrak{A}_{n+1}|, |\mathfrak{A}_n|)$ is t-connected. $\square$

Theorem 4.17. *The Brin group on 2 colours each of arity 2 i.e. $2V$, is of type F_∞.*

Proof. By Lemmas 3.5 and 3.7, Remark 3.6 and Theorem 4.16 we may apply [8, Cor. 3.3]. $\square$

Remark 4.18. As a by-product, we get by [8, Cor. 3.3] a new proof of the fact that $2V$ is finitely presented. This was first proved in [7], where an explicit finite presentation was constructed.

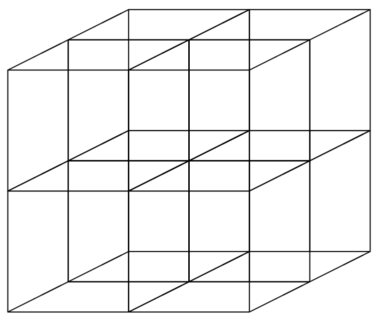
5. The case $s = 3$

In this section we consider Brin's group sV for $s = 3$. Our objective is to show that $3V$ is of type F_∞ by adapting the construction of the function M of Lemma 4.11 to the case $s = 3$. In particular we show that Theorem 4.6 holds with $M \in C_{8t}$. This immediately leads to a modification of Proposition 4.12 that $f_t(\sigma) \in \Sigma^t_{8t}$. The rest of the proof will be analogous to the previous case.
As before, we fix a Y and prove that K_Y is t-connected if $|Y|$ is sufficiently large. For $A < Y$ we consider the coloured graph Γ_A as in Definition 4.5. This time the graph is embedded in 3 dimensional real space and the three possible colours $\{a, b, c\}$ correspond to the axes of the standard coordinate system of $\mathbb{R}^3$. For any subgraph $\Delta \subseteq \Gamma_A$ we put

$$\mathrm{glb}_A(\Delta) := \mathrm{glb}_A\{\text{Simple contractions associated to the edges of } \Delta\}.$$

Consider a connected component Δ of Γ_A. The vertices of Δ correspond, via the geometric realisation of $3V$, to subparallelepipeds of the unit cube I, all of the same shape and size. For simplicity, we draw them as cubes and call them subcubes. Let i be an element (i.e. a vertex) of Δ. By some

abuse of notation we shall also label by i the subcube corresponding to the element i of Δ.
We claim that the vertices of Δ are inside a stack of 8 subcubes, see Figure 14. Obviously one of these subcubes corresponds to i. Observe that we do not claim that all the subcubes in the stack correspond to elements of Y, only that Δ is a set consisting of some of the subcubes in the stack. To see that the claim holds, let i be $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \times [\gamma_1, \gamma_2]$. The interval $A_0 = [\alpha_1, \alpha_2]$ comes from a set of binary subdivisions of $[0,1]$. The left descendant of an interval $[x,y]$ is $[x,(x+y)/2]$ and the right descendant of $[x,y]$ is $[(x+y)/2, y]$. Then A_0 is a descendant of some interval J_A that is subdivided into A_0 and A_1 in the binary subdivision. Recall, see for example Lemma 4.4, that each cube in a connected set can only have one neighbour of each colour/direction. Define B_1 and C_1 analogously. Then the cubes in the stack containing Δ are precisely the cubes $A_i \times B_j \times C_k$, where $i,j,k \in \{0,1\}$.



A stack of 8 cubes

Figure 14

For a connected component Δ of Γ_A we define the enveloping stack of Δ to be the smallest set $U(\Delta)$ of some subcubes from the 8 cube stack defined above such that $U(\Delta)$ contains all $i \in \Delta$, and the union of the elements of $U(\Delta)$ is a cube.
Note that if one of the vertices of Δ is locally maximal with respect to some $C < Y$ such that $A \leq C$ then every vertex of Δ is locally maximal with respect to C. This leads to the following definition.

Definition 5.1. A connected component Δ of Γ_A is called $*$-connected if there is some admissible set C such that $A \leq C < Y$ and every vertex of Δ is locally maximal with respect to C.

The following diagram exhibits possible $*$-connected components of the graph Γ_A for $A < Y$. Note that parallel edges are labeled by the same colour.

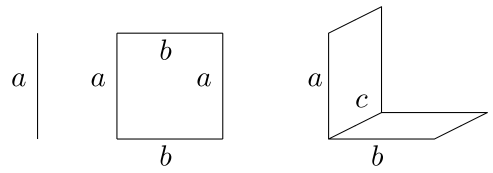


Figure 15

We call the graphs in Figure 15 an edge, a square, an open book and a cube respectively.

Lemma 5.2. *Let Δ be a $*$–connected component of Γ_A. Then, up to changing the colours, Δ is one of the graphs in Figure 15. Moreover, if Δ is not an open book, then for $M = glb_A(\Delta)$ the vertices involved in M lie inside Δ. In particular, $M \in C_8$.*

Proof. We argue as in Lemma 4.7. We consider the element $m \in M$ which yields Δ, i.e. the vertices of Δ are obtained from m by the halving operations. Observe that $M = \{m\} \cup (M \cap Y)$. Consider the geometric realisation of M. Then m is a subcube of the unitary cube and the enveloping stack $U(\Delta)$ lies inside m. Since $M < Y$ we may choose some simple expansion $M < M_1 \leq Y$ of colour a, say. The expansion $M < M_1$ corresponds to halving the cube m by a hyperplane of direction (i.e. colour) a. Furthermore, this halving also yields a halving of the enveloping stack $U(\Delta)$. In other words, not all the vertices of Δ are in the same half of m, as that would mean that $M = M_1$. Moreover, as Δ is connected, this halving can be inverted, by using the commutativity relations, to give a simple contraction of Y of direction a. If $M_1 = Y$, then Δ is an edge and $M \in C_2$.
Note, that since the halving operation of m in direction a halves $U(\Delta)$, we have an edge e in Δ with label a and vertices i, j. In particular, the elements i and j represent neighbouring cubes in $U(\Delta)$, one contained in $m\alpha_a^1$ and the other in $m\alpha_a^2$. Since $e \in \Gamma_A$ there is a contraction of Y contracting precisely i and j. This implies that in the process of obtaining Y from M via halving operations on m, there is another chain of halving operations starting with halving in a direction different from a, say b. Hence, by the commutativity relations, there exists M_2 with $M_1 < M_2 \leq Y$ such that M_2 consists of halving both $m\alpha_a^1$ and $m\alpha_a^2$ in direction b. Clearly, this allows inversion and therefore the above procedure for a can also be applied to b . After performing these two subdivisions we get a stack S of four cubes. Moreover, we may assume that there are vertices of Δ lying in at least three of those four cubes. Otherwise Δ would be either disconnected or $M \neq glb_A(\Delta)$. Note also that, to obtain Δ, only halving of those four cubes in a direction c different from directions a and b is possible. So it remains to consider the three possibilities below. Recall, we are assuming that Δ is $*$-connected.
(1) If none of the cubes is halved, then $M_2 = Y$, Δ is a square and $M \in C_4$.
(2) Suppose all four cubes are halved at least once. Then the rooted tree representing the way Δ is obtained from m, starts as the first tree in Figure 16 below. In this case we may use the commutativity relations to get a rooted tree with halving in direction c at the beginning. Therefore, the

assumptions that Δ is connected and that $M = \mathrm{glb}_A(\Delta)$ imply that in fact there is only one halving in direction c. In particular, the rooted tree is precisely the first tree in Figure 16. Thus Δ is a cube, m yields the whole stack of 8 cubes, $M \in C_8$ and M involves precisely the vertices of Δ.
(3) Finally, assume that only three of the four cubes are halved at least once in direction c. Then we may assume that the rooted tree representing the halving operations done on m, begins exactly as the second tree in Figure 16 below. Note that at this point, and as a consequence of the geometric interpretation, we know that Δ is a subgraph of the open book B containing the three edges labeled c. Also, B lies inside the 8 cube stack associated to Δ. Furthermore, the elements of B correspond to elements of Y. We shall show that Δ is exactly the open book B. Since Δ is connected it suffices to show that any two neighbouring cubes in the open book B can be contracted in Y. Consider the admissible set M_a obtained as follows: First, halve m in direction a and assume that the second half of m, i.e. $m\alpha_a^2$, contains the only one of the cubes not cut in direction c. Then perform in $m\alpha_a^2$ all halvings needed to reach those elements of Y that are descendents of $m\alpha_a^2$. The first half of m, $m\alpha_a^1$, is not cut anymore. Then $M \leq M_a$, $M_a = \{m\alpha_a^1\} \cup (M_a \cap Y)$.

Observe that, in the first half of m, there are only two colours in the path needed to obtain the elements of $\Delta \cap \Gamma_{M_a}$ from M_a. As $\Delta \cap \Gamma_{M_a}$ is $*$-connected in Γ_{M_a}, we may apply Lemma 4.7 and deduce that the square of the open book B with edges labeled by b and c is in Δ. The same argument with b substituted by a implies that the square of the open book B with edges labeled by c and a is in Δ. Thus Δ is the open book B.

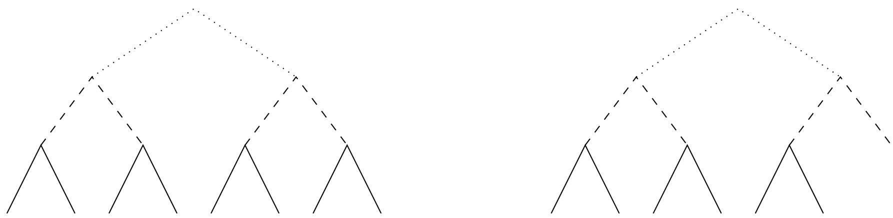


Figure 16

□

We are now ready to prove the analogue to Theorem 4.6 with $M \in C_{8t}$.

Theorem 5.3. *Let $s = 3$. There exists an order reversing poset map*

$$M : \{\textit{Poset of simplices of } |K_Y|\} \to K_Y$$

such that for any t-simplex $\sigma : A_t < A_{t-1} < \ldots < A_0$ we have

$$A_t \leq M(\sigma) \in C_{8t}.$$

Proof. We split the proof into three steps. Fix a linear ordering on the colours a, b, c.

(1) The definition of M on vertices of K_Y. For each admissible $A < Y$ we define a designated edge $M(A)$ as follows:

Consider the graph Γ_A. We define $M(A)$ as an edge of Γ_A such that if $\Gamma_A = \Gamma_B$ for some $B < Y$, then $M(A) = M(B)$. If there is some open book between the $*$-connected components of Γ_A, we define $M(A)$ to be the middle edge of the open book with middle edge of smallest possible colour amongst those open books which are $*$-connected components of Γ_A.

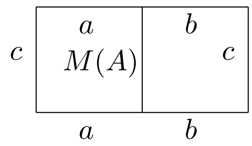


Figure 17: The open book extended

If Γ_A does not have an open book as a $*$-connected component, but contains a $*$-connected component, which is a separate edge e, i.e. case 1 of Figure 15, we define $M(A) = e$. Again, there might be more than one such edge e and we choose e of smallest possible colour.
If Γ_A does not contain $*$-connected components, which are open books or separate edges, we choose $M(A)$ to be an edge of the smallest possible colour of a $*$-connected component of Γ_A.

From now on we write Δ_A for the $*$-connected component of Γ_A such that $M(A) \in \Delta_A$. We can further assume that if $\Delta_A = \Delta_B$ then $M(A) = M(B)$.

(2) Let $A = A_r < A_{r-1} < \ldots < A_0$ be contractions of Y such that all $M(A_i)$ belong to Δ_A. Recall that each $M(A_i)$ is a simple contraction of Y. Let $\Omega = \{M(A_r), \ldots, M(A_0)\}$ and put $N = \mathrm{glb}_A(\Omega)$. We aim to show that $N \in C_8$ and that the vertices of Y involved in N are inside Δ_A.
Observe that Δ_A is $*$-connected. So it must be one of the graphs of Figure 15. If it is an edge, a square or a cube then our claim that $N \in C_8$ follows from Lemma 5.2. So we may assume that Δ_A is an open book. We have

$$\Delta_A = \Delta_A \cap \Gamma_{A_r} \supseteq \ldots \supseteq \Delta_A \cap \Gamma_{A_0}.$$

The definition of M yields that if $\Delta_A = \Delta_A \cap \Gamma_{A_r} = \ldots = \Delta_A \cap \Gamma_{A_0}$ then $M(A_r) = \ldots = M(A_0)$. In this case $N = M(A_r) \in C_2$. So we may assume that there is some $0 \leq i < r$ such that

$$\Delta_A = \Delta_A \cap \Gamma_{A_r} = \ldots = \Delta_A \cap \Gamma_{A_{i+1}} \supsetneq \Delta_A \cap \Gamma_{A_i}.$$

Denote $B = A_i$. We have

$$\Delta_B \subseteq \Delta_A \cap \Gamma_B \subsetneq \Delta_A.$$

Moreover, by the definition of M, $M(A) = M(A_r) = \ldots = M(A_{i+1})$ is the middle edge of the open book Δ_A.
We claim that $\Delta_A \cap \Gamma_B$ is a subgraph of one of the following two graphs:

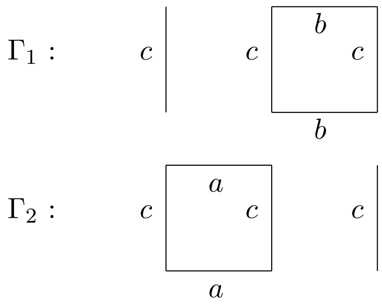


Figure 18

Observe that $\Delta_A \cap \Gamma_B$ is not connected. Indeed, in the process of obtaining B from A there was a cutting of a cube containing $U(\Delta_A)$ which halved $U(\Delta_A)$. The structure of Δ_A as an open book with three parallel edges c implies that such a halving cannot be in direction c. The case when the direction of this halving is a corresponds to Γ_1 (see Fig. 18), i.e. $\Delta_A \cap \Gamma_B \subseteq \Gamma_1$ and the case when the direction is b corresponds to Γ_2, i.e. $\Delta_A \cap \Gamma_B \subseteq \Gamma_2$. Alternatively, consider the second tree in Figure 16. The commutativity relations do not allow us to move c to the top, whereas having a or b at the top yields a disconnected graph. A similar argument shows that there is an expansion $A_{i+1} < \widetilde{B}$ such that $\Delta_A \cap \Gamma_{\widetilde{B}} = \Gamma_k$, where we have fixed one $k \in \{1, 2\}$ such that $\Delta_A \cap \Gamma_B \subseteq \Gamma_k$.

For any $0 \le j \le i$ we also have $M(A_j) \in \Delta_{A_j} \subseteq \Delta_A \cap \Gamma_B$. Then since $\Delta_A \cap \Gamma_B \subseteq \Gamma_k$ we have $\Omega \subset (\Delta_A \cap \Gamma_B) \cup \{M(A)\} \subseteq \Gamma_k = \Delta_A \cap \Gamma_{\widetilde{B}} \subseteq \Gamma_{\widetilde{B}}$. Hence $A < \widetilde{B} \le \Omega$ and so

$$\mathrm{glb}_{\widetilde{B}}(\Gamma_k) \le \mathrm{glb}_{\widetilde{B}}(\Omega) = N.$$

Now split $\Gamma_k = D_1 \cup D_2$ into its connected components, where D_1 is the edge and D_2 is the square. Note that D_1 and D_2 are $*$-connected components of $\Gamma_{\widetilde{B}}$, hence Lemma 5.2 implies thet $glb_{\widetilde{B}}(D_i)$ involves (i.e. contracts) 2^i vertices (i.e. elements) of Y. Then by Lemma 3.11 $glb_{\widetilde{B}}(D_1 \cup D_2)$ contracts $2 + 4 = 6$ vertices of Y. Hence $N \in C_6 \subseteq C_8$.

(3) The definition of M on a simplex of K_Y:
Let $\sigma : A_t < A_{t-1} < \ldots < A_0$ be a simplex of K_Y and $t \ge 1$. Thus $\Gamma_{A_0} \le \ldots \le \Gamma_{A_{t-1}} \le \Gamma_{A_t}$ and we have already defined $M(A_i)$ as an edge of Γ_{A_i} for all i. Let $\Omega = \{M(A_t), M(A_{t-1}), \ldots, M(A_0)\}$, which is a set of edges of Γ_{A_t}.
Consider the following partition of Ω:
Put $\alpha_1 = t$ and

$$\Omega_1 = \Omega \cap \Delta_{A_{\alpha_1}}.$$

Assume Ω_{r-1} is defined. If $\bigcup_{i=1}^{r-1} \Omega_i \neq \Omega$, choose the largest $j \in \{0, ..., t\}$ such that

$$M(A_j) \in \Omega \setminus (\bigcup_{i=1}^{r-1} \Omega_i).$$

Rename A_j to A_{α_r} and put $\Omega_r = \Omega \cap \Delta_{A_{\alpha_r}}$. Hence at each step we have a subchain (i.e. subsimplex) of σ satisfying the conditions of (2).

At some point we will have $\Omega = \bigcup_{i=1}^{k} \Omega_i$. Let

$$N_i := \mathrm{glb}_{A_{\alpha_i}}(\Omega_i).$$

By step (2), $N_i \in C_8$ and the vertices of Y involved in N_i are contained in $\Delta_{A_{\alpha_i}}$. Now we claim that these N_i are pairwise disjoint contractions of Y. To see this, let $i \neq j$. We may assume that $A_{\alpha_i} \leq A_{\alpha_j}$ and therefore $\Gamma_{A_{\alpha_i}} \supseteq \Gamma_{A_{\alpha_j}}$. As $\Delta_{A_{\alpha_i}}$ is a $*$-connected component in $\Gamma_{A_{\alpha_i}}$, we deduce that either $\Delta_{A_{\alpha_i}}$ and $\Delta_{A_{\alpha_j}}$ are disjoint (and in this case N_i and N_j are also disjoint) or $\Delta_{A_{\alpha_j}} \subseteq \Delta_{A_{\alpha_i}}$. But the second case is impossible by the construction of the partition above.
Next we define

$$M(\sigma) = \mathrm{glb}_A(\Omega).$$

Clearly,

$$M(\sigma) = \mathrm{glb}_A(\{N_1, \ldots, N_k\})$$

and, if $k \leq t$, then

$$M(\sigma) \in C_{8k} \subseteq C_{8t}.$$

Finally, if $k = t + 1$ then all Ω_i contain precisely one edge, so for all i we have $N_i = M(A_i)$ and so $M(\sigma) \in C_{2(t+1)} \subseteq C_{8t}$.

□

As a corollary we get the following modified version of Proposition 4.12.

Corollary 5.4. *For any t there is a map*

$$f_t : |K_Y|^t \to |K_Y|$$

which is homotopy equivalent to the inclusion $i_t : |K_Y|^t \to |K_Y|$ such that $f_t(\sigma) \subseteq \Sigma_{8t}^t$.

From now on we can proceed analogously to the case $s = 2$. As a first step we have a three-dimensional analogue to Theorem 4.14:

Corollary 5.5. *Let $s = 3$. There exists a function $\alpha(t)$ such that if $|Y| \geq \alpha(t)$, the inclusion of $|K_Y|^t$ in $|K_Y|$ is null-homotopic.*

Proof. Follow the proofs of Theorem 4.14 and Lemma 4.13 substituting Proposition 4.12 with Corollary 5.4. □

Theorem 5.6. *The Brin group $3V$ on 3 colours of arity 2 is of type F_∞.*

Proof. The proof follows the proof of Theorem 4.17. The main point is the construction of the poset map M of Theorem 5.3. Applying Corollary 5.5, the rest follows as before. □

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Dessislava H. Kochloukova, Department of Mathematics, University of Campinas, Cx. P. 6065, 13083-970 Campinas, SP, Brazil
E-mail address: desi@unicamp.br

Conchita Martínez-Pérez, Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail address: conmar@unizar.es

Brita E. A. Nucinkis, School of Mathematics, University of Southampton, Southampton, SO17 1BJ, United Kingdom
E-mail address: bean@soton.ac.uk